Application of Davies–Petersen Lemma

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Abstract In this paper we have shown how use of a simple lemma first proved by Davies and Petersen and later extended by Mohapatra and Russell can be used effectively to prove three main results which can yield integral inequalities of Hardy and Copson. We have also shown how those results can be used to obtain many known results obtained by Levinson, Pachpatte, Chan, etc. by carefully manipulating these three theorems. A look at this paper will also reveal that there can be simple proofs of sophisticated results after they have been proved by exploiting the important points that make things work. It does not take away the value of the original contributions. We have also mentioned that it has not been possible to deduce other known results by using our results.

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1 Introduction

With a view to providing an alternative proof of the discrete version of Hilbert's inequality G. H. Hardy proved the following inequality:

Theorem 1.1 ([13] p. 239, Theorem 326) *If* $p > 1$, $a_n > 0$, $n = 0, 1, 2, \ldots,$ *then*

$$\sum_{n=0}^{\infty} \left((n+1)^{-1} \sum_{k=0}^{n} a_k \right)^p \leq q^p \sum_{n=0}^{\infty} a_n^p, \tag{1}$$

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with $\frac{1}{p} + \frac{1}{q} = 1$. The constant q^p on the right-hand side of (1) is best possible. In [23, Theorem 1] Davies and Petersen proved the following result.

Theorem 1.2 ([7], Theorem 1) *Suppose $A = (a_{mn})$ be an infinite matrix with*

$$a_{mn} > 0 \quad (n \leq m), a_{mn} = 0 \quad (n > m), m, n = 1, 2, \ldots, \tag{2}$$

Further assume that

$$0 \leq \frac{a_{mn}}{a_{kn}} \leq K, \quad (0 \leq n \leq k \leq m) \tag{3}$$

and $\frac{a_{mn}}{a_{kn}}$ is a decreasing sequence as n increases in (3) with $0 \leq n \leq k \leq m$.

Let us also assume that there exists an $f(m)$ ($f(m) \to \infty$ as $m \to \infty$) such that the matrix (c_{mn}) defined by $c_{mn} = f(m)a_{mn}$ $(n = 1, 2, \ldots)$ has properties (2) and (3) mentioned before with perhaps a different constant k in (2).

If $x_n \geq 0$ $(n = 1, 2, \ldots,)$ and if

$$\sum_{k=1}^{\infty} a_{k1} \{f(k)\}^{1-p} \tag{4}$$

converges and

$$\sum_{k=n}^{\infty} a_{k1} \{f(k)\}^{1-p} \leq M a_{n1} \{f(n)\}^{1-p}, \tag{5}$$

then

$$\sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{m} a_{mn} x_n \right\}^p \leq C \sum_{m=1}^{\infty} \{x_m f(m) a_{mn}\}^p, \tag{6}$$

where p is an integer and C is an arbitrary constant.

In [9, Theorem 2], Davies and Petersen extended Theorem 1.2 to all real $p > 1$. This extension was, in fact, a consequence of the following lemma which we name as Davies–Petersen lemma for sequences.

Lemma 1.1 ([9], Lemma 1) *If $p > 1$ and $z_n \geq 0$ $(n = 1, 2, \ldots,)$, then*

$$\left(\sum_{k=1}^{n} z_k \right)^p \leq p \sum_{k=1}^{n} z_k \left(\sum_{j=1}^{k} z_j \right)^{p-1}. \tag{7}$$

Using this lemma, Davies and Petersen proved an analogue of the Theorem 1.2 for all real $p > 1$. Johnson and Mohapatra [15] proved discrete inequalities for a class of matrices and called such inequalities as Hardy–Davies–Petersen inequality. For details, we refer the reader to [15].

An analogue of the Lemma 1.1 for integrals was proved in [9, Lemma 2] by Davies and Petersen.

Lemma 1.2 ([9], Lemma 2) *Let $p > 1$ and $z(x)$ be any positive integral function of x. Then*

$$\left(\int_0^x z(x)dx\right)^p = p\int_0^x z(x)\left(\int_0^x z(t)dt\right)^{p-1} dx. \tag{8}$$

Davies and Petersen used Lemma 1.2 to prove an integral inequality (see [9, Theorem 4]) involving μ-kernel which is defined below.

Let an μ-kernel $a(x, y)$ satisfy the following conditions:

$$\begin{cases} a(x, y) > 0 \ (y \le x) \\ a(x, y) = 0 \ (y > x) \end{cases} \qquad (x, y) \ge 0. \tag{9}$$

Also

$$0 \le \frac{a(x_0, y)}{a(x_1, y)} \le K \qquad (0 \le y \le x_1 \le x_0), \tag{10}$$

where K is an absolute constant. Further let there exist a function $f(x)$ ($f(x) \to \infty$ as $x \to \infty$) such that $c(x, y) = f(x)a(x, y)$ is an μ-kernel. Davies and Petersen proved

Theorem 1.3 ([9], Theorem 4) *Let $a(x, y)$ be an μ-kernel and $u(y) \ge 0$ $(y \ge 0)$. Then if*

$$\int_0^\infty \{f(x)\}^{-p}\, dx \tag{11}$$

exists and

$$\int_{x_0}^\infty \{f(x)\}^{-p}\, dx \le M\,[f(x_0)]^{1-p}\,, \tag{12}$$

we have

$$\int_0^\infty \left\{\int_0^x a(x, y)u(y)dy\right\}^p dx \le C\int_0^\infty \{u(x)f(x)a(x, x)\}^p\, dx, \tag{13}$$

where $p \ge 1$ and C is a constant which depends on p.

