The Contribution of Information and Communication Technology to the Teaching of Proof

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1 Introduction

Research on proof and proving in mathematics education has been carried out for a long time. It began with criticizing old models of teaching proof (Herbst, 2002) for their inefficiency in understanding the role of proof in mathematics and the development of students' skills in producing conjectures and constructing proofs. In many countries the reaction to this criticism led to abandoning the practice of proof at school, sometimes getting rid of theorems, and in general reducing the importance of proving in secondary school curricula. Nevertheless, studies in the field of teaching and learning proof have opened new perspectives and generated a lively stream of research. On the one hand, research has focused on epistemological reflections about the relationship between proof and proving in mathematical practice and education (e.g., Hanna, 1989; Hanna & Janke, 2007; Duval, 2007; Balacheff, 2008 Hanna, Jahnke, & Pulte, 2009). On the other, it has focused on student practices related to proof and proving (e.g., Harel & Sawder, 1998) from both a cognitive and didactic point of view. An overview and detailed discussion of these issues can be found in Mariotti (2006), Reid and Knipping (2010), Hanna and De Villiers (2012), and Stylianides, Bieda and Morselli (2016).[1]

Most of the new proposals are based on short-term experiments limited to a very specific proof task (see, for instance, Miyazaki, Fujita, & Jones, 2015). Only a few, such as the examples found in Boero (2007), have been rooted in long-term design-based experiments. What I present here are some findings from teaching experiments that investigate the use of ICT to introduce secondary school students to proof and proving and, more generally, to developing specific mathematical meanings related

[1] A rich source of references can be found at http://lettredelapreuve.org.

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to proof and proving. Some of these experiments were carried out by the author, sometimes in collaboration with other colleagues. Some of them lasted for years and involved different classes for a whole academic year while others were more circumscribed, involving individuals or pairs of students. The interviews aimed at observing students' behaviours in great detail. In elaborating on this wide base of findings, my objective is not to provide an overview of the rich research on proof in relation to technological settings, but to illustrate the potentials of a specific kind of software environment for fostering a sense of proof and, more widely, a theoretical perspective. Some of the results presented here have been published elsewhere; for instance, in Mariotti (2012, 2014) and Baccaglini-Frank, Antonini, Leung and Mariotti (2018). In the following, I offer a synthesis of contributions presented in the past years based on the unifying lens of the theory of semiotic mediation (TSM) and, specifically, the construct of the semiotic potential of an artefact.

In the first section, I introduce the educational and epistemological perspectives that shape my discussion. I use the theoretical construct of semiotic potential in order to explain why and how certain designed activities in a Dynamic Geometry Environment[2] (DGE) may contribute to the construction of specific meanings constituting the mathematical meaning of proof. After introducing a specific characterization of a theorem that sheds light on the mathematical meaning of proof I start to discuss the potentialities offered by a DGE. I firstly elaborate on construction problems discussing how the solution of such problems may be related to the key meanings of axioms and theorems. In the following sections, I discuss how open construction problems and the related conjecturing process may give sense to the notion of conditional statement when carried out in a DGE, offering a context to relate spontaneous arguments to mathematical proof. The last issue concerns the potential offered by exploring impossible figures in a DGE with respect to indirect proof.

2 Theoretical Background

In this section, I discuss the theoretical foundations of my research on the use of ICT to foster student engagement with mathematical proof. These foundations include an educational perspective based on the theory of semiotic mediation, an epistemological perspective based on key mathematical meanings related to proof, and the notion of mathematical theorem.

[2]Following Sinclair and Robutti (2012), I use the term "dynamic geometry environment." As the authors write, since at least 1996, this term has been used over *dynamic geometry software* "to underscore the fact that we are dealing with microworlds (including pre-existing sketches and designed tasks) and not just a software program." (p. 571).

2.1 Educational Perspective

The Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008; Mariotti, 2009) combines a semiotic and an educational perspective. It elaborates on the Vigotskian notion of semiotic mediation, which considers the role of human mediation in the teaching-learning process as crucial. Starting from the key notion of artefact,[3] TSM interprets the teaching and learning process through a semiotic lens. It focusses on students' production of signs and the evolution of these signs from personal meanings emerging from the use of an artefact to the mathematical meanings that are the goal of teaching. A basic assumption is that personal meanings that emerge in accomplishing a task may be related to specific mathematical meanings, but also that such a relationship cannot be taken for granted. On the contrary, the *intentional* intervention of the teacher is needed to promote students' conscious construction of this relationship in social interaction. According to a Vygotskian approach, TSM sees both the individual production of signs and their collective elaboration within social activities as fundamental, and in particular within mathematical discussions (Bartolini & Mariotti, 2008).

In the past years, several long-term teaching experiments have been conducted to check, refine, and elaborate our assumptions. Different artefacts were involved, either concrete or digital, as well different school levels (see Bartolini Bussi, 1996; Mariotti, 2007, 2010). The theoretical framework of TSM originated and developed around two key elements: the notion of the semiotic potential of an artefact and the notion of a didactic cycle (Bartolini Bussi & Mariotti, 2008). Here I focus on the notion of @@@ semiotic potential which will be used in the following discussion.

When an artefact is used to accomplish a task, it may happen that an expert—a mathematician—recognizes the echo of specific mathematical notions. For instance, the use of an abacus may immediately bring to mind the mathematical notion of positional notation and the polynomial notation of numbers. As I discuss later, drawing a figure in a dynamic geometry system may evoke the classic notion of geometric construction by ruler and compass. However, if the user is not an expert, the meanings emerging from use of the artefact may not be immediately and consciously related to mathematical meanings. Instead, they are related to the specific context and the specific individual. They are 'personal meanings.'

