General Solution and Hyers–Ulam Stability of DuoTrigintic Functional Equation in Multi-Banach Spaces

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Abstract In this paper, we introduce the general form of a new duotrigintic functional equation. Then, we find the general solution and study the generalized Hyers–Ulam stability of such functional equation in multi-Banach spaces by employing fixed point technique. Also, we give an example for non-stability cases for this new functional equation.

1 Introduction

Stability problem of a functional equation was first posed by Ulam [36] and that was partially answered by Hyers [14] and then generalized by Aoki [1] and Rassias [26] for additive mappings and linear mappings, respectively. In 1994, a generalization of Rassias theorem was obtained by Gâvruta [13], who replaced $\epsilon \left(\|x\|^p + \|y\|^p\right)$ by a general control function $\phi(x, y)$. This idea is known as generalized Hyers–Ulam–Rassias stability. After that, the general stability problems of various functional equations such as quadratic [8], cubic [3, 5, 17, 28], quartic [3, 4, 27], quintic [39], sextic [39], septic and octic [38], nonic [6, 29, 30], decic [2], undecic [32], quattuordecic [33], hexadecic [22], octadecic [23], vigintic [25], viginticduo [19], quattuorvigintic [15, 24, 31], octavigintic [16] and trigintic [7] functional equations have been investigated by a number of authors with more general domains and co-domains.

2 Preliminaries

In this section, we recall some basic concepts concerning Multi-Banach Spaces. The Multi-Banach Spaces were first investigated by Dales and Polyakov [10]. Theory of Multi-Banach Spaces is similar to operator sequence space and has some

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connections with operator spaces and Banach Spaces. In 2007 Dales and Moslehian [9] first proved the stability of mappings and also gave some examples on multi-normed spaces. The asymptotic aspects of the quadratic functional equations in multi-normed spaces were investigated by Moslehian et al. [21] in 2009. In the last two decades, the stability of functional equations on multi-normed spaces was proved by many mathematicians (see [12, 18, 34, 35, 37, 40]). Let $(\wp, \|\cdot\|)$ be a complex normed space, and let $k \in \mathbb{N}$. We denote by $\wp^k$ the linear space $\wp \oplus \wp \oplus \wp \oplus \ldots \oplus \wp$ consisting of k- tuples $(x_1, \ldots, x_k)$ where $x_1, \ldots, x_k \in \wp$. The linear operations on $\wp^k$ are defined coordinate wise. The zero element of either $\wp$ or $\wp^k$ is denoted by 0. We denote by $\mathbb{N}_k$ the set $\{1, 2, \ldots, k\}$ and by Ψ_k the group of permutations on k symbols.

Definition 1 ([9]) A multi-norm on $\left\{\wp^k : k \in \mathbb{N}\right\}$ is a sequence $(\|.\|) = (\|.\|_k : k \in \mathbb{N})$ such that $\|.\|_k$ is a norm on $\wp^k$ for each $k \in \mathbb{N}, \|x\|_1 = \|x\|$ for each $x \in \wp$, and the following axioms are satisfied for each $k \in \mathbb{N}$ with $k \geq 2$:

1. $\left\|\left(x_{\sigma(1)}, \ldots, x_{\sigma(k)}\right)\right\|_k = \|(x_1 \ldots x_k)\|_k$, for $\sigma \in \Psi_k, x_1, \ldots, x_k \in \wp$;
2. $\|(\alpha_1 x_1, \ldots, \alpha_k x_k)\|_k \leq \left(\max_{i \in \mathbb{N}_k} |\alpha_i|\right) \|(x_1 \ldots x_k)\|_k$ for $\alpha_1 \ldots \alpha_k \in \mathbb{C}, x_1, \ldots, x_k \in \wp$;
3. $\|(x_1, \ldots, x_{k-1}, 0)\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$, for $x_1, \ldots, x_{k-1} \in \wp$;
4. $\|(x_1, \ldots, x_{k-1}, x_{k-1})\|_k = \|(x_1, \ldots, x_{k-1})\|_{k-1}$ for $x_1, \ldots, x_{k-1} \in \wp$.

In this case, we say that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space.

Suppose that $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-normed space, and take $k \in \mathbb{N}$. We need the following two properties of multi-norms. They can be found in [9].

$$(a)\ \|(x, \ldots x)\|_k = \|x\|, \forall x \in \wp,$$

$$(b)\ \max_{i \in \mathbb{N}_k} \|x_i\| \leq \|(x_1, \ldots, x_k)\|_k \leq \sum_{i=1}^{k} \|x_i\| \leq k \max_{i \in \mathbb{N}_k} \|x_i\|, \forall x_1, \ldots, x_k \in \wp.$$

It follows from (b) that if $(\wp, \|.\|)$ is a Banach Space, then $(\wp^k, \|.\|_k)$ is a Banach Space for each $k \in \mathbb{N}$;

In this case, $\left((\wp^k, \|.\|_k) : k \in \mathbb{N}\right)$ is a multi-Banach space.

3 The Fixed Point Method

The fixed point method is one of the most dynamic areas of research during the last 60 years with lot of applications in various fields of pure and applied mathematics. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a generalized metric on X if d satisfies

1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

The following fixed point theorem proved by Diaz and Margolis [11] plays an important role in proving our theorem:

Theorem 1 ([11]) *Let $(\mathcal{X}, d)$ be a complete generalized metric space and let $\mathcal{J} : \mathcal{X} \to \mathcal{X}$ be a strictly contractive mapping with Lipschitz constant $\mathcal{L} < 1$. Then for each given element $x \in \mathcal{X}$, either*

$$d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(i) $d(\mathcal{J}^n x, \mathcal{J}^{n+1} x) < \infty$ for all $n \geq n_0$;
(ii) The sequence $\{\mathcal{J}^n x\}$ is convergent to a fixed point y^* of $\mathcal{J}$;
(iii) y^* is the unique fixed point of T in the set $Y = \{y \in \mathcal{X} : d(\mathcal{J}^{n_0} x, y) < \infty\}$;
(iv) $d(y, y^*) \leq \frac{1}{1-\mathcal{L}} d(y, \mathcal{J} y)$ for all $y \in Y$.

Let X and Y be real vector spaces. For convenience, we use the following abbreviation for a mapping $f : X \to Y$

$$\begin{aligned}
Df(x, y) &= f(x + 16y) - 32f(x + 15y) + 496f(x + 14y) - 4960f(x + 13y) \\
&+35960f(x+12y) - 201376f(x+11y) + 906192f(x+10y) - 3365856f(x+9y) \\
&+10518300f(x + 8y) - 28048800f(x + 7y) + 64512240f(x + 6y) \\
&-129024480f(x + 5y) + 225792840f(x + 4y) - 347373600f(x + 3y) \\
&+471435600f(x + 2y) - 565722720f(x + y) + 601080390f(x) \\
&-565722720f(x - y) + 471435600f(x - 2y) - 347373600f(x - 3y) \\
&+225792840f(x - 4y) - 129024480f(x - 5y) + 64512240f(x - 6y) \\
&-28048800f(x - 7y) + 10518300f(x - 8y) - 3365856f(x - 9y) \\
&+906192f(x - 10y) - 201376f(x - 11y) + 35960f(x - 12y)
\end{aligned}$$

$$-4960f(x-13y) + 496f(x-14y) - 32f(x-15y) + f(x-16y) - 32!f(y) \quad (1)$$

for all $x, y \in X$, where $32! = 2.631308369 \times 10^{35}$.

