Error Estimates of Approximations for the Complex Valued Integral Transforms

Andrea Aglić Aljinović

Abstract In this survey paper error estimates of approximations in complex domain for the Laplace and Mellin transform are given for functions f which vanish beyond a finite domain $[a,b] \subset [0,\infty\rangle$ and whose derivative belongs to $L_p[a,b]$. New inequalities involving integral transform of f, integral mean of f and exponential and logarithmic mean of the endpoints of the domain of f are presented. These estimates enable us to obtain two associated numerical quadrature rules for each transform and error bounds of their remainders.

1 Introduction

1.1 Laplace and Mellin Transform

The **Laplace transform** $\mathscr{L}(f)$ of Lebesgue integrable mapping $f:[a,b]\to\mathbb{R}$ is defined by

$$\mathscr{L}(f)(z)=\int_0^\infty f(t)\,e^{-zt}\mathrm{d}t \tag{1}$$

for every $z\in\mathbb{C}$ for which the integral on the right-hand side of (1) exists, i.e. $\left|\int_0^\infty f(t)\,e^{-zt}\mathrm{d}t\right|<\infty$.

The **Mellin transform** $\mathscr{M}(f)$ of Lebesgue integrable mapping $f:[0,\infty\rangle\to\mathbb{R}$ is defined by

$$\mathscr{M}(f)(z)=\int_0^\infty f(t)\,t^{z-1}\mathrm{d}t \tag{2}$$

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for every $z \in \mathbb{C}$ for which the integral on the right-hand side of (2) exists, i.e. $\left|\int_0^\infty f(t)\, t^{z-1}\mathrm{d}t\right| < \infty$.

If $f : [a,b] \to \mathbb{R}$ is Lebesgue integrable mapping which vanishes beyond a finite domain, where $[a,b] \subset [0,\infty\rangle$ instead of (1) and (2), we have the finite Laplace and finite Mellin transform

$$\mathscr{L}(f)(z) = \int_a^b f(t)\, e^{-zt}\mathrm{d}t \qquad \mathscr{M}(f)(z) = \int_a^b f(t)\, t^{z-1}\mathrm{d}t.$$

The Laplace and Mellin transform not only are widely used in various branches of mathematics (for instance, for solving boundary value problem or Laplace equation, for summation of infinite series) but also have significant applications in the field of physics and engineering, particularly in computer science (in image recognition because of its scale invariance property). More about the Laplace, Mellin, and other integral transforms can be found in [5].

1.2 Weighted Montgomery Identity for a Complex Valued Weight Function

Montgomery identity states (see [6]):

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \int_a^b P(x,t)\, f'(t)\,\mathrm{d}t, \tag{3}$$

where $P(x,t)$ is the Peano kernel, defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \le t \le x, \\ \frac{t-b}{b-a} & x < t \le b. \end{cases}$$

The **weighted Montgomery identity** states (given by Pečarić in [7])

$$f(x) - \frac{1}{\int_a^b w(t)\,\mathrm{d}t}\int_a^b f(t)\, w(t)\,\mathrm{d}t = \int_a^b P_w(x,t)\, f'(t)\,\mathrm{d}t \tag{4}$$

where $w : [a,b] \to \mathbb{R}$ is a weight function, i.e. integrable function such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$, $W(x) = \int_a^x w(t)\,\mathrm{d}t$, $x \in [a,b]$ and $P_w(x,t)$ the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \le t \le x, \\ \frac{W(t)}{W(b)} - 1, & x < t \le b. \end{cases} \tag{5}$$

Obviously, weighted Montgomery identity (4) for uniform normalized weight function $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ reduces to the Montgomery identity (3).

It is easy to check that the weighted Montgomery identity holds also for a complex valued weight function $w : [a, b] \to \mathbb{C}$ such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$.

Let us check the last condition for the kernels $w(t) = e^{-zt}$ $t \in [a, b]$ and $w(t) = t^{z-1}$, $t \in [a, b]$ of the Laplace and Mellin transform. Since $\int_a^b e^{-zt}\mathrm{d}t = \frac{1}{z}\left(e^{-za} - e^{-zb}\right)$, by using notation $z = x + iy$ we have

$$
\begin{aligned}
e^{-za} &= e^{-zb} \\
e^{-xa}\left(\cos(-ya) + i\sin(-ya)\right) &= e^{-xb}\left(\cos(-yb) + i\sin(-yb)\right) \\
a &= b
\end{aligned}
$$

and obviously $\int_a^b w(t)\,\mathrm{d}t \neq 0$ holds for the kernel of the Laplace transform.

Also, it holds that $\frac{d}{dt}t^z = zt^{z-1}$ for $z \in \mathbb{C}$ and $\int_a^b t^{z-1}\mathrm{d}t = \frac{b^z - a^z}{z}$. Using notation $z = x + iy$ we have

$$
\begin{aligned}
b^z &= a^z \\
e^{z\ln a} &= e^{z\ln b} \\
e^{x\ln a}\left(\cos(y\ln a) + i\sin(y\ln a)\right) &= e^{x\ln b}\left(\cos(y\ln b) + i\sin(y\ln b)\right) \\
a &= b.
\end{aligned}
$$

For the kernel of the Mellin transform $w(t) = t^{z-1}$, $t \in [a, b]$ we can also conclude $\int_a^b w(t)\,\mathrm{d}t \neq 0$.

1.3 Difference Between Two Weighted Integral Means

By subtracting two weighted Montgomery identities (4), one for the interval $[a, b]$ and the other for $[c, d] \subseteq [a, b]$, the next result is obtained (see [2, 3]).

Lemma 1 *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$, $w : [a, b] \to \mathbb{C}$ and $u : [c, d] \to \mathbb{C}$ some weight functions, such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$, $\int_c^d u(t)\,\mathrm{d}t \neq 0$ and*

$$
W(x) = \begin{cases} 0, & t < a, \\ \int_a^x w(t)\,\mathrm{d}t, & a \le t \le b, \\ \int_a^b w(t)\,\mathrm{d}t, & t > b, \end{cases} \quad U(x) = \begin{cases} 0, & t < c, \\ \int_c^x u(t)\,\mathrm{d}t & c \le t \le d, \\ \int_c^d u(t)\,\mathrm{d}t, & t > d, \end{cases}
$$

and $[c,d] \subseteq [a,b]$. *Then the next formula is valid*

$$\frac{1}{\int_a^b w(t)\,\mathrm{d}t}\int_a^b w(t)\,f(t)\,\mathrm{d}t - \frac{1}{\int_c^d u(t)\,\mathrm{d}t}\int_c^d u(t)\,f(t)\,\mathrm{d}t = \int_a^b K(t)\,f'(t)\,\mathrm{d}t \tag{6}$$

where

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{U(t)}{U(d)}, & t \in \langle c,d\rangle, \\ 1 - \frac{W(t)}{W(b)}, & t \in [d,b]. \end{cases} \tag{7}$$

Remark 1 The result of the previous lemma for real-valued weight functions has been proved in [4].