To express the double relation linking the artefact and its use with, on the one hand, possible personal meanings and, on the other, with mathematical meanings, TSM introduces the notion of @@@semiotic potential (Bartolini Bussi & Mariotti, 2008, p. 754). This notion also expresses the double use of an artefact in an educational context. The artefact is used by the students to accomplish a task, but is simultaneously used by the teacher to exploit its semiotic potential and foster students' mathematical meanings. Consequently, analysing the semiotic potential of an artefact is at the core of designing and teaching a sequence of lessons. Such an analysis involves both cognitive and epistemological aspects. The former identifies

[3]The term artefact refers to any generic product of human culture purposefully designed to act or interact in a human setting.

meanings that can emerge in accomplishing a given task while the latter identifies the possible mathematical meanings evoked. In this chapter, I use the notion of semiotic potential to illustrate and discuss the educational potential of a DGE with respect to mathematical meanings related to proof.

2.2 Epistemological Perspective

Proof is one of the key elements of mathematics. It is the product of a process of validation that allows the inclusion of any new statement in a specific theory given that such a statement can be logically derived from the set of axioms previously assumed. Such a formal perspective (Arzarello, 2007) makes a proof independent of any interpretation and factual verification of the statements involved. In this respect, the specificity of proof contrasts with argumentation and any action or process of reasoning aimed at convincing others (or oneself) that something is true or false.

As Duval clearly stated (2007), argumentation and proof must be distinguished. Further, a cognitive gap might exist between the two processes in spite of their possible contiguity. The epistemological gap concerns the unbridgeable distance between the semantic level, where the interpretation of any statement finds reasons for its acceptability, and the theoretical level where the theoretical validity of a statement must be stated in accordance with laws of logic within a *hypothetical–deductive* approach. The cognitive gap concerns the distinction between the main function of argumentation—convincing oneself and others that a statement is true—and the main function of mathematical proof—logical validation of a statement within a specific theory.

According to this analysis, the educational challenge concerns the possibility of mistaking the two processes. In principle, a mathematical proof should not refer to any interpretation of the statement involved. However, it is not realistic for mathematicians or students to think that such interpretation does not play a crucial role in both producing and/or accepting a proof. Indeed, it is the interpretation given to the statements that determines the final epistemic value attributed to the statement that has been proved. Moreover, it is on the semantic plane that the explicative function any proof is expected to provide is based (Hanna, 1989). The semantic plane develops understanding of 'why' what has been proved is true (Dreyfus & Hadas, 1996).

In summary, the didactic question of proof requires resolving the potential conflict between the two main functions of proof: theoretically validating, and explaining why. This means developing a teaching intervention that enables students to develop a coherent intertwining of argumentation and proof, though preserving their specificity. In this perspective, in the following I elaborate a bit more on the notion of proof.

The term proof is often used, both in the current literature and in textbooks, without any clear reference to the other key elements involved. As said above, distinguishing argumentation from mathematical proof is based on differentiating their aims: on the one hand, stating the epistemic value of a statement and, on the other, validating a statement with a theory. Though these two aims may overlap, no clear idea of mathematical proof is possible without explicitly linking it to the idea of *theory*—both the theoretical system defined by the axioms, definitions, and already proven theorems and the meta-theoretical system of the inference rules stating what is meant by *logically derived*.

Very often, when discussing the issue of proof, we take the perspective of mathematics experts and leave reference to a statement and a theory implicit. However, if we take the students' point of view, we realize that neither of these perspectives can be taken for granted and the complexity of the notion of theory cannot be underestimated. On the one hand, meanings must be developed in relation to the status and role of the different statements involved in a proof. That is, the mathematical meaning of terms like theory, axioms, definitions, and theorems must be developed. On the other hand, consciousness of the means of supporting any single step of a proof—the specific 'logical means' that can be used to validate a new statement—must also be developed. The centrality of the latter has been clearly pointed out by Sierpinska (2005):

> Theoretical thinking asks not only, Is this statement true? but also What is the validity of our methods of verifying that it is true? Thus theoretical thinking always takes a distance towards its own results. […] theoretical thinking is thinking where thought and its object belong to distinct planes of action. (pp. 121–23)

In the school context, the complexity of this meta-theoretical level seems to be ignored. It is commonly taken for granted that students' ways of reasoning are spontaneously adaptable to the sophisticated functioning of a theoretical system. Therefore, not much is said about it, and inference rules in particular and their functioning in the development of a theory are rarely made explicit.[4]

In fact, at least two aspects at the meta-level should be made explicit and discussed in the classroom: (1) the acceptability of some specific inference means, and (2) the fact that, except for those explicitly shared, no other inference means are acceptable. If meta-theoretical aspects remain implicit, students have no control of their arguments. Control remains totally in the hands of the teacher, with the consequence that students feel confusion, uncertainty, and lack of understanding. Awareness of a reference theory as a system of shared principles and inference rules is needed if we are to speak of proof in a mathematical sense. Indeed, "what characterises a Mathematical Theorem is the system of statement, proof, and theory" (Mariotti et al. 1997, p. 182).

Developing the interrelated meanings of the three components of the notion of Mathematical Theorem therefore becomes a crucial pedagogical objective. In the following, I discuss how teaching can be designed to address this objective.

[4]An exception is that of mathematical induction. But mathematical induction is very rarely presented in comparison to other modalities of proving, which are commonly considered natural and spontaneous ways of reasoning.

