Chapter 18
Recurrence and Transience of Continuous-Time Open Quantum Walks

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Abstract This paper is devoted to the study of continuous-time processes known as continuous-time open quantum walks (CTOQW). A CTOQW represents the evolution of a quantum particle constrained to move on a discrete graph, but which also has internal degrees of freedom modeled by a state (in the quantum mechanical sense). CTOQW contain as a special case continuous-time Markov chains on graphs. Recurrence and transience of a vertex are an important notion in the study of Markov chains, and it is known that all vertices must be of the same nature if the Markov chain is irreducible. In the present paper we address the corresponding result in the context of irreducible CTOQW. Because of the "quantum" internal degrees of freedom, CTOQW exhibit non standard behavior, and the classification of recurrence and transience properties obeys a "trichotomy" rather than the classical dichotomy. Essential tools in this paper are the so-called "quantum trajectories" which are jump stochastic differential equations which can be associated with CTOQW.

18.1 Introduction

Open quantum walks (OQW) have been developed originally in [1, 2]. They are natural quantum extensions of classical Markov chains and, in particular, any classical discrete-time Markov chain on a finite or countable set can be obtained as a particular case of OQW. Roughly speaking, OQW are random walks on a

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graph where, at each step, the walker jumps to the next position following a law which depends on an internal degree of freedom, the latter describing a quantum-mechanical state. From a physical point of view, OQW are simple models offering different possibilities of applications (see [28, 29]). From a mathematical point of view, their properties can been studied in analogy with those of classical Markov chain. In particular, usual notions such as irreducibility, period, ergodicity, have been investigated in [3, 8–10, 20]. For example, the notions of transience and recurrence have been studied in [5], proper definitions of these notions have been developed in this context and the analogues of transient or recurrent points have been characterized. An interesting feature is that the internal degrees of freedom introduce a source of memory which gives rise to a specific non-Markovian behavior. Recall that, in the classical context (see [22]), an exact dichotomy exists for irreducible Markov chains: a point is either recurrent or transient, and the nature of a point can be characterized in terms of first return time, or in terms of number of visits. In contrast, irreducible open quantum walks exhibit three possibilities regarding the behavior of return time and number of visits. In this article, we study the recurrence and transience, as well as their characterizations, for continuous-time versions of OQW.

In the same way that open quantum walks are quantum extensions of discrete-time Markov chains, there exist natural quantum extensions of continuous-time Markov processes. One can point to two different types of continuous-time evolutions with a structure akin to open quantum walks. The first (see [6]) is a natural extension of classical Brownian motion and is called open quantum Brownian motion; it is obtained by considering OQW in the limit where both time and space are properly rescaled to continuous variables. The other type of such evolution (see [25]) is an analogue of continuous-time Markov chains on a graph, is obtained by rescaling time only, and is called continuous-time open quantum walks (CTOQW). In this article we shall concentrate on the latter.

Roughly speaking CTOQW represents a continuous-time evolution on a graph where a "walker" jumps from node to node at random times. The intensity of jumps depends on the internal degrees of freedom; the latter are modified by the jump, but also evolve continuously between jumps. In both cases the form of the intensity, as well as the evolution of the internal degrees of freedom at jump times and between them, can be justified from a quantum mechanical model.

As is well-known, in order to study a continuous-time Markov chain, it is sufficient to study the value of the process at the jump times. Indeed, the time before a jump depends exclusively on the location of the walker, and the destination of the jump is independent of that time. As a consequence, the process restricted to the sequence of jump times is a discrete-time Markov chain, and all the properties of that discrete-time Markov chain such as irreducibility, period, transience, recurrence, are transferred to the continuous-time process. This is not the case for OQW. In particular, a CTOQW restricted to its jump times is not a (discrete-time) open quantum walk. Therefore, the present study of recurrence and transience cannot be directly derived from the results in [5]. Nevertheless, we can still adopt a similar approach and, for instance, we study irreducibility of CTOQW in connection to that of quantum dynamical systems as in [11]. Note that general notions of recurrence and transience are developed in [17] for general quantum Markov semigroups with unbounded generators. The work elaborated in [17] is based on potential theory and we explicit the connection between the notions of recurrence and transience of CTOQW and those in [17]. Finally, as in the discrete case, we obtain a trichotomy, in the sense that irreducible CTOQW can be classified into three different classes, depending on the properties of the associated return time and number of visits.

The paper is structured as follows: in Sect. 18.2, we recall the definition of continuous-time open quantum walks and in particular introduce useful classical processes attached to CTOQW; Sect. 18.3 is devoted to the notion of irreducibility for CTOQW; in Sect. 18.4, we address the question of recurrence and transience and give the classification of CTOQW mentioned above.

18.2 Continuous Time Open Quantum Walks and Their Associated Classical Processes

This section is devoted to the introduction of continuous-time open quantum walks (CTOQW). In Sect. 18.2.1, we introduce CTOQW as a special instance of quantum Markov semigroups (QMS) with generators preserving a certain block structure. Section 18.2.2 is devoted to the exposition of the Dyson expansion associated with a QMS, which will be a relevant tool in all remaining sections. It also allows us to introduce the relevant probability space. Finally, in Sect. 18.2.3 we associate to this stochastic process a Markov process called *quantum trajectory* which has an additional physical interpretation, and which will be useful in its analysis.

18.2.1 Definition of Continuous-Time Open Quantum Walks

Let V denotes a set of vertices, which may be finite or countably infinite. CTOQW are quantum analogues of continuous-time Markov semigroups acting on the set $\mathrm{L}^\infty(V)$ of bounded functions on V. They are associated with stochastic processes evolving in the composite system

$$\mathcal{H} = \bigoplus_{i \in V} \mathfrak{h}_i \,, \tag{18.1}$$

where the $\mathfrak{h}_i$ are separable Hilbert spaces. This decomposition has the following interpretation: the label i in V represents the position of a particle and, when the particle is located at the vertex $i \in V$, its internal state is encoded in the space $\mathfrak{h}_i$ (see below). Thus, in some sense, the space $\mathfrak{h}_i$ describes the internal degrees of freedom of the particle when it is sitting at site $i \in V$. When $\mathfrak{h}_i$ does not depend on i, that is if $\mathfrak{h}_i = \mathfrak{h}$, for all $i \in V$, one has the identification $\mathcal{H} \simeq \mathfrak{h} \otimes \ell^2(V)$ and then it is natural to write $\mathfrak{h}_i = \mathfrak{h} \otimes |i\rangle$ (we use here Dirac's notation where the *ket* $|i\rangle$ represents the i-th vector in the canonical basis of $\ell^\infty(V)$, the *bra* $\langle i|$ represents the associated linear form, and $|i\rangle\langle j|$ represents the linear map $\varphi \mapsto \langle j|\varphi\rangle\,|i\rangle$). We will adopt the notation $\mathfrak{h}_i \otimes |i\rangle$ to denote $\mathfrak{h}_i$ in the general case (i.e. when $\mathfrak{h}_i$ depends on i) to emphasize the position of the particle, using the identification $\mathfrak{h}_i \otimes \mathbb{C} \simeq \mathfrak{h}_i$. We thus write:

$$\mathcal{H} = \bigoplus_{i \in V} \mathfrak{h}_i \otimes |i\rangle \ . \tag{18.2}$$

Last, we denote by $\mathcal{I}_1(\mathcal{K})$ the two-sided ideal of trace-class operators on a given Hilbert space $\mathcal{K}$ and by $\mathcal{S}_{\mathcal{K}}$ the space of density matrices on $\mathcal{K}$, defined by:

$$\mathcal{S}_{\mathcal{K}} = \{\rho \in \mathcal{I}_1(\mathcal{K}) \mid \rho^* = \rho, \rho \geq 0, \mathrm{Tr}(\rho) = 1\}.$$

A *faithful* density matrix is an invertible element of $\mathcal{S}_{\mathcal{K}}$, which is therefore a trace-class and positive-definite operator. Following quantum mechanical fashion, we will use the word "state" interchangeably with "density matrix".

We recall that a quantum Markov semigroup (QMS) on $\mathcal{I}_1(\mathcal{K})$ is a semigroup $\mathcal{T} := (\mathcal{T}_t)_{t\geq 0}$ of completely positive maps on $\mathcal{I}_1(\mathcal{K})$ that preserve the trace. The QMS is said to be uniformly continuous if $\lim_{t\to 0}\|\mathcal{T}_t - \mathrm{Id}\| = 0$ for the operator norm on $\mathcal{B}(\mathcal{K})$. It is then known (see [21]) that the semigroup $(\mathcal{T}_t)_{t\geq 0}$ has a generator $\mathcal{L} = \lim_{t\to\infty}(\mathcal{T}_t - \mathrm{Id})/t$ which is a bounded operator on $\mathcal{I}_1(\mathcal{K})$, called the Lindbladian, and Lindblad's theorem characterizes the structure of such generators. One consequently has $\mathcal{T}_t = \mathrm{e}^{t\mathcal{L}}$ for all $t \geq 0$, where the exponential is understood as the limit of the norm-convergent series.

Continuous-time open quantum walks are particular instances of uniformly continuous QMS on $\mathcal{I}_1(\mathcal{H})$, for which the Lindbladian has a specific form. To make this more precise, we define the following set of block-diagonal density matrices of $\mathcal{H}$:

$$\mathcal{D} = \Big\{\mu \in \mathcal{S}(\mathcal{H})\,;\ \mu = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|\Big\} \ .$$

In particular, for $\mu \in \mathcal{D}$ with the above definition, one has $\rho(i) \in \mathcal{I}_1(\mathfrak{h}_i)$, $\rho(i) \geq 0$ and $\sum_{i\in V} \mathrm{Tr}\big(\rho(i)\big) = 1$. In the sequel, we use the usual notations $[X, Y] = XY - YX$ and $\{X, Y\} = XY + YX$, which stand respectively for the commutator and anticommutator of two operators $X, Y \in \mathcal{B}(\mathcal{H})$.

Definition 18.1 Let $\mathcal{H}$ be a Hilbert space that admits a decomposition of the form (18.1). A *continuous-time open quantum walk* is a uniformly continuous quantum Markov semigroup on $\mathcal{I}_1(\mathcal{H})$ such that its Lindbladian $\mathcal{L}$ can be written:

$$\begin{aligned} \mathcal{L} : \mathcal{I}_1(\mathcal{H}) &\to \mathcal{I}_1(\mathcal{H}) \\ \mu \quad &\mapsto -\mathrm{i}[H, \mu] + \sum_{i,j\in V} \mathbb{1}_{i\neq j}\big(S_i^j \mu S_i^{j*} - \frac{1}{2}\{S_i^{j*} S_i^j, \mu\}\big) \ , \end{aligned} \tag{18.3}$$

where H and $(S_i^j)_{i,j}$ are bounded operators on $\mathcal{H}$ that take the following form:

- $H = \sum_{i\in V} H_i \otimes |i\rangle\langle i|$, with H_i bounded self-adjoint operators on $\mathfrak{h}_i$, i in V;
- for every $i \neq j \in V$, S_i^j is a bounded operator on $\mathcal{H}$ with support included in $\mathfrak{h}_i$ and range included in $\mathfrak{h}_j$, and such that the sum $\sum_{i,j\in V} S_i^{j*} S_i^j$ converges in the strong sense. Consistently with our notation, we can write $S_i^j = R_i^j \otimes |j\rangle\langle i|$ for bounded operators $R_i^j \in \mathcal{B}(\mathfrak{h}_i, \mathfrak{h}_j)$.