2 Error Estimates of Approximations in Complex Domain for the Laplace Transform

In this chapter error estimates of approximations complex domain for the Laplace transform are given for functions which vanish beyond a finite domain $[a,b] \subset [0,\infty\rangle$ and such that $f' \in L_p[a,b]$. New inequalities involving Laplace transform of f, integral mean of f and exponential mean of the endpoints of the domain of f are presented. In the next chapter these inequalities are used to obtain two associated numerical rules and error bounds of their remainders in each case. These results are published in [1].

Here and hereafter the symbol $L_p[a,b]$ $(p \geq 1)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

and $L_\infty[a,b]$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = ess\sup_{t\in[a,b]} |f(t)|.$$

Exponential mean $E(z,w)$ of z and w is given by

$$E(z,w) = \begin{cases} \frac{e^z - e^w}{z-w}, & \text{if } z \neq w, \\ e^w, & \text{if } z = w. \end{cases} \qquad z,w \in \mathbb{C} \tag{8}$$

Definition 1 We say (p,q) is a pair of conjugate exponents if $1 < p,q < \infty$ and $\frac{1}{p}+\frac{1}{q}=1$; or if $p=1$ and $q=\infty$; or if $p=\infty$ and $q=1$.

The next theorem was proved in [5]:

Theorem 1 *Let $g:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping on $[a,b]$. Then for all $x\neq 0$ we have the inequality*

$$\left|\mathscr{F}(g)(x)-E(-2\pi ixa,-2\pi ixb)\int_a^b g(s)\,\mathrm{d}s\right|$$
$$\leq\begin{cases}\frac{1}{3}(b-a)^2\left\|g'\right\|_\infty, & \text{if } g'\in L_\infty[a,b],\\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}}(b-a)^{1+\frac{1}{q}}\left\|g'\right\|_p, & \text{if } g'\in L_p[a,b],\\ (b-a)\left\|g'\right\|_1 & \text{if } g'\in L_1[a,b].\end{cases}$$

where $\mathscr{F}(g)(x)$ is Fourier transform

$$\mathscr{F}(g)(x)=\int_a^b g(t)\,e^{-2\pi ixt}\mathrm{d}t.$$

and $E(z,w)$ is given by (8).

Next, we apply identity for the difference of the two weighted integral means (6) with two special weight functions: uniform weight function and kernel of the Laplace transform. In such a way new generalizations of the previous results are obtained.

Theorem 2 *Assume (p,q) is a pair of conjugate exponents. Let $f:[a,b]\to\mathbb{R}$ be absolutely continuous, $f'\in L_p[a,b]$ and $c,d\in[a,b]$, $c<d$. Then for* $\operatorname{Re} z\geq 0$ *and $1<p\leq\infty$ we have inequalities*

$$\left|\frac{d-c}{b-a}\mathscr{L}(f)(z)-E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t\right|$$
$$\leq e^{-a\operatorname{Re}z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p$$
$$\leq(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p,$$

while for $p=1$ we have

$$\left|\frac{d-c}{b-a}\mathscr{L}(f)(z)-E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t\right|$$
$$\leq 2e^{-a\operatorname{Re}z}(d-c)\left\|f'\right\|_1\leq 2(d-c)\left\|f'\right\|_1,$$

where $E(z,w)$ is exponential mean of z and w given by (8).

Proof If we apply identity (6) with $w(t) = e^{-zt}$, $t \in [a, b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c, d]$, we have $W(t) = (t-a) E(-za, -zt)$, $t \in [a, b]$; $U(t) = \frac{t-c}{d-c}$, $t \in [c, d]$ and

$$\frac{1}{(b-a) E(-za, -zb)} \mathscr{L}(f)(z) - \frac{1}{d-c} \int_c^d f(t)\,\mathrm{d}t = \int_a^b K(t) f'(t)\,\mathrm{d}t.$$

Since $[c, d] \subseteq [a, b]$ we use (7) so

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a, c], \\ -\frac{W(t)}{W(b)} + \frac{t-c}{d-c}, & t \in \langle c, d\rangle, \\ 1 - \frac{W(t)}{W(b)}, & t \in [d, b]. \end{cases}$$

Thus

$$\frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\,\mathrm{d}t = \frac{d-c}{b-a} W(b) \int_a^b K(t) f'(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\,\mathrm{d}t \right| \le \left\| \frac{d-c}{b-a} W(b) K(t) \right\|_q \left\| f' \right\|_p.$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_q = \left(\int_a^c \left| \frac{d-c}{b-a} W(t) \right|^q \mathrm{d}t \right.$$
$$\left. + \int_c^d \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|^q \mathrm{d}t + \int_d^b \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right|^q \mathrm{d}t \right)$$

and since $\operatorname{Re} z \ge 0$ we have $|W(t)| = \left| \int_a^t e^{-zs} \mathrm{d}s \right| \le \int_a^t \left| e^{-zs} \right| \mathrm{d}s = \int_a^t \left| e^{-s \operatorname{Re} z} \right| \mathrm{d}s \le (t-a) e^{-a \operatorname{Re} z}$ for $t \in [a, b]$, thus

$$\int_a^c \left| \frac{d-c}{b-a} W(t) \right|^q \mathrm{d}t \le \int_a^c \left(e^{-a \operatorname{Re} z} \frac{d-c}{b-a} (t-a) \right)^q \mathrm{d}t = e^{-aq \operatorname{Re} z} \left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\int_c^d \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|^q \mathrm{d}t \le \int_c^d \left(\left| \frac{d-c}{b-a} W(t) \right| + \left| \frac{t-c}{b-a} W(b) \right| \right)^q \mathrm{d}t$$

$$\leq e^{-aq\operatorname{Re} z}\int_c^d\left(\frac{d-c}{b-a}(t-a)+t-c\right)^q \mathrm{d}t$$
$$=\left(\frac{e^{-a\operatorname{Re} z}}{b-a}\right)^q\int_c^d((b-a+d-c)t-c(b-a)-a(d-c))^q\,\mathrm{d}t.$$

