Chapter 6
Teaching the Multiplication Table and Its Properties for Learning How to Learn

Raimundo Olfos and Masami Isoda

Why do the Japanese traditionally introduce multiplication up to the multiplication table in the second grade? There are four possible reasons. The first reason is that it is possible to teach. The second reason is that Japanese teachers plan the teaching sequence to teach the multiplication table as an opportunity to teach learning how to learn. The third reason is that memorizing the table itself has been recognized as a cultural practice. The fourth reason is to develop the sense of wonder with appreciation of its reasonableness. The second and the fourth reasons are discussed in Chap. 1 of this book as "learning how to learn" and "developing students who learn mathematics by and for themselves in relation to mathematical values, attitudes, ways of thinking, and ideas." This chapter describes these four reasons in this order to illustrate the Japanese meaning of teaching content by explaining how the multiplication table and its properties are taught under the aims of mathematics education. In Chap. 1, the aims are described by the three pillars: human character formation for mathematical values and attitudes, mathematical thinking and ideas, and mathematical knowledge and skills.

6.1 Revisiting the Japanese Educational Principle

For explaining the Japanese content of teaching, we have to revisit Chap. 1 of this book first and provide some necessary information on the manner of teaching. The Japanese educational principle in mathematics (MEXT, 2008; Shimizu, 1984) is to develop students who learn mathematics by and for themselves based on what they

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have already learned. In accordance with this principle, learning how to learn in itself becomes the content of teaching. Indeed, Japanese students learn how to extend the multiplication table after they have been introduced to the meaning of multiplication in the same grade. The extension of the multiplication table is one of the best opportunities to develop students in accordance with this principle.

Learning the multiplication table is a facilitated activity which includes extension of the table and coordination of the processes of memorization and application. For students to be able to learn mathematics by and for themselves, Japanese teachers plan well-sequenced activities and think of several strategies for teaching. In the given teaching sequence (task sequence), the students are able to engage in activities in which they need to remember what they have learned and appreciate the advantage of those methods for development in the lesson. Major activities in class usually include solving a given unknown task, with a discussion of the unknown as a problematic, and communication of ideas to solve the problematic by challenging the unknown to be known.[1] At every necessary moment throughout the class, the teachers provide opportunities for students to compare what is learned and what is unknown, and to reflect on what they have learned before and during the class. In the classroom, the teachers hang posters or printouts on the walls in an organized way showing content related to what has already been learned as hints so that the weaker students can use them as needed. This way, the students not only learn knowledge and skills but also learn how to learn, including values, attitudes, ideas, and ways of thinking in mathematics. From this process, the students gain a rich opportunity for understanding and connecting various ideas.

6.2 A Survey of Appropriate Grades to Introduce the Multiplication Table

In Japan, after World War II, under the USA occupation through the General Headquarters (GHQ) of the Allied Powers, there was a discussion on whether to introduce multiplication and the multiplication table in the second or third grade. Traditionally, the Japanese used to introduce it in the second grade; however, the GHQ recommended the third grade or upper grades in relation to the experience in the USA, known as progressivism. In 1957, Tatsuya Matsubara surveyed the appropriate grade for memorizing the multiplication table in relation to mental age with the support of Yoshinobu Wada.[2]

[1] Japanese teachers call problem solving for a problematic originating from a given unknown task a "problem-solving approach" (Isoda, 2015; Isoda and Katagiri, 2012).

[2] Yoshinobu Wada, a professor at the Tokyo University of Education, was known as a curriculum specialist in the Ministry of Education, who tried to defend the order from the General Headquarters of the Allied Powers. He introduced mathematical activity as the reorganization of the organism by J. Dewey (1916) which is currently known as the radical constructivism by Glasersfeld (1995) in relation to Piaget (1970). The mathematical activity at that time was the new view of the mathematization principle described in the 1943 textbooks during World War II. It was revised as the mathematical thinking and attitude principle, and the extension and integration principle in Japanese national curriculum reform (Isoda, 2018, 2019).

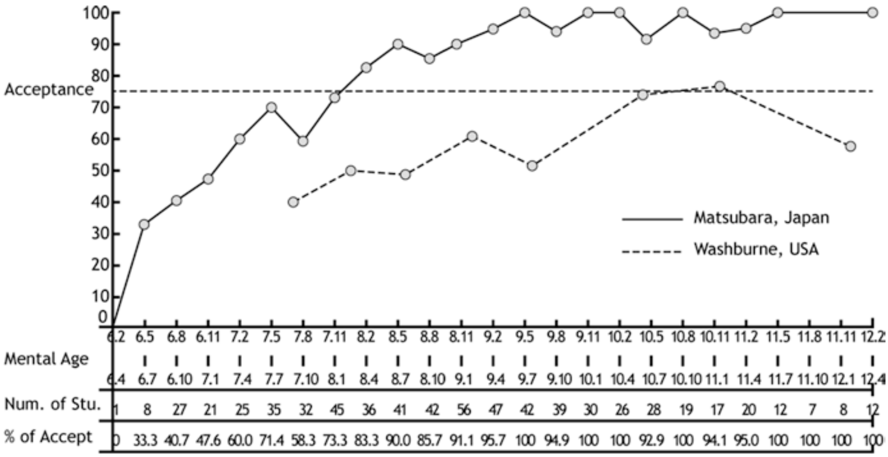

Fig. 6.1 The mental age for successful learning of the multiplication table according to the 75% acceptance (*Accept*) line among Japanese students and USA students (*Stu.*). *Num.* number

In his survey, he adopted the research by Carleton Washburne[3] (1931) in Japanese settings, such as the ways of teaching, and he compared the difference in students' success between Japan and the USA, as shown in Fig. 6.1. The Japanese setting meant the Japanese method of teaching under the cultural tradition of memorizing the multiplication table.[4] The teaching content and methods involved 36 hours of lessons which were developed under the supervision of Wada and the teachers from the Elementary School at the Tokyo University of Education.[5] The US setting studied by Washburne was the Winnetka schools in the USA which were influenced by progressivist education.

From the obtained results, shown in Fig. 6.1, Matsubara (1969) concluded that a mental age of 8.1 years is a possible age to learn multiplication, which implies that it might be suitable to teach the multiplication table from the later semester in the second grade. From the viewpoint of curriculum reform, the USA setting was influenced by progressivism. The results were related to differences in the curriculum and teaching culture. This implies that the lower achievements in the USA at an older age may have been relevant to the curriculum and education in that setting in that era.