3 Introducing Students to Theorems

For some time different research studies have highlighted the potentials and pitfalls (de Villliers, 1998) of DGEs in offering powerful resources for introducing students to proof. As Hadas, Hershkowitz and Schwarz (2000) have pointed out:

> [The] findings concerning the failure to teach proofs, the recognition of the multiple aspects of proving, and the existence of DG tools lead naturally to the design of investigative situations in which DG tools may foster these multiple aspects. (p. 130)

In the following, I describe the potentials of a DGE in relation to geometrical construction and situate it within the theoretical framework of TSM; that is, in terms of semiotic potential.

3.1 Geometrical Construction in a DGE

Let us start from the relationship, immediately evoked in the mind of any mathematician, between drawing a figure in a DGE and the mathematical meaning of geometrical construction; that is, drawing a figure by ruler and compass. In terms of TSM, such a relationship can be articulated through both an epistemological and cognitive analysis, and leads to outlining the semiotic potential of the artefact 'DGE' with respect to the meaning of Theorem.

Euclidean geometry is traditionally referred to as 'ruler-and-compass geometry.' However, despite referring to a concrete objective—e.g., producing a graphic trace on a sheet of paper or other surface—a geometrical construction has a pure theoretical nature. Solving a construction problem corresponds to proving a theorem that validates the construction procedure (Mariotti, 2007).

The use of ruler and compass generates a set of axioms defining the theoretical system of Euclid's Elements. To appreciate the key role played by construction problems in Euclidean Geometry, it suffices to remember that the very first proposition of the first book of the Elements deals with the construction of an equilateral triangle, and that the solution to the long puzzling problem of trisecting an angle was definitely proved impossible to solve by ruler and compass. The constructability or non-constructability of a figure has been a central issue in mathematics (Arzarello et al., 2012). Although, as classic research studies have shown (Schoenfeld, 1985), the theoretical meaning of geometrical construction is complex and difficult to grasp, the centrality of its role in the history of geometry and the revival triggered by the advent of DGE make it worthy of consideration.

On the one hand, the use of virtual tools simulates the concrete use of traditional tools like the ruler and compass. On the other, the digital architecture of a DGE embedding the theoretical framework of Euclidean Geometry (Laborde & Sträßer, 1990) enables the user to implement the logical relationships between the geometrical properties constructed by the tools and the geometrical properties that are their consequences. Moreover, any DGE offers a dragging modality which represents the core of the technological environment. The dragging modality allows the user to move any constructed figure after clicking and dragging one of its basic points. After a selected point has been dragged, the figure on the screen is redrawn and recalculated from the subsequent new positions, but maintains all the properties defined by the constructing procedure. As a consequence, the stability of dragging constitutes the standard test of correctness for any drawn figure. Thus, a solution is acceptable if and only if the figure on the screen is stable under the dragging test. Because any DGE embodies a system of relationships consistent with the broad system of a geometrical theory, solving construction problems in a DGE means not only accepting all the facilities of the software, but also accepting a logic system within which to make sense of the geometrical phenomena that occur in that environment.

A dynamic figure behaves according to its intrinsic logic: its elements are related by the hierarchical relationships stated by the constructing procedure. Such a hierarchy corresponds to a relationship of logical dependence among the properties in the sense that the final figure will show not only the constructed properties, but also all the properties that can be derived from them according to Euclidean Geometry. Specific tools on the DGE menu correspond to a set of theoretical construction tools in Euclidean Geometry (Laborde & Laborde, 1991). This makes it possible to state a correspondence between the control of dragging (dragging test) and validation by theorems (e.g., validation by mathematical proof within Euclidean Geometry theory).[5]

3.2 The Semiotic Potential of DGE Construction Tools

Interpreting the previous analysis in terms of semiotic potential we can recognize a double relationship between some tools of a DGE and, on the one hand, meanings emerging from their use in solving a construction task and, on the other, specific mathematical meanings related to the notion of Theorem. Specific construction tools can be related to a virtual dynamic drawing representing a geometrical figure whose acceptability as a solution of the construction problem can be controlled by checking its stability by dragging. At the same time, the use of these specific construction tools may evoke specific geometrical axioms and theorems that can be used for validating the construction procedure within Euclidean Geometry theory. In other words, the solution of a construction problem within a DGE can evoke the theoretical meaning of geometrical constructions. Exploiting the semiotic potential of a DGE may thus lead to developing the mathematical meaning of MT and specifically the meanings of proof referring to a particular theory.

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[5] Actually a DGE provides a larger set of tools, including for instance “measure of an angle,” “rotation of an angle,” and the like. This implies that the whole set of possible constructions does not coincide with that attainable only with ruler and compass. See Stylianides and Stylianides (2005) for a full discussion.

This was the design principle of a number of the long-term teaching experiments I conducted. It involved developing a sequence of didactic cycles using specific construction tools and semiotic activities aimed at the individual and social elaboration of signs (see Mariotti, 2001, 2009). As explained above, the semiotic potential of an artefact concerns the relationship between the meanings emerging from the activities with the artefact and the mathematical meanings evoked.

A *construction task* consists of:

- producing a DGE figure that should be stable by dragging;
- writing a description of the procedure used to obtain the DGE figure and producing a validation of the 'correctness' of such a procedure.

Thus a construction task consists of two types of requests. The first asks for interaction with the artefact, the second for producing a written text referring to the interaction. The request for *validating* the correctness of the procedure acquires its meaning in relation to the DGE environment: the construction problem is solved if the figure obtained on the screen passes the dragging test. Validating such a construction means explaining and gaining insight into the reason why it passes the test.

