Best Constants for Weighted Poincaré-Type Inequalities

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Abstract In this paper we will determine the best constant for a class of (weighted and non-weighted) new Poincaré-type inequalities. In particular, we obtain sharp inequalities under the concavity, convexity of the weight function. We also establish a family of sharp Poincaré inequalities involving the second derivative.

1 Introduction

If Ω is a sufficiently regular bounded convex domain in $\mathbb{R}^d$ and f a smooth function defined on Ω with vanishing mean value over Ω, then a well-known form of the Poincaré inequality states that there is a constant $c_{p,\Omega}$ independent of f such that

$$\|f\|_p \leq c_{p,\Omega} \|\nabla f\|_p \,, \tag{1}$$

where $1 \leq p < \infty$ and $\|.\|_p$ denotes the classical $L^p(\Omega)$-norm, see, e.g., [12]. Poincaré-type inequalities are a key tool in the study of many problems of partial differential equations and numerical analysis, see [3, 7, 10, 13]. In the estimate (1), the constant $c_{p,\Omega}$ is finite for any p and generally it is not explicitly known. For practical purposes it is important to know an explicit expression for this constant (see, e.g., [11]). The determination of analytical expression of the Poincaré constant $c_{p,\Omega}$ as function of p and Ω is a difficult task. Specific estimates related to $c_{p,\Omega}$ have been obtained only in very special cases. For example, for $p = 2$, by some elementary considerations, Payne and Weinberger [9] showed that in order to determine the best Poincaré constant $c_{2,\Omega}$ in (1), it is basically enough to consider *weighted* Poincaré inequalities in one dimension. In this way, the one-dimensional case is essential since the case of several dimensions can be reduced to it. The main

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idea is to decompose the original domain into smaller subdomains, keeping the same mean value of f on each subdomain, and then to apply sharp one-dimensional *weighted* Poincaré inequalities. Payne and Weinberger have exploited this method in [9] to prove that the optimal constant in (1) is d/π when the L^2-norm is used. For $p = 1$, this technique was also developed by Acosta and Durán in [1]. For general p, Chua and Wheede [2] have successfully used this method for estimating the constants $c_{p,\Omega}$.

Some simple generalizations of (1) are well known, see, e.g., [8, Theorems 8.11 and 8.12]. One involves replacing the condition $\int_\Omega f(\mathbf{x})d\mathbf{x} = 0$ by the α-weighted average of f over Ω, $\int_\Omega \alpha(\mathbf{x})f(\mathbf{x})d\mathbf{x} = 0$, where α is any weight function from $L_1(\Omega)$ satisfying $\int_\Omega \alpha(\mathbf{x})d\mathbf{x} = 1$. We do not know the exact value of the best possible constants appearing in such general inequalities. We must add at this place that determining exact constants in inequalities between norms of functions and their derivatives is a very difficult problem that usually requires delicate considerations. Each case when exact constants are found is a great achievement.

In this paper, continuing the previous line of research we will discuss just the 1-dimensional case of the above. To be specific, we are interested in finding the smallest constant which is admissible in the following Poincaré-type inequality:

$$\int_0^1 |f(t)|\,dt \leq c(\alpha)\int_0^1 \left|f'(t)\right|dt, \tag{2}$$

where f is such that f is absolutely continuous, $f, f' \in L^1[0, 1]$ and

$$\int_0^1 \alpha(t)f(t)dt = 0, \tag{3}$$

with α a weight function on [0, 1] whose integral over [0, 1] is one. Our first main result provides an explicit expression for the best constant in (2). We will show that $c(\alpha) = 1/2$ is the best possible value. Moreover, we shall characterize all weight functions α for which (2) holds with best possible value 1/2. In [1, Theorem 3.1], Acosta and Durán showed that if, in addition, α is *concave* on [0, 1] then the following weighted version of (2)

$$\int_0^1 \alpha(t)\,|f(t)|\,dt \leq c(\alpha)\int_0^1 \alpha(t)\left|f'(t)\right|dt, \tag{4}$$

holds true with the constant $c(\alpha) = 1/2$. Moreover, they also showed that the constant $1/2$ cannot be improved in the case when $\alpha \equiv 1$. We shall give a new proof of this result, and, under specified conditions, prove that inequality (4) continues to hold when concavity of the weight function α is replaced by convexity. It is also shown, under suitable conditions on the weighted function, that Poincaré inequality (4) still holds with the best constant $c(\alpha) = 1/2$ for this general class of functions. Sharp Poincaré inequalities involving the second derivative are also considered.

The present paper is organized as follows. In Sect. 2, we shall first determine the best constant in (2), and characterize all weight functions α for which (2) holds with best possible value. We then give a new proof of the Acosta and Durán's inequality (4) when the weight function is concave. Furthermore, under appropriate assumptions on the weight function like convexity or monotonicity we generalize it, showing that inequality (4) still holds with the best constant $c(\alpha) = 1/2$ for this class of functions. Finally, in Sect. 3 we show how our arguments can be used to establish new sharp Poincaré inequality involving the second derivative.

2 Sharp Inequalities for the First Derivative

For the sake of clarity of our presentation, we shall first consider inequalities which involve only the first derivative of a function and the function itself. We first discuss optimal weight functions corresponding to the best constant in (2) with respect to the choice to some large class of weight functions. We also establish several weighted Poincaré-type inequalities under some appropriate assumptions on the weight function α like concavity, convexity.

