Topological Degree Theory and Ulam's Stability Analysis of a Boundary Value Problem of Fractional Differential Equations

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Abstract In this article, we study the existence and uniqueness of positive solution to a class of nonlinear fractional order differential equations with boundary conditions. By using fixed point theorems on contraction mapping together with topological degree theory, we investigate some sufficient conditions in order to obtain the existence and uniqueness of positive solution for the considered problem. Further we also investigate different kinds of Ulam stability for the considered problem. Moreover, we also provide an example to justify the whole results.

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1 Introduction

The study of fractional differential equations is an important area for research in recent time, because of its wide range of applications in describing the real-word problems. These applications can be found in various scientific and engineering disciplines such as physics, chemistry, optimization theory, biology, viscoelasticity, control theory, signal processing, etc. For details, we refer [1–8]. Due to large number of applications of fractional differential equations, researchers are giving much attention to study fractional order differential equations, we refer the readers to [9–13] and the references therein for the recent development in the theory of fractional differential equations. It is worthwhile to mention that Caputo's fractional

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order derivative plays important roles in applied problems as it provides known physical interpretation for initial and boundary value problems of differential equations of arbitrary order. On the other hand, the Riemann–Liouville derivative of fractional order does not provide physical interpretations in most of the cases for initial and boundary problems. Existence theory of differential equations of classical as well as arbitrary order with multi-point boundary conditions has attracted the attention of many researchers and is a rapidly growing area of research, because such problems occurred in applications, we refer the readers to [14–19]. The area devoted to study boundary value problems of classical order differential equations has been studied and plenty of work is available on it by means of degree theory, however, for differential equations of fractional order, the area is quite recent and very few papers are available on it. As in [20], the authors studied the following problems by using topological degree theory

$$\begin{aligned} {}^cD^q u(t) &= f(t,\ u(t)),\ 0 < q < 1,\ t \in [0, T],\\ u(0) + g(u) &= u_0, \end{aligned}$$

and

$$\begin{aligned} {}^cD^q u(t) &= f(t,\ u(t)),\ 0 < q < 1,\ t \in [0, T],\\ au(0) + bu(T) &= c,\ a + b \neq 0,\\ {}^cD^q u(t) &= f(t,\ u(t)),\ 0 < q < 1,\ t \in [0, T],\\ u(0) &= u_0, \Delta u(t_k) = I_k(u(t_k)), k = 1, 2, \cdots n. \end{aligned}$$

where $g \in C([0, T], \mathbf{R})$, $f \in C([0, T] \times \mathbf{R}, \mathbf{R})$, $I_k : \mathbf{R} \to \mathbf{R}$ is continuous function called impulse. Similarly in [21], the authors studied the following multi-point boundary value problems by topological degree theory given by

$$\begin{aligned} {}^cD^q u(t) &= f(t,\ u(t)),\ 1 < q \le 2,\ t \in [0, 1],\\ u(0) &= g(u),\ \sum_{k=0}^{m-2} \lambda_k u(\eta_k) + h(u) = u(1). \end{aligned}$$

where $g, h \in C([0, 1], \mathbf{R})$, $f \in C([0, 1] \times \mathbf{R}, \mathbf{R})$. In very recent times, Shah et al. [22] developed sufficient conditions for the existence and uniqueness of positive solution to a coupled system with four-point boundary conditions via topological degree.

Motivated by the above work, in this article, we study the following class of nonlinear fractional order differential equations with given boundary conditions as

$$\begin{aligned} {}^cD^q u(t) &= f(t,\ u(t)),\ 1 < q \le 2,\ t \in J = [0, 1],\\ \lambda_1 u(0) + \mu_1 u(1) &= g_1(u),\\ \lambda_2 u'(0) + \mu_2 u'(1) &= g_1(u). \end{aligned}$$

where $g_k : C(J, \mathbf{R}) \to \mathbf{R}\ for\ k = 1, 2$, are continuous functions and $f : J \times \mathbf{R} \to \mathbf{R}$ is nonlinear continuous function and $\lambda_k, \mu_k (k = 1, 2)$ are real constants with $\lambda_k + \mu_k \neq 0, k = 1, 2$.

Here we remark that existence theory together with stability analysis is very important from numerical as well as optimization point of view. Beside from existence theory of solutions to the nonlinear fractional differential equations, the aspect devoted to stability analysis has been attracted the attention, see [23–26]. Different kinds of stability including exponential, Mittag–Leffler, and Lyapunov stability have been studied for the said differential equations, for details see [27–29]. Another kind of stability which greatly attracted the researchers' attention has been recently considered for nonlinear and linear fractional differential equations, we refer [30–33]. This important form of stability was first pointed out by Ulam in 1940 and was brilliantly explained by Hyers in 1940. After that valuable contributions have been done in this regard. In 1997, Rassias extended the aforementioned stability to some other forms known as Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stability. The concerned stability results have been investigated recently for fractional differential equations, ordinary and functional equations, see [34]. The aforementioned stability has been investigated for functional, integral, and differential equations very well, see [35–37]. In the last few years significant contribution has been done in the aforementioned aspects. Problems devoted to integral, functional, and differential equations have been evaluated for the aforesaid stability, see [38–50, 55, 56].

Therefore in this work, the considered class of differential equations of fractional order is investigated for positive solutions by means of contraction mapping principle coupled with topological degree theory. Sufficient conditions are developed under which the considered class of boundary value problem has at least one and unique solution. Then using nonlinear analysis we develop sufficient conditions for different kinds of Ulam stability.

We present the rest of the paper in four sections, in Sect. 2, we present some of the basic results and theorems, which are helpful in this paper. Also, we give some assumptions which are needed for this study. Section 3 is devoted to the main results. In Sect. 4, we provide a detailed analysis for stability theory. In Sect. 5, we give an example for verification of the established results. In the last section, we give a brief conclusion.

2 Preliminaries Results

In this section, we recall some definitions and basic results which are helpful throughout in this article, for details see [51–54].

The notation $C(J, \mathbf{R})$ is used for Banach space for all continuous function defined for J into $\mathbf{R}$ with norm

$$\|u\|_c = \sup\{|u(t)| : 0 \leq t \leq 1\}.$$

We denote $X = C[0, 1]$, we recall the following results from degree theory.

Definition 1 Let $C \subset X$ and $T : C \to X$ be a continuous bounded map, then T is α-condensing if $\alpha(T(V)) < \alpha(V)$ for all V bounded subset of C with $\alpha(V) > 0$.

The following theorem given by Isaia is important for our main results

Theorem 1 *Let $T : V \to X$ be α-condensing and*

$$V = \{u \in X : \ thereexists \quad \lambda \in [0, 1] \quad suchthat \quad u = \lambda T u\}.$$

If V is a bounded subset of X and there exists $r > 0$ such that $V \subset \mathcal{U}_r(0)$, then the degree

$$D(I - \lambda T, \mathcal{U}_r(0), 0) = 1, \quad forall \quad \lambda \in [0, 1].$$

Consequently, T has at least one fixed point and the set of the fixed points of T lies in $\mathcal{U}_r(0)$.

