Stability of an n-Dimensional Functional Equation in Banach Space and Fuzzy Normed Space

Sandra Pinelas, V. Govindan, and K. Tamilvanan

Abstract In this paper, the authors investigate the general solution of a new additive functional equation

$$f\left(\sum_{i=1}^{n} x_i\right)+\sum_{j=1;i\neq j}^{n} f\left(-x_j-x_i+\sum_{1\le i<j<k\le n} x_k\right)=\left(\frac{n^2-5n+6}{2}\right)\sum_{i=1}^{n} f(x_i)$$

where n is a positive integer with $\mathbb{N}-\{1,2,3,4\}$ and discuss its generalized Hyers–Ulam stability in Banach spaces and stability in fuzzy normed spaces using two different methods.

1 Introduction

In 1940, Ulam [26] raised the following question. Under what conditions does there exist an additive mapping near an approximately addition mapping? The case of approximately additive functions was solved by Hyers [11] under the assumption that for $\epsilon > 0$ and $f : E_1 \to E_2$ be such that $\|f(x+y)-f(x)-f(y)\| \le \epsilon$ for all $x, y \in E_1$ then there exists a unique additive mapping $T : E_1 \to E_2$ such that $\|f(x)-T(x)\| \le \epsilon$ for all $x \in E_1$.

In 1978, a generalized version of the theorem of Hyers for approximately linear mapping was given by Rassias [20]. He proved that for a mapping $f : E_1 \to E_2$ be

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such that $f(tx)$ is continuous in $t \in \mathbb{R}$ and for each fixed $x \in E_1$ assume that there exist constant $\epsilon > 0$ and $p \in [0, 1)$ with

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(\|x\|^p + \|y\|^p) \tag{1}$$

for all $x, y \in E_1$ then there exists a unique R-Linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}\|x\|^p \tag{2}$$

for all $x \in E_1$.

A number of mathematicians were attracted by the result of Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers–Ulam–Rassias stability.

During the last decades, the stability problems of several functional equations have been extensively investigated by a number of authors [1, 5, 8, 12, 23, 24].

In 1982–1989, Rassias [21, 22] replaced the sum appeared in the right-hand side of the Eq. (1) by the product of powers of norms. In modelling applied problems only partial information may be known (or) there may be a degree of uncertainty in the parameters used in the model or some measurements may be imprecise. Due to such features, we are tempted to consider the study of functional equations in the fuzzy setting.

For the last 40 years, fuzzy theory has become a very active area of research and a lot of development has been made in the theory of fuzzy sets to find the fuzzy analogues of the classical set theory. This branch finds a wide range of applications in the field of science and engineering.

Katsaras [13] introduced an idea of fuzzy norm on a linear space in 1984, in the same year Wu and Fang [27] introduced a notion of fuzzy normed space to give a generalization of the Kolmogoroff normalized theorem for fuzzy topological linear spaces. In 1991, Biswas [4] defined and studied fuzzy inner product spaces in linear space. In 1991, Felbin [7] introduced an alternative definition of a fuzzy norm on a linear topological structures of a fuzzy normed linear spaces. In 1994, Cheng and Mordeson [6] introduced a definition of fuzzy norm on a linear space in such a manner that the corresponding induced fuzzy metric is of Kramosil and Michalek [14]. In 2003, Bag and Samanta [2] modified the definition of Cheng and Mordeson [6] by removing a regular condition. Recently various results have been investigated by numerous authors, one can refer to [3, 9, 10, 15–19, 25].

Before we proceed to the main theorems, we will introduce some definitions and an example to illustrate the idea of fuzzy norm.

Definition 1 Let X be a real linear space. A function $N : X \times \mathbb{R} \longrightarrow [0, 1]$ is said to be fuzzy norm on X if for all $x, y \in X$ and $a, b \in \mathbb{R}$.

(N_1) $N(x, a) = 0$ for $a \leq 0$;
(N_2) $x = 0$ iff $N(x, a) = 1$ for all $a > 0$;

(N_3) $N(ax, b) = N\left(x, \frac{b}{|a|}\right)$ if $a \neq 0$;
(N_4) $N(x + y, a + b) \geq min\{N(x, a), N(y, b)\}$;
(N_5) $N(x, .)$ is a non-decreasing function on $\mathbb{R}$ and $lim_{a \longrightarrow \infty} N(x, a) = 1$.
(N_6) For $x \neq 0$, $N(x, .)$ is continuous on $\mathbb{R}$.

The pair (X, N) is called a fuzzy normed linear space. One may regard $N(x, a)$ as the truth value of the statement the norm of x is less than or equal to the real number a.

Definition 2 Let (X, N) be a fuzzy normed linear space. Let x_n be a sequence in X. Then x_n is said to be convergent if there exists $x \in X$ such that $lim_{n \to \infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence x_n and we denote it by $N - lim_{n \to \infty} x_n = x$.

Definition 3 A sequence x_n in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists n_0 such that for all $n \geq n_0$ and all $p > 0$, we have $N\left(x_{n+p} - x_n, t\right) > 1 - \epsilon$.

Definition 4 Every convergent sequence in a fuzzy normed space is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

Definition 5 A mapping $f : X \to Y$ between fuzzy normed spaces X and Y is continuous at a point x_0 if for each sequence $\{x_n\}$ converging to x_0 in X, the sequence $f\{x_n\}$ converges to $f\{x_0\}$. If f is continuous at each point of $x_0 \in X$, then f is said to be continuous on X.

Example Let $(X, \|.\|)$ be a normed linear space. Then

$$N(x, a) = \begin{cases} \frac{a}{a + \|x\|}, & a > 0, \quad x \in X \\ 0, & a \leq 0, \quad x \in X \end{cases}$$

is a fuzzy norm on X.

In the following we will suppose that $N(x, .)$ is left continuous for every x. A fuzzy normed linear space is a pair(X, N), where X is a real linear space and N is a fuzzy norm on X. Let (X, N) be a fuzzy normed linear space. A sequence $\{x_n\}$ in X is said to be convergent if there exist $x \in X$ such that $\lim_{n \to \infty} N(x_n - x, t) = 1$ $(t > 0)$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N - \lim_{n \to \infty} x_n = x$. A sequence $\{x_n\}$ in fuzzy normed space (X, N) is called cauchy if for each $\epsilon > 0$ and $\delta > 0$, there exist $n_0 \in N$ such that

$$N(x_m - x_n, \delta) > 1 - \epsilon, \quad (m, n \geq n_0).$$

If each cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed space is called a fuzzy Banach space.