We now set down some of the notation which will be used throughout. Let $W^{1,1}[0,1]$ denote the space of absolutely continuous functions on [0, 1] such that f and $f' \in L^1\,[0,1]$. Consider a linear functional of the form

$$T_\alpha[f] := \int_0^1 \alpha(t) f(t) dt \tag{5}$$

where α is a weight on $[0,1]$. Throughout this paper, by weight function we mean a nonnegative *integrable* function on [0, 1]. We assume in addition that α is normalized in the sense that its integral is equal to one:

$$\int_0^1 \alpha(t) dt = 1. \tag{6}$$

To ease the notation let us denote by $\mathscr{N}$ the class of weight functions on [0, 1] satisfying (6). This condition simply means that $T_\alpha[f] = 1$ for the constant function f of value 1 on [0, 1]. In what follows, $(.)_+^0 : \mathbb{R} \to \mathbb{R}$ will denote the function defined by

$$(x)_+^0 = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Applying integration by parts, we deduce

$$T_\alpha[f] = f(0) + \int_0^1 \left(\int_0^1 (s-t)_+^0 \alpha(s) ds \right) f'(t) dt.$$

Hence if we denote

$$K_\alpha(x,t) = (x-t)_+^0 - \int_0^1 (s-t)_+^0 \alpha(s)ds$$

then it follows from Taylor's formula

$$f(x) = f(0) + \int_0^1 (x-t)_+^0 f'(t)dt$$

that

$$f(x) - T_\alpha[f] = \textstyle\int_0^1 K_\alpha(x,t) f'(t)dt. \tag{7}$$

Thus, if we assume that $T_\alpha[f] = 0$, we may use the representation formula (7) to estimate L_1-norm of f in terms of the L_1-norm of f' as follows:

$$\begin{aligned}\textstyle\int_0^1 |f(x)|\,dx &= \textstyle\int_0^1 \left|\int_0^1 K_\alpha(x,t)f'(t)dt\right| dx \\ &\le \left(\sup_{0\le t\le 1} \textstyle\int_0^1 |K_\alpha(x,t)|\,dx\right) \textstyle\int_0^1 \left|f'(t)\right| dt.\end{aligned} \tag{8}$$

Hence, denoting by

$$c(\alpha) := \sup_{0\le t\le 1} \int_0^1 |K_\alpha(x,t)|\,dx \tag{9}$$

we get that for any function $f \in W^{1,1}[0,1]$ satisfying $T_\alpha[f] = 0$, the following Poincaré inequality holds:

$$\int_0^1 |f(t)|\,dt \le c(\alpha) \int_0^1 \left|f'(t)\right| dt. \tag{10}$$

An interesting problem is to know the dependence of the constant $c(\alpha)$, where $c(\alpha)$ is given by (9), on the weight function and, in particular, to find the best constant $c_{\min} = \min_{\alpha\in\mathcal{N}} c(\alpha)$. Therefore, there are two important questions that arise:

(1) What is the exact value of the best possible constant $c_{\min}$?
(2) What are all the normalized weight functions α for which the best constant $c_{\min}$ is achieved?
(3) What is the exact value of $c(\alpha)$ under the concavity or convexity of the weight function α ?

Our first result will be fundamental for the remainder of this paper. Indeed, it permits us to determine the exact value of the best possible constant $c_{\min}$ and also to establish a complete characterization of the weight functions for which inequality (10) is satisfied with the best possible value $c_{\min}$. More precisely, we have the following characterization:

Lemma 1 *For any weight function* $\alpha \in \mathcal{N}$,

$$c(\alpha) \geq 1/2.$$

The inequality $c(\alpha) = 1/2$ *holds if and only if* α *satisfies the additional condition*

$$\int_0^{1/2} \alpha(t)dt = 1/2. \tag{11}$$

Moreover, if $c(\alpha) = 1/2$*, then the constant* $\frac{1}{2}$ *in* (10) *is optimal.*

Proof Let α be a weight function belonging to $\mathcal{N}$. From the above considerations, we know that inequality (10) holds with the constant

$$c(\alpha) = \sup_{0 \leq t \leq 1} \int_0^1 |K_\alpha(x,t)|\,dx. \tag{12}$$

But, the integral appearing in (12) is easily calculated. Indeed, we have

$$\begin{aligned} F(t) &:= \textstyle\int_0^1 |K_\alpha(x,t)|\,dx \\ &= \textstyle\int_0^1 \left|(x-t)_+^0 - \int_0^1 (s-t)_+^0 \alpha(s)ds\right| dx \\ &= \left(\textstyle\int_t^1 \alpha(s)ds\right) t + \left(1 - \int_t^1 \alpha(s)ds\right)(1-t) \\ &= \left(1 - \textstyle\int_0^t \alpha(s)ds\right) t + \left(\int_0^t \alpha(s)ds\right)(1-t) \\ &= \left(1 - 2\textstyle\int_0^t \alpha(s)ds\right) t + \int_0^t \alpha(s)ds. \end{aligned}$$

Moreover, we clearly observe that

$$F(\frac{1}{2}) = \frac{1}{2}.$$

Therefore, by continuity of F on $[0, 1]$ it becomes obvious that

$$c(\alpha) = \sup_{0 \leq t \leq 1} F(t) \geq \frac{1}{2}. \tag{13}$$

This proves the first statement of the lemma.