We will say that the open quantum walk is *semifinite* if $\dim \mathfrak{h}_i < \infty$ for all $i \in V$.

From now on we will use the convention that $S_i^i = 0$, $R_i^i = 0$ for any $i \in V$. As one can immediately check, the Lindbladian $\mathcal{L}$ of a CTOQW preserves the set $\mathcal{D}$. More precisely, for $\mu = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i| \in \mathcal{D}$, we have $\mathcal{T}_t(\mu) =: \sum_{i\in V} \rho_t(i) \otimes |i\rangle\langle i|$ for all $t \geq 0$, with

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t(i) = -\mathrm{i}[H_i, \rho_t(i)] + \sum_{j\in V} \left(R_j^i \rho_t(j) R_j^{i*} - \frac{1}{2}\{R_i^{j*} R_i^j, \rho_t(i)\} \right) .$$

18.2.2 Dyson Expansion and Associated Probability Space

In this article, our main focus is on a stochastic process $(X_t)_{t\geq 0}$ that informally represents the position of a particle or walker constrained to move on V. In order to rigorously define this process and its associated probability space, we need to introduce the *Dyson expansion* associated with a CTOQW. In particular, this allows to define a probability space on the possible trajectories of the walker. We will recall the result for general QMS as we will use it in the next section. The application to CTOQW is described shortly afterwards.

Let $(\mathcal{T}_t)_{t\geq 0}$ be a uniformly continuous QMS with Lindbladian $\mathcal{L}$ on $\mathcal{I}_1(\mathcal{K})$ for some separable Hilbert space $\mathcal{K}$. By virtue of Lindblad's Theorem [21], there exists a bounded self-adjoint operator $H \in \mathcal{B}(\mathcal{K})$ and bounded operators L_i on $\mathcal{K}$ $(i \in I)$ such that for all $\mu \in \mathcal{I}_1(\mathcal{K})$,

$$\mathcal{L}(\mu) = -\mathrm{i}[H, \mu] + \sum_{i\in I}\left(L_i \mu L_i^* - \frac{1}{2}\{L_i L_i^*, \mu\}\right),$$

where I is a finite or countable set and where the series is strongly convergent. The first step is to give an alternative form for the Lindbladian. First introduce

$$G := -\mathrm{i}H - \frac{1}{2}\sum_{i\in I} L_i^* L_i \ ,$$

so that for any $\mu \in \mathcal{D}$,

$$\mathcal{L}(\mu) = G\mu + \mu G^* + \sum_{i\in I} L_i \, \mu \, L_i^* \, . \tag{18.4}$$

Remark that $G + G^* + \sum_{i\in I} L_i^* L_i = 0$ (the form described in (18.4) is actually the general form of the generator of a QMS given by Lindblad [21]). The operator $-(G+G^*)$ is positive semidefinite and $t \mapsto \mathrm{e}^{tG}$ defines a one-parameter semigroup of contractions on $\mathcal{K}$ by a trivial application of the Lumer–Phillips theorem (see e.g. Corollary 3.17 in [14]). We are now ready to give the Dyson expansion of the QMS.

Proposition 18.1 *Let $(\mathcal{T}_t)_{t\geq 0}$ be a QMS with Lindbladian $\mathcal{L}$ as given above. For any initial density matrix $\mu \in \mathcal{S}_{\mathcal{K}}$, one has*

$$\mathcal{T}_t(\mu) = \sum_{n=0}^{\infty} \sum_{i_1,\ldots,i_n \in I} \int_{0<t_1<\cdots<t_n<t} \zeta_t(\xi)\, \mu\, \zeta_t(\xi)^* \, \mathrm{d}t_1 \cdots \mathrm{d}t_n \, , \tag{18.5}$$

where $\zeta_t(\xi) = \mathrm{e}^{(t-t_n)\,G} L_{i_n} \cdots L_{i_1} \mathrm{e}^{t_1\,G}$ for $\xi = (i_1, \ldots, i_n; t_1, \ldots, t_n)$.

We now turn to applying this to CTOQW. Due to the block decomposition of H and of the S^i_j, one can write $G = \sum_{i\in V} G_i \otimes |i\rangle\langle i|$, where (recall that $R_i^i = 0$)

$$G_i = -\mathrm{i}H_i - \frac{1}{2} \sum_j R_i^{j*} R_i^j \, , \tag{18.6}$$

so that $G_i + G_i^* = -\sum_j R_i^{j*} R_i^j$. From Proposition 18.1 we then get the following expression for the Lindbladian: for all $\mu = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$ in $\mathcal{D}$,

$$\mathcal{L}(\mu) = \sum_{i\in V} \Big(G_i \rho(i) + \rho(i) G_i^* + \sum_j R_j^i \, \rho(j) \, R_j^{i*} \Big) \otimes |i\rangle\langle i| \, . \tag{18.7}$$

Corollary 18.1 *Let $(\mathcal{T}_t)_{t\geq 0}$ be a CTOQW with Lindbladian $\mathcal{L}$ given by (18.7). For any initial density matrix $\mu \in \mathcal{D}$, one has*

$$\mathcal{T}_t(\mu) = \sum_{n=0}^{\infty} \sum_{i_0,\ldots,i_n \in V} \int_{0<t_1<\cdots<t_n<t} T_t(\xi)\, \rho(i_0) T_t(\xi)^* \mathrm{d}t_1 \cdots \mathrm{d}t_n \, \otimes |i_n\rangle\langle i_n| \, , \tag{18.8}$$

where, for $\xi = (i_0, \ldots, i_n; t_1, \ldots, t_n)$ with $i_0, \ldots, i_n \in V^{n+1}$ and $0 < t_1 < \ldots < t_n$,

$$T_t(\xi) := \mathrm{e}^{(t-t_k)G_{i_k}} R_{i_{k-1}}^{i_k} \mathrm{e}^{(t_k-t_{k-1})G_{i_{k-1}}} \cdots \mathrm{e}^{(t_2-t_1)G_{i_1}} R_{i_0}^{i_1} \mathrm{e}^{t_1 G_{i_0}} . \tag{18.9}$$

if k is the largest element such that $t_k \leq t$.

Note the small discrepancy between ξ in (18.5) and ξ in (18.8): ξ contains an additional index i_0, which is due to the decomposition of μ.

Remark 18.1 Equation (18.5) is also called an unravelling of the QMS. It was first introduced in [12, 30], with the heuristic interpretation of giving an expression for $\mathcal{T}_t(\mu)$ as the average, over all possible trajectories $\xi = (i_0, \dots, i_n; t_1, \dots, t_n)$, of the evolution of μ "when it follows the trajectory ξ". We will discuss connections with an operational interpretation of $T_t(\xi)\rho(i_0)T_t(\xi)^*$ in Sect. 18.2.4.

The decomposition described in (18.9) will allow us to give a rigorous definition of the probability space associated with the evolution of the particle on V. The goal is to introduce the probability measure $\mathbb{P}_\mu$ that models the law of the position of the particle, when the initial density matrix is $\mu \in \mathcal{D}$. The following is inspired by [4, 7, 19].

First define the set of all possible trajectories up to time $t \in [0, \infty]$ as $\Xi_t := \bigsqcup_{n \in \mathbb{N}} \Xi_t^{(n)}$, where $\Xi_t^{(n)}$ is the set of trajectories on V up to time t comprising n jumps:

$$\Xi_t^{(n)} := \{\xi = (i_0, \dots, i_n; t_1, \dots, t_n) \in V^{n+1} \times \mathbb{R}^n,\ 0 < t_1 < \cdots < t_n < t\}\ .$$

For $t \in \mathbb{R}_+$, the set $\Xi_t^{(n)}$ is equipped with the σ-algebra $\Sigma_t^{(t)}$ and with the measure $\nu_t^{(n)}$, which is induced by the map

$$I_n : \begin{array}{ccc} \left(V^{n+1} \times [0, t)^n, \mathcal{P}(V^{n+1}) \times \mathcal{B}([0, t)^n), \delta^{n+1} \times \frac{1}{n!}\lambda_n\right) & \to & \left(\Xi_t^{(n)}, \Sigma_t^{(n)}, \nu_t^{(n)}\right) \\ (i_0, \dots, i_n; s_1, \dots, s_n) & \mapsto & (i_0, \dots, i_n; s_{\min}, \dots, s_{\max}) \end{array}\ ,$$

where δ is the counting measure on V, $\mathcal{B}([0, t)^n)$ is the Borel σ-algebra on $[0, t)^n$ and λ_n is the Lebesgue measure on $\mathcal{B}([0, t)^n)$ for all $n \geq 0$. These measures are σ-finite and this allows us to apply Carathéodory's extension Theorem. We first define the σ-algebra $\Sigma_t := \sigma(\Sigma_t^{(t)}, n \in \mathbb{N})$ and the measure ν_t on Ξ_t such that $\nu_t = \nu_t^{(n)}$ on $\Xi_t^{(n)}$. For a given $\mu = \sum_{i \in V} \rho(i) \otimes |i\rangle\langle i|$ in $\mathcal{D}$, one can then define the probability measure $\mathbb{P}_\mu^t$ on (Ξ_t, Σ_t) such that, for all $E \in \Sigma_t$,

$$\begin{aligned} \mathbb{P}_\mu^t(E) &:= \int_E \operatorname{Tr}\big(T_t(\xi)\, \mu\, T_t(\xi)^*\big)\, \mathrm{d}\nu_t(\xi) \\ &= \sum_{n=0}^{\infty} \sum_{i_0, \dots, i_n \in V} \int_{0 < t_1 < \cdots < t_n < t} \mathbb{1}_{\xi \in E} \operatorname{Tr}\big(T_t(\xi)\rho(i_0)T_t(\xi)^*\big)\, \mathrm{d}t_1 \cdots \mathrm{d}t_n\ , \end{aligned}$$

where $\xi = (i_0, \dots, i_n; t_1, \dots, t_n)$ and where $T_t(\xi)$ is defined by Eq. (18.9). The measure $\mathbb{P}_\mu^t$ is a probability measure as one can check that $\mathbb{P}_\mu^t(\Xi_t) = \operatorname{Tr}\big(\mathrm{e}^{t\mathcal{L}}(\mu)\big) = 1$. The family of probability measures $\big(\mathbb{P}_\mu^t\big)_{t \geq 0}$ is consistent, as (18.9) and (18.2.2) show that if $E \in \Sigma_t$,

$$\mathbb{P}_\mu^{t+s}(E) = \sum_{n=0}^{\infty} \sum_{i_0,\dots,i_n \in V} \int_{0<t_1<\dots<t_n<t} \mathbb{1}_{\xi \in E} \operatorname{Tr}\big(e^{s\mathcal{L}}\big(T_t(\xi)\,\rho(i_0)\,T_t(\xi)^*\big)\big)\, dt_1 \cdots dt_n$$
$$= \mathbb{P}_\mu^t(E)$$

for all $t, s \geq 0$. Hence, Kolmogorov's consistency Theorem allows us to extend $(\mathbb{P}_\mu^t)_{t\geq 0}$ to a probability measure $\mathbb{P}_\mu$ on $(\Xi_\infty, \Sigma_\infty)$ where $\Sigma_\infty = \sigma(\Sigma_t, t \in \mathbb{R}_+)$.

