Chapter 10
Building Opportunities for Learning Multiplication

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10.1 Introduction

There have been fewer research studies on multiplication than studies on addition and subtraction (Fuson, 2003; Verschaffel, Greer, and De Corte, 2007). Verschaffel et al. (2007) state that there is really a scarcity of research regarding strategies used by students to solve multiplication and division problems.

Brocardo and Serrazina (2008) point out a curricular "big idea" to approach numbers and operations with the perspective of number sense development, articulating numbers, operations, and applications. For example, the authors note that decomposition of numbers can be learned through number expertise articulating addition and multiplication: 80 is 20 + 20 + 20 + 20, four times 20, or 4×20. They also refer to the importance of articulating the meanings and the structures of the operations.

Focusing on number sense and the role of mental calculation, Brocardo (2011) stresses the importance of being able to look at numbers as the center of a web of relationships. For example, the number 48 may be represented as $2 \times 2 \times 12$, 2×24, 4×12, $50 - 2$, $100/2 - 2$, or 6×8. When solving numerical problems, students with number sense use the representation of 48 that is more suitable to mental manipulation of the numbers or that best suits the context of the problem.

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The comprehension-based approach for learning multiplication introduced in this chapter includes these ideas of curricular integration and working with numbers and operations in the perspective of number sense development, which are briefly referred to here and will be detailed in the following sections.

10.2 Learning Trajectories

Planning multiplication teaching implies more than structuring the mathematical ideas involved in this operation. It is equally important to think how students can learn and progress in their learning, and to bear in mind that not all of them learn at the same pace and in the same way.

Simon (1995) uses the sailor metaphor to explain the concept of learning trajectory, which we consider paramount for thinking about teaching multiplication. The sailor has a global plan that includes precise milestones and a clear definition of the place of arrival at the end of the trip. However, this plan has to be successively adjusted according to different events—weather conditions, boat performance, or unforeseen situations that may arise. Such adjustments may also include unanticipated stages. Like the sailor, teachers also need a global plan to guide the proposals they prepare for students. They have to change their global plan to take into account the capacity of each student to learn, the ideas or doubts that arise, and the unforeseen situations they encounter. Like the sailor, teachers plan each stage of their journey, bearing in mind the hypothetical trajectory and the conditions resulting from the implementation of the previous stages.

Setting the global plan of the "journey" (learning multiplication) involves starting by clarifying the key milestones that determine the stages of a nonlinear path. At a macro level—the global plan of the journey—the hypothetical learning trajectory includes setting the progression of mathematical ideas and of strategies and models associated with multiplication. It also includes a flexible sequential vision, since the trajectory that is actually undertaken determines the adjustments and the paths to follow at the next stage. Lastly, it includes progression and interconnection as aspects that always underlie the design/selection of tasks for students.

10.3 A Hypothetical Trajectory in the Third Grade

The Portuguese educational curriculum states that in the third grade, (8- to 9-year-old) students should complete their studies on multiplication tables, develop their knowledge of whole and decimal numbers, and learn to build multiplication algorithms. During the previous grades, they have started the transition from repeated addition to multiplication, the exploration of multiplication meanings, and comprehension and memorization of facts—namely, the ones arising from study of the multiplication tables of 2, 5, and 10.

In the third grade, the key milestones for multiplication learning are the following (Fosnot, 2007; Fosnot and Dolk, 2001):

- Consolidation of understanding of a group as a unit
- Distributive property of multiplication in relation to addition and subtraction
- Commutative property of multiplication
- Position values pattern associated with multiplication by 10
- Associative property of multiplication
- Understanding of the inverse relationship between multiplication and division
- Understanding of the proportional reasoning of multiplication

This last one—although emerging in the third grade—is further developed in later grades.

The models associated with multiplication that students can build by exploring each task are also important milestones for setting the hypothetical learning trajectory. They are closely related to the models and procedures used in addition (see Figs. 10.1 and 10.2):

- Decomposition of terms used in repeated addition allows moving from a linear model to a two-dimensional model- the array model.
- The linear model, which supports repeated addition by "jumps," becomes a proportional model, such as the double line or proportion tables.

Considering that our focus is the third grade and that the work with proportions is developed in later grades, the trajectory we introduce here favors the use of the array model. The choice of the array model is supported by authors such as Barmby, Harries, Higgins, and Suggate (2009), who consider it an important support in the evolution of multiplicative reasoning. It should be noted that this is the model that helps to build and consolidate the use of distributive and associative properties, as Figs. 10.3 and 10.4 illustrate.

The rectangular model also allows understanding of the commutative property (Fig. 10.5), which cannot be understood from the linear model of successive addition: 4 rows of 5 elements have the same number of elements as 5 rows of 4.

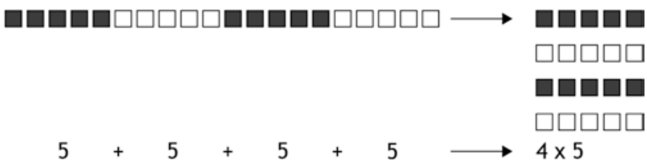
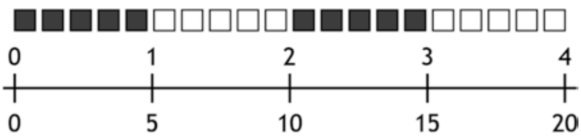


Fig. 10.1 From a linear model to an array model

Fig. 10.2 From a linear model to a double line model



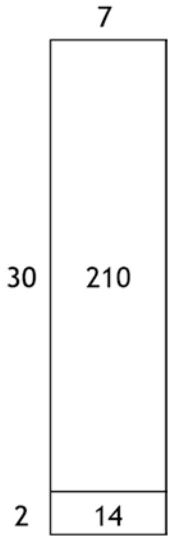


Fig. 10.3 Array model for supporting the use of the distributive property of multiplication in relation to addition in calculation

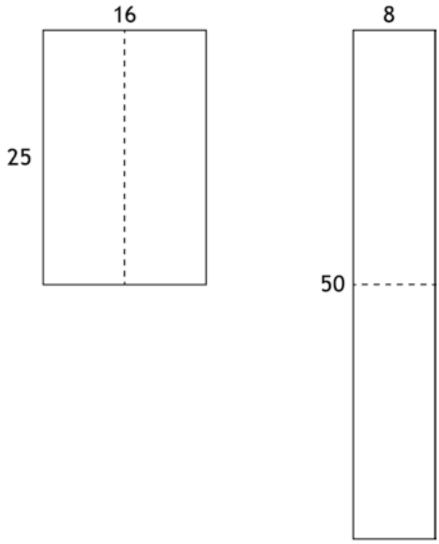


Fig. 10.4 Array model for supporting the use of the associative property of multiplication

Combining the learning milestones with the array model, the numerical universe that is used, and the order of magnitude of the numerical values, several learning trajectories can be built according to choices depending on the specific curricular nature, the characteristics of the students, and also the specific context of each school.