The following propositions are needed.

Proposition 1 *If $T_1, T_2 : V \to X$ are α-Lipschitz maps with constants κ_1 and κ_2 respectively, then $T_1 + T_2 : V \to X$ are α-Lipschitz with constants $\kappa_1 + \kappa_2$.*

Proposition 2 *If $T_1 : V \to X$ is compact, then T is α-Lipschitz with constant $\kappa = 0$.*

Proposition 3 *If $T_1 : V \to X$ is Lipschitz with constant κ, then T_1 is α-Lipschitz with the same constant κ.*

Definition 2 The fractional(arbitrary) order integral of a function $u \in L^1([0, b], \mathbf{R})$ of order $q \in \mathbf{R}^+$ is defined by

$$I^q u(t) = \frac{1}{\Gamma(q)} \int_0^t (t - s)^{q-1} u(s) ds, \quad n - 1 < q \leq n.$$

Definition 3 The Caputo's fractional order derivative of a function u on the interval $[0, b]$ is defined by

$${}^c D^q u(t) = \frac{1}{\Gamma(n - q)} \int_0^t (t - s)^{n-q-1} u^{(n)}(s)\, ds, \ n = [q] + 1,$$

where $[q]$ represents integer part of q.

For the existence of solutions to the considered problem, we need the following results:

Theorem 2 *The fractional order differential eqnarray of order $q > 0$ of the form*

$${}^c D^q u(t) = 0, \ n - 1 < q \leq n,$$

has a solution of the form

$$u(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

where $c_i \in \mathbf{R}$, *for* $i = 0, 1, \ldots, n-1$.

Theorem 3 *The following result holds for a fractional order differential equation q*

$$I^q[{}^cD^q u](t) = u(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1},$$

for arbitrary $c_i \in \mathbf{R}$, *for* $i = 0, 1, 2, \ldots, n-1$.

The consequence of Theorems 2 and 3 leads us to the following useful result.

Theorem 4 *Let* $u \in CJ$ *and* $y \in C(J \times \mathbf{R}, \mathbf{R})$, *then the solution of linear fractional differential equation*

$$\begin{aligned} {}^cD^q u(t) &= f(t, u(t)),\ 1 < q \leq 2,\ t \in [0, 1], \\ \lambda_1 u(0) + \mu_1 u(1) &= g_1(u), \\ \lambda_2 u'(0) + \mu_2 u'(1) &= g_1(u). \end{aligned} \tag{1}$$

where $g_k (k = 1, 2) : C(J, \mathbf{R}) \to \mathbf{R}$ *are nonlocal continuous functions and the real constant* λ_k, μ_k *satisfy the relations* $\lambda_k + \mu_k \neq 0$, *for* $k = 1, 2$, *is given by*

$$u(t) = g(u) + \int_0^1 \mathscr{G}(t, s) f(s, u(s)) ds,$$

where

$$g(u) = \frac{1}{\lambda_1 + \mu_1} g_1(u) + \frac{1}{\lambda_2 + \mu_2} \left[t - \mu_1\right] g_2(u)$$

and $\mathscr{G}(t, s)$ *is the Green's function provided by*

$$\mathscr{G}(t, s) = \begin{cases} \frac{(t-s)^{q-1}}{\Gamma(q)} + \frac{\mu_1 (1-s)^{q-1}}{(\lambda_1+\mu_1)\Gamma(q)} + \frac{\mu_2}{\lambda_2+\mu_2} \left(\frac{\mu_1}{\lambda_1+\mu_1} - t\right) \frac{(1-s)^{q-2}}{\Gamma(q-1)}, & 0 \leq s \leq t \leq 1, \\ \frac{\mu_1 (1-s)^{q-1}}{(\lambda_1+\mu_1)\Gamma(q)} + \frac{\mu_2}{\lambda_2+\mu_2} \left(\frac{\mu_1}{\lambda_1+\mu_1} - t\right) \frac{(1-s)^{q-2}}{\Gamma(q-1)}, & 0 \leq t \leq s \leq 1, \end{cases} \tag{2}$$

Proof Consider the following linear problem of FDES subject to the given boundary condition for $y \in C(J, \mathbf{R})$

$$\begin{aligned} {}^cD^q u(t) &= y(t),\ 1 < q \leq 2,\ t \in [0, 1], \\ \lambda_1 u(0) + \mu_1 u(1) &= g_1(u), \\ \lambda_2 u'(0) + \mu_2 u'(1) &= g_1(u). \end{aligned} \tag{3}$$

In view of Lemma 2, (3) can be written as

$$u(t) = c_0 + c_1 t - I^q y(t),\ c_0,\ c_1 \in \mathbf{R}, \tag{4}$$

using $\lambda_1 u(0) + \mu_1 u(1) - g_1(u) = 0$ in (4), we get

$$\lambda_1 c_0 + \mu_1 I^q y(t) + \mu_1 c_0 + \mu_1 c_1 = g_1(u)$$

which yields

$$c_0 = -\frac{\mu_1}{\lambda_1 + \mu_1} c_1 - \frac{\mu_1}{\lambda_1 + \mu_1} I^q y(t) + \frac{1}{\lambda_1 + \mu_1}.$$

Now using $\lambda_2 u'(0) + \mu_2 u'(1) - g_2(u) = 0$ in (4), we get

$$\lambda_2 c_1 + \mu_2 I^{q-1} y(1) + \mu_2 c_1 = g_2(u)$$

implies that

$$c_1 = \frac{1}{\lambda_2 + \mu_2}\left[g_2(u) - \mu_2 I^{q-1} y(1)\right].$$

By simple calculation, we get

$$c_0 = \frac{1}{\lambda_1 + \mu_1}\left[g_1(u) - \frac{\mu_1}{\lambda_2 + \mu_2} g_2(u)\right] + \frac{\mu_1}{\lambda_1 + \mu_1}\left[\frac{\mu_2}{\lambda_2 + \mu_2} I^{q-1} y(1) - I^q y(1)\right].$$

Hence (4) becomes

$$\begin{aligned} u(t) = {} & \frac{1}{\lambda_1 + \mu_1} g_1(u) + \frac{1}{\lambda_2 + \mu_2}(t - \mu_1)\, g_2(u) + \frac{\mu_1}{\lambda_1 + \mu_1} I^q y(1) \\ & + \frac{\mu_2}{\lambda_2 + \mu_2}\left(\frac{\mu_1}{\lambda_1 + \mu_1} - t\right) I^{q-1} y(1) + I^q y(t). \end{aligned}$$

$$hencewehave\ u(t) = g(u) + \int_0^1 \mathcal{G}(t,s) f(s, u(s)) ds, \tag{5}$$

where

$$g(u) = \frac{1}{\lambda_1 + \mu_1} g_1(u) + \frac{1}{\lambda_2 + \mu_2}\left[t - \mu_1\right] g_2(u) \tag{6}$$

Thus in view of (5), our considered problem (1) is written as in the form of Fredholm integral eqnarray given by

$$u(t) = g(u) + \int_0^1 \mathcal{G}(t,s) f(s,u(s))ds, \ t \in [0,1], \tag{7}$$

where $\mathcal{G}(t,s)$ is Green's function defined as in (2) and $g(u)$ is defined in (6).

