Multiple Hardy–Littlewood Integral Operator Norm Inequalities

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Abstract How to obtain the sharp constant of the Hardy–Littlewood inequality remains unsolved. In this paper, the new analytical technique is to convert the exact constant factor to the norm of the corresponding operator, the multiple Hardy–Littlewood integral operator norm inequalities are proved. As its generalizations, some new integral operator norm inequalities with the radial kernel on n-dimensional vector spaces are established. The discrete versions of the main results are also given.

Mathematics Subject Classification (2000) 47A30, 26D10

1 Introduction

Throughout this paper, we write

$$E_n(\alpha) = \{x = (x_1, x_2, \cdots, x_n) : x_k \geq 0, 1 \leq k \leq n, \|x\|_\alpha = (\sum_{k=1}^{n} |x_k|^\alpha)^{1/\alpha}, \alpha > 0\},$$

$E_n(\alpha)$ is an n-dimensional vector space, when $1 \leq \alpha < \infty$, $E_n(\alpha)$ is a normed vector space. In particular, $E_n(2)$ is an n-dimensional Euclidean space $\mathbb{R}^n_+$.

$$\|f\|_{p,\omega} = (\int_{E_n(\alpha)} |f(x)|^p \omega(x) dx)^{1/p},$$

$$L^p(\omega) = \{f : f \ is \ measurable, \ and \|f\|_{p,\omega} < \infty\},$$

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D. Andrica, T. M. Rassias (eds.), *Differential and Integral Inequalities*, Springer Optimization and Its Applications 151,
https://doi.org/10.1007/978-3-030-27407-8_17

where, ω is a non-negative measurable function on $E_n(\alpha)$. If $\omega(x) \equiv 1$, we will denote $L^p(\omega)$ by $L^p(E_n(\alpha))$, and $\|f\|_{p,1}$ by $\|f\|_p$. $\Gamma(\alpha)$ is the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \ (\alpha > 0).$$

$B(\alpha, \beta)$ is the Beta function:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \ (\alpha, \beta > 0).$$

The celebrated Hardy–Littlewood inequality (see [1], Theorem 401 and [2–4]) asserts that if f and g are non-negative, and $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} \geq 1$, $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $\delta < 1 - \frac{1}{p}$, $\beta < 1 - \frac{1}{q}$, $\delta + \beta \geq 0$, and $\delta + \beta > 0$, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\delta y^\beta |x-y|^{\lambda-\delta-\beta}} dxdy \leq c(\int_0^\infty f^p dx)^{1/p} (\int_0^\infty g^q dx)^{1/q}. \tag{1}$$

Here, c denotes a positive number depending only on the parameters of the theorem (here p, q, δ, β). But Hardy was unable to fix the constant c in (1). We note that (1) is equivalent to

$$\|T_0 f\|_p \leq c\|f\|_p, \tag{2}$$

where,

$$T_0(f, x) = \int_0^\infty \frac{1}{x^\delta y^\beta |x-y|^{\lambda-\delta-\beta}} f(y) dy. \tag{3}$$

Hence, $c = \|T_0\|$ in (2) is the sharp constant for (1) and (2). Under the above conditions, Hardy–Littlewood [2] proved that there exists a positive constant c_1 such that

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\delta y^\beta |x-y|^{\lambda-\delta-\beta}} dxdy \leq c_1 \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda} dxdy. \tag{4}$$

The following Hardy–Littlewood–Pólya inequality was proved in [5] and [6]:

Theorem 1 *Let* $f \in L^p(0, \infty)$, $g \in L^q(0, \infty)$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$, $0 < \lambda < 1$, $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq c_2 \|f\|_p \|g\|_q, \tag{5}$$

where,

$$c_2 = c_2(p, q, \lambda) = \frac{1}{1-\lambda}\{(\frac{p}{p-1})^{p(1-\frac{1}{q})} + (\frac{q}{q-1})^{q(1-\frac{1}{p})}\}. \tag{6}$$

Let

$$T_1(f, x) = \int_0^\infty \frac{1}{|x-y|^\lambda} f(y)dy. \tag{7}$$

Then (5) is equivalent to

$$\|T_1 f\|_{p_1} \le c_2 \|f\|_p, \tag{8}$$

where, $1 < p < \infty, 1 - \frac{1}{p} < \lambda < 1$, $\frac{1}{p_1} = \frac{1}{p} + \lambda - 1$, c_2 is given by (6). For a function $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, define its potential of order λ as

$$T_2(f, x) = \int_{\mathbb{R}^n} \frac{1}{\|x-y\|_2^\lambda} f(y)dy,\ \ 0 < \lambda < n. \tag{9}$$

Theorem 2 ([6, pp. 412–413]) *There exists a constant c_3 depending only upon n, p, and λ, such that*

$$\|T_2 f\|_{p_2} \le c_3 \|f\|_p, \tag{10}$$

where, $\frac{1}{p_2} = \frac{1}{p} + \frac{\lambda}{n} - 1$.

Theorem 3 ([7–10]) *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty, 0 < \lambda < n$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, then there exists a constant $c_4 = c_4(p, \lambda, n)$ (depending only upon n, p, and λ), such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x-y\|_2^\lambda} dxdy \le c_4 \|f\|_p \|g\|_q, \tag{11}$$

where,

$$c_4 \le \frac{n}{pq(n-\lambda)}(\frac{S_n}{n})^{\lambda/n}\{(\frac{\lambda/n}{1-(1/p)})^{\lambda/n} + (\frac{\lambda/n}{1-(1/q)})^{\lambda/n}\},$$

and S_n is the surface areas of the unit sphere in $\mathbb{R}^n$. In particular, for $p = q = \frac{2n}{2n-\lambda}$,

$$c_4 = \pi^{\lambda/2} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n-\frac{\lambda}{2})}\{\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\}^{\frac{\lambda}{n}-1}$$

is the best possible constant.

But when $p \ne q$, the best possible value of c_4 is also unknown.

In 2017, the author Kuang [14] established the norm inequality of operator T_2.