The intent of Davies and Petersen in proving Theorem 1.3 was to provide a generalization of Hardy's integral inequality:

Theorem 1.4 ([13], Theorem 327) *If* $p > 1$, $f(x) \geq 0$ *for* $0 < x < \infty$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < q^p \int_0^\infty f(x)^p dx \tag{14}$$

unless $f(x)$ *is identically zero, with* $q = p/(p-1)$.

Over the years Theorem 1.4 has been generalized in several directions and many research papers have been written on this inequality (see [1–12, 14–18, 20–32] and all the references in those papers).

Mohapatra and Russell [21] mentioned Lemma 1.2 as Davies–Petersen lemma and remarked what happens when $0 < p < 1$. They wrote the following:

Lemma 1.3 ([21], Lemma 1)

(i) Let $1 \leq p < \infty$ *and* $Z(t)$ *be non-negative and integrable over* $0 < t < x$. *Then*

$$\left(\int_0^x Z(t)dt\right)^p = p\int_0^x Z(t)\left\{\int_0^t Z(u)du\right\}^{p-1} dt. \tag{15}$$

The result holds for $0 < p < 1$ *provided* $\int_0^t Z(u)du > 0$ *for* $0 < t < x$.

(ii) Let $1 \leq p < \infty$ *and* $Z(t)$ *be an integrable function for* $x < t < \infty$. *Then*

$$\left(\int_x^\infty Z(t)dt\right)^p = p\int_x^\infty Z(t)\left\{\int_t^\infty Z(u)du\right\}^{p-1} dt. \tag{16}$$

The result holds for $0 < p < 1$ *provided that* $\int_t^\infty Z(u)du > 0$ *for* $x < t < \infty$.

Proof of (15) was given by Davies and Petersen [9] and proof of (16) was given by Mohapatra and Russell (see [21, Lemma 1, p. 201]).

The main objective of this chapter is to use the above lemma (hereafter called as Davies–Petersen lemma) to obtain three theorems which will yield many known results as corollaries.

2 Known Integral Inequalities

In this section we give a number of generalizations of Hardy's, Copson's and Levinson's integral inequalities. The results of Copson and Levinson were proved to provide generalization of Hardy's integral inequality. We state these below:

Theorem 2.1 (Hardy [13]) *Let* $p > 1$, $c \neq 1$, *and* $h(x)$ *be non-negative and Lebesgue integrable on* $[0, a]$ *or* $[a, \infty]$ *for every* $a > 0$ *according as* $c > 1$ *or*

$c < 1$. *If, we define,*

$$F(x) = \begin{cases} \int_0^x h(t)dt, & c > 1; \\ \int_x^\infty h(t)dt, & 0 < c < 1; \end{cases}$$

then

$$\int_0^\infty x^{-c} \{F(x)\}^p \, dx \le \left(\frac{p}{|c-1|} \right)^p \int_0^\infty x^{-c} \{xh(x)\}^p \, dx. \tag{17}$$

In [8] Copson has proved integral inequality with a view to generalizing Hardy's inequality. One such result is

Theorem 2.2 ([8], Theorem 1) *Let* $\phi(x)$, $f(x)$ *be non-negative for* $x \ge 0$ *and be continuous in* $[0, \infty)$. *Let* $p \ge 1, c > 1$. *If* $0 < b \le \infty$ *and*

$$\int_0^b F(x)^p \Phi(x)^{-c} \phi(x) dx \tag{18}$$

converges at the lower limit of integration, then

$$\int_0^b F(x)^p \Phi(x)^{-c} \phi(x) dx \le \left(\frac{p}{c-1} \right)^p \int_0^b f(x)^p \Phi(x)^{p-c} \phi(x) dx \tag{19}$$

where $\Phi(x) = \int_o^x \phi(t)dt$, $F(x) = \int_o^x f(t)\phi(t)dt$.

Remark 2.1 The case $c = p > 1$ and $\phi(x) = 1$ is Hardy's classical integral inequality [13, Theorem 327] which inspired numerous researchers including Copson. In fact, Theorem 2.2 mentioned above is one of the six inequalities established in [8]. Beesack [2] has proved six similar inequalities two of which provide alternative proofs of [8, Theorem 5 and Theorem 6]. Independent generalization of Copson's inequalities has been done by Love [18], Mohapatra and Russel [21], and Mohapatra and Vajravelu [22].

