Operator Inequalities Involved Wiener–Hopf Problems in the Open Unit Disk

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Abstract In this effort, we employ some of the linear differential inequalities to achieve integral inequalities of the type Wiener–Hopf problems (WHP). We utilize the concept of subordination and its applications to gain linear integral operators in the open unit disk that preserve two classes of analytic functions with a positive real part. Linear second-order differential inequalities play a significant role in the field of complex differential equations. Our study is based on a neighborhood containing the origin. Therefore, the Wiener–Hopf problem is decomposed around the origin in the open unit disk using two different classes of analytic functions. Moreover, we suggest a generalization for WHP by utilizing some classes of entire functions. Special cases are given in the sequel as well. A necessary and sufficient condition for WHP to be averaging operator on a convex domain (in the open unit disk) is given by employing the subordination relation (inequality).

1 Introduction

The Wiener–Hopf problems (WHP) [1] is a mathematical method to solve systems of integral equations extensively used in the field of applied mathematics [2], specifically in optimization theory [3], control systems [4], electromagnetics [5], image processing [6], and cloud computing system [7]. The technique acts by developing the complex-holomorphic properties of transforming functions. The Wiener operator of absolutely convergent Taylor series of a complex variable is given by the formal

$$w(z) = \sum_{n\in\mathbb{N}} \omega_n z^n, \quad \texttt{with} \quad \|w\|_{\mathfrak{W}} = \sum_{n\in\mathbb{N}} |\omega_n| < \infty.$$

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D. Andrica, T. M. Rassias (eds.), *Differential and Integral Inequalities*, Springer Optimization and Its Applications 151,
https://doi.org/10.1007/978-3-030-27407-8_13

It has been studied in many infinite spaces such as Hilbert spaces [8] and Banach spaces [9]. The main stage in many WHP is to decompose an arbitrary function into two functions. Overall, this can be done by putting

$$\Psi_+(\zeta) = \frac{1}{2\pi i}\int_{\Omega_1} \Psi(z)\frac{dz}{z-\zeta} \tag{1}$$

and

$$\Psi_-(\zeta) = -\frac{1}{2\pi i}\int_{\Omega_2} \Psi(z)\frac{dz}{z-\zeta}, \tag{2}$$

where the contours Ω_1, Ω_2 are parallel to the real line, but move above and below the point $z = \zeta$, respectively.

In this paper, we investigate some of the linear differential inequalities involving WHP. Our discussion is based on the concept of subordination: $\phi(z) \prec \psi(z)$, where $z \in U = \{z \in \mathbb{C} : |z| < 1\}$ (the open unit disk), if there occurs a Schwartz function $\sigma(z), \sigma(0) = 0, |\sigma(z)| < 1$ such that $\phi(z) = \psi(\sigma(z))$. We shall show that the integrals (1) and (2) preserve analytic functions with a positive real part. Special generalizations are provided involving entire functions. Moreover, we illustrate a necessary and sufficient condition for some convex inequalities containing (1) and (2).

Let $\mathfrak{H} = \mathfrak{H}(U)$ indicate the class of analytic functions in U. For a positive integer n and a complex number ϕ, let

$$\mathfrak{H}[\phi, n] = \{\varphi \in \mathfrak{H} : \varphi(z) = \phi + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots\}.$$

Define special classes of analytic functions

$$\mathfrak{P}_n = \{\varphi \in \mathfrak{H}[1, n] : \Re(\varphi(z)) > 0, \texttt{ for } z \in U\}$$

$$\mathfrak{H}[0, n] = \{\varphi \in \mathfrak{H} : \varphi(z) = \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots\},$$

and

$$\mathfrak{A}_n = \{\varphi \in \mathfrak{H} : \varphi(z) = z + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots\},$$

where $\mathfrak{A}_1 = \mathfrak{A}$ is called the normalized class satisfying the normalized condition $\varphi(0) = \varphi'(0) - 1 = 0$ and taking the form

$$\mathfrak{A} = \{\varphi \in \mathfrak{H} : \varphi(z) = z + \phi_2 z^2 + \ldots\}.$$

Since our study is in the open unit disk, we need to define the following W-H operator (WHO)

$$W_\zeta(\varphi)(z) = \frac{1}{2\pi i}\int_0^z \varphi(\xi)\frac{d\xi}{\xi - \zeta}, \tag{3}$$

where $\varphi \in \mathfrak{H}[1, n]$ taking the expansion

$$\varphi(z) = 1 + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots, \quad z \in U$$

Denote $W_0(\varphi)(z) = W(\varphi)(z)$.

