Some New Hermite–Hadamard Type Integral Inequalities for Twice Differentiable Mappings and Their Applications

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Abstract The authors discover a general fractional integral identity regarding Hermite–Hadamard type inequality for twice differentiable functions. By using this integral equation, the authors derive some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings defined on $\mathbf{m}$-invex set. It is pointed out that some new special cases can be deduced from main results. At the end, some applications to special means for different positive real numbers are provided as well.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

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The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1) through various classes of convex functions, interested readers are referred to [1–33, 35, 38, 39, 41–45, 49, 51, 52]. Let us recall some special functions and evoke some basic definitions as follows.

Definition 1 The incomplete beta function is defined for $a, b > 0$ as

$$\beta_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 < x \leq 1. \tag{2}$$

Definition 2 ([50]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.

The invex set S is also termed an η-connected set.

Definition 3 ([34]) Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y, x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{3}$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 3, f becomes a preinvex function [40]. If the mapping $\eta(y, x) = y - x$ in Definition 3, then the non-negative function f reduces to h-convex mappings [47].

Definition 4 ([48]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s \in (0, 1]$, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f\big(x + t\eta(y, x)\big) \leq (1-t)^s f(x) + t^s f(y). \tag{4}$$

Definition 5 ([37]) A function $f : K \longrightarrow \mathbb{R}$ is said to be s-Godunova-Levin-Dragomir-preinvex of second kind, if

$$f\big(x + t\eta(y, x)\big) \leq (1-t)^{-s} f(x) + t^{-s} f(y), \tag{5}$$

for each $x, y \in K$, $t \in (0, 1)$ and $s \in (0, 1]$.

Definition 6 ([46]) A non-negative function $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be tgs-convex on K if the inequality

$$f\big((1-t)x + ty\big) \leq t(1-t)[f(x) + f(y)] \tag{6}$$

grips for all $x, y \in K$ and $t \in (0, 1)$.

Definition 7 ([31]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be MT-convex functions, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{7}$$

Definition 8 ([39]) Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y, mx)\big) \leq mh_1(t) f(x) + h_2(t) f(y) \tag{8}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

Definition 9 ([32]) Let $f \in L[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^0_{a+} f(x) = J^0_{b-} f(x) = f(x)$.

Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Motivated by the above works and the references therein, the main objective of this article is to apply the notion of generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings and an interesting lemma to establish some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings defined on **m**-invex set. Also, some new special cases will be deduced. At the end, some applications to special means for different positive real numbers will be given as well.

2 Main Results

The following definitions will be used in this section.

Definition 10 Let $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as **m**-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi\eta(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0, 1]$.

Remark 1 In Definition 10, under certain conditions, the mapping $\eta(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0, 1]$ could reduce to $\eta(y, mx)$. For example, when $\mathbf{m}(t) = m$ for all $t \in [0, 1]$, then the **m**-invex set degenerates an m-invex set on K.

Definition 11 ([24]) Let $K \subseteq \mathbb{R}$ be an open **m**-invex set with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, $\varphi : I \longrightarrow K$ are continuous functions and $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. A mapping $f : K \longrightarrow (0, +\infty)$ is said to be generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex, if

$$f\big(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x))\big) \leq \Big[\mathbf{m}(\xi)h_1(\xi)f^r(x) + h_2(\xi)f^r(y)\Big]^{\frac{1}{r}} \tag{9}$$

holds for all $x, y \in I$ and $t, \xi \in [0, 1]$, where $r \neq 0$.

Remark 2 In Definition 11, if we choose $\mathbf{m} = m = r = 1$, this definition reduces to the definition considered by Noor in [36] and Fulga and Preda in [13].

Remark 3 In Definition 11, if we choose $\mathbf{m} = m = r = 1$ and $\varphi(x) = x$, then we get Definition 8.

Remark 4 Let us discuss some special cases in Definition 11 as follows.