If we denote

$$\lambda(t)=(b-a+d-c)t-c(b-a)-a(d-c)$$

we have $\lambda(c)=(d-c)(c-a)$ and $\lambda(d)=(d-c)(b+d-2a)$ so

$$\left(\frac{e^{-a\operatorname{Re} z}}{b-a}\right)^q\int_c^d((b-a+d-c)t-c(b-a)-a(d-c))^q\,\mathrm{d}t$$
$$=\frac{e^{-aqs\operatorname{Re} z}\left(\lambda(d)^{q+1}-\lambda(c)^{q+1}\right)}{(b-a)^q(q+1)(b-a+d-c)}$$
$$=\frac{e^{-aq\operatorname{Re} z}(d-c)^{q+1}\left((b+d-2a)^{q+1}-(c-a)^{q+1}\right)}{(b-a)^q(q+1)(b-a+d-c)}\leq\frac{e^{-aq\operatorname{Re} z}2^q(d-c)^q(b-a)}{(q+1)}.$$

Also

$$\int_d^b\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q\mathrm{d}t=\int_d^b\left|\frac{d-c}{b-a}\int_t^b e^{-zs}\mathrm{d}s\right|^q\mathrm{d}t$$
$$\leq e^{-aq\operatorname{Re} z}\int_d^b\left(\frac{d-c}{b-a}(b-t)\right)^q\mathrm{d}t=e^{-aq\operatorname{Re} z}\left(\frac{d-c}{b-a}\right)^q\frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q$$
$$\leq e^{-a\operatorname{Re} z}\left(\left(\frac{d-c}{b-a}\right)^q\frac{(c-a)^{q+1}}{(q+1)}+\frac{2^q(d-c)^q(b-a)}{(q+1)}+\left(\frac{d-c}{b-a}\right)^q\frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}}$$
$$\leq e^{-a\operatorname{Re} z}\left(\left(\frac{d-c}{b-a}\right)^q\frac{(b-a)^{q+1}}{(q+1)}+\frac{2^q(d-c)^q(b-a)}{(q+1)}\right)^{\frac{1}{q}}$$
$$=e^{-a\operatorname{Re} z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}$$

and since $e^{-a\operatorname{Re}z} \le 1$ inequalities in case $1 < p \le \infty$ are proved. For $p = 1$ we have

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_{\infty} = \max \left\{ \sup_{t\in[a,c]} \left| \frac{d-c}{b-a} W(t) \right|, \right.$$

$$\left. \sup_{t\in[c,d]} \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|, \sup_{t\in[d,b]} \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right| \right\}$$

and

$$\sup_{t\in[a,c]} \left| \frac{d-c}{b-a} W(t) \right| \le e^{-a\operatorname{Re}z} \frac{(d-c)(c-a)}{(b-a)},$$

$$\sup_{t\in[c,d]} \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right| \le \sup_{t\in[c,d]} \left\{ \left| \frac{d-c}{b-a} W(t) \right| + \left| \frac{t-c}{b-a} W(b) \right| \right\}$$

$$\le e^{-a\operatorname{Re}z} \frac{d-c}{b-a} (d-a) + e^{-a\operatorname{Re}z} (d-c) = e^{-a\operatorname{Re}z} (d-c) \frac{b+d-2a}{b-a},$$

$$\sup_{t\in[d,b]} \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right| \le e^{-a\operatorname{Re}z} \frac{(d-c)(b-d)}{(b-a)}.$$

Thus

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_{\infty} \le e^{-a\operatorname{Re}z} \frac{d-c}{b-a} \max \{(c-a), (b+d-2a), (b-d)\}$$

$$\le e^{-a\operatorname{Re}z} 2(d-c)$$

and since $e^{-a\operatorname{Re}z} \le 1$ the proof is completed.

Remark 2 The inequalities from the previous theorem hold for $\operatorname{Re}z \ge 0$. Similarly it can be proved that in case $\operatorname{Re}z < 0$ and $1 < p \le \infty$ we have the inequality

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\,\mathrm{d}t \right|$$

$$\le e^{-b\operatorname{Re}z} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \|f'\|_p,$$

while for $\operatorname{Re} z < 0$ and $p = 1$ we have

$$\left|\frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za, -zb)\int_c^d f(t)\,\mathrm{d}t\right| \le e^{-b\operatorname{Re} z} 2(d-c)\left\|f'\right\|_1.$$

Theorem 3 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous, $f' \in L_p[a, b]$ and $c, d \in [a, b]$, $c < d$. Then for $\operatorname{Re} z \ge 0$ and $1 < p \le \infty$, we have inequalities*

$$\begin{aligned}
&\left|\frac{d-c}{b-a}E(-zc, -zd)\int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t\right| \\
&\le e^{-c\operatorname{Re} z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p \\
&\le (d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p,
\end{aligned}$$

while for $p = 1$ we have

$$\begin{aligned}
&\left|\frac{d-c}{b-a}E(-zc, -zd)\int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t\right| \\
&\le e^{-c\operatorname{Re} z} 2(d-c)\left\|f'\right\|_1 \\
&\le 2(d-c)\left\|f'\right\|_1,
\end{aligned}$$

where $E(z, w)$ is exponential mean of z and w given by (8).

Proof By applying identity (6) with $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ and $u(t) = e^{-zt}$, $t \in [c, d]$ and proceeding in the similar manner as in the proof of the Theorem 2.

Remark 3 The inequalities from the previous theorem hold for $\operatorname{Re} z \ge 0$. Similarly it can be proved that in case $\operatorname{Re} z < 0$ and $1 < p \le \infty$ we have the inequality

$$\begin{aligned}
&\left|\frac{d-c}{b-a}E(-zc, -zd)\int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t\right| \\
&\le e^{-d\operatorname{Re} z}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p,
\end{aligned}$$

while for $\operatorname{Re} z < 0$ and $p = 1$ we have

$$\left|\frac{d-c}{b-a}E(-zc, -zd)\int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t\right| \le e^{-d\operatorname{Re} z} 2(d-c)\left\|f'\right\|_1.$$

Corollary 1 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous and* $f' \in L_p[a, b]$*. Then for all* $\operatorname{Re} z \geq 0$ *and* $1 < p \leq \infty$*, we have the inequality*

$$\left| E(-za, -zb) \int_a^b f(t)\,\mathrm{d}t - \mathscr{L}(f)(z) \right| \leq (b-a)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left\| f' \right\|_p,$$

while for $p = 1$ *we have*

$$\left| E(-za, -zb) \int_a^b f(t)\,\mathrm{d}t - \mathscr{L}(f)(z) \right| \leq 2(b-a) \left\| f' \right\|_1.$$

Proof By applying Theorem 2 or 3 in the special case when $c = a$ and $d = b$.