In other words, we need the following assumptions to be hold, which are needed for our main results:

(B_1) For $u, v \in C[0,1]$, there exist $k_g \in [0,1)$, such that

$$|g(u) - g(v)| \leq k_g \|u - v\|_c;$$

(B_2) For arbitrary $u \in C(J, \mathbf{R})$, there exist $C_g, M_g > 0$ and $r_1 \in [0,1)$,

$$| f(u) | \leq C_g ||u||_c^{r_1} + M_g;$$

(B_3) For arbitrary $u \in C(J, \mathbf{R})$, there exist $C_f, M_f > 0$ and $r_2 \in [0,1)$,

$$| f(t,u) | \leq C_f ||u||_c^{r_2} + M_f;$$

(B_4) There exists a constant $L_f > 0$, such that

$$| f(t,u) - f(t,\bar{u}) | \leq L_f \|u - \bar{u}\|_c, \ for any \ u, \bar{u} \in \mathbf{R}.$$

Let an operator $T : C(J, \mathbf{R}) \longrightarrow C(J, \mathbf{R}) be defined$. Then 7 in the form of operator equation as

$$Tu(t) = Fu(t) + Gu(t), \tag{8}$$

where

$$Fu(t) = \frac{1}{\lambda_1 + \mu_1} g_1(u) + \frac{1}{\lambda_2 + \mu_2}[t - \mu_1] g_2(u), \ Gu(t) = \int_0^1 \mathcal{G}(t,s) f(s,u(s))ds.$$

The solution of operator equation (8) is the corresponding solution of the considered problem (1).

3 Main Results

Theorem 5 *The operator $F : C(J, \mathbf{R}) \to C(J, \mathbf{R})$ is Lipschitz with constant $k_g \in [0,1)$. Consequently F is α-Lipschitz with constant k_g. Moreover F obeys the growth condition given by*

$$\|Fu\|_c \leq C_g \|u\|_c^{r_1} + M_g, \quad for every \ u \in C(J, \mathbf{R}). \tag{9}$$

Proof By (B_1)

$$\begin{aligned}\|Fu - Fv\|_c &= \sup\left|\frac{1}{\lambda_1+\mu_1}(g_1(u)-g_1(v)) + \left(\frac{t}{\lambda_2+\mu_2} - \frac{\mu_1}{\lambda_2+\mu_2}\right)(g_2(u)-g_1(v))\right| \\ &\le \frac{1}{|\lambda_1+\mu_1|}|g_1(u)-g_1(v)| + \left|\frac{1}{\lambda_2+\mu_2} - \frac{\mu_1}{\lambda_2+\mu_2}\right| |g_2(u)-g_2(v)|,\end{aligned}$$

using $t \le 1$

$$\begin{aligned}\|Fu - Fv\|_c &\le \frac{k_{g_1}}{|\lambda_1+\mu_1|}\|u-v\|_c + \frac{k_{g_2}}{|\lambda_2+\mu_2|}\|u-v\|_c \ using\ k_{g_1},\ k_{g_2} \in [0,1) \\ &\le \left[\frac{k_{g_1}}{|\lambda_1+\mu_1|} + \frac{k_{g_2}}{|\lambda_2+\mu_2|}\right]\|u-v\|_c\,,\end{aligned}$$

using

$$\left[\frac{k_{g_1}}{|\lambda_1+\mu_1|} + \frac{k_{g_2}}{|\lambda_2+\mu_2|}\right] = k_g.$$

Thus

$$\|Fu - Fv\| \le k_g\|u-v\|_c\,.$$

Hence in view of Proposition 1, F is α-Lipschitz with constant k_g. For growth condition, consider

$$\begin{aligned}\|Fu\|_c &= \sup\left|\frac{1}{\lambda_1+\mu_1}g_1(u) + \frac{1}{\lambda_2+\mu_2}(t-\mu_1)g_2(u)\right| \\ &\le \sup\left|\frac{1}{\lambda_1+\mu_1}g_1(u)\right| + \sup\left|\frac{t-\mu_1}{\lambda_2+\mu_2}\right||g_2(u)| \\ &\le \frac{C_{g_1}}{|\lambda_1+\mu_1|}\|u\|_c^{r_1} + M_{g_1} + \frac{1}{|\lambda_2+\mu_2|}C_{g_2}\|u\|_c^{r_1} + M_{g_2} \\ &= \left[\frac{C_{g_1}}{|\lambda_1+\mu_1|} + \frac{C_{g_2}}{|\lambda_2+\mu_2|}\right]\|u\|_c^{r_1} + M_{g_1} + M_{g_2}\end{aligned}$$

which implies that

$$\|Fu\|_c \le C_g\|u\|_c^{r_1} + M_g,\ C_g = \frac{C_{g_1}}{|\lambda_1+\mu_1|} + \frac{C_{g_2}}{|\lambda_2+\mu_2|},\ M_g = M_{g_1} + M_{g_2}.$$

which is the growth condition (9).

Theorem 6 *The operator $G : C(J,\mathbf{R}) \longrightarrow C(J,\mathbf{R})$ is continuous, moreover G satisfies the following growth condition:*

$$\|Gu_n\|_c \leq M\frac{C_f\,\|u\|_c^{r_2}+M_f}{\Gamma(q)},\ \ for\,every\ u\in C(J,\mathbf{R}),$$

where $M = 1+\left|\frac{\mu_1}{\lambda_1+\mu_1}\right|+\left|\frac{\mu_2}{\lambda_2+\mu_2}\left(\frac{\mu_1}{\lambda_1+\mu_1}\right)\right|.$

Proof Consider that $\{u_n\}$ be the sequence of bounded set $\mathscr{B}_k=\{\|u\|_c\leq k : u\in C(J,\mathbf{R})\}$.
Where $\mathscr{B}_k\subseteq C(J,\mathbf{R})$ and $u_n\to u$ as $n\to\infty$ in $\mathscr{B}_k$. We have to show that $\|Gu_n-Gu\|_c\to 0$ as $n\to\infty$.
Consider

$$\begin{aligned}|Gu_n(t)-Gu(t)| &\leq \int_0^1 |\mathscr{G}(t,s)|\,|f(s,u_n(s)-f(s,u(s))|\,ds\\ &\leq \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\,|f(s,u_n(s))-f(s,u(s))|\,ds\\ &\quad+\left|\frac{\mu_1}{\lambda_1+\mu_1}\right|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\,|f(s,u_n(s))-f(s,u(s))|\,ds\\ &\quad+\left|\frac{\mu_2}{\lambda_2+\mu_2}\right|\left[\frac{\mu_1}{\lambda_1+\mu_1}\right]\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\\ &\quad\times|f(s,u_n(s))-f(s,u(s))|\,ds.\end{aligned}$$