1. Taking $h_1(t) = h(1-t)$, $h_2(t) = h(t)$, then we get generalized relative semi-$(\mathbf{m}, h)$-preinvex mappings.
2. Taking $h_1(t) = (1-t)^s$, $h_2(t) = t^s$ for $s \in (0, 1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Breckner-preinvex mappings.
3. Taking $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Godunova–Levin–Dragomir-preinvex mappings.
4. Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-$(\mathbf{m}, tgs)$-preinvex mappings.
5. Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-**m**-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

For establishing our main results regarding some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity via fractional integrals, we need the following lemma.

Lemma 1 *Let $\varphi : I \longrightarrow K$ be a continuous function and* $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. *Suppose $K \subseteq \mathbb{R}$ be an open* **m***-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ where $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) \neq 0$ and $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) \neq 0$ for all $t \in [0, 1]$. If $f : K \longrightarrow \mathbb{R}$ is a twice differentiable mapping on K° such that $f'' \in L(K)$, then for any $\alpha > 0$, the following identity holds:*

$$-\frac{(\alpha+1)}{2^{\alpha-1}}\frac{1}{\eta^2(\varphi(y),\mathbf{m}(t)\varphi(x))}f\left(\mathbf{m}(t)\varphi(x)+\frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)$$
$$-\frac{(\alpha+1)}{2^{\alpha-1}}\frac{1}{\eta^2(\varphi(x),\mathbf{m}(t)\varphi(y))}f\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)$$
$$+\frac{\Gamma(\alpha+2)}{\eta^{\alpha+2}(\varphi(y),\mathbf{m}(t)\varphi(x))}$$
$$\times\Big[J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x)+\frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^+}f(\mathbf{m}(t)\varphi(x)+\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))$$
$$+J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x)+\frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^-}f(\mathbf{m}(t)\varphi(x))\Big]$$
$$+\frac{\Gamma(\alpha+2)}{\eta^{\alpha+2}(\varphi(x),\mathbf{m}(t)\varphi(y))}$$
$$\times\Big[J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^+}f(\mathbf{m}(t)\varphi(y)+\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))$$
$$+J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^-}f(\mathbf{m}(t)\varphi(y))\Big]$$
$$=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))) \tag{10}$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi$$
$$+\int_{\frac{1}{2}}^1(1-\xi)^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi.$$

We denote

$$T_f^{\alpha}(\eta,\varphi,\mathbf{m};x,y):=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))) \tag{11}$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi$$
$$+\int_{\frac{1}{2}}^1(1-\xi)^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi.$$

Proof

$$T_f^{\alpha}(\eta,\varphi,\mathbf{m};x,y)=T_{11}+T_{12}+T_{21}+T_{22},$$

where

$$T_{11}=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi;$$

$$T_{12}=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))\right)d\xi;$$

$$T_{21}=\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi;$$

$$T_{22}=\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))\right)d\xi.$$

Now, using twice integration by parts, we have

$$T_{11}=\frac{\xi^{\alpha+1}f'\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\Bigg|_0^{\frac{1}{2}}$$

$$-\frac{(\alpha+1)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\int_0^{\frac{1}{2}}\xi^{\alpha}f'\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi$$

$$=\frac{1}{2^{\alpha+1}}\frac{f'\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}-\frac{(\alpha+1)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}$$

$$\times\left\{\frac{\xi^{\alpha}f\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\Bigg|_0^{\frac{1}{2}}\right.$$

$$\left.-\frac{\alpha}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\int_0^{\frac{1}{2}}\xi^{\alpha-1}f\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi\right\}$$

$$=\frac{1}{2^{\alpha+1}}\frac{f'\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}-\frac{(\alpha+1)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))} \tag{12}$$

$$\times\left\{\frac{1}{2^{\alpha}}\frac{f\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(x),\mathbf{m}(t)\varphi(y))}\times J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^-}f(\mathbf{m}(t)\varphi(y))\right\}.$$