Theorem 4 ([11]) *Let* $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty$, $0 < \lambda < n$, $\delta + \beta \geq 0$, $1 - \frac{1}{p} - \frac{\lambda}{n} < \frac{\delta}{n} < 1 - \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda+\delta+\beta}{n} = 2$, *then there exists a constant* $c_5 = c_5(p, \delta, \beta, \lambda, n)$, *such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda}}dxdy \leq c_5\|f\|_p\|g\|_q. \tag{12}$$

Remark 1 Inequality (12) can be given an equivalent form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta}}dxdy \leq c_5\|f\|_p\|g\|_q, \tag{13}$$

then the conditions $1 - \frac{1}{p} - \frac{\lambda}{n} < \frac{\delta}{n} < 1 - \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda+\delta+\beta}{n} = 2$ are replaced by

$$\frac{\delta}{n} < 1 - \frac{1}{p} < \frac{\lambda}{n} - \frac{\beta}{n}, \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2.$$

The multiple Hardy–Littlewood integral operator T_3 defined by

$$T_3(f, x) = \int_{\mathbb{R}^n} \frac{f(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta}}dy. \tag{14}$$

Then (13) is equivalent to

$$\|T_3 f\|_p \leq c_5\|f\|_p. \tag{15}$$

But, the problem of determining the best possible constants in (13) and (15) remains unsolved. In this paper, the new analytical technique is to convert the exact constant factor to the norm $c_5 = \|T_3\|$ of the corresponding operator T_3. Hence, we consider operator norm inequality (15). Without loss of generality, we may consider that the multiple Hardy–Littlewood integral operator T_4 defined by

$$T_4(f, x) = \int_{\mathbb{R}^n_+} \frac{f(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta}}dy \tag{16}$$

and f be a nonnegative measurable function on $\mathbb{R}^n_+$, thus, by the triangle inequality, we have

$$|\|x\|_2 - \|y\|_2| \leq \|x-y\|_2 \leq \|x\|_2 + \|y\|_2.$$

Let

$$K_4(x, y) = (\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta})^{-1},$$

$$K_5(x,y) = (\|x\|_2^{\delta}\|y\|_2^{\beta}(\|x\|_2 + \|y\|_2)^{\lambda-\delta-\beta})^{-1},$$

$$K_6(x,y) = (\|x\|_2^{\delta}\|y\|_2^{\beta}(|\|x\|_2 - \|y\|_2|)^{\lambda-\delta-\beta})^{-1},$$

$$T_j(f,x) = \int_{\mathbb{R}_+^n} K_j(x,y)f(y)dy, \tag{17}$$

$$\|T_j\| = \sup_{f\neq 0} \frac{\|T_j f\|_{p,\omega}}{\|f\|_p}, \;\; j = 4,5,6,$$

where, ω is a nonnegative measurable weight function on $\mathbb{R}_+^n$. If $\delta > 0$, $\beta > 0$, $\lambda - \delta - \beta > 0$, then

$$T_5(f,x) \leq T_4(f,x) \leq T_6(f,x),$$

and therefore,

$$\|T_5\| \leq \|T_4\| \leq \|T_6\|. \tag{18}$$

Thus, we may use the norms $\|T_5\|$, $\|T_6\|$ of the operator T_5, T_6 with the radial kernels to find the norm inequality of the multiple Hardy–Littlewood integral operator T_4. As their generalizations, we introduce the new integral operator T defined by

$$T(f,x) = \int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha)f(y)dy, \;\; x \in E_n(\alpha), \tag{19}$$

where, the radial kernel $K(\|x\|_\alpha, \|y\|_\alpha)$ is a nonnegative measurable function defined on $E_n(\alpha) \times E_n(\alpha)$, which satisfies the following condition:

$$K(\|x\|_\alpha, \|y\|_\alpha) = \|x\|_\alpha^{-\lambda} K(1, \|y\|_\alpha \|x\|_\alpha^{-1}), \;\; x, y \in E_n(\alpha), \lambda > 0. \tag{20}$$

Equation (19) includes many famous operators as special cases. In particular, for $n = 1$, we have

$$T(f,x) = \int_0^\infty K(x,y)f(y)dy, \;\; x > 0, \tag{21}$$

and

$$K(x,y) = x^{-\lambda}K(1, yx^{-1}), \;\; x, y > 0, \lambda > 0. \tag{22}$$

The kernel in (3)

$$K(x, y) = \frac{1}{x^{\delta} y^{\beta} |x - y|^{\lambda-\delta-\beta}}$$

satisfies (22). In 2016, the author Kuang [12] proved that if $f \in L^p(\omega_0)$, $g \in L^q(\omega_0)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega_0(x) = x^{1-\lambda}$, and

$$\max\{\frac{1}{p}, \delta + \beta + \frac{1}{q}\} < \lambda < 1 + \delta + \beta < 1 + \frac{1}{p},$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^{\delta} y^{\beta} |x - y|^{\lambda-\delta-\beta}} dxdy \leq c_0 \|f\|_{p,\omega_0} \|g\|_{q,\omega_0},$$

which is equivalent to

$$\|T_0 f\|_p \leq c_0 \|f\|_{p,\omega_0},$$

where, T_0 is defined by (3) and

$$c_0 = B(\lambda - \frac{1}{p}, 1 - \lambda + \delta + \beta) + B(\frac{1}{p} - \delta - \beta, 1 - \lambda + \delta + \beta)$$
$$+B(\frac{1}{q}, 1 - \lambda + \delta + \beta) + B(\lambda - \delta - \beta - \frac{1}{q}, 1 - \lambda + \delta + \beta). \quad (23)$$

We define $\omega_1 = x^{\lambda-1}$, then the above norm inequality is also equivalent to

$$\|T_0 f\|_{p,\omega_1} \leq c_0 \|f\|_p. \quad (24)$$

The celebrated Hardy–Littlewood inequality (1) and (2) are important in analysis mathematics and its applications. In this paper, we give some new improvements and extensions of (24). As some further generalizations of the above results, the norm inequalities of the multiple integral operators with the radial kernels on n-dimensional vector spaces $E_n(\alpha)$ are established. In particular, using new analytical techniques, we convert the exact constant factor we are looking for into the norm of the corresponding operator, under a somewhat different hypothesis, we get lower and upper bounds of the sharp constant of the multiple Hardy–Littlewood inequality. Finally, the discrete versions of the main results are also given in Sect. 6.

2 Main Results

Our main results read as follows.