We now prove the second part of this lemma. To this end, a simple calculation shows that the derivative of F is given by

$$F'(t) = 1 - 2\int_0^t \alpha(s)ds + \alpha(t)(1-2t). \tag{14}$$

To establish the "if" part, let us assume that $c(\alpha) = 1/2$. Since $F(1/2) = 1/2$ then (13) implies $F'(\frac{1}{2}) = 0$. Hence, substituting $t = 1/2$ into Eq. (14) yields condition (11).

For the "only if" part, let us assume that $\int_0^{1/2} \alpha(x)dx = 1/2$. Then for every $\alpha \in \mathscr{N}$ that satisfies the last condition, we clearly have, by Eq. (14),

$$F'(t) > 0 \text{ on } [0, 1/2), \qquad \text{and} \qquad F'(t) < 0 \text{ on } (1/2, 1],$$

and thus $\sup_{0\le t\le 1} F(t) = F(1/2)$. Remembering that $F(1/2) = 1/2$ we easily get $c(\alpha) = \sup_{0\le t\le 1} F(t) = 1/2$, as required. To see that the constant $\frac{1}{2}$ is optimal in the case where $c(\alpha) = 1/2$, take $f = f_\varepsilon$, where $\varepsilon > 0$, and f_ε is defined by

$$f_\varepsilon(x) = \begin{cases} -1 & \text{if } x \in \left[0, \frac{1}{2} - \varepsilon\right] \\ \frac{1}{\varepsilon}\left(x - \frac{1}{2}\right) & \text{if } x \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right] \\ 1 & \text{if } x \in \left[\frac{1}{2} + \varepsilon, 1\right]. \end{cases} \tag{15}$$

We clearly have

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon\to 0}\left(-\int_0^{1/2-\varepsilon} \alpha(x)dx + \int_{1/2+\varepsilon}^1 \alpha(x)dx\right) \\ &\quad + \lim_{\varepsilon\to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= -\int_0^{1/2} \alpha(x)dx + \int_{1/2}^1 \alpha(x)dx \\ &\quad + \lim_{\varepsilon\to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx. \end{aligned}$$

Now since $c(\alpha) = \frac{1}{2}$, we have that $\int_0^{1/2} \alpha(x)dx = 1/2$ which yields

$$-\int_0^{1/2} \alpha(x)dx + \int_{1/2}^1 \alpha(x)dx = 0.$$

Hence, we get

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon\to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= 0. \end{aligned}$$

Observe that in the case where $\alpha \equiv 1$ condition $\int_0^1 f_\varepsilon(x)\alpha(x)dx = 0$ is fulfilled for arbitrary ε. On the other hand,

$$\begin{aligned} \int_0^1 |f_\varepsilon(x)|\,dx &= 1 - \varepsilon \\ \int_0^1 \left|f_\varepsilon'(x)\right| dx &= \ 2. \end{aligned}$$

Then,we arrive at

$$\frac{\int_0^1 |f_\varepsilon(x)|\,dx}{\int_0^1 \left|f_\varepsilon'(x)\right|dx} = \frac{1-\varepsilon}{2} \to \frac{1}{2} \text{ when } \varepsilon \to 0,$$

which concludes the proof.

Now we state our first result whose proof follows immediately from Lemma 1.

Theorem 1 *If the weight function* α *from* $\mathscr{N}$ *satisfies* (11)*, then*

$$\int_0^1 |f(t)|\,dt \le \tfrac{1}{2}\int_0^1 \left|f'(t)\right|dt \tag{16}$$

for every function $f \in W^{1,1}[0,1]$ *such that* $\int_0^1 \alpha(x)f(x)dx = 0$. *Moreover, if* α *does not satisfy (11), then the constant* $\frac{1}{2}$ *is optimal.*

Proof The first part of the theorem is a direct consequence of Lemma 1. For the second part, without loss of generality, assume that $\int_0^{1/2}\alpha(x)dx > 1/2$. Take an arbitrary $\epsilon \in (0,1)$ and consider the sequence of functions

$$f_\varepsilon(x) = \begin{cases} -1 & \text{if } x \in \left[0, \frac{1}{2}-\varepsilon\right] \\ l(x) & \text{if } x \in \left[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right] \\ 1+2\delta & \text{if } x \in \left[\frac{1}{2}+\varepsilon, 1\right], \end{cases} \tag{17}$$

where l is the affine function

$$l(x) = -1 + \frac{1+\delta}{\varepsilon}\left(x - \frac{1}{2} + \varepsilon\right) \text{ and } \delta = \frac{2\int_0^{1/2}\alpha(x)dx - 1}{2\int_{1/2}^1 \alpha(x)dx}.$$

Observe that δ is positive and it satisfies

$$-\int_0^{1/2}\alpha(x)dx + (1+2\delta)\int_{1/2}^1 \alpha(x)dx = 0. \tag{18}$$

A simple calculation shows that

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon\to 0}\left(-\int_0^{1/2-\varepsilon}\alpha(x)dx + (1+2\delta)\int_{1/2+\varepsilon}^1 \alpha(x)dx\right) \\ &\quad + \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= -\int_0^{1/2}\alpha(x)dx + (1+2\delta)\int_{1/2}^1\alpha(x)dx \\ &\quad + \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx. \end{aligned}$$