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[3] Washburne was known for the Winnetka Plan for progressive education and was the president of the Progressive Education Association.

[4] Since the sixteenth century, the Japanese "3Rs" (reading, writing, and arithmetic) have included memorization of the multiplication table. In the East, it was normal to memorize the division table in the past. See also Chap. 7 in this book.

[5] It is a kind of lesson study under the collaboration of Wada Group and the Elementary School Mathematics Group of Tokyo university of Education. Tokyo University of Education was the predecessor of the University of Tsukuba, which originated lesson study in 1873. The elementary schools established a national-level lesson study group as a society in 1904. Wada also established his own lesson study group as a society and its still exist after Wada passed away.

6.3 The Multiplication Table in Japanese Textbooks for Learning How to Learn

This section illustrates how Japanese teachers teach the multiplication table and learning how to learn in order to develop students who learn mathematics by and for themselves. In the case of Japan, elementary school mathematics textbooks are part of the results of lesson study as well as a major reference for lesson study.[6] Here, these textbooks are preferred for illustration of the teaching.[7]

The four sets of textbooks analyzed were Gakko Tosho (Hitotsumatsu, 2005; Isoda and Murata, 2011),[8] Tokyo Shoseki (Hironaka and Sugiyama, 2006), and PROMETAM (2005)[9] (Secretaría de Educación, 2007).[10] The objective of the analysis was to know the aims of constructing, extending, memorizing, and applying the multiplication table of the numbers from 1 to 9.

For teaching the meaning of multiplication and the multiplication table, around 33–35 hours of lessons with exercise and tests are allotted, which is distributed as described in the sample shown in Table 6.1.

The activities employed in the various books for teaching the multiplication table are similar. For example, Gakko Tosho textbooks present seven activities for introducing the multiplication table of 2, and these same activities are used with minimal variation in addressing the tables of 5, 3, and 4. The activities proposed in the Gakko Tosho books for presenting the multiplication table of 2 are shown in Fig. 6.2.

Table 6.1 Sample for teaching multiplication in the second grade

Content of subunits	Number of hours of lessons
1. Meaning of multiplication	4 hours of lessons
2.1 Multiplication tables of 2, 5, 3, and 4	9 hours (including time for memorizing) + 3 hours of exercise, application
2.2 Multiplication tables of 6 to 9	9 hours + 1 hour of practice
2.3. Multiplication by 1	1 hour
3. Properties of the multiplication table	3 hours + 2 hours of practice and challenges

[6] In Japan, teachers must preferentially use textbooks authorized by the national government.

[7] This alternative might be not understandable for countries that usually use worksheets for teaching, such as the USA and the UK. The roles of textbooks differ depending on the country. Worksheet culture has originated from textbooks that are applicable for different curriculum and do not only follow the official curriculum. In the East, textbooks traditionally represent the official curriculum well.

[8] The chapters on multiplication for the second grade in the 2005 and 2011 editions are similar.

[9] Here, we chose various textbooks to shows the similarity and consistency on the different curriculum standards in Japan and the country in Japan Overseers Cooperation. See the English translation of three generations of Japanese Curriculum Standards: Isoda, 2005, Isoda and Chino, 2006 and Isoda, 2010.

[10] PROMETAM [Proyecto Mejoramiento en la Enseñanza Técnica en el Área de Matemática] was a textbook developmental project conducted in Honduras by JICA [the Japan International Cooperation Agency].

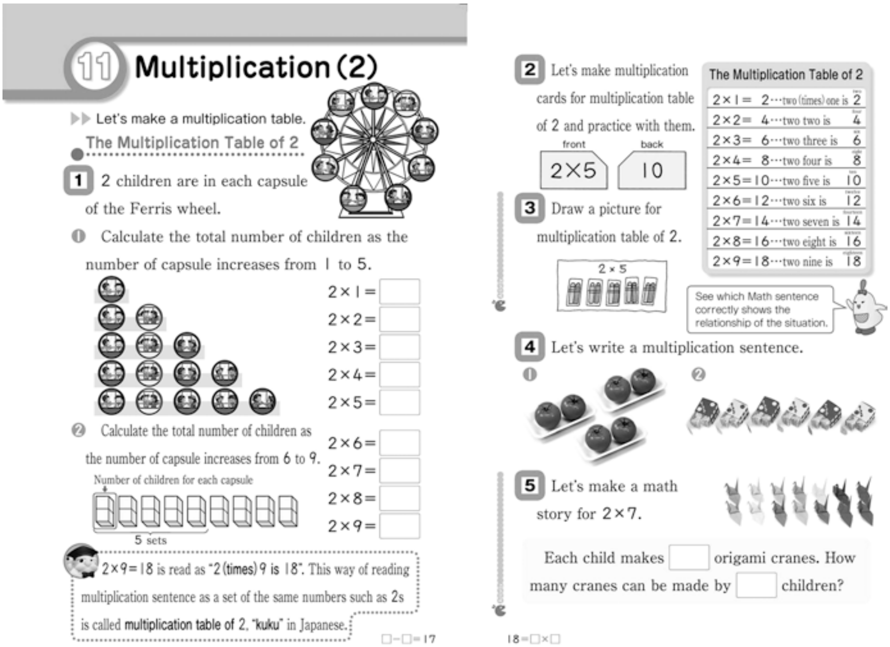


Fig. 6.2 Isoda and Murata (2011), Grade 2, Vol. 2, pp. 17–18, Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 13–14

As exemplified in Fig. 6.2, both editions are almost the same and include the following activities:

1. A situation with discrete quantities which can be extended
2. An activity for extension with a diagram and tape (consecutive antiquity) at the back, so that students can extend it with a block model and see the pattern, and can continue by reading the expression and its interpretation (the expression of multiplication and the multiplication table)
3. The manner of reading the row of 2 for comparing expressions and memorizing
4. Using cards with the product written on the back for memorizing
5. Representing situations as multiplication
6. Determining multiplication from the picture
7. Developing a situational problem from an expression like 2 × 7

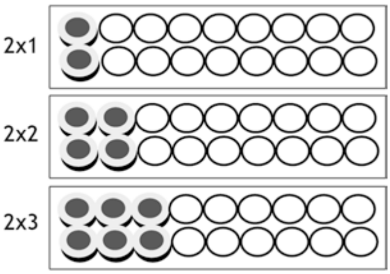
The activities proposed in the Gakko Tosho textbook for presenting the multiplication tables of 2, 5, 3, and 4 are shown in Table 6.2. Similar teaching of content and sequence are repeated in every extension of each row for enabling students to learn how to extend the multiplication table.