With a view to generalizing Hardy's inequality, Levinson established the following results

Theorem 2.3 ([17], Theorem 4, p. 329) *Let* $p > 1$, $f(x) \ge 0$ *and let* $r(x)$ *be positive and locally absolutely continuous in* $(0, \infty)$. *In addition, let*

$$\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \ge \frac{1}{\lambda} \tag{20}$$

for some $\lambda > 0$ *and for almost all* x. *If*

$$H(x) = \frac{\int_0^x r(t) f(t) dt}{x r(x)}, \tag{21}$$

then

$$\int_0^\infty H(x)^p dx \le \lambda^p \int_0^\infty f(x)^p dx. \tag{22}$$

Theorem 2.4 ([1], Theorem 5, p. 393) *Let $p > 1$, $f(x) \ge 0$, $r(x)$ be locally absolutely continuous for $x > 0$. Let*

$$\frac{x r'(x)}{r(x)} - \frac{p-1}{p} \ge \frac{1}{\lambda} \tag{23}$$

for some $\lambda > 0$. If

$$J(x) = \frac{r(x)}{x} \int_x^\infty \frac{f(t)}{r(t)} dt, \tag{24}$$

then

$$\int_0^\infty J(x)^p dx \le \lambda^p \int_0^\infty f(x)^p dx, \tag{25}$$

Remark 2.2 If in Theorem 2.4, we take $r(x) = 1$ and $\lambda = \frac{p}{p-1}$, then we get Hardy's integral inequality [13, Theorem 327]. If $r(x) = x$ and $\lambda = p$, then (25) reduces to the dual inequality related to that of Hardy.

3 Main Results

In this section, we shall prove three integral inequalities from which we shall be able to deduce a number of known results as corollaries. These theorems will be proved by using Davies–Petersen lemma and careful use of Hölder inequalities.

Since some of the research papers consider the interval of integration as (a, b) in place of $(0, \infty)$, our next theorem will be established for (a, b). Although these were proved in [6], we give complete proofs and apply them to get many known results as corollaries.

Thus, this chapter shows how simple techniques can yield nice results.

Theorem 3.1 (See [6], Theorem A) *Let $0 \le a < b \le \infty$ and h be a non-negative function which is Lebesgue integrable in (x, b), and u is a positive function with*

$$U(x) = \int_a^x u(t) dt \tag{26}$$

is finite for each x in $a < x < b$. Then the following inequality holds:

(i) When $1 < p < \infty$,

$$\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx \le p^p \int_a^b u(x)\left(\frac{h(x)U(x)}{u(x)}\right)^p dx. \tag{27}$$

(ii) If $0 < p < 1$, *then the inequality* $\le$ *is replaced by* $\ge$. *If* $p = 1$, *then the inequality* (27) *reduces to an equality.*

Proof

Case 1 If $p = 1$,

$$\int_a^b u(x)\left(\int_x^b h(t)dt\right)dx = \int_a^b h(t)\left(\int_a^t u(x)dx\right)dt = \int_a^b h(t)U(t)dt. \tag{28}$$

This completes the proof for the case $p = 1$.

Case 2 If $1 < p < \infty$, if the left-hand side of (27) is infinite, then apply the following with b replaced by c with $a < c < b$, and the let c approach b from below.

Using Davies–Petersen lemma (Lemma 1.7),

$$\begin{aligned}\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx &= p\int_a^b u(x)\int_x^b h(t)dt\left(\int_t^b h(s)ds\right)^{p-1} dx\\ &= p\int_a^b h(t)\left(\int_t^b h(s)ds\right)^{p-1} dt\int_a^t u(x)dx\\ &= p\int_a^b h(t)U(t)\left(\int_t^b h(s)ds\right)^{p-1} dt. \end{aligned}\tag{29}$$

Now, let us write the expressing on the right-hand side of (29) as

$$p\int_a^b \frac{h(t)U(t)}{u(t)}u(t)\left(\int_t^b h(s)ds\right)^{p-1} dt \tag{30}$$

and apply Hölder's inequality to (30).

Hence, (30) is not greater than

$$p\left[\int_a^b u(t)\left(\int_t^b h(s)ds\right)^p dt\right]^{\frac{1}{p'}}\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}. \tag{31}$$

Now, collecting results from (29)–(31), we have, after dividing both sides by $\left[\int_a^b \left(\int_t^b h(s)ds\right)^p u(t)dt\right]^{\frac{1}{p'}}$

$$\left[\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx\right]^{\frac{1}{p'}} \leq p\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}, \quad (32)$$

which yields the required result for Theorem 1.1, when $1 < p < \infty$.

Case 3 When $0 < p < 1$. In this case the Hölder inequality applied to the expression (30) in case 2 yields

$$\left[\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx\right]^{\frac{1}{p}} \geq p\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}$$

and the result follows.

Note that in cases 2 and 3, if the expression by which we are dividing both sides is zero, then the inequality is automatically satisfied because both sides of the inequality to be proved are zero. □

Theorem 3.2 (See [6], Theorem B) *Let* $0 \leq a < b \leq \infty$, *h be a non-negative function which is Lebesgue integrable in* $a < x < b$ *and u be a positive function such that*

$$U(x) = \int_x^b u(t)dt$$

is finite for $a < x < b$. *Then for* $1 \leq p < \infty$,

$$\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx \leq p^p \int_a^b u(x)\left(\frac{h(x)U(x)}{u(x)}\right)^p dx. \quad (33)$$

If $0 < p < 1$, *then the inequality in* (33) *becomes* $\geq$.

Proof

Case 1 $p = 1$. In this case change of order of integration for the double integral on the left-hand side of (33) yields the equality.