In most of our discussions below we will specialize to the case where μ is of the form $\mu = \rho \otimes |i\rangle\langle i|\ Q_{<2\alpha}ii$. In such a case, we denote by $\mathbb{P}_{i,\rho}$ the probability $\mathbb{P}_\mu$.

18.2.3 Quantum Trajectories Associated with CTOQW

Quantum trajectories are another convenient way to describe the distribution of the process $(X_t, \rho_t)_{t\geq 0}$ associated with the CTOQW. Actually, the combination of quantum trajectories and of the Dyson expansion will be essential tools for the main result of this article. Formally speaking, quantum trajectories model the evolution of the state when a continuous measurement of the position of the particle is performed. The state at time t can be described by a pair (X_t, ρ_t) with $X_t \in V$ the position of the particle at time t (as recorded by the measuring device) and $\rho_t \in \mathcal{S}_\mathcal{H}$ the density matrix describing the internal degrees of freedom, given by the wave function collapse postulate and thus constrained to have support on $\mathfrak{h}_i$ alone. The stochastic process $(X_t, \rho_t)_{t\geq 0}$ is then a Markov process, and this will allow us to use the standard machinery for such processes. However, their rigorous description is less straightforward than the one for discrete-time OQW. It makes use of stochastic differential equations driven by jump processes. We refer to [25] for the justification of the below description and for the link between discrete and continuous-time models. Remark that we denote by the same symbol the stochastic process $(X_t)_{t\geq 0}$ appearing in this and the previous section. This will be justified in Remark 18.2.4 below.

In order to present the stochastic differential equation satisfied by the pair $(X_t, \rho_t)_{t\geq 0}$ we need a usual filtered probability space $\big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\big)$, where we consider independent Poisson point processes $N^{i,j}$, $i, j \in V, i \neq j$ on $\mathbb{R}^2$ (again we take $N^{i,i} = 0$ by convention). These Poisson point processes will govern the jump from site i to site j on the graph V.

Definition 18.2 Let $(\mathcal{T}_t)_{t\geq 0}$ be a CTOQW with Lindbladian $\mathcal{L}$ of the form (18.3) and let $\mu = \sum_{i\in V} \rho(i) \otimes |i\rangle\langle i|$ be an initial density matrix in $\mathcal{D}$. The quantum trajectory describing the indirect measurement of the position of the CTOQW is the Markov process $(\mu_t)_{t\geq 0}$ taking values in the set $\mathcal{D}$ such that

$$\mu_0 = \rho_0 \otimes |X_0\rangle\langle X_0|\ ,$$

where X_0 and ρ_0 are random with distribution

$$\mathbb{P}\Big((X_0,\rho_0)=\big(i,\frac{\rho(i)}{\mathrm{Tr}(\rho(i))}\big)\Big)=\mathrm{Tr}\big(\rho(i)\big) \text{ for all } i\in V$$

and such that $\mu_t =: \rho_t \otimes |X_t\rangle\langle X_t|$ satisfies for all $t \geq 0$ the following stochastic differential equation:

$$\mu_t = \mu_0 + \int_0^t \mathcal{M}(\mu_{s-})\,\mathrm{d}s + \sum_{i,j}\int_0^t\int_{\mathbb{R}}\left(\frac{S_i^j\,\mu_{s-}\,S_i^{j*}}{\mathrm{Tr}(S_i^j\mu_{s-}S_i^{j*})}-\mu_{s-}\right)\mathbb{1}_{0<y<\mathrm{Tr}(S_i^j\mu_{s-}S_i^{j*})}N^{i,j}(\mathrm{d}y,\mathrm{d}s) \tag{18.10}$$

where

$$\mathcal{M}(\mu)=\mathcal{L}(\mu)-\sum_{i,j}\big(S_i^j\,\mu\,S_i^{j*}-\mu\,\mathrm{Tr}(S_i^j\,\mu\,S_i^{j*})\big)$$

so that for $\mu=\sum_i \rho(i)\otimes|i\rangle\langle i|\in\mathcal{D}$,

$$\mathcal{M}(\mu)=\sum_i\big(G_i\rho(i)+\rho(i)G_i^*-\rho(i)\,\mathrm{Tr}\big(G_i\rho(i)+\rho(i)G_i^*\big)\big)\otimes|i\rangle\langle i|\;.$$

Remark 18.2 An interesting fact has been pointed out in [25]: continuous-time classical Markov chains can be realized within this setup by considering $\mathfrak{h}_i=\mathbb{C}$ for all $i\in V$.

Let us briefly describe the evolution of the solution $(\mu_t)_{t\geq 0}$ of (18.10), and in particular explain why μ_t is of the form $\rho_t\otimes|X_t\rangle\langle X_t|$. Assume that $X_0=i_0$ for some $i_0\in V$ and consider ρ_0 a state on $\mathfrak{h}_{i_0}$. Remark that for any state ρ on $\mathfrak{h}_{i_0}$, $\mathcal{M}(\rho\otimes|i_0\rangle\langle i_0|)$ is of the form $\rho'\otimes|i_0\rangle\langle i_0|$. We then consider the solution, for all $t\geq 0$, of

$$\eta_t=\rho_0+\int_0^t\big(G_{i_0}\eta_s+\eta_s G_{i_0}^*-\eta_s\mathrm{Tr}(G_{i_0}\eta_s+\eta_s G_{i_0}^*)\big)\,\mathrm{d}s\;.$$

We stress the fact that the solution of this equation takes its values in the set on states of $\mathfrak{h}_{i_0}$ (this nontrivial fact is well-known in the theory of quantum trajectories, see [24] for further details). Now let us define the first jump time. To this end we introduce for $j\neq i_0$

$$T_1^j=\inf\Big\{t\geq 0\,;\;N^{i_0,j}\big(\{u,y\,|\,0\leq u\leq t,\;0\leq y\leq \mathrm{Tr}(R_{i_0}^j\eta_u R_{i_0}^{j\,*})\}\big)\geq 1\Big\}\;.$$

The random variables T_1^j are nonatomic, and mutually independent. Therefore, if we let $T_1 = \inf_{j\neq i_0}\{T_1^j\}$ then there exists a unique $j \in V$ such that $T_1^j = T_1$. In addition,

$$\begin{aligned}\mathbb{P}\big(T_1 \leq \varepsilon\big) &\leq \sum_{j\neq i_0} \mathbb{P}\big(T_1^j \leq \varepsilon\big)\\ &= \sum_{j\neq i_0} (1 - \mathrm{e}^{-\int_0^\varepsilon \mathrm{Tr}(R_{i_0}^j \eta_u R_{i_0}^{j*})\,\mathrm{d}u})\\ &\leq \sum_{j\neq i_0} \int_0^\varepsilon \mathrm{Tr}(R_{i_0}^j \eta_u R_{i_0}^{j*})\,\mathrm{d}u\\ &\leq \varepsilon \sum_{j\neq i_0} \|R_{i_0}^{j*} R_{i_0}^j\| \end{aligned} \tag{18.11}$$

where the sums are over all j in V with $j \neq i_0$. Now remark that our assumption that $\sum_{i,j} S_i^{j*} S_i^j$ converges strongly implies that the sum $\sum_{j\neq i} \|R_i^{j*} R_i^j\|$ is finite for all i in V, so that Eq. (18.11) implies $\mathbb{P}(T_1 > 0) = 1$. On $[0, T_1]$ we then define the solution $(X_t, \rho_t)_{t\geq 0}$ as

$$(X_t, \rho_t) = (i_0, \eta_t) \text{ for } t \in [0, T_1) \quad \text{and}$$

$$(X_{T_1}, \rho_{T_1}) = \big(j, \frac{R_i^j \eta_{T_1-} R_i^{j*}}{\mathrm{Tr}(R_i^j \eta_{T_1-} R_i^{j*})}\big) \text{ if } T_1 = T_1^j.$$

We then solve

$$\eta_t = \rho_{T_1} + \int_0^t \big(G_j \eta_s + \eta_s G_j^* - \eta_s \mathrm{Tr}(G_j \eta_s + \eta_s G_j^*)\big)\,\mathrm{d}s\ ,$$

and define a new jumping time T_2 as above. By this procedure we define an increasing sequence $(T_n)_n$ of jumping times. We show that $T := \lim_n T_n = +\infty$ almost surely: we introduce

$$N_t = \sum_{i,j} \big(\int_0^{t\wedge T} \int_{\mathbb{R}} \mathbb{1}_{0<y<\mathrm{Tr}(S_i^j \mu_{s-} S_i^{j*})}\, N^{i,j}(\mathrm{d}y, \mathrm{d}s)\big)$$

(the sum is over all i, j with $i \neq j$) which counts the number of jumps before t. In particular $N_{T_p} = p$ for all $p \in \mathbb{N}$. Now from the properties of the Poisson processes we have for all $p \in \mathbb{N}$ and all $m \in \mathbb{N}$,

$$\mathbb{E}(N_{T_p \wedge m}) \leq \mathbb{E}\big(N_m\big) = \sum_{i,j} \mathbb{E}\big(\int_0^{m\wedge T} \mathrm{Tr}(S_i^j \mu_{s-} S_i^{j*})\,\mathrm{d}s\big) \leq m \sum_{i,j} \|S_i^{j*} S_i^j\|\ .$$

Denoting $C = \sum_{i,j} \|S_i^{j*} S_i^j\|$ (which is finite) the inequality $p\,\mathbb{P}(T_p \le m) \le \mathbb{E}(N_{T_p \wedge m})$ implies

$$\mathbb{P}(T_p \le m) \le \frac{m}{p} C\ .$$

This implies that $\mathbb{P}(\lim_p T_p \le m) = 0$ for all $m \in \mathbb{N}$ so that $\lim_p T_p = +\infty$ almost surely. Therefore, the above considerations define (X_t, ρ_t) for all $t \in \mathbb{R}_+$.