In this paper, we introduce the Duotrigintic functional equation:

$$Df(x, y) = 0.$$

for all $x, y \in X$. Moreover, we prove the stability of the Duotrigintic functional equation (1) in Multi-Banach Spaces by using fixed point method. It is easy to show that the function $f(x) = x^{32}$ satisfies the functional equation (1), which is called as duotrigintic functional equation and every solution of the duotrigintic functional equation is said to be a duotrigintic mapping.

4 General Solution of Duotrigintic Functional Equation in (1)

Theorem 2 *Let X and Y be vector spaces. If $f : X \to Y$ is the function (1) for all $x, y \in X$, then f is a Duotrigintic mapping.*

Proof Substituting $x = 0$ and $y = 0$ in (1), we obtain that $f(0) = 0$. Substituting (x, y) with (x, x) and $(x, -x)$ in (1), respectively, and subtracting two resulting equations, we can arrive at $f(-x) = f(x)$, that is to say, f is an even function.

Letting (x, y) by $(16x, x)$ and $(0, 2x)$ respectively in (1), and subtracting the two resulting equations, we arrive at
$32f(31x) - 528f(30x) + 4960f(29x) - 35464f(28x) + 201376f(27x)$
$-911152f(26x) + 3365856f(25x) - 10482340f(24x)$
$+28048800f(23x) - 64713616f(22x) + 129024480f(21x)$
$-224886648f(20x) + 347373600f(19x) - 474801456f(18x)$
$+565722720f(17x) - 590562090f(16x) + 565722720f(15x)$
$-499484400f(14x)+347373600f(13x)-161280600f(12x)+129024480f(11x)$
$-193536720f(10x) + 28048800f(9x) + 215274540f(8x) + 3365856f(7x)$
$-348279792f(6x) + 201376f(5x) + 471399640f(4x) + 4960f(3x)$

$$-\frac{32!}{2}f(2x) + 32!f(x) = 0 \tag{2}$$

for all $x \in X$. Substituting $x = 15x$ and $y = x$ in (1), further multiplying the resulting equation by 32, and subtracting the obtained result from (2), we get
$496f(30x) - 10912f(29x) + 123256f(28x) - 949344f(27x) + 5532880f(26x)$
$-25632288f(25x) + 97225052f(24x) - 308536800f(23x) + 852847984f(22x)$
$-1935367200f(21x) + 3903896712f(20x) - 6877997280f(19x)$
$+1.064115374 \times 10^{10}f(18x) - 1.452021648 \times 10^{10}f(17x)$
$+1.751256495 \times 10^{10}f(16x) - 1.866884976 \times 10^{10}f(15x)$
$+1.760364264 \times 10^{1}0f(14x) - 1.47385656 \times 10^{10}f(13x)$
$+1.09546746 \times 10^{10}f(12x) - 7096346400f(11x) + 3935246640f(10x)$
$-2036342880f(9x) + 1112836146f(8x) - 333219744f(7x)$
$-240572400f(6x) - 28796768f(5x) + 477843672f(4x) - 1145760f(3x)$

$$-\frac{32!}{2}f(2x) + 32!(33)f(x) = 0 \tag{3}$$

for all $x \in X$. Replacing (x, y) with $(14x, x)$ in (1), further multiplying the resulting equation by 496, and subtracting the obtained result from (3), we have
$4960f(29x) - 122760f(28x) + 1510816(27x) - 12303280f(26x)$
$+74250208f(25x) - 352246180f(24x) + 1360927776f(23x)$
$-43842288f(22x) + 1.19768376 \times 10^{10}f(21x)$
$-2.809417433 \times 10^{10}f(20x) + 5.71181448 \times 10^{10}f(19x)$
$-1.013520949 \times 10^{11}f(18x) + 1.577770891 \times 10^{11}f(17x)$

$-2.163194927 \times 10^{11} f(16x) + 2.619296193 \times 10^{11} f(15x)$
$-2.805322308 \times 10^{11} f(14x) + 2.658599035 \times 10^{11} f(13x)$
$-2.22877383 \times 10^{11} f(12x) + 1.652009592 \times 10^{11} f(11x)$
$-1.08058002 \times 10^{11} f(10x) + 6.19597992 \times 10^{1}0 f(9x)$
$-3.08852349 \times 10^{10} f(8x) + 1.357898506 \times 10^{10} f(7x)$
$-5457649200 f(6x) + 1640667808 f(5x) + 28372440 f(4x) + 98736736 f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(529) f(x) = 0 \quad (4)$$

for all $x \in X$. Replacing (x, y) with $(13x, x)$ in (1), further multiplying the resulting equation by 4960, and subtracting the obtained result from (4), we get
$35960 f(28x) - 949344 f(27x) + 12298320 f(26x) - 104111392 f(25x)$
$+646578780 f(24x) - 3133784544 f(23x)$
$+1.231041694 \times 10^{10} f(22x) - 4.01939304 \times 10^{10} f(21x)$
$+1.110278737 \times 10^{11} f(20x) - 2.628625656 \times 10^{11} f(19x)$
$+5.386093259 \times 10^{11} f(18x) - 9.621553973 \times 10^{11} f(17x)$
$+1.506653563 \times 10^{12} f(16x) - 2.076390957 \times 10^{12} f(15x)$
$+2.52545246 \times 10^{12} f(14x) - 2.715498831 \times 10^{12} f(13x)$
$+2.583107308 \times 10^{12} f(12x) - 2.173119617 \times 10^{12} f(11x)$
$+1.614915054 \times 10^{12} f(10x) - 1.057972687 \times 10^{12} f(9x)$
$+6.090761859 \times 10^{11} f(8x) - 3.064017253 \times 10^{11} f(7x)$
$+1.336643988 \times 10^{11} f(6x) - 5.053010019 \times 10^{10} f(5x)$
$+1.67230182 \times 10^{10} f(4x) - 4395980544 f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(5489) f(x) = 0 \quad (5)$$