Theorem 5 *Let* $1 < p, q < \infty, \lambda \geq n > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, $0 \leq \delta < 1 - \frac{1}{q}, 0 \leq \beta < 1 - \frac{1}{p}$, *and*

$$\max\{\beta + 1 - \frac{1}{q}, \delta + n(1 - \frac{1}{p})\} < \lambda < \min\{\frac{\delta + \beta}{1 - (1/n)}, \frac{\delta + \beta}{1 - \frac{1}{pn(1-(1/q))}}\}.$$

If $f \in L^p(\mathbb{R}^n_+)$, $f(x) \geq 0, x \in \mathbb{R}^n_+$, $\omega(x) = \|x\|_2^{p(\lambda-n)}$, *then the multiple Hardy–Littlewood integral operator* T_4 *is defined by (16):* $T_4 : L^p(\mathbb{R}^n_+) \to L^p(\omega)$ *exists as a bounded operator and*

$$c_3 \leq \|T_4\| \leq c_1^{1-(1/p)} c_2^{1/p},$$

where,

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))$$
$$+ B(\frac{n}{\lambda}(\lambda - \delta - \frac{1}{q} + 1) - n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))\},$$
$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{pn}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))$$
$$+ B(\frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - 1 + \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))\},$$
$$c_3 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(n(1 - \frac{1}{p}) - \beta, \lambda - \delta - n(1 - \frac{1}{p})).$$

For $n = 1$, we have

Theorem 6 *Let* $1 < p, q < \infty, \lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $0 \leq \beta < 1 - \frac{1}{p}, 0 \leq \delta < 1 - \frac{1}{q}$, *and*

$$\max\{\beta + 1 - \frac{1}{q}, \delta + 1 - \frac{1}{p}\} < \lambda < \frac{\delta + \beta}{1 - \frac{1}{p(1-(1/q))}}.$$

If $f \in L^p(0, \infty)$, $f(x) \geq 0, x \in (0, \infty)$, $\omega(x) = x^{p(\lambda-1)}$, *then the Hardy–Littlewood integral operator* T_0 *is defined by (3):*$T_0 : L^p(0, \infty) \to L^p(\omega)$ *exists as a bounded operator and*

$$c_3 \leq \|T_0\| \leq c_1^{1-(1/p)} c_2^{1/p}$$

where,

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q} - 1 - \beta) + 1, \frac{\delta + \beta}{\lambda}) + B(\frac{1}{\lambda}(1 - \delta - \frac{1}{q}), \frac{\delta + \beta}{\lambda})$$

$$c_2 = B(\frac{p}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - (1 - \frac{\delta + \beta}{\lambda})p(1 - \frac{1}{q}))$$
$$+ B(p(1 - \frac{1}{q})(1 - \frac{1}{\lambda}(\delta + 1 - \frac{1}{p})), 1 - (1 - \frac{\delta + \beta}{\lambda})p(1 - \frac{1}{q})),$$

$$c_3 = B(1 - \beta - \frac{1}{p}, \lambda - \delta - 1 + \frac{1}{p})$$

Corollary 1 *Let* $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 \leq \delta < \frac{1}{p}$, $0 \leq \beta < \frac{1}{q}$, $\delta+\beta > 0$, $\lambda = 1$. *If* $f \in L^p(0, \infty)$, $f(x) \geq 0$, $x \in (0, \infty)$, *then the integral operator* T_0 *is defined by (3):* $T_0 : L^p(0, \infty) \to L^p(0, \infty)$ *exists as a bounded operator and*

$$B(\frac{1}{p} - \delta, \frac{1}{q} - \beta) \leq \|T_0\| \leq B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta).$$

As some further generalizations of the above results, we have

Theorem 7 *Let* $1 < p < \infty$, $1 < q < \infty$, $\delta, \beta \geq 0$, $\lambda \geq n$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, $\omega(x) = \|x\|_\alpha^{p(\lambda-n)}$, *the radial kernel* $K(\|x\|_\alpha, \|y\|_\alpha)$ *satisfies (20).*

(i) If

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty (K(1, u))^{n/\lambda} u^{\frac{n}{\lambda}(\frac{1}{q}-1)+n-1} du < \infty, \tag{25}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty (K(1, u))^{\frac{pn}{\lambda}(1-\frac{1}{q})} u^{\frac{n(p-1)(q-1)}{\lambda q}-1} du < \infty, \tag{26}$$

then the integral operator T *is defined by (19):*$T : L^p(E_n(\alpha)) \to L^p(\omega)$ *exists as a bounded operator and*

$$\|Tf\|_{p,\omega} \leq c\|f\|_p. \tag{27}$$

This implies that

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \leq c, \tag{28}$$

where,

$$c = c_1^{(1-(1/p))} c_2^{1/p}. \tag{29}$$

(ii) If

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1,u)u^{(1-\frac{1}{p})n-1} du < \infty, \tag{30}$$

then

$$\|T\| \geq c_3. \tag{31}$$

In particular, for $n = 1$, by Theorem 7, we get

Theorem 8 *Let* $1 < p < \infty, 1 < q < \infty$, $\delta, \beta \geq 0$, $1 \leq \lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $\omega(x) = x^{p(\lambda-1)}$, *the radial kernel* $K(x, y)$ *satisfies (22).*

(i) If

$$c_1 = \int_0^\infty (K(1,u))^{1/\lambda} u^{\frac{1}{\lambda}(\frac{1}{q}-1)} du < \infty, \tag{32}$$

$$c_2 = \int_0^\infty (K(1,u))^{\frac{p}{\lambda}(1-\frac{1}{q})} u^{\frac{(p-1)(q-1)}{\lambda q}-1} du < \infty, \tag{33}$$

then the integral operator T *is defined by (21):*$T : L^p(0,\infty) \to L^p(\omega)$ *exists as a bounded operator and*

$$\|Tf\|_{p,\omega} \leq c\|f\|_p. \tag{34}$$

This implies that

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \leq c, \tag{35}$$

where,

$$c = c_1^{(1-(1/p))} c_2^{1/p}. \tag{36}$$

(ii) If

$$c_3 = \int_0^\infty K(1,u) u^{-\frac{1}{p}} du < \infty, \tag{37}$$

then

$$\|T\| \geq c_3. \tag{38}$$

For $\lambda = n$, we have $\frac{1}{p} + \frac{1}{q} = 1$, and by Theorem 7, we get

Theorem 9 *Let* $1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1, \delta, \beta \geq 0$*, the radial kernel* $K(\|x\|_\alpha, \|y\|_\alpha)$ *satisfies:*

$$K(\|x\|_\alpha, \|y\|_\alpha) = \|x\|_\alpha^{-n} K(1, \|y\|_\alpha \|x\|_\alpha^{-1}),\ \ x, y \in E_n(\alpha).$$