Definition 1 The integral operator WHO is called averaging operator, if $\varphi \in \mathfrak{K}$ (the class of convex function) satisfies

$$W(\varphi)(0 = \varphi(0)), \quad W(\varphi)(U) \subset co\varphi(U).$$

Remark 1 For the function $\varphi \in \mathfrak{A}$ which is starlike (S^*) on U, the operator WHO is also starlike. This result comes from equation (2.5–28) [10] when $\alpha = 1$.

2 Results

Our first result is that $W(\varphi)$ is closed in the space $\mathfrak{H}[1, n]$.

Proposition 1 *For analytic function* $\varphi \in \mathfrak{H}[1, n]$, *the operator* $W(\varphi) \in \mathfrak{H}[1, n]$.

Proof Let $\varphi(z) = 1 + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots$

$$\begin{aligned} W(\varphi)(z) &= \frac{1}{2\pi i}\int_0^z \varphi(\xi)\frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i}\int_0^z [1 + \phi_n \xi^n + \phi_{n+1}\xi^{n+1} + \ldots]\frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i}\int_0^z [\frac{1}{\xi} + \phi_n \xi^{n-1} + \phi_{n+1}\xi^n + \ldots]d\xi. \end{aligned}$$

Since dz/z is accurate in a cut plane, which means a plane eliminates some line moving from the origin to ∂U, we have

$$\int_0^z \frac{1}{\xi}d\xi = \int_{\partial U}\frac{1}{z}dz = 2\pi i.$$

Moreover, we have

$$\int_0^z \xi^{m-1}d\xi = \frac{\xi^m}{m}\Big|_0^z = \frac{z^m}{m}.$$

Hence, we attain

$$W(\varphi)(z) = \frac{1}{2\pi i}[2\pi i + \sum_{m \geq n} \frac{\phi_m z^m}{m}]$$
$$= 1 + \sum_{m \geq n} \frac{\phi_m}{2m\pi i} z^m,$$

which proves that $W(\varphi)$ is analytic in U. In other words $W(\varphi) \in \mathfrak{H}[1, n]$ taking the expansion

$$W(\varphi)(z) = 1 + \omega_n z^n + \omega_{n+1} z^{n+1} + \ldots, \quad z \in U.$$

Proposition 2 *Let* $\lambda \neq 0$ *be a complex number with* $\Re(\lambda) > 0$ *and let* n *be a positive integer. If* $\varphi \in \mathfrak{P}_n$ *such that*

$$\left| \Im\Big(\frac{\lambda\, W(\varphi) + zW(\varphi)'}{\lambda\, W(\varphi)}\Big) \right| \leq n\Re(\frac{1}{\lambda}).$$

Then $W(\varphi) \in \mathfrak{P}_n$.

Proof Set the following functions

$$B(z) = \frac{1}{\lambda}, \quad C(z) = \frac{\lambda\, W(\varphi) + zW(\varphi)'}{\lambda\, W(\varphi)}.$$

Now,

$$\begin{aligned}
\Re\Big(B(z)\, zW(\varphi)' + C(z)W(\varphi)\Big) &= \Re\Big(\frac{1}{\lambda}\, zW(\varphi)' + \frac{\lambda\, W(\varphi) + zW(\varphi)'}{\lambda\, W(\varphi)} W(\varphi)\Big) \\
&= \Re\Big(\frac{\lambda\, W(\varphi) + 2zW(\varphi)'}{\lambda}\Big) \\
&= \Re\Big(W(\varphi)\Big) + 2\Re\Big(\frac{zW(\varphi)'}{\lambda}\Big) \\
&= \Re\Big(1 + \omega_n z^n + \omega_{n+1} z^{n+1} + \ldots\Big) \\
&+ 2\Re\Big(\frac{n\omega_n}{\lambda} z^n + \frac{(n+1)\omega_{n+1}}{\lambda} z^{n+1} + \ldots\Big) \\
&= 1 + \Re\Big((1 + \frac{2n}{\lambda})\omega_n z^n + (1 + \frac{2(n+1)}{\lambda})\omega_{n+1} z^{n+1} + \ldots\Big).
\end{aligned}$$

By setting

$$\lambda = \frac{2m}{2\pi\, im - 1}, \quad m \geq n,$$

we have

$$\Re\Big(B(z)\,zW(\varphi)' + C(z)W(\varphi)\Big) = \Re(\varphi(z)) > 0.$$

Hence, in view of Corollary 4.1a.1 in [10], we obtain $W(\varphi) \in \mathfrak{P}_n$.