In view of continuity of f, we have

$$f(t,u_n(s))\to f(t,u(s))\ as\ n\to\infty,\ \ for\,each\ t\in J.$$

Applying (B_3), and using Lebesgue dominated convergent theorem, we have

$$\int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\left[C_f\,\|u\|_c^{r_2}+M_f\right]ds\to 0\ as\ n\to\infty.$$

Consequently, $\left|\frac{\mu_1}{\lambda_1+\mu_1}\right|\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\left[C_f\,\|u\|_c^{r_2}+M_f\right]ds\to 0\ as\ n\to\infty$
and

$$\left|\frac{\mu_2}{\lambda_2+\mu_2}\right|\left[\frac{\mu_1}{\lambda_1+\mu_1}\right]\int_0^1\frac{(1-s)^{q-2}}{\Gamma(q-1)}\left[C_f\,\|u\|_c^{r_2}+M_f\right]ds\to 0\ as\ n\to\infty.$$

From which it is followed that $\|Gu_n-Gu\|\to 0\ as\ n\to\infty$. Thus G is continuous.

For growth condition, consider

$$
\begin{aligned}
|Gu_n(t)| &= \left|\int_0^1 \mathscr{G}(t,s) f(s,u_n(s))ds\right| \\
&\le \int_0^1 \mathscr{G}(t,s)\,|f(s,u_n(s))|\,ds \\
&\le \int_0^1 \frac{(t-s)^{q-1}}{\Gamma(q)}\left[C_f\,\|u\|_c^{r_2}+M_f\right]ds \\
&\quad + \left|\frac{\mu_1}{\lambda_1+\mu_1}\right| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\left[C_f\,\|u\|_c^{r_2}+M_f\right]ds \\
&\quad + \frac{1}{\Gamma(q-1)}\left|\frac{\mu_2}{\lambda_2+\mu_2}\right|\left(\left|\frac{\mu_1}{\lambda_1+\mu_1}-1\right|\right) \\
&\times \int_0^1 (1-s)^{q-2}\left[C_f\,\|u\|_c^{r_2}+M_f\right]ds.
\end{aligned}
$$

From which, we have

$$
\|Gu_n\| \le \frac{C_f\|u\|_c^{r_2}+M_f}{\Gamma(q)}\left[1+\left|\frac{\mu_1}{\lambda_1+\mu_1}\right|\left(\left|\frac{\mu_2}{\lambda_2+\mu_2}\right|+1\right)\right]. \tag{10}
$$

Hence

$$
\|Gu_n\| \le \frac{M}{\Gamma(q)}\left(C_f\|u\|_c^{r_2}+M_f\right). \tag{11}
$$

Theorem 7 *The operator $G : C(J,\mathbf{R}) \to C(J,\mathbf{R})$ is completely continuous and α-Lipschitz with constant zero.*

Proof For the compactness of G, we consider $\mathscr{D} \subseteq \mathscr{B}_k \subseteq C(J,\mathbf{R})$ is bounded set. We have to show that $G(\mathscr{D})$ is relatively compact in $C(J,\mathbf{R})$ with the help of Arzel$\grave{a}$ Ascali theorem.
Let $\{u_n\}$ be sequence in $\mathscr{D} \subseteq \mathscr{B}_k$ for every $u_n \in \mathscr{D}$. Then from Growth condition (11), it is obvious that $G(\mathscr{D})$ is bounded in $C(J,\mathbf{R})$.

Let $0 \le t_1 \le t_2 \le 1$, then for equi-continuity, we discuss two cases from Green's function (2) as:

Case I $0 \le s \le t \le 1$.

$$
\begin{aligned}
|Gu_n(t_1)-Gu_n(t_2)| &\le \frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right]|f(s,u_n(s))|\,ds \\
&\quad + \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\,|f(s-u_n(s))|\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+(t_2-t_1)\left|\frac{\mu_2}{\lambda_2+\mu_2}\right|\frac{1}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}\,|f(s,u_n(s))|\,ds\\
&\le \frac{1}{\Gamma(q)}\int_0^{t_1}\left[(t_1-s)^{q-1}-(t_2-s)^{q-1}\right]\left(C_f\,\|u_n\|_c^{r_2}+M_f\right)ds\\
&+\frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\left(C_f\,\|u_n\|_c^{r_2}+M_f\right)ds\\
&+(t_2-t_1)\frac{|\mu_2|}{|\lambda_2+\mu_2|}\frac{1}{\Gamma(q-1)}\int_0^1(1-s)^{q-2}\left(C_f\|u_n\|_c^{r_2}+M_f\right)ds.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\|Gu_n(t_1)-Gu_n(t_2)\| = &\left[\frac{(t_1^q-t_2^q)}{\Gamma(q+1)}+\frac{(t_2-t_1)^q}{\Gamma(q+1)}+\frac{(t_2-t_1)\,|\mu_2|}{|\lambda_2+\mu_2|\,\Gamma(q)}\right]\\
&\times\left(C_f\,\|u\|_c^{r_2}+M_f\right). \qquad (12)
\end{aligned}
$$

Clearly $t_1\to t_2$, then the right-hand side of (12) tends to zero. So

$$
\|Gu_n(t_1)-Gu_n(t_2)\|\to 0 \;\; as \;\; t_1\to t_2.
$$

Thus in this case G is equi-continuous.

Case II if $0\le t\le s\le 0$, then

$$
\begin{aligned}
|Gu_n(t_1)-Gu_n(t_2)| &\le \frac{\mu_2\,(t_1-t_2)}{|\lambda_2+\mu_2|\,\Gamma(q-1)}\int_0^1(1-s)^{q-2}\,|f(s,u_n(s))|\,ds\\
&\le \frac{\mu_2\,(t_1-t_2)}{|\lambda_2+\mu_2|\,\Gamma(q)}\left(C_f\,\|u\|_c^{r_2}+M_f\right)\to 0,\;\, as\; t_1\to t_2.
\end{aligned}
$$

So G in this case is also equi-continuous. Hence G is equi-continuous and $G(\mathscr{D})\subseteq C(J,\mathbf{R})$, which satisfies the hypothesis of Arzela Ascali theorem. So $G(\mathscr{D})$ is relatively compact in $C(J,\mathbf{R})$. G is completely continuous. It is easy to show that G is $\alpha-$Lipschitz with constant zero by using Proposition 2.