(i) If

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, u) u^{-(1/p)+n-1} du < \infty, \tag{39}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, u) u^{-(1/p)} du < \infty, \tag{40}$$

then the integral operator T *is defined by (19):* $T : L^p(E_n(\alpha)) \to L^p(E_n(\alpha))$ *exists as a bounded operator and*

$$\|Tf\|_p \leq c\|f\|_p. \tag{41}$$

This implies that

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|_p}{\|f\|_p} \leq c, \tag{42}$$

where,

$$c = c_1^{(1/q)} c_2^{(1/p)}. \tag{43}$$

(ii) If

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, u) u^{(n/q)-1} du < \infty, \tag{44}$$

then,

$$\|T\| \geq c_3. \tag{45}$$

In particular, for $n = 1$, by Theorem 9, we get

$$c = c_1 = c_2 = c_3 = \int_0^\infty K(1, u) u^{-(1/p)} du, \tag{46}$$

then by (42), (45), and (46), we get

$$\|T\| = c = \int_0^\infty K(1, u) u^{-(1/p)} du. \tag{47}$$

Thus, we get the following

Corollary 2 *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *the kernel* $K(x, y)$ *satisfies (22). Then the integral operator* T *is defined by (21):* $T : L^p(0, \infty) \to L^p(0, \infty)$ *exists as a bounded operator and*

$$\|Tf\|_p \leq c\|f\|_p, \tag{48}$$

where $\|T\| = c = \int_0^\infty K(1, u)u^{-(1/p)}du$ *is the sharp constant.*

3 Proofs of Theorems

We require the following lemmas to prove our results:

Lemma 1 ([4, 13]) *If* $a_k, b_k, p_k > 0, 1 \leq k \leq n, f$ *is a measurable function on* $(0, \infty)$, *then*

$$\int_{\mathbb{R}_+^n} f\Big(\sum_{k=1}^n (\frac{x_k}{a_k})^{b_k}\Big) x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n$$
$$= \frac{\prod_{k=1}^n a_k^{p_k}}{\prod_{k=1}^n b_k} \times \frac{\prod_{k=1}^n \Gamma(\frac{p_k}{b_k})}{\Gamma(\sum_{k=1}^n \frac{p_k}{b_k})} \int_0^\infty f(t)t^{(\sum_{k=1}^n \frac{p_k}{b_k} - 1)} dt.$$

We get the following Lemma 2 by taking $a_k = 1, b_k = \alpha > 0, p_k = 1, 1 \leq k \leq n$, in Lemma 1.

Lemma 2 *Let* f *be a measurable function on* $(0, \infty)$, *then*

$$\int_{E_n(\alpha)} f(\|x\|_\alpha^\alpha)dx = \frac{(\Gamma(1/\alpha))^n}{\alpha^n \Gamma(n/\alpha)} \int_0^\infty f(t)t^{(n/\alpha)-1}dt. \tag{49}$$

Proof of Theorem 7

(i) Let

$$p_1 = \frac{p}{p-1}, \; q_1 = \frac{q}{q-1},$$

thus, we have

$$\frac{1}{p_1} + \frac{1}{q_1} + (1 - \frac{\lambda}{n}) = 1, \; \frac{p}{q_1} + p(1 - \frac{\lambda}{n}) = 1.$$

By Hölder's inequality, we get

$$\begin{aligned}T(f,x) &= \int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha) f(y)dy \\ &= \int_{E_n(\alpha)} \{\|y\|_\alpha^{(\frac{n}{p_1\lambda})} K^{n/\lambda}(\|x\|_\alpha, \|y\|_\alpha) f^p(y)\}^{1/q_1} \\ &\times \{K^{n/\lambda}(\|x\|_\alpha, \|y\|_\alpha)\|y\|_\alpha^{-(\frac{n}{q_1\lambda})}\}^{1/p_1}\{f(y)\}^{p(1-\frac{\lambda}{n})}dy \\ &\le \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} K^{n/\lambda}(\|x\|_\alpha, \|y\|_\alpha)|f(y)|^p dy\}^{1/q_1} \\ &\times \{\int_{E_n(\alpha)} \|y\|_\alpha^{-(\frac{n}{q_1\lambda})} K^{n/\lambda}(\|x\|_\alpha, \|y\|_\alpha)dy\}^{1/p_1} \|f\|_p^{p(1-\frac{\lambda}{n})} \\ &= I_1^{1/q_1} \times I_2^{1/p_1} \times \|f\|_p^{p(1-\frac{\lambda}{n})}, \end{aligned} \tag{50}$$

where,

$$I_1 = \int_{E_n(\alpha)} \|y\|_\alpha^{(\frac{n}{p_1\lambda})} K^{n/\lambda}(\|x\|_\alpha, \|y\|_\alpha)|f(y)|^p dy,$$

$$I_2 = \int_{E_n(\alpha)} \|y\|_\alpha^{-(\frac{n}{q_1\lambda})} K^{n/\lambda}(\|x\|_\alpha, \|y\|_\alpha)dy.$$

In I_2, by using Lemma 2, and letting $u = \|x\|_\alpha^{-1} t^{1/\alpha}$, and use (20), (49), and (25), we get

$$\begin{aligned}I_2 &= \|x\|_\alpha^{-n} \int_{E_n(\alpha)} \|y\|_\alpha^{-(\frac{n}{q_1\lambda})} K^{n/\lambda}(1, \|y\|_\alpha \cdot \|x\|_\alpha^{-1})dy \\ &= \|x\|_\alpha^{-n} \frac{(\Gamma(1/\alpha))^n}{\alpha^n \Gamma(n/\alpha)} \int_0^\infty t^{-(\frac{n}{q_1\lambda\alpha})} K^{n/\lambda}(1, t^{1/\alpha}\|x\|_\alpha^{-1}) \times t^{\frac{n}{\alpha}-1}dt \\ &= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \|x\|_\alpha^{-(\frac{n}{q_1\lambda})} \int_0^\infty K^{\frac{n}{\lambda}}(1,u) u^{-(\frac{n}{q_1\lambda})+n-1} du \\ &= c_1 \|x\|_\alpha^{-(\frac{n}{q_1\lambda})}. \end{aligned} \tag{51}$$

Hence, by (50) and (51), we conclude that

$$\begin{aligned}\|Tf\|_{p,\omega} &= (\int_{E_n(\alpha)} |T(f,x)|^p \omega(x)dx)^{1/p} \le (\int_{E_n(\alpha)} I_1^{\frac{p}{q_1}} I_2^{\frac{p}{p_1}} \|f\|_p^{p^2(1-\frac{\lambda}{n})} \omega(x)dx)^{1/p} \\ &= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} (\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} K^{\frac{n}{\lambda}}(\|x\|_\alpha, \|y\|_\alpha)|f(y)|^p dy)^{\frac{p}{q_1}} \\ &\times \|x\|_\alpha^{-\frac{pn}{p_1 q_1 \lambda}} \omega(x)dx\}^{1/p}. \end{aligned} \tag{52}$$