Proposition 3 *Let $\lambda \neq 0$ be a complex number with $\Re(\lambda) > -n$, where n is a positive integer. Let $\varphi \in \mathfrak{A}_n$ and*

$$\Re\Big(\lambda + n - \frac{zW(\lambda\varphi)'(z)}{W(\lambda\varphi)(z)}\Big) > 0.$$

If $|\varphi(z)| < M,\ M > 0$, then $W(\lambda\varphi) \in \mathfrak{A}_n$ and $|W(\lambda\varphi)(z)| < N,\ N > 0$.

Proof First, we show that $W(\varphi) \in \mathfrak{A}_n$. Let $\varphi(z) = z + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots$

$$\begin{aligned} W(\lambda\varphi)(z) &= \frac{1}{2\pi i}\int_0^z \lambda\varphi(\xi)\frac{d\xi}{\xi} \\ &= \frac{\lambda}{2\pi i}\int_0^z [\xi + \phi_n\xi^n + \phi_{n+1}\xi^{n+1} + \ldots]\frac{d\xi}{\xi} \\ &= \frac{\lambda}{2\pi i}\int_0^z [1 + \phi_n\xi^{n-1} + \phi_{n+1}\xi^n + \ldots]d\xi. \end{aligned}$$

By letting $\lambda = 2\pi i$, we have

$$W(\lambda\varphi)(z) = z + \omega_n z^n + \ldots \in \mathfrak{A}_n.$$

Assume the following functions:

$$B(z) = 1, \quad C(z) = \frac{\lambda W(\lambda\varphi)(z) - zW(\lambda\varphi)'(z)}{W(\lambda\varphi)(z)}, \quad D(z) = \varphi(z) - \lambda W(\lambda\varphi)(z)$$

$$\begin{aligned} &|B(z)zW(\lambda\varphi)'(z) + C(z)W(\lambda\varphi)(z) + D(z)| \\ &= \Big|zW(\lambda\varphi)'(z) + \frac{\lambda W(\lambda\varphi)(z) - zW(\lambda\varphi)'(z)}{W(\lambda\varphi)(z)}W(\lambda\varphi)(z) + \varphi(z) - \lambda W(\lambda\varphi)(z)\Big| \\ &= |\varphi(z)| < M. \end{aligned}$$

Hence, in view of Corollary 4.1b.1 in [10], we have

$$|W(\lambda\varphi)| < sup_{z\in U}\Big\{\frac{M + |D(z)|}{|nB(z) + C(z)|}\Big\} := N.$$

This completes the proof.

Proposition 4 *Let n be a positive integer and let* $\varphi \in \mathfrak{H}[0, n]$ *achieving*

$$\Re\Big(n - \frac{zW(\varphi)'(z)}{W(\varphi)(z)}\Big) \geq 0.$$

If $|\varphi(z)| < M, \ M > 0$ *and*

$$\Big|n - \frac{zW(\varphi)'(z)}{W(\varphi)(z)}\Big| \geq \frac{2M}{N}$$

then $W(\varphi) \in \mathfrak{H}[0, n]$ *and* $|W(\varphi)(z)| < N, \ N > 0.$

Proof First, we show that $W(\varphi) \in \mathfrak{H}[0, n]$. Let $\varphi(z) = \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots$

$$\begin{aligned} W(\varphi)(z) &= \frac{1}{2\pi i} \int_0^z \varphi(\xi) \frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i} \int_0^z [\phi_n \xi^n + \phi_{n+1} \xi^{n+1} + \ldots] \frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i} \int_0^z [\phi_n \xi^{n-1} + \phi_{n+1} \xi^n + \ldots] d\xi. \end{aligned}$$