Case 2 $1 < p < \infty$, if the left-hand side of (33) is infinite, we apply the proof given below with a replaced by c, $a < c < b$ and then let $c \to a$ from above. Hence we assume, for the rest of the proof, the left-hand side of (33) as finite.

By Davies–Petersen Lemma (Lemma 1.2), we have

$$\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx = p\int_a^b u(x)\left[\int_a^x h(t)\left(\int_a^t h(s)ds\right)^{p-1} dt\right]dx$$
$$= p\int_a^b h(t)\left(\int_a^t h(s)ds\right)^{p-1}\int_t^b u(x)dx\, dt \quad (34)$$

By applying Hölder's inequality in the same manner, as in the proof of Theorem 3.1, we obtain with $p' = p/(p-1)$,

$$\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx \le p\left[\int_a^b u(t)\left(\int_a^t h(s)ds\right)^p dt\right]^{\frac{1}{p'}}$$
$$\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}. \quad (35)$$

Since the integral $\int_a^b u(t)\left(\int_a^t h(s)ds\right)^p dt$ is finite, we divide both sides of (35) by that integral to get

$$\left[\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx\right]^{\frac{1}{p}} \le p\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}. \quad (36)$$

Raising both sides of (36) to power p the result follows.

Case 3 $0 < p < 1$. In this case the Hölder's inequality yields the required results because $\le$ is replaced by $\ge$. □

Our next theorem is Levinson type generalization of Hardy's inequality. We first state Levinson's result for the reader to appreciate our next theorem.

Theorem 3.3 (See Levinson [17], Theorem 2) *Let* $0 \le a < b \le \infty$, $\phi(u) \ge 0$ *and* $\phi''(u) \ge 0$ *when* $0 \le u \le b$, $p > 1$. *Let*

$$\phi(x)\phi''(x) \ge \left(1 - \frac{1}{p}\right)\left(\phi'(x)\right)^2 \quad for \quad a < x < b. \quad (37)$$

At end points of the interval $a < x < b$, *let* $\phi(x)$ *take its limiting values, finite or infinite. For* $x > 0$, *let* $r(x)$ *be continuous and non-decreasing, and let*

$$R(x) = \int_0^x r(t)dt. \quad (38)$$

Then

$$\int_0^\infty \phi\left(\int_0^x \frac{r(t)f(t)}{R(x)}\right)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \phi(f(x))dx. \quad (39)$$

Remark 3.1 When $\phi(u) = u^p$, $p > 1$, (39) is automatically satisfied and the inequality (39) reduces to Hardy's inequality when $r(t) \equiv 1$.

We prove the following generalization of Theorem 3.3.

Theorem 3.4 (See [6] Theorem C) *Let $p \geq 1$ and let ϕ, R, r, and f be defined as in Theorem 3.3 so that the hypotheses of Theorem 3.3 are satisfied. Further assume that g is a positive function which is Lebesgue integrable over the interval $(0, b)$, and*

$$U(x) = \int_x^b \frac{g(t)}{(R(t))^p} dt. \tag{40}$$

Then

$$\int_0^b g(x)\, \phi\left(\int_0^x \frac{r(x) f(t)}{R(x)}\right) dx$$
$$\leq p^p \int_a^b (g(x))^{p-1} \left\{R(x)^{p-1} r(x) U(x)\right\}^p \phi(f(x)) dx \tag{41}$$

Proof Let us write $\eta(t)^p = \phi(t)$. Then

$$\phi(x)\phi''(t) \geq \left(1 - \frac{1}{p}\right) \left(\phi'(t)\right)^2$$

means

$$p(p-1)\eta^{2(p-1)} \left(\eta'(t)\right)^2 + p\eta(t)^{2p-1}\eta''(t) \geq (p-1)p\eta^{2(p-1)} \left(\eta'(t)\right)^2.$$

This implies that $\eta(t)\eta''(t) \geq 0$. Since $\phi(t) \geq 0$, $\eta(t) \geq 0$ and consequently, the condition (37) amounts to $\eta''(t) \geq 0$. Hence, the function $\eta(t)$ is convex. Now, by Jensen's in equality applied to $\int_0^x \frac{r(t) f(t)}{R(x)} dt$ yields

$$\eta\left(\int_0^x \frac{r(t) f(t)}{R(x)} dt\right) \leq \int_0^x \frac{r(t)\eta(f(t))}{R(x)} dt \tag{42}$$

Substituting $\phi(t)^{1/p}$ for $\eta(t)$ in (42) and raising both sides to power p, we get

$$\phi\left(\int_0^x \frac{r(t) f(t)}{R(x)} dt\right) \leq \left(\int_0^x \frac{r(t)\phi(f(t))^{1/p}}{R(x)} dt\right)^p \tag{43}$$

since g is a positive function (43) yields

$$\int_0^b g(x)\phi\left(\int_0^x \frac{r(t)f(t)}{R(x)}dt\right)dx \le \int_0^b g(x)\left(\int_0^x \frac{r(t)\phi(f(t))^{1/p}}{R(x)}dt\right)^p \quad (44)$$