Theorem 8 *Assume that* (B_1)–(B_3) *hold, then boundary value problem (1) has at least one positive solution* $u\in C(J,\mathbf{R})$ *and the set of the solutions is bounded in* $C(J,\mathbf{R})$.

Proof As $F,G,T\;:\;C(J,\mathbf{R})\;\to\;C(J,\mathbf{R})$ have been defined previously are continuous in view of continuity of f,g. Moreover F and G are α-Lipschitz. Thus T is strict α-contraction. Consider

$$
W_0=\{u\in C(J,\mathbf{R}):\; there exist\;\lambda\in[0,1],\lambda u=\lambda Tu\}.
$$

To prove that W_0 is bounded subset of $C(J, \mathbf{R})$. Let $u \in W_0$ such that $u = \lambda T u$, one can see

$$\begin{aligned}\|u\|_c &= \|\lambda T u\|_c = \lambda \left(\|Fu + Gu\|_c\right)\\ &\leq \lambda \left(\|Fu\|_c + \|Gu\|_c\right)\end{aligned}$$

In view of Growth conditions of F, G, we get

$$\|u\|_c \leq \left(C_g \|u\|_c^{r_1} + M_g + M \frac{C_f \|u\|_c^{r_2} + M_f}{\Gamma(q)}\right), \quad r_1, r_2 \in [0, 1). \tag{13}$$

Thus W_0 is bounded. If not, let $\mathscr{R} = \|u\|_c$, taking $\mathscr{R} = \|u\|_c$ such that $\mathscr{R} \to \infty$.
Then from (13), we have

$$1 \leq \lim_{\mathscr{R}\to\infty} \lambda \left[\frac{C_g \|u\|_c^{r_1} + M_g}{\mathscr{R}} + M \frac{C_f \|u\|_c^{r_2} + M_f}{\mathscr{R}\Gamma(q)}\right] = 0.$$

which is contraction.

This implies that W_0 is bounded and T has at least one fixed point by means of Theorem 1, which is the corresponding positive solution of boundary value problem therefore (1).

Theorem 9 *Under the assumption* (B_1) *to* (B_4)*, boundary value problem (1) has a unique solution if* $G^* < 1$*, where*

$$G^* = \frac{k_{g_1}}{|\lambda_1 + \mu_1|} + \frac{k_{g_2}}{|\lambda_2 + \mu_2|} + L_f \int_0^1 \mathscr{G}(t, s) ds.$$

Proof From (B_1)–(B_3), we have

$$\begin{aligned}\|Tu - Tv\| \leq\ & \frac{1}{|\lambda_1 + \mu_1|} \|g_1(u) - g_1(v)\|_c + \frac{1}{|\lambda_2 + \mu_2|} \|g_2(u) - g_2(v)\|_c\\ &+ \int_0^1 \mathscr{G}(t, s) \|f(s, u) - f(s, v)\| ds\\ &\leq \frac{k_{g_1}}{|\lambda_1 + \mu_1|} \|u - v\|_c + \frac{k_{g_2}}{|\lambda_2 + \mu_2|} \|u - v\|_c\\ &+ L_f \int_0^1 \mathscr{G}(t, s) \|u - v\|_c ds\\ &\leq \left[\frac{k_{g_1} + k_{g_2}}{|\lambda_1 + \mu_1| + |\lambda_2 + \mu_2|} + L_f \int_0^1 \mathscr{G}(1, s) ds\right] \|u - v\|_c\end{aligned}$$

$$\|Tu - Tv\|_c \leq G^* \|u - v\|_c$$

where

$$G^* = \frac{k_{g_1}}{|\lambda_1 + \mu_1|} + \frac{k_{g_2}}{|\lambda_2 + \mu_2|} + L_f \int_0^1 \mathscr{G}(1,s)ds < 1.$$

Hence T has a unique fixed point, which is the corresponding positive solution to the considered problem (1).

4 Ulam's Stability Analysis of Boundary Value Problem (1)

In this section, we prove necessary and sufficient conditions for various types of Ulam's stability like Ulam–Hyers, generalized Ulam–Hyers stability, Ulam–Hyers–Rassias, and generalized Ulam–Hyers–Rassias stability of the solutions to the considered problem (1) of nonlinear fractional differential equations. In this regard we review the following definitions and results for further analysis.

Definition 4 The solution $u \in C([0,1])$ of the fractional differential equation given by

$$^cD^q u(t) = f(t, u(t)),\ t \in J, \tag{14}$$

is Ulam–Hyers stable if we can find a real number $\hat{C}_{L_f,k_g,\mathscr{G}^*} > 0$ with the property that for every $\epsilon > 0$ and for every solution $u \in C[0,1]$ of the inequality

$$\left|{}^cD^q u(t) - f(t, u(t))\right| \leq \epsilon,\ t \in [0,1], \tag{15}$$

there exists unique solution $v \in C[0,1]$ of the given fractional differential equation (1) with a constant $\hat{C}_{L_f,k_g,\mathscr{G}^*} > 0$ with

$$\|u - v\|_c \leq \hat{C}_{L_f,k_g,\mathscr{G}^*}\epsilon.$$

Definition 5 The solution $u \in C[0,1]$ of the fractional differential equation (1) is called to be generalized Ulam–Hyers stable , if we can find

$$\theta_{f,q} : (0,\infty) \to \mathbf{R}^+,\ \theta_{f,q}(0) = 0,$$

such that for each solution $u \in C[0,1]$ of the inequality (15), we can find a unique solution $v \in C[0,1]$ of the fractional differential equation (1) with

$$\|u - v\|_c \leq \hat{C}_{L_f,k_g,\mathscr{G}^*}\theta_{f,q}.$$

Next we recall the definitions of Ulam–Hyers–Rassias and generalized Ulam–Hyers–Rassias stability [34] for our considered problem (1) as below:

Definition 6 Fractional differential equation (1) is said to be Ulam–Hyers–Rassias stable with respect to $\varphi \in C([0,1],\mathbf{R})$ if there exists a nonzero positive real constant $\hat{C}_{L_f,k_g,\mathcal{G}^*}$ such that for each $\epsilon > 0$ and for every solution $u \in C[0,1]$ of the inequality

$$\left|{}^cD^q u(t) - f(t,u(t))\right| \le \varphi(t)\epsilon,\ t \in [0,1], \tag{16}$$

there exists a solution $v \in C[0,1]$ of the Eq. (1), such that

$$|u(t) - v(t)| \le \hat{C}_{L_f,k_g,\mathcal{G}^*}\epsilon\varphi(t),\ t \in [0,1].$$

Definition 7 Equation (1) is said to be generalized Ulam–Hyers–Rassias stable with respect to $\varphi \in C[0,1]$, if there exists a real number $\hat{C}_{L_f,k_g,\mathcal{G}^*} > 0$ such that for each solution $u \in C[0,1]$ of the inequality

$$\left|{}^cD^q u(t) - f(t,u(t))\right| \le \varphi(t),\ t \in [0,1], \tag{17}$$

there exists a solution $v \in C[0,1]$ of the Eq. (1) such that $|u(t) - v(t)| \le \hat{C}_{L_f,k_g,\mathcal{G}^*}\theta(\epsilon)\varphi(t)$, $t \in [0,1]$.