Corollary 2 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous and* $f' \in L_p[a, b]$*. Then for all* $\operatorname{Re} z \geq 0$*, for any* $c \in [a, b]$ *and* $1 < p \leq \infty$*, we have the inequality*

$$\begin{aligned} &\left| \mathscr{L}(f)(z) - (b-a) E(-za, -zb) f(c) \right| \\ &\leq (b-a)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left\| f' \right\|_p, \end{aligned}$$

while for $p = 1$ *we have*

$$\left| \mathscr{L}(f)(z) - (b-a) E(-za, -zb) f(c) \right| \leq 2(b-a) \left\| f' \right\|_1.$$

Proof By applying the proof of the Theorem 2 in the special case when $c = d$. Since f is absolutely continuous, it is continuous, thus as a limit case we have $\lim_{c \to d} \frac{1}{d-c} \int_c^d f(t)\,\mathrm{d}t = f(c)$.

3 Numerical Quadrature Rules for the Laplace Transform

In this section we use two previous corollaries to obtain two numerical quadrature rules.

Let $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a division of the interval $[a, b]$, $h_k := t_{k+1} - t_k,\ k = 0, 1, \ldots, n-1$ and $\nu(h) := \max_k \{h_k\}$. Define the sum

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} E(-zt_k, -zt_{k+1}) \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t \tag{9}$$

where $\operatorname{Re} z \geq 0$.

The following approximation theorem holds.

Theorem 4 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$*,* $f' \in L_p[a, b]$*. Then we have the quadrature rule*

$$\mathscr{L}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$*,* $\mathscr{E}(f, I_n, z)$ *is given by (9) and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2\nu(h) \left\|f'\right\|_1.$$

Proof For $1 < p \leq \infty$ by applying the Corollary 1 with $a = t_k, b = t_{k+1}$ we have

$$\begin{aligned} &\left|E(-zt_k, -zt_{k+1}) \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} e^{-zt} f(t)\,\mathrm{d}t\right| \\ &\leq (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}$$

Summing over k from 0 to $n-1$ and using generalized triangle inequality, we obtain

$$\begin{aligned} |R(f, I_n, z)| &= |\mathscr{L}(f)(z) - \mathscr{E}(f, I_n, z)| \\ &\leq \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}. \end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned} &\left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left(\left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} \\ &= \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p \end{aligned}$$

and the first inequality is proved. For $p = 1$ we have

$$|R(f, I_n, z)| \leq \sum_{k=0}^{n-1} 2h_k \left(\int_{t_k}^{k+1} \left| f'(t) \right| \mathrm{d}t \right)$$
$$\leq 2\nu(h) \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} \left| f'(t) \right| \mathrm{d}t \right) = 2\nu(h) \left\| f' \right\|_1$$

and the proof is completed.

Corollary 3 *Suppose that all assumptions of Theorem 4 hold. Additionally suppose*

$$\mathscr{E}(f, I_n, z) = \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} f(t)\,\mathrm{d}t$$
$$\cdot \sum_{k=0}^{n-1} E\left(-z\left(a + k \cdot \frac{b-a}{n} \right), -z\left(a + (k+1) \cdot \frac{b-a}{n} \right) \right).$$

Then we have the quadrature rule

$$\mathscr{L}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p,$$

while for $p = 1$ *we have*

$$|R(g, I_n, z)| \leq \frac{2(b-a)}{n} \left\| f' \right\|_1.$$

Proof If we apply Theorem 4 with equidistant partition of $[a, b]$.

Now, define the sum

$$\mathscr{A}(f, I_n, z) = \sum_{k=0}^{n-1} (t_{k+1} - t_k) E(-zt_k, -zt_{k+1}) f\left(\frac{t_{k+1} + t_k}{2} \right) \tag{10}$$

where $\operatorname{Re} z \geq 0$.

The following approximation theorem also holds.

Theorem 5 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$*,* $f' \in L_p[a, b]$*. Then we have the quadrature rule*

$$\mathscr{L}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$*,* $\mathscr{A}(f, I_n, z)$ *is given by (10) and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2\nu(h) \left\|f'\right\|_1.$$

Proof By applying the Corollary 2 with $a = t_k$, $b = t_{k+1}$, $c = \frac{t_{k+1}+t_k}{2}$ and then summing over k from 0 to $n - 1$, we obtain results similarly as in the proof of the Theorem 4.

Corollary 4 *Suppose that all assumptions of Theorem 5 hold. Additionally suppose*

$$\mathscr{A}(f, I_n, z) = \frac{b-a}{n} f\left(a + \frac{k(k+1)(b-a)}{2n}\right)$$
$$\cdot \sum_{k=0}^{n-1} E\left(-z\left(a + k \cdot \frac{b-a}{n}\right), -z\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right).$$

Then we have the quadrature rule

$$\mathscr{L}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p,$$

while for $p = 1$ *we have*

$$|R(g, I_n, z)| \leq \frac{2(b-a)}{n} \left\|f'\right\|_1.$$

Proof By applying Theorem 5 with equidistant partition of $[a, b]$.

Remark 4 For both numerical quadrature formulae in case $\operatorname{Re} z < 0$, for $1 < p \leq \infty$, the reminder $R(f, I_n, z)$ satisfies the estimate

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1} \right]^{\frac{1}{q}} \left\| f' \right\|_p ,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} 2\nu(h) \left\| f' \right\|_1 .$$

For equidistant partition of $[a, b]$ and for $1 < p \leq \infty$ we have

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \frac{(b - a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p ,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} \frac{2(b - a)}{n} \left\| f' \right\|_1 .$$

4 Error Estimates of Approximations in Complex Domain for the Mellin Transform

In this chapter error estimates of approximations complex domain for the Laplace transform are given for functions which vanish beyond a finite domain $[a, b] \subset [0, \infty\rangle$ and such that $f' \in L_p[a, b]$. New inequalities involving Laplace transform of f, integral mean of f, exponential and logarithmic means of the endpoints of the domain of f are presented. In the next section these inequalities are used to obtain two associated numerical rules and error bounds of their remainders in each case. These results are published in [3].