In this paper, the authors investigate the general solution and generalized Hyers–Ulam stability of a new type of n-dimensional functional equation of the form

$$f\left(\sum_{i=1}^{n} x_i\right) + \sum_{j=1;i\neq j}^{n} f\left(-x_j - x_i + \sum_{1\leq i<j<k\leq n} x_k\right) = \left(\frac{n^2-5n+6}{2}\right)\sum_{i=1}^{n} f\left(x_i\right) \tag{3}$$

where n is a positive integer with $\mathbb{N}-\{1,2,3,4\}$, in the setting of Banach space and fuzzy normed space using direct and fixed point methods.

Theorem 1 (Banach's Contraction Principle) *Let (X, d) be a complete metric space and consider a mapping $T : X \longrightarrow X$ which is strictly contractive mapping, that is*

(A1) $d(Tx, Ty) \leq Ld(x, y)$ *for some (Lipschitz constant) $L < 1$, then*

(i) The mapping T has one and only fixed point $x^* = T(x^*)$;
(ii) The fixed point for each given element x^* is globally attractive that is

(A2) $lim_{n\longrightarrow\infty} T^n x = x^*$, *for any starting point $x \in X$;*

(iii) One has the following estimation inequalities:

(A3) $(T^n x, x^*) \leq \frac{1}{1-L} d(T^n x, T^{n+1}x)$, *for all $n \geq 0, x \in X$.*
(A4) $(x, x^*) \leq \frac{1}{1-L} d(x, x^*), \forall x \in X$.

Theorem 2 (The Alternative of Fixed Point) *Suppose that for a complete generalized metric space (X, d) and a strictly contractive mapping $T : X \longrightarrow Y$ with Lipschitz constant L. Then, for each given element $x \in X$ either*

(B1) $(T^n x, T^{n+1}x) = +\infty$, *for all $n \geq 0$, or*
(B2) there exists natural number n_0 such that:

(i) $d(T^n x, T^{n+1}x) < \infty$ *for all $n \geq n_0$;*
(ii) The sequence $(T^n x)$ is convergent to a fixed point y^* of T;
(iii) y^* is the unique fixed point of T in the set $Y = \{y \in X; d(T^{n_0}x, y) < \infty\}$;
iv) $d(y^*, y) \leq \frac{1}{1-L} d(y, Ty)$ *for all $y \in L$.*

2 General Solution of the Functional Equation (3)

In this section, we obtain the general solution of the functional equation (3). Throughout this section , let X and Y be real vector spaces.

Theorem 3 *A function $f : X \longrightarrow Y$ satisfies the functional equation (3) then $f : X \longrightarrow Y$ satisfies the functional equation (1).*

Proof Let $f : X \longrightarrow Y$ satisfy the functional equation (3). Replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(0, 0, \ldots, 0), (x, 0, \ldots, 0)$ and $(x, x, 0 \ldots, 0)$ in (3) we obtain

$$f(0) = 0, \ f(-x) = -f(x) \ and \ f(2x) = 2f(x) \tag{4}$$

for all $x \in X$. It is easy to verify from (3) that

$$f\left(\frac{x}{2^i}\right) = \frac{1}{2^i} f(x), i = 1, 2, 3, \ldots, n \tag{5}$$

for all $x \in X$. Setting $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, y, 0, \ldots, 0)$ in (3) and using oddness of f, we obtain the result of (1).

Define a mapping $f : X \to Y$ by

$$D_f(x_1, x_2, \ldots, x_n) = f(\sum_{i=1}^{n} x_i) + \sum_{j=1; i \neq j}^{n} f\left(-x_j - x_i + \sum_{1 \le i < j < k \le n} x_k\right) - \left(\frac{n^2 - 5n + 6}{2}\right) \sum_{i=1}^{n} f(x_i)$$

for all$x_1, x_2, \ldots, x_n \in X$.

2.1 Stability Result for (3) in Banach Space Using Direct Method

In this section, we consider X to be a real vector space and Y to be a Banach space, we present the Hyers–Ulam stability of the functional equation (3).

Theorem 4 *Let* $\psi : X^n \longrightarrow [0, \infty)$ *be a function such that* $\sum_{k=0}^{\infty} \frac{\psi(2^{kj}x, 2^{kj}x, 0, \ldots, 0)}{2^{kj}}$ *converges in* $\mathbb{R}$ *and*

$$\lim_{k \to \infty} \frac{\psi(2^{kj}x_1, 2^{kj}x_2, \ldots, 2^{kj}x_n)}{2^{kj}} = 0 \tag{6}$$

for all $x_1, x_2, \ldots, x_n \in X$. *If a function* $f : X \longrightarrow Y$ *satisfies*

$$||D_f(x_1, x_2, \ldots, x_n)|| \le \psi(x_1, x_2, x_3, \ldots, x_n) \tag{7}$$

for all $x_1, x_2, \ldots, x_n \in X$, *then there exists a unique additive function* $A : X \longrightarrow Y$ *which satisfies the functional equation (3) and*

$$||f(x) - A(x)|| \le \frac{1}{(n^2 - 5n + 6)} \sum_{k=0}^{\infty} \frac{\psi(2^{kj}x, 2^{kj}x, 0, \ldots, 0)}{2^{kj}} \tag{8}$$

for all $x \in X$*. The function* A *is given by*

$$A(x) = \lim_{k \longrightarrow \infty} \frac{f(2^{kj} x)}{2^{kj}} \tag{9}$$

for all $x \in X$*.*

Proof Setting $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, 0, \ldots, 0)$ in (7), we obtain

$$|| \left(\frac{n^2 - 5n + 6}{2} \right) f(2x) - (n^2 - 5n + 6) f(x) || \leq \psi(x, x, 0, ..., 0) \tag{10}$$

for all $x \in X$. It follows from (10) that

$$|| \frac{f(2x)}{2} - f(x) || \leq \frac{\psi(x, x, 0, \ldots, 0)}{(n^2 - 5n + 6)} \tag{11}$$

for all $x \in X$. Setting x by $2x$ in (11), we obtain

$$|| \frac{f(2^2 x)}{2} - f(2x) || \leq \frac{\psi(2x, 2x, 0, \ldots, 0)}{(n^2 - 5n + 6)} \tag{12}$$

for all $x \in X$. It follows from (12) we get

$$|| \frac{f(2^2 x)}{2^2} - \frac{f(2x)}{2} || \leq \frac{\psi(2x, 2x, 0, \ldots, 0)}{2(n^2 - 5n + 6)} \tag{13}$$