Table 6.3 shows that Gakko Tosho, Tokyo Shoseki, and PROMETAM have chosen the same manner of presenting the multiplication table. The similarity between the learning activities and problem situations in the books from the different publishers implies consistency of the Japanese approach. The reason is explained in the next section.

Table 6.2 Gakko Tosyo teaching sequence

Activity	Multiplication table of 2	Multiplication table of 5	Multiplication table of 3	Multiplication table of 4
1. A situation	Yes	Yes	Yes	Yes
2. Finding products and extending	Yes	Yes, variation	Yes, variation	Yes, variation
3. Continuing the row for memorizing	Yes	Yes	No	No
4. Using cards	Yes	Yes	Yes, variation	Yes, variation
5. Representing with drawings	Yes	No	Yes, variation	Yes, variation
6. Determining the expression	Yes	Yes	Yes, variation	Yes, variation
7. Constructing problems	Yes	Yes	Yes	Yes
8. Others	No	Yes	Yes	Yes, variation

Table 6.3 Comparison of Gakko Tosyo, Tokyo Syoseki, PROMETAM

Activities for learning the multiplication table of 2	Publisher		
	Gakko Tosho	Tokyo Shoseki	PROMETAM
1. A situation	Yes	Yes	Yes
2. Finding products and extending	Yes	Yes	Yes
3. Continuing the row for memorizing	Yes	Yes	Yes
4. Using cards	Yes	Yes	Yes
5. Representing with drawings	Yes	No	No
6. Determining the expression	Yes	Yes	Yes
7. Constructing problems	Yes	Yes	No
Practicing with rows of an array 	No (instead of the array, it uses blocks with covering and uncovering sheet to show it as variable like Fig. 6.9)	Yes	Yes

6.3.1 Developing Multiplication Tables for the Rows of 2, 5, 3, and 4

There is consistency in developing the rows of the multiplication table in Japanese textbooks, which is the repetition of the format shown in Fig. 6.2 from the row of 2 to the other rows. The repetition provides the students with the opportunity for learning how to construct and extend the rows: Students are able to imagine the ways of learning at the next rows.

The teaching sequence for the rows of 2 to 5 is 2, 5, 3, and 4, instead of 2, 3, 4 and 5 because the products in the rows of 2 and 5 are known through counting by twos and fives. Students feel the necessity for memorization of the products in the rows of 3 and 4, likely through counting by 2s and 5s.

Uniquely, the Gakko Tosho (Hitotsumatsu, 2005) textbook for the second grade has the following activity between the rows of 2–5 and the rows of 6–9 (see Table 3.1 in Chap. 3). The idea embedded in Fig. 6.3 is the distribution which makes it possible for the students to extend the rows of 2–5 to the rows of 6–9. For example, addition of the row of 2 and the row of 4 produces the row of 6. Students can predict further rows for extension of the table by themselves. The way of extending multiplication based on their prediction encourages them to develop further rows by and for themselves.

In the case of the PROMETAM project for the Central American country of Honduras, the teachers' guide recommends that students need to practice for about 5 minutes each day without fail. For example, they can recite the table being studied when they arrive at school, before starting class, before leaving for recess, before leaving school, etc. The students should memorize the tables appropriately to solidify the base for understanding multidigit multiplication, which will be discussed in the next grade.

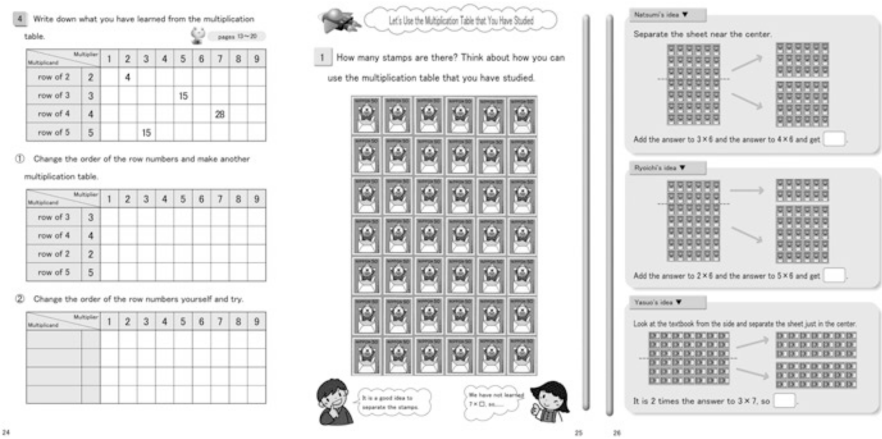


Fig. 6.3 Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 22–24

6.3.2 Transferring the Responsibility for Construction and Memorization of the Multiplication Table

The responsibility for the construction and memorization of the table is transferred from teachers to students in the following teaching sequence and materials (see Brousseau, 1997).

The study of the multiplication tables of 2 and 5 guided by the teacher includes the way to learn. Based on counting by 2s and 5s, the students can easily know the product of the rows of 2 and 5. Then, the study process for the tables of 3 and 4 should be planned so that the students will manage concrete situations and build these tables by applying what they have learned. The students can find each product by adding the multiplicand to the previous product in the table, so they do not need to add from the beginning to find the next product in the table. By repetition of the same ways of learning (Fig. 6.2), the students are able to imagine what they need to do next. As shown in Fig. 6.3, the students have a hypothesis for the extension of the table, which they want to check by themselves. By repetition in Table 6.2 and use of the hypothesis, they are able to generate and confirm new rows in a learned manner.

As shown in Fig. 6.2 and in Tables 6.1 and 6.2, the teacher and the students can use arrays or blocks, multiplication cards, and manner of reading pattern for every row as a means for constructing, extending, practicing, and memorizing the multiplication table. To make the students responsible for constructing, extending, and memorizing, the teaching sequence and materials are prepared in the textbooks and by the teachers.