Using the Minkowski's inequality for integrals (see [3]):

$$\{\int_X (\int_Y |f(x,y)|dy)^p \omega(x)dx\}^{1/p} \le \int_Y \{\int_X |f(x,y)|^p \omega(x)dx\}^{1/p} dy,\ 1 \le p < \infty,$$

and letting $v = \|y\|_\alpha t^{-(1/\alpha)}$, we obtain

$$\begin{aligned}
\|Tf\|_{p,\omega} &\le c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times (\int_{E_n(\alpha)} K^{\frac{pn}{q_1\lambda}}(\|x\|_\alpha, \|y\|_\alpha) \|x\|_\alpha^{-\frac{pn}{p_1q_1\lambda}} \omega(x)dx)^{\frac{q_1}{p}} dy\}^{1/q_1} \\
&= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times (\int_{E_n(\alpha)} K^{\frac{pn}{q_1\lambda}}(1, \|y\|_\alpha \cdot \|x\|_\alpha^{-1}) \|x\|_\alpha^{-\frac{pn}{q_1} - \frac{pn}{p_1q_1\lambda} + p(\lambda-n)} dx)^{\frac{q_1}{p}} dy\}^{\frac{1}{q_1}} \\
&= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times \Big(\frac{(\Gamma(1/\alpha))^n}{\alpha^n \Gamma(n/\alpha)} \int_0^\infty K^{\frac{pn}{q_1\lambda}}(1, \|y\|_\alpha \cdot t^{-\frac{1}{\alpha}}) t^{-\frac{pn}{q_1\alpha} - \frac{pn}{\lambda\alpha p_1 q_1} + \frac{p(\lambda-n)}{\alpha}} t^{\frac{n}{\alpha}-1} dt\Big)^{\frac{q_1}{p}} dy\}^{\frac{1}{q_1}} \\
&= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times \Big(\|y\|_\alpha^{-\frac{pn}{q_1p_1\lambda}} \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K^{\frac{pn}{q_1\lambda}}(1, v) v^{\frac{pn}{p_1q_1\lambda}-1} dv\Big)^{\frac{q_1}{p}} dy\}^{\frac{1}{q_1}} \\
&= c_1^{1/p_1} c_2^{1/p} \|f\|_p^{p(1-\frac{\lambda}{n})} \|f\|_p^{\frac{p}{q_1}} = c_1^{(1-(1/p))} c_2^{1/p} \|f\|_p.
\end{aligned}$$

Thus,

$$\|Tf\|_{p,\omega} \le c\|f\|_p. \tag{53}$$

(ii) For proving (31), we take

$$\begin{aligned}
f_\varepsilon(x) &= \|x\|_\alpha^{-(n/p)-\varepsilon} \varphi_{B^c}(x), \\
g_\varepsilon(x) &= (p\varepsilon)^{1/p_1} \{\frac{\alpha^{n-1}\Gamma(n/\alpha)}{(\Gamma(1/\alpha))^n}\}^{1/p_1} \|x\|_\alpha^{-\frac{n}{p_1}-(p-1)\varepsilon} \varphi_{B^c}(x),
\end{aligned}$$

where, $\varepsilon > 0, B = B(0,1) = \{x \in E_n(\alpha) : \|x\|_\alpha < 1\}$, φ_{B^c} is the characteristic function of the set $B^c = \{x \in E_n(\alpha) : \|x\|_\alpha \ge 1\}$, that is

$$\varphi_{B^c}(x) = \begin{cases} 1, x \in B^c \\ 0, x \in B. \end{cases}$$

Thus, we get

$$\|f_\varepsilon\|_p = \Big(\frac{(\Gamma(1/\alpha))^n}{p\varepsilon\alpha^{n-1}\Gamma(n/\alpha)}\Big)^{1/p},$$

$$\|g_\varepsilon\|_{p_1}^{p_1} = (p\varepsilon)\Big(\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)}\Big)^{-1}\int_{B^c}\|x\|_\alpha^{-n-(p-1)p_1\varepsilon}dx$$

$$= (p\varepsilon)\frac{1}{\alpha}\int_1^\infty t^{-\frac{p\varepsilon}{\alpha}-1}dt = 1.$$

Using the sharpness in Hölder's inequality (see [13]):

$$\|Tf\|_{p,\omega} = \sup\{|\int_{E_n(\alpha)} T(f,x)g(x)(\omega(x))^{1/p}dx| : \|g\|_{p_1} \le 1\},$$

where, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p_1} = 1$, thus, if $\|g\|_{p_1} \le 1$, then

$$|\int_{E_n(\alpha)} T(f,x)g(x)\{\omega(x)\}^{1/p}dx| \le \|Tf\|_{p,\omega}. \tag{54}$$

By (54) and (19), we get

$$\|Tf_\varepsilon\|_{p,\omega} \ge \int_{E_n(\alpha)} T(f_\varepsilon,x)g_\varepsilon(x)\{\omega(x)\}^{1/p}dx$$

$$= \int_{E_n(\alpha)}\int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha)f_\varepsilon(y)g_\varepsilon(x)\|x\|_\alpha^{\lambda-n}dydx$$

$$= (p\varepsilon)^{1/p_1}\{\frac{\alpha^{n-1}\Gamma(n/\alpha)}{(\Gamma(1/\alpha))^n}\}^{1/p_1}$$

$$\times \int_{B^c}\{\int_{B^c} K(\|x\|_\alpha, \|y\|_\alpha)\|y\|_\alpha^{-(n/p)-\varepsilon}dy\}\|x\|_\alpha^{-\frac{n}{p_1}-(p-1)\varepsilon+\lambda-n}dx. \tag{55}$$

Letting $u = t^{1/\alpha}\|x\|_\alpha^{-1}$, and using (20), we have

$$\int_{B^c} K(\|x\|_\alpha, \|y\|_\alpha)\|y\|_\alpha^{-(n/p)-\varepsilon}dy$$

$$= \frac{(\Gamma(1/\alpha))^n}{\alpha^n\Gamma(n/\alpha)}\|x\|_\alpha^{-\lambda}\int_1^\infty K(1, t^{1/\alpha}\|x\|_\alpha^{-1})t^{-(\frac{n}{p\alpha})-\frac{\varepsilon}{\alpha}+\frac{n}{\alpha}-1}dt$$

$$= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\|x\|_\alpha^{-\lambda+\frac{n}{p_1}-\varepsilon}\int_{\|x\|_\alpha^{-1}}^\infty K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}du. \tag{56}$$