Logarithmic mean $L(a, b)$ is given by

$$L(a, b) = \begin{cases} \frac{a-b}{\ln a - \ln b}, & \text{if } a \neq b, \\ a, & \text{if } a = b, \end{cases} \qquad a, b \in \mathbb{R}. \tag{11}$$

Theorem 6 *Assume (p, q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous, $[a, b] \subset \langle 0, \infty\rangle$, $f' \in L_p[a, b]$ and $[c,d] \subseteq [a, b]$. Then for* $\operatorname{Re} z \geq 1$ *and* $1 < p \leq \infty$ *the following inequality holds:*

$$\left|\frac{d-c}{b-a}\mathscr{M}(f)(z)-\frac{E(z\ln a, z\ln b)}{L(a,b)}\int_c^d f(t)\,\mathrm{d}t\right|$$
$$\leq b^{(\operatorname{Re} z)-1}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p$$

while for $p=1$ it holds

$$\left|\frac{d-c}{b-a}\mathscr{M}(f)(z)-\frac{E(z\ln a, z\ln b)}{L(a,b)}\int_c^d f(t)\,\mathrm{d}t\right|\leq 2b^{(\operatorname{Re} z)-1}(d-c)\left\|f'\right\|_1.$$

Here $E(z,w)$ is exponential mean given by (8) and $L(a,b)$ is logarithmic mean given by (11).

Proof Taking $w(t)=t^{z-1}$, $t\in[a,b]$ and $u(t)=\frac{1}{d-c}$, $t\in[c,d]$, we have

$$W(t)=\int_a^t t^{z-1}\mathrm{d}t=\frac{t^z-a^z}{z}=\frac{e^{z\ln t}-e^{z\ln a}}{z}$$
$$=\frac{e^{z\ln t}-e^{z\ln a}}{z\ln t-z\ln a}\cdot\frac{\ln t-\ln a}{t-a}(t-a)=\frac{E(z\ln a, z\ln t)}{L(a,t)}(t-a)$$

for all $t\in[a,b]$ and $U(t)=\frac{t-c}{d-c}$ for all $t\in[c,d]$. Now, we apply identity (6) with these weight functions

$$\frac{L(a,b)}{(b-a)E(z\ln a, z\ln b)}\mathscr{M}(f)(z)-\frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t=\int_a^b K(t)f'(t)\,\mathrm{d}t.$$

Since $[c,d]\subseteq[a,b]$ we use (7) so

$$K(t)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in[a,c],\\ -\frac{W(t)}{W(b)}+\frac{t-c}{d-c}, & t\in\langle c,d\rangle,\\ 1-\frac{W(t)}{W(b)}, & t\in[d,b].\end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathscr{M}(f)(z)-\frac{E(z\ln a, z\ln b)}{L(a,b)}\int_c^d f(t)\,\mathrm{d}t=\frac{d-c}{b-a}W(b)\int_a^b K(t)f'(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathscr{M}(f)(z)-\frac{E(z\ln a,z\ln b)}{L(a,b)}\int_c^d f(t)\,\mathrm{d}t\right|\le\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q\|f'\|_p.$$

Now, for $1<p\le\infty$ (for $1\le q<\infty$) we have

$$\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q=\left(\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q\mathrm{d}t\right.$$

$$\left.+\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q\mathrm{d}t+\int_d^b\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.$$

Using notation $x=\mathrm{Re}\,z$, $y=s\,\mathrm{Im}\,z$, since $x\ge1$ we have $|W(t)|=\left|\int_a^t e^{(x-1+iy)\ln s}\mathrm{d}s\right|\le\int_a^t\left|e^{(x-1+iy)\ln s}\right|\mathrm{d}s=\int_a^t\left|e^{(x-1)\ln s}\right|\mathrm{d}s\le(t-a)\,e^{(x-1)\ln b}$ $=(t-a)\,b^{(x-1)}$ for $t\in[a,b]$, thus

$$\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q\mathrm{d}t\le\int_a^c\left(b^{(x-1)}\frac{d-c}{b-a}(t-a)\right)^q\mathrm{d}t=b^{q(x-1)}\left(\frac{d-c}{b-a}\right)^q\frac{(c-a)^{q+1}}{(q+1)},$$

$$\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q\mathrm{d}t\le\int_c^d\left(\left|\frac{d-c}{b-a}W(t)\right|+\left|\frac{t-c}{b-a}W(b)\right|\right)^q\mathrm{d}t$$

$$\le b^{q(x-1)}\int_c^d\left(\frac{d-c}{b-a}(t-a)+t-c\right)^q\mathrm{d}t$$

$$=\left(\frac{b^{q(x-1)}}{b-a}\right)^q\int_c^d((b-a+d-c)\,t-c\,(b-a)-a\,(d-c))^q\,\mathrm{d}t.$$

If we denote

$$\lambda(t)=(b-a+d-c)\,t-c\,(b-a)-a\,(d-c)\tag{12}$$

we have $\lambda(c)=(d-c)(c-a)\ge0$ and $\lambda(d)=(d-c)(b+d-2a)\ge0$ so

$$\left(\frac{b^{q(x-1)}}{b-a}\right)^q\int_c^d((b-a+d-c)\,t-c\,(b-a)-a\,(d-c))^q\,\mathrm{d}t$$

$$=\frac{b^{q(x-1)}\left(\lambda(d)^{q+1}-\lambda(c)^{q+1}\right)}{(b-a)^q\,(q+1)\,(b-a+d-c)}$$

$$=\frac{b^{q(x-1)}\,(d-c)^{q+1}\left((b+d-2a)^{q+1}-(c-a)^{q+1}\right)}{(b-a)^q\,(q+1)\,(b-a+d-c)}\le\frac{b^{q(x-1)}2^q\,(d-c)^q\,(b-a)}{(q+1)}.$$

Also

$$\int_d^b \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right|^q \mathrm{d}t = \int_d^b \left| \frac{d-c}{b-a} \int_t^b e^{(x-1+iy)\ln s} ds \right|^q \mathrm{d}t$$
$$\leq b^{q(x-1)} \int_d^b \left(\frac{d-c}{b-a} (b-t) \right)^q \mathrm{d}t = b^{q(x-1)} \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_q$$
$$\leq b^{(x-1)} \left(\left(\frac{d-c}{b-a} \right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} + \left(\frac{d-c}{b-a} \right)^q \frac{(b-d)^{q+1}}{(q+1)} \right)^{\frac{1}{q}}$$
$$\leq b^{(x-1)} \left(\left(\frac{d-c}{b-a} \right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} \right)^{\frac{1}{q}}$$
$$= b^{(x-1)} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}}$$

and the first inequality is proved. For $p = 1$ we have

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_\infty = \max \left\{ \sup_{t\in[a,c]} \left| \frac{d-c}{b-a} W(t) \right|, \right.$$
$$\left. \sup_{t\in[c,d]} \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|, \sup_{t\in[d,b]} \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right| \right\}$$

and

$$\sup_{t\in[a,c]} \left| \frac{d-c}{b-a} W(t) \right| \leq b^{(x-1)} \frac{(d-c)(c-a)}{(b-a)},$$

$$\sup_{t\in[c,d]} \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right| \leq \sup_{t\in[c,d]} \left\{ \left| \frac{d-c}{b-a} W(t) \right| + \left| \frac{t-c}{b-a} W(b) \right| \right\}$$
$$\leq b^{(x-1)} \frac{d-c}{b-a} (d-a) + b^{(x-1)} (d-c) = b^{(x-1)} (d-c) \frac{b+d-2a}{b-a},$$

$$\sup_{t\in[d,b]} \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right| \leq b^{(x-1)} \frac{(d-c)(b-d)}{(b-a)}.$$

Thus

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_{\infty} \leq b^{(x-1)} \frac{d-c}{b-a} \max \{(c-a), (b+d-2a), (b-d)\}$$
$$\leq b^{(x-1)} 2(d-c)$$

and the proof is completed.