In a similar way, we find

$$T_{12} = \frac{1}{2^{\alpha+1}} \frac{f'\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} - \frac{(\alpha+1)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} \tag{13}$$

$$\times\left\{\frac{1}{2^{\alpha}} \frac{f\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))}\right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(y), \mathbf{m}(t)\varphi(x))} \times J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^{-}} f(\mathbf{m}(t)\varphi(x))\right\}.$$

$$T_{21} = -\frac{1}{2^{\alpha+1}} \frac{f'\left(\mathbf{m}(t)\varphi(y) + \frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x), \mathbf{m}(t)\varphi(y))} + \frac{(\alpha+1)}{\eta(\varphi(x), \mathbf{m}(t)\varphi(y))} \tag{14}$$

$$\times\left\{-\frac{1}{2^{\alpha}} \frac{f\left(\mathbf{m}(t)\varphi(y) + \frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x), \mathbf{m}(t)\varphi(y))} + \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(x), \mathbf{m}(t)\varphi(y))} \times\right.$$

$$\left. J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y) + \frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^{+}} f(\mathbf{m}(t)\varphi(y) + \eta(\varphi(x), \mathbf{m}(t)\varphi(y)))\right\}.$$

$$T_{22} = -\frac{1}{2^{\alpha+1}} \frac{f'\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} + \frac{(\alpha+1)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} \tag{15}$$

$$\times\left\{-\frac{1}{2^{\alpha}} \frac{f\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} + \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(y), \mathbf{m}(t)\varphi(x))} \times\right.$$

$$\left. J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^{+}} f(\mathbf{m}(t)\varphi(x) + \eta(\varphi(y), \mathbf{m}(t)\varphi(x)))\right\}.$$

Adding Eqs. (12)–(15), we get our lemma.

Remark 5 In Lemma 1, if we take $\alpha = 1$, $\mathbf{m}(t) \equiv 1$ for all $t \in [0, 1]$, $a < b$, $x = \mu a + (1-\mu)b$, $y = \mu b + (1-\mu)a$, where $\mu \in [0, 1]\backslash\left\{\frac{1}{2}\right\}$ and $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) = \varphi(x) - \mathbf{m}(t)\varphi(y)$, $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) = \varphi(y) - \mathbf{m}(t)\varphi(x)$, where $\varphi(x) = x$ for all $x \in I$, in identity (10), then it becomes identity of Lemma 2.1 in [41].

Theorem 2 *Let* $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ *and* $\varphi : I \longrightarrow K$ *are continuous functions and* $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. *Suppose* $K \subseteq \mathbb{R}$ *be an open* $\mathbf{m}$*-invex subset, where* $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) \neq 0$ *and* $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) \neq 0$ *for all* $t \in [0, 1]$. *Assume that* $f : K \longrightarrow (0, +\infty)$ *is a twice differentiable mapping on* K° *such that*

$f'' \in L(K)$. If f''^q is generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mapping, $0 < r \leq 1$ and $q > 1$, $p^{-1} + q^{-1} = 1$, then for any $\alpha > 0$, the following inequality for fractional integrals hold:

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right| \leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{16}$$

$$\times\Bigg\{\Big[\left(f''(x)\right)^{rq} I^r(h_1(\xi); \mathbf{m}(\xi), r) + \left(f''(y)\right)^{rq} I^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} I^r(h_1(\xi); \mathbf{m}(\xi), r) + \left(f''(x)\right)^{rq} I^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(x)\right)^{rq} J^r(h_1(\xi); \mathbf{m}(\xi), r) + \left(f''(y)\right)^{rq} J^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} J^r(h_1(\xi); \mathbf{m}(\xi), r) + \left(f''(x)\right)^{rq} J^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}\Bigg\},$$

where

$$I(h_1(\xi); \mathbf{m}(\xi), r) := \int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi) h_1^{\frac{1}{r}}(\xi) d\xi, \quad I(h_2(\xi); r) := \int_0^{\frac{1}{2}} h_2^{\frac{1}{r}}(\xi) d\xi;$$

and

$$J(h_1(\xi); \mathbf{m}(\xi), r) := \int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi) h_1^{\frac{1}{r}}(\xi) d\xi, \quad J(h_2(\xi); r) := \int_{\frac{1}{2}}^1 h_2^{\frac{1}{r}}(\xi) d\xi.$$