Thus, we obtain

$$W(\varphi)(z) = \omega_n z^n + \ldots \in \mathfrak{H}[0, n].$$

Assume the following functions:

$$B(z) = 1, \quad C(z) = -\frac{zW(\varphi)'(z)}{W(\varphi)(z)}, \quad D(z) = \varphi(z)$$

$$\begin{aligned} &|B(z)zW(\lambda\varphi)'(z) + C(z)W(\varphi)(z) + D(z)| \\ &= \Big|zW(\lambda\varphi)'(z) - \frac{zW(\varphi)'(z)}{W(\varphi)(z)} W(\varphi)(z) + \varphi(z)\Big| \\ &= |\varphi(z)| < M. \end{aligned}$$

Hence, in view of Theorem 4.1b in [10], we have $|W(\varphi)(z)| < N$. This completes the proof.

Proposition 5 *Let* $M > 0, \ N > 0$ *and let* $\varphi \in \mathfrak{H}[0, 1]$ *achieving*

$$\Big|\Im(\frac{zW(\varphi)'(z)}{W(\varphi)(z)})\Big| \geq \frac{M}{N}.$$

Then $W(\varphi) \in \mathfrak{H}[0, 1]$ *and* $|W(\varphi)(z)| < N.$

Proof It is clear that $W(\varphi) \in \mathfrak{H}[0, 1]$. Consider the following functions:

$$B(z) = 1, \quad C(z) = -\frac{zW(\varphi)'(z)}{W(\varphi)(z)}$$

$$\begin{aligned}
&|B(z)zW(\lambda\varphi)'(z) + C(z)W(\varphi)(z)| \\
&= \left| zW(\lambda\varphi)'(z) - \frac{zW(\varphi)'(z)}{W(\varphi)(z)} W(\varphi)(z) \right| \\
&= 0 < M.
\end{aligned}$$

Hence, in view of Theorem 4.1c in [10], we have

$$|W(\varphi)(z)| < N := \sup_{z \in U} \{ \frac{M}{|B(z).||\Im C(z)/B(z)|} \}.$$

This completes the proof.

Next, we discuss the upper bound of the WHO with respect to convex analytic function, by using the second-order differential subordination.

Theorem 1 *Let h be convex in U and let $\varphi \in \mathfrak{H}[h(0), 1]$ satisfying the subordination*

$$z^2 W(\varphi)''(z) + zW(\varphi)'(z) + W(\varphi)(z) \prec h(z)$$

then $W(\varphi)(z) \prec h(z)$.

Proof Since h is convex then it has the normalized property $h(0) = 0$ then we have $W(\varphi)(z) \in \mathfrak{H}[0, 1]$ (Proposition 5). Consider the following functions:

$$A = 1, \quad B(z) = 1, \quad D(z) = 0.$$

Since $\Re(B(z)) = A = 1$ then in view of Theorem 4.1f [10], we have $W(\varphi)(z) \prec h(z)$.

Theorem 2 *Let $\varphi \in \mathfrak{H}[0, 1]$ satisfying the subordination*

$$z^2 W(\varphi)''(z) + zW(\varphi)'(z) + W(\varphi)(z) \prec z$$

then $W(\varphi)(z) \prec \frac{z}{2}$ and $z/2$ is the best (0,1)-dominant.

Proof It is clear that $W(\varphi)(z) \in \mathfrak{H}[0, 1]$ (Proposition 5). Consider the following real numbers:

$$A = 1, \quad B = 1, \quad C = 1.$$

Then in view of Theorem 4.1g [10], we have $W(\varphi)(z) \prec \dfrac{z}{2}$ and $z/2$ is the best (0,1)-dominant.

Theorem 3 *Let n be a positive integer and $\varphi \in \mathfrak{H}[1,n]$ satisfying the linear first differential subordination*

$$zW(\varphi)'(z) + W(\varphi)(z) \prec [\frac{1+z}{1-z}]^{\alpha}$$

then

$$W(\varphi)(z) \prec [\frac{1+z}{1-z}]^{\beta}$$

where $\alpha := \beta + o(n) > 0$.