Remark 5 The inequalities from the previous theorem hold for $\operatorname{Re} z \geq 1$. Similarly it can be proved that in case $\operatorname{Re} z < 1$ and $1 < p \leq \infty$ the following inequality holds:

$$\left| \frac{d-c}{b-a} \mathscr{M}(f)(z) - \frac{E(z \ln a, z \ln b)}{L(a,b)} \int_c^d f(t)\, \mathrm{d}t \right|$$
$$\leq a^{(\operatorname{Re} z)-1} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$

while for $\operatorname{Re} z < 1$ and $p = 1$ it holds

$$\left| \frac{d-c}{b-a} \mathscr{M}(f)(z) - \frac{E(z \ln a, z \ln b)}{L(a,b)} \int_c^d f(t)\, \mathrm{d}t \right| \leq 2a^{(\operatorname{Re} z)-1} (d-c) \left\| f' \right\|_1 .$$

Remark 6 In case $a = 0$ and $\operatorname{Re} z \geq 1$ proceeding in the same way as in the previous proof and using the fact that $0^z = 0$ and thus $\frac{b^z - a^z}{z(b-a)} = \frac{b^{z-1}}{z}$ we obtain

$$\left| \frac{d-c}{b} \mathscr{M}(f)(z) - \frac{b^{z-1}}{z} \int_c^d f(t)\, \mathrm{d}t \right| \leq b^{(\operatorname{Re} z)-1} (d-c) \left(\frac{(2^q+1)(b)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$

and

$$\left| \frac{d-c}{b} \mathscr{M}(f)(z) - \frac{b^{z-1}}{z} \int_c^d f(t)\, \mathrm{d}t \right| \leq 2b^{(\operatorname{Re} z)-1} (d-c) \left\| f' \right\|_1 .$$

Theorem 7 *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, $[a,b] \subset \langle 0, \infty \rangle$, $f' \in L_p[a,b]$ and $c, d \in [a,b]$, $c < d$. Then for* $\operatorname{Re} z \geq 1$ *and $1 < p \leq \infty$ the following inequality holds:*

$$\left| \frac{(d-c) E(z \ln c, z \ln d)}{(b-a) L(c,d)} \int_a^b f(t)\, \mathrm{d}t - \int_c^d t^{z-1} f(t)\, \mathrm{d}t \right|$$
$$\leq d^{(\operatorname{Re} z)-1} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p ,$$

while for $p = 1$ we have

$$\left|\frac{(d-c)\,E(z\ln c, z\ln d)}{(b-a)\,L(c,d)}\int_a^b f(t)\,\mathrm{d}t - \int_c^d t^{z-1} f(t)\,\mathrm{d}t\right| \le d^{(\operatorname{Re} z)-1} 2(d-c)\left\|f'\right\|_1,$$

where $E(z,w)$ is given by (8) and $L(a,b)$ is logarithmic mean given by (11).

Proof If we apply identity (6) with $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u(t) = t^{z-1}$, $t \in [c,d]$, we have $W(t) = \frac{t-a}{b-a}$, $t \in [a,b]$; $U(t) = \frac{E(z\ln c, z\ln t)}{L(c,t)}(t-c)$, $t \in [c,d]$ and

$$\frac{1}{(b-a)}\int_a^b f(t)\,\mathrm{d}t - \frac{L(c,d)}{(d-c)\,E(z\ln c, z\ln d)}\int_c^d t^{z-1} f(t)\,\mathrm{d}t = \int_a^b K(t)\,f'(t)\,\mathrm{d}t.$$

Since $[c,d] \subseteq [a,b]$ we use (7) so

$$K(t) = \begin{cases} -\frac{t-a}{b-a}, & t \in [a,c], \\ \frac{U(t)}{U(d)} - \frac{t-a}{b-a}, & t \in \langle c,d\rangle, \\ \frac{b-t}{b-a}, & t \in [d,b]. \end{cases}$$

Thus

$$\frac{(d-c)\,E(z\ln c, z\ln d)}{(b-a)\,L(c,d)}\int_a^b f(t)\,\mathrm{d}t - \int_c^d t^{z-1} f(t)\,\mathrm{d}t = U(d)\int_a^b K(t)\,f'(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{(d-c)\,E(z\ln c, z\ln d)}{(b-a)\,L(c,d)}\int_a^b f(t)\,\mathrm{d}t - \int_c^d t^{z-1} f(t)\,\mathrm{d}t\right| \le \left\|U(d)\,K(t)\right\|_q \left\|f'\right\|_p.$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\left\|U(d)\,K(t)\right\|_q = \left(\int_a^c \left|\frac{t-a}{b-a}U(d)\right|^q \mathrm{d}t \right.$$
$$\left. + \int_c^d \left|U(t) - \frac{t-a}{b-a}U(d)\right|^q \mathrm{d}t + \int_d^b \left|\frac{b-t}{b-a}U(d)\right|^q \mathrm{d}t\right)^{\frac{1}{q}}.$$