Now, we can apply Theorem 3.2 to the left-hand side of (44) with $h(t) = r(t)\phi(f(t))^{1/p}$ and $u(x) = \frac{g(x)}{(R(x))^p}$, and $a = 0$. We will get

$$\int_0^b g(x)\phi\left(\int_0^x \frac{r(t)f(t)}{R(x)}dt\right)dx \le \int_0^b \frac{g(x)}{R(x)^p}\left(\frac{r(x)\phi(f(x))^{1/p}U(x)R(x)^p}{g(x)}\right)^p dx$$

$$\le p^p \int_0^b g(x)^{1-p}\left\{R(x)^{p-1}r(x)U(x)\right\}^p \phi(f(x))dx, \quad (45)$$

after simplification. This completes the proof of Theorem 3.4. □

Remark 3.2 Theorem 3.3 can be obtained from Theorem 3.4 by setting $g(x) \equiv 1$. Since $r(x)$ is non-decreasing, we estimate $u(x)$ from

$$U(x) \le \frac{1}{r(x)}\int_x^b \frac{r(t)}{R(t)^p}dt \le \frac{1}{p-1}\left(\frac{1}{r(x)R(x)^{p-1}}\right) \quad (46)$$

Then (39) follows.

4 Corollaries from Theorems 3.1 and 3.2

Corollary 4.1 (See Chan [7], Theorem 1, p. 165) *If $h(t)$ is Lebesgue integrable in (x, ∞) for every $x \in (1, \infty)$, and $h(t) > 0$ for all $t \in (1, \infty)$, then for $1 < p < \infty$,*

$$\int_1^\infty \frac{1}{x}\left(\int_0^\infty h(t)dt\right)^p dx \le p^p \int_1^\infty \frac{1}{x}(x \ \ln x \ h(x))^p \, dx. \quad (47)$$

The inequality is reversed if $0 < p < 1$ and yields equality when $p = 1$.

Proof Equation (47) follows from Theorem 3.1 by setting $u(x) = \frac{1}{x}$ and $U(x) = \ln(x)$, $a = 1$ and $b = \infty$. □

Corollary 4.2 (See Chan [7], Theorem 2, p. 166) *Suppose $h(t)$ is Lebesgue integrable over $(0, x)$ for each $x \in (0, 1)$, and $h(t) \ge 0$ for all $t \in (0, 1)$. Then for $1 < p < \infty$,*

$$\int_0^1 \frac{1}{x}\left(\int_0^x h(t)dt\right)^p dx \le p^p \int_0^1 \frac{1}{x}(x\,|\ln x|\,h(x))^p \, dx. \quad (48)$$

The inequality is reversed if $0 < p < 1$ and yields equality when $p = 1$.

Proof In Theorem 3.2, take $a = 0$, $b = 1$, $u(x) = \frac{1}{x}$, $0 < x < 1$. Then

$$U(x) = \int_x^1 \frac{1}{x} dx = -\ln x = |\ln x|, \tag{49}$$

with these, we see that the Corollary 4.2 holds. □

Corollary 4.3 (See Chen [7], Theorem 3, p. 166) *Suppose $h(t)$ is integrable in the sense of Lebesgue over $(1, x)$ for each $x \in (1, \infty)$, and $h(t) \geq 0$ for all $t \in (1, \infty)$. Then for $1 < p < \infty$,*

$$\int_1^\infty \frac{1}{x} (\ln x)^{-p} \left(\int_1^x h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_1^\infty \frac{1}{x} (x h(x))^p \, dx. \tag{50}$$

Proof In Theorem 3.2, set $u(x) = x^{-1}(\ln x)^{-p}$, $1 < p < \infty$, $a = 1$, and $b = \infty$. Then $U(x) = (p-1)^{-1}(\ln x)^{-p+1}$ and Corollary 4.3 follows. □

Corollary 4.4 (See Chen [7], Theorem 4, p. 167) *Let $h(t)$ be Lebesgue integrable over $(x, 1)$ for each $x \in (0, 1)$. Then for $1 < p < \infty$,*

$$\int_0^1 x^{-1} (\ln x)^{-p} \left(\int_x^1 h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^1 x^{-1} (x h(x))^p \, dx. \tag{51}$$

Proof In Theorem 3.1, set $u(x) = x^{-1}(\ln x)^{-p}$ and $1 < p < \infty$. Let a approach zero from above and $b = 1$. Clearly,

$$U(x) = \left| \int_0^x \frac{dt}{t |\ln t|^p} \right| = \frac{1}{(p-1)|\ln x|^{p-1}}, \tag{52}$$

Equation (51) follows from Theorem 3.1. □

Corollary 4.5 *Let $h(t)$ be as in Corollary 4.3. Then for $1 < p, q < \infty$,*

$$\int_0^b \frac{1}{x |\ln x|^q} \left(\int_a^b h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^b x^{p-1} |\ln x|^{p-q} h(x)^p dx \tag{53}$$

Corollary 4.6 *Let $h(t)$ be as in corollary 4.4. Then for $1 < p, q < \infty$,*

$$\int_a^\infty \frac{1}{x |\ln x|^q} \left(\int_a^x h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_a^\infty x^{p-1} |\ln x|^{p-q} h(x)^p dx \tag{54}$$

Remark 4.1 Corollaries 4.5 and 4.6 are obtained from Theorems 3.1 and 3.2 by choosing $u(x) = \frac{1}{x |\ln x|^q}$. Also both inequalities are reversed when $0 < p < 1$.