We insert (56) into (55) and use Fubini's theorem to obtain

$$
\begin{aligned}
\|Tf_\varepsilon\|_{p,\omega} &\geq (p\varepsilon)^{1/p_1}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{1/p} \\
&\times \int_{B^c} \|x\|_\alpha^{-p\varepsilon-n}(\int_{\|x\|_\alpha^{-1}}^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}du)dx \\
&= (p\varepsilon)^{1/p_1}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{1/p} \\
&\times \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\int_{\beta(u)}^{\infty} \|x\|_\alpha^{-p\varepsilon-n}dx)du \\
&= (p\varepsilon)^{1/p_1}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{(1/p)+1} \times \frac{1}{\alpha} \\
&\times \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\int_{\beta(u)}^{\infty} t^{-(p\varepsilon)/\alpha-1}dt)du \\
&= (p\varepsilon)^{-(1/p)}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{(1/p)+1} \\
&\times \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\beta(u))^{-(p\varepsilon)/\alpha}du,
\end{aligned}
$$

where, $\beta(u) = \max\{1, u^{-1}\}$. Thus, we get

$$
\begin{aligned}
\|T\| &= \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \geq \frac{\|Tf_\varepsilon\|_{p,\omega}}{\|f_\varepsilon\|_p} \\
&\geq \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\beta(u))^{-(p\varepsilon)/\alpha}du. \qquad (57)
\end{aligned}
$$

By letting $\varepsilon \to 0^+$ in (57) and using Fatou lemma, we get

$$
\|T\| \geq \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-1}du = c_3.
$$

The proof is complete.

4 Some Applications

As applications, a large number of known and new results have been obtained by proper choice of kernel K. In this section we present some model applications which display the importance of our results.

Example 1 Let $h : E_n(\alpha) \times E_n(\alpha) \to \mathbb{R}_+$ be a measurable function. K_7 is defined by

$$K_7(\|x\|_\alpha, \|y\|_\alpha) = \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta |\|x\|_\alpha - \|y\|_\alpha|^{\lambda-\delta-\beta}}, \tag{58}$$

and let

$$T_7(f, x) = \int_{E_n(\alpha)} \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta |\|x\|_\alpha - \|y\|_\alpha|^{\lambda-\delta-\beta}} f(y)dy.$$

If p, q, λ, and ω satisfy the conditions of Theorem 7, and

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta |1-u|^{\lambda-\delta-\beta}}\}^{\frac{n}{\lambda}} u^{\frac{n}{\lambda}(\frac{1}{q}-1)+n-1} du < \infty, \tag{59}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta |1-u|^{\lambda-\delta-\beta}}\}^{\frac{pn}{\lambda}(1-\frac{1}{q})} u^{\frac{n}{\lambda}(p-1)(1-\frac{1}{q})-1} du < \infty, \tag{60}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{h(u)}{u^\beta |1-u|^{\lambda-\delta-\beta}} u^{n(1-\frac{1}{p})-1} du < \infty, \tag{61}$$

then by Theorem 7, we get

$$c_3 \le \|T_7\| \le c_1^{(1-(1/p))} c_2^{1/p}. \tag{62}$$

Setting $h(u) = 1$, we distinguish four cases:

(i) The case $n > 1$. Let $0 \le \delta < 1 - \frac{1}{q}$, $0 \le \beta < 1 - \frac{1}{p}$, and

$$\max\{\beta+1-\frac{1}{q}, \delta+n(1-\frac{1}{p})\} < \lambda < \min\{1+\delta+\beta, \frac{\delta+\beta}{1-(1/n)}, \frac{\delta+\beta}{1-\frac{1}{pn(1-(1/q))}}\},$$

then by (59), (60), and (61), we get

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \{B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta)) + B(\frac{n}{\lambda}(\lambda - \delta - \frac{1}{q} + 1) - n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))\}, \tag{63}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \{B(\frac{pn}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta)) + B(\frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - 1 + \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))\}, \tag{64}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(n(1-\frac{1}{p})-\beta, 1-\lambda+\delta+\beta)$$
$$+ B(\lambda-\delta-n(1-\frac{1}{p}), 1-\lambda+\delta+\beta)\}. \quad (65)$$

(ii) The case $n = 1$. Let $0 \le \beta < 1 - \frac{1}{p}$,$0 \le \delta < 1 - \frac{1}{q}$,$\delta + \beta > 0$, and

$$\max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\} < \lambda < \min\{1 + \delta + \beta, \frac{\delta + \beta}{1 - \frac{1}{p(1-(1/q))}}\},$$

then by (59), (60), and (61), we get

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q} - 1 - \beta) + 1, \frac{\delta + \beta}{\lambda}) + B(\frac{1}{\lambda}(1 - \delta - \frac{1}{q}), \frac{\delta + \beta}{\lambda}), \quad (66)$$

$$c_2 = B(\frac{p}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - (1 - \frac{\delta + \beta}{\lambda})p(1 - \frac{1}{q}))$$
$$+ B(p(1 - \frac{1}{q})(1 - \frac{1}{\lambda}(\delta + 1 - \frac{1}{p})), 1 - (1 - \frac{\delta + \beta}{\lambda})p(1 - \frac{1}{q})), \quad (67)$$

$$c_3 = B(1-\beta-\frac{1}{p}, 1-\lambda+\delta+\beta) + B(\lambda-\delta-1+\frac{1}{p}, 1-\lambda+\delta+\beta). \quad (68)$$

(iii) The case $\lambda = n$, this implies that $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le \delta < \min\{\frac{1}{p}, n - \frac{1}{q}\}$, $0 \le \beta < \min\{\frac{1}{q}, n - \frac{1}{p}\}$, $n - 1 < \delta + \beta$, then by (59), (60), and (61), we get

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(n - \frac{1}{p} - \beta, 1 - n + \delta + \beta)$$
$$+ B(\frac{1}{p} - \delta, 1 - n + \delta + \beta)\}, \quad (69)$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(\frac{1}{q} - \beta, 1 - n + \delta + \beta)$$
$$+ B(n - \delta - \frac{1}{q}, 1 - n + \delta + \beta)\}, \quad (70)$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(\frac{n}{q} - \beta, 1 - n + \delta + \beta)$$
$$+ B(\frac{n}{p} - \delta, 1 - n + \delta + \beta)\}. \quad (71)$$

(iv) The case $\lambda = n = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$, $\delta+\beta > 0$, then by (69), (70), and (71), we get

$$\|T_7\| = B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta). \tag{72}$$