for all $x \in X$. Plugging (x, y) into $(12x, x)$ in (1), further multiplying the resulting equation by 35960, and subtracting the obtained result from (5), we have
$201376 f(27x) - 5537840 f(26x) + 74250208 f(25x)$
$-646542820 f(24x) + 4107696416 f(23x)$
$-2.027624738 \times 10^{10} f(22x) + 8.084225136 \times 10^{10} f(21x)$
$-2.672101943 \times 10^{11} f(20x) + 7.457722824 \times 10^{11} f(19x)$
$-1.781250825 \times 10^{12} f(18x) + 3.677564904 \times 10^{12} f(17x)$
$-6.612856963 \times 10^{12} f(16x) + 1.04151637 \times 10^{13} f(15x)$
$-1.442737172 \times 10^{13} f(14x) + 1.762789018 \times 10^{13} f(13x)$
$-1.903174351 \times 10^{13} f(12x) + 1.817026939 \times 10^{13} f(11x)$
$-1.533790913 \times 10^{13} f(10x) + 1.43358197 \times 10^{13} f(9x)$
$-7.51043434 \times 10^{12} f(8x) + 4.333318575 \times 10^{12} f(7x)$
$-2.186195751 \times 10^{12} f(6x) + 9.581047478 \times 10^{11} f(5x)$
$-3.615150858 \times 10^{11} f(4x) + 1.16641352 \times 10^{11} f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(41449) f(x) = 0 \quad (6)$$

for all $x \in X$. Plugging (x, y) into $(11x, x)$ in (1), further multiplying the resulting equation by 201376, and subtracting the obtained result from (6), we arrive at
$906192 f(26x) - 25632288 f(25x) + 352282140 f(24x) - 3133784544 f(23x)$
$+2.0276046 \times 10^{10} f(22x) - 1.016430688 \times 10^{11} f(21x)$
$+4.105924235 \times 10^{11} f(20x) - 1.372360898 \times 10^{12} f(19x)$
$+3.867104325 \times 10^{12} f(18x) - 9.313651939 \times 10^{12} f(17x)$
$+1.936957672 \times 10^{13} f(16x) - 3.505409525 \times 10^{13} f(15x)$
$+5.552533436 \times 10^{13} f(14x) - 7.730792521 \times 10^{13} f(13x)$
$+9.489123499 \times 10^{13} f(12x) - 1.028728952 \times 10^{14} f(11x)$
$+9.858506937 \times 10^{13} f(10x) - 8.350223342 \times 10^{13} f(9x)$
$+6.244227173 \times 10^{13} f(8x) - 4.113594037 \times 10^{13} f(7x)$
$+2.379623793 \times 10^{13} f(6x) - 1.203311299 \times 10^{13} f(5x)$
$+5.286846507 \times 10^{12} f(4x) - 2.001591711 \times 10^{12} f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(242825) f(x) = 0 \tag{7}$$

for all $x \in X$. Plugging (x, y) into $(10x, x)$ in (1), further multiplying the resulting equation by 906192, and subtracting the obtained result from (7), we have
$3365856 f(25x) - 97189092 f(24x) + 1360927776 f(23x)$
$-1.231061832 \times 10^{10} f(22x) + 8.084225136 \times 10^{10} f(21x)$
$-4.105915173 \times 10^{11} f(20x) + 1.677750882 \times 10^{12} f(19x)$
$-5.664494989 \times 10^{12} f(18x) + 1.610394623 \times 10^{13} f(17x)$
$-3.909089907 \times 10^{13} f(16x) + 8.186685635 \times 10^{13} f(15x)$
$-1.490863309 \times 10^{14} f(14x) + 2.374792521 \times 10^{14} f(13x)$
$-3.323199342 \times 10^{14} f(12x) + 4.097805079 \times 10^{14} f(11x)$
$-4.461091714 \times 10^{14} f(10x) + 4.291511697 \times 10^{14} f(9x)$
$-3.647688975 \times 10^{14} f(8x) + 2.73651237 \times 10^{14} f(7x)$
$-1.808154283 \times 10^{14} f(6x) + 1.048878683 \times 10^{14} f(5x)$
$-5.317407875 \times 10^{13} f(4x) + 2.342050117 \times 10^{13} f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(1149017) f(x) = 0 \tag{8}$$

for all $x \in X$. Replacing (x, y) with $(9x, x)$ in (1), further multiplying the resulting equation by 3365856, and subtracting the obtained result from (8), we get
$10518300 f(24x) - 308536800 f(23x) + 4384027440 f(22x) - 4.01939304 \times 10^{10} f(21x) + 2.672111005 \times 10^{11} f(20x) - 1.372360898 \times 10^{12} f(19x)$
$+5.664491624 \times 10^{12} f(18x) - 1.929913693 \times 10^{13} f(17x)$
$+5.53173227 \times 10^{13} f(16x) - 1.352720538 \times 10^{14} f(15x)$
$+2.851914892 \times 10^{14} f(14x) - 5.225069332 \times 10^{14} f(13x)$
$+8.368895818 \times 10^{14} f(12x) - 1.177003885 \times 10^{15} f(11x)$
$+1.45803204 \times 10^{15} f(10x) - 1.593998867 \times 10^{15} f(9x)$
$+1.539372314 \times 10^{15} f(8x) - 1.313133109 \times 10^{15} f(7x)$
$+9.883941957 \times 10^{14} f(6x) - 6.550999864 \times 10^{14} f(5x)$

$+3.811204361 \times 10^{14} f(4x) - 1.938394451 \times 10^{14} f(3x)$

$$- \frac{32!}{2} f(2x) + 32!(4514873) f(x) = 0 \tag{9}$$

for all $x \in X$. Replacing (x, y) with $(8x, x)$ in (1), further multiplying the resulting equation by 10518300, and subtracting the obtained result from (9), we arrive at
$28048800 f(23x) - 833049360 f(22x) 1.19768376 \times 10^{10} f(21x)$
$-1.110269675 \times 10^{11} f(20x) + 7.457722824 \times 10^{11} f(19x)$
$-3.867107693 \times 10^{12} f(18x) + 1.610394623 \times 10^{13} f(17x)$
$-5.53173122 \times 10^{13} f(16x) + 1.597536392 \times 10^{14} f(15x)$
$-3.933676047 \times 10^{14} f(14x) + 8.346112548 \times 10^{14} f(13x)$
$-1.538067247 \times 10^{15} f(12x) + 2.476775902 \times 10^{15} f(11x)$
$-3.500669031 \times 10^{15} f(10x) + 4.356442419 \times 10^{15} f(9x)$
$-4.782971563 \times 10^{15} f(8x) + 4.637308513 \times 10^{15} f(7x)$
$-3.970312093 \times 10^{15} f(6x) + 2.998731922 \times 10^{15} f(5x)$
$-1.994214631 \times 10^{15} f(4x) + 1.165396876 \times 10^{15} f(3x)$