Corollary 4.7 (Hardy [12]) *Let $p > 1$, $r \neq 1$, and $h(t)$ be a non-negative function which is integrable on the interval $(0, a]$ or $[a, \infty)$ for every $a > 0$, according as $r > 1$ or $r < 1$. If $F(x)$ is defined by*

$$F(x) := \begin{cases} \int_0^x h(t)dt & r > 1; \\ \int_x^\infty h(t)dt & r < 1; \end{cases}$$

then

$$\int_0^\infty x^{-r} F(x)^p dx \leq \left(\frac{p}{|r-1|} \right)^p \int_0^\infty x^{-r} (xh(x))^p dx \tag{55}$$

Proof Inequality (55) is deducible from Theorems 3.1 and 3.2 by taking $u(x) = x^{-r}$ according to the value of r. If $r < 1$, it is deduced from Theorem 3.1 but when $r > 1$, it can be obtained using Theorem 3.2. $U(x)$ is easily calculated to complete the proofs for both cases. □

Corollary 4.8 (Levinson [17], Theorem 4) *Suppose $1 < p < \infty$, $f(x) \geq 0$ and $r(x)$ is positive and locally absolutely continuous in $(0, \infty)$. Further assume that $r(x)$ satisfies the following:*

$$\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda}, \tag{56}$$

for some $\lambda > 0$ and for almost all x.

If we define

$$H(x) = \frac{1}{xr(x)} \int_0^x r(t) f(t) dt, \tag{57}$$

then

$$\int_0^\infty H(x)^p dx \leq \lambda^p \int_0^\infty f(x)^p dx. \tag{58}$$

Remark 4.2 If $r(x) \equiv 1$, $\lambda = \frac{p}{p-1}$, then (58) reduces to well-known Hardy's inequality [12, Theorem 327].

Proof (of Corollary 4.8) We shall use Theorem 3.2 with $a = 0$, $h(x) = r(x)f(x)$ and $u(x) = (xr(x))^{-p}$.

Applying integration by parts,

$$U(x) = \int_x^b \frac{dt}{t^p r(t)^p} = \left[\left(\frac{t^{-p+1}}{-p+1} \frac{1}{r(t)^p} \right) \right]_x^b - \frac{p}{p-1} \int_x^b \frac{t^{-p+1}}{r(t)^{p+1}} r'(t) dt. \tag{59}$$

Since $r(b) > 0$, $b > 0$ and $1 < p < \infty$.

$$\frac{b^{-p+1}}{(-p+1)r(b)^p} < 0. \tag{60}$$

Hence, (59) yields

$$U(x) \leq \frac{x}{(p-1)x^p r(x)^p} - \frac{p}{(p-1)} \int_x^b \left[\frac{tr'(t)}{r(t)}\right] \frac{dt}{(tr(t))^p}. \tag{61}$$

Using

$$-\frac{tr'(t)}{r(t)} \leq \frac{p-1}{p} - \frac{1}{\lambda} \tag{62}$$

in the integral on the right-hand side of (61), we get by (59) and (61)

$$U(x) \leq \frac{x}{(p-1)(xr(x))^p} + U(x) + \frac{p}{(p-1)\lambda} U(x)$$

or

$$U(x) \leq \frac{\lambda x}{p(xr(x))^p} = \frac{\lambda x U(x)}{p}, \tag{63}$$

since $u(x) = (xr(x))^{-p}$. Now, for $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$u(x)^{-1/p'} U(x) r(x) f(x) \leq u(x)^{-1/p'} \frac{\lambda x u(x)}{p} r(x) f(x) = \frac{\lambda}{p} f(t), \tag{64}$$

since $r(x)^{-1} = u(x)^{1/p}$. Now applying Theorem 3.2 and letting $b \to \infty$, the result follows. □

Corollary 4.9 (See Levinson [17], Theorem 5) *Suppose $f(x) \geq 0$, $r(x) > 0$, $1 < p < \infty$, and $r(x)$ be locally absolutely continuous for $x > 0$.*

Let

$$\frac{xr'(x)}{r(x)} - \frac{p-1}{p} \geq \frac{1}{\lambda} \tag{65}$$

for some $\lambda > 0$. If, we write

$$J(x) = \frac{r(x)}{x} \int_x^\infty \frac{f(t)}{r(t)} dt \tag{66}$$

then

$$\int_0^\infty J(x)^p dx \leq \lambda^p \int_0^\infty f(x)^p dx \tag{67}$$

Proof In Theorem 3.1, take $b = \infty$ and $u(x) = (r(x)/x)^p$, $1 < p < \infty$ and $h(t) = f(t)/r(t)$. Now follow the method of proof of Corollary 4.8 and use Theorem 3.1 with $a \to 0$. □

Remark 4.3 If, in Corollary 4.9, $r(x) = x$ and $\lambda = p$, then Corollary 4.9 reduces to the dual inequality related to Hardy's inequality.

Pachpatte [27] followed the method of Levinson [17] to obtain generalization of two theorems of Chan [7]. We can deduce the results of Pachpatte from our Theorems 3.1 and 3.2 by appropriate choice of $u(x)$.