for all $x \in X$. It follows from (11) and (13) that

$$|| \frac{f(2^2 x)}{2^2} - f(x) || \leq \frac{1}{(n^2 - 5n + 6)} \left[\psi(x, x, 0, \ldots, 0) + \frac{\psi(2x, 2x, 0, \ldots, 0)}{2} \right] \tag{14}$$

for all $x \in X$. Generalizing, we get

$$|| \frac{f(2^n x)}{2^n} - f(x) || \leq \frac{1}{(n^2 - 5n + 6)} \sum_{k=0}^{n-1} \frac{\psi(2^k x, 2^k x, 0, \ldots, 0)}{2^k}$$

$$\leq \frac{1}{(n^2 - 5n + 6)} \sum_{k=0}^{\infty} \frac{\psi(2^k x, 2^k x, 0, \ldots, 0)}{2^k} \tag{15}$$

for all $x \in X$. Now we have to prove that the sequence $\left\{ \frac{f(2^k x)}{2^k} \right\}$ is a cauchy sequence for all $x \in X$. For every positive integer n, m and for all $x \in X$, consider

$$||\frac{f(2^{n+m}x)}{2^{n+m}} - \frac{f(2^n x)}{2^n}|| = \frac{1}{2^n}||f(2^n x) - \frac{f(2^{n+m}x)}{2^m}||$$
$$\leq \frac{1}{(n^2-5n+6)} \sum_{i=0}^{m-1} \frac{\psi(2^{i+n}x, 2^{i+n}x, 0, \ldots, 0)}{2^{i+n}}$$
$$\leq \frac{1}{(n^2-5n+6)} \sum_{i=0}^{\infty} \frac{\psi(2^{i+n}x, 2^{i+n}x, 0, \ldots, 0)}{2^{i+n}} \tag{16}$$

for all $x \in X$. By condition (6), the right-hand side approaches 0 as $n \to \infty$. Thus, the sequence is a cauchy sequence due to the completeness of the Banach space Y

$$A(x) = \lim_{k \longrightarrow \infty} \frac{f(2^k x)}{2^k} \quad \forall x \in X,$$

is well-defined. We can see that (9) holds. To show that A satisfies (3), we set $(x, y) = (2^n x_1, 2^n x_2, \ldots, 2^n x_n)$ in (7) and divide the resulting equation by 2^n, we obtain

$$\frac{1}{2^k}||D_f(2^k x_1, 2^k x_2, \ldots, 2^k x_n)|| \leq \frac{1}{2^k}\psi(2^k x_1, 2^k x_2, \ldots, 2^k x_n).$$

Taking the limit as $n \to \infty$, using (6) and (9), A satisfies (3). To prove the uniqueness of A, suppose that there exist another cubic function $B : X \to Y$ such that B satisfies (3) and (8), we have

$$||A(x) - B(x)|| \leq \frac{1}{2^l}||A(2^l x) - f(2^l x)|| + ||f(2^l x) - B(2^l x)||$$
$$\leq \frac{2}{(n^2-5n+6)} \sum_{k=0}^{\infty} \frac{\psi(2^{k+1}x, 2^{k+1}x, 0, \ldots., 0)}{2^{k+1}} \quad \forall x \in X.$$

By condition (6), the right-hand side approaches 0 as $n \to \infty$, and it follows that $A(x) = B(x)$ for all $x \in X$. Hence, A is unique. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 4, concerning the stability of (3).

Corollary 1 *Let λ and s be a non-negative real numbers. Let $f : X \longrightarrow Y$ be a function satisfying the inequality*

$$||D_f(x_1, x_2, x_3, \ldots, x_n)|| \leq \begin{cases} \lambda \\ \lambda(\sum_{i=1}^n ||x_i||^s) \\ \lambda(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}) \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$. *Then there exists a unique additive function* $A : X \longrightarrow Y$ *such that*

$$||f(x) - A(x)|| \le \begin{cases} \frac{|2|\lambda}{(n^2-5n+6)} \\ \frac{4\lambda||x||^s}{(n^2-5n+6)|2-2^s|} & ; \quad s \ne 1 \\ \frac{4\lambda||x||^{ns}}{(n^2-5n+6)|2-2^{ns}|} & ; \quad s \ne \frac{1}{n} \end{cases}$$

2.1.1 Stability Result for (3) in Banach Space Using Fixed Point Method

In this segment, the authors presented the generalized Ulam–Hyers stability of the functional equation (3) in Banach space and using fixed point method.

Theorem 5 *Let* $f : X \longrightarrow Y$ *be a mapping for which there exists a function* $\psi : X^n \longrightarrow [0, \infty)$ *with the condition*

$$lim_{k\longrightarrow\infty} \frac{\psi(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n)}{\eta_i^k} = 0 \tag{17}$$

where

$$\eta_i = \begin{cases} 2, \; if & i = 0 \\ \frac{1}{2}, \; if & i = 1 \end{cases}$$

such that the functional inequality

$$||D_f(x_1, x_2, \ldots, x_n)|| \le \psi(x_1, x_2, \ldots, x_n) \tag{18}$$

for all $x_1, x_2, \ldots, x_n \in W$. *If there exists* $L = L(i)$ *such that the function*

$$x \longrightarrow \beta(x) = \frac{\psi(x/2, x/2, 0, \ldots, 0)}{(n^2 - 5n + 6)}$$

has the property,

$$\frac{1}{\eta_i}\beta(\eta_i x) = L\beta(x) \tag{19}$$

for all $x \in X$. *Then there exists a unique additive function* $A : X \longrightarrow Y$ *satisfying the functional equation (3) and*

$$||f(x) - A(x)|| \le \frac{L^{1-i}}{1-L}\beta(x) \tag{20}$$

holds for all $x \in X$.

Proof Consider the set $d = \{u/u : X \longrightarrow Y, u(0) = 0)\}$ and introduce the generalized metric on M. $d(u, v) = inf \{k \in (0, \infty) : ||u(x) - v(x)|| \leq k\beta(x), x \in X\}$. It is easy to see that (M, d) is complete. Define $T : M \longrightarrow M$ by

$$Tu(x) = \frac{1}{\eta_i}u(\eta_i x)$$

for all $x \in M$. Now $u, v \in M$,

$$d(u, v) \leq k \Rightarrow ||u(x) - v(x)|| \leq k\beta(x) \quad \forall x \in X;$$

$$\Rightarrow ||\frac{1}{\eta_i}u(\eta_i x) - \frac{1}{\eta_i}v(\eta_i x)|| \leq \frac{1}{\eta_i}k\beta(\eta_i x) \quad \forall x \in X;$$

$$\Rightarrow ||Tu(x) - Tv(x)|| \leq k\beta(x) \quad \forall x \in X;$$

$$\Rightarrow d(Tu, Tv) \leq Lk$$

This implies $d(Tu, Tv) \leq Ld(u, v)$ for all $u, v \in M$. (i.e.,) T is strictly contractive mapping on with Lipschitz constant L. Replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, 0, \ldots, 0)$ in (18), we obtain

$$||\frac{(n^2 - 5n + 6)}{2}f(2x) - (n^2 - 5n + 6)f(x)|| \leq \psi(x, x, 0, \ldots, 0) \tag{21}$$

for all $x \in X$. It follows from (21) that

$$||f(x) - \frac{f(2x)}{2}|| \leq \frac{\psi(x, x, 0, \ldots, 0)}{(n^2 - 5n + 6)} \tag{22}$$

for all $x \in X$. Using (19) for the case $i = 0$, it reduces to

$$||f(x) - \frac{f(2x)}{2}|| \leq \beta(x)$$

for all $x \in X$.