The hypothetical trajectory introduced below (see Table 10.1) is thus one among many possible trajectories. This is an example of an actual trajectory implemented in a third-grade class (Mendes, Brocardo, and Oliveira, 2013), which includes adjustments resulting from an experiment in a classroom of ten sequences of tasks and some particular conditions of the class and the school. We also point out the contexts used in the tasks, which take into account aspects related to multiplication learning.

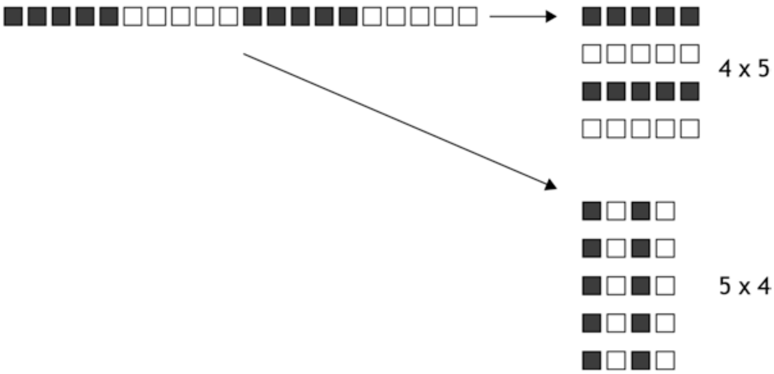


Fig. 10.5 Array model for supporting the commutative property of multiplication

Table 10.1 Multiplication learning trajectory in the third grade

Sequences of tasks	Learning milestones	Contexts and numbers
Sequences 1 and 2: multiplication tasks where the calculation by groups is made evident (6 tasks)	Consolidation of understanding a group as one unit Distributive property of multiplication in relation to addition and subtraction Commutative property of multiplication	Items displayed in a grocery store interconnected with use of multiples of 5, 3, and 6 Packs with 4, 6, and 12 stickers
Sequences 3 and 4: tasks whose context is related to the rectangular array (6 tasks)	Consolidation of understanding a group as one unit Distributive property of multiplication in relation to addition and subtraction Commutative and associative properties of multiplication	Patterns in curtains and yard pavements interconnected with use of multiples of 5 and 10 Stacks of boxes interconnected with use of multiples of 5 and 10
Sequences 5 and 6: tasks with numbers in decimal representation (6 tasks)	Distributive property of multiplication in relation to addition and subtraction Commutative property of multiplication	Filling and emptying of bottles, relating their capacities to use of reference decimal numbers and relating them to each other (0.5, 1.5, 2.5, and 0.25) Using and relating reference decimal numbers associated with values of different coins (0.1, 0.2, 0.5, and 0.99)
Sequences 7 and 9: division tasks where multiplication is favored, revealing the relation between two operations (8 tasks)	Understanding of the inverse relationship between multiplication and division	Collecting cards and using beverage vending machines to divide using multiplication, using multiples of 6 and 8
Sequence 10: multiplication tasks where multiplication is favored, revealing the relation between two operations (3 tasks)	Understanding of proportional reasoning	Filling in and using prices from tables for grocery item costs and a trip to the theater, using multiples of 1.25, 1.10, 1.60, and 0.99

Looking at Table 10.1 we can see that the learning milestones are the support for the trajectory and they emerge from the contexts of the tasks. When a new numerical set is introduced, the learning milestones are "revisited": by starting the study of multiplication with decimal numbers, the learning milestones previously considered when studying the natural numbers are reworked.

This "revisiting" process is also present in the numbers included in each task. It starts by using situations involving multiples of 2, 3, 5, and 6. Then, it "revisits" the use of those multiples in order to work with multiples of 4, 10, and 12. This numerical "revisiting" is a sequential chain that repeats itself when introducing new learning milestones: when developing the idea of the inverse relationship between multiplication and division, the numerical set is restricted to the natural numbers and the groups of 6, 8, and 10 are used again (sequences 7 and 9). When introducing the proportional sense of multiplication, 1.25 and multiples of 10 are used, which are numbers that were previously considered as a reference (sequence 10). This is the starting point to build relationships with new numerical values.

10.4 Specifying the Hypothetical Trajectory: A Sequence of Tasks

Setting a learning trajectory like the one we showed in the previous section implies paying great attention to the specific characteristics of each of its sequence of tasks. We will now analyze sequence 4, composed of four tasks, as shown in Figs. 10.6, 10.7, 10.8, and 10.9.

Task 1: Stacks of Boxes

The Piedade Grocery Store received boxes, each containing 24 apples as shown in Fig. 10.8. The 25 boxes were stacked as shown in the Fig. 10.6.

In total, how many apples are there?

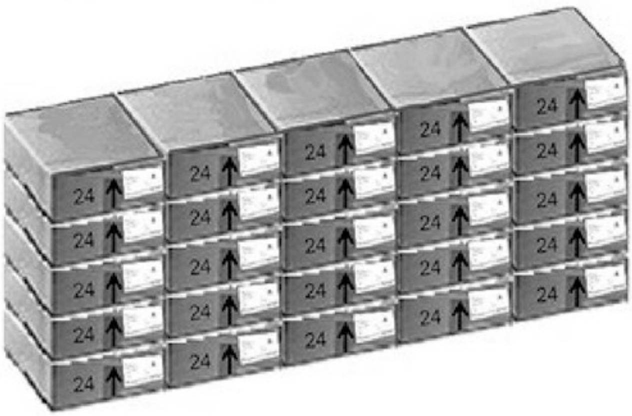
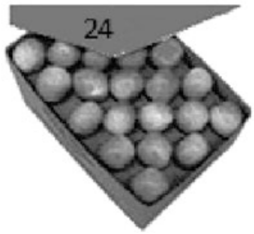


Fig. 10.6 25 boxes with 24 apples each

Fig. 10.7 25 boxes with 48 apples each

Fig. 10.8 Box with 24 apples



Task 2: Stacks of Boxes
In the Bairro Supermarket, there is also a stack of 25 apple boxes. These boxes are bigger, and each contains 48 apples as shown in the Fig. 10.7.