Corollary 4.10 (See Pachpatte [27], Theorem 1) *Let f be a non-negative and Lebesgue integrable function over the interval $[1, b)$, $1 < b \le \infty$. Let $1 < p < \infty$ and $r(x)$ be a positive and locally absolutely continuous function on the interval $[1, b)$. Let*

$$1 - px(\ln x)\frac{r'(x)}{r(x)} \ge \frac{1}{\alpha} \tag{68}$$

for almost all x in $[1, b)$ and for some constant $\alpha > 0$. If $F(x)$ is given by

$$F(x) = \frac{1}{r(x)} \int_x^b \frac{r(t)f(t)}{t} dt, \quad x \in [1, b), \tag{69}$$

then

$$\int_1^b x^{-1} F(x)^p dx \le (\alpha p)^p \int_1^b x^{-1} [\ln x f(x)]^p \, dx. \tag{70}$$

Proof In Theorem 3.1 take $u(x) = x^{-1}(r(x))^{-p}$, $h(x) = \frac{r(x)f(x)}{x}$ and $a = 1$. Then, by integrating by parts,

$$U(x) = \int_1^x \frac{dt}{t(r(t))^p} = \frac{\ln x}{(r(x))^p} - (-p)\int_1^x \frac{(lnx)r'(t)}{r(t)^{p+1}} dt. \tag{71}$$

Using (68) in the integral on the right-hand side of (71), we get, after some calculation,

$$U(x) \le \int_1^x \frac{dt}{t(r(t))^p} = \frac{\ln x}{r(x)^p} + p\int_1^x \frac{(\ln t)r'(t)}{r(t)^{p+1}} dt \tag{72}$$

using (68) to the integral on the right-hand side of (72), and observing that

$$U(x) = \int_1^x \frac{dt}{t(r(t))^p},$$

we will get

$$U(x) \leq \frac{\alpha \ln x}{(r(x))^p}. \tag{73}$$

Now the corollary can be deduced from Theorem 3.1. □

Remark 4.4 If, in Corollary 4.10, we take $r(t) \equiv 1$ in $[1, b)$ and $f(t) = tg(t)$, then $\alpha = 1$ yield Theorem 1 (1a) of Chan [7].

Corollary 4.11 (See Pachpatte [27], Theorem 2) *Let $p \geq 1$ and f be a non-negative and integrable function on* $(0, 1)$. *Let r be a positive and locally absolutely continuous function on* $(0, 1)$. *Suppose further that*

$$1 - px(\ln x)\frac{r'(x)}{r(x)} \geq \frac{1}{\alpha} \tag{74}$$

for almost all x in $(0, 1)$ *and for some constant* $\alpha > 0$.

If, we define,

$$F(x) = \frac{1}{r(x)} \int_0^x \frac{r(t)f(t)}{t}dt, \quad x \in (0, 1), \tag{75}$$

then

$$\int_0^1 \frac{(F(x))^p}{x}dx \leq (\alpha p)^p \int_0^1 x^{-1}\left[|\ln x| f(x)\right]^p dx. \tag{76}$$

Proof Use Theorem 3.2 with $a = 0$, $b = 1$, $h(t) = r(t)f(t)/t$ and $u(x) = x^{-1}r(t)^{-p}$. Then follow the method used in the proof of Corollary 4.10 to get the required results of Corollary 4.11. We leave the details to the interested reader. □

Remark 4.5 The case $r(x) = 1$ for all x in $(0, 1)$, $\alpha = 1$ and $f(x) = xg(x)$, reduces Corollary 4.11 to Theorem 2 of the Chan [7].

Our next set of corollaries will be concerned with inequalities of the type proved by Copsen [8] and Beesack [2]. When those results were proved one felt that they are unique in their findings and could not be unified. We shall show that they follow from our Theorems 3.1 and 3.2 by choosing $u(x)$ appropriately.

Let f and ϕ be positive and measurable functions on $(0, \infty)$ and let us suppose that $\Phi(x) = \int_0^x \phi(t)dt$ exists for all x in $0 < x < \infty$. Whenever the integrals written below have finite values, we can write

$$G_1(x) = \int_0^x f(t)\phi(t)dt, \tag{77}$$

and

$$G_2(x) = \int_x^\infty f(t)\phi(t)dt. \tag{78}$$

Corollary 4.12 (Copson [8], Theorem 1) *If* $0 < b < \infty$, $1 < p < \infty$, *then*

$$\int_0^b G_1(x)\Phi(x)^{-c}\phi(x)dx < \left(\frac{p}{c-1}\right)^p \int_0^b f(x)^p \Phi(x)^{p-c}\phi(x)dx \tag{79}$$

if $0 \le a < \infty$, $0 < p \le 1$, $c > 1$, *and* $\lim_{x\to\infty} \Phi(x) = \infty$, *then*

$$\int_a^\infty G_1(x)^p\Phi(x)^{-c}\phi(x)dx \ge \left(\frac{p}{p-1}\right)^p \int_a^\infty f(x)^p \Phi(x)^{p-c}\phi(x)dx. \tag{80}$$

Proof In Theorem 3.2, take $u(x) = \frac{\phi(x)}{\Phi(x)^c} \quad (x > 0)$. Then

$$U(x) = \frac{1}{(c-1)}\left[\Phi(x)^{1-c} - \Phi(b)^{1-c}\right].$$

□

Since $c > 1$ and $\Phi(x)$ is a monotonically increasing function of x, we can conclude that

$$U(x) \le \frac{1}{(c-1)}\left[\Phi(x)\right]^{1-c}.$$

Now (79) follows from Theorem 3.2 when we substitute for $u(t)$ and $h(t)$.