$$- \frac{32!}{2} f(2x) + 32!(15033173) f(x) = 0 \tag{10}$$

for all $x \in X$. Replacing (x, y) with $(7x, x)$ in (1),further multiplying the resulting equation 28048800, and subtracting the obtained result from (10), we arrive at
$64512240 f(22x) - 1935367200 f(21x) + 2.809508052 \times 10^{10} f(20x)$
$-2.6286625656 \times 10^{11} f(19x) + 1.781247459 \times 10^{12} f(18x) - 9.313651939 \times 10^{12} f(17x)$
$+3.909090958 \times 10^{13} f(16x) - 1.352720538 \times 10^{14} f(15x)$
$+3.933675767 \times 10^{14} f(14x) - 9.748796622 \times 10^{14} f(13x)$
$+2.080914588 \times 10^{15} f(12x) - 3.856442309 \times 10^{15} f(11x)$
$+6.242743601 \times 10^{15} f(10x) - 8.866760471 \times 10^{15} f(9x)$
$+1.108487277 \times 10^{16} f(8x) - 1.222228904 \times 10^{16} f(7x)$
$+1.189767046 \times 10^{16} f(6x) - 1.022547957 \times 10^{16} f(5x)$
$+7.754846356 \times 10^{15} f(4x) - 5.193238933 \times 10^{15} f(3x)$

$$- \frac{32!}{2} f(2x) + 32!(43081973) f(x) = 0 \tag{11}$$

for all $x \in X$. Setting (x, y) by $(6x, x)$ in (1), further multiplying the resulting equation by 64512240, and subtracting the obtained result from (11), we arrive at
$129024480 f(21x) - 3902990520 f(20x) + 5.71181448 \times 10^{10} f(19x)$
$-5.386126918 \times 10^{11} f(18x) + 3.677564904 \times 10^{12} f(17x)$
$-1.936956622 \times 10^{13} f(16x) + 8.186685633 \times 10^{13} f(15x)$
$-2.851915174 \times 10^{14} f(14x) + 8.346112548 \times 10^{14} f(13x)$
$-2.080914522 \times 10^{15} f(12x) + 4.467215911 \times 10^{15} f(11x)$
$-8.323658349 \times 10^{15} f(10x) + 1.354309065 \times 10^{16} f(9x)$
$-1.93285258 \times 10^{16} f(8x) + 2.427407082 \times 10^{16} f(7x)$

$-2.688169178 \times 10^{16} f(6x) + 2.628355153 \times 10^{16} f(5x)$
$-2.271698069 \times 10^{16} f(4x) + 1.743374903 \times 10^{16} f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(107594213) f(x) = 0 \tag{12}$$

for all $x \in X$. Replacing (x, y) with $(5x, x)$ in (1), further multiplying the resulting equation by 129024480, and subtracting the obtained result from (12), we have
$225792840 f(20x) - 6877997280 f(19x) 1.013487293 \times 10^{11} f(18x)$
$-9.621553973 \times 10^{11} f(17x) + 6.612867481 \times 10^{12} f(16x)$
$-3.505409525 \times 10^{13} f(15x) + 1.490863029 \times 10^{14} f(14x)$
$-5.225069332 \times 10^{14} f(13x) + 1.538067313 \times 10^{15} f(12x)$
$-3.856442438 \times 10^{15} f(11x) + 8.323662221 \times 10^{15} f(10x)$
$-1.558977711 \times 10^{16} f(9x) + 2.549181227 \times 10^{16} f(8x)$
$-3.655730204 \times 10^{16} f(7x) + 4.613637043 \times 10^{16} f(6x)$
$-5.138745418 \times 10^{16} f(5x) + 5.07093769 \times 10^{16} f(4x)$
$-4.47501023 \times 10^{16} f(3x)$

$$-\frac{32!}{2} f(2x) f(2x) + 32!(236618693) f(x) = 0 \tag{13}$$

for all $x \in X$. Plugging (x, y) into $(4x, x)$ in (1), further multiplying the resulting equation by 225792840, and subtracting the obtained result from (13), we have
$347373600 f(19x) - 1.064451964 \times 10^{10} f(18x)$
$1.577770891 \times 10^{11} f(17x) - 1.506643045 \times 10^{12} f(16x) + 1.041516365 \times 10^{13} f(15x)$
$-5.552536241 \times 10^{13} f(14x) + 2.374792521 \times 10^{14} f(13x)$
$-8.36889742 \times 10^{14} f(12x) + 2.476782998 \times 10^{15} f(11x)$
$-6.242851659 \times 10^{15} f(10x) + 1.354414659 \times 10^{16} f(9x)$
$-2.549871384 \times 10^{16} f(8x) + 4.19226389 \times 10^{16} f(7x)$
$-6.051502427 \times 10^{16} f(6x) + 7.710867162 \times 10^{16} f(5x)$
$-8.73852283 \times 10^{16} f(4x) + 8.93192555 \times 10^{16} f(3x)$

$$-\frac{32!}{2} f(2x) + 32!(462411533) f(x) = 0 \tag{14}$$

for all $x \in X$. Replacing (x, y) with $(3x, x)$ in (1), further multiplying the resulting equation by 347373600, and subtracting the obtained result from (14), we have
$471435600 f(18x) - 1.45202165 \times 10^{10} f(17x) + 2.16330011 \times 10^{11} f(16x)$
$-2.076390956 \times 10^{12} f(15x) + 1.442734367 \times 10^{13} f(14x)$
$-7.73082726 \times 10^{13} f(13x) + 3.3233089 \times 10^{14} f(12x)$
$-1.77169036 \times 10^{15} f(11x) + 3.502283946 \times 10^{15} f(10x)$
$-8.87819402 \times 10^{15} f(9x) + 1.939093697 \times 10^{16} f(8x)$
$-3.682661996 \times 10^{16} f(7x) + 6.132260323 \times 10^{16} f(6x)$
$-9.030938968 \times 10^{16} f(5x) + 1.88753222 \times 10^{17} f(4x)$
$-1.418900525 \times 10^{17} f(3x)$

$$-\frac{32!}{2}f(2x)+32!(809785133)f(x)=0 \tag{15}$$

for all $x \in X$. Replacing (x, y) with $(2x, x)$ in (1), further multiplying the resulting equation by 471435600, and subtracting the obtained result from (15), we get
$565722700f(17x)-1.75020466\times10^{10}f(16x)+2.619296198\times10^{11}f(15x)$
$-2.525951944\times10^{12}f(14x)+1.764262872\times10^{13}f(13x)$
$-9.51141113\times10^{13}f(12x)+4.11953627\times10^{14}f(11x)$
$-1.47336995\times10^{15}f(10x)+4.43994465\times10^{15}f(9x)$
$-1.144964077\times10^{16}f(8x)+2.558689753\times10^{16}f(7x)$
$-5.008288087\times10^{16}f(6x)+8.667539472\times10^{16}f(5x)$
$-1.337895693\times10^{17}f(4x)+1.856385106\times10^{17}f(3x)$