In this supermarket, how many apples are stored in the boxes?

Task 3: Stacks of Boxes
In the Girassol Supermarket, the total number of apples is the same as that in the Bairro Supermarket, but each box contains only 24 apples as shown in Figure.

In total, how many boxes of 24 apples are there in the Girassol supermarket?

10.4.1 Connected Calculations

Regarding mathematical ideas about multiplication, with this sequence it is intended that students progressively drop the idea of repeated addition and evolve toward multiplicative reasoning. It is also intended that they use the properties of multiplication to calculate products. Therefore, the context of tasks 1 and 2 facilitates a progression towards the array use, in combination with use of the properties of multiplication. The stacks of boxes can be seen in different ways, as different groups of rows or columns. For example, the stack with 25 boxes in task 1 may be seen as having 5 columns, each one with 5 boxes. It may also be seen as having 2 rows of 5 boxes, plus another 2 rows of 5 boxes, plus 1 row of 5 boxes. In the first case, we “see” the stack of boxes organized into 5 columns, and it is the calculation of the number of apples in each column that sets the total number of apples. In the second case, we observe that in 2 columns there are 10 boxes, and we look at the stack with the biggest possible number of groups of 2 columns. The total number of apples is obtained from the number of apples in each of the groups (of 2 columns and of 1 column) that are considered.

When students use these two strategies and make groups to calculate the requested value, they do not think about the properties of multiplication, nor the array model. However, the analysis of these strategies and the solution for other situations based on the same type of context can lead to an understanding of the properties, drawing the conclusion that $5 \times 24 + 5 \times 24 = 10 \times 24$ and that $25 \times 24 = 10 \times 24 + 10 \times 24 + 5 \times 24$. Besides exploring the different groups of boxes and the corresponding use of the distributive property of multiplication, students can also associate the total number of rows and columns with the total number of boxes. They start using the array model in similar situations, where each “cell” of the rectangle corresponds to a set of objects—in this case, a set of apples—and not just to an object, as happened at an earlier stage of learning multiplication.

Two important ideas regarding the learning trajectory, progression, and interconnection are achieved either in the numerical values involved or in the possibility of using results and relationships from previous tasks:

- In task 2, each box has twice the number of apples as each box in task 1.
- In tasks 2 and 3, the total number of apples is equal and the quantity that fits in the boxes in task 2 is twice that in task 3.
- In task 2, by moving 2 boxes (the ones in the last column), we get an arrangement similar to the one in task 1.
- Task 4 helps to consolidate relations and properties used in the previous tasks—double, distributive, commutative, and associative properties—and the use of numerical values present along the chain, such as groups of 24 and 48, and products of factors that result in 600 and 1200.

In parallel with matters of progression and interconnection between the tasks in each sequence, which are essentially oriented by the fundamental ideas linked to multiplication learning and to the overall design of the global hypothetical trajectory, it is also important to bear in mind other aspects like diversity and the characteristics of each task. We will analyze those aspects in the next section.

10.5 The Tasks

In this hypothetical learning trajectory, we have included tasks of different natures: not just problems and investigations, nor just exercises. Each type of task has its own potential. It is fundamental to select the more appropriate ones according to the teaching objectives.

The sequence shown in Figs. 10.6, 10.7, and 10.8 includes problems (tasks 1, 2, and 3) and exercises (task 4). The stickers packs task (Fig. 10.10) is an example of another type of task (investigation) which can also be included when building a learning trajectory.

10.5.1 Task 4: Stickers Packs

Eva, Luís, and Leandra collect stickers. The stickers are sold in packs of 4, 6, and 12 stickers. The 12-stickers packs are sold out. Raquel bought stickers, and she got 48.

Which stickers pack might she have bought? Explain your thoughts.

The selection of problem and investigation contexts—i.e., the characteristics of the situations that may be mathematized by the students (Fosnot and Dolk, 2001)—should be oriented so that the contexts (i) allow construction of models, (ii) make it possible for students to really understand and act upon them, and (iii) inspire students to ask questions and find solutions.

Tasks 1, 2, and 3 (Figs. 10.7 and 10.8) explore a context of fruit boxes and the different ways they can be stacked. There are other contexts that also allow students to progressively build and refine the models underlying the multiplication [characteristic (i)]. They are based on organizing groups of objects in packages (eggs, balls, drink cans), rectangular patterns in curtains, or collecting and organizing the necessary data to inventory the objects stored in a certain place. The boxes and the way they are stacked (Fig. 10.6) can help certain ways of thinking associated with the rectangular arrangement; i.e., they allow students to model situations using such an arrangement. Earlier, students should have had the opportunity to explore contexts that allowed them to model a situation such as repeated addition on a numerical line. At a later stage of multiplication learning, they should, for example, explore situations whose context allows them to model multiplication as an area or a proportion, using a corresponding double numerical line.

By analyzing the tasks included in Figs. 10.6, 10.7, 10.8, 10.9 and 10.10, we can easily see that they suggest that students seek ways to find solutions using different knowledge and relations [characteristic (ii)] in accordance with their level of mathematical development. For example, in task 1, to calculate 25×24, they can observe the image and start to calculate the total number of apples in each column, determining 5×24. Others can make groups of 10 boxes, calculating 10×24. Yet others, less familiar with multiplication procedures, can repeatedly add 24.