In the framework of TSM, students' development of a theoretical perspective can be witnessed by the evolution of the sign "construction." At the beginning, the term construction makes sense only in relation to using particular tools to draw a DGE figure and having that figure pass the dragging test. Later on, the meaning of the term construction acquires the theoretical meaning of geometrical construction (Mariotti, 2001) validated by a proof within a geometry theory. In other words, the evolution of meanings, accomplished in the mathematical discussion led by the teacher, occurs through the elaboration of a correspondence between specific DGE tools and their modes of use on the one hand, and Euclidean axioms and derived theorems and definitions on the other.

At the very beginning, starting from an empty menu, students are invited to discuss the choice of appropriate tools to introduce in the menu. At the same time a corresponding set of @@@construction axioms are formulated and stated as the first core of the geometry theory that any validation should refer to. I want to stress the power of the semiotic potential of a DGE with respect to the possibility of selecting the tools that are available; in other words, the semiotic potential that the artefact "*available menu*" has with respect to the mathematical meaning of theory and specifically to the property of growth—adding new theorems and definitions—that is crucial to understanding the hypothetical–deductive structure of any mathematical theory. As the results of a number of teaching experiments showed, students not only produced new statements and their proofs, but also became aware of the theory within which the proofs made sense, and how such a theory is developed. As long as new problems are solved and new constructions are produced, the corresponding theorems can be validated, added to the set of shared validating principles, and reported in students' notebooks. The students participated in two parallel processes of evolution: the enlargement of the available menu in the DGE and the corresponding development of a geometry theory.

In summary, a DGE offers a rich and powerful context for introducing students to a theoretical perspective. It provides an environment for phenomenological experiences of the mathematical meanings of:

- axioms that correspond to the use of specific construction tools
- geometrical theorems that validate specific geometrical constructions
- meta-theoretical actions related to the development of the theory by adding new theorems and definitions.

Experiences in the classroom over the course of our teaching experiments confirmed both the unfolding of the semiotic potential and the evolution of an interlaced sense of proof and theory. Different aspects of this evolution are presented and discussed in several papers (Mariotti, 2001, 2007, 2009). The following two excerpts are examples from our findings that show the theoretical meaning of construction and its relationship with the mathematical meaning of theorem.

Example. The theorem of the angle bisector.
The first excerpt shows a student's answer to the task of constructing the angle bisector of an angle using only specific DGE tools such as line, ray, segment (point, point), and compass (point, segment). The students had already learned the correspondence between the use of these tools and the classic three criteria of congruence that constitute the germ of the available theory.

Excerpt 1
Max produces a stable figure in the DGE and the list of the construction steps. Then he writes:

> Prove that the angle bisector by construction is an angle bisector by congruence criterion (ita. criterio di uguaglianza)
>
> AB = AC by circle
>
> AO is in common
>
> OB = OC by circle
>
> center A and B
>
> The two triangles are equal because of the third criterion of congruence (Δ ABO = Δ AOC)
>
> Equal sides correspond to equal angles and thus
>
> $<$ OAC = $<$ BAO
>
> AO is the angle bisector of BAC
>
> As was to be proved.

From the point of view of mathematics, this text is still very rough. However, it is possible to recognize the germ of a proof and, overall, to see how the student is explicitly relating the construction steps to the theoretical elements available. What is particularly significant is the reformulation of the task at the beginning of the proof text ("Prove that the angle bisector…"). It demonstrates the student's need to anticipate interpreting the list of 'theoretical statements' according to the construction.

Excerpt 2
After a first sequence of activities, the teacher opened a collective discussion with the aim of revising the students' personal notebooks. From a comparison of the notebooks, the teacher then guided a mathematical discussion on ordering the sequence of the theoretical elements and giving them the right status: are they axioms, theorems, or definitions?

After the discussion, each student was asked to write a report on the activity. During the discussion some time was devoted to the construction of angle bisectors and the proof of the corresponding "*bisector theorem*." Different proofs were proposed based on applying different theorems. Traces of this part of the discussion can be found in the following report by another student, Stefano. It shows how he grasped the sense of theory both in terms of conventionality and a logically ordered system of statements. He writes:

> We then switched to examine the proof of the *bisector theorem*. One of my classmates stated that the *bisector theorem* could be proved also with the isosceles triangle, but to do that we would have needed to have the last theorem concerning the perpendicular. If I say that even having the theorem, we couldn't use it, it doesn't mean that we are fools but simply that when we began [the proof] we didn't have it, and our means for proving were in minor quantity.

In commenting on the intervention of one of his classmates, Stefano explicitly states the need for a proof to refer to the theory available.

4 More About the Semiotic Potential of a DGE

In this section I elaborate a bit more on the potential of a DGE with respect to the notion of mathematical theorem. In particular, I consider the semiotic potential offered by dragging in relation to the third component of MT—the statement.

Difficulties often arise in the interpretation of a given statement to be proved. These difficulties concern the meaning of the premise and the conclusion, as well the meaning of the logical dependency between them. Not many studies have been devoted to this specific issue. However, an interesting exception is the work of Selden and Selden (1995) which discussed the specific phenomenon of "unpacking an informal statement." This refers to the challenge that students often face of making the formal elements—for instance, the logical quantifiers—of the statement to be proved explicit.

This difficulty has a parallel in the challenges students face when asked to formulate a conjecture in the form of a conditional statement—"if … then …" (Boero, Garuti, & Lemut, 1999). The failure to manage conditionality and to grasp the different status of premises and conclusions may be a true obstacle to developing a correct meaning of MT. Developing the mathematical meaning of a conditional statement can therefore be considered a crucial issue in the general context of developing the meaning of MT.