$$i.e., d(f, Tf) \leq L \Rightarrow d(f, Tf) \leq 1 = L = L^0 < \infty.$$

Again replacing $x = \frac{x}{2}$ in (21) and (22), we get

$$||\left(\frac{n^2 - 5n + 6}{2}\right)f(x) - (n^2 - 5n + 6)f\left(\frac{x}{2}\right)|| \leq \psi(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0)$$

and

$$||f(x) - 2f\left(\frac{x}{2}\right)|| \leq \frac{2}{(n^2 - 5n + 6)}\psi(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0) \tag{23}$$

for all $x \in X$. Using (19) for the case $i = 0$, it reduces to

$$||f(x) - \frac{f(2x)}{2}|| \leq L\beta(x) \tag{24}$$

for all $x \in X$. (i.e.,) $d(f, Tf) \leq 2 \Rightarrow d(f, Tf) \leq 2 = L^0 < \infty$. In the above case, we arrive

$$d(f, Tf) \leq L^{1-i}.$$

Therefore $(B_2(i))$holds. By $(B_2(ii))$, it follows that there exists a fixed point A of T in X, such that

$$A(x) = lim_{k \longrightarrow \infty} \frac{f(\eta_i^k x)}{\eta_i^k} \tag{25}$$

for all $x \in X$. In order to prove $A : X \longrightarrow Y$ is additive. Replacing $(x_1, x_2, \ldots, x_n)$ by $(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n)$ in (18) and dividing by η_i^k, it follows from (17) and (25), we see that A satisfies (3) for all $x_1, x_2, \ldots, x_n \in X$. Hence A satisfies the functional equation (3). By $(B_2(iii))$, A is the unique fixed point of T in the set, $Y = \{f \in M; d(Tf, A) < \infty\}$. Using the fixed point alternative result, A is the unique function such that

$$||f(x) - A(x)|| \leq k\beta(x)$$

for all $x \in W$ and $k > 0$.Finally by $(B_2(iv))$, we obtain

$$d(f, A) \leq \frac{1}{1-L}d(f, Tf)$$

$$(i.e.,)\ d(f, A) \leq \frac{L^{1-i}}{1-L}.$$

Hence, we conclude that

$$||f(x) - A(x)|| \leq \frac{L^{1-i}}{1-L}\beta(x)$$

for all $x \in X$. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 5 concerning the stability of (3).

Corollary 2 *Let $f : X \longrightarrow Y$ be a mapping and there exist real numbers λ and s such that*

$$||D_f(x_1, x_2, \ldots, x_n)|| \leq \begin{cases} \lambda \\ \lambda(\sum_{i=1}^n ||x_i||^s) \\ \lambda(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}) \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$. Then there exists a unique additive function $A : X \longrightarrow Y$ such that

$$||f(x) - A(x)|| \leq \begin{cases} \frac{|2|\lambda}{(n^2-5n+6)} \\ \frac{4\lambda||x||^s}{(n^2-5n+6)|2-2^s|} & ; \quad s \neq 1 \\ \frac{4\lambda||x||^{ns}}{(n^2-5n+6)|2-2^{ns}|} & ; \quad s \neq \frac{1}{n} \end{cases}$$

for all $x \in X$.

Proof Setting

$$\psi(x_1, x_2, x_3, \ldots, x_n) \leq \begin{cases} \lambda \\ \lambda(\sum_{i=1}^n ||x_i||^s) \\ \lambda(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}) \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$. Now

$$\frac{\psi(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n)}{\eta_i^k} \leq \begin{cases} \frac{\lambda}{\eta_i^k} \\ \frac{\lambda}{\eta_i^k} \left\{\sum_{i=1}^n ||\eta_i^k x_i||^s\right\} \\ \frac{\lambda}{\eta_i^k} \left\{\prod_{i=1}^n ||\eta_i^k x_i||^{ns} + \sum_{i=1}^n ||\eta_i^k x_i||^{ns}\right\} \end{cases}$$

$$= \begin{cases} \longrightarrow \quad 0 \quad as \quad k \longrightarrow \infty \\ \longrightarrow \quad 0 \quad as \quad k \longrightarrow \infty \\ \longrightarrow \quad 0 \quad as \quad k \longrightarrow \infty \end{cases}$$

i.e., (21) holds. But we have $\beta(x) = \frac{2}{(n^2-5n+6)} \psi(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0)$.

Hence

$$\beta(x) = \frac{1}{(n^2-5n+6)}\psi(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0) = \begin{cases} \frac{2\lambda}{(n^2-5n+6)} \\ \frac{4\lambda||x||^s}{(n^2-5n+6)2^s} \\ \frac{4\lambda||x||^{ns}}{(n^2-5n+6)2^{ns}} \end{cases}$$

$$\frac{1}{\eta_i}\beta(\eta_i x) = \begin{cases} \frac{1}{\eta_i}\frac{2\lambda}{(n^2-5n+6)} \\ \frac{1}{\eta_i}\frac{4\lambda||x||^s}{(n^2-5n+6)2^s} \\ \frac{1}{\eta_i}\frac{4\lambda||x||^n s}{(n^2-5n+6)2^{ns}} \end{cases}$$

$$= \begin{cases} \eta_i^{-1}\beta(x) \\ \eta_i^{s-1}\beta(x) \\ \eta_i^{ns-1}\beta(x) \end{cases}$$

for all $x \in X$. Hence the inequality (3) holds for

$L = 2^{-1}\ \ if\ \ i = 0\ \ and\ \ L = \frac{1}{2^{-1}}\ \ if\ \ i = 1$
$L = 2^{s-1}\ \ for\ \ s < 1 if i = 0\ \ and\ \ L = \frac{1}{2^{s-1}}\ \ for\ \ s > 1\ \ if\ \ i = 1.$
$L = 2^{ns-1}\ \ for\ \ s < \frac{1}{n}\ \ if\ \ i = 0\ \ and\ \ L = \frac{1}{2^{ns-1}}\ \ for\ \ s > \frac{1}{n}\ \ if\ \ i = 1.$