Fig. 10.9 Task 4: Three numerical chains

50 × 10 =	10 × 60 =	12 × 50 =
25 × 20 =	20 × 30 =	24 × 50 =
25 × 4 =	40 × 15 =	50 × 24 =
25 × 24 =	40 × 30 =	25 × 48 =
50 × 12 =	20 × 60 =	50 × 48 =

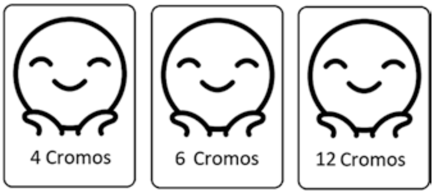


Fig. 10.10 The stickers packs task

The context of the tasks should also challenge students to analyze possibilities, find patterns, ask questions, or compare different forms of reasoning [characteristic (iii)]. Therefore, the tasks should “refer to” situations that students know or can imagine. They should also allow analysis from different points of view. For example, in the stickers packs task (Fig. 10.10), students could discuss several purchase possibilities for Raquel and decide which one would be the most inexpensive.

When designing and implementing a learning trajectory, tasks that focus on appropriation of certain facts and numerical relations should be included—usually called practice exercises. We highlight the ones we call numerical chains, as described by Fosnot and Dolk (2001). They aim to develop students’ mental calculation using important properties and relations of multiplication. Considering the specific characteristics of numerical chains (and according to the authors above), when exploring them, teachers should maintain a lively pace (not spending more than 15 minutes on them), and should favor oral skills (instead of written records).

The sequence shown in Fig. 10.9 includes three chains aiming to highlight powerful strategies of mental calculation based on application of the associative property of multiplication in the particular case of relations with doubles and halves, and the distributive property of multiplication in relation to addition, using reference numbers. Each chain should be explored on different days, since the aim is to focus on relations one at a time.

The exploration of a numerical chain has particular characteristics that we illustrate with the case of a teacher, Isabel, when she was working on the second chain in task 4 (Fig. 10.9). Isabel wrote an expression on the board and gave some time for students to think about it. She first wrote "10 × 60" on the board. Only after several students had raised their hand, stating that they already knew what the result of 10 × 60 was, did she ask one of them to give his or her answer. After analyzing it, Isabel moved on to the next numerical chain, writing "20 × 30" on the board. The following dialogue shows how Isabel explored the various students' answers for the 10 × 60 calculation.

Leandra: "It is 10 × 60 or 60 × 10; it is 600."
Isabel (writing "20 × 30" on the board): "And now?"
Duarte: "20 × 30 is 600 because it is 20 × 10 × 3. And 20 × 10 is 200, and × 3 is 600."
Bernardo: "And it can also be 10 × 30 times 10 × 30, which is 300 plus 300."
Raquel: "It is 600 because it is equal to the last one! 40 is the double of 20 and 15 is half of 30."
Gustavo: "We can also do 40 × 10 plus 40 × 5. It is 400 + 200, which is 600."
(Isabel wrote "20 × 60" and a lot of arms are raised). Isabel - "And now?"
Guilherme: "It is 1200 because 60 × 10 is 600 and plus 60 × 10 is 600, so it is 1200."
David: "I thought of 20 × 30 two times."
José: "It is 1200 because it is the same as 40 × 30."
Duarte: "We also can do 60 × 2 and then × 10."

According to the objectives of the chain and the way the students reacted, the teacher would decide the level of freedom for justifications for different procedures to be analyzed and which processes to highlight. In the previous episode, Isabel chose not to ask for a justification for the 10 × 60 result since it was an answer most students already knew by heart. Regarding other calculations, she gave them opportunities to explain their ways of thinking that revealed application of different properties of multiplication.

10.6 Enacting the Tasks: Planning and Exploring

After selecting each task, teachers still must consider two very important moments: planning how to organize the class and putting such planning into action by exploring the task in the classroom (Stein, Engle, Smith, and Hughes, 2008). These two moments should be oriented by the learning trajectory, taking a global and flexible route to be followed according to the learning purposes and the students' reactions.

At these two moments, teachers' attention should be focused on the students. Keeping the learning trajectory always in mind, teachers should be able to plan and explore from what students can understand, do, and ask.

10.6.1 Planning the Tasks' Enactment

This is a stage in teachers' work that can involve different aspects. We consider that the aspects involving the preparation of the tasks' enactment in the classroom—bearing in mind what students will be able to do and the doubts they might have—are particularly relevant. Therefore, we give great importance to anticipation of their strategies and difficulties.

Anticipating the strategies associated with a task involves in-depth knowledge of their potential and mainly thorough knowledge of the way students think. Teachers have to put themselves in the place of their students and foresee the ways they find solutions at different levels of sophistication, according to different levels of learning and distinct ways of reasoning. This anticipation will help teachers recognize and understand the strategies used in the classroom and understand which ones are related to their teaching objectives, i.e., the mathematical ideas they intend students to learn (Stein et al., 2008).

By anticipating students' strategies, teachers will be able to identify their difficulties in the classroom according to the solutions that are found and understand why these exist. Thus, it will be easier to find a way to help students to overcome such difficulties. At the same time as they foresee students' strategies, teachers should also anticipate possible difficulties linked to the interpretation of the task itself.

We will now show the strategies students could use in task 2 of the sequence illustrated in Fig. 10.7. In parallel with this anticipation, we will also identify some difficulties that students may have in each strategy.

We will organize the possible ways to find solutions into three categories: (i) ones based on additive reasoning, (ii) ones that use multiplicative reasoning and that take into account the context of the task, and (iii) ones that use multiplicative reasoning but do not take into account the context of the task. For each of these categories, we sequentially list the strategies from the least to the most sophisticated.

(i) *Strategies based on additive reasoning*: Task 2 is included in sequence 4, so it is expected that students will use strategies underlying the properties of multiplication. However, some students can still use additive strategies like the ones listed in Table 10.2.

(ii) *Strategies using multiplicative reasoning and taking into account the context of the task*: The context "stacks of boxes" favors use of strategies that take advantage of the properties and multiplicative relations, like the ones listed in Table 10.3.

(iii) *Strategies using multiplicative reasoning but not taking into account the context of the task*: Students may use multiplicative strategies and not be influenced by the way the boxes are organized. Still, some expected strategies take advantage of the previous task (see task 1 in Fig. 10.6) by establishing numerical relations between them (Table 10.4).