Example 2 Let $h : E_n(\alpha) \times E_n(\alpha) \to \mathbb{R}_+$ be a measurable function. K_8 is defined by

$$K_8(\|x\|_\alpha, \|y\|_\alpha) = \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta (\|x\|_\alpha + \|y\|_\alpha)^{\lambda-\delta-\beta}}, \tag{73}$$

and let

$$T_8(f, x) = \int_{E_n(\alpha)} \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta (\|x\|_\alpha + \|y\|_\alpha)^{\lambda-\delta-\beta}} f(y)dy.$$

If p, q, λ, and ω satisfy the conditions of Theorem 7, and

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta(1+u)^{\lambda-\delta-\beta}}\}^{\frac{n}{\lambda}} u^{\frac{n}{\lambda}(\frac{1}{q}-1)+n-1} du < \infty, \tag{74}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta(1+u)^{\lambda-\delta-\beta}}\}^{\frac{pn}{\lambda}(1-\frac{1}{q})} u^{\frac{n}{\lambda}(p-1)(1-\frac{1}{q})-1} du < \infty, \tag{75}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta(1+u)^{\lambda-\delta-\beta}}\} u^{n(1-\frac{1}{p})-1} du < \infty, \tag{76}$$

then by Theorem 7, we get

$$c_3 \le \|T_8\| \le c_1^{(1-(1/p))} c_2^{1/p}.$$

Setting $h(u) = 1$, we distinguish four cases:

(i) The case $n > 1$. Let $0 \le \delta < 1 - \frac{1}{q}$, $0 \le \beta < 1 - \frac{1}{p}$, and

$$\lambda > \max\{\beta + 1 - \frac{1}{q}, \delta + n(1 - \frac{1}{p})\},$$

then by (74), (75), and (76), we get

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, \frac{n}{\lambda}(\lambda - \delta + 1 - \frac{1}{q}) - n), \tag{77}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{pn}{\lambda}(1-\frac{1}{q})(1-\frac{1}{p}-\beta), \frac{pn}{\lambda}(1-\frac{1}{q})(\lambda-\delta-1+\frac{1}{p})), \tag{78}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(n(1-\frac{1}{p})-\beta, \lambda-\delta-n(1-\frac{1}{p})). \tag{79}$$

(ii) The case $n = 1$. Let $0 \le \beta < 1 - \frac{1}{p}$, $0 \le \delta < 1 - \frac{1}{q}$, and $\lambda > \max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\}$, then by (74), (75), and (76), we get

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q}-1-\beta)+1, \frac{1}{\lambda}(1-\delta-\frac{1}{q})), \tag{80}$$

$$c_2 = B(\frac{p}{\lambda}(1-\frac{1}{q})(1-\frac{1}{p}-\beta), \frac{p}{\lambda}(1-\frac{1}{q})(\lambda-\delta-1+\frac{1}{p})), \tag{81}$$

$$c_3 = B(1-\beta-\frac{1}{p}, \lambda-\delta-1+\frac{1}{p}). \tag{82}$$

(iii) The case $\lambda = n$, this implies that $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le \beta < \frac{1}{q}$, $0 \le \delta < \min\{\frac{n}{p}, n - \frac{1}{q}\}$, then by (74), (75), and (76), we get

$$c_1 = c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{1}{q}-\beta, n-\delta-\frac{1}{q}), \tag{83}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{n}{q}-\beta, \frac{n}{p}-\delta), \tag{84}$$

and

$$c_3 \le \|T_8\| \le c_1. \tag{85}$$

(iv) The case $\lambda = n = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$, $\alpha + \beta > 0$, and $\max\{\beta + \frac{1}{p}, \delta + \frac{1}{q}\} < 1$, then by (83), (84), and (85), we get

$$\|T_8\| = B(\frac{1}{p}-\delta, \frac{1}{q}-\beta). \tag{86}$$

5 Multiple Hardy–Littlewood Integral Operator Norm Inequalities

In Examples 1 and 2, setting $h(u) = 1, \alpha = 2$, thus, $E_n(\alpha)$ reduces to $E_n(2) = \mathbb{R}_+^n$, T_7, T_8 reduces to T_6, T_5, respectively. Assume $f \in L^p(\mathbb{R}_+^n)$, $f(x) \geq 0, x \in \mathbb{R}_+^n$, $1 < p, q < \infty, \lambda \geq n, \delta, \beta \geq 0, \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$. The multiple Hardy–Littlewood integral operator T_4 is defined by (16):$T_4 : L^p(\mathbb{R}_+^n) \to L^p(\omega)$, where $\omega(x) = \|x\|_2^{p(\lambda-n)}$ and

$$\|T_4\| = \sup_{f \neq 0} \frac{\|T_4\|_{p,\omega}}{\|f\|_p}.$$

We distinguish four cases:

(i) The case $n > 1$. Let $0 \leq \delta < 1 - \frac{1}{q}, 0 \leq \beta < 1 - \frac{1}{p}$, and

$$\max\{\beta + 1 - \frac{1}{q}, \delta + n(1 - \frac{1}{p})\} < \lambda < \min\{\frac{\delta + \beta}{1 - (1/n)}, \frac{\delta + \beta}{1 - \frac{1}{pn(1-(1/q))}}\},$$

then by (18), (63), (64) and (79), we get

$$c_3 \leq \|T_4\| \leq c_1^{1-(1/p)} c_2^{1/p}, \tag{87}$$

where

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta)) + B(\frac{n}{\lambda}(\lambda - \delta - \frac{1}{q} + 1) - n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))\}, \tag{88}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{pn}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta)) + B(\frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - 1 + \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))\}, \tag{89}$$

$$c_3 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(n(1 - \frac{1}{p}) - \beta, \lambda - \delta - n(1 - \frac{1}{p})). \tag{90}$$

(ii) The case $n = 1$. Let $0 \leq \beta < 1 - \frac{1}{p}, 0 \leq \delta < 1 - \frac{1}{q}$, and

$$\max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\} < \lambda < \frac{\delta + \beta}{1 - \frac{1}{p(1-(1/q))}},$$

then by (18), (66), (67) and (82), we get

$$c_3 \le \|T_4\| \le c_1^{1-(1/p)} c_2^{1/p},$$

where,

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q} - 1 - \beta) + 1, \frac{\delta+\beta}{\lambda}) + B(\frac{1}{\lambda}(1 - \delta - \frac{1}{q}), \frac{\delta+\beta}{\lambda}), \tag{91}$$

$$c_2 = B(\frac{p}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - (1 - \frac{\delta+\beta}{\lambda})p(1 - \frac{1}{q}))$$
$$+ B(p(1 - \frac{1}{q})(1 - \frac{1}{\lambda}(\delta + 1 - \frac{1}{p})), 1 - (1 - \frac{\delta+\beta}{\lambda})p(1 - \frac{1}{q})), \tag{92}$$

$$c_3 = B(1 - \beta - \frac{1}{p}, \lambda - \delta - 1 + \frac{1}{p}). \tag{93}$$