Remark 1 A function $u \in C[0,1]$ is said to be the solution of inequality given in (15) if and only if there exists a function $\varpi \in C[0,1]$ that depends on u only such that

(i) $|\varpi(t)| \le \epsilon,\ forall\ t \in [0,1]$;
(ii) ${}^cD^q u(t) = f(t,u(t)) + \varpi(t),\quad forall\ t \in [0,1]$.

Lemma 1 *Under the assumption given in Remark 1, the solution* $u \in C[0,1]$ *of the boundary value problem given by*

$$\begin{cases} {}^cD^q u(t) = f(t,\ u(t)) + \varpi(t),\ 1 < q \le 2,\ t \in [0,1], \\ \lambda_1 u(0) + \mu_1 u(1) = g_1(u), \\ \lambda_2 u'(0) + \mu_2 u'(1) = g_1(u) \end{cases} \tag{18}$$

satisfies the following relation:

$$\left|u(t) - \left(g(u) + \int_0^1 \mathcal{G}(t,s)f(s,u(s))ds\right)\right| \le \epsilon\mathcal{G}^*,\ where\ \max_{t\in[0,1]}\int_0^1 |\mathcal{G}(t,s)|ds = \mathcal{G}^*. \tag{19}$$

Proof In view of Theorem 4, the solution of the problem (18) is given by

$$u(t) = g(u) + \int_0^1 \mathcal{G}(t,s)f(s,u(s))ds + \int_0^1 \mathcal{G}(t,s)\varpi(s)ds,$$

where $\mathscr{G}$ is the Green's function as defined in Theorem 4. Using Remark 1 we can easily get the result given in (19).

Theorem 10 *Under the assumption* $(B_1), (B_4)$ *and Lemma 1, the solution of the considered problem (1) is Ulam's stable and consequently generalized Ulam–Hyers stable if the condition* $[k_g + L_f\mathscr{G}^*] < 1$ *holds.*

Proof Let $u \in C[0, 1]$ be any solution of boundary value problem (1) and $v \in C[0, 1]$ be the unique solution of the considered problem (1), then take

$$\begin{aligned}
|u(t) - v(t)| &= \left|u(t) - \left(g(v) + \int_0^1 \mathscr{G}(t, s) f(s, v(s))ds\right)\right| \\
&\leq \left|u(t) - \left(g(u) + \int_0^1 \mathscr{G}(t, s) f(s, u(s))ds\right)\right| \\
&+ \left|g(u) - g(v) + \int_0^1 \mathscr{G}(t, s)[f(s, u(s)) - f(s, v(s))]\right| \\
&\leq \epsilon\mathscr{G}^* + k_g\|u - v\|_c + L_f\mathscr{G}^*\|u - v\|_c.
\end{aligned}$$

From which we have

$$\|u - v\|_c \leq \epsilon\mathscr{G}^* + [k_g + L_f\mathscr{G}^*]\|u - v\|_c,$$

where k_g is defined in Theorem 5 which yields

$$\|u - v\|_c \leq \hat{C}_{L_f,k_g,\mathscr{G}^*}\epsilon, \ \ where \ \frac{\mathscr{G}^*}{1 - [k_g + L_f\mathscr{G}^*]} = \hat{C}_{L_f,k_g,\mathscr{G}^*}. \tag{20}$$

Hence the solution of the considered problem (1) is Ulam–Hyers stable. Further if we set $\theta(\epsilon) = \epsilon$ such that $\theta(0) = 0$, then we get

$$\|u - v\|_c \leq \hat{C}_{L_f,k_g,\mathscr{G}^*}\theta(\epsilon) \tag{21}$$

which implies that the solution of the proposed problem is generalized Ulam–Hyers stable.

(B_5) Let for $\delta_\varphi > 0$ there exists a nondecreasing function $\varphi \in ([0, 1], \mathbf{R}^+)$ such that

$$\int_0^t \frac{(t - s)^{q-1}}{\Gamma(q)}\varphi(s)ds \leq \delta_\varphi\varphi(t), \ \ for\ t \in [0, 1].$$

Theorem 11 *Under the assumptions* $(B_1), (B_4), (B_5)$*, the solution of the considered problem (1) is Ulam–Hyers–Rassias stable if* $k_g + L_f\mathscr{G}^* < 1$.

Proof Let $u \in C[0,1]$ be any solution of the inequality (16) and $v \in C[0,1])$ be the unique solution of the problem (1), then the solution of

$$\begin{aligned}
&{}^{c}D^{q}u(t) = f(t,\ u(t)) + \varpi(t),\ 1 < q \leq 2,\ t \in [0,1],\\
&\lambda_1 u(0) + \mu_1 u(1) = g_1(u),\\
&\lambda_2 u'(0) + \mu_2 u'(1) = g_1(u)
\end{aligned}$$

is given by

$$\begin{aligned}
u(t) = {}& \frac{1}{\lambda_1+\mu_1} g_1(u) + \frac{1}{\lambda_2+\mu_2}(t-\mu_1)\, g_2(u) + \frac{\mu_1}{\lambda_1+\mu_1}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s,u(s))\\
&+ \frac{\mu_2}{\lambda_2+\mu_2}\left(\frac{\mu_1}{\lambda_1+\mu_1} - t\right)\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s,u(s)) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s,u(s))\\
&+ \frac{\mu_1}{\lambda_1+\mu_1}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\varpi(s) + \frac{\mu_2}{\lambda_2+\mu_2}\left(\frac{\mu_1}{\lambda_1+\mu_1} - t\right)\\
&\times \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\varpi(s) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\varpi(s)\\
u(t) = {}& g(u) + \int_0^1 \mathscr{G}(t,s) f(s,u(s))ds \qquad (22)\\
&+ \frac{\mu_1}{\lambda_1+\mu_1}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\varpi(s) + \frac{\mu_2}{\lambda_2+\mu_2}\left(\frac{\mu_1}{\lambda_1+\mu_1} - t\right)\\
&\times \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\varpi(s) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\varpi(s).
\end{aligned}$$

Then from (22) we have

$$\begin{aligned}
&\left| u(t) - g(u) - \int_0^1 \mathscr{G}(t,s) f(s,u(s))ds \right|\\
&\quad = \left| \frac{\mu_1}{\lambda_1+\mu_1}\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\varpi(s) + \frac{\mu_2}{\lambda_2+\mu_2}\left(\frac{\mu_1}{\lambda_1+\mu_1} - t\right)\right.\\
&\quad\quad \left. \times \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\varpi(s) + \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\varpi(s) \right|\\
&\quad \leq \epsilon\delta_{\varphi}\varphi(t).
\end{aligned}$$