$$-\frac{32!}{2}f(2x)+32!(1281220733)f(x)=0 \tag{16}$$

for all $x \in X$. Replacing (x, y) with (x, x) in (1), further multiplying the resulting equation by 565722720, and subtracting the obtained result from (16), we arrive
$601080390f(16x)-1.923457248\times10^{10}f(15x)$
$+2.981358734\times10^{11}f(14x)-2.981358734\times10^{12}f(13x)$
$+2.161485082\times10^{13}f(12x)-1.210431646\times10^{14}f(11x)$
$+5.446942408\times10^{14}f(10x)-2.023150037\times10^{15}f(9x)$
$+6.32234387\times10^{15}f(8x)-1.685958364\times10^{16}f(7x)$
$+3.877704238\times10^{16}f(6x)-7.755678478\times10^{16}f(5x)$
$+1.357196483\times10^{17}f(4x)-2.08799458\times10^{17}f(3x)$

$$-\frac{32!}{2}f(2x)+32!(1846943453)f(x)=0 \tag{17}$$

for all $x \in X$. Taking $x = 0$ and $y = x$ in (1), further multiplying the resulting equation by 601080390, and subtracting the obtained result from (17), we arrive

$$-\frac{32!}{2}f(2x)+32!(2147483648)f(x)=0 \tag{18}$$

for all $x \in X$. It follows from (18), we get

$$f(2x)=2^{32}f(x) \tag{19}$$

for all $x \in X$. □

5 Hyers–Ulam Stability of Functional Equation (1) in Multi-Banach Spaces

Theorem 3 *Let X be a linear space and let* $\left((Y^k, \|.\|_k) : k \in \mathbb{N}\right)$ *be a Multi-Banach Space. Suppose that* δ *is a non-negative real number and* $f : X \to Y$ *be a function fulfills*

$$\sup_{k\in\mathbb{N}} \|(f(x_1, y_1), \ldots, f(x_k, y_k))\|_k \leq \delta \tag{20}$$

$\forall x_1, \ldots, x_k, y_1, \ldots, y_k \in X$*. Then there exists a unique Duotrigintic mapping* $\mathcal{T} : X \to Y$ *such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{T}(x_1), \ldots, f(x_k) - \mathcal{T}(x_k))\|_k \leq \frac{4294967297}{32!(4294967295)}\delta. \tag{21}$$

$\forall x_i \in X$*, where* $i = 1, 2, \ldots, k$.

Proof Letting (x_i, y_i) by $(16x_i, x_i)$ and $(0, 2x_i)$ respectively in (20), and subtracting the two resulting equations, we arrive at

$$\begin{aligned}
&\sup_{k\in\mathbb{N}} \|(32f(31x_1) - 528f(30x_1) + 4960f(29x_1) - 35464f(28x_1)\\
&+201376f(27x_1) - 911152f(26x_1) + 3365856f(25x_1) - 10482340f(24x_1)\\
&+28048800f(23x_1) - 64713616f(22x_1) + 129024480f(21x_1)\\
&-224886648f(20x_1) + 347373600f(19x_1) - 474801456f(18x_1)\\
&+565722720f(17x_1) - 590562090f(16x_1) + 565722720f(15x_1)\\
&-499484400f(14x_1)\\
&+347373600f(13x_1) - 161280600f(12x_1) + 129024480f(11x_1)\\
&-193536720f(10x_1) + 28048800f(9x_1) + 215274540f(8x_1) + 3365856f(7x_1)\\
&-348279792f(6x_1) + 201376f(5x_1) + 471399640f(4x_1) + 4960f(3x_1)\\
&-\frac{32!}{2}f(2x_1) + 32!f(x_1), \ldots, 32f(31x_k) - 528f(30x_k) + 4960f(29x_k)\\
&-35464f(28x_k)\\
&+201376f(27x_k) - 911152f(26x_k) + 3365856f(25x_k)\\
&-10482340f(24x_k) + 28048800f(23x_k) - 64713616f(22x_k)\\
&+129024480f(21x_k)\\
&-224886648f(20x_k) + 347373600f(19x_k) - 474801456f(18x_k)\\
&+565722720f(17x_k) - 590562090f(16x_k) + 565722720f(15x_k)\\
&-499484400f(14x_k) + 347373600f(13x_k) - 161280600f(12x_k)\\
&+129024480f(11x_k)\\
&-193536720f(10x_k) + 28048800f(9x_k) + 215274540f(8x_k) + 3365856f(7x_k)\\
&-348279792f(6x_k) + 201376f(5x_k) + 471399640f(4x_k)
\end{aligned}$$

$$+4960f(3x_k) - \frac{32!}{2}f(2x_k) + 32!f(x_k)\Big)\Big\|_k \leq \frac{3}{2}\delta \tag{22}$$

for all $x_i \in X$, where $i = 1, 2, .., k$. Substituting $x_i = 15x_i$ and $y_i = x_i$ in (20), further multiplying the resulting equation by 32, and subtracting the obtained result from (22), we arrive at

$$\sup_{k\in\mathbb{N}} \|(496f(30x_1) - 10912f(29x_1) + 123256f(28x_1) - 949344f(27x_1)$$
$$+5532880f(26x_1) - 25632288f(25x_1) + 97225052f(24x_1)$$
$$-308536800f(23x_1) \quad +852847984f(22x_1) - 1935367200f(21x_1)$$
$$+3903896712f(20x_1)$$
$$-6877997280f(19x_1) + 1.064115374 \times 10^{10} f(18x_1)$$
$$-1.452021648 \times 10^{10} f(17x_1) + 1.751256495 \times 10^{10} f(16x_1)$$
$$-1.866884976 \times 10^{10} f(15x_1) + 1.760364264 \times 10^{1}0 f(14x_1)$$
$$-1.47385656 \times 10^{10} f(13x_1) + 1.09546746 \times 10^{10} f(12x_1)$$
$$-7096346400f(11x_1) + 3935246640f(10x_1) - 2036342880f(9x_1)$$
$$+1112836146f(8x_1) - 333219744f(7x_1) - 240572400f(6x_1)$$
$$-28796768f(5x_1) + 477843672f(4x_1) - 1145760f(3x_1) - \frac{32!}{2} f(2x_1)$$
$$+32!(33)f(x_1), ..,$$
$$496f(30x_k) - 10912f(29x_k) + 123256f(28x_k) - 949344f(27x_k)$$
$$+5532880f(26x_k) - 25632288f(25x_k) + 97225052f(24x_k)$$
$$-308536800f(23x_k)$$
$$+852847984f(22x_k) - 1935367200f(21x_k) + 3903896712f(20x_k)$$
$$-6877997280f(19x_k) + 1.064115374 \times 10^{10} f(18x_k)$$
$$-1.452021648 \times 10^{10} f(17x_k) + 1.751256495 \times 10^{10} f(16x_k)$$
$$-1.866884976 \times 10^{10} f(15x_k) + 1.760364264 \times 10^{1}0 f(14x_k)$$
$$-1.47385656 \times 10^{10} f(13x_k) + 1.09546746 \times 10^{10} f(12x_k)$$
$$-7096346400f(11x_k) + 3935246640f(10x_k) - 2036342880f(9x_k)$$
$$+1112836146f(8x_k) - 333219744f(7x_k) - 240572400f(6x_k)$$
$$-28796768f(5x_k) + 477843672f(4x_k) - 1145760f(3x_k)$$
$$-\frac{32!}{2} f(2x_k) + 32!(33)f(x_k)\Big)\Big\|_k \leq \frac{67}{2}\delta \tag{23}$$