Proof From Lemma 1, generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvexity of f''^q, Hölder inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right|$$

$$\leq \int_0^{\frac{1}{2}} \xi^{\alpha+1}\Big[|f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))|$$

$$+|f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))|\Big]d\xi$$

$$+\int_{\frac{1}{2}}^1 (1-\xi)^{\alpha+1}\Big[|f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))|$$

$$+|f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))|\Big]d\xi$$

$$\leq \left(\int_0^{\frac{1}{2}} \xi^{p(\alpha+1)} d\xi\right)^{\frac{1}{p}} \times \left\{ \left(\int_0^{\frac{1}{2}} \left(f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^{\frac{1}{2}} \left(f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}} \right\}$$

$$+ \left(\int_{\frac{1}{2}}^1 (1-\xi)^{p(\alpha+1)} d\xi\right)^{\frac{1}{p}} \times \left\{ \left(\int_{\frac{1}{2}}^1 \left(f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \left(f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}} \right\}$$

$$\leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}$$

$$\times \left\{ \left(\int_0^{\frac{1}{2}} \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq} + h_2(\xi)\left(f''(y)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}} \right.$$

$$+ \left(\int_0^{\frac{1}{2}} \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq} + h_2(\xi)\left(f''(x)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^1 \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq} + h_2(\xi)\left(f''(y)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}}$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq} + h_2(\xi)\left(f''(x)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}} \right\}$$

$$\leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}$$

$$\times \left\{ \left[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^q h_1^{\frac{1}{r}}(\xi) d\xi\right)^r + \left(\int_0^{\frac{1}{2}} \left(f''(y)\right)^q h_2^{\frac{1}{r}}(\xi) d\xi\right)^r \right]^{\frac{1}{rq}} \right.$$

$$+ \left[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^q h_1^{\frac{1}{r}}(\xi) d\xi\right)^r + \left(\int_0^{\frac{1}{2}} \left(f''(x)\right)^q h_2^{\frac{1}{r}}(\xi) d\xi\right)^r \right]^{\frac{1}{rq}}$$

$$+\left[\left(\int_{\frac{1}{2}}^{1}\mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^{q}h_1^{\frac{1}{r}}(\xi)d\xi\right)^{r}+\left(\int_{\frac{1}{2}}^{1}\left(f''(y)\right)^{q}h_2^{\frac{1}{r}}(\xi)d\xi\right)^{r}\right]^{\frac{1}{rq}}$$

$$+\left[\left(\int_{\frac{1}{2}}^{1}\mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^{q}h_1^{\frac{1}{r}}(\xi)d\xi\right)^{r}+\left(\int_{\frac{1}{2}}^{1}\left(f''(x)\right)^{q}h_2^{\frac{1}{r}}(\xi)d\xi\right)^{r}\right]^{\frac{1}{rq}}\Bigg\}$$

$$=\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}$$

$$\times\Bigg\{\left[\left(f''(x)\right)^{rq}I^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(y)\right)^{rq}I^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(y)\right)^{rq}I^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(x)\right)^{rq}I^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(x)\right)^{rq}J^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(y)\right)^{rq}J^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(y)\right)^{rq}J^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(x)\right)^{rq}J^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 2.

Corollary 1 *In Theorem 2, if we take* $\mathbf{m}(\xi)\equiv m\in(0,1]$ *for all* $\xi\in[0,1]$, $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ *and* $f''(x)\le L$, $\forall x\in I$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m},h)$*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right|\le 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{17}$$

$$\times\left\{\left[mI^{r}(h(t);r)+I^{r}(h(1-t);r)\right]^{\frac{1}{rq}}+\left[mI^{r}(h(1-t);r)+I^{r}(h(t);r)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2 *In Corollary 1 for* $h_1(t)=(1-t)^s$ *and* $h_2(t)=t^s$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m},s)$*-Breckner-preinvex mappings*

$$\begin{aligned}\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right|&\le 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}\left(\frac{r}{(s+r)2^{\frac{s}{r}+1}}\right)^{\frac{1}{q}}\\&\times\left\{\left[m+\left(2^{\frac{s}{r}+1}-1\right)^{r}\right]^{\frac{1}{rq}}+\left[m\left(2^{\frac{s}{r}+1}-1\right)^{r}+1\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{18}$$