Table 10.2 Additive strategies and expected difficulties

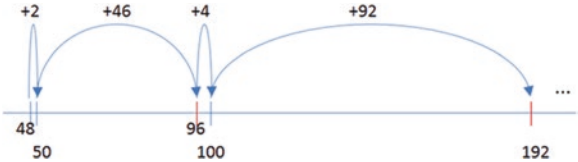
Expected strategies	Possible difficulties
Thinking of 25 boxes, each one with 48 apples, and using additive procedures: Horizontally representing addition, by doing $48 + 48 + \ldots + 48$ (25 addends) Repeatedly adding 48, by doing $48 + 48 = 96$, $96 + 48 = 144, \ldots$ Adding addends 2 by 2, by doing $48 + 48 = 96$, $96 + 96 = 192$, $192 + 192 = 384, \ldots$	In doing calculations with large numbers correctly, taking into account the number of addends (25) and the number we need to add (48)
Thinking of 25 boxes, each with 48 apples, and using the numerical line to sum them  	In doing calculations with large numbers correctly, it is easy to lose count of the number of times 48 is added
Adding the number of boxes to the number of apples in each box, by doing $25 + 48$ (an incorrect strategy)	Interpreting and understanding the task

Table 10.3 Multiplication strategies taking into account the context and expected difficulties

Expected strategies	Possible difficulties
Observing how the boxes are stacked and from there calculating by rows using multiplication, thinking: 2 rows, each with 6 boxes, is 12 boxes 2 rows, each with 5 boxes, is 10 boxes 1 row with 3 boxes $12 \times 48 + 10 \times 48 + 3 \times 48$ Or thinking row by row: $6 \times 48 + 6 \times 48 + 5 \times 48 + 5 \times 48 + 3 \times 48$	Calculating the product 12×48 Forgetting to add some partial products
Observing how the boxes are stacked and from there calculating by columns using multiplication, thinking: 2 columns, each with 4 boxes, is 8 boxes 3 columns, each with 5 boxes, is 15 boxes 1 column with 2 boxes $8 \times 48 + 15 \times 48 + 2 \times 48$ Or thinking column by column: $4 \times 48 + 4 \times 48 + 5 \times 48 + 5 \times 48 + 5 \times 48 + 2 \times 48$	Calculating the product 15×48 Forgetting to add some partial products
Observing how the boxes are stacked, mentally understanding that such an arrangement is the same as having a rectangular layout with 5 columns and 5 rows of boxes, then calculating by columns or by rows using multiplication, thinking: $5 \times 5 \times 48$ or 25×48 First calculating 5 x 48 and then multiplying by 5, which is the same as $5 \times (5 \times 48)$ or $(5 \times 5) \times 48$ Calculating 25×48, by doing 20×48 plus 5×48	Doing the calculations linking factors to multiples of 10

Table 10.4 Multiplication strategies not taking into account the context and the expected difficulties

Expected solutions found by students	Possible difficulties
Identifying the situation as being multiplicative and the numerical values to use, then calculating using the decimal decomposition of 48: $25 \times 48 = 25 \times 40 + 25 \times 8$	Doing the calculations linking factors to multiples of 10
Identifying the situation as being multiplicative and the numerical values to use, then calculating using the decimal decomposition of 25: $25 \times 48 = 20 \times 48 + 5 \times 48$	Doing the calculations linking factors to multiples of 10
Relating this task to the previous one and thinking that the number of boxes is the same, but now each box has 48 apples; i.e., it has double the number of apples that were in the boxes in the previous task. If the total of apples was 600 before, now it is doubled: $2 \times 600 = 1200$	
Relating this task to the previous one and thinking that the number of boxes is the same, but now each box has 48 apples; instead of immediately thinking of doubling it, duplicating the number of apples in each box, by doing: $25 \times 48 = 25 \times (2 \times 24)$ $25 \times (2 \times 24) = 2 \times (25 \times 24)$—i.e., $2 \times 600 = 1200$	
Identifying the situation as being multiplicative and the numerical values to use, then using doubles and halves relations: $25 \times 48 = 50 \times 24$ because 50 is the double of 25 and 24 is half of 48 $50 \times 24 = 100 \times 12$ because 100 is the double of 50 and 12 is half of 24 $100 \times 12 = 1200$ because I know how to multiply by 100	Doing the calculations linked to doubles and halves
Identifying the situation as being multiplicative and the numerical values to use, then using reference numbers (in this case, the number 50, close to 48) and compensating: $25 \times 48 = 25 \times 50 - 25 \times 2$	Forgetting that the compensation implies subtracting 25×2 and not just 2

Foreseeing the strategies students will use to find solutions for a certain task is very demanding and difficult for teachers. However, as this practice progresses, anticipation of different solutions becomes easier since the level of knowledge of the way students think about multiplicative reasoning is increasingly deeper.

Besides improving the knowledge of expected solutions when using this practice, it is also fundamental that teachers be able to list and discuss with other teachers the possible solutions to a given task. Possibly the task has already been explored in previous years, so it would be interesting to see the solutions found by those students, and to interpret and understand them, thus increasing the knowledge of the way students reason and what different representations they use to explain it.

Although teachers try to list, as thoroughly as possible, the expected solutions students may use, it is possible that unexpected strategies emerge in the classroom. Still, the fact that teachers have thought about different task solutions in advance may later prove useful for recognizing and understanding the ones that have not been thought about before in the classroom.

10.7 Exploring and Discussing Tasks

All work carried out in class has to consider two fundamental aspects. The first one is related to the teacher's purpose for exploring a given task, considering the mathematical ideas they expect students to develop and without losing sight of the task's objectives and the learning trajectory set. The second aspect, directly related to predicting the strategies to be used by students, is how teachers manage the interactions between them, ensuring that "bridges" are built between strategies with different levels of sophistication. In this way, it is possible for students who use less powerful strategies to be able to understand the more efficient strategies of their colleagues, which will allow them to progress in their learning.

The two aspects identified above prove that the teacher's action in the classroom is strongly supported by the preparation that has been done beforehand regarding selection of tasks and prediction of the strategies that students may use.

In the classroom, after a brief presentation of the selected task, students start to solve it individually or in pairs. At that moment, the teacher's role is to monitor the students' work, which is facilitated by the preparation made in anticipating the students' strategies. So, teachers initially have to have an idea if students understand the task and interpret it correctly. From there, each one works at his or her own level of knowledge.