6.3.3 Extension of the Multiplication Tables of 6–9 and 1

Based on learning how to learn by repetition of the same learning sequence for the multiplication tables of 2 to 5 and the expectation of extension, students can extend the multiplication tables of 6 to 9 in every two class hours by themselves. In every class, the teachers ask the students to develop every row in the same manner.

The row of 1 is not easy to learn in the same manner because students do not feel any necessity for learning it. In the Gakko Tosho textbook, it is introduced as shown in Fig. 6.4. The necessity of the row of 1 exists for permanence of form (see Chaps. 3 and 4, Peacock (1880)). As long as the students use their previously learned knowledge, the numbers of candies and oranges should be expressed by multiplication. In this context, the piece of cake on the dish is expressed as 1×1. Realizing its necessity, the students can develop the row of 1 in the same manner for permanence of form.

Japanese teachers usually allot about two class hours for every row because it takes time for memorization as well as construction of the multiplication table by the students.

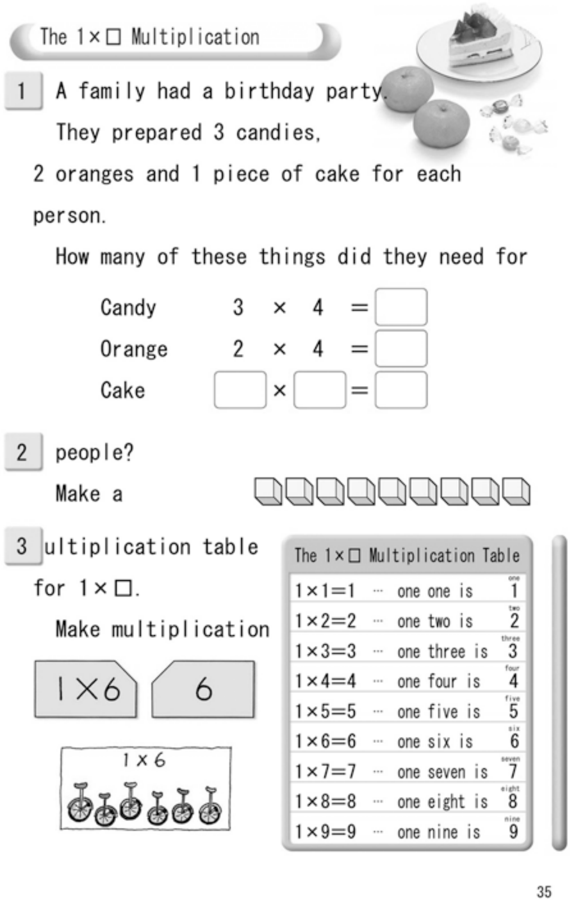


Fig. 6.4 Hitotsumatsu (2005), Grade 2, Vol. 2, p. 35

6.3.4 Properties of the Multiplication Table for Discovering the World of Multiplication with a Sense of Wonder

After the construction of every row and memorization, the Japanese textbook treats the multiplication table as a world of multiplication and as an operation without situations (Figs. 6.5 and 6.6). It is remarkable difference when we compared it with other countries such as Chile, Mexico and Singapore which use several grades to extend multiplication table up to row of 9 (see Table 2.4 in Chap. 2). Even if the students have not memorized the multiplication table well, they can fill in the products using the property of every row by adding the same number to the next column. After completing the table, the students can find several patterns hidden in the multiplication table. Commutativity of multiplication is discovered at this moment. As discussed in Chap. 3, there is no contradiction in the Japanese definition and the

Fig. 6.5 Patterns in the multiplication table of 3, as demonstrated by Y. Yamamoto (Rasmussen and Isoda, 2019)

What can you find?

$3 \times 1 = 3$	$3 \times 11 = 33$
$3 \times 2 = 6$	$3 \times 12 = 36$
$3 \times 3 = 9$	$3 \times 13 = 39$
$3 \times 4 = 12$	$3 \times 14 = 42$
$3 \times 5 = 15$	$3 \times 15 = 45$
$3 \times 6 = 18$	$3 \times 16 = 48$
$3 \times 7 = 21$	$3 \times 17 = 51$
$3 \times 8 = 24$	$3 \times 18 = 54$
$3 \times 9 = 27$	$3 \times 19 = 57$

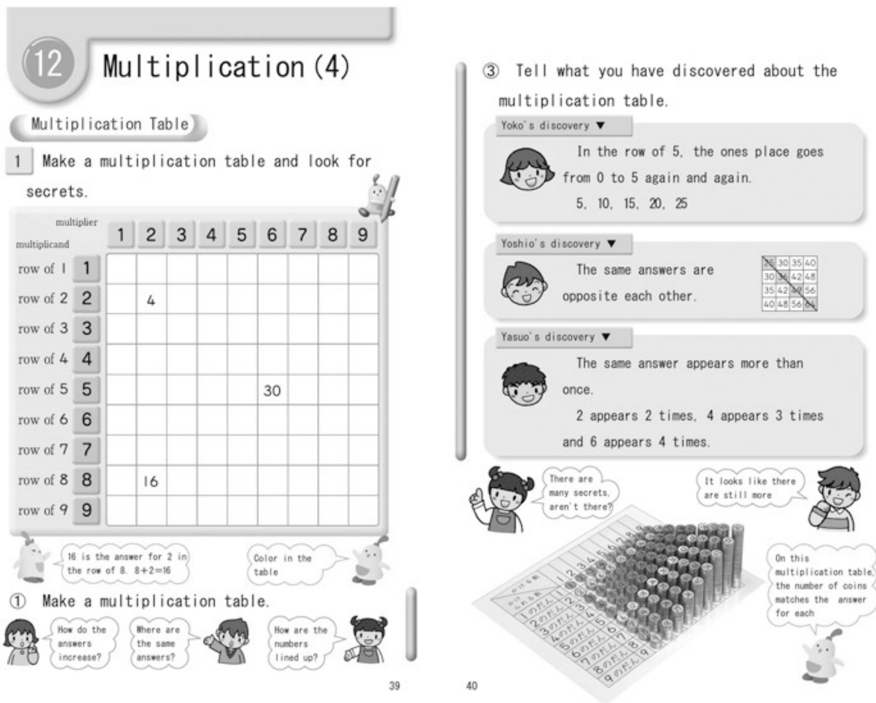


Fig. 6.6 Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 39–40

multiplication table; thus, it is not necessary to discuss commutativity from the introduction of multiplication (see Chap. 5).