18.2.4 Connection Between Dyson Expansion and Quantum Trajectories

The connection between the process $(X_t, \rho_t)_{t\ge 0}$ defined in this section and the Dyson expansion has been deeply studied in the literature. We do not give all the details of this construction and instead refer to [4, 7] for a complete and rigorous justification. The main point is that the process $(X_t, \rho_t)_{t\ge 0}$ defined in Sect. 18.2.3 can be constructed explicitly on the space $(\Xi^\infty, \Sigma^\infty, \mathbb{P})$, as we now detail.

Recall the interpretation of $\xi = (i_0, \dots, i_n; t_1, \dots, t_n)$ as the trajectory of a particle, initially at i_0 and jumping to i_k at time t_k. First, on $(\Xi_\infty, \Sigma_\infty, \mathbb{P})$ define the random variable $\tilde{N}_t^{i,j}$ by

$$\tilde{N}_t^{i,j}(\xi) = \mathrm{card}\big\{k = 0, \dots, n-1 \,|\, t_{k+1} \le t \text{ and } (i_k, i_{k+1}) = (i, j)\big\}$$

for $\xi = (i_0, \dots, i_n; t_1, \dots, t_n)$ as above. Now, let

$$\begin{aligned} \tilde{X}_t(\xi) &= \begin{cases} i_k & \text{if } t_k \le t < t_{k+1} \\ i_n & \text{if } t_n \le t. \end{cases} \\ \tilde{\rho}_t(\xi) &= \frac{T_t(\xi)\rho(i_0)\, T_t(\xi)^*}{\mathrm{Tr}(T_t(\xi)\rho(i_0) T_t(\xi)^*)} \end{aligned} \tag{18.12}$$

(recall that $T_t(\xi)$ is defined in (18.9)) and

$$\tilde{\mu}_t = \tilde{\rho}_t \otimes |\tilde{X}_t\rangle\langle\tilde{X}_t|\ .$$

Differentiating (18.12), one can show that the process $(\tilde{\mu}_t)_{t\ge 0}$ satisfies

$$\mathrm{d}\tilde{\mu}_t = \mathcal{M}(\tilde{\mu}_{t-})\,\mathrm{d}t + \sum_{i,j} \Big(\frac{S_i^j\, \tilde{\mu}_{s-} S_i^{j*}}{\mathrm{Tr}(S_i^j\, \tilde{\mu}_{t-} S_i^{j*})} - \tilde{\mu}_{t-}\Big)\, \mathrm{d}\tilde{N}^{i,j}(t)\ . \tag{18.13}$$

It is proved in [4] that the processes

$$(\tilde{N}_t^{i,j})_{t\geq 0} \quad \text{and} \quad \Big(\int_0^t \int_{\mathbb{R}} \mathbb{1}_{0<y<\mathrm{Tr}(S_i^j \mu_{s-} S_i^{j*})} N^{i,j}(\mathrm{d}y, \mathrm{d}s)\Big)_{t\geq 0}$$

(for $(\mu_t)_{t\geq 0}$ and $N^{i,j}$ defined in the previous section) have the same distribution. Therefore, $(\tilde{\mu}_t)_{t\geq 0}$ and $(\mu_t)_{t\geq 0}$ have the same distribution. For this reason, we will denote the random variables $\tilde{\eta}_t$, $\tilde{X}_t$, $\tilde{\rho}_t$ by η_t, X_t, ρ_t, i.e. we identify the random variables obtained by the construction in Sect. 18.2.3 and those defined by (18.12). In addition, from expression (18.8) for $\mathcal{T}_t$ and (18.12) for ρ_t, X_t we recover immediately that $\mu_t = \rho_t \otimes |X_t\rangle\langle X_t|$ satisfies

$$\mathbb{E}_{\mu_0}(\mu_t) = \mathcal{T}_t(\mu_0)$$

where $\mathbb{E}_{\mu_0}$ is the expectation with respect to the probability $\mathbb{P}_{\mu_0}$ defined in Sect. 18.2.2. This identity shows that the quantum Markov semigroup $(\mathcal{T}_t)_{t\geq 0}$ plays for the process $(X_t, \rho_t)_{t\geq 0}$ the same role as the Markov semigroup in the classical case. Because a notion of irreducibility is naturally associated with such a semigroup (see [11] for the original definition and [16] for general considerations on the irreducibility of Lindbladians), this will allow us to associate a notion of irreducibility to a continuous-time open quantum walk.

Now note that expressions (18.12) give an interpretation of X_t and ρ_t in terms of quantum measurement. Indeed, one can see the operator $T_t(\xi)$ for $\xi = (i_0, \dots, i_n; t_1, \dots, t_n)$ (or, rather, the map $\rho \mapsto T_t(\xi)\rho T_t(\xi)^*$) as describing the effect of the trajectory where jumps (up to time t) occur at times $t_1,\dots,t_n$ and $i_0,\dots,i_n$ is the sequence of updated positions: as long as the particle sits at $i_k \in V$, the evolution of its internal degrees of freedom is given by the semigroup of contraction $(\mathrm{e}^{t\,G_{i_k}})_{t\geq 0}$ and, as the particle jumps to i_{k+1}, it undergoes an instantaneous transformation governed by $R_{i_k}^{i_{k+1}}$ (this $T_t(\xi)$ is then the analogue for continuous-time OQW of the operator L_π of [9]). Therefore, the expression for $\rho_t(\xi)$ in (18.12) encodes the effect of the reduction postulate, or postulate of the collapse of the wave function, on the state of a particle initially at i_0 and with internal state ρ_0. This rigorous connection of the unravelling (18.9) to (indirect) measurement was first described in [4] (see also [23, 24], as well as [13] for a connection to two-time measurement statistics).

To summarize this section and the preceding one, we have defined a Markov process $(\mu_t)_t$ as $\mu_t = \rho_t \otimes |X_t\rangle\langle X_t|$, where $X_t \in V$ and $\rho_t \in \mathcal{S}_{\mathfrak{h}_{X_t}}$, of which the law can be computed in two ways: either by the Dyson expansion of the CTOQW as in (18.2.2) or by use of the stochastic differential equation (18.10).

18.3 Irreducibility of Quantum Markov Semigroups

In this section, we state the equivalence between different notions of irreducibility for general quantum Markov semigroup. Our main motivation is the fact that we could not find a complete proof in the case of an infinite-dimensional Hilbert space, as is required e.g. for CTOQW with infinite V. We then discuss irreducibility for CTOQW.

Theorem 18.1 *Let $\mathcal{T} := (\mathcal{T}_t)_{t\geq 0}$ be a quantum Markov semigroup with Lindbladian*

$$\mathcal{L}(\mu) = G\mu + \mu G^* + \sum_{i\in I} L_i\, \mu\, L_i^* \,. \tag{18.14}$$

The following assertions are equivalent:

1. *$\mathcal{T}$ is* positivity improving*: for all $A \in \mathcal{I}_1(\mathcal{K})$ with $A \geq 0$ and $A \neq 0$, there exists $t > 0$ such that* $\mathrm{e}^{t\mathcal{L}}(A) > 0$.
2. *For any $\varphi \in \mathcal{K}\backslash\{0\}$, the set $\mathbb{C}[\mathcal{L}]\,\varphi$ is dense in $\mathcal{K}$ where $\mathbb{C}[\mathcal{L}]$ is the set of polynomials in* e^{tG} *for $t > 0$ and in L_i for $i \in I$.*
3. *For any $\varphi \in \mathcal{K}\backslash\{0\}$, the set $\mathbb{C}[G, L]\,\varphi$ is dense in $\mathcal{K}$ where $\mathbb{C}[G, L]$ is the set of polynomials in G and in L_i for $i \in I$.*
4. *$\mathcal{T}$ is irreducible, i.e. there exists $t > 0$ such that $\mathcal{T}_t$ admits no non-trivial projection $P \in \mathcal{B}(\mathcal{K})$ with $\mathcal{T}_t\big(P\mathcal{I}_1(\mathcal{K})P\big) \subset P\mathcal{I}_1(\mathcal{K})P$.*

From now on, any quantum Markov semigroup which satisfies any one of the equivalent statements of Theorem 18.1 is simply called *irreducible*.

Remark 18.3 Positivity improving maps are also called primitive. We therefore call *primitivity* the property of being positivity improving. Remark also that one can replace "there exists $t > 0$" by "for all $t > 0$" in assertions 1. and 4. above to get another equivalent formulation of irreducibility and primitivity. This follows from the observation that assertion 3. does not depend on t.

Proof We first prove the equivalence of 1. and 2. Note that 1. holds if and only if for every $\varphi_0 \neq 0$, there exists $t_0 > 0$ such that $\langle \varphi, \mathrm{e}^{t\mathcal{L}}(|\varphi_0\rangle\langle\varphi_0|)\varphi\rangle > 0$ for all $\varphi \neq 0$. Now remark that from Eq. (18.8),

$$\langle \varphi, \mathrm{e}^{t\mathcal{L}}(|\varphi_0\rangle\langle\varphi_0|)\varphi\rangle = \sum_{n=0}^{\infty} \sum_{i_0,\ldots,i_n\in I} \int_{0<t_1<\cdots<t_n<t} |\langle \varphi, \zeta_t(\xi)\varphi_0\rangle|^2 \,\mathrm{d}t_1\cdots\mathrm{d}t_n \tag{18.15}$$

where $\xi = (i_1, \ldots, i_n; t_1, \ldots, t_n)$. Assume 1. and fix $\varphi_0 \neq 0$. If for some $t \geq 0$, the left-hand side of (18.15) is positive for any $\varphi \neq 0$, then for any such $\varphi \neq 0$ there exists ξ with $\langle \varphi, \zeta_t(\xi)\varphi_0\rangle \neq 0$. Since $\zeta_t(\xi)\varphi_0 \in \mathbb{C}[\mathcal{L}]\varphi_0$ and the latter is a vector space, this implies that $\mathbb{C}[\mathcal{L}]\varphi_0$ is dense in $\mathcal{K}$. Now assume

2. and fix $\varphi_0 \neq 0$. Since $\mathbb{C}[\mathcal{L}]\varphi_0$ is dense in $\mathcal{K}$, for any $\varphi \neq 0$ there exists an element $\psi = \mathrm{e}^{s_n G} L_{i_n} \cdots L_{i_1} \mathrm{e}^{s_1 G}\varphi_0$ such that $\langle \varphi, \psi\rangle \neq 0$. However, for $t \geq s_1 + \ldots + s_n$, ψ is of the form $\zeta_t(\xi)\varphi_0$ for some $\xi = (i_1, \ldots, i_n; t_1, \ldots, t_n)$. By continuity of ζ in $t_1, \ldots, t_n$, the right-hand side of (18.15) is positive and this proves 1.