Corollary 3 *In Corollary 1 for $h_1(t) = (1-t)^{-s}$ and $h_2(t) = t^{-s}$ and $0 < s < r$, we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-$(\mathbf{m}, s)$-Godunova–Levin–Dragomir-preinvex mappings*

$$\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right| \leq 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}\left(\frac{r}{(r-s)2^{1-\frac{s}{r}}}\right)^{\frac{1}{q}}$$
$$\times\left\{\left[m+\left(2^{1-\frac{s}{r}}-1\right)^{r}\right]^{\frac{1}{rq}}+\left[m\left(2^{1-\frac{s}{r}}-1\right)^{r}+1\right]^{\frac{1}{rq}}\right\}. \tag{19}$$

Corollary 4 *In Theorem 2, if we take $\mathbf{m}(\xi) \equiv m \in (0,1]$ for all $\xi \in [0,1]$, $h_1(t) = h_2(t) = t(1-t)$ and $f''(x) \leq L$, $\forall x \in I$, we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-$(\mathbf{m}, tgs)$-preinvex mappings*

$$\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right| \leq 4L(m+1)^{\frac{1}{rq}}\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{20}$$
$$\times\beta_{1/2}^{\frac{1}{q}}\left(1+\frac{1}{r},1+\frac{1}{r}\right).$$

Corollary 5 *In Corollary 1 for $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ and $r \in \left(\frac{1}{2}, 1\right]$, we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-$\mathbf{m}$-MT-preinvex mappings*

$$\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right| \leq 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{rq}} \tag{21}$$
$$\times\left\{\left[m\beta_{1/2}^{r}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)+\beta_{1/2}^{r}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right.$$
$$\left.+\left[m\beta_{1/2}^{r}\left(1+\frac{1}{2r},1-\frac{1}{2r}\right)+\beta_{1/2}^{r}\left(1-\frac{1}{2r},1+\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right\}.$$

Theorem 3 *Let $h_1, h_2 : [0,1] \longrightarrow [0,+\infty)$ and $\varphi : I \longrightarrow K$ are continuous functions and $\mathbf{m} : [0,1] \longrightarrow (0,1]$. Suppose $K \subseteq \mathbb{R}$ be an open $\mathbf{m}$-invex subset, where $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) \neq 0$ and $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) \neq 0$ for all $t \in [0,1]$. Assume that $f : K \longrightarrow (0,+\infty)$ is a twice differentiable mapping on K° such that $f'' \in L(K)$. If f''^{q} is generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mapping,*

$0 < r \leq 1$ *and* $q \geq 1$, *then for any* $\alpha > 0$, *the following inequality for fractional integrals hold:*

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right| \leq \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{22}$$

$$\times\Bigg\{\Big[\left(f''(x)\right)^{rq} F^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(y)\right)^{rq} F^r(h_2(\xi); \alpha, r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} F^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(x)\right)^{rq} F^r(h_2(\xi); \alpha, r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(x)\right)^{rq} G^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(y)\right)^{rq} G^r(h_2(\xi); \alpha, r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} G^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(x)\right)^{rq} G^r(h_2(\xi); \alpha, r)\Big]^{\frac{1}{rq}}\Bigg\},$$

where

$$F(h_1(\xi); \mathbf{m}(\xi), \alpha, r) := \int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\xi^{\alpha+1}h_1^{\frac{1}{r}}(\xi)d\xi;$$

$$F(h_2(\xi); \alpha, r) := \int_0^{\frac{1}{2}} \xi^{\alpha+1}h_2^{\frac{1}{r}}(\xi)d\xi,$$

and

$$G(h_1(\xi); \mathbf{m}(\xi), \alpha, r) := \int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi)(1-\xi)^{\alpha+1}h_1^{\frac{1}{r}}(\xi)d\xi;$$

$$G(h_2(\xi); \alpha, r) := \int_{\frac{1}{2}}^1 (1-\xi)^{\alpha+1}h_2^{\frac{1}{r}}(\xi)d\xi.$$