Students find a number of different properties in the multiplication table and feel a sense of wonder.[11] Such mathematical structures of multiplication table enable

[11] It is related with mathematical value in relation to mathematical thinking (see Chap. 1, Table 1.1 and Mangao, Ahmad, and Isoda (2017).

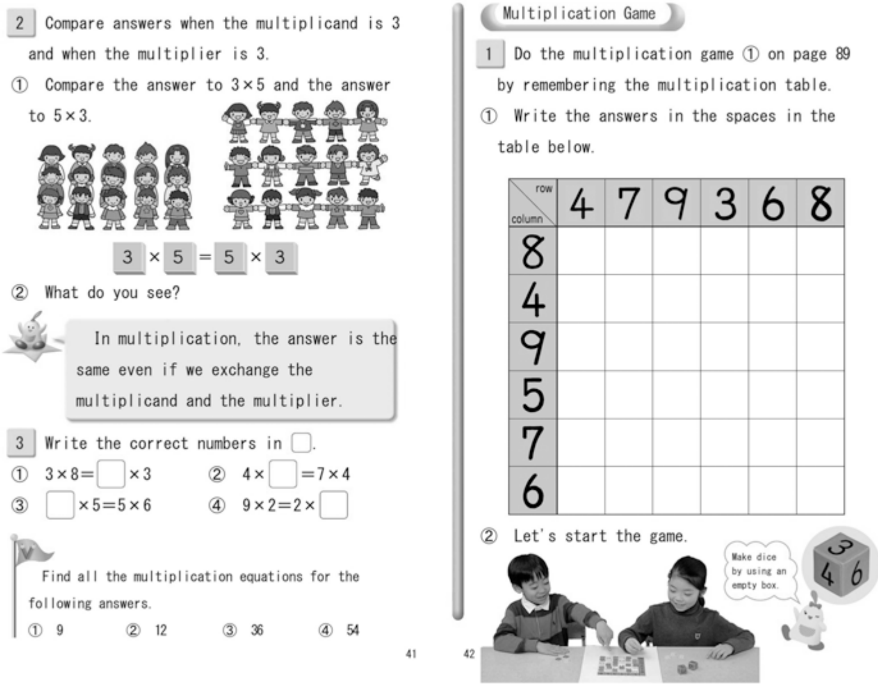


Fig. 6.7 Hitotsumatsu (2005), Grade 2, Vol. 2, pp. 41–42 (Row and column should be alternate)

students to recognize the existence of the world of multiplication with the harmonious feeling of beautifulness.[12] Some of them are revealed in later grades, as discussed below (Fig. 6.7).

Example 1 In the third grade (grade 2 in the 2017 curriculum), multiplication will be extended beyond 9 × 9. In Fig. 6.5, various patterns in the numbers can be found: products in the tens place: 0 0 0 (blanks of tens), 1 1 1, 2 2 2, 3 3 3, 4 4 4, and 5 5 5; 3 × 18 = 3 × (10 + 8) = 3 × 10 + 3 × 8 = 30 + 3 × 8); products in the units place (3, 6, 9, 2, 5, 8, 1, 4, 7 (if we change the order, we see 1 in the 7th line, 2 in the 4th line, 3 in the 1st line, 4 in the 8th line, 5 in the 5th line, 6 in the 2nd line, 7 in the 9th line, 8 in the 6th line, and 9 in the 3rd line). How do we explain these patterns? Can we find similar properties in other rows? (This example was provided by Yoshikazu Yamamoto from the Elementary School at the University of Tsukuba (Rasmussen and Isoda, 2019).)

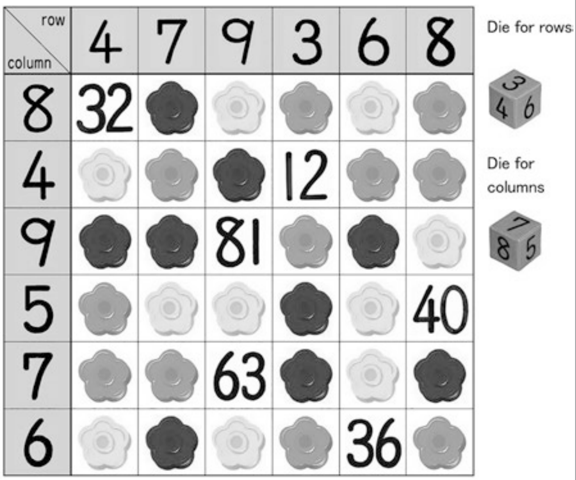
[12] In Japan, students have the opportunity to learn the world of addition using an addition table and the world of subtraction using a subtraction table by finding their properties. See Isoda and Katagiri (2012), Dizon M., D., Ahumad, N., J., Isoda, M. (2017), and the lesson study videos by Takao Seiyama at https://www.youtube.com/watch?v=7TY_SHTgmFQ, https://www.youtube.com/watch?v=TR34ZdBXmz8, https://www.youtube.com/watch?v=NNWtmIQ7YNs&t=426s, and https://www.youtube.com/watch?v=njNK6xoAkwQ.

Example 2 In the upper grades, after students have learned the concept of averages, some teachers ask the students to find the total products in the multiplication table up to 9 × 9. There are various ways to find the total value of the multiplication table. Two beautiful and wonderful ways are 45 × (1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9) and 5 × 5 × 81.[13] The explanation of this property requires the ability to see the decomposition of a number with factors (multiplication) and addition.

Rasmussen and Isoda (2019) have analyzed example 1 using anthropological theory and noted that the Japanese extension of the multiplication table is fruitful teaching material to develop mathematical thinking.