To obtain (80), we need to note that $\Phi(\infty) = \infty$ and that gives

$$u(t) \le \frac{1}{(c-1)}\left[\Phi(t)\right]^{1-c}, \quad c > 1.$$

Then we use Theorem 3.1 to get the required result.

Corollary 4.13 (Copson [8], Theorems 3 and 4)

(i) If $1 \le p < \infty$, $c > 1$, $0 \le a < \infty$, *then*

$$\int_a^\infty G_2(x)^p\Phi(x)^{-c}\phi(x)dx < \left(\frac{p}{1-c}\right)^p \int_a^\infty f(x)^p \Phi(x)^{p-c}\phi(x)dx \tag{81}$$

(ii) if $0 < p \le 1$, $c < 1$, $0 < b \le \infty$, *then*

$$\int_0^b G_2(x)^p \Phi(x)^{-c}\phi(x)dx \geq \left(\frac{p}{1-c}\right)^p \int_0^b f(x)^p \Phi(x)^{p-c}\phi(x)dx. \quad (82)$$

Proof Use the method outlined in the proof of Corollary 4.12 and use Theorem 3.1 instead of Theorem 3.2. □

Corollary 4.14 (Beesack [2], Results (33))

(i) If $0 < a < \infty$, $1 < p < \infty$, *then*

$$\int_a^\infty G_2(x)^p \Phi(x)^{-1}\phi(x)dx \leq p^p \int_a^\infty f(x)^p \Phi(x)^{p-1} \left\{\ln \frac{\Phi(x)}{\phi(x)}\right\}^p \phi(x)dx \quad (83)$$

(ii) [Copson [8], Theorem 6 and Beesack [2], result (33)]. If $0 < p < 1$, *and* $0 < a < \infty$, *then the inequality in* (83) *is reserved. We also get equality when* $p = 1$.

Proof In Theorem 3.1, let $b \to \infty$, and $u(x) = \phi(x)/\Phi(x)$ and $h(x) = f(x)\phi(x)$. Then

$$U(x) = \int_a^x \frac{\phi(x)}{\Phi(x)}dt = \ln\left\{\frac{\Phi(x)}{\Phi(a)}\right\},$$

since $\phi(t) = \Phi'(t)$ almost everywhere. The results can now be obtained. □

Corollary 4.15 (See Copson [8], Theorem 5 and Beesack [2], Result (28)) *If* $0 < b < \infty$, $1 < p < \infty$, *then*

$$\int_0^b G_1(x)^p \Phi(x)^{-1}\phi(x)dx \leq p^p \int_0^b f(x)^p \Phi(x)^{p-1} \left\{\ln \frac{\Phi(b)}{\Phi(x)}\right\}^p \Phi(x)dx, \quad (84)$$

if $0 < p < 1$ *and* $0 < b < \infty$, *the inequality* (84) *is reserved. If* $p = 1$, (84) *reduces to an equality.*

Proof In Theorem 3.2, let us take $a = 0$, $u(x) = \frac{\phi(x)}{\Phi(x)}$ and $h(t) = f(x)\phi(x)$. Then

$$U(x) = \int_x^b \phi(t)\Phi(t)^{-1}dt = \ln\left\{\frac{\Phi(b)}{\Phi(x)}\right\} \quad (85)$$

since $\Phi'(x) = \phi(x)$ almost everywhere. Now the corollary follows. □

5 Conclusion

As we have obtained many known results from Theorem 3.1–3.3 in Sect. 4, we can also obtain results proved in Mohapatra and Russell [21] and Mohapatra and

Vajravelu [22]. We can also obtain results proved by same authors given in the references. However at this time we are unable to deduce results proved by Love [18] and [19] and Bicheng et al. [3].

A look at Theorem 2.1 of [3] shows that it is an improvement of Hardy's inequality. It will be instructive to see how are can generalize the results proved in [3] and many other papers in the reference so that a number of inequalities can be unified.

The objective of this chapter is to demonstrate that simple use of integration by parts and Hӧlder's inequality which led to Davies–Peterson lemma can deliver fairly general results which unify variants of inequalities of Hardy, Copson, and Levinson types.

One can think of generalizing the results of Sect. 3 to Orlicz spaces and try to obtain interesting results. A look at the result of Love [20] where he has proved Hardy inequalities in Orlicz and Luxemburg norms shows that one can think of generalizations of theorems in Sect. 3 to more general norms as a field for future research. We should also look at the paper of Andersen and Heinig [1] where nice results involving integral operators have been proved. Equally instructive are also the papers of Boas [4], Boas and Imoru [5], and Nemeth [23] where nice results are established. It will be interesting to see if some unification of these results is possible.

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