Now, from (21) we prove the following cases:

Case1: $L = 2^{-1}\ \ if\ \ i = 0$
$||f(x) - A(x)| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{(2^{-1})}{1-2^{-1}}\frac{2\lambda}{(n^2-5n+6)} = \frac{2\lambda}{(n^2-5n+6)}$
Case2: $L = \frac{1}{3^{-1}}\ \ if\ \ i = 1$
$||f(x) - A(x)|| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{1}{1-2}\frac{2\lambda}{(n^2-5n+6)} = \frac{-2\lambda}{(n^2-5n+6)}$
Case3: $L = 2\ \ for\ \ s < 1\ \ if\ \ i = 0$
$||f(x) - A(x)|| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{2^{s-1}}{1-2^{s-1}}\frac{4\lambda||x||^s}{(n^2-5n+6)2^s} = \frac{4\lambda||x||^s}{(n^2-5n+6)(2-2^s)}$
Case4: $L = \frac{1}{2^{s-1}}\ \ for\ \ s > 1\ \ if\ \ i = 1$
$||f(x) - A(x)|| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{1}{1-\frac{1}{2^{s-1}}}\frac{4\lambda||x||^s}{(n^2-5n+6)2^s} = \frac{4\lambda||x||^s}{(n^2-5n+6)(2^s-2)}$
Case5: $L = 2^{ns-1}\ \ for\ \ s < 1\ \ if\ \ i = 0$
$||f(x) - A(x)|| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{2^{ns-1}}{1-2^{ns-1}}\frac{4\lambda||x||^{ns}}{(n^2-5n+6)2^{ns}} = \frac{4\lambda||x||^{ns}}{(n^2-5n+6)(2-2^{ns})}$
Case6: $L = \frac{1}{2^{ns-1}}\ \ for\ \ s > \frac{1}{n}\ \ if\ \ i = 1$
$||f(x) - A(x)|| \leq \frac{L^{1-i}}{1-L}\beta(x) = \frac{1}{1-\frac{1}{2^{ns-1}}}\frac{4\lambda||x||^{ns}}{(n^2-5n+6)2^{ns}} = \frac{4\lambda||x||^{ns}}{(n^2-5n+6)(2^{ns}-2)}.$

3 Stability Result for (3) in Fuzzy Normed Space Using Direct Method

Throughout this section, assume that X, $\left(Z, N^{'}\right)$, (Y, N) are linear space, Banach space, and fuzzy normed space, respectively, we now investigate the fuzzy stability of the functional equation (3).

Theorem 6 *Let $\beta \in \{1, -1\}$ be fixed and let $\psi : X^n \longrightarrow Z$ be a mapping such that for some $d > 0$ with $0 < \left(\frac{d}{2}\right)^{\beta} < 1$.*

$$N'\left(\psi(2^{\beta}x, 2^{\beta}x, 0, \ldots, 0), r\right) \geq N'(d^{\beta}\psi(x, x, 0, \ldots, 0), r) \tag{26}$$

for all $x \in X$ and all $r > 0$, $d > 0$, and

$$lim_{k\longrightarrow\infty}N'\left(\psi(2^{\beta k}x_1, 2^{\beta k}x_2, \ldots, 2^{\beta k}x_n), 2^{\beta k}r\right) = 1 \tag{27}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $r > 0$. Suppose an odd mapping $f : X \longrightarrow Y$ with $f(0) = 0$ satisfies the inequality

$$N(D_f(x_1, x_2, \ldots, x_n), r) \geq N'(\psi(x_1, x_2, \ldots, x_n), r) \tag{28}$$

for all $r > 0$ and all $x_1, x_2, \ldots, x_n \in X$. Then the limit

$$A(x) = N - lim_{k\longrightarrow\infty}\frac{f(2^{\beta k}x)}{2^{\beta k}} \tag{29}$$

exists for all $x \in X$ and the mapping $A : X \longrightarrow Y$ is the unique additive mapping such that

$$N(f(x) - A(x), r) \geq N'(\psi(x, x, 0, \ldots, 0), \frac{(n^2 - 5n + 6)}{2}r|2 - d|) \tag{30}$$

for all $x \in X$ and for all $r > 0$.

Proof Let $\beta = 1$. Replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, 0, \ldots, 0)$ in (28), we get

$$N\left((n^2 - 5n + 6)f(x) - \frac{(n^2 - 5n + 6)}{2}f(2x), r\right) \geq N'(\psi(x, x, 0, \ldots, 0), r) \tag{31}$$

for all $x \in X$ and all $r > 0$. Replacing x by $2^k x$ in (31), we obtain

$$N\left(\frac{f(2^{k+1}x)}{2} - f(2^k x), \frac{r}{(n^2 - 5n + 6)}\right) \geq N'(\psi(2^k x, 2^k x, 0, \ldots, 0), r) \tag{32}$$

for all $x \in X$ and for all $r > 0$. Using (26), we get

$$N\left(\frac{f(2^{k+1}x)}{2} - f(2^k x), \frac{r}{(n^2-5n+6)}\right) \geq N'(\psi(x,x,0,\ldots,0), \frac{r}{d^k}) \tag{33}$$

for all $x \in X$ and for all $r > 0$. It is easy to verify from (33) that

$$N\left(\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}, \frac{r}{(n^2-5n+6)2^k}\right) \geq N'(\psi(x,x,0,\ldots,0), \frac{r}{d^k}) \tag{34}$$

holds for all $x \in X$ and for all $r > 0$. Replacing r by $d^k r$ in (34),we get

$$N\left(\frac{f(2^{k+1}x)}{2^{k+1}} - \frac{f(2^k x)}{2^k}, \frac{d^k r}{(n^2-5n+6)2^k}\right) \geq N'(\psi(x,x,0,\ldots,0), r) \tag{35}$$

for all $x \in X$ and for all $r > 0$. It follows from

$$\frac{f(2^k x)}{2^k} - f(x) = \sum_{i=0}^{k-1}\left[\frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i}\right] \tag{36}$$

and (35) that

$$\begin{aligned} &N\left(\frac{f(2^k x)}{2^k} - f(x), \sum_{i=0}^{k-1}\frac{d^i r}{(n^2-5n+6)2^i}\right)\\ &\geq min\left\{N\left(\frac{f(2^{i+1}x)}{2^{i+1}} - \frac{f(2^i x)}{2^i}, \frac{d^i r}{(n^2-5n+6)2^i}\right) : i = 0,1,2,\ldots,k-1\right\}\\ &\geq N'(\psi(x,x,0,\ldots,0), r) \end{aligned} \tag{37}$$