To prove the equivalence of 2. and 3., we use the fact that $G = \lim_{t\to 0}(\mathrm{e}^{tG} - \mathrm{Id})/t$, which implies that for any $\varphi \in \mathcal{K}\backslash\{0\}$,

$$\mathbb{C}[G, L]\varphi \subset \overline{\mathbb{C}[\mathcal{L}]}\varphi \subset \overline{\mathbb{C}[\mathcal{L}]\varphi}\ .$$

Since $\mathrm{e}^{tG} = \lim_{n\to\infty} \sum_{k=0}^{n} t^k G^k/k!$, for any $\varphi \in \mathcal{K}\backslash\{0\}$ we also have

$$\mathbb{C}[\mathcal{L}]\varphi \subset \overline{\mathbb{C}[G, L]}\varphi \subset \overline{\mathbb{C}[G, L]\varphi}\ .$$

Therefore, for any $\varphi \in \mathcal{K}\backslash\{0\}$,

$$\mathbb{C}[\mathcal{L}]\varphi \text{ is dense in } \mathcal{K} \Leftrightarrow \mathbb{C}[G, L]\varphi \text{ is dense in } \mathcal{K}\ . \tag{18.16}$$

That 1. implies 4. is obvious. It remains to prove that 4. implies 2. To this end, suppose that $\mathcal{T}$ is irreducible. Let $\varphi \in \mathcal{K}\backslash\{0\}$ and denote by P the orthogonal projection on $\overline{\mathbb{C}[\mathcal{L}]\varphi}$. The goal is to prove that $P = \mathrm{Id}$. For all $\psi \in \mathcal{K}\backslash\{0\}$,

$$\begin{aligned}
\mathrm{e}^{t\mathcal{L}}(P|\psi\rangle\langle\psi|P) &= \sum_{n=0}^{\infty} \sum_{i_0,\ldots,i_n\in I} \int_{0<t_1<\cdots<t_n<t} \zeta_t(\xi)P|\psi\rangle\langle\psi|P\zeta_t(\xi)^*\mathrm{d}t_1\cdots\mathrm{d}t_n\\
&= \sum_{n=0}^{\infty} \sum_{i_0,\ldots,i_n\in I} \int_{0<t_1<\cdots<t_n<t} |\zeta_t(\xi)P\psi\rangle\langle\zeta_t(\xi)P\psi|\,\mathrm{d}t_1\cdots\mathrm{d}t_n\ ,
\end{aligned}$$

Since $\zeta_t(\xi) \in \mathbb{C}[\mathcal{L}]$ and $P\psi \in \overline{\mathbb{C}[\mathcal{L}]\varphi}$, we have $\zeta_t(\xi)P\psi \in \overline{\mathbb{C}[\mathcal{L}]\varphi}$ and thus

$$\mathcal{T}_t(P|\psi\rangle\langle\psi|P) = P\,\mathcal{T}_t(P|\psi\rangle\langle\psi|P)P\ .$$

Since $\mathcal{T}_t$ is irreducible by assumption, P must be trivial. As it is non-zero, $P = \mathrm{Id}$. Since P is the orthogonal projection on $\overline{\mathbb{C}[\mathcal{L}]\varphi}$, this shows that $\mathbb{C}[\mathcal{L}]\varphi$ is dense in $\mathcal{K}$. □

Remark 18.4 An immediate corollary of Theorem 18.1 is that a quantum Markov semigroup $\mathcal{T} = (\mathcal{T}_t)_t$ is irreducible if and only if its adjoint $\mathcal{T}^* = (\mathcal{T}_t^*)_t$ is irreducible.

We now introduce the notion of irreducibility of a CTOQW, focusing on the trajectorial formulation. Let $\mathcal{T} := (\mathcal{T}_t)_{t\geq 0}$ be a CTOQW on a set V. For i, j in V

and $n \in \mathbb{N}$, we denote by $\mathcal{P}^n(i, j)$ the set of continuous-time trajectories going from i to j in n jumps:

$$\mathcal{P}^n(i, j) = \{\xi = (i_0, \ldots, i_n; t_1, \ldots, t_n) \in \Xi_\infty^{(n)} \mid i_0 = i,\ i_n = j\}$$

and we set $\mathcal{P}(i, j) = \cup_{n\in\mathbb{N}}\mathcal{P}^n(i, j)$. For any $\xi = (i, \ldots, j; t_1, \ldots, t_n)$ in $\mathcal{P}(i, j)$, we recall that the operator $T_t(\xi)$ from $\mathfrak{h}_i$ to $\mathfrak{h}_j$ is defined by

$$T_t(\xi) = \mathrm{e}^{(t-t_n)G_{i_n}} R^j_{i_{n-1}} \mathrm{e}^{(t_n-t_{n-1})G_{i_{n-1}}} \cdots \mathrm{e}^{(t_2-t_1)G_{i_1}} R^{i_1}_i \mathrm{e}^{t_1 G_i} .$$

The following proposition is a direct application of Theorem 18.1, and will constitute our definition of irreducibility for continuous-time open quantum walks. The criterion here is equivalent to any other formulation proposed in Theorem 18.1.

Proposition 18.2 *A CTOQW with Lindbladian (18.3) is* irreducible *if and only if, for every i and j in V, and for any φ in $\mathfrak{h}_i\backslash\{0\}$, the set*

$$\left\{T_t(\xi)\,\varphi,\ t \ge 0,\ \xi \in \mathcal{P}(i, j)\right\}$$

is total in $\mathfrak{h}_j$.

Remark 18.5 From Theorem 18.1, an equivalent condition of irreducibility is that for every i and j in V and for any φ in $\mathfrak{h}_i\backslash\{0\}$, the set of all $G^{k_n}_{i_n} R^j_{i_{n-1}} G^{k_{n-1}}_{i_{n-1}} \cdots G^{k_1}_{i_1} R^{i_1}_i G^{k_0}_i \varphi$ for any $i_0, i_1, \ldots, i_n$ with $i_0 = i$ and $i_n = j$, and any $k_0, \ldots, k_n$ in $\mathbb{N}_0$, is total in $\mathfrak{h}_j$. This immediately implies that a CTOQW is irreducible if, for every i and j in V and φ in $\mathfrak{h}_i\backslash\{0\}$, the set

$$\left\{R^{i_n}_{i_{n-1}} \ldots R^{i_1}_{i_0}\varphi,\ i_0, i_1, \ldots, i_n \in V,\ i_0 = i, i_n = j,\ n \in \mathbb{N}_0\right\} \tag{18.17}$$

is total in $\mathfrak{h}_j$. This is equivalent to saying that the completely positive map induced by the off-diagonal terms of $\mathcal{L}$ (i.e. the map $\mu \mapsto \sum_{i,j}(R^i_j \otimes |i\rangle\langle j|)\mu(R^i_j \otimes |i\rangle\langle j|)^*$) is irreducible as a (discrete-time) completely positive map (see [11, 15]). This of course is true for continuous-time Markov chains, which are irreducible if the discrete-time map induced by the off-diagonal terms is irreducible. In the case of CTOQW, however, this is not true, as the next example shows.

Example 18.1 Consider the OQW with $V = \{1, 2\}$ and $\mathfrak{h}_1 = \mathfrak{h}_2 = \mathbb{C}^2$, and Lindbladian defined by (18.7) with:

$$G_1 = G_2 = \frac{1}{2}\begin{pmatrix} -1 & 2 \\ -2 & -1 \end{pmatrix},\ R^2_1 = R^1_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

One can easily check that $\left\{T_t(\xi)\,\varphi,\ t \ge 0,\ \xi \in \mathcal{P}(i, j)\right\} = \mathfrak{h}_j$ for all $i, j \in \{1, 2\}$ and $\varphi \in \mathfrak{h}_i \setminus \{0\}$, so that the CTOQW is irreducible, but the criterion in (18.17) in terms of R^2_1 and R^1_2 is not satisfied.

18.4 Transience and Recurrence of Irreducible CTOQW

In the classical theory of Markov chains on a finite or countable graph, an irreducible Markov chain can be either transient or recurrent. Transience and recurrence issues are central to the study of Markov chains and help describe the Markov chain's overall structure. In the case of CTOQW, transience and recurrence notions are made more complicated by the fact that the process $(X_t)_{t\geq 0}$ alone is not a Markov chain.

In the present section, we define the notion of recurrence and transience of a vertex in our setup and prove a dichotomy similar to the classical case, based on the average occupation time at a vertex. However, compared to the classical case, the relationship between the occupation time and the first passage time at the vertex is less straightforward. Recall that the first passage time at a given vertex $i \in V$ is defined as

$$\tau_i = \inf\{t \geq T_1 | X_t = i\}$$

where T_1 is defined in Sect. 18.2.3. Similarly the occupation time is given by

$$n_i = \int_0^\infty \mathbb{1}_{X_t=i} \, \mathrm{d}t \ .$$

In the discrete-time and irreducible case (Theorem 3.1. of [5]), the authors prove that there exists a trichotomy rather than the classical dichotomy. We state a similar result for continuous-time semifinite open quantum walks (we recall that an OQW is semifinite if $\dim \mathfrak{h}_i < \infty$ for all $i \in V$).

Theorem 18.2 *Consider a semifinite irreducible continuous-time open quantum walk. Then we are in one (and only one) of the following situations:*

1. *For any i, j in V and ρ in $\mathcal{S}_{\mathfrak{h}_i}$, one has $\mathbb{E}_{i,\rho}(n_j) = \infty$ and $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$.*
2. *For any i, j in V and ρ in $\mathcal{S}_{\mathfrak{h}_i}$, one has $\mathbb{E}_{i,\rho}(n_j) < \infty$ and $\mathbb{P}_{i,\rho}(\tau_i < \infty) < 1$.*
3. *For any i, j in V and ρ in $\mathcal{S}_{\mathfrak{h}_i}$, one has $\mathbb{E}_{i,\rho}(n_j) < \infty$, but there exist i in V and ρ, ρ' in $\mathcal{S}_{\mathfrak{h}_i}$ (ρ necessarily non-faithful) such that $\mathbb{P}_{i,\rho}(\tau_i < \infty) = 1$ and $\mathbb{P}_{i,\rho'}(\tau_i < \infty) < 1$.*

Note that in the sequel we only focus on the semifinite case. Recall that when $\mathfrak{h}_i$ is one-dimensional for all $i \in V$, we recover classical continuous-time Markov chains. In this case, the Markov chain falls in one of the first two categories of this theorem; that is, the third category is a specifically quantum situation.

The rest of this section is dedicated to the proof of Theorem 18.2. More precisely, in Sect. 18.4.1 we prove the dichotomy between infinite and finite average occupation time. This allows us to define transience and recurrence of CTOQW. We also give examples of CTOQW that fall in each of the three classes of Theorem 18.2. In Sect. 18.4.2 we state technical results that give closed expressions for the occupation time and the first passage time. Finally, the proof of Theorem 18.2 is given in Sect. 18.4.3.