Now using (18), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= \lim_{\varepsilon\to 0}\varepsilon\delta = 0.\end{aligned}$$

As we have

$$\begin{aligned}\int_0^1 |f_\varepsilon(x)|\,dx &= 1+\delta-\varepsilon-\tfrac{\delta\varepsilon}{1+\delta} \\ \int_0^1 \left|f_\varepsilon'(x)\right|dx &= \quad 2(1+\delta),\end{aligned}$$

it follows

$$\frac{\int_0^1 |f_\varepsilon(x)|\,dx}{\int_0^1 \left|f_\varepsilon'(x)\right|dx} = \frac{1+\delta-\varepsilon-\frac{\delta\varepsilon}{1+\delta}}{2(1+\delta)} \to \frac{1}{2} \text{ when } \varepsilon\to 0.$$

This proves the theorem.

Remark 1 A scaling argument shows that for a general interval $[a, b]$ the inequality (16) takes the form

$$\int_a^b |f(t)|\,dt \le \tfrac{(b-a)}{2}\int_a^b \left|f'(t)\right|dt.$$

We have the following result as a corollary to Theorem 1.

Corollary 1 *Suppose that α is any weight function from $\mathcal{N}$. Then, for every function $f \in W^{1,1}[0, 1]$ satisfying $\int_0^1 \alpha(t)f(t)dt = 0$ the following inequality holds:*

$$\int_0^1 \alpha(t)\,|f(t)|\,dt \le \tfrac{1}{2}\int_0^1 \left|f'(t)\right|dt. \tag{19}$$

The constant $\frac{1}{2}$ cannot be improved.

Proof For a given function α in $\mathcal{N}$, we define

$$v(x) = \int_0^x \alpha(s)ds, \qquad x \in [0, 1].$$

Clearly $v(0) = 0$, $v(1) = 1$ and v is non-decreasing on $[0, 1]$. We associate with any function f, belonging to the class $W^{1,1}[0, 1]$ on $[0, 1]$ and satisfying the condition $\int_0^1 \alpha(x)f(x)dx = 0$, the following function

$$\tilde{f}(t) := f(v^{-1}(t)), \qquad t \in [0, 1],$$

where v^{-1} is the inverse of v. Clearly $\tilde{f} \in W^{1,1}[0, 1]$ and it satisfies the condition

$$\int_0^1 \tilde{f}(t)dt = \int_0^1 \tilde{f}(v(x))v'(x)dx = \int_0^1 \alpha(x)f(x)dx = 0.$$

Then, by Theorem 1, $\tilde{f}$ satisfies the inequality

$$\textstyle\int_0^1 \left|\tilde{f}(t)\right| dt \leq \frac{1}{2} \int_0^1 \left|\tilde{f}'(t)\right| dt.$$

Now by making the change of variables $t = v(x)$, we see that

$$\int_0^1 \alpha(x) \left|f(x)\right| dx \leq \frac{1}{2} \int_0^1 \left|\tilde{f}'(v(x))\right| v'(x)dx = \frac{1}{2} \int_0^1 \left|f'(x)\right| dx,$$

which is the announced statement.

We observe that the integral on the right-hand side in the estimate (19) is not weighted. In view of this result the following question arises naturally:

- Does inequality (19) also hold with a weight function in the integral appearing on the right-hand side?

Acosta and Durán, see [1, Theorem 3.1], provide a positive answer to this question if the weight function is concave. We shall give a new and simpler proof of this result, and generalize it to a wide class of weight functions. From now on, we would like to consider the case when the weight function α is chosen in such a way that it belongs to the special class $\mathscr{M}$ defined by:

$$\mathscr{M} := \left\{\alpha \in \mathscr{N} : \int_0^{1/2} \alpha(t)dt = 1/2\right\}. \tag{20}$$

Recall that the best constant in the Poincaré inequality (2) is attained for this class of weight functions. This subset is nonempty, since it clearly contains the following two “broken” functions:

$$\alpha_2(x) = \begin{cases} 2 - 4x & \text{if } x \in [0, 1/2] \\ 4x - 2 & \text{if } x \in [1/2, 1], \end{cases}$$

$$\alpha_3(x) = \begin{cases} 4x & \text{if } x \in [0, 1/2] \\ 4 - 4x & \text{if } x \in [1/2, 1]. \end{cases}$$

Moreover, since $\mathscr{M}$ is convex it is an infinite set. Note that the function α_2 is convex on [0, 1], while α_3 is concave on [0, 1].

Without concavity assumption on α the weighted Poincaré-type inequality (4) does not hold true in general. The question we now want to address is that of determining some appropriate assumptions on the weight function α like convexity, monotonicity, which ensure that (4) holds. To the best of our knowledge the problem involving finding the weight functions that produce the best constant $1/2$ in (4) was not considered previously. In doing so, we shall provide a new version and a different proof of the Acosta and Durán result for a general class of weight functions belonging to $\mathscr{M}$.

To this end we first make the following observation that gives a sufficient condition on the weight functions for inequality (4) to hold.