for all $x_i \in X$, where $i = 1, 2, \ldots, k$. Applying the same procedure of Theorem 2 and using (19), we get

$$\sup_{k\in\mathbb{N}} \left\| \left(f(x_1) - \frac{1}{2^{32}} f(2x_1), \ldots, f(x_k) - \frac{1}{2^{32}} f(2x_k) \right) \right\|_k \leq \frac{4294967297}{(32!)(4294967295)} \delta \tag{24}$$

for all $x_i \in X$, where $i = 1, 2, \ldots, k$.

Let $\Lambda = \{g : X \to Y | g(0) = 0\}$ and introduce the generalized metric d defined on Λ by

$$d(u, v) = \inf \left\{ \lambda \in [0, \infty] | \sup_{k\in\mathbb{N}} \|(u(x_1) - v(x_1), \ldots, u(x_k) - v(x_k))\|_k \leq \lambda \right\}$$

forall $x_1, \ldots, x_k \in X$. Then it is easy to show that (Λ, d) is a generalized complete metric space. See [20].

We define an operator $\mathcal{J} : \Lambda \to \Lambda$ by

$$\mathcal{J}u(x) = \frac{1}{2^{32}} u(2x) \ \forall x \in X.$$

We assert that $\mathcal{J}$ is a strictly contractive operator. Given $u, v \in \Lambda$, let $\lambda \in (0, \infty)$ be an arbitrary constant with $d(u, v) \leq \lambda$. By the definition of d, it follows that

$$\sup_{k \in \mathbb{N}} \|(u(x_1) - v(x_1), \ldots, u(x_k) - v(x_k))\|_k \leq \lambda,$$

for all $x_1, \ldots, x_k \in X$. Therefore,

$$\begin{aligned}
&\sup_{k \in \mathbb{N}} \|(\mathcal{J}u(x_1) - \mathcal{J}v(x_1), \ldots, \mathcal{J}u(x_k) - \mathcal{J}v(x_k))\|_k \\
&\quad \leq \sup_{k \in \mathbb{N}} \left\| \left(\frac{1}{2^{32}} u(2x_1) - \frac{1}{2^{32}} v(2x_1), \ldots, \frac{1}{2^{32}} u(2x_k) - \frac{1}{2^{32}} v(2x_k) \right) \right\|_k \\
&\quad \leq \frac{1}{2^{32}} \lambda
\end{aligned}$$

for all $x_1, \ldots, x_k \in X$. Hence, it holds that $d(\mathcal{J}u, \mathcal{J}v) \leq \frac{1}{2^{32}} \lambda$ *i.e.*, $d(\mathcal{J}u, \mathcal{J}v) \leq \frac{1}{2^{32}} d(u, v) \ \forall u, v \in \Lambda$. This means that $\mathcal{J}$ is strictly contractive operator on Λ with the Lipschitz constant $\mathcal{L} = \frac{1}{2^{32}}$.

By (24), we have $d(\mathcal{J}h, h) \leq \frac{4294967297}{(32!)(4294967296)} \delta$. According to Theorem 1, we deduce the existence of a fixed point of $\mathcal{J}$ that is the existence of mapping $\mathcal{T} : X \to Y$ such that

$$\mathcal{T}(2x) = 2^{32} \mathcal{T}(x) \quad \forall x \in X.$$

Moreover, we have $d\left(\mathcal{J}^n h, \mathcal{T}\right) \to 0$, which implies

$$\mathcal{T}(x) = \lim_{n \to \infty} \mathcal{J}^n h(x) = \lim_{n \to \infty} \frac{h(2^n x)}{2^{32n}}$$

for all $x \in X$.

Also, $d(h, \mathcal{T}) \leq \frac{1}{1 - \mathcal{L}} d(\mathcal{J}h, h)$ implies the inequality

$$d(h, \mathcal{T}) \leq \frac{1}{1 - \frac{1}{2^{32}}} d(\mathcal{J}h, h)$$
$$\leq \frac{4294967297}{(32!)(4294967295)} \delta.$$

Setting $x_1 = \ldots = x_k = 2^n x$, $y_1 =, \ldots, = y_k = 2^n y$ in (20) and divide both sides by 2^{32^n}. Then, using property (a) of multi-norms, we obtain

$$\|D\mathcal{T}(x, y)\| = \lim_{n\to\infty} \frac{1}{2^{32^n}} \left\| Dh\left(2^n x, 2^n y\right)\right\| = 0$$

for all $x, y \in X$. Hence $\mathcal{T}$ is Duotrigintic mapping.

The uniqueness of $\mathcal{T}$ follows from the fact that $\mathcal{T}$ is the unique fixed point of $\mathcal{J}$ with the property that there exists $\ell \in (0, \infty)$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{T}(x_1), \ldots, f(x_k) - \mathcal{T}(x_k))\|_k \leq \ell$$

for all $x_1, \ldots, x_k \in X$.

This completes the proof of the theorem. □

Corollary 1 *Let X be a linear space, and let* $(Y^k, \|.\|_k)$ *be a Multi-Banach space. Let* $\theta > 0, 0 < p < 32$ *and* $f : X \to Y$ *be a mapping satisfying* $f(0) = 0$

$$\sup_{k\in\mathbb{N}} \|\mathcal{D}f(x_1, y_1), \ldots, \mathcal{D}f(x_k, y_k)\|_k \leq \theta\left(\|x_1\|^p + \|y_1\|^p, .., \|x_k\|^p + \|y_k\|^p\right) \tag{25}$$

for all $x_1, \ldots, x_k, y_1 \ldots, y_k \in X$. *Then there exists a unique mapping* $\mathcal{T} : X \to Y$ *such that*

$$\sup_{k\in\mathbb{N}} \|(f(x_1) - \mathcal{T}(x_1), \ldots, f(x_k) - \mathcal{T}(x_k))\| \leq \frac{1}{2^{32} - 2^p} \Psi(\|x_1\|^p, \ldots, \|x_k\|^p) \tag{26}$$

where

$$\begin{aligned}\Psi = \frac{2}{32!}\theta\Big[&\frac{1}{2}2^p + (16^p + 1) + 32(15^p + 1) + 496(14^p + 1) + 4960(13^p + 1)\\ &+35960(12^p + 1) + 201376(11^p + 1) + 906192(10^p + 1) + 3365856(9^p + 1)\\ &+10518300(8^p + 1) + 28048800(7^p + 1)\\ &+64512240(6^p + 1) + 129024480(5^p + 1) + 225792840(4^p + 1)\\ &+347373600(3^p + 1) + 471435600(2^p + 1) + 866262915\Big]\end{aligned}$$