18.4.1 Definition of Recurrence and Transience

We begin by proving that for an irreducible CTOQW, the average occupation time $\mathbb{E}_{i,\rho}(n_j)$ of site j starting from site i is either finite for all i, j or infinite for all i, j.

Proposition 18.3 *Consider a semifinite irreducible continuous-time open quantum walk. Suppose furthermore that there exist $i_0, j_0 \in V$ and $\rho_0 \in \mathcal{S}_{\mathfrak{h}_{i_0}}$ such that $\mathbb{E}_{i_0,\rho_0}(n_{j_0}) = \infty$. Then, for all $i, j \in V$ and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$ one has $\mathbb{E}_{i,\rho}(n_j) = \infty$.*

Proof Fix $i, j \in V$ and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$. Then one has

$$\mathbb{E}_{i,\rho}(n_j) = \int_0^\infty \mathbb{P}_{i,\rho}(X_t = j)\,\mathrm{d}t = \int_0^\infty \mathrm{Tr}\big(\mathrm{e}^{t\mathcal{L}}(\rho \otimes |i\rangle\langle i|)(\mathrm{Id} \otimes |j\rangle\langle j|)\big)\,\mathrm{d}t\ .$$

By hypothesis, $(\mathcal{T}_t)_{t\geq 0}$ is irreducible and thus positivity improving by Theorem 18.1; by Remark 18.4 the same is true of $(\mathcal{T}_t^*)_{t\geq 0}$. Therefore, since for any $i \in V$, $\mathfrak{h}_i$ is finite-dimensional, for any $s > 0$ there exist scalars $\alpha, \beta > 0$ such that

$$\mathrm{e}^{s\mathcal{L}}(\rho \otimes |i\rangle\langle i|) \geq \alpha\, \rho_0 \otimes |i_0\rangle\langle i_0| \ \text{ and } \ \mathrm{e}^{s\mathcal{L}^*}(\mathrm{Id} \otimes |j\rangle\langle j|) \geq \beta\, \mathrm{Id} \otimes |j_0\rangle\langle j_0|\ .$$

We then have, fixing $s > 0$,

$$\begin{aligned}\mathbb{E}_{i,\rho}(n_j) &\geq \int_{2s}^\infty \mathrm{Tr}\Big(\mathrm{e}^{(t-2s)\mathcal{L}}\big(\mathrm{e}^{s\mathcal{L}}(\rho \otimes |i\rangle\langle i|)\big)\,\mathrm{e}^{s\mathcal{L}^*}(\mathrm{Id} \otimes |j\rangle\langle j|)\Big)\,\mathrm{d}t \\ &\geq \alpha\beta \int_0^\infty \mathrm{Tr}\big(\mathrm{e}^{u\mathcal{L}}(\rho_0 \otimes |i_0\rangle\langle i_0|)(\mathrm{Id} \otimes |j_0\rangle\langle j_0|)\big)\,\mathrm{d}u \\ &\geq \alpha\beta\, \mathbb{E}_{i_0,\rho_0}(n_{j_0})\ .\end{aligned}$$

This concludes the proof. □

This proposition leads to a natural definition of recurrent and transient vertices of V:

Definition 18.3 For any continuous-time open quantum walk, we say that a vertex i in V is:

- recurrent if for any $\rho \in \mathcal{S}_{\mathfrak{h}_i}$, $\mathbb{E}_{i,\rho}(n_i) = \infty$;
- transient if there exists $\rho \in \mathcal{S}_{\mathfrak{h}_i}$ such that $\mathbb{E}_{i,\rho}(n_i) < \infty$.

Thus, by Proposition 18.3, for an irreducible CTOQW, either all vertices are recurrent, in which case we say that the CTOQW is recurrent; or all vertices are transient, in which case we say that it is transient. Furthermore, in the transient case, $\mathbb{E}_{i,\rho}(n_i) < \infty$ for all ρ in $\mathcal{S}_{\mathfrak{h}_i}$.

As already mentioned in the introduction, a general notion of recurrence and transience of quantum dynamical semigroups has been defined by Fagnola and Rebolledo in [17] (see also [18]). It is natural to wonder if this general notion reduces to ours in the case of CTOQW. When applied to the semigroup $(\mathcal{T}_t)_{t\geq 0}$, the definition of recurrence in [17] (denoted FR-recurrence in [5]) is that for any positive semidefinite operator A of $\mathcal{B}(\mathcal{H})$, the set

$$D(\mathfrak{U}(A)) = \Big\{\varphi = \sum_{i\in V} \varphi_i \otimes |i\rangle \ \text{ s.t. } \int_0^\infty \langle \varphi, \mathcal{T}_s^*(A)\,\varphi\rangle \, \mathrm{d}s < \infty \Big\} .$$

equals $\{0\}$. As we have

$$\mathbb{E}_{i,\rho}(n_i) = \int_0^\infty \mathrm{Tr}\big(\rho\, \mathcal{T}_s^*(\mathrm{Id}_{\mathfrak{h}_i} \otimes |i\rangle\langle i|)\big)\, \mathrm{d}s \ ,$$

we see that our definition of recurrence for CTOQW is equivalent to the fact that for any $i \in V$, $D(\mathfrak{U}(\mathrm{Id}_{\mathfrak{h}_i} \otimes |i\rangle\langle i|)) = \{0\}$. Consequently, it is clear that if the CTOQW is FR-recurrent, then it is recurrent in our sense. Conversely, if the CTOQW is recurrent in our sense, then for any definite-positive A and any $\varphi = \sum_{i\in V} \varphi_i \otimes |i\rangle$, there exists i such that $\varphi_i \neq 0$, and if the CTOQW is semifinite, then $A \geq \lambda_i \mathrm{Id}_{\mathfrak{h}_i} \otimes |i\rangle\langle i|$ for some $\lambda_i > 0$. We then have

$$\int_0^\infty \langle \varphi, \mathcal{T}_s^*(A)\,\varphi\rangle \, \mathrm{d}s \geq \lambda_i \, \mathbb{E}_{i,|\varphi_i\rangle\langle\varphi_i|}(n_i) = +\infty \ .$$

By Theorem 2 of [17], the quantum dynamical semigroup $(\mathcal{T}_t)_{t\geq 0}$ is not transient, and by Proposition 5 of the same reference, it must be recurrent if $(\mathcal{T}_t)_{t\geq 0}$ is irreducible. Therefore, for irreducible semifinite CTOQW our notion of recurrence and FR-recurrence are equivalent. We refer to [5] for a more complete discussion of the different notions of recurrence that appear in the literature for OQW. Note that the notion of FR-recurrence is more general since it encompasses the case of unbounded generators (the approach of [17] derives from potential theory); here we are essentially interested in semifinite CTOQW in order to have a clear trichotomy, so that our S_i^j are automatically bounded.

We conclude this section by illustrating Theorem 18.2 with simple examples. The n-th example below corresponds to the n-th situation in Theorem 18.2.

Example 18.2

1. For $V = \{0, 1\}$ and $\mathfrak{h}_0 = \mathfrak{h}_1 = \mathbb{C}$, consider the CTOQW characterized by the following operators:

$$G_0 = G_1 = -\frac{1}{2}\,, \qquad R_0^1 = R_1^0 = 1 \ .$$

   Then the process $(X_t)_{t\geq 0}$ is a classical continuous Markov chain on $\{0, 1\}$, where the walker jumps from one site to the other after an exponential time of parameter 1.

2. For $V = \mathbb{Z}$ and $\mathfrak{h}_i = \mathbb{C}$ for all $i \in \mathbb{Z}$, consider the CTOQW described by the transition operators:

$$G_i = -\frac{1}{2}, \quad R_i^{i+1} = \frac{\sqrt{3}}{2}, \quad R_i^{i-1} = \frac{1}{2} \text{ for all } i \in \mathbb{Z} .$$

The process $(X_t)_{t\geq 0}$ is a classical continuous Markov chain on $\mathbb{Z}$ where after an exponential time of parameter 1, the walker jumps to the right with probability $\frac{3}{4}$ or to the left with probability $\frac{1}{4}$.

3. Consider the CTOQW defined by $V = \mathbb{N}$ with $\mathfrak{h}_1 = \mathbb{C}^2$ and $\mathfrak{h}_0 = \mathfrak{h}_i = \mathbb{C}$ for $i \geq 2$, and

$$G_0 = -\frac{1}{2}, \quad G_1 = -\frac{1}{2} I_2 ,$$

$$R_0^1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad R_1^0 = \begin{pmatrix} 0 & 1 \end{pmatrix}, \quad R_1^2 = \begin{pmatrix} 1 & 0 \end{pmatrix}, \quad R_2^1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

$$G_i = -\frac{1}{2}, \quad R_i^{i+1} = \frac{\sqrt{3}}{2}, \quad R_{i+1}^i = \frac{1}{2} \text{ for } i \geq 2 .$$

This is an example of positivity improving CTOQW where, for $\rho = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, one has $\mathbb{P}_{1,\rho}(\tau_1 < \infty) = 1$ but $\mathbb{P}_{i,\rho'}(\tau_i < \infty) < 1$ for any $\rho' \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. This example therefore exhibits “specifically quantum” behavior. It is inspired from [5].

18.4.2 Technical Results

Proposition 18.4 below is essential, as it expresses the probability of reaching a site in finite time as the trace of the initial state, evolved by a certain operator.

Proposition 18.4 *For any continuous-time open quantum walk, there exists a completely positive linear operator $\mathfrak{P}_{i,j}$ from $\mathcal{I}(\mathfrak{h}_i)$ to $\mathcal{I}(\mathfrak{h}_j)$ such that for every $i, j \in V$ and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$,*

$$\mathbb{P}_{i,\rho}(\tau_j < \infty) = \operatorname{Tr}\big(\mathfrak{P}_{i,j}(\rho)\big) .$$

Furthermore, the map $\mathfrak{P}_{i,j}$ can be expressed by:

$$\mathfrak{P}_{i,j}(\rho) = \sum_{n=0}^{\infty} \sum_{\substack{i_1,\dots,i_{n-1} \in V\setminus\{j\} \\ i_0=i, i_n=j}} \int_{0<t_1<\dots<t_n<\infty} R(\xi)\, \rho\, R(\xi)^* \mathrm{d}t_1 \dots \mathrm{d}t_n ,$$

where $\xi=(i_0,\dots,i_n;t_1,\dots,t_n)$ *and* $R(\xi)=R_{i_{n-1}}^{i_n}\,\mathrm{e}^{(t_n-t_{n-1})G_{i_{n-1}}}\,R_{i_{n-2}}^{i_{n-1}}\dots R_{i_0}^{i_1}\,\mathrm{e}^{t_1G_{i_0}}$.

Note that we do not require the $\mathfrak{h}_i$ to be finite-dimensional here.