Then using the same fashion as in Theorem 10, we have

$$|u(t) - v(t)| \leq \left| u(t) - \left(g(u) + \int_0^1 \mathscr{G}(t,s) f(s,u(s))ds \right) \right|$$

$$+ |g(u) - g(v)| + \left| \int_0^1 \mathscr{G}(t,s) f(s, u(s)) ds - \int_0^1 \mathscr{G}(t,s) f(s, v(s)) ds \right|$$
$$\leq \epsilon \delta_\varphi \varphi(t) + k_g \|u - v\|_c + L_f \mathscr{G}^* \|u - v\|_c$$

$$\textit{which further gives} \|u - v\|_c \leq \epsilon \delta_\varphi \varphi(t) + [k_g + L_f \mathscr{G}^*] \|u - v\|_c. \tag{23}$$

Hence we have

$$\|u - v\|_c \leq \hat{C}_{L_f, k_g, \mathscr{G}^*} \epsilon \delta_\varphi \varphi(t), \; t \in [0, 1], \; where \; \hat{C}_{L_f, k_g, \mathscr{G}^*} = \frac{1}{1 - [k_g + L_f \mathscr{G}^*]}. \tag{24}$$

Hence from (24) we concluded that the solution of the considered problem (1) is Ulam–Hyers–Rassias stable. Further it is easy to prove that the solution of the considered problem (1) is generalized Ulam–Hyers Rassias stable.

5 Example

Example 1 Consider the boundary value problem

$$\begin{aligned} {}^c D^{\frac{3}{2}} u(t) &= \frac{|u(t)|}{(1 + e^t)(1 + 9u(t))}, \quad t \in [0, 1], \\ u(0) + u(1) &= g_1(u) = \sum_{k=1}^{5} \delta_k u(t_k), \; t_k \in (0, 1), \; \sum_{k=1}^{5} \delta_k \leq \frac{1}{20}, \\ \frac{1}{2} \acute{u}(0) + \frac{1}{2} \acute{u}(1) &= g_2(u) = \sum_{k=1}^{3} \acute{\delta}_k u(t_k), \; t_k \in (0, 1), \; \sum_{k=1}^{3} \acute{\delta}_k \leq \frac{1}{10}. \end{aligned} \tag{25}$$

Then $\lambda_1 = \mu_1 = 1, \lambda_2 = \mu_2 = \frac{1}{2}, \; g_1(u) = \sum_{k=1}^{5} \delta_k u(t_k), g_2(u) = \sum_{k=1}^{3} \acute{\delta}_k u(t_k)$, and $f(t, u) = \frac{|u(t)|}{(1+e^t)(1+9u(t))}$.

We have

$$|f(t, u) - f(t, v)| \leq \left| \frac{|u| (1 + 9 |v|) - |v| (1 + 9 |u|)}{(1 + e^t)(1 + 9 |u|)(1 + 9 |v|)} \right| \leq \frac{1}{200} |u - v|.$$

Clearly

$$M_f = 0, C_f = \frac{1}{2},\ k_{g_1} = \frac{1}{20},\ k_{g_2} = \frac{1}{10},\ from\ which,\ we\ have$$
$$k_g = \frac{3}{20},\ r_1 = r_2 = \frac{1}{2}, L_f = \frac{1}{200}, q = \frac{3}{2}.$$

By simple computation, one can show that

$$G^* = \frac{\frac{1}{20}}{2} + \frac{\frac{1}{10}}{1} + \frac{1}{200}\int_0^1 \mathscr{G}(1,s)ds = \frac{1}{40} + \frac{1}{10} + \frac{1}{200}\int_0^1 \mathscr{G}(1,s)ds < 1.$$

Thus in view of Theorem 9, (25) has unique solution. Further, it is easy to show that the set of solution is bounded by using Theorem 8. Further the condition $k_g + L_f\mathscr{G}^* < 1$ obviously holds so by Theorem 10, the solution of the given problem is Ulam–Hyers stable and consequently generalized Ulam–Hyers stable. Let $\varphi(t) = t$, then the conditions of Ulam–Hyers Rassias and generalized Ulam–Hyers Rassias stability can be easily received by using Theorem 11.

6 Conclusion

Considering the Caputo fractional derivative we have successfully established existence theory of at least one solution to a boundary value problem of fractional differential equations by using topological degree theory. Further by using nonlinear functional analysis we have developed appropriate conditions for different kinds of Ulam stability theory including Ulam–Hyers, generalized Ulam–Hyers, Ulam–Hyers Rassias, and generalized Ulam–Hyers–Rassias stability. The whole results have been demonstrated by a proper example.

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References

1. R. Hilfer, *Applications of Fractional Calculus in Physics* (World Scientific, Singapore, 2000)
2. A.A. Kilbas, O.I. Marichev, S.G. Samko, *Fractional Integrals and Derivatives (Theory and Applications)* (Gordon and Breach, Switzerland, 1993)
3. A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, vol. 204 (Elsevier, Amsterdam, 2006)
4. V. Lakshmikantham, S. Leela, J. Vasundhara, *Theory of Fractional Dynamic Systems* (Cambridge Academic, Cambridge, 2009)