As students develop their work, and facing the different strategies that come out, teachers should be able to relate such strategies to the ones they have anticipated. The way students represent and explain their reasoning is not always clearly noticeable, even when teachers have foreseen a strategy based on similar reasoning. To facilitate their action at this exploration stage, it is important to ask themselves questions such as:

- "Do most students understand the problem? Are there any difficulties?"
- "Are the strategies used in line with the ones I anticipated?"
- "Are there any strategies I did not foresee?"

In this particular case, when monitoring the students' work, the teacher Isabel realized that they were not using additive strategies. In fact, although these strategies were expected due to the previous experience of the students in other tasks covered by the multiplication trajectory, they only used procedures whose underlying reasoning was a multiplicative one. Based on the foreseen strategies, Isabel identified that a pair of students had suggested a way of solving the task that she had not thought of beforehand.

While monitoring the students' work, teachers begin to prepare the collective discussion. They ask themselves questions about the objectives that have been set, and they identify the potential of the strategies that are used in order to select those that should be presented and discussed with the whole class. They ask themselves questions such as:

- "Considering the purpose of the task I have chosen and the strategies I have anticipated, which solutions will be presented and discussed with all students?"
- "In what order will they be presented and discussed?"

Isabel's aim was that students use the array model associated with the context and relate it to the properties of multiplication. So, looking at the students' strategies, she chose two of them in line with the ideas she intended to highlight. The choice she made was facilitated by the work she had done in advance regarding the strategies, since this allowed her to make a quick decision in class and in accordance with her intentions. It is interesting to note that one of the chosen strategies had not been initially anticipated by Isabel, to her surprise. However, when questioning the students about the way they were thinking, she decided this solution was worth sharing with all class.

The two solutions chosen by Isabel were from the pair Eva and Guilherme (Fig. 10.11) and from the pair Duarte and Tiago (Fig. 10.12).

While Eva and Guilherme showed a sketch to support their reasoning, Duarte and Tiago did not explicitly show something that could ground the reasoning they made.

In order to decide the order of the presentations and their discussions, Isabel used the criterion of progressive presentation of the strategies from less to more abstract. Therefore, Eva and Guilherme were the first pair to do their presentation, followed by Duarte and Tiago. The pairs were supported by the A3 sheet of paper on which they had solved the task, which was put up on the blackboard.

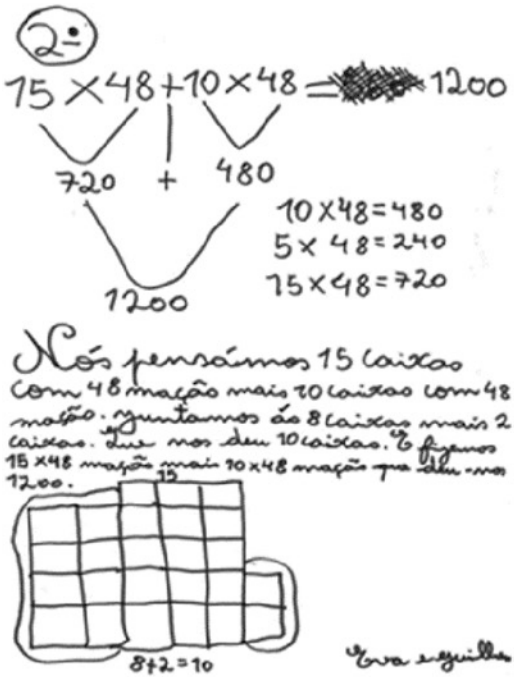


Fig. 10.11 The solution found by Eva and Guilherme

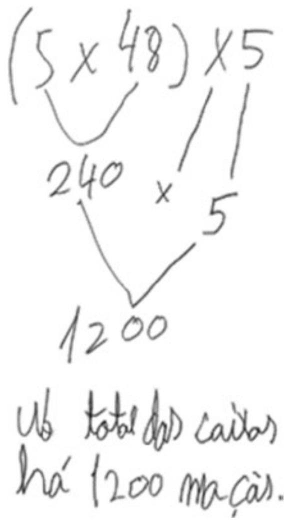


Fig. 10.12 The solution found by Duarte and Tiago

After choosing which students' solutions will support the discussions with the whole class, teachers have a decisive role in the key moments that follow. Indeed, the moment when the teacher guides the discussion with the whole class—facilitating the interactions between the students—is fundamental in the whole process. This is when the ideas associated with the learning trajectory set are pointed out, and "bridges" should be built at several levels: between different solutions with more or less sophisticated levels of reasoning; between solutions, ideas, and mathematical concepts; and also between the solutions that have been found and the purposes of the class. Teachers can guide their actions by asking themselves questions such as:

- "How should I guide the presentations and the sharing of the solutions that have been found, so as to facilitate the interactions between students?"
- "How should I manage the collective discussions in order to build 'bridges' between different solutions—some more informal and others more powerful?"
- "How should I guide the discussion so that students at lower levels of learning may evolve?"
- "How should I manage the collective discussion so that all students may learn in light of the class objectives?"

Isabel chose to alternate the presentations by the selected pairs with discussions with the whole class. She started by asking Eva and Guilherme to explain their solution. Eva's oral presentation was very close to the written records made by the pair.

Eva: "We thought of 15 boxes with 48 apples plus 10 boxes with 48 apples. We added to the 8 boxes 2 more boxes," (here she pointed to the sketch they made) "which gave us 10 boxes. And we did 15 × 48 + 10 × 48."

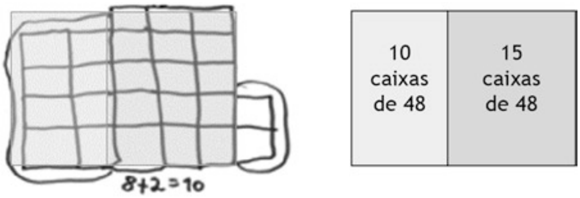


Fig. 10.13 Representation of a stack of 10 boxes plus 15 boxes (*caixas* in Portuguese)

Fig. 10.14 Representation of 10 × 48 + 15 × 48 = 25 × 48

These students took advantage of the rectangular arrangement, transforming the “stack of boxes” into two “rectangles” with 10 and 15 boxes (see Fig. 10.13), and then calculated the corresponding partial products—10 × 48 and 15 × 48—considering that each box had 48 apples.