Proof We have the trivial identity:

$$\mathbb{P}_{i,\rho}(\tau_j<t)=\sum_{n=0}^{\infty}\sum_{\substack{i_1,\dots,i_{n-1}\in V\setminus\{j\}\\ i_0=i,i_n=j}}\times\int_{0<t_1<\dots<t_n<t}\mathrm{Tr}\Big(\mathrm{e}^{(t-t_n)\mathcal{L}}\big(R(\xi)\,\rho\,R(\xi)^*\otimes|j\rangle\langle j|\big)\Big)\,\mathrm{d}t_1\dots\mathrm{d}t_n. \tag{18.18}$$

Then, since $\mathrm{e}^{(t-t_n)\mathcal{L}}$ is trace preserving,

$$\mathbb{P}_{i,\rho}(\tau_j<t)=\sum_{n=0}^{\infty}\sum_{\substack{i_1,\dots,i_{n-1}\in V\setminus\{j\}\\ i_0=i,i_n=j}}\int_{0<t_1<\dots<t_n<t}\mathrm{Tr}\big(R(\xi)\,\rho\,R(\xi)^*\big)\,\mathrm{d}t_1\dots\mathrm{d}t_n\ ,$$

and since both sides of the identity are nondecreasing in t, taking the limit $t\to+\infty$ yields

$$\mathbb{P}_{i,\rho}(\tau_j<\infty)=\sum_{n=0}^{\infty}\sum_{\substack{i_1,\dots,i_{n-1}\in V\setminus\{j\}\\ i_0=i,i_n=j}}\int_{0<t_1<\dots<t_n<\infty}\mathrm{Tr}\big(R(\xi)\,\rho\,R(\xi)^*\big)\,\mathrm{d}t_1\dots\mathrm{d}t_n\ .$$

It remains to show that $\mathfrak{P}_{i,j}$ is well defined. Let us denote by $(V_n)_{n\in\mathbb{N}}$ an increasing sequence of subsets of V such that $|V_n|=\min(n,|V|)$ and $\bigcup_{n\in\mathbb{N}}V_n=V$. For any $X\in\mathcal{I}(\mathfrak{h}_i)\setminus\{0\}$ write the canonical decomposition $X=X_1-X_2+\mathrm{i}X_3-\mathrm{i}X_4$ of X as a linear combination of four nonnegative operators. We get

$$\begin{aligned}
&\mathrm{Tr}\Big|\sum_{n=0}^{N}\sum_{\substack{i_1,\dots,i_{n-1}\in V_N\setminus\{j\}\\ i_0=i,i_n=j}}\int_{0<t_1<\dots<t_n<N}R(\xi)\,X\,R(\xi)^*\,\mathrm{d}t_1\dots\mathrm{d}t_n\Big|\\
&\le\sum_{m=1}^{4}\mathrm{Tr}\Big|\sum_{n=0}^{N}\sum_{\substack{i_1,\dots,i_{n-1}\in V_N\setminus\{j\}\\ i_0=i,i_n=j}}\int_{0<t_1<\dots<t_n<N}R(\xi)\,X_m\,R(\xi)^*\,\mathrm{d}t_1\dots\mathrm{d}t_n\Big|\\
&\le\sum_{m=1}^{4}\mathrm{Tr}\,X_m\times\sum_{n=0}^{N}\sum_{\substack{i_1,\dots,i_{n-1}\in V_N\setminus\{j\}\\ i_0=i,i_n=j}}\int_{0<t_1<\dots<t_n<N}\mathrm{Tr}\Big(R(\xi)\,\frac{X_m}{\mathrm{Tr}(X_m)}\,R(\xi)^*\Big)\\
&\quad\times\mathrm{d}t_1\dots\mathrm{d}t_n
\end{aligned}$$

$$\leq \sum_{m=1}^{4} \operatorname{Tr} X_m \times \mathbb{P}_{i,\frac{X_m}{\operatorname{Tr}(X_m)}}(\tau_j < N)$$

$$\leq \sum_{m=1}^{4} \operatorname{Tr} X_m$$

$$\leq 2\operatorname{Tr}|X|$$

(alternatively apply Theorem 5.17 in [31] to $X_1 - X_2$ and $X_3 - X_4$). Then

$$\sup_N \operatorname{Tr}\Bigg|\sum_{n=0}^{N} \sum_{\substack{i_1,\dots,i_{n-1}\in V_N\setminus\{j\}\\ i_0=i, i_n=j}} \int_{0<t_1<\dots<t_n<N} R(\xi)\, X\, R(\xi)^*\, \mathrm{d}t_1 \dots \mathrm{d}t_n\Bigg| < \infty\ .$$

Consequently, by the Banach–Steinhaus Theorem, the operator on $\mathcal{I}(\mathfrak{h}_i)$ to $\mathcal{I}(\mathfrak{h}_j)$ defined by

$$\mathfrak{P}_{i,j}(X) = \sum_{n=0}^{\infty} \sum_{\substack{i_1,\dots,i_{n-1}\in V\setminus\{j\}\\ i_0=i, i_n=j}} \int_{0<t_1<\dots<t_n<\infty} R(\xi)\, X\, R(\xi)^*\, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$

is everywhere defined and bounded. □

As a corollary, using the definition of the operator $\mathfrak{P}_{i,j}$ for $i, j \in V$, we obtain a useful expression for $\mathbb{E}_{i,\rho}(n_j)$:

Corollary 18.2 *For every $i, j \in V$ and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$, we have*

$$\mathbb{E}_{i,\rho}(n_j) = \sum_{k=0}^{\infty} \operatorname{Tr}\big(\mathfrak{P}_{j,j}^k \circ \mathfrak{P}_{i,j}(\rho)\big)\ . \tag{18.19}$$

Proof Let $i, j \in V$ and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$. Then

$$\mathbb{E}_{i,\rho}(n_j) = \int_0^\infty \mathbb{P}_{i,\rho}(X_t = j)\,\mathrm{d}t = \int_0^\infty \operatorname{Tr}\big(\mathrm{e}^{t\mathcal{L}}(\rho\otimes|i\rangle\langle i|)(\mathrm{Id}\otimes|j\rangle\langle j|)\big)\,\mathrm{d}t$$

$$= \operatorname{Tr}\Big(\sum_{n=0}^{\infty}\sum_{k=0}^{n} \sum_{\substack{m_1,\dots,m_k\in\mathbb{N}\\ m_1<\cdots<m_k=n}} \sum_{\substack{i_1,\dots,i_{m_1-1},i_{m_1+1},\dots,i_{m_k-1}\neq j\\ i_0=i,\, i_{m_1}=i_{m_2}=\cdots=i_{m_k}=j}} \times \int_{0<t_1<\cdots<t_n<t} \Upsilon\rho\Upsilon^*\,\mathrm{d}t_1\cdots\mathrm{d}t_n\,\mathrm{d}t\Big),$$

where $\Upsilon = R_{i_{n-1}}^{i_n} e^{(t_n - t_{n-1})G_{i_{n-1}}} R_{i_{n-2}}^{i_{n-1}} \cdots R_{i_{m_1}}^{i_{m_1+1}} e^{(t_{m_1+1}-t_{m_1})G_{i_{m_1}}} R_{i_{m_1-1}}^{i_{m_1}} \cdots R_{i_0}^{i_1} e^{t_1 G_{i_0}}$. The above expression corresponds to a decomposition of any path from i to j as a concatenation of a path from i to j, and k paths from j to j which do not go through j except at their start- and endpoints. This yields Eq. (18.19). □

The next corollary allows us to link the quantity $\mathbb{P}_{i,\rho}(\tau_j < \infty)$ to the adjoint of the operator $\mathfrak{P}_{i,j}$. In particular, as we shall see, it is a first step towards linking the properties of $\mathbb{P}_{i,\rho}(\tau_j < \infty)$ and $\mathbb{E}_{i,\rho}(n_j)$.

Corollary 18.3 *Let i and j be in V. One has*

$$\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1 \Leftrightarrow \mathfrak{P}_{i,j}^*(\mathrm{Id}) = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & * \end{pmatrix}$$

in the decomposition $\mathfrak{h}_i = \mathrm{Ran}\,\rho \oplus (\mathrm{Ran}\,\rho)^\perp$.

In particular, if there exists a faithful ρ in $\mathcal{S}_{\mathfrak{h}_i}$ such that $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$, then one has $\mathbb{P}_{i,\rho'}(\tau_j < \infty) = 1$ for any ρ' in $\mathcal{S}_{\mathfrak{h}_i}$.

Proof By Proposition 18.4, one has $\mathbb{P}_{i,\rho}(\tau_j < \infty) = \mathrm{Tr}\big(\rho\, \mathfrak{P}_{i,j}^*(\mathrm{Id})\big)$. Therefore, if $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$, then $\mathfrak{P}_{i,j}^*(\mathrm{Id})$ has the following form in the decomposition $\mathfrak{h}_i = \mathrm{Ran}\,\rho \oplus (\mathrm{Ran}\,\rho)^\perp$:

$$\mathfrak{P}_{i,j}^*(\mathrm{Id}) = \begin{pmatrix} \mathrm{Id} & A \\ A & B \end{pmatrix}.$$

Besides, the fact that $\mathrm{Id} \geq \mathfrak{P}_{i,j}^*(\mathrm{Id})$ forces A to be null. In particular, if ρ is faithful, then $\mathfrak{P}_{i,j}^*(\mathrm{Id}) = \mathrm{Id}$ and therefore $\mathbb{P}_{i,\rho'}(\tau_j < \infty) = 1$ for any ρ' in $\mathcal{S}_{\mathfrak{h}_i}$. □

18.4.3 Proof of Theorem 18.2

Let i and j be in V. As we can see in Corollary 18.3, if we suppose that $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$ for a faithful density matrix ρ, we necessarily have $\mathfrak{P}_{i,j}^*(\mathrm{Id}) = \mathrm{Id}$. This will be used in the following proposition, which in turn explains the statement regarding non-faithfulness in the third category of Theorem 18.2.