for all $x \in X$ and for all $r > 0$. Replacing x by $2^m x$ in (37), we get

$$N\left(\frac{f(2^{k+m}x)}{2^{k+m}} - \frac{f(2^m x)}{2^m}, \sum_{i=m}^{m+k-1}\frac{d^i r}{(n^2-5n+6)2^i}\right) \geq N'(\psi(x,x,0,\ldots,0), \frac{r}{d^m}) \tag{38}$$

for all $x \in X$ and for all $r > 0$ and all $m, k \geq 0$. Replacing r by $d^m r$ in (38), we get

$$N\left(\frac{f(2^{k+m}x)}{2^{k+m}} - \frac{f(2^m x)}{2^m}, \sum_{i=0}^{k-1}\frac{d^i r}{(n^2-5n+6)2^i}\right) \geq N'(\psi(x,x,0,\ldots,0), r) \tag{39}$$

for all $x \in X$ and for all $r > 0$ and all $m, k \geq 0$. Using (N_3) in (38), we obtain

$$N\left(\frac{f(2^{k+m}x)}{2^{k+m}} - \frac{f(2^m x)}{2^m}, r\right) \geq N'\left(\psi(x,x,0,\ldots,0), \frac{r}{\sum_{i=m}^{m+k-1}\frac{d^i}{(n^2-5n+6)2^i}}\right) \tag{40}$$

for all $x \in X$, $r > 0$ and all $m, k \geq 0$. Since $0 < d < 2$ and $\sum_{i=0}^{k} \left(\frac{d}{2}\right)^i < \infty$, the Cauchy criterion for convergence and (N_5) implies that $\left\{\frac{f(2^k x)}{2^k}\right\}$ is a Cauchy sequence in (Y, N). Since (Y, N) is a fuzzy Banach space, this sequence converges to some point $A(x) \in Y$. So one can define the mapping $A : X \longrightarrow Y$ by

$$A(x) := N - \lim_{k \longrightarrow \infty} \frac{f(2^k x)}{2^k}$$

for all $x \in X$. Letting $m = 0$ in (40), we get

$$N\left(\frac{f(2^k x)}{2^k} - f(x), r\right) \geq N'\left(\psi(x, x, 0, \ldots, 0), \frac{r}{\sum_{i=0}^{k-1} \frac{d^i}{(n^2-5n+6)2^i}}\right) \tag{41}$$

for all $x \in X$. Taking the limits as $k \longrightarrow \infty$ and using (N_6), we arrive

$$N(f(x) - A(x), r) \geq N'(\psi(x, x, 0, \ldots, 0), (n^2 - 5n + 6)r.(2 - d))$$

for all $x \in X$ and for all $r > 0$. Now, we claim that A is additive. Replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(2^k x_1, 2^k x_2, \ldots, 2^k x_n)$ in (28), respectively, we get

$$N\left(\frac{1}{2^k} D_f(2^k x_1, 2^k x_2, \ldots, 2^k x_n), r\right) \geq N'(\psi(2^k x_1, 2^k x_2, \ldots, 2^k x_n), 2^k r) \tag{42}$$

for all $r > 0$ and for all $x_1, x_2, \ldots, x_n \in X$. Since

$$\lim_{k \longrightarrow \infty} N'\left(\psi(2^{\beta k} x_1, 2^{\beta k} x_2, \ldots, 2^{\beta k} x_n), 2^{\beta k} r\right) = 1.$$

A satisfies the additive functional equation (3). Hence $A : X \to Y$ is additive. To prove the uniqueness of A, let A' be another additive mapping satisfying (30). Fix $x \in X$, clearly $A(2^n x) = 2^n A(x)$ and $A^{'}(2^n x) = 2^n A^{'}(x)$ for all $x \in X$ and all $n \in N$. It follows from (30) that $N(A(x) - A'(x), r) = N\left(\frac{A(2^k x)}{2^k} - \frac{A'(2^k x)}{2^k}, r\right)$

$$\geq min\left\{N\left(\frac{A(2^k x)}{2^k} - \frac{f(2^k x)}{2^k}, \frac{r}{2}\right), N\left(\frac{f(2^k x)}{2^k} - \frac{A'(2^k x)}{2^k}, \frac{r}{2}\right)\right\}$$

$$\geq N'\left(\psi(2^k x, 2^k x, 0, \ldots, 0), \frac{r(n^2 - 5n + 6)2^k(2 - d)}{2}\right)$$

$$\geq N'\left(\psi(2^k x, 2^k x, 0, \ldots, 0), \frac{r(n^2 - 5n + 6)2^k(2 - d)}{2d^k}\right)$$

for all $x \in X$ and $r > 0$. Since $lim_{k \longrightarrow \infty} \frac{r(n^2-5n+6)2^k(2-d)}{2d^k} = \infty$, we obtain

$$lim_{k \to \infty} N'\left(\psi(x, x, 0, \ldots, 0), \frac{r(n^2 - 5n + 6)2^k(2 - d)}{2d^k}\right) = 1.$$

Thus $N(A(x) - A'(x), r) = 1$ for all $x \in X$ and $r > 0$ and so $A(x) = A'(x)$. For $\beta = -1$, we can prove the result by a similar method.

The following corollary is an immediate consequence of Theorem 6, concerning the stability for the functional equation (3).

Corollary 3 *Suppose that the function $f : X \longrightarrow Y$ satisfies the inequality*

$$N(Df(x_1.x_2, \ldots., x_n), r) \geq \begin{cases} N'(\theta, r) \\ N'(\theta \sum_{i=1}^{n} ||x_i||^s, r) \\ N'(\theta(\sum_{i=1}^{n} ||x_i||^{ns} + \Pi_{i=1}^{n}||x_i||^s), r) \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$ and all $r > 0$, where θ, s are constants then there exists a unique additive mapping $A : X \to Y$ such that

$$N(f(x) - A(x), r) \geq \begin{cases} N'(\theta, \frac{r(n^2-5n+6)}{|2|} \\ N'\left(2\theta||x||^s, \frac{r(n^2-5n+6)|2-2^s|}{2}\right) & ; s \neq 1 \\ N'\left(2\theta||x||^{ns}, \frac{r(n^2-5n+6)|2-2^{ns}|}{2}\right) & ; s \neq \frac{1}{n} \end{cases}$$

3.1 Stability Result for (3) in Fuzzy Normed Space Using Fixed Point Method

Throughout this section, the authors investigated the generalized Ulam–Hyers stability of the functional equation (3) in fuzzy normed space using fixed point method.