In the current literature, there is a shared opinion about the fundamental role that open problems and conjecturing activities play in developing a sense of proof and fostering a productive relationship between 'spontaneous' argumentation processes and theoretical validation (Arsac & Mante, 1983; Arsac, 1992; Pedemonte, 2002). Different contexts allow for open questions in different ways, thereby offering different potentials for posing and solving open problems and, consequently, for formulating conjectures. In the following section, I focus on conjecturing tasks and the very particular context of Dynamic Geometry. Specifically, I illustrate the semiotic potential of specific dragging modalities performed in a DGE context while solving conjecturing tasks.

Previous studies carried out by Boero and his colleagues focused on different aspects of students' real world experience, but showed how dynamic aspects of the phenomena under investigation were fundamental. Their studies confirmed what other studies claimed (Simon, 1996; Harel & Sawder, 1998)—that dynamicity seemed to foster transformational mental processes that are key to producing conditional statements. The formulation of a conjecture can be described as a "crystallization" of a dynamic exploration—a specific moment, and a specific position, when the occurrence of one fact in a conditional statement has the occurrence of another fact as a consequence (Boero et al., 1999, 2007). This makes it reasonable to address the role of modes of dragging in conjecture production and to consider the solution of conjecturing open problems in a DGE.

4.1 Conjecturing in a DGE: Dragging as a Semiotic Mediator of Conditionality

I use the term 'conjecture open problem' in the following to refer to a task that explicitly asks the solver to formulate a conjecture (Mariotti, 2014). This is a very common case in geometry and involves asking the solver to formulate a conditional statement expressing a possible logical dependency between the geometrical properties of a given configuration. In a DGE, preliminary explorations are expected that involve, firstly, the construction of a dynamic figure implementing the initial configuration and, then, active transformations of the figure in search of a possible answer. This means that while observing the dynamic image on the screen, the solver has to interpret the perceptual data coming from the screen and transform them into geometrical properties that formulate a statement expressing a conditional relationship between the properties.

Several studies done on students' exploration processes show both different dragging modalities and the potential of such modalities in assisting the conjecturing process (Arzarello, Olivero, Paola, & Robutti, 2002; Olivero, 2003; Hölzl, 1996; Leung & Lopez-Real, 2002; Lopez-Real & Leung, 2006). Elaborating on these results, it is possible to outline the semiotic potential of particular modalities of dragging with respect to the mathematical meaning of *conditional statement* in a geometry context. Dragging modalities can be considered as specific artefacts used to solve an open problem, and the meanings emerging from their use can be related to the mathematical meanings of premise, conclusion, and the logical dependency between them.

4.2 Invariants by Dragging and Their Relationship

The notion of *invariant* by dragging is at the core of any DGE. As discussed above, when a figure is acted upon, two kinds of properties simultaneously appear as invariants—those stated by the commands used in the construction and all the resulting properties within Euclidean Geometry. This means that a specific *relationship between invariants* is preserved by dragging, and this relationship corresponds to the validity of a logical implication between properties of a geometrical figure. This becomes a crucial element when solving an open problem asking for a conjecture.

Because of their simultaneity, it may be difficult to maintain control of the logical hierarchy between the different invariants. Nevertheless, a careful analysis of the movement of the different elements of a figure (see Mariotti, 2014) reveals an asymmetry between the two kinds of invariants. In other words, two different movements occur that are worth distinguishing and analysing carefully. One movement—*direct motion*—is the variation of an element in the plane under the direct control of the mouse. The second movement—*indirect motion*—is the variation of any other element as a consequence of direct motion.

During a dynamic exploration, the solver can 'feel' motion dependency through a conscious use of the dragging tool. This allows him/her to distinguish between *direct invariants* and *indirect invariants* and interpret their dynamic relationship in terms of the logical consequences between geometrical properties, and eventually express it as a conditional statement between a premise and a conclusion.

Let us consider the following conjecturing open problem: *given a quadrilateral and the midpoint of its sides, what can we say about the quadrilateral that has these midpoints as vertices*.

Once the quadrilateral and its midpoints have been constructed, explorations of the possible configurations make rather evident the emergence of new properties concerning both the parallelism and the equality between the sides of the new quadrilateral. This may lead to the conjecture (Varignon's Theorem): "Given a quadrilateral and the midpoint of its sides, the quadrilateral that has these midpoints as vertices is a parallelogram."

The distinction between direct and indirect movement produces a new interpretation of some of the classic results on different dragging explorations. The modality of dragging previously described as *Dummy locus dragging (*or *Lieu muet dragging*) is especially worthy of attention. This modality consists of dragging a configuration with the intention of maintaining a specific property; that is, achieving a constrained movement of the original figure *as if* a specific property were 'invariant.'. This type of invariant named *Indirectly Induced Invariant* (Baccaglini-Frank & Mariotti, 2010) corresponds to the consequence of the combination of all the properties given by the construction plus a new hypothesis corresponding to the constrained dragging. In other words, via the constrained dragging that we call *Maintaining Dragging* (MD) (Baccaglini-Frank & Mariotti, 2010), a new property is added to the initial premises. This corresponds to what mathematicians commonly refer to as exploring "under which condition…a certain property occurs."

What is meaningful for my purpose here is that using MD to solve a conjecturing open problem, the student can directly and intentionally control the distinction between which property is maintained and which property is searched. This distinction corresponds to the distinction between the premises and conclusion of a conditional statement: the conclusion is the property the solver decides to maintain, the premise is the property corresponding to the constrained movement, and the conditional relationship between these properties corresponds to the simultaneity of their occurrence.