Proof From Lemma 1, generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity of f''^q, the well-known power mean inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right|$$

$$\leq \int_0^{\frac{1}{2}} \xi^{\alpha+1}\big[|f''\left(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x))\right)|$$

$$+|f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))|\Big]d\xi$$

$$+\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\Big[|f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))|$$

$$+|f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))|\Big]d\xi$$

$$\leq\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}\times\left\{\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}}\right\}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq}+h_2(\xi)\left(f''(y)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq}+h_2(\xi)\left(f''(x)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq}+h_2(\xi)\left(f''(y)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq}+h_2(\xi)\left(f''(x)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\Bigg[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^q \xi^{\alpha+1} h_1^{\frac{1}{r}}(\xi)d\xi\right)^r+\left(\int_0^{\frac{1}{2}} \left(f''(y)\right)^q \xi^{\alpha+1} h_2^{\frac{1}{r}}(\xi)d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^q \xi^{\alpha+1} h_1^{\frac{1}{r}}(\xi)d\xi\right)^r+\left(\int_0^{\frac{1}{2}} \left(f''(x)\right)^q \xi^{\alpha+1} h_2^{\frac{1}{r}}(\xi)d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[\left(\int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^q (1-\xi)^{\alpha+1} h_1^{\frac{1}{r}}(\xi)d\xi\right)^r$$

$$+\left(\int_{\frac{1}{2}}^1 \left(f''(y)\right)^q (1-\xi)^{\alpha+1} h_2^{\frac{1}{r}}(\xi)d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[\left(\int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^q (1-\xi)^{\alpha+1} h_1^{\frac{1}{r}}(\xi)d\xi\right)^r$$

$$+\left(\int_{\frac{1}{2}}^1 \left(f''(x)\right)^q (1-\xi)^{\alpha+1} h_2^{\frac{1}{r}}(\xi)d\xi\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}$$

$$= \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}$$

$$\times\Big\{\Big[\left(f''(x)\right)^{rq} F^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r)+\left(f''(y)\right)^{rq} F^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} F^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r)+\left(f''(x)\right)^{rq} F^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(x)\right)^{rq} G^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r)+\left(f''(y)\right)^{rq} G^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} G^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r)+\left(f''(x)\right)^{rq} G^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}\Big\}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 3.

Corollary 6 *In Theorem 3, if we take* $\mathbf{m}(\xi) \equiv m \in (0, 1]$ *for all* $\xi \in [0, 1]$, $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ *and* $f''(x) \leq L$, $\forall x \in I$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, h)$*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L\left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{23}$$

$$\times\left\{\left[mF^r(h(1-t); \alpha, r) + F^r(h(t); \alpha, r)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[mG^r(h(1-t); \alpha, r) + G^r(h(t); \alpha, r)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 7 *In Corollary 6 for* $h_1(t) = (1-t)^s$ *and* $h_2(t) = t^s$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, s)$*-Breckner-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L\left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{24}$$

$$\times\left\{\left[m\left(\frac{r}{(s+r(\alpha+2))2^{\frac{s}{r}+\alpha+2}}\right)^r + \beta_{1/2}^r\left(\alpha+2, 1+\frac{s}{r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m\beta_{1/2}^r\left(\alpha+2, 1+\frac{s}{r}\right) + \left(\frac{r}{(s+r(\alpha+2))2^{\frac{s}{r}+\alpha+2}}\right)^r\right]^{\frac{1}{rq}}\right\}.$$

Corollary 8 *In Corollary 6 for* $h_1(t) = (1-t)^{-s}$ *and* $h_2(t) = t^{-s}$ *and* $0 < s < r$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, s)$*-Godunova–Levin–Dragomir-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L\left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{25}$$

$$\times\left\{\left[m\left(\frac{r}{(r(\alpha+2)-s)2^{\alpha+2-\frac{s}{r}}}\right)^r + \beta_{1/2}^r\left(\alpha+2, 1-\frac{s}{r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m\beta_{1/2}^r\left(\alpha+2, 1-\frac{s}{r}\right) + \left(\frac{r}{(r(\alpha+2)-s)2^{\alpha+2-\frac{s}{r}}}\right)^r\right]^{\frac{1}{rq}}\right\}.$$

Corollary 9 *In Theorem 3, if we take* $\mathbf{m}(\xi) \equiv m \in (0,1]$ *for all* $\xi \in [0,1]$, $h_1(t) = h_2(t) = t(1-t)$ *and* $f''(x) \leq L$, $\forall x \in I$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, tgs)$*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 4L(m+1)^{\frac{1}{rq}} \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{26}$$

$$\times \beta_{1/2}^{\frac{1}{q}}\left(\alpha+2+\frac{1}{r}, 1+\frac{1}{r}\right).$$