After they have studied the multiplication table, the students are engaged in a game to know the significance of memorizing the table. Table 6.4 is a sample lesson plan given in the Annex of the Gakko Tosho textbook. And then, on the 2nd grade

Table 6.4 A Lesson for enjoying to use memorized table: Row and column should be alternate

<table>
<tr><td colspan="2">Objective: That students have fun using the multiplication tables of 6 to 9 and learn
Point of assessments: Do the students enjoy playing by using the multiplication tables from memory? In the game, can students predict the values of the dice from the remaining numbers on the game board?</td></tr>
<tr><td>Teacher: We are going to play with a game board and two dice. We will use 30 chips to cover some of the spaces on the board
Today we are going to play first with dice and then with cards


Teacher: To play, roll the dice (or use a substitute). Multiply the two numbers and say the answer. If the answer is correct, you win the chip from that space. If there is no chip in that space, you have to put one of yours in that space. Decide how many times you will roll the dice. Play by taking turns. The student who gets the most chips wins</td>
<td>Specific materials are needed. The game board can be made by the students in their notebooks, or a photocopy can be used. The dice can be made from pieces of wood. A spinner or a deck of multiplication cards made by the students can be used as substitutes.
To make the game board, write the numbers in the first row and the first column. In the inner spaces of the game board, write the products of the numbers in the first row and the first column.
Then, place 30 chips on the game board, leaving some spaces uncovered. Students realize that the patterns of multiplication table do not appear if we change the order of numbers on the multiplication. They cannot fill in without memorizing the multiplication table. To answer this task they recognize the significance of memorizing</td></tr>
</table>

[13] 5 × 5 is at the center of the multiplication table.

Gakko Tosyo textbook, the 2005 edition extend the multiplication to the case of ten times and the 2011 edition additionally extend the multiple beyond the multiplication table 9 × 9 to multiplied simple two digit numbers by ones.

6.4 Memorizing the Multiplication Table as a Cultural Practice

Memorization of the multiplication table is a cultural practice that favors learning the multiplication table. In some countries, memorizing has a negative meaning because it seems to be forced by teachers without appropriate understanding and express it as a part of number sense instead of explaining it as memorizing. However, it does not have such a negative meaning in the East. The Japanese have been engaging in this cultural practice since the sixteenth century for using the abacus. In the sixteenth century, even though the knowledge of the division table for the abacus was necessary to be an accountant. *Jinkoki*, by Yosoda (1627), as shown in Fig. 6.8, was the most popular and standard textbook until the middle of the nineteenth century which mentioned up to extraction of the square root and Pythagoras theorem. It became popular for everyone to memorize the multiplication table like songs. In this book, the multiplication table was read as *ni ni no shi* ("2 2, 4"), *ni san no roku* ("2 3, 6"), etc. In English, this means "two multiplied by two equals four" (in short, "two two is four") and "two multiplied by three equals six" (in short, "two three is six"). There were no algebraic expressions yet in that era.

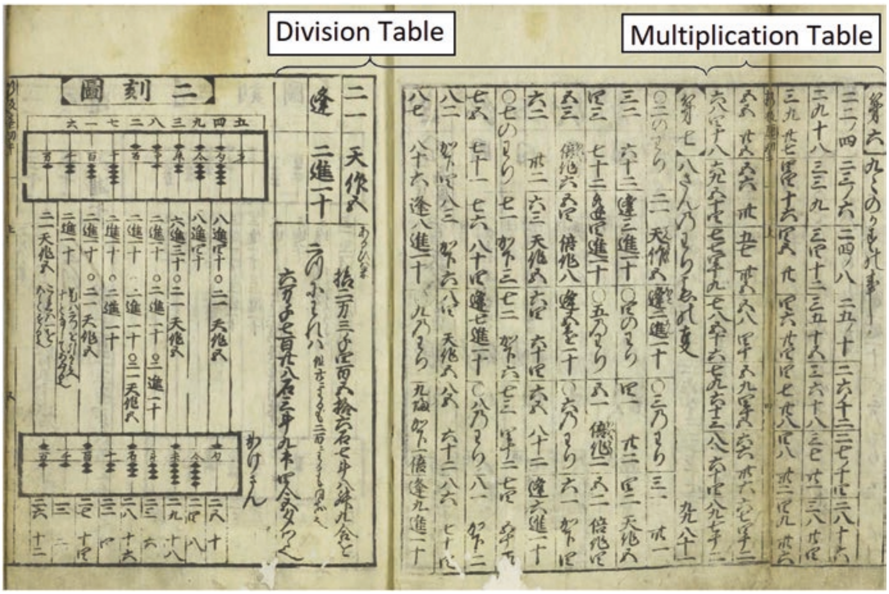


Fig. 6.8 Yoshida, M. (1627) *Jinkoki*, pp. 3–4

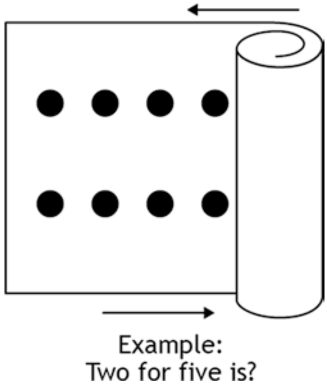


Fig. 6.9 A paper role model to extend the row of 2 to see the multiple as variable

At present, the recitation begins *ni ichi ga ni* ("2 1, 2"), *ni nin ga shi* ("2 2, 4"), *ni san ga roku* ("2 3, 6"), etc. For the row of 2, the students can recite it like a song in 10 seconds. The majority of 8- or 7-year-old students can memorize it, as already mentioned. As for second-grade students, it is a milestone for their learning in their culture. Historically, there was a tradition to memorize not only the multiplication table but also the division table, memorizing multiplication table was basics and the people who mercerized the division table recognized experts for using the abacus.[14] In the case of the division table, the practice of memorization was lost because we do not need it if we know multiplication and we do not use the abacus anymore for calculation.

In Chaps. 4 and 5 of this book, we mentioned that the multiplication table is introduced with the rows of 2 and 5 because the products of both can be found through counting by 2s and 5s. Additionally, teachers use some sequences for memorization practice. The following is an example from a Japanese class:

1. After constructing the row of 2 with meaning, ask the students to say and repeat it from "2 × 1" to "2 × 9" on the board.
2. Cover the product of "2 × 1" with a piece of paper and ask them to say what it is (then lift the piece of paper to verify the answer).
3. Ask the students: Two multiplied by one? Two one is two. Ask them to visualize and repeat the sequence, counting by 2s up to 10 to promote memorization.
4. As in Fig. 6.9, covering the products of "2 × 1" and "2 × 2", get the students to recite the multiplication table from "2 × 1" to "2 × 9" with counting by 2s and adding 2 every time. Repeat the activity, covering up more products, until they are all covered.
5. Ask the students to stand up and recite the multiplication table quietly and to sit down once they are finished. (The teacher observes who among the students takes a longer time, who is faster, and who needs additional practice.)