Taking the perspective of semiotic mediation, I claim that the different dragging modalities, together with the different types of invariants, offer rich semiotic potential with respect to the mathematical notion of conjecture and specifically to the mathematical meaning of a conditional statement as the logical relationship between premises and a conclusion. The asymmetry of the relationship between invariants offers the possibility of distinguishing the logical status of the properties of a DGE figure; that is, their status as premise or conclusion. Thus, according to the previous analysis, it is possible to outline the following semiotic potential of the different means of dragging in solving a conjecture-production task in respect to the mathematical meaning of conditionality. The semiotic potential is recognizable in the relationship between:

- the indirectly induced invariant (the property the solver intends to achieve) and the mathematical meaning of the conclusion of the conjecture statement
- the invariant constrained by the specific goal-oriented movement (the property that must be assumed in order to obtain the induced invariant) and the mathematical meaning of the premise of the conjecture statement
- the haptic sensation of causality relating the direct and the indirect movement and the mathematical meaning of logical dependence between premise and conclusion.

Results from several studies show how different meanings related to the notion of conjecture may emerge and how the different kinds of invariant can be characterized by their specific status in the activity of exploration. These results can be used by teachers to exploit the semiotic potential of dragging and specifically of MD (see Baccaglini-Frank & Mariotti, 2010).

5 Impossible Figures and Proof by Contradiction

In the previous sections, I discussed specific aspects of the didactic potential of a DGE for introducing students to mathematical theorems. In this section, I focus on the potential offered by a DGE with respect to indirect proof—that is, proof by contradiction and proof by contraposition. Before showing examples, I present a short account of the model of indirect proof.

Given a *principal statement*, there are two levels at which a proof develops: the *theoretical* level and the *meta-theoretical* level. The very beginning of the proving process consists of a shift to a new statement characterized by new premises. It is usually introduced by the claim "let us start from negating the conclusion." We call this new statement the *secondary statement.* This new statement is related to the principal statement by the fact that its premise includes the negation of the conclusion of the principal statement. The validation of the secondary statement is reached through a direct deductive proof.

The relationship between the validation of the secondary statement and the expected proof of the principal statement is usually taken for granted—commonly ratified by the generic assertion "thus the theorem is proved." However, after proving the secondary statement, something remains unsolved, as clearly explained by Leron (1985):

> Formally, we must be satisfied that the contradiction has indeed established the truth of the theorem (having falsified its negation), but psychologically, many questions remain unanswered. What have we really proved in the end? What about the beautiful constructions we built while living for a while in this false world? Are we to discard them completely? And what about the mental reality we have temporarily created? I think this is one source of frustration, of the feeling that we have been cheated, that nothing has been really proved, that it is merely some sort of a trick—a sorcery—that has been played on us. (p. 323)

The crucial point lies in the final laconic assertion: "thus the theorem is proved." Validating the principal statement pertains to the meta-theoretical level and condenses a meta-theorem relating the validation of the secondary statement to the validation of the principal statement. What is often missing is something that could bridge the gap between the validation of the principal statement and the absurd conclusion resulting from the proof of the secondary statement. In order to clarify the source of such difficulties for students, investigations focused on posing problems in a DGE. The aim was to explore if and how a DGE offers a base for bridging the gap.

5.1 Dragging Impossible Figures

The appearance of conflicting or impossible configurations is one of the critical elements of the production of an indirect proof (Fischbein, 1993). In the case of DGE figures, Leung & Lopez-Real (2002) introduced the notion of *pseudo object* to refer to a figure on which the user forces an assumption so that it is "biased with extra meaning." "This biased DGE," they maintain, "exists as a kind of hybrid state between the visual-true DGE (a virtual representation of the Euclidean world) and a pseudo-true interpretation" (p. 22).

Interesting behaviours are described when the solution of a conjecturing task involves impossible robust figures. They show how the aim of restoring harmony between the figural and theoretical aspects (Fischbein, 1993) can help not only to overcome a possible impasse, but also construct the argument providing the missing step for validating the falsity of an assumption (Antonini & Mariotti, 2008). The link between premises and conclusions, expressed by the relationship between invariant properties observed on the screen after dragging a figure, may contribute to bridging the (logical) gap between the absurd conclusion coming from the proof of the secondary statement and the validation of the principal statement. In other words, in a DGE, this potential bridge can be realized with the support of dragging modes that induce the solver to conceive a pseudo object. The dynamic of the figure induces the solver to interpret the constructed figure as simultaneously representing properties that are contradictory within Euclidean theory.

In the following, I report some results from our classroom research via two exemplary cases. They refer to two different formulations of an open problem, both leading to a conjecture that is expected to be supported by an indirect argument, and consequently providing an introduction to indirect proof.

Case 1. The case of Paolo and Riccardo

This first case concerned the task: What can you say about the angle formed by two angle-bisectors in a triangle?

Exploring the possible configurations can lead the solver to consider the case of orthogonality between two angle-bisectors, and to the conjecture that this case is impossible. Among the protocols of solving processes collected in our studies (Mariotti & Antonini, 2009), we found examples of indirect arguments leading to a contradictory conclusion. The following example is drawn from the interview of a pair of grade 12 students, Paolo and Riccardo.

In the first part of the exploration, Paolo and Riccardo consider the case that the angle between the angle bisectors is an obtuse angle. They then exclude this possibility and move on to consider the case of orthogonality.

61 P: As for 90 [degree], it would be necessary that […] K/2 = 45, H/2 = 45 […].
62 I: In fact, it is sufficient that […] K/2 + H/2 is 90.
63 R: Yes, but it cannot be.
64 P: Yes, but it would mean that K + H is … a square […]
65 R: It surely should be a square, or a parallelogram
66 P: (K-H)/2 would mean that […] K + H is 180°…
67 R: It would be impossible. Exactly, I would have with these two angles already 180, that surely it is not a triangle. […]
71 R: We can exclude that [the angle] $\frac{\pi}{2}$ is [right] because it would become a quadrilateral. […]
81 R: [the angle] is not 90° because I would have a quadrilateral, in fact the sum of the two angles would be already 180, without the third angle.