Remark 2 Let us first observe that if $T_\alpha[f] = 0$ then multiplying (7) by $\alpha \in \mathscr{N}$ and applying Fubini's theorem, we obtain

$$\int_0^1 \alpha(x)\,|f(x)|\,dx \le 2\int_0^1 \left(\left(1-\int_0^x \alpha(t)dt\right)\int_0^x \alpha(t)dt\right)\left|f'(x)\right|dx. \tag{21}$$

As a consequence of the above inequality, the weighted Poincaré inequality (4) obviously holds for all $\alpha \in \mathscr{N}$ satisfying

$$\left(1-\int_0^x \alpha(t)dt\right)\int_0^x \alpha(t)dt \le \frac{\alpha(x)}{4},\ \ x \in (0,1)\,. \tag{22}$$

Acosta and Durán have shown that the above inequality holds for every concave weight function $\alpha \in \mathscr{N}$, see [1, Lemma 3.1].

In what follows we will show that large subclasses of weight functions in $\mathscr{N}$ satisfy inequality (22). We now give a useful fact:

Remark 3 By using the identity $ab \le \frac{(a+b)^2}{4}$, when $a, b \ge 0$, together with the fact that, for all x in the interval $(0, 1)$, $0 \le \int_0^x \alpha(t)dt \le 1$, we obtain that

$$\left(1-\int_0^x \alpha(t)dt\right)\int_0^x \alpha(t)dt \le \frac{1}{4}. \tag{23}$$

Therefore, inequality (22) holds for every x on $(0, 1)$ such that $\alpha(x) \ge 1$.

To prove our next weighted Poincaré-type inequality, we make use of two general lemmas:

Lemma 2 *Let $\alpha \in \mathscr{N}$ be a non-decreasing weight function on I_0, where $I_0 := (0, t_\alpha)$ is any nonempty sub-interval of $(0, 1)$, then (22) holds for all values $x \in I_0 \cap (0, 1/2)$ satisfying*

$$\frac{4x-1}{4x^2} \le \alpha(x). \tag{24}$$

If α does not satisfy (24), then there is a function from the described class for which (22) does not hold.

Proof We first observe that since α is a non-decreasing function on I_0 we have

$$\int_0^x \alpha(t)dt \leq x\alpha(x), x \in I_0 \cap (0, 1/2) . \tag{25}$$

According to Remark 3, it is enough to consider only those values of $x \in I_0$ for which $0 < \alpha(x) < 1$. Thus, under this condition, inequality (25) yields

$$\int_0^x \alpha(t)dt \leq x\alpha(x) \leq 1/2, x \in I_0 \cap (0, 1/2) .$$

Therefore, since the function $h(x) = x(1-x)$ is non-decreasing on $[0, 1/2]$, we get

$$\left(1 - \int_0^x \alpha(t)dt\right) \int_0^x \alpha(t)dt \leq x\alpha(x)(1 - x\alpha(x)), x \in I_0 \cap (0, 1/2) ,$$

and so to prove inequality (22) for a given fixed $x \in I_0 \cap (0, 1/2)$ it is enough to show that $x(1 - x\alpha(x)) \leq 1/4$, which is in turn equivalent to $\frac{4x-1}{4x^2} \leq \alpha(x)$ as was assumed. This proves the first part of the lemma.

For the second part, we may use a geometrically evident idea to construct, through any fixed point t in $(0, 1)$ such that

$$0 < \alpha(t) < \frac{4t - 1}{4t^2},$$

the function

$$f_t(x) = \begin{cases} \alpha(t) & \text{if } x \in [0, t] \\ l(x) & \text{if } x \in [t, 1] , \end{cases}$$

where l is the linear polynomial passing through the points

$$(t, \alpha(t)) \qquad and \qquad (1, \frac{2 - (1 + t)\alpha(t)}{1 - t}).$$

It is easy to see that f_t is a non-decreasing weight function that belongs to $\mathscr{N}$. Also, we have

$$\left(1 - \int_0^t f_t(x)dx\right) \int_0^t f_t(x)dx = t\alpha(t)(1 - t\alpha(t)) > \frac{f_t(t)}{4} \; (:= \frac{\alpha(t)}{4}).$$

This shows that (22) is not satisfied at the point t for the function f_t and this completes the proof of our lemma.

If the weight function $\alpha \in \mathscr{N}$ and non-increasing, then we have the following:

Lemma 3 *Let $\alpha \in \mathscr{N}$ be a non-increasing weight function on I_1, where $I_1 := (t_\alpha, 1)$ is any nonempty sub-interval of $(0, 1)$, then (22) holds for all values $x \in I_1 \cap (1/2, 1)$ satisfying*

$$\frac{3-4x}{4(1-x)^2} \leq \alpha(x). \tag{26}$$

If α does not satisfy (26), then there is a function from the described class for which (22) does not hold.

Proof We prove only the first statement since the proof of the second is essentially the same as that of Lemma 2. We will show that this case can be reduced the one treated in the previous lemma. Indeed, let us fix a non-increasing weight function α from $\mathscr{N}$. Define $\tilde{\alpha}$ the function in the interval $(0, 1)$ by $\tilde{\alpha}(t) = \alpha(1-t)$. Then, a straightforward inspection shows that $\tilde{\alpha}$ is a non-decreasing function belonging to $\mathscr{N}$. Hence the desired result follows by applying Lemma 2 to $\tilde{\alpha}$. This completes the proof of Lemma 3.

Now we make some comments, containing consequences of Lemmas 2 and 3.