When the lesson ends, sometimes the printout of the multiplication table with the products covered can be left on the classroom wall. The students can practice freely

[14] *Jinkoki* did not address addition and subtraction because it is a kind of visualized counting if we use an abacus.

and with satisfaction at confirming their answers by uncovering the products. This practice is competitive but enjoyable for second-grade students.

In Eastern culture, teachers have the responsibility to make students memorize the multiplication table. Thus, teachers place a lot of opportunity for providing activities to support the students. An array sheet like that shown in Fig. 6.9 is used in building the multiplication table and also in practicing it. The amount of the vertical array diagram represents the multiplicand or the quantity in each group. The situation and the product can be presented by moving the paper that covers the groups horizontally.

Practicing the multiplication table includes four activities: (1) correctly recite the table observing the expression or the collection of arrays; (2) reciting from 2×1 to 2×9; (3) reciting the table from the bottom up and from the top down; and (4) reciting the table in random order.

Teachers assess students' degrees of understanding by observing whether they can relate the mathematical expression to the meaning of other expressions. The group of groups represented by collections of balls also suggests plates with fruit, columns with cubes, etc. The student gains understanding by relating each expression to the expression in the table; for example, $2 \times 4 + 2$ is 2×5. (Mr. Tsubota's class in Chap. 5 of this book is also an exemplar.)

6.4.1 Using the Cards

As shown in Figs. 6.2 and 6.4, each card has on its front the expression (binary operation) of multiplication and on its back the product. The Gakko Tosho textbooks include them in the Annex. Otherwise, the teachers or students prepare them in an appropriate size. They are used not only to practice memorization but also to find patterns in the multiplication table. The fundamental ways of memorizing the multiplication table using multiplication cards are as follows:

Individual use: (A) Place the cards in random on the table. Say the product while looking at the expression on the front of the card. (The students can place a mark on the cards they have incorrectly answered and practice more with them.) (B) Place the cards in random. Say the expression while looking at the product on the back of the card. (C) Carry out the practice of (A) or (B) with various multiplication tables.

Use in pairs: (A) One student shows the front of a card to another student, who gives the product while looking at the expression on the front of the card. Repeat this activity, taking turns. (The roles can also be changed when one student answers incorrectly, or each student can continue until he or she has correctly answered five times). (B) Each student prepares cards for one of the multiplication tables in random order. (It is best that they use cards for only one or two rows.) Each student places a card face up on the table at the same time, reads the expression, and gives the product while looking at the card. The student with a greater product wins. (C) Place the cards on the table, face up. A student chooses one, reads the expression, and gives the product. To check the answer given, look at the product on the back of the card. If the student has answered correctly, he or she can keep this card and continue with another card. If he or she has answered incorrectly, he or she lose his turn and does not keep the card (they can also take turns). The student who collects the greatest number of cards wins.

Use in pairs or in a group: (A) Place the cards face down on the table. A student quizzes his or her classmates by saying an expression from the multiplication tables in use. The others look for the product of this expression and pick up the cards that have this product. The student who gets the greatest number of cards wins. (B) Place the cards face up on the table. A student quizzes his or her classmates by saying a product from the multiplication tables in use. The others look for the expression of this product and pick up the cards that have this expression. The student who gets the greatest number of cards wins.

The teachers should help the students to invent other ways and to use the cards considering the students' real situation (see Fig. 4.2, Chap. 2). For this kind of activity, Japanese teachers usually use the first 3–5 minutes of each class to practice all together. Enjoyable daily cultural practice is the key to memorization.

When we say real situation, some of teachers and math-educators usually imagine the dichotomy to distinguish mathematics and real world. However, as explained Chap. 1, Japanese Approach usually consider on the curriculum sequence under the extension and integration principle (see Fig. 1.1). It is the reorganization process of mathematization. On this context, Japanese Approach enhance sense making (see Chap. 5), and it means change the intuition (see Fig. 5.22) and reality itself. What is the reality for students in these activities on memorizing and using multiplication table? To the terminology of horizontal and vertical mathematization by Treffers, A. (1987), Freduental, H. (1994) expressed uncomfortableness from his perspective of mathematization (1973) and redefined mathematization with levels by the terminology of living and life. He also mentioned mathematical object as entity (1983) to explain existence. On these context, reality, here, means the reality for second grade students on their life. For second grade students, reality is also existed on their enjoying games to think about and explore the rule and the behind structure for winning the game, as well as their narrow economical experience. With comparison of second grade students' economical-arithmetical life, these kinds of games provide the real situation for their world of multiplication within classroom. On this reality, these activities to memorize multiplication table is a kind of cultural practice with enthusiasm in Japanese classroom. The tools for these cultural practices has been developed by teachers. Followings are further examples.

6.4.2 Using Area-Array Cards

Mr. Hiroshi Tanaka (2007) designed new illustrated multiplication cards which include area-array images (see Fig. 6.10).

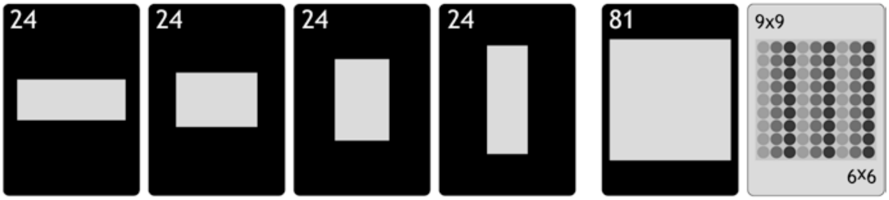


Fig. 6.10 Multiplication cards by Hiroshi Tanaka

Fig. 6.11 A notebook: The left is student activity and the right is progress of every row and three step assessment with stamps and signs

6.4.3 Using a Notebook and Journal Writing at Home

In Japan, to develop children's custom of self-learning at home, teachers usually use a notebook for homework and have them exchange journals/diaries.

These activities are not only for memorization but also for making it enjoyable for students, as shown in Fig. 6.11.

6.5 The Sense of Wonder in the Multiplication Table

During the middle of the second semester in three semesters per year, the second-grade students in all schools in Japan can be seen reciting the multiplication table in front of their teachers. What kind of actual practice does the teacher provide when the students are learning the multiplication table?