Then the only possible case is that I have a quadrilateral, that is, the sum of the angles is 360 (Fig. 1).

Using an abductive argument, implicitly based on the theorem about the sum of the measures of the angles of a triangle, Paolo and Riccardo arrive at the conjecture: "*it is sufficient that […] K/2 + H/2 is 90.*" But Riccardo acknowledges the

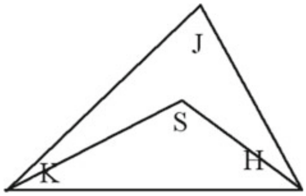

Fig. 1 Paolo and Riccardo's figure

impossibility of this condition (63), immediately followed by Paolo who identifies an immediate consequence of the configuration: "it would mean K + N is …" Realizing the absurd, he seeks a figural interpretation of the 'absurd' conclusion that generates the adaptation of the figure: "*it surely should be a square, …*" (65). The falsity of the original assumption is now acceptable "*because it would become a quadrilateral,*" i.e., not a triangle. More specifically, a new interpretation of the image on the screen is achieved that fulfils the given properties, but also leads to a new conclusion allowing the students to overcome the previous contradictory conclusion. This new interpretation gives sense and opens the development of an indirect argument.

This protocol shows the dynamics of a pseudo object: the initial appearance of an impossible figure is overcome by the image of a square, immediately generalized into a parallelogram. This new image solves the distress of the absurd without canceling its origin.

Because of its nature as pseudo object—representing a contradictory relation and likely to turn into a coherent representation—the dynamic figure acted upon by the solver has the potential, on the one hand, to offer support to the proof of the secondary statement and, on the other, to maintain the relationship between the secondary statement and the principal statement.

According to the model above, the secondary statement "If S (the angle between the angle bisectors) is right, then the configuration becomes a quadrilateral" can be interpreted as "It is not possible that S is right because otherwise the triangle would become a quadrilateral." This interpretation allows the solver to relate the secondary statement to the principal statement.

Case 2. The Case of Stefano and Giulio

Let us now consider the second case (Baccaglini-Frank et al., 2013). The structure of the problem is still a conjecturing open problem, but the text explicitly requests a geometrical construction and an explanation for a negative answer.

The task sounds like this: Is it possible to construct a triangle with two perpendicular angle bisectors? If so, provide steps for a construction. If not, explain why.

In the following example, similar to what happened in the previous case, we can observe how the emergence of a pseudo object can be related to a first awareness about the impossibility of a construction. However, we can also observe the role played by the DGE figure in that emergence. The excerpt is drawn from the interview of a pair of high school students (grade 12), Stefano and Giulio.

1. Stefano: No, the only way is to have 90 degree angles… [unclear which angles these are as he was not constructing the figure or looking at the screen]

2. Giulio: That for a triangle is a bit difficult! [giggling]… So… they have to be…
3. Stefano: If triangles have 4 angles…
4. Giulio: No, I was about to say something silly…

Stefano immediately states that there is only a possibility. Some elements of an impossible configuration are mentioned and then the students quickly move on to constructing a figure in the DGE. Giulio constructs two perpendicular lines and refers to them as the bisectors of the triangle (Fig. 2a).

5. Stefano: Yes, these are bisectors, right?
6. Interviewer: Yes.
7. Giulio: So, now we need to get… bisectors… how can we have an angle from the bisector?
8. Giulio: the symmetric image?… It's enough to do the symmetric of this one.

The solvers have constructed a figure with two robust angle bisectors that intersect perpendicularly (Fig. 2b).

9. Stefano: The only thing is that this (Fig. 2b) isn't a triangle!
10. Giulio: Therefore now we could do like this here [drawing the lines through the symmetric points and the two drawn vertices of the triangle]
11. Interviewer: Yes.
12. Stefano: It's that something atrocious comes out!
13. Giulio: And here… theoretically the point of intersection should be … the points… very small detail…hmmm
14. Stefano: No, we proved that this is equal to this [pointing to angles], and this is equal to this because they are bisectors… these two are equal so these are parallel.
15. Stefano: These two [referring to the two parallel lines] have a hole so it is not a triangle.

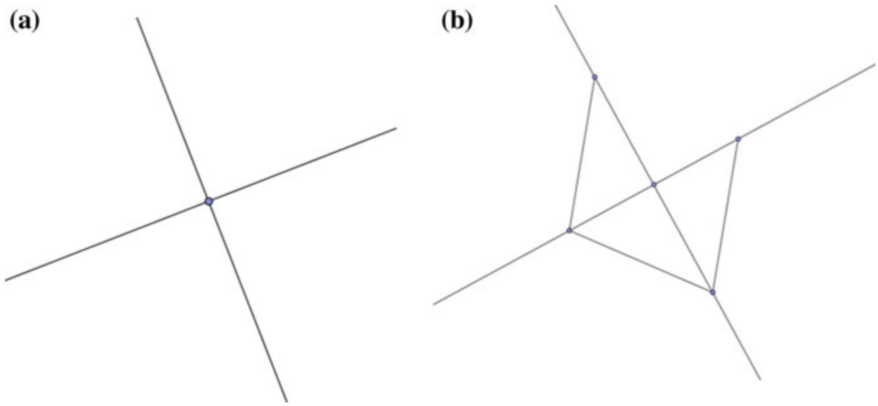

Fig. 2 **a** Giulio's construction of the angles' bisectors. **b** The completed figure

The solvers use the DGE to construct two perpendicular lines. Then the property of being bisectors is realized through constructing the symmetric image of a generic segment that has its extremes each on a different line. Once the construction is completed, it is possible to observe the properties that are consequences of these two robustly constructed properties. The students notice that “the figure must have two adjacent angles with two parallel sides” (Lines 9–14) and, at the same time, that there is “a hole” in what was expected to be a triangle (Line 15). The pseudo object emerges. It has a base as a triangle should have and two parallel sides as a parallelogram has. The figure on the screen has the property “triangle” projected onto it, though is it clearly not a triangle. This pseudo object shows that it is not possible to robustly construct what they were asked to construct, but at the same time shows why it is not possible. This figure, because of its nature as a pseudo object, enables connecting the principal statement “the requested construction is not possible” with the secondary statement “if the angle bisectors are perpendicular, then the triangle is a quadrilateral.”