Proof The proof is similar to that of Theorem 3, replacing δ by $\theta\left(\|x_1\|^p + \|y_1\|^p, .., \|x_k\|^p + \|y_k\|^p\right)$. □

Corollary 2 *Let X be a linear space, and let $(Y^k, \|.\|_k)$ be a Multi-Banach Space. Let $\theta > 0, 0 < r + s = p < 32$ and $f : X \to Y$ be a mapping satisfying $f(0) = 0$*

$$\sup_{k\in\mathbb{N}} \|\mathcal{D}f(x_1, y_1), \ldots, \mathcal{D}f(x_k, y_k)\|_k \leq \theta\left(\|x_1\|^r . \|y_1\|^s, \ldots, \|x_k\|^r . \|y_k\|^s\right) \tag{27}$$

for all $x_1, \ldots, x_k, y_1 \ldots, y_k \in X$. Then there exists a unique mapping $\mathcal{T} : X \to Y$ such that

$$\sup_{k\in\mathbb{N}} \|(f(x_1)-\mathcal{T}(x_1), \ldots, f(x_k)-\mathcal{T}(x_k))\| \leq \frac{1}{2^{32} - 2^p}\Psi_{32}(\|x_1\|^{r+s}, \ldots, \|x_k\|^{r+s}) \tag{28}$$

where

$$\begin{aligned}\Psi_{32} = \frac{2}{32!}\theta\big[&16^r + 32(15^r) + 496(14^r) + 4960(13^r) + 35960(12^r)\\ &+201376(11^r) + 906192(10^r) + 3365856(9^r) + 10518300(8^r) + 28048800(7^r)\\ &+64512240(6^r) + 129024480(5^r) + 225792840(4^r) + 347373600(3^r)\\ &+471435600(2^r) + 565722720\big]\end{aligned}$$

Proof The proof is similar to that of Theorem 3, replacing δ by $\theta\left(\|x_1\|^r . \|y_1\|^s, .., \|x_k\|^r . \|y_k\|^s\right)$. □

Example Let $k \in \mathbb{N}$. We define $\phi : \mathbb{R} \to \mathbb{R}$, by

$$\phi(x) = \begin{cases} 1 & x \in [1, \infty) \\ x^{32} & x \in (-\infty, \infty) \\ -1 & x \in (-\infty, -1]. \end{cases}$$

We consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{\phi(4^n x)}{4^{32n}}, \quad (x \in \mathbb{R}).$$

Then f satisfies the following functional inequality:

$$\|Df(x_1, y_1)), .., Df(x_k, y_k)\|_k \leq \frac{2^{32} + 32!}{4^{32} - 1}4^{96}$$
$$\left(|x_1|^{32} +, .., |x_k|^{32} + |y_1|^{32} +, .., |y_k|^{32}\right)$$

for all $x_1, \ldots, x_k, y_1, .., y_k \in \mathbb{R}$.

Proof We have

$$|f(x)| \leq \frac{4^{32}}{4^{32}-1}$$

for all $x \in \mathbb{R}$. Therefore, we see that f is bounded. Let $x, y \in \mathbb{R}$. If $|x|^{32}+|y|^{32}=0$ or $|x|^{32}+|y|^{32} \geq \frac{1}{4^{32}}$, then

$$|Df(x,y)| \leq \frac{\left(2^{32}+32!\right)4^{32}}{4^{32}-1} \leq \frac{\left(2^{32}+32!\right)4^{32}}{4^{32}-1}4^{32}\left(|x|^{32}+|y|^{32}\right).$$

Now, suppose that $0<|x|^{32}+|y|^{32} \leq \frac{1}{4^{32}}$. Then there exists a non-negative integer k such that

$$\frac{1}{4^{32(k+2)}} \leq |x|^{32}+|y|^{32} < \frac{1}{4^{32(k+1)}}.$$

Hence, $4^k x < \frac{1}{4}$ and $4^k y < \frac{1}{4}$, and
$4^n(x+16y), 4^n(x+15y), 4^n(x+14y), 4^n(x+13y), 4^n(x+12y),$
$4^n(x+11y), 4^n(x+10y), 4^n(x+9y), 4^n(x+8y), 4^n(x+7y)$
$4^n(x+6y), 4^n(x+5y), 4^n(x+4y), 4^n(x+3y), 4^n(x+2y)$
$4^n(x+y)4^n(x), 4^n(x-y), 4^n(x-2y), 4^n(x-3y),$
$4^n(x-4y), 4^n(x-5y), 4^n(x-6y), 4^n(x-7y),$
$4^n(x-8y), 4^n(x-9y), 4^n(x-10y), 4^n(x-11y), 4^n(x-12y)$
$4^n(x-13y), 4^n(x-14y), 4^n(x-15y), 4^n(x-16y) \in (-1,1)$
for all $n=0,1,\ldots,k-1$. Thus we get

$$\begin{aligned}
\frac{|Df(x,y|)}{|x|^{32}+|y|^{32}} &\leq \sum_{n=k}^{\infty} \frac{2^{32}+32!}{4^{32n}(|x|^{32}+|y|^{32})} \\
&\leq \sum_{n=0}^{\infty} \frac{2^{32}+32!}{4^{32n}4^{32(k+2)}(|x|^{32}+|y|^{32})}4^{64} \\
&\leq \sum_{n=0}^{\infty} \frac{2^{32}+32!}{4^{32n}}4^{64} = \frac{2^{32}+32!}{4^{32}-1}4^{96},
\end{aligned}$$