Proof It is clear that $W(\varphi)(z) \in \mathfrak{H}[1,n]$ (Proposition 1). According to Theorem 3.1c [10], we have

$$W(\varphi)(z) \prec [\frac{1+z}{1-z}]^{\beta}.$$

Theorem 4 *Let λ be a real number with $|\lambda| \leq 1$. If $\varphi \in \mathfrak{H}[1,n]$ satisfying $\Re(\varphi(z)) > 0$, then the generalized WHO achieves*

$$\Re(W_{\lambda}(\varphi)(z)) = \Re(\frac{1}{2\pi i e^{\lambda z^n}} \int_0^z \varphi(\xi) e^{\lambda \xi^n} \frac{d\xi}{\xi}) > 0$$

such that

$$|\Im(\frac{1}{2\pi i e^{\lambda z^n}})'| \leq n\Re(\frac{1}{2\pi i z\, e^{\lambda z^n}}).$$

Proof According to the relation 4.2–6 [10], we have the desire inequality.

Note that $W_0(\varphi)(z)) = W(\varphi)(z))$.

Theorem 5 *Let λ be a real number with $|\lambda| \leq 1$ and $\gamma > 0$. If $\varphi \in \mathfrak{H}[1,n]$ satisfying $\Re(\varphi(z)) > 0$ then the generalized WHO achieves*

$$\Re(W_{\lambda,\gamma}(\varphi)(z)) = \Re(\frac{1}{2\pi i z^{\gamma-1}\, e^{\lambda z^n}} \int_0^z \varphi(\xi)\xi^{\gamma-1}\, e^{\lambda \xi^n} \frac{d\xi}{\xi}) > 0.$$

Proof A direct application of the relation 4.2–4 [10], we have the desire inequality.

Note that $W_{0,1}(\varphi)(z)) = W(\varphi)(z))$. Theorems 4 and 5 show that the generalized WHO satisfies the relation

$$\varphi(z) \in \mathfrak{P}_n \Rightarrow W_{\lambda,\gamma}(\varphi)(z)) \in \mathfrak{P}_n.$$

Theorem 6 *Let φ be an analytic function in U with $\varphi(0) = 1$ $(\varphi \in \mathfrak{H}[1, n])$. If either of the following three conditions is achieved:*

- $$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec e^z, \quad \lambda > 1$$

- $$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec \frac{1 + Az}{1 + Bz}$$

$$\Big(-1 < B < A \leq 1, \ \ |\lambda| \geq \frac{A - B}{1 - |B|} \Big)$$

- $$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec \sqrt{1 + z}, \quad \lambda \geq 1$$

then

$$W(\varphi)(z) \prec e^z.$$

Proof According to Proposition 1, we have $W(\varphi)(z) \in \mathfrak{H}[1, n]$. Let $h(z)$ be the convex univalent function defined by $h(z) = e^z$. Then, obviously $\lambda\, z\, (h(z))'$ is starlike. The main aim of the proof reads on the information that if the subordination

$$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec 1 + \lambda \frac{z\, (h(z))'}{h(z)} = 1 + \lambda z := \Theta(z)$$

is achieved, then $W(\varphi)(z) \prec h(z)$ (see Corollary 3.4h.1, p. 135 [10]). By Remark 2.1 in [11] and the first condition, we obtain

$$h(z) \prec \Theta(z) \Longrightarrow W(\varphi)(z) \prec h(z).$$

Now, let $\psi(z) := \dfrac{1 + Az}{1 + Bz}$ then $\psi^{-1}(\eta) = \dfrac{\eta - 1}{A - B\eta}$. But $\psi(z) \prec h(z)$ means $z \prec \psi^{-1}(\Theta(z))$ and

$$|\psi^{-1}(\Theta(e^{it})| = |\frac{\lambda\, e^{it}}{(A - B) - \lambda\, B\, e^{it}}| \geq \frac{\lambda}{A - B + \lambda |B|} \geq 1$$

for $\lambda \geq (A - B)(1 - |B|)$. Hence,

$$h(z) \prec \Theta(z) \Longrightarrow W(\varphi)(z) \prec h(z).$$

Finally, let $\Lambda(z) = \sqrt{1 + z}$, where $\Lambda(U) \subset \Theta(U)$ then if $\lambda \geq 1$, we attain

$$h(z) \prec \Theta(z) \Longrightarrow W(\varphi)(z) \prec h(z).$$

A direct application of Lemma 4.4b in [10], we get the following outcome:

Theorem 7 *Let $\varphi \in \mathfrak{K}$ such that $\varphi(0) = 0$ and $h \in \mathfrak{K}$ such that $\varphi(z) \prec h(z)$. Then the WHO is averaging operator on $\mathfrak{K}$ satisfying $W(\varphi)(z) \prec h(z)$.*

Next, we discuss the case φ is not convex.