The following lesson plan was developed by Mr. Kozo Tsubota (2007), a teacher at the Elementary School at the University of Tsukuba. It uses the voice and ideas of a real teacher—the one who is designing and leading the lesson study community. The theme of his lesson study on the multiplication table is "Teaching the properties of the multiplication table to encourage students to discover patterns in the multiplication table with a sense of wonder and to appreciate the patterns in the table." The task is related with judicious using of calculator if we ask it at second grade students, and if not it become upper grade task.

6.5.1 Focusing on Beautiful Patterns with a Sense of Wonder and Appreciation

When the multiplication table is being taught, it is usually with the following sequence of steps:

1. The meaning of multiplication is built through known situations: ways of counting and iterated sums.
2. The multiplication table is developed up to 9. It is extended up to 9 × 9, through explorations.
3. Students are asked to recite the multiplication table and apply it.
4. The multiplication table as a whole is used with the goal of identifying patterns of addition, subtraction, and multiplication.

In these activities, many teachers usually focus on step 3. However, students should not simply memorize the multiplication table as if it were a song. In step 4, students should be given activities so they can discover the beautiful patterns in the numbers—in several rows of results—that make up the multiplication table. For example, the sum of the digits in the units place and in the tens place for any product of 9 is equal to 9; thus, 9 × 7 = 63 and 6 + 3 = 9. Moreover, if we take any product from the first half of the row of 9 and add it to the corresponding product from the opposite side of the second half of the row, the result will be 90; for example, 9 × 1 = 9 and 9 × 9 = 81, and 9 + 81 = 90. Similarly, 9 × 2 = 18 and 9 × 8 = 72, and 18 + 72 = 90.

6.5.2 Preparing a Problematic: "Why"

Students develop a sense of wonder based on the awareness of problematics in relation to given tasks (see Chap. 1). The lessons should be designed to allow the students to follow up on these kinds of questions and investigate the "why."

Materials have been developed so that students can see two multiplicative expressions in class and be amazed by the fact that the results are the same. They ask why, carefully observing the expressions, transforming them and hypothesizing a response, and find relationships among the numbers. The expressions given to the students are:

$4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4 \times 4$
$8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8$

The students are asked which expression will give a larger result.

The answer is not easy for them to find, even when they make the calculations on paper. The students are allowed to use a calculator to find the answer. At this point, they can use the repetition function for arithmetic operations. The function consists of pressing 4 × = = . . . and 8 × = = ... When the calculator displays the results, it is confirmed that they are exactly the same. The result of both expressions is 16,777,216.

At this moment, the question "Why are the results the same?" appears in the students' minds. The students spend the rest of the lesson trying to answer the question and discussing the problems among themselves.

The teacher should allow interaction among the students and guide the discussion toward mathematical thinking. For example, the teacher should try to get the students to reach an understanding of the numbers 4 and 8. The students should realize that $4 \times 4 \times 4 = 64$ and $8 \times 8 = 64$ are equal, or that the numbers can be decomposed into $4 = 2 \times 2$ and $8 = 2 \times 2 \times 2$. The structure of this problem uses the power that $4^{12} = 8^8$; in other words, $4^{12} = (2^2)^{12}$, and $8^8 = (2^3)^8$.

6.5.3 *How to Begin the Class?*

"Now I will write two mathematical expressions on the board. As soon as I finish, I will ask you which of the two gives a larger result. I want you to give an intuitive prediction, so raise your hand for the expression you think is greater."

The teacher then writes the two following expressions silently on the board. The students look attentively at the board while the teacher writes the expressions. They are thinking about the results of the two addition problems:

(A) $4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4 + 4$
(B) $8 + 8 + 8 + 8 + 8 + 8 + 8 + 8$

After writing on the board, the teacher says, "OK, now I will ask. First, who thinks the result of A is greater?"

A few students raise their hands. The teacher continues with "Who thinks that B has a greater result?" Now, many students raise their hands. The majority of the students think that 8 is greater. The teacher then asks, "Why do you think so?" The students will probably give many different answers. The teacher asks one of the students who raised his hand.

Student: "I calculated the answer. I thought of a simple addition."

The teacher asks: “Good, so, how did you calculate the answer?” The student replies that he used multiplication. When the teacher asks them to write the expression, the students write:

(A) $4 \times 12 = 48$
(B) $8 \times 8 = 64$

The majority of the students agree that this is correct. The teacher then asks, “Any other reason?” Another student gives another reason. He goes to the board and tries to explain it by drawing line segments between the two expressions.

(4 + 4)	+	(4 + 4)	+	(4 + 4)	+	(4 + 4)	+	(4 + 4)
|		|		|		|		|
8	+	8	+	8	+	8	+	8

This is fast. After grouping and connecting the numbers, he asks the other students to explain it. Can anyone explain the meaning of the groupings?

After some interactions, the teacher says: “Now comes the principal question of the lesson. I will change the plus signs to multiplication signs, and you must respond quickly to the initial question: Which of the expressions do you think is greater?”

6.6 Final Remarks

In Chap. 2 of this book, we confirmed that the multiplication table is taught in different grades around the world and posed the question as to the choice of grade for introducing it. In Japan, it is taught in the second grade, and this chapter has explained four reasons for this. The first reason is that it is possible. The second reason is that students are able to extend the multiplication table by themselves in an appropriate teaching sequence. To do so, they study ways to produce the table for the rows of 2 to 5 at first, and then they adapt ways of extension to other rows. They learn the meaning of a situation, producing the row with models and patterns, and creating situations for multiplication expressions. At the last stage, the structure of the multiplication table is analyzed and the properties of the table are established. The third reason is that memorizing the table is an enjoyable activity for students. The fourth reason is to develop a sense of wonder by exploring the patterns in the table and appreciate the reasonableness of the world of multiplication.

In the Japanese approach, students are able to learn the skill to extend what they have learned and the significance of their learning. Japanese teachers try to set the tasks and activities for memorizing and using the table through the various activities for sense making on the world of multiplication as a part of enjoyable cultural practice. The consequence of further Japanese students’ achievements in relation to the number sense, but not only limited multiplication, are known by surveys such as Reys, Reys, Nohda, Ishida, Yoshikawa, & Shimizu (1991) and Reys, Reys, Nohda and Emori (1995).

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