Isabel stressed that the way these students had used the rectangular layout to calculate the total number of apples by adding the two products 15 × 48 and 10 × 48 was the same as calculating 25 × 48.

This relation allowed comparison between the strategy used by Eva and Guilherme and those used by other students who determined the total number of apples by calculating the product of 25 × 48 (see Fig. 10.14). Supported by a sketch, students could understand that calculating the number of apples in 10 boxes plus the number of apples in 15 boxes is the same as calculating, all at once, the number of apples in 25 boxes.

Confronting strategies enables the teacher to point out mathematical ideas that are key to multiplication learning. The fact that 10 × 48 + 15 × 48 and 25 × 48 are equal illustrates the distributive property of multiplication in relation to addition.

Encouraging students to orally explain their way of thinking, along with their written records, may help other colleagues with different levels of understanding about multiplication to progress in terms of ideas and relations that can be established.

For example, the part of the solution from Eva and Guilherme that is shown in Fig. 10.11 could also help support a collective discussion, where it is pointed out that the calculations made were based on numerical relations.

Using the knowledge of multiples of 10, 10 × 48 is mentally calculated first. Considering that 5 is half of 10, the corresponding product is also half of the prior product. Lastly, underlying the distributive property of multiplication in relation to addition, 15 × 48 is calculated by adding the previous partial products (see Fig. 10.15).

$$10 \times 48 = 480$$
$$5 \times 48 = 240$$
$$15 \times 48 = 720$$

Fig. 10.15 Part of the solution found by Eva and Guilherme

We will now see how Isabel guided the collective discussion about the presentation by the other selected pair, Duarte and Tiago. The level of abstraction of their strategy, which was noticeable in their written records and in the way they explained it, initially led their classmates to ask for clarifications. The episode transcribed below shows how difficult it was for their classmates to understand this solution and how Isabel guided the pair in order for them to explain it in other words, considering that the first attempt had not been successful.

Gustavo: "I don't understand! Can you explain it better?"

Isabel: "Can one of you two try to explain it in another way so your classmates can understand it?"

Duarte: "We took these two boxes," (he pointed to the two last boxes on the right of the figure) "and we put them over here," (he pointed to the upper layer on the left) "and they disappeared from here," (he pointed to the two last boxes on the right) "then we did 5 × 48 because they were the boxes in one column. Since there were 5 columns, we then did times 5." (He wrote on the blackboard "(5 × 48) × 5)".)

Unlike the previous pair, these students did not draw a sketch to support the visualization of the transformation of the box stack into a rectangle; they only did it mentally. Then they calculated the number of apples by column, by doing 5 × 48. As they identified 5 columns, they then calculated 5 times the number of apples in each column. However, because they wrote their calculations as they were reasoning, they put factor 5 corresponding to 5 columns on the right since they wrote sequentially from left to right.

In case there were still students who did not understand this way of representing and thinking, it was important to clarify the expression "(5 × 48) × 5." The intermediate calculation of 5 × 48 allowed its translation according to the context of the task. Considering a rectangle with 5 columns (and 5 rows), the teacher might ask students for the meaning of 240, i.e., the number of apples in each column of boxes. From there, the meaning of 5 × 240 might be quickly associated with the total number of apples since there were 5 columns, each one with 240 apples. The relation between 5 × 240 and 240 × 5 (the expression used by Duarte and Tiago), which was not supported by the context itself, can be understood if we take into account these students' previous experiences. In mathematical terms, the equality between 5 × 240 and 240 × 5 is justified by the commutative property of multiplication, which the students already knew about—namely, when they did calculations associated with multiplication tables.

Still focusing on this solution, Isabel encouraged the class to ask for clarifications from Duarte and Tiago:

Enzo: "I would like to know how Duarte and Tiago did 240 × 5 so quickly."

Duarte (answering, thinking of 5 × 240): "We know that 5 × 4 is 20, so 5 × 40 is 200. And as we know that 5 × 2 is 10, we know that 5 × 200 is 1000. That's why we wrote 1200."

Duarte's explanation, besides underlying the distributive property of multiplication in relation to addition, is related to another fundamental idea of multiplication learning: the use of multiples of 10. By stimulating questions about powerful strategies of calculation and its corresponding explanation, Isabel promoted the students' development in terms of their level of learning multiplication.

Isabel's action led to possible answers to the questions that could guide the collective discussions mentioned above. Regarding the presentation and sharing of the selected solutions, the teacher organized two moments associated with each presentation. After the presentation by the first pair, she generalized the discussion with the whole class, giving an opportunity for students to contribute to it. The teacher highlighted aspects she considered relevant to the targeted solution. After the presentation by the second pair, Isabel organized a second collective discussion, which was another important moment of interaction and in which she had also a key role.

Regarding "bridge" building (Stein et al., 2008), Isabel picked up Eva and Guilherme's presentation and related it to other students' solutions, highlighting the equality between the two expressions that were the basis for each group to initiate the calculation. In the case of Duarte and Tiago, she encouraged them to clarify their solution and explain the way they thought it was associated with the rectangular arrangement. She was trying to understand if the other students understood it and, when doubts remained, she guided the discussion so the context could be used to facilitate the explanation. She also used the students' previous experience with the commutative property.

To allow students at lower learning levels to evolve, Isabel requested Eva and Guilherme to explain orally how they did some of their calculations, where powerful numerical relations associated with properties of multiplication were evident. She also encouraged Duarte and Tiago to clarify, when asked by a classmate, how they "quickly" did a certain calculation, highlighting important relations associated with multiples of 10.

The objectives of the task—to use the array model associated with the context and relate it to properties of multiplication—were highlighted throughout the discussion. Therefore, the teacher selected some solutions and, using the students' voices, related the strategies to the array model. From there, important multiplication ideas related to its properties emerged. These were explained by the students or highlighted by the teacher.

10.8 Implementing Learning Trajectories and Lesson Study: Perspectives on the Teacher's Role

By building a multiplication learning trajectory, we have sought to exemplify central elements of the teachers' action. Next, we will show some convergent aspects between the approach introduced here and the lesson study approach (Isoda and Olfos, 2009) to the teacher's role, and we will discuss the possible contributions of such contexts to teachers' professional development.