Corollary 10 *In Corollary 6 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r \in \left(\frac{1}{2}, 1\right]$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$\mathbf{m}$*-*MT*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \left(\frac{1}{2}\right)^{\frac{1}{rq}} \tag{27}$$

$$\times \left\{\left[m\beta_{1/2}^{r}\left(\alpha+2-\frac{1}{2r}, 1+\frac{1}{2r}\right) + \beta_{1/2}^{r}\left(\alpha+2+\frac{1}{2r}, 1-\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left. + \left[m\beta_{1/2}^{r}\left(\alpha+2+\frac{1}{2r}, 1-\frac{1}{2r}\right) + \beta_{1/2}^{r}\left(\alpha+2-\frac{1}{2r}, 1+\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right\}.$$

Remark 6 By applying our Theorems 2 and 3 for $\alpha = 1$, we can deduce some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings via classical integrals. The details are left to the interested reader.

3 Applications to Special Means

Definition 12 A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for different positive real numbers α, β.

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Let us consider continuous functions $\varphi : I \longrightarrow K$, $\eta : K \times K \longrightarrow \mathbb{R}$ and $\overline{M} := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above-mentioned means. Therefore one can obtain various inequalities using the results of Sect. 2 for these means as follows. If we take $\mathbf{m}(t) \equiv 1$, $\forall t \in [0, 1]$ and replace $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) = \eta(\varphi(y), \mathbf{m}(t)\varphi(x)) = M(\varphi(x), \varphi(y))$ for all $x, y \in I$, in (16) and (22), one can obtain the following interesting inequalities involving means:

$$\left|T_f^{\alpha}(\overline{M}, \varphi, 1; a, b)\right| \leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{28}$$

$$\times\left\{\left[\left(f''(a)\right)^{rq} I^r(h_1(\xi); r) + \left(f''(b)\right)^{rq} I^r(h_2(\xi); r)\right]^{\frac{1}{rq}}\right.$$

$$+\left[\left(f''(b)\right)^{rq} I^r(h_1(\xi); r) + \left(f''(a)\right)^{rq} I^r(h_2(\xi); r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(a)\right)^{rq} J^r(h_1(\xi); r) + \left(f''(b)\right)^{rq} J^r(h_2(\xi); r)\right]^{\frac{1}{rq}}$$

$$\left.+\left[\left(f''(b)\right)^{rq} J^r(h_1(\xi); r) + \left(f''(a)\right)^{rq} J^r(h_2(\xi); r)\right]^{\frac{1}{rq}}\right\},$$

$$\left|T_f^{\alpha}(\overline{M}, \varphi, 1; a, b)\right| \leq \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{29}$$

$$\times\left\{\left[\left(f''(a)\right)^{rq} F^r(h_1(\xi); \alpha, r) + \left(f''(b)\right)^{rq} F^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}\right.$$

$$+\left[\left(f''(b)\right)^{rq} F^r(h_1(\xi); \alpha, r) + \left(f''(a)\right)^{rq} F^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(a)\right)^{rq} G^r(h_1(\xi); \alpha, r) + \left(f''(b)\right)^{rq} G^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}$$

$$\left.+\left[\left(f''(b)\right)^{rq} G^r(h_1(\xi); \alpha, r) + \left(f''(a)\right)^{rq} G^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}\right\}.$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p$ in (28) and (29), we get the inequalities involving means for a particular choices of f''^q that are generalized relative semi-1-$(r; h_1, h_2)$-preinvex mappings.

4 Conclusion

In this article, we first presented a new general fractional integral identity concerning twice differentiable mappings defined on **m**-invex set. By using the notion of generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity and lemma as an auxiliary result, some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings are established. It is pointed out that some new special cases are deduced from main results. At the end, some applications to special means for different positive real numbers are provided. Motivated by this interesting class we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite–Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of convex and preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p, q)-calculus, time scale calculus, and conformable fractional integrals.

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