To prove the stability result, we define the following μ_i is a constant such that

$$\eta_i = \begin{cases} 2 & if \quad i = 0 \\ \frac{1}{2} & if \quad i = 1 \end{cases}$$

and Ω is the set such that $\Omega = \{t/t : W \longrightarrow B, t(0) = 0\}$.

Theorem 7 *Let $f : X \longrightarrow Y$ be a mapping for which there exists a function $\psi : X^n \longrightarrow Z$ with condition*

$$lim_{k \longrightarrow \infty} N'\left(\psi(\eta^k x_1, \eta^k x_2, \ldots, \eta^k x_n), \eta^k r\right) = 1 \tag{43}$$

for all $x_1, x_2, \ldots, x_n \in X$ *and all* $r > 0$ *and satisfying the inequality*

$$N(D_f(x_1, x_2, \ldots, x_n), r) \geq N'(\psi(x_1, x_2, \ldots, x_n), r) \tag{44}$$

for all $x \in X$ *and* $r > 0$. *If there exist* $L = L[i]$ *such that the function* $x \longrightarrow \beta(x) = \frac{1}{(n^2-5n+6)}\psi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$ *has the property*

$$N'\left(L\frac{1}{\eta_i}\beta(\eta_i x), r\right) = N'(\beta(x), r) \tag{45}$$

for all $x \in X$ *and* $r > 0$, *then there exists a unique additive function* $A : X \longrightarrow Y$ *satisfying the functional equation (3) and*

$$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right)$$

for all $x \in X$ *and* $r > 0$.

Proof Let d be a general metric on Ω such that

$$d(t, u) = inf\left\{k \in (0, \infty) | N(t(x) - u(x), r) \geq N'(\beta(x), kr), x \in X, r > 0\right\}$$

It is easy to see that (Ω, d) is complete. Define $T : \Omega \longrightarrow \Omega$ by $Tt(x) = \frac{1}{\eta_i}t(\eta_i x)$ for all $x \in X$,for $t, u \in \Omega$, we have

$$\begin{aligned} d(t, u) = k &\Rightarrow N(t(x) - u(x), r) \geq N'(\beta(x), kr) \\ &\Rightarrow N\left(\frac{t(\eta_i x)}{\eta_i} - \frac{u(\eta_i x)}{\eta_i}, r\right) \geq N'(\beta(\eta_i x), k\eta_i r) \qquad (46) \\ &\Rightarrow N(Tt(x) - Tu(x), r) \geq N'(\beta(\eta_i x), k\eta_i r) \\ &\Rightarrow N(Tt(x) - Tu(x), r) \geq N'(\beta(x), kLr) \\ &\Rightarrow d(Tt(x) - Tu(x)) \geq kL \\ &\Rightarrow d(Tt - Tu, r) \geq Ld(t, u) \end{aligned}$$

for all $t, u \in \Omega$.Therefore T is strictly contractive mapping on Ω with Lipschitz constant L, replacing $(x_1, x_2, x_3, \ldots, x_n)$ by $(x, x, 0, \ldots, 0)$ in (44), we get

$$N\left(\frac{(n^2 - 5n + 6)}{2}) f(2x) - (n^2 - 5n + 6) f(x), r\right) \geq N'(\psi(x, x, 0, \ldots, 0), r) \tag{47}$$

for all $x \in X$ and $r > 0$. Using (N_3) in (47), we arrive

$$N\left(\frac{f(2x)}{2} - f(x), r\right) \geq N'\left(\frac{\psi(x, x, 0, \ldots, 0)}{(n^2 - 5n + 6)}, r\right) \tag{48}$$

for all $x \in X$ and $r > 0$ with the help of (45) when $i = 0$, it follows from (48) that

$$\Rightarrow N\left(\frac{f(2x)}{2} - f(x), r\right) \geq N'(L\beta(x), r)$$

$$\Rightarrow d(Tf, f) \geq L = L^1 = L^{1-i}. \tag{49}$$

Replacing x by $\frac{x}{2}$ in (47), we obtain

$$N\left(f(x) - 2f\left(\frac{x}{2}\right), r\right) \geq N'\left(\frac{2}{(n^2 - 5n + 6)}\psi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right), r\right)$$

for all $x \in X$ and $r > 0$, when $i = 1$, it follows from (49), we get

$$\Rightarrow N\left(f(x) - 2f\left(\frac{x}{2}\right), r\right) \geq N'(\beta(x), r)$$

$$\Rightarrow T(f, Tf) \leq 1 = L^0 = L^{1-i}. \tag{50}$$

Then from (49) and (50), we can conclude

$$\Rightarrow T(f, Tf) \leq L^{1-i} < \infty.$$

Now from the fixed point alternative in both cases, it follows that there exists a fixed point A of T in Ω such that

$$A(x) = N - \lim_{k \longrightarrow \infty} \frac{f(\eta^k x)}{\eta^k}$$

for all $x \in W$ and $r > 0$. Replacing $(x_1, x_2, \ldots, x_n)$ by $(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n)$ in (44), we arrive

$$N\left(\frac{1}{\eta_i^k} Df(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n), r\right) \geq N'(\psi(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n), \eta_i^k r)$$

for all $r > 0$ and all $x_1, x_2, \ldots, x_n \in X$. By proceeding the same procedure of the Theorem 5.1 , we can prove the function $A : X \longrightarrow Y$ is additive and it satisfies the functional equation (3). By a fixed point alternative, since A is a unique fixed point of T in the set

$$\Delta = \{f \in \Omega / d(f, A) < \infty\}.$$

Therefore A is a unique function such that

$$N(f(x) - A(x), r) \geq N'(\beta(x), kr)$$

for all $x \in W$ and $r > 0$. Again using the fixed point alternative, we obtain

$$d(f, A) \leq \frac{1}{1-L} d(f, Tf)$$

$$\Rightarrow d(f, A) \leq \frac{L^{1-i}}{1-L}$$

$$\Rightarrow N(f(x) - A(x), r) \geq N'\left(\beta(x)\frac{L^{1-i}}{1-L}, r\right)$$

for all $x \in X$ and $r > 0$. This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 7 concerning the stability of (3).