Using notation $x = \operatorname{Re} z$, $y = s \operatorname{Im} z$, we have $x \geq 1$. Since $|U(t)| = \left|\int_c^t e^{(x-1+iy)\ln s} \mathrm{d}s\right| \leq \int_c^t \left|e^{(x-1+iy)\ln s}\right| \mathrm{d}s = \int_c^t \left|e^{(x-1)\ln s}\right| \mathrm{d}s \leq (t-c)\, e^{(x-1)\ln d} = (t-c)\, d^{(x-1)}$ for $t \in [c,d]$, we have

$$\int_a^c \left|\frac{t-a}{b-a} U(d)\right|^q \mathrm{d}t \leq d^{(x-1)q} \int_a^c \left(\frac{t-a}{b-a}(d-c)\right)^q \mathrm{d}t = d^{(x-1)q} \left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned}
&\int_c^d \left|U(t) - \frac{t-a}{b-a} U(d)\right|^q \mathrm{d}t \leq \int_c^d \left(|U(t)| + \left|\frac{t-a}{b-a} U(d)\right|\right)^q \mathrm{d}t \\
&\leq d^{(x-1)q} \int_c^d \left(t - c + \frac{d-c}{b-a}(t-a)\right)^q \mathrm{d}t \\
&\leq \frac{d^{(x-1)q}}{(b-a)^q} \int_c^d \left((b-a+d-c)\, t - c\,(b-a) - a\,(d-c)\right)^q \mathrm{d}t \\
&= d^{(x-1)q} \frac{\left(\lambda(d)^{q+1} - \lambda(c)^{q+1}\right)}{(b-a)^q (q+1)(b-a+d-c)} \\
&= d^{(x-1)q} \frac{(d-c)^{q+1} \left((b+d-2a)^{q+1} - (c-a)^{q+1}\right)}{(b-a)^q (q+1)(b-a+d-c)} \leq d^{(x-1)q} \frac{2^q (d-c)^q (b-a)}{(q+1)},
\end{aligned}$$

where $\lambda(t)$ is given by (12) and

$$\int_d^b \left|\frac{b-t}{b-a} U(d)\right|^q \mathrm{d}t \leq d^{(x-1)q} \int_d^b \left(\frac{b-t}{b-a}(d-c)\right)^q \mathrm{d}t = d^{(x-1)q} \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\begin{aligned}
&\|U(d)\, K(t)\|_q \\
&\leq d^{(x-1)} \left(\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} + \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \\
&\leq d^{(x-1)} \left(\left(\frac{d-c}{b-a}\right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)}\right)^{\frac{1}{q}} \\
&= d^{(x-1)} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}
\end{aligned}$$

and the first inequality is proved. For $p = 1$ we have

$$\|U(d)\, K(t)\|_\infty = \max\left\{\sup_{t\in[a,c]} \left|\frac{t-a}{b-a} U(d)\right|,\right.$$

$$\sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right|,\ \sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right|\right\}$$

and

$$\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}U(d)\right|\le d^{(x-1)}\frac{(c-a)(d-c)}{(b-a)},$$

$$\sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right|=\sup_{t\in[c,d]}\left\{|U(t)|+\left|\frac{t-a}{b-a}U(d)\right|\right\}$$
$$\le d^{(x-1)}\sup_{t\in[c,d]}\left|d-c+\frac{d-a}{b-a}(d-c)\right|=d^{(x-1)}(d-c)\frac{b+d-2a}{b-a},$$

$$\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right|\le d^{(x-1)}\frac{(b-d)(d-c)}{(b-a)}.$$

Thus

$$\|U(d)K(t)\|_\infty\le d^{(x-1)}\frac{d-c}{b-a}\max\{(c-a),(b+d-2a),(b-d)\}\le d^{(x-1)}2(d-c)$$

and the proof is completed.

Remark 7 The inequalities from the previous theorem hold for $\operatorname{Re} z\ge 1$. Similarly it can be proved that in case $\operatorname{Re} z<1$ and $1<p\le\infty$ the following inequality holds:

$$\left|\frac{(d-c)E(z\ln c,z\ln d)}{(b-a)L(c,d)}\int_a^b f(t)\,\mathrm{d}t-\int_c^d t^{z-1}f(t)\,\mathrm{d}t\right|$$
$$\le c^{(\operatorname{Re} z)-1}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p,$$

while for $\operatorname{Re} z<1$ and $p=1$ it holds

$$\left|\frac{(d-c)E(z\ln c,z\ln d)}{(b-a)L(c,d)}\int_a^b f(t)\,\mathrm{d}t-\int_c^d t^{z-1}f(t)\,\mathrm{d}t\right|\le c^{(\operatorname{Re} z)-1}2(d-c)\left\|f'\right\|_1.$$

Remark 8 In case $a=c=0$ and $\operatorname{Re} z\ge 1$ all the inequalities from the Theorem 7 holds with a term $\frac{d^{z-1}}{z}$ instead of $\frac{E(z\ln c,z\ln d)}{L(c,d)}$.

Corollary 5 *Assume (p, q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $[a, b] \subset \langle 0, \infty \rangle$, $f' \in L_p [a, b]$. Then for all $\operatorname{Re} z \geq 1$ and $1 < p \leq \infty$ we have the inequality*

$$\left| \mathscr{M}(f)(z) - \frac{E(z \ln a, z \ln b)}{L(a, b)} \int_a^b f(t)\,\mathrm{d}t \right| \leq b^{(\operatorname{Re} z)-1} (b-a)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$

while for $p = 1$ we have

$$\left| \mathscr{M}(f)(z) - \frac{E(z \ln a, z \ln b)}{L(a, b)} \int_a^b f(t)\,\mathrm{d}t \right| \leq 2b^{(\operatorname{Re} z)-1} (b-a) \left\| f' \right\|_1 .$$

Proof By applying the proof of the Theorem 6 or 7 in the special case when $c = a$ and $d = b$.

Corollary 6 *Assume (p, q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $[a, b] \subset \langle 0, \infty \rangle$, $f' \in L_p [a, b]$. Then for all $\operatorname{Re} z \geq 1$, for any $c \in [a, b]$ and $1 < p \leq \infty$ we have the inequality*

$$\left| \mathscr{M}(f)(z) - (b-a) \frac{E(z \ln a, z \ln b)}{L(a, b)} f(c) \right| \leq b^{(\operatorname{Re} z)-1} (b-a)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$

while for $p = 1$ we have

$$\left| \mathscr{M}(f)(z) - (b-a) \frac{E(z \ln a, z \ln b)}{L(a, b)} f(c) \right| \leq 2b^{(\operatorname{Re} z)-1} (b-a) \left\| f' \right\|_1 .$$

Proof By applying the proof of the Theorem 6 in the special case when $c = d$. Since f is absolutely continuous, it is continuous, thus as a limit case we have $\lim_{c \to d} \frac{1}{d-c} \int_c^d f(t)\,\mathrm{d}t = f(c)$.

5 Numerical Quadrature Rules for the Mellin Transform

Since the exponents of the term $(b-a)^{1+\frac{1}{q}}$ in the inequalities from the last two corollaries are greater than 1, these inequalities can be useful to obtain numerical quadrature formulae. Using Corollaries 5 and 6 we obtain the following two numerical rules.