Proposition 18.5 *Let i be in V. If there exists a faithful ρ in $\mathcal{S}_{\mathfrak{h}_i}$ such that $\mathbb{P}_{i,\rho}(\tau_i < \infty) = 1$, then one has $\mathbb{E}_{i,\rho'}(n_i) = \infty$ for any ρ' in $\mathcal{S}_{\mathfrak{h}_i}$.*

Proof We set $\tau_1^i = \tau_i$ and, for all $n > 1$, we define $\tau_i^{(n)}$ as the time at which $(X_t)_{t\geq 0}$ reaches i for the n-th time:

$$\tau_i^{(n)} = \inf\{t > \tau_{n-1}^i \,|\, X_t = i \text{ and } X_{t-} \neq i\}.$$

From Corollary 18.3, one has $\mathbb{P}_{i,\rho'}(\tau_i < \infty) = 1$ for all ρ' in $\mathcal{S}_{\mathfrak{h}_i}$. This implies that for all $n > 0$, $\tau_i^{(n)}$ is $\mathbb{P}_{i,\rho'}$-almost finite for any $\rho' \in \mathcal{S}_{\mathfrak{h}_i}$. For $n \geq 0$, let T_n^i be the occupation time in i between $\tau_i^{(n)}$ and $\tau_i^{(n+1)}$:

$$T_n^i = \inf\{u > 0 \,|\, X_{\tau_i^{(n)}+u} \neq i\}$$

with the convention that $\tau_i^{(0)} = 0$. Since we have

$$\mathbb{E}_{i,\rho'}(n_i) \geq \mathbb{E}_{i,\rho'}\Big(\sum_{n\geq 1} T_n^i\Big) \geq \sum_{n\geq 1} \inf_{\hat\rho\in\mathcal{S}_{\mathfrak{h}_i}} \mathbb{E}_{i,\hat\rho}(T_n^i)\ ,$$

it will be enough to obtain a lower bound for $\mathbb{E}_{i,\hat\rho}(T_n^i)$ which is uniform in n and in $\hat\rho$. To this end, we use the quantum trajectories defined in (18.10). We first compute $\mathbb{P}_{i,\hat\rho}(T_n^i > t)$ for all $t \geq 0$. To treat the case of $n = 1$ we consider the solution of

$$\eta_t^{\hat\rho} = \hat\rho + \int_0^t \big(G_i\, \eta_s^{\hat\rho} + \eta_s^{\hat\rho}\, G_i^* - \eta_s^{\hat\rho}\, \mathrm{Tr}(G_i\, \eta_s^{\hat\rho} + \eta_s^{\hat\rho}\, G_i^*)\big)\, \mathrm{d}s\ . \tag{18.20}$$

Using the independence of the Poisson processes $N^{i,j}$ involved in (18.10) we get

$$\begin{aligned}
\mathbb{P}_{i,\hat\rho}(T_1^i > t) &= \mathbb{P}_{i,\hat\rho}\big(\text{no jump has occurred before time t}\big)\\
&= \mathbb{P}_{i,\hat\rho}\Big(N^{i,j}\big(\{u, y\,|\,0\leq u\leq t,\ 0\leq y\leq \mathrm{Tr}(R_i^j\,\eta_u^{\hat\rho}\,R_i^{j\,*})\}\big) = 0\ \forall j\neq i\Big)\\
&= \prod_{j\neq i} \mathbb{P}_{i,\hat\rho}\Big(N^{i,j}\big(\{u, y\,|\,0\leq u\leq t,\ 0\leq y\leq \mathrm{Tr}(R_i^j\,\eta_u^{\hat\rho}\,R_i^{j\,*})\}\big) = 0\Big)\\
&= \prod_{j\neq i} \exp\big(-\int_0^t \mathrm{Tr}(R_i^j\,\eta_s^{\hat\rho}\,R_i^{j*})\,\mathrm{d}s\big)\\
&= \exp\Big(\int_0^t \mathrm{Tr}\big((G_i + G_i^*)\,\eta_s^{\hat\rho}\big)\,\mathrm{d}s\Big)
\end{aligned} \tag{18.21}$$

where we used relation (18.6). Similarly, using the strong Markov property,

$$\begin{aligned}
\mathbb{P}_{i,\hat\rho}(T_n^i > t) &= \mathbb{E}_{i,\hat\rho}(\mathbb{1}_{T_n^i>t})\\
&= \mathbb{E}_{i,\hat\rho}\big(\mathbb{E}_{i,\rho_i^{(n)}}(\mathbb{1}_{T_1^i>t})\big)\ \text{ where } \rho_i^{(n)} := \rho_{\tau_i^{(n)}}\\
&= \mathbb{E}_{i,\hat\rho}\Big(\exp\big(\int_0^t \mathrm{Tr}\big((G_i + G_i^*)\,\eta_s^{\rho_i^{(n)}}\big)\,\mathrm{d}s\big)\Big)\\
&\geq \mathrm{e}^{-t\|G_i+G_i^*\|_\infty}\ .
\end{aligned}$$

Now, using the fact that $\mathbb{E}_{i,\hat{\rho}}(T_n^i) = \int_0^\infty \mathbb{P}_{i,\hat{\rho}}(T_n^i > t)\,\mathrm{d}t$, this gives us the expected lower bound:

$$\mathbb{E}_{i,\hat{\rho}}(T_n^i) \geq \frac{1}{\|G_i + G_i^*\|_\infty} .$$

This concludes the proof. □

The next proposition is connected to the first point of Theorem 18.2.

Proposition 18.6 *Consider a semifinite irreducible continuous-time open quantum walk. If there exist i, j in V and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$ such that $\mathbb{E}_{i,\rho}(n_j) = \infty$, then one has $\mathbb{P}_{j,\rho'}(\tau_j < \infty) = 1$ for any ρ' in $\mathcal{S}_{\mathfrak{h}_j}$.*

Proof By Proposition 18.2, there is no nontrivial invariant subspace of $\mathfrak{h}_j$ left invariant by $R(\xi)$ for all $\xi \in \mathcal{P}(j, j)$. Since any such ξ is a concatenation of paths from j to j that remain in $V \setminus \{j\}$ except for their start- and endpoints, there is also no nontrivial projection P_j of $\mathfrak{h}_j$ such that $\mathfrak{P}_{j,j}(P_j\mathcal{I}_1(\mathfrak{h}_j)P_j) \subset P_j\mathcal{I}_1(\mathfrak{h}_j)P_j$ (where $\mathfrak{P}_{j,j}$ is the operator of Proposition 18.4). The latter is therefore a completely positive irreducible map acting on the set of trace-class operators on $\mathfrak{h}_j$. By the Russo–Dye Theorem (see [26]), one has $\|\mathfrak{P}_{j,j}\| = \|\mathfrak{P}_{j,j}^*(\mathrm{Id})\| \leq 1$, so that the spectral radius λ of $\mathfrak{P}_{j,j}$ satisfies $\lambda \leq 1$. By the Perron–Frobenius Theorem of Evans and Hoegh-Krøhn (see [15] or alternatively Theorem 3.1 in [27]), there exists a faithful density matrix ρ' on $\mathfrak{h}_j$ such that $\mathfrak{P}_{j,j}(\rho') = \lambda\rho'$. If $\lambda < 1$, then by Corollary 18.2 one has $\mathbb{E}_{j,\rho'}(n_j) < \infty$, but then Proposition 18.3 contradicts our running assumption that $\mathbb{E}_{i,\rho}(n_j) = \infty$. Therefore $\lambda = 1$ and ρ' is a faithful density matrix such that $\mathbb{P}_{j,\rho'}(\tau_j < \infty) = \mathrm{Tr}\big(\mathfrak{P}_{j,j}(\rho')\big) = \mathrm{Tr}(\rho') = 1$. We then conclude by Corollary 18.3. □

Proposition 18.7 *Consider a semifinite irreducible continuous-time open quantum walk; if there exists $i \in V$ such that for all $\rho' \in \mathcal{S}_{\mathfrak{h}_i}$ one has $\mathbb{P}_{i,\rho'}(\tau_i < \infty) = 1$, then $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$ for any $j \in V$ and $\rho \in \mathcal{S}_{\mathfrak{h}_i}$.*

Proof Fix i and j in V. Observe first that, by irreducibility, for any ρ in $\mathcal{S}_{\mathfrak{h}_i}$, there exists

$$\xi = (i = i_0, i_1, \ldots, i_{n-1}, i_n = j; t_1, \ldots, t_n)$$

such that $\mathrm{Tr}\big(R(\xi)\rho R(\xi)^*\big) > 0$. We denote by $t(\xi)$ the element t_n of ξ. Using the continuity of $\mathrm{Tr}\big(R(\xi)\rho R(\xi)^*\big)$ in ρ and the compactness of $\mathcal{S}_{\mathfrak{h}_i}$, we obtain a finite family $\xi_1, \ldots, \xi_p$, of paths, again going from i to j, such that

$$\inf_{\rho\in\mathcal{S}_{\mathfrak{h}_i}} \max_{k=1,\ldots,p} \mathrm{Tr}\big(R(\xi_k)\rho R(\xi_k)^*\big) > 0 .$$

Let $\delta > \max_{k=1,\dots,p} t(\xi_k)$. By continuity of each $\mathrm{Tr}\big(R(\xi_i)\rho R(\xi_i)^*\big)$ in the underlying jump times $t_1, \dots, t_n$ and using expression (18.18), we have

$$\alpha := \inf_{\rho \in \mathcal{S}_{\mathfrak{h}_i}} \mathbb{P}_{i,\rho}(\tau_j \leq \delta) > 0 \,.$$

Now, if $\mathbb{P}_{i,\rho}(\tau_i < \infty) = 1$ for all ρ in $\mathcal{S}_{\mathfrak{h}_i}$, then the discussion in Sect. 18.2.3 implies that almost-surely one can find an increasing sequence $(\tau_{i,n})_n$ of times with $\tau_{i,n} \to \infty$ and $x_{\tau_{i,n}} = i$. Choose a subsequence $(\tau_{i,\varphi(n)})_n$ such that $\tau_{i,\varphi(n)} - \tau_{i,\varphi(n-1)} > \delta$ for all n. Since never reaching j means in particular not reaching j between $\tau_{i,\varphi(n)}$ and $\tau_{i,\varphi(n+1)}$ for $n = 1, \dots, k$, the Markov property of $(X_t, \rho_t)_{t\geq 0}$ and the lower bound $\tau_{i,\varphi(n)} - \tau_{i,\varphi(n-1)} > \delta$ imply that for all $\rho \in \mathcal{S}_{\mathfrak{h}_i}$,

$$\begin{aligned}
\mathbb{P}_{i,\rho}(\tau_j = \infty) &\leq \mathbb{P}_{i,\rho}\big(\forall n \in \{0, \dots, k\},\ \forall t \in [\tau_{i,\varphi(n)}, \tau_{i,\varphi(n+1)}]\,,\ X_t \neq j\big) \\
&\leq \big(\sup_{\rho \in \mathfrak{h}_i} \mathbb{P}_{i,\rho}(\tau_j > \delta)\big)^k \\
&\leq (1-\alpha)^k \,.
\end{aligned}$$

Since the above is true for all k, we have $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$. □

Now we combine all the results of Sect. 18.4.2 to prove Theorem 18.2.

Proof (Proof of Theorem 18.2) Proposition 18.3 shows that either $\mathbb{E}_{i,\rho}(n_j) = \infty$ for all i, j and ρ, or $\mathbb{E}_{i,\rho}(n_j) < \infty$ for all i, j and ρ. Proposition 18.6 combined with Proposition 18.7 shows that in the former case, $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$ for all i, j and ρ as well. Proposition 18.5 shows that, in the latter case, $\mathbb{P}_{i,\rho}(\tau_j < \infty) = 1$ may only occur for non-faithful ρ, and this concludes the proof. □

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