Remark 4 Let $\alpha \in \mathscr{N}$. Since α is nonnegative function on the interval $(0, 1)$ then (24) is automatically satisfied on $(0, 1/4]$, and hence, by Lemma 2, inequality (22) holds if α is a non-decreasing function on any interval $I_0 \subset (0, 1/4)$. The same is true if α is a non-increasing function on any interval $I_1 \subset (3/4, 1)$.

Hence, we have the following weighted Poincaré inequality:

Theorem 2 *Let α be any concave function belonging to $\mathscr{M}$. Then, for every function $f \in W^{1,1}[0, 1]$ satisfying $\int_0^1 \alpha(t) f(t) dt = 0$, the following weighted Poincaré-type inequality holds*

$$\int_0^1 \alpha(t) \left|f(t)\right| dt \leq \frac{1}{2} \int_0^1 \alpha(t) \left|f'(t)\right| dt. \tag{27}$$

The constant $\frac{1}{2}$ cannot be improved.

In order to prove the above theorem, we shall apply the results of the previous two lemmas, and some preliminary facts about concave functions on $[0, 1]$ that will often be used without explicit reference. A somewhat known result that we can use as a starting point is the following form of the right-hand side of the Hermite–Hadamard inequality. It says: If the function α is concave on $[0, 1]$, then

$$\int_0^x \alpha(t) dt \leq x\alpha(x/2), x \in [0, 1]. \tag{28}$$

In particular, for any concave weight function belonging to $\mathscr{N}$, the following inequality holds:

$$1 \leq \alpha(1/2).$$

Moreover, if α is concave and $\alpha \in \mathscr{M}$, then substituting $x = 1/2$ in (28) and taking account of the fact that $\int_0^{1/2} \alpha(t)dt = 1/2$, we have

$$1 \leq \alpha(1/4).$$

It is also easy to check that

$$1 \leq \alpha(3/4), \tag{29}$$

in fact since α is concave the following inequality holds

$$\int_x^1 \alpha(t)dt \leq (1-x)\alpha((1+x)/2), x \in [0, 1],$$

and so (29) follows immediately by substituting $x = 1/2$ in the above equation. Inequalities (28) have been extensively studied in the literature, see, e.g., [4–6].

We now turn to the proof of Theorem 2.

Proof Let us fix a function α from $\mathscr{M}$. We begin the proof of Theorem 2 by noting that since α is concave, then, there exists an t_α in $[0, 1]$ such that α is non-decreasing on $[0, t_\alpha]$ and non-increasing on $[t_\alpha, 1]$. We will distinguish the following two cases:

Case 1 $t_\alpha \in (0, 1/2)$. If $t_\alpha \in (0, 1/4]$, this case is easier to handle, indeed by Remark 4, there is nothing to prove, since (24) is automatically satisfied on $(0, t_\alpha]$, and hence inequality (22) holds on $(0, t_\alpha)$. We may therefore assume that $t_\alpha \in (1/4, 1/2)$. Let us denote I_0 the sub-interval $(0, t_\alpha)$. Then, since $1 \leq \alpha(1/4)$ and α is non-decreasing function on $(1/4, t_\alpha)$, it follows that $1 \leq \alpha(t)$ for all t in $(1/4, t_\alpha)$. Observe that for all $x \in (1/4, t_\alpha)$, $\frac{4x-1}{4x^2} \leq 1$, then inequality (24) is satisfied for all $x \in (0, t_\alpha]$. Hence, Lemma 2 applies, so inequality (22) holds on $(0, t_\alpha)$.

Now, we will use I_1 to denote the sub-interval $(1/2, 1)$. Then, since $1 \leq \alpha(1/2)$, $1 \leq \alpha(3/4)$, and α is non-increasing function on I_1, it follows that $1 \leq \alpha(t)$ for all t in $(1/2, 3/4)$. Then, arguing as before, we see that (26) is satisfied for all $x \in I_1$. Lemma 3 applies, consequently, inequality (22) also holds on I_1.

Finally, in the sub-interval $(t_\alpha, 1/2)$ we have nothing to prove, since $1 \leq \alpha(t)$ for all t in $(t_\alpha, 1/2)$.

Case 2 $t_\alpha \in [1/2, 1)$. The proof is similar to the proof of the above case and hence is omitted.

So altogether, inequality (22) holds on $(0, 1)$, then the weighted Poincaré constant in (27) follows from Remark 2.

We note that our proof given in case $\alpha \in \mathscr{M}$ is completely different from the one given in [1]. We observe also that if (27) is unweighted and we allow α to very freely over $\mathscr{N}$, then Lemma 2 and Theorem 2 inform us that the value $1/2$ of the best weighted Poincaré constant in (27) and it is attained if and only if $\alpha \in \mathscr{M}$.

We may now ask the following question:

- Do we get similar result to Theorem 2 if concavity is replaced by convexity assumption?

We first make the following observation:

Remark 5 The inequality (27) cannot hold in Theorem 2 if concavity is replaced by convexity. Indeed, if α is a non-decreasing convex function on [0, 1], then applying the Hermite–Hadamard inequality on [0, 1], we have $\alpha(1/2) \leq 1$, and since α is non-decreasing we deduce that $\alpha(0) \leq 1$. These two conditions together imply that $\int_0^{1/2} \alpha(t)dt = 1/2$, which can only happen if α is the one constant weight. To see this is quite simple and becomes obvious on drawing a figure. So the weighted Poincaré inequality does not hold for non-constant weight convex functions, which are non-decreasing.