10.8.1 The Teacher's Role

One central aspect in both approaches is the importance given to careful lesson planning. Within the scope of lesson study, Isoda and Olfos (2009) set six challenges associated with lesson planning, where we can see some similarities with the learning multiplication trajectory developed by us: (i) description of the mathematical situations in context to be addressed in the lesson, (ii) characterization of the different tasks assigned to students and to teachers at different moments of the lesson, (iii) time limits and organization of the different moments of the lesson, (iv) anticipation of the students' behaviors and products, (v) preparation of possible interventions by the teacher to guide the class toward the proposed goal, and (vi) selection and preparation of the materials and means for the lesson. Next, we will explain how each of these challenges set by Isoda and Olfos (2009) are similar to the perspectives guiding our work.

In connection with the first challenge, and in the case of the set learning trajectory, both the sequences of the mathematical tasks and each task itself are carefully thought through. For each task, there is a clear description of its objectives, with identification of the learning milestones and models involved in the solution of a problem in context. Their preparation follows the criterion of the interconnection between tasks, considering, for example, the numbers involved from task to task and the contexts promoting the use of certain models or strategies. The contexts of the tasks are also important, since they should be a challenge for the students and lead them to want to explore them.

Teachers bring well-planned and previously explored tasks to the classroom. We also see these aspects reflected in a similar way in lesson study—namely, in challenges (i) and (vi). As shown earlier in this chapter, planning the learning trajectory involves, among other aspects, anticipation of the strategies used by students to find solutions. This aspect is also included in the characteristics referred to in challenge (iv), as well as the possible difficulties the students may face, according to their different levels of mathematical development during the trajectory in question.

Another similarity between the two approaches is related to the nature of the mathematical tasks proposed. In both cases, the selection of suitable problems and how they are explored in the classroom demands a very particular focus from the teacher. In fact, according to Isoda and Olfos (2009), by solving good problems, the students may gain new knowledge by applying previously learned knowledge. This characteristic of progressing in knowledge by solving problems is also included in the set trajectory, since the aim is to integrate, in each new task, knowledge and strategies developed in previous tasks. Therefore, in both approaches, problem solving is seen not simply as application of knowledge but as an opportunity to generate new knowledge.

However, for this to happen, we have to consider the way the problem is explored in class and how teachers and students' activities are organized (challenges (iii) and (v)). Both approaches have a social dimension, since, for each task, time is reserved for presenting and discussing the students' solutions. These moments are seen as opportunities to deepen the students' learning, favoring the understanding of concepts. Thus, in both approaches there is clearly a time limitation and organization of the different moments of the lesson (challenge (iii)) and characterization of the different tasks assigned to students and to teachers at different moments of the lesson (challenge (ii)), particularly at the moment of the discussion of students' exploration of the task.

This crucial moment in the lesson has to be prepared beforehand. Considering the task's objectives and the mathematical ideas that students develop in finding a solution, the teacher will select and order the solutions that will be presented in front of the class. So, as Isoda and Olfos (2009) state, the teacher has to "study the students' possible answers beforehand in order to ensure a flow and a progression pace and avoiding inactivity" (p. 166). As shown in the previous section, these options also aim to promote communication between students presenting strategies with different levels of mathematical sophistication, favoring not only progression of those still at less developed stages but also learning improvement of the other students. In fact, by becoming aware of the various possible solution processes and by reflecting on them under the teacher's guidance, the students can develop a deeper understanding of the mathematical knowledge involved.

In this respect, we stress again the importance of the teacher's role, since teachers have to lead students to make connections among the various solutions found in class and to highlight the most powerful representations and the mathematical ideas underlying the strategies presented. In one of the examples shown above, we saw clearly how the teacher could relate a less sophisticated solution to one of the most powerful ideas associated with the topic: the distributive property of multiplication in relation to addition. This is an aspect also pointed out by Isoda and Olfos (2009) regarding the teacher's role at this stage of the lesson: "The main task of the teacher is to listen to students, understand their point of view, connect it to the class objective, and guide the next moments" (p. 165). Thus, when assuming this role, teachers focus on listening and understanding students' reasoning, building "bridges" with the learning milestones of the ongoing trajectory.

10.8.2 Opportunities for Professional Development

The implementation of the learning trajectory presented here occurred in a collaborative context between one primary classroom teacher and one researcher. We could imagine a similar scenario in a curricular development project, a research project, or a teacher training program where, despite the different roles, collaborative work could be developed (Goodchild, 2014). We also find here some overlapping points with lesson study, which (according to Isoda, Arcavi and Mena (2007)) usually includes a cycle with the following steps: planning, the research lesson, and the reviewing lessons. These can then be repeated in two or more implementation cycles with other teachers. As pointed out by these authors, all these processes occur in collaboration with other teachers, higher education teacher educators, and, possibly, supervisors from local educational authorities.

In the case presented here, the teacher and the researcher undertook a very meaningful and extended process to adapt and constantly improve the learning trajectory. They met every week to reflect on each class and to plan the next ones. The researcher attended the classes and followed up on the students' work, also contributing to the teacher's decision making in the lessons—namely, the decisions related to the collective discussion moments. The teacher is a professional who was always willing to learn and reflect on topics she considered could improve her performance and her students' learning quality, seeing this experience as an important opportunity for professional development.

Still, we have to consider that the workload involved in the preparation and implementation of a learning trajectory like the one shown here is huge. It is an ambitious project that needs to be accomplished with the support of one or more experts. While recognizing that this work cannot be developed by the teacher alone, teachers may adapt and put this idea into practice under certain conditions; namely, they can collaboratively develop learning trajectories that are more limited in time, involving the preparation of fewer tasks or tasks already tested by themselves or others, which will allow acquisition of knowledge of students' strategies and difficulties.

The set of ideas for teaching multiplication developed in this chapter can be adapted to particular contexts and to the specific curricular guidelines of each country and each grade. Furthermore, we consider that the materials presented here could be used in initial and in-service teacher education. These can allow future teachers to get to know several students' strategies and reasoning, helping them to understand all of their potential. Such materials can also lead teachers to question and discuss the learning of multiplication, thus becoming a starting point for reflection about their practice and for motivating themselves to discuss and improve it.

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