5. K.S. Miller, B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations* (Wiley, New York, 1993)
6. I. Podlubny, *Fractional Differential Equations*. Mathematics in Science and Engineering (Academic, New York, 1999)
7. V.E. Tarasov, *Fractional Dynamics: Application of Fractional Calculus to Dynamics of Particles, Fields and Media* (Springer, HEP, Heidelberg, Germany, 2010)
8. I. Podlubny, *Fractional differential equations. An introduction to fractional derivative, fractional differential equations, to methods of their solution and some of their applications*. Mathematics in Science and Engineering, vol. 198 (Academic, San Diego, 1999)
9. M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order and nonlocal conditions. Nonlinear Anal. **71**, 2391–2396 (2009)
10. B. Ahmad, S. Sivasundaram, On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order. Appl. Math. Comput. **217**, 480–487 (2010)
11. Z. Bai, On positive solutions of a nonlocal fractional boundary value problem. Nonlinear Anal. **72**, 916–924 (2010)
12. M. El-Shahed, J.J. Nieto, Nontrivial solutions for a nonlinear multi-point boundary value problem of fractional order. Comput. Math. Appl. **59**, 3438–3443 (2010)
13. A. Belarbi, M. Benchohra, A. Ouahab, Existence results for functional differential equations of fractional order. Appl. Anal. **85**, 1459–1470 (2006)
14. R.P. Agarwal, M. Benchohra, S. Hamani, Boundary value problems for differential inclusions with fractional order. Adv. Stud. Contemp. Math. **12**(2), 181–196 (2008)
15. M. Benchohra, J.R. Graef, S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations. Appl. Anal. **87**, 851–863 (2008)
16. M. Benchohra, S. Hamani, S.K. Ntouyas, Boundary value problems for differential equations with fractional order. Surv. Math. Appl. **3**, 1–12 (2008)
17. K. Shah, R.A. Khan, Existence and uniqueness of positive solutions to a coupled system of nonlinear fractional order differential equations with anti periodic boundary conditions. Differ. Equ. Appl. **7**(2), 245–262 (2015)
18. K. Shah, H. Khalil, R.A. Khan, Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations. J. Chaos Solitons Fractals **77**, 240–246 (2015)
19. C.F. Li, X.N. Luo, Y. Zhou, Existence of positive solutions of the boundary value problem for nonlinear fractional differential equations. Comput. Math. Appl. **59**, 1363–1375 (2010)
20. J. Wang , Y. Zhou, W. Wei, Study in fractional differential equations by means of topological degree methods. Numer. Funct. Anal. Optim. **33**(2), 216–238 (2012)
21. R.A. Khan, K. Shah, Existence and uniqueness of positive solutions to fractional order multi-point boundary value problems. Commun. Appl. Anal. **19**, 515–526 (2015)
22. K. Shah, A. Ali, R.A. Khan, Degree theory and existence of positive solutions to coupled systems of multi-point boundary value problems. Bound. Value Probl. **2016**, 43 (2016)
23. I.A. Rus, Ulam stabilities of ordinary differential equations in a Banach space. Carpathian J. Math. **26**, 103–107 (2010)
24. M. Obloza, Hyers stability of the linear differential equation. Rocznik Nauk.-Dydakt. Prace Mat. **13**, 259–270 (1993)
25. J. Wang, X. Li, Ulam-Hyers stability of fractional Langevin equations. Appl. Math. Comput. **258**, 72–83 (2015)
26. S. Tang, A. Zada, S. Faisal, M.M.A. El-Sheikh, T. Li, Stability of higher order nonlinear impulsive differential equations. J. Nonlinear Sci. Appl. **9**, 4713–4721 (2016)
27. G. Lijun, D. Wang, G. Wang, Further results on exponential stability for impulsive switched nonlinear time-delay systems with delayed impulse effects. Appl. Math. Comput. **268**, 186–200 (2015)
28. I. Stamova, Mittag-Leffler stability of impulsive differential equations of fractional order. Q. Appl. Math. **73**(3), 525–535 (2015)
29. J.C. Trigeassou, et al., A Lyapunov approach to the stability of fractional differential equations. Signal Process. **91**(3), 437–445 (2011)

30. J. Wang, L. Lv, W. Zhou, Ulam stability and data dependence for fractional differential equations with Caputo derivative. Electron. J. Qual. Theory Differ. Equ. **63**, 1–10 (2011)
31. K. Shah, D. Vivek, K. Kanagarajan, Dynamics and stability of Ψ-fractional pantograph equations with boundary conditions. Bol. Soc. Paran. Mat. **22**(2), 1–13 (2018)
32. A. Zada, S. Faisal, Y. Li, On the Hyers–Ulam stability of first order impulsive delay differential equations. J. Funct. Spaces **2016**, 6 pages (2016)
33. A. Zada, O. Shah, R. Shah, Hyers–Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems. Appl. Math. Comput. **271**, 512–518 (2015)
34. T.M. Rassias, On the stability of the linear mapping in Banach spaces. Proc. Am. Math. Soc. **72**, 297–300 (1978)
35. S.M. Ulam, *Problems in Modern Mathematics* (Wiley, New York, 1940)
36. S.M. Ulam, *A Collection of Mathematical Problems* (Interscience, New York, 1960)
37. P. Kumama, A. Ali, K. Shah, R.A. Khan, Existence results and Hyers-Ulam stability to a class of nonlinear arbitrary order differential equations. J. Nonlinear Sci. Appl. **10**, 2986–2997 (2017)
38. J. Wang, L. Lv, Y. Zhou, Ulam stability and data dependece for fractional differential equations with Caputo derivative. Electron. J. Qual. Theory Differ. Equ. **63**, 1–10 (2011)
39. F. Haq, K. Shah, G. Rahman, M. Shahzad, Hyers-Ulam stability to a class of fractional differential equations with boundary conditions. Int. J. Appl. Comput. Math. **2017**, 1–13 (2017)
40. J. Vanterler da C. Sousa, E. Capelas de Oliveira, Ulam–Hyers stability of a nonlinear fractional Volterra integro-differential equation. Appl. Math. Lett. **81**, 50–56 (2018)
41. A. Zada, S. Ali, Y. Li, Ulam's type stability for a class of implicit fractional differential equations with non-instantaneous integral impulses and boundary condition. Adv. Difference Equ. **2017**, 317 (2017)
42. J. Wang, M. Feckan, Y. Zhou, Ulam's type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. **395**, 258–264 (2012)
43. S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Nonlinear Analysis* (Springer, New York, 2011)
44. J. Wang, M. Feckan, Y. Zhou, Ulam's type stability of impulsive ordinary differential equations. J. Math. Anal. Appl. **395**, 258–264 (2012)
45. D.H. Hyers, G. Isac, T.M. Rassias, *Stability of Functional Equations in Several Variables* (Birkhäuser, Basel, 1998)
46. S.M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis* (Hadronic Press, Palm Harbor, 2001)
47. S.M. Jung, Hyers-Ulam stability of linear differential equations of first order. Appl. Math. Lett. **17**, 1135–1140 (2004)
48. S.E. Takahasi, T. Miura, S. Miyajima, On the Hyers-Ulam stability of the Banach space-valued differential equation $y' = \lambda y$. Bull. Korean Math. Soc. **39**, 309–315 (2002)
49. D.S. Cimpean, D. Popa, Hyers-Ulam stability of Euler's equation. Appl. Math. Lett. **24**, 1539–1543 (2011)
50. J. Wang, Y. Zhou, Mittag-Leffler-Ulam stabilities of fractional evolution equations. Appl. Math. Lett. **25**, 723–728 (2012)
51. J. Wang, L. Lv, Y. Zhou, Boundary value problems for fractional differential equations involving Caputo derivative in Banach spaces. J. Appl. Math. Comput. **38**(1–2), 209–224 (2012)
52. K. Deimling, *Nonlinear Functional Analysis* (Springer, New York, 1985)
53. K. Diethelm, *The Analysis of Fractional Differential Equations*. Lecture Notes in Mathematics (Springer, New York, 2010)
54. F. Isaia, On a nonlinear integral equation without compactness. Acta. Math. Univ. Comenianae. **75**, 233–240 (2006)
55. S.M. Ulam, *A Collection of the Mathematical Problems* (Interscience, New York, 1960)
56. T.M. Rassias, On the stability of functional equations and a problem of Ulam. Acta. Appl. Math. **62**, 23–130 (2000)