Our investigations of indirect argumentation and indirect proof produced by students have shown that they may be supported by dragging exploration leading to perceiving what we have called pseudo objects (Baccaglini-Frank et al. 2013, p. 65). Nevertheless, the effectiveness of the appearance of a pseudo object for triggering an indirect argument is related to the logical control of the dynamic figure. In other words, what is essential is awareness of the logical meaning of the dynamic figure—awareness that allows the solver at the same time to project on it the expected properties and to recognize in it the consequences of the constructed properties. Such a double logical interpretation of the dynamic figure allows the solver to make sense of the “absurd” and provides him/her with a bridge to fill the logical gap between the principal and the secondary statement.

Based on these results, it seems possible to claim that a DGE offers a suitable context for handling indirect proof because dragging to produce an impossible configuration or pseudo object provides an informal language to talk about the absurd. This reminds us of Thompson’s (1996) claim:

> If such indirect proofs are encouraged and handled informally, then when students study the topic more formally, teachers will be in a position to develop links between this informal language and the more formal indirect-proof structure. (p. 480)

Specifically, within a DGE context, open problems that ask about the construction of geometrically impossible figures may play a crucial role in unfolding the semiotic potential of dragging with respect to the mathematical meanings of deriving the absurd. In this way, they contribute to developing meanings related to indirect proof.

6 Conclusions

I begin my concluding arguments by reflecting on the distance between the way in which experts and novices approach the discovery of a new theorem. As Polya wrote (De Villiers, 2001), mathematicians are highly motivated to search for a proof:

> [H]aving verified the theorem in several particular cases, we gathered strong inductive evidence for it. The inductive phase overcame our initial *suspicion* and gave us a strong *confidence* in the theorem. Without such *confidence* we would have scarcely found the courage to undertake the proof which did not look at all a routine job. When you have satisfied yourself that the theorem is *true*, you start *proving* it. (Polya, 1954, pp. 83–84, emphasis added)

Such experiences, so natural for experts, do not belong to the reality of novices and students who commonly gravitate toward empirical approaches to truth and often disagree only on the number of confirmations needed. In spite of the fact that proof lies at the heart of mathematics, research on mathematics education has shown the complexity of fostering students' sense of proof and, more generally, of introducing a theoretical perspective in school mathematics. The advent and development of new technological devices has opened new directions of research and posed crucial questions about the possibilities and challenges of supporting the transition from informal to formal proof in mathematics.

In this chapter, I selected the specific technological context of a DGE and, elaborating on results from my investigations, discussed its potential for teaching at the secondary school level. I used TSM to frame the educational context and focused on the notion of semiotic potential to describe the relationship between using specific DGE tools for specific tasks, the situated meanings that are expected to emerge, and their connection with the mathematical meanings related to proof or, more broadly, mathematical theorem. I explained the semiotic potential that emerged in solving a construction task and showed how it relates, on the one hand, to the meaning of drawing constrained by the use of specific tools and, on the other, to the mathematical meaning of proving a statement within a set of shared assumptions and theorems. Exploiting this semiotic potential allows teachers to guide students in constructing and intertwining different meanings of MT. Specifically, it allows teachers to guide the co-emergence of the mathematical meaning of proof and theory.

Further analysis of dragging modalities highlighted the semiotic potential of conjecturing open problems in a DGE. Different dragging modalities can be related to different types of invariants and also to the different logical status of the properties of the geometrical object represented on the screen. Some properties correspond to the premise of a statement and others to the conclusion, while their simultaneity corresponds to the logical relationship between them. In other words, the mathematical meaning of a conditional statement can be related to the relationship between the specific invariant elements that emerge in exploring a configuration and the intention of producing a conjecture.

Additional semiotic potentials of dragging emerged in the solving of non constructability tasks. The semiotic potential of a pseudo object was shown to relate to the mathematical meaning of indirect proof. Specifically, I claim that, in a DGE, rigid construction combined with intentional and controlled dragging may lead the solver to perceive the figure as degenerating into a pseudo object. The hybrid nature of pseudo objects seems to support making sense of indirect proof and creates a bridge between the principal statement to be proved and the absurd configuration emerging from the proof of the secondary statement.

Beyond illustrating the educational potential of a specific technology, the discussion offers insight into the complexity of managing problem solving productively in a DGE. The strict dependence between the experience of acting *geometrically* in a DGE and the development of a coherent web of mathematical meanings enabling theoretical control over what is drawn and moved on the screen gives us a key for interpreting both the strength and the fragility of using DGE activities in school practice. In particular, experimenting with conjecturing and proving in a DGE requires adequate training in a mathematician's eye and feel for theory. After all, interpreting one's own perceptions as mathematical evidence in terms of geometric properties and logical relationships is not a spontaneous or immediate process. On the contrary, developing a theoretical eye is the result of a complex learning process that is inconceivable without a teacher's expertise in setting specific tasks and guiding student awareness of the link between their personal experience and mathematical knowledge.

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