Thus, our aim, of course, is to find some subclasses of convex functions for which inequality (27) remains valid. The last observation motivates us to introduce the subsets of weight functions of $\mathscr{N}$ (satisfying the Dirichlet conditions)

$$\mathscr{N}_0 = \{\alpha \in \mathscr{N} : \alpha(0) = 0\}$$
$$\mathscr{N}_1 = \{\alpha \in \mathscr{N} : \alpha(1) = 0\}.$$

A weighted Poincaré inequality with a nonnegative weight function in $\mathscr{N}_0$ or $\mathscr{N}_1$ is sometimes referred to as Poincaré–Friedrichs inequality. In order to state our next theorem, we would like to point out that the Hermite–Hadamard inequality for convex functions gives

$$\left(1 - \int_0^x \alpha(t)dt\right) \int_0^x \alpha(t)dt \leq x(1-x)\frac{\alpha(1)+\alpha(x)}{2}\frac{\alpha(0)+\alpha(x)}{2}$$
$$\leq \frac{1}{4}\frac{\alpha(1)+\alpha(x)}{2}\frac{\alpha(0)+\alpha(x)}{2}.$$

By Remark 2 is enough to prove that (22) holds for $x \in (0, 1)$ such that $\alpha(x) \leq 1$. Therefore, we conclude that if $\alpha \in \mathscr{N}_0$ with $\alpha(1) \leq 3$ or $\alpha \in \mathscr{N}_1$ with $\alpha(0) \leq 3$ then we have

$$\left(1 - \int_0^x \alpha(t)dt\right) \int_0^x \alpha(t)dt \leq \frac{1}{4}\alpha(x).$$

Hence, by exactly the same argument as before we can show our modest extension of Theorem 2 when concavity is replaced by convexity.

Theorem 3 *Let α be any convex function belonging to $\mathscr{N}_0$ with $\alpha(1) \leq 3$, or $\mathscr{N}_1$ with $\alpha(0) \leq 3$. Then, for every function $f \in W^{1,1}[0, 1]$ such that $\int_0^1 \alpha(t) f(t)dt = 0$, the following weighted Poincaré-type inequality holds*

$$\int_0^1 \alpha(t)\,|f(t)|\,dt \leq \tfrac{1}{2}\int_0^1 \alpha(t)\left|f'(t)\right| dt. \tag{30}$$

The constant $\frac{1}{2}$ *cannot be improved.*

Conditions $\alpha(1) \leq 3$ or $\alpha(0) \leq 3$, required in Theorem 3, are not optimal. Indeed, let us consider the weight function

$$\alpha_m(x) = \begin{cases} mx & \text{if } x \in [0, 1/m]\,, \\ \frac{mx+m(m-2)}{(m-1)^2} & \text{if } x \in [1/m, 1]\,, \end{cases}$$

where $m \in (1, 2]$. Then it is easy to see that α_m is convex and that belongs to $\mathcal{N}_0$ for any $m \in (1, 2]$. Note also that $\alpha_m(x) < 1$ if $x \in [0, 1/m)$, and $\alpha_m(x) \geq 1$ if $x \in [1/m, 1]$. According to Remark 3, it is sufficient to consider interval $[0, 1/m)$. Therefore, after performing the integration inequality (22) simplifies to

$$\frac{x}{2}\left(1 - \frac{m}{2}x^2\right) \leq \frac{1}{4}, \;\; (x \in (0, 1/m)).$$

But now we can check that the latter holds if $m \geq \frac{32}{27}$. Hence, for any $m \in \left[\frac{32}{27}, 2\right]$ the weighted Poincaré inequality (30) holds for α_m. However, we have $\alpha_m(1) > 3$ for any $m \in [\frac{32}{27}, \frac{3}{2})$.

3 Sharp Inequalities for the Second Derivatives

In this section we discuss sharp weighted Poincaré inequality involving the second derivative. More precisely, we want to take advantage of a possible second order regularity of f and consider in this section estimates of the L^1-norm of a function f in terms of the L^1-norm of its second-order derivative. To state our main result we will use the following notation:

$$W^{2,1}[0, 1] = \left\{ f : f', f'' \; abs.\, cont., \; f'' \in L^1[0, 1) \right\}.$$

By applying twice inequality (16) for functions $f \in W^{2,1}[0, 1]$ which satisfy the conditions

$$\int_0^1 f(t)dt = \int_0^1 f'(t)dt = 0, \tag{31}$$

we conclude that

$$\int_0^1 |f(t)|\, dt \leq \frac{1}{2}\int_0^1 \left|f'(t)\right| dt \leq \frac{1}{4}\int_0^1 \left|f''(t)\right| dt. \tag{32}$$

As the next result shows the constant $1/4$, obtained by iteration from the Poincaré inequality of first order, is far from being the best one. The following alternative approach leads the optimal constant.