Corollary 4 *Suppose a function $f : X \longrightarrow Y$ satisfies the inequality*

$$N(D_f(x_1, x_2, \ldots, x_n), r) \geq \begin{cases} N'(\theta, r) \\ N'(\theta \sum_{i=1}^n ||x_i||^s, r) \\ N'(\theta(\sum_{i=1}^n ||x_i||^{ns} + \Pi_{i=1}^n ||x_i||^s), r) \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$ and $r > 0$, where θ, s are constants with $\theta > 0$. Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$N(f(x) - A(x), r) \geq \begin{cases} N'(\theta, \frac{r(n^2-5n+6)}{|2|} \\ N'\left(2\theta ||x||^s, \frac{r(n^2-5n+6)|2-2^s|}{2}\right) & ; \quad s \neq 1 \\ N'\left(2\theta ||x||^{ns}, \frac{r(n^2-5n+6)|2-2^{ns}|}{2}\right) & ; \quad s \neq \frac{1}{n} \end{cases}$$

for all $x \in X$ and $r > 0$.

Proof Setting

$$\psi(x_1, x_2, x_3, \ldots, x_n) \leq \begin{cases} \theta \\ \theta(\sum_{i=1}^n ||x_i||^s) \\ \theta(\prod_{i=1}^n ||x_i||^s + \sum_{i=1}^n ||x_i||^{ns}) \end{cases}$$

for all $x_1, x_2, \ldots, x_n \in X$. Then

$$N'\left(\psi\left(\eta_i^k x_1, \eta_i^k x_2, \ldots, \eta_i^k x_n\right), \eta_i^k r\right) = \begin{cases} N'(\theta, \eta_i^k r) \\ N'\left(\theta \sum_{i=1}^n ||x_i||^s, \eta_i^{(1-s)k} r\right) \\ N'\left(\theta(\sum_{i=1}^n ||x_i||^{ns} + \Pi_{i=1}^n ||x_i||^s), \eta_i^{(1-ns)k} r\right) \end{cases}$$

$$= \begin{cases} \longrightarrow \quad 1 \quad as \quad k \longrightarrow \infty, \\ \longrightarrow \quad 1 \quad as \quad k \longrightarrow \infty, \\ \longrightarrow \quad 1 \quad as \quad k \longrightarrow \infty. \end{cases}$$

Thus, (6) holds. But we have

$$\beta(x) = \frac{2}{(n^2 - 5n + 6)} \psi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right)$$

has the property

$$N'\left(L \frac{1}{\eta_i} \beta(\eta_i x), r\right) \geq N'(\beta(x), r)$$

for all $x \in X$ and $r > 0$. Hence

$$N'(\beta(x), r) = N'\left(\psi\left(\frac{x}{2}, \frac{x}{2}, 0, \ldots, 0\right), (n^2 - 5n + 6) r\right)$$

$$= \begin{cases} N'(\theta, r(n^2 - 5n + 6)) \\ N'\left(\frac{2}{2^s} \theta ||x||^s, r(n^2 - 5n + 6)\right) \\ N'\left(\frac{2}{2^{ns}} \theta ||x||^{ns}, r(n^2 - 5n + 6)\right). \end{cases}$$

Now,

$$N'\left(\frac{1}{\eta_i} \beta(\eta_i x), r\right) = \begin{cases} N'\left(\frac{\theta}{\eta_i}, r(n^2 - 5n + 6)\right) \\ N'\left(\frac{\theta}{\eta_i}\left(\frac{2}{2^s}\right) ||\eta_i x||^s, r(n^2 - 5n + 6)\right) \\ N'\left(\frac{\theta}{\eta_i}\left(\frac{2}{2^{ns}}\right) ||\eta_i x||^{ns}, r(n^2 - 5n + 6)\right) \end{cases}$$

$$= \begin{cases} N'(\eta_i^{-1} \beta(x), r) \\ N'(\eta_i^{s-1} \beta(x), r) \\ N'(\eta_i^{ns-1} \beta(x), r) \end{cases}$$

Now from the following cases for the conditions (i) and (ii)

Case(i): $L = 2^{-1} \quad for \quad s = 0 \quad if \quad i = 0$
$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{2^{-1}}{1-2^{-1}}\frac{2\theta}{(n^2-5n+6)}, r\right) \geq N'\left(\theta, \frac{r(n^2-5n+6)}{2}\right)$

Case(ii): $L = \left(\frac{1}{2}\right)^{-1} \quad for \quad s = 0 \quad if \quad i = 1$

$N(f(x) - A(x), r) \geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \geq N'\left(\frac{1}{1-\left(\frac{1}{2}\right)^{-1}}\frac{2\theta}{(n^2-5n+6)}, r\right) \geq N'\left(\theta, \frac{-r(n^2-5n+6)}{2}\right)$

Case(iii): $L = (2)^{s-1} \quad for \quad s < 1 \quad if \quad i = 0$

$$\begin{aligned} N(f(x) - A(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \\ &\geq N'\left(\frac{2^{s-1}}{1-2^{s-1}}\frac{2\theta||x||^s}{(n^2-5n+6)2^s}, r\right) \\ &\geq N'\left(2\theta||x||^s, \frac{r(n^2-5n+6)(2-2^s)}{2}\right) \end{aligned}$$

Case(iv): $L = (2)^{1-s} \quad for \quad s > 1 \quad if \quad i = 1$

$$\begin{aligned} N(f(x) - A(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \\ &\geq N'\left(\frac{2^{1-s}}{1-2^{1-s}}\frac{2\theta||x||^s}{(n^2-5n+6)2^s}, r\right) \\ &\geq N'\left(2\theta||x||^s, \frac{r(n^2-5n+6)(2^s-2)}{2}\right) \end{aligned}$$

Case(v): $L = (2)^{ns-1} \quad for \quad s < \frac{1}{n} \quad if \quad i = 0$

$$\begin{aligned} N(f(x) - A(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \\ &\geq N'\left(\frac{2^{ns-1}}{1-2^{ns-1}}\frac{2\theta||x||^{ns}}{(n^2-5n+6)2^{ns}}, r\right) \\ &\geq N'\left(2\theta||x||^{ns}, \frac{r(n^2-5n+6)(2-2^{ns})}{2}\right) \end{aligned}$$

Case(vi): $L = (2)^{1-ns} \quad for \quad s < \frac{1}{n} \quad if \quad i = 1$

$$\begin{aligned} N(f(x) - A(x), r) &\geq N'\left(\frac{L^{1-i}}{1-L}\beta(x), r\right) \\ &\geq N'\left(\frac{2^{1-ns}}{1-2^{1-ns}}\frac{2\theta||x||^{ns}}{(n^2-5n+6)2^{ns}}, r\right) \\ &\geq N'\left(2\theta||x||^{ns}, \frac{r(n^2-5n+6)(2^{ns}-2)}{2}\right) \end{aligned}$$

Hence the proof is completed.

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