Let $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a division of the interval $[a, b]$, $h_k := t_{k+1} - t_k,\ k = 0, 1, \ldots, n-1$ and $\nu(h) := \max_k \{h_k\}$. Define the sum

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} \frac{E(z \ln t_k, z \ln t_{k+1})}{L(t_k, t_{k+1})} \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t \tag{13}$$

where $\operatorname{Re} z \geq 1$.

The following approximation theorem holds.

Theorem 8 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$, $[a, b] \subset \langle 0, \infty\rangle$, $f' \in L_p[a, b]$. *Then we have the quadrature rule*

$$\mathscr{M}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 1$, $\mathscr{E}(f, I_n, z)$ *is given by (13) and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p, \tag{14}$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2b^{(\operatorname{Re} z)-1} \nu(h) \left\|f'\right\|_1. \tag{15}$$

Proof For $1 < p \leq \infty$ by applying the Corollary 5 with $a = t_k$, $b = t_{k+1}$ we have

$$\begin{aligned}
&\left|\frac{E(z \ln t_k, z \ln t_{k+1})}{L(t_k, t_{k+1})} \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} t^{z-1} f(t)\,\mathrm{d}t\right| \\
&\leq (t_{k+1})^{x-1} (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq b^{x-1} (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}
\end{aligned}$$

where $x = \operatorname{Re} z$. Summing over k from 0 to $n - 1$ and using generalized triangle inequality, we obtain

$$\begin{aligned}
|R(f, I_n, z)| &= |\mathscr{M}(f)(z) - \mathscr{E}(f, I_n, z)| \\
&\leq b^{x-1} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}.
\end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned}
&\sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\
&\leq \left[\sum_{k=0}^{n-1} \left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left(\left(\int_{t_k}^{t_{k+1}} \left|f'(t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} = \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p
\end{aligned}$$

and the inequality (14) is proved. For $p = 1$ we have

$$|R(f, I_n, z)| \leq \sum_{k=0}^{n-1} 2b^{x-1} h_k \left(\int_{t_k}^{k+1} |f'(t)| \, \mathrm{d}t \right)$$
$$\leq 2b^{x-1} \nu(h) \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} |f'(t)| \, \mathrm{d}t \right) = 2b^{x-1} \nu(h) \|f'\|_1$$

and the proof is completed.

Corollary 7 *Suppose that all assumptions of Theorem 8 hold. Additionally suppose*

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} f(t) \, \mathrm{d}t \cdot \frac{E\left(z \ln\left(a + k \cdot \frac{b-a}{n}\right), z \ln\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}{L\left(\left(a + k \cdot \frac{b-a}{n}\right), \left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}. \tag{16}$$

Then we have the quadrature rule

$$\mathscr{M}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 1$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|f'\|_p, \tag{17}$$

while for $p = 1$ *we have*

$$|R(g, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \frac{2(b-a)}{n} \|f'\|_1. \tag{18}$$

Proof If we apply Theorem 8 with equidistant partition of $[a, b]$, $t_j = a + j \cdot \frac{b-a}{n}$, $j = 0, 1, \ldots, n$, we have (16) and $h_k = \frac{b-a}{n}$, $k = 0, 1, \ldots, n-1$. For $1 < p \leq \infty$ we obtain

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1} \right]^{\frac{1}{q}} \|f'\|_p = \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \|f'\|_p,$$

while for $p = 1$, $\nu(h) = \frac{b-a}{n}$ and the claim immediately follows.

Now, define the sum

$$\mathscr{A}(f, I_n, z) = \sum_{k=0}^{n-1} (t_{k+1} - t_k) \frac{E(z \ln t_k, z \ln t_{k+1})}{L(t_k, t_{k+1})} f\left(\frac{t_{k+1} + t_k}{2}\right) \tag{19}$$

where $\operatorname{Re} z \geq 1$.

Also the following approximation theorem holds.

Theorem 9 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous function on $[a, b]$, $[a, b] \subset \langle 0, \infty \rangle$, $f' \in L_p[a, b]$. Then we have the quadrature rule*

$$\mathscr{M}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 1$, $\mathscr{A}(f, I_n, z)$ is given by (19) and for $1 < p \leq \infty$ the reminder $R(f, I_n, z)$ satisfies the estimate

$$|R(f, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p, \tag{20}$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2b^{(\operatorname{Re} z)-1} \nu(h) \left\|f'\right\|_1. \tag{21}$$

Proof By applying the Corollary 6 with $a = t_k$, $b = t_{k+1}$, $c = \frac{t_{k+1}+t_k}{2}$ and then summing over k from 0 to $n-1$, we obtain results similarly as in the proof of the Theorem 8.

Corollary 8 *Suppose that all assumptions of Theorem 9 hold. Additionally suppose*

$$\mathscr{A}(f, I_n, z) = \sum_{k=0}^{n-1} \frac{b-a}{n} f\left(a + \frac{k(k+1)(b-a)}{2n}\right) \cdot \frac{E\left(z \ln\left(a + k \cdot \frac{b-a}{n}\right), z \ln\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}{L\left(\left(a + k \cdot \frac{b-a}{n}\right), \left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}.$$

Then we have the quadrature rule

$$\mathscr{M}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 1$ and for $1 < p \leq \infty$ the reminder $R(f, I_n, z)$ satisfies the estimate

$$|R(f, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p, \tag{22}$$

while for $p = 1$ we have

$$|R(g, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \frac{2(b-a)}{n} \left\| f' \right\|_1. \tag{23}$$

Proof By applying Theorem 9 with equidistant partition of $[a, b]$.

Remark 9 Both numerical quadrature formulae hold also in case $a = 0$ with the term $\frac{t_1^{z-1}}{z}$ instead of $\frac{E(z \ln a, z \ln t_1)}{L(a, t_1)}$ in the first approximation sum (13) and the second approximation sum (19).

Remark 10 For both numerical quadrature formulae in case $\operatorname{Re} z < 1$, for $1 < p \leq \infty$, the reminder $R(f, I_n, z)$ satisfies the estimate

$$|R(f, I_n, z)| \leq a^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1} \right]^{\frac{1}{q}} \left\| f' \right\|_p,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2a^{(\operatorname{Re} z)-1} \nu(h) \left\| f' \right\|_1.$$

For equidistant partition of $[a, b]$ and for $1 < p \leq \infty$ we have

$$|R(f, I_n, z)| \leq a^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq a^{(\operatorname{Re} z)-1} \frac{2(b-a)}{n} \left\| f' \right\|_1.$$

Remark 11 It is easy to see that in all these numerical rules estimate tends to zero as n tends to infinity.

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