(iii) The case $\lambda = n$, this implies that $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$, and

$$\max\{\beta + \frac{1}{p}, \delta + \frac{1}{q}\} < n < 1 + \delta + \beta,$$

then by (18), (69), (70), and (84), we get

$$c_3 \le \|T_4\| \le c_1^{1-(1/p)} c_2^{1/p},$$

where

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(n - \frac{1}{p} - \beta, 1 - n + \delta + \beta)$$
$$+ B(\frac{1}{p} - \delta, 1 - n + \delta + \beta)\}, \tag{94}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{1}{q} - \beta, 1 - n + \delta + \beta)$$
$$+ B(n - \delta - \frac{1}{q}, 1 - n + \delta + \beta)\}, \tag{95}$$

$$c_3 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(\frac{n}{q} - \beta, \frac{n}{p} - \delta). \tag{96}$$

(iv) The case $\lambda = n = 1$. Let $0 \leq \delta < \frac{1}{p}$, $0 \leq \beta < \frac{1}{q}$, $\delta + \beta > 0$, then by (18), (72), and (86), we get

$$B(\frac{1}{p} - \delta, \frac{1}{q} - \beta) \leq \|T_4\| \leq B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta). \tag{97}$$

We have thus also proved that Theorems 5 and 6 are correct.

6 The Discrete Versions of the Main Results

Let $a = \{a_m\}$ be a sequence of real numbers, we define

$$\|a\|_{p,\omega} = \{\sum_{m=1}^{\infty} |a_m|^p \omega(m)\}^{1/p}, \ l^p(\omega) = \{a = \{a_m\} : \|a\|_{p,\omega} < \infty\}.$$

If $\omega(m) \equiv 1$, we will denote $l^p(\omega)$ by l^p, and $\|a\|_{p,1}$ by $\|a\|_p$. Defining f, K by $f(x) = a_m$, $K(x, y) = K(m, n)(m - 1 \leq x < m, n - 1 \leq y < n)$, respectively, we obtain the corresponding series form of (21):

$$T(a, m) = \sum_{n=1}^{\infty} K(m, n) a_n. \tag{98}$$

Then by Theorem 8, we get

Theorem 10 *Let* $1 < p < \infty$, $1 < q < \infty$, $\delta, \beta \geq 0$, $\delta + \beta > 0$, $1 \leq \lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $\omega(m) = m^{p(\lambda-1)}$, *the kernel* $K(m, n)$ *satisfies*

$$K(m, n) = m^{-\lambda} K(1, nm^{-1}). \tag{99}$$

(i) If

$$c_1 = \int_0^{\infty} (K(1, u))^{\frac{1}{\lambda}} u^{\frac{1-q}{\lambda q}} du < \infty, \tag{100}$$

$$c_2 = \int_0^{\infty} (K(1, u))^{\frac{p}{\lambda}(1 - \frac{1}{q})} u^{\frac{(p-1)(q-1)}{\lambda q} - 1} du < \infty, \tag{101}$$

then the integral operator T *is defined by (98):* $T : l^p \to l^p(\omega)$ *exists as a bounded operator and*

$$\|Ta\|_{p,\omega} \leq c\|a\|_p. \tag{102}$$

This implies that

$$\|T\| = \sup_{a\neq 0} \frac{\|Ta\|_{p,\omega}}{\|a\|_p} \leq c, \tag{103}$$

where

$$c = c_1^{(1-(1/p))} c_2^{1/p}. \tag{104}$$

(ii) If

$$c_3 = \int_0^\infty K(1,u)u^{-\frac{1}{p}}du < \infty, \tag{105}$$

then

$$\|T\| \geq c_3. \tag{106}$$

For $\lambda = 1$, we have $\frac{1}{p} + \frac{1}{q} = 1$ and by Theorem 10, we get

$$\|Ta\|_p \leq c\|a\|_p, \tag{107}$$

where $c = \|T\| = \int_0^\infty K(1,u)u^{-(1/p)}du$ is the sharp constant. In particular, let

$$K(m,n) = \frac{1}{m^\delta n^\beta |m-n|^{\lambda-\delta-\beta}},$$

if $0 \leq \beta < 1 - \frac{1}{p}$, $0 \leq \delta < 1 - \frac{1}{q}$, $\delta + \beta > 0$, and

$$\max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\} < \lambda < \min\{1 + \delta + \beta, \frac{\delta+\beta}{1 - \frac{1}{p(1-(1/q))}}\},$$

then by Example 1, we get

$$c_3 \leq \|T\| \leq c_1^{1-(1/p)} c_2^{1/p}, \tag{108}$$

where c_1, c_2, and c_3 are defined by (66), (67), and (68), respectively.

If $\lambda = 1$, that is, $0 \leq \delta < \frac{1}{p}$, $0 \leq \beta < \frac{1}{q}$, $\delta + \beta > 0$, then by (72), we have

$$\|T\| = B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta). \tag{109}$$

Remark 2 In 2016, the author Kuang [12] proved that if $a = \{a_m\} \in l^p(\omega_0)$, $b = \{b_n\} \in l^q(\omega_0)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega_0(m) = m^{1-\lambda}$, and

$$\max\{\frac{1}{p}, \delta + \beta + \frac{1}{q}\} < \lambda < 1 + \beta + \delta < 1 + \frac{1}{p}$$

then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m^{\delta} n^{\beta} |m-n|^{\lambda-\delta-\beta}} \leq c_0 \|a\|_{p,\omega_0} \|b\|_{q,\omega_0}, \tag{110}$$

where c_0 is defined by (23). Inequality (110) is equivalent to

$$\|T_9(a)\|_p \leq c_0 \|a\|_{p,\omega_0}, \tag{111}$$

where,

$$T_9(a, m) = \sum_{n=1}^{\infty} \frac{a_n}{m^{\delta} n^{\beta} |m-n|^{\lambda-\delta-\beta}}$$

is the Hardy–Littlewood operator. We define $\omega_1(m) = m^{\lambda-1}$, then the above norm inequality is also equivalent to

$$\|T_9(a)\|_{p,\omega_1} \leq c_0 \|a\|_p. \tag{112}$$

Hence, (108) and (109) are new improvements and extensions of (112).

Acknowledgement The author wishes to express his thanks to Professor Bicheng Yang for his careful reading of the manuscript and for his valuable suggestions.

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