or

$$|Df(x,y)| \leq\leq \frac{2^{32}+32!}{4^{32}-1}4^{96}(|x|^{32}+|y|^{32}).$$

□

References

1. T. Aoki, On the stability of the linear transformation in Banach spaces. J. Math. Soc. Jpn. **2**, 64–66 (1950)
2. M. Arunkumar, A. Bodaghi, J.M. Rassias, E. Sathiya, The general solution and approximations of a decic type functional equation in various normed spaces. J. Chungcheong Math. Soc. **29**(2) 287–328 (2016)
3. A. Bodaghi, Intuitionistic fuzzy stability of the generalized forms of cubic and quartic functional equations. J. Intel. Fuzzy Syst. **30**(4), 2309–2317 (2016)
4. A. Bodaghi, Stability of a mixed type additive and quartic function equation. Filomat **28**(8), 1629–1640 (2014)
5. A. Bodaghi, S.M. Moosavi, H. Rahimi, The generalized cubic functional equation and the stability of cubic Jordan $*$-derivations. Ann. Univ. Ferrara **59**(2), 235–250 (2013)
6. A. Bodaghi, C. Park, J.M. Rassias, Fundamental stabilities of the nonic functional equation in intuitionistic fuzzy normed spaces. Commun. Korean Math. Soc. **31**(4), 729–743 (2016)
7. P. Choonkil, R. Murali, A. Antony Raj, Stability of trigintic functional equation in multi-banach spaces: a fixed point approach. Korean J. Math. **26**, 615–628 (2018)
8. S. Czerwik, On the stability of the quadratic mapping in normed spaces. Abh. Math. Sem. Univ. Hamb. **62**(1), 59–64 (1992)
9. H.G. Dales, M. Moslehian, Stability of mappings on multi-normed spaces. Glasg. Math. J. **49**, 321–332 (2007)
10. H.G. Dales, M.E. Polyakov, Multi-normed spaces and multi-Banach algebras (2011). arXiv:1112.5148
11. J.B. Diaz, B. Margolis, A fixed point theorem of the alternative for contractions on generalized complete metric space. Bull. Am. Math. Soc. **74**, 305–309 (1968)
12. M. Fridoun, Approximate Euler-Lagrange-Jensen type additive mapping in multi-banach spaces: a fixed point approach. Commun. Korean Math. Soc. **28**, 319–333 (2013)
13. P. Gâvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings. J. Math. Anal. Appl. **184**, 431–436 (1994)
14. D.H. Hyers, On the stability of the linear functional equation. Proc. Natl. Acad. Sci. USA **27**, 222–224 (1941)
15. R. John Michael, R. Murali, J.R. Matina, A. Antony Raj, General solution, stability and non-stability of quattuorvigintic functional equation in multi-banach spaces. Int. J. Math. Appl. **5**(2-A), 181–194 (2017)
16. R. John Michael, R. Murali, A. Antony Raj, Stability of octavigintic functional equation in multi-banach spaces: a fixed point approach (submitted)
17. K.W. Jun, H.M. Kim, The generalized Hyers-Ulam-Rassias stability of a cubic functional equation. J. Math. Anal. Appl. **274**(2), 867–878 (2002)
18. W. Liguang, L. Bo, B. Ran, Stability of a mixed type functional equation on multi-Banach spaces: a fixed point approach. Fixed Point Theory Appl. **2010**(1), 9 (2010)
19. R. Murali, P. Sandra, A. Antony Raj, General solution and a fixed point approach to the Ulam-Hyers stability of Viginti Duo functional equation in multi-banach spaces. IOSR J. Math. **13**(4), 48–59 (2017)
20. D. Mihet, V. Radu, On the stability of the additive Cauchy functional equation in random normed spaces. J. Math. Anal. Appl. **343**, 567–572 (2008)
21. M.S. Moslehian, K. Nikodem, D. Popa, Asymptotic aspects of the quadratic functional equations in multi-normed spaces. J. Math. Anal. Appl. **355**, 717–724 (2009)
22. R. Murali, P. Sandra, A. Antony Raj, Ulam- Hyers stability of hexadecic functional equations in multi-Banach spaces. Analysis **34**(4) (2017)
23. M. Nazarianpoor, J.M. Rassias, G. Sadeghi, Stability and nonstability of octadecic functional equation in multi-normed spaces. Arab. J. Math. **7**(3) (2017)
24. M. Nazarianpoor, J.M. Rassias, G. Sadeghi, Solution and Stability of Quattuorvigintic Functional Equation in Intuitionistic Fuzzy Normed Spaces. Iran. J. Fuzzy Syst. **15**(4), 13–30 (2018)

25. M. Ramdoss, B. Abasalt, A. Antony Raj, General solution and Ulam-Hyers stability of Viginti functional equations in multi-banach spaces. J. Chungcheong Math. Soc. **31**(2) (2018)
26. T.M. Rassias, On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
27. J.M. Rassias, Solution of the Ulam stability problem for quartic mappings. Glas. Mat. Ser. III **34**(2), 243–252 (1999)
28. J.M. Rassias, Solution of the Ulam stability problem for cubic mappings. Glas. Mat. Ser. III **36**(1), 63–72 (2001)
29. J.M. Rassias, M. Eslamian, Fixed points and stability of nonic functional equation in quasi-β-normed spaces. Contin. Anal. Appl. Math. **3**, 293–309 (2015)
30. J.M. Rassias, M. Arunkumar, E. Sathya, T. Namachivayam, Various generalized Ulam-Hyers Stabilities of a nonic functional equations. Tbilisi Math. J. **9**(1), 159–196 (2016)
31. J.M. Rassias, R. Murali, M.J. Rassias, V. Vithya, A. Antony Raj, General solution and stability of Quattuorvigintic functional equation in matrix normed spaces. Tbilisi Math. J. **11**(2), 97–109 (2018)
32. K. Ravi, J.M. Rassias, B.V. Senthil Kumar, Ulam-Hyers stability of undecic functional equation in quasi-β normed spaces fixed point method. Tbilisi Math. Sci. **9**(2), 83–103 (2016)
33. K. Ravi, J.M. Rassias, S. Pinelas, S. Suresh, General solution and stability of quattuordecic functional equation in quasi β-normed spaces. Adv. Pure Math. **6**(12) 921–941 (2016)
34. A. Sattar, M. Fridoun, Approximate a quadratic mapping in multi-banach spaces, a fixed point approach. Int. J. Nonlinear Anal. Appl. **7**, 63–75 (2016)
35. X. Tian Zhou, R. John Michael, X. Wan Xin, Generalized Ulam-Hyers stability of a general mixed AQCQ functional equation in Multi-Banach Spaces: a fixed point approach. Eur. J. Pure Appl. Math. **3**, 1032–1047 (2010)
36. S.M. Ulam, *A Collection of the Mathematical Problems* (Interscience, New York, 1960)
37. W. Xiuzhong, C. Lidan, L. Guofen, Orthogonal stability of mixed Additive-Quadratic Jenson type functional equation in multi-banach spaces. Adv. Pure Math. **5**, 325–332 (2015)
38. T.Z. Xu, J.M. Rassias, Approximate septic and octic mappings in quasi-β-normed spaces. J. Comput. Anal. Appl. **15**(6), 1110–1119 (2013)
39. T.Z. Xu, J.M. Rassias, M.J. Rassias, W.X. Xu, A fixed point approach to the stability of quintic and sextic functional equations in quasi-β-normed spaces. J. Inequal. Appl. **2010**, 23 (2010)
40. W. Zhihua, L. Xiaopei, T.M. Rassias, Stability of an additive-cubic-quartic functional equation in multi-Banach spaces. Abstr. Appl. Anal. **2011**, 11 (2011)