Theorem 8 *Let $\varphi \in \mathfrak{H}(U)$ and $h \in \mathfrak{K}$ such that $\varphi(z) \prec h(z)$ and*

$$\Re\Big(- \frac{W(\varphi)(z) - \varphi(z)}{zW(\varphi)(z)'}\Big) > 0.$$

Then the WHO is averaging operator on $\mathfrak{K}$ satisfying $W(\varphi)(z) \prec h(z)$.

Proof Since $\varphi \in \mathfrak{H}(U)$ then we obtain $W(\varphi) \in \mathfrak{H}(U)$. A computation leads to

$$W(\varphi)(z) - \frac{W(\varphi)(z) - \varphi(z)}{zW(\varphi)(z)'}.zW(\varphi)(z)' = \varphi(z) \prec h(z).$$

In view of Theorem 3.1a in [10], we get

$$\varphi(z) \prec h(z) \Longrightarrow W(\varphi)(z) \prec h(z),$$

which implies that WHO is averaging operator on $\mathfrak{K}$.

Theorem 9 *Let $\varphi \in \mathfrak{H}(U)$ and h is starlike on U. If $\varphi(z) \prec h(z)$, then*

$$W(\varphi)(z) \prec W(h)(z).$$

Proof By Remark 1, $W(h)(z)$ is starlike on U. Suppose that $W(\varphi)(z) \not\prec W(h)(z)$, then there occur some points $z_0 \in U$ and $\eta_0 \in \partial U$ such that $W(\varphi)(z_0) = W(h)(\eta_0)$ and $W(\varphi)(U_0) \subset W(h)(U)$. Thus, by Lemma 2.2c [10], we obtain

$$z_0\, W(\varphi)(z_0)' = k\eta_0 W(h)'(\eta_0), \quad k \geq 1.$$

This implies that

$$\varphi(z_0) = k\, h(\eta_0) \notin h(U),$$

which contradicts the assumption $\varphi(z) \prec h(z)$. Hence, $W(\varphi)(z) \prec W(h)(z)$.

References

1. N. Wiener, Tauberian theorems. Ann. Math. **33**, 1–100 (1932)
2. J.B. Lawrie, I.D. Abrahams, A brief historical perspective of the Wiener–Hopf technique. J. Eng. Math. **59**(4), 351–358 (2007)
3. F.B. Vicente et al., Wiener–Hopf optimal control of a hydraulic canal prototype with fractional order dynamics. ISA Trans. **82**, 130–144 (2018)
4. B.P. Lampe, E.N. Rosenwasser, H_2-optimization of sampled-data systems with a linear periodic plant. II. H_2-optimization of system St based on the Wiener–Hopf method. Autom. Remote Control **77**(9), 1524–1543 (2016)
5. V.G. Daniele, R. Zich, *The Wiener–Hopf Method in Electromagnetics* (SciTech Publishing, Chennai, 2014)
6. C.H. Hsia, T.C. Wu, J.S. Chiang, A new method of moving object detection using adaptive filter. J. Real-Time Image Proc. **13**(2), 311–325 (2017)
7. R.W. Ibrahim, A. Ghani, Hybrid cloud entropy systems based on Wiener process. Kybernetes **45**(7), 1072–1083 (2016)
8. O.V. Lopushansky, A.V. Zagorodnyuk, Hilbert spaces of analytic functions of infinitely many variables. Ann. Polon. Math. **81**(2), 111–122 (2003)
9. A. Lopushansky, Sectorial operators on Wiener algebras of analytic functions. Topology **48**, 105–110 (2009)
10. S.S. Miller, P.T. Mocanu. Differential subordinations: theory and applications (CRC Press, Boca Raton, 2000)
11. M. Rajni, S. Nagpal, V. Ravichandran. On a subclass of strongly starlike functions associated with exponential function. Bull. Malaysian Math. Sci. Soc. **38**(1), 365–386 (2015)