Theorem 4 *For all* $f \in W^{2,1}[0,1]$ *satisfying* (31) *the following inequality holds*

$$\int_0^1 |f(t)|\,dt \le \frac{1}{16}\int_0^1 \left|f''(t)\right|dt. \tag{33}$$

The constant $1/16$ *is the smallest possible.*

Proof It can be easily verified that the operator

$$l_1\,[f]\,(x) := \int_0^1 f(t)dt + (x-\frac{1}{2})\int_0^1 f'(t)dt$$

reproduces the linear polynomials, that is,

$$l_1\,[f]\,(x) \equiv f(x)\ for\ f(t)=1\ and\ f(t)=t, t\in[0,1]\,.$$

Then, by Peano's kernel theorem, each function in $W^{2,1}[0,1]$ satisfying (31) can be represented in the form

$$f(x) = \int_0^1 \left[(x-t)_+ - l_1\,[(.-t)_+]\,(x)\right] f''(t)dt,$$

where $x_+ := \frac{1}{2}(x+|x|)$. From this, in a standard way we derive the inequality

$$\int_0^1 |f(x)|\,dx \le \max_{0\le t\le 1}\int_0^1 |K_1(x,t)|\,dx \int_0^1 \left|f''(t)\right|dt,$$

where the kernel $K_1(x,t)$ is defined by

$$\begin{aligned} K_1(x,t) &= (x-t)_+ - l_1\,[(.-t)_+]\,(x) \\ &= (x-t)_+ - \textstyle\int_0^1 (s-t)_+ ds - (x-\frac{1}{2})\int_0^1 (s-t)_+^0 ds. \end{aligned}$$

In order to compute the integral $\int_0^1 |K_1(x,t)|\,dx$ we first note that

$$\begin{aligned} l_t(x) &:= \textstyle\int_0^1 (s-t)_+ ds + (x-\frac{1}{2})\int_0^1 (s-t)_+^0 ds \\ &= \textstyle\int_t^1 (s-t)ds + (x-\frac{1}{2})\int_t^1 ds \\ &= \textstyle\frac{(1-t)^2}{2} + (x-\frac{1}{2})(1-t) \\ &= (1-t)(x-\frac{t}{2}). \end{aligned}$$

Therefore the line l_t crosses the x-axis at the point $x_1 = \frac{t}{2}$. We easily find also the point x_2 at which l_t intersects the truncated power function $(x - t)_+$. We should have

$$x - t = l_t(x) \; at \; x = x_2.$$

This immediately implies that

$$x_2 = \frac{1+t}{2} > t.$$

It is seen that $\int_0^1 |K_1(x,t)|\,dx$, which is the area between the functions l_t and $(. - t)_+$, equals 2 times the area $\mathscr{A}$, which satisfies the following equation

$$\begin{aligned} 2\mathscr{A} &= (x_2 - x_1)(x_2 - t) - (x_2 - t)^2 \\ &= \tfrac{1}{2}\tfrac{1-t}{2} - \left(\tfrac{1-t}{2}\right)^2 \\ &= \tfrac{(1-t)t}{4}. \end{aligned}$$

Since $(1-t)t \leq 1/4$ for all $t \in [0, 1]$, we obtain

$$\max_{0 \leq t \leq 1} \int_0^1 |K_1(x,t)|\,dx = \frac{1}{16}$$

and thus the desired inequality (33) holds.

It remains to show that the constant $1/16$ cannot be improved. In order to see this we consider the function

$$f_0(x) := \begin{cases} \frac{1}{4} - x \quad on \quad x \in \left[0, \frac{1}{2}\right] \\ x - \frac{3}{4} \quad on \quad x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We have $\int_0^1 |f_0(x)|\,dx = \frac{1}{8}$ while the variation $V(f_0')$ of f_0' on $[0, 1]$ is equal to 2. Thus, a smoothing of f_0 in a neighborhood $[-\varepsilon, \varepsilon]$ of the zero will produce a function f_ε from $W^{2,1}[0, 1]$ for which

$$\int_0^1 |f_\varepsilon(x)|\,dx \geq \frac{1}{8} - \varepsilon,\; V(f_\varepsilon') = \int_0^1 \left|f''(x)\right|dx = 2.$$

Thus inequality (33) with a constant smaller that $\frac{1}{16}$ does not hold true for $\varepsilon > 0$ sufficiently small. The proof is complete.

The common restrictions on f to estimate a certain norm of f in terms of a norm of higher derivative are of the form

$$f^{(i)}(\xi) = 0, \, j = 0, 1, \ldots, m,$$

with an appropriate m. Usually ξ is taken to be the middle of the interval considered. We can see as we did in Theorem 4 that the conditions

$$f(a) = f'(a) = 0 \; or \; f(b) = f'(b) = 0$$

imply the estimate

$$\int_a^b |f(t)| \, dt \leq \frac{(b-a)^2}{2} \int_a^b \left| f''(t) \right| dt.$$

Applying this inequality twice, on [0, 1/2] and [1/2, 1], we get under the conditions

$$f(\frac{1}{2}) = f'(\frac{1}{2}) = 0 \tag{34}$$

that

$$\int_0^1 |f(t)| \, dt \leq \frac{1}{8} \int_0^1 \left| f''(t) \right| dt.$$

Therefore the conditions we considered in Theorem 4 yield a better estimation of $\|f\|_1$ than the standard ones (34).

This arises the following question:

- Are there other functionals that produce the smallest constant?

It is difficult to characterize all of them as we did in the case $W^{1,1}[0, 1]$. Even if we restricted ourselves to the study of functionals of the form

$$\int_a^b \alpha(x) f(x) dx = 0, \int_a^b \alpha(x) f'(x) dx = 0,$$

with a certain weight function α on [0, 1], we would arrive at the problem of investigation of the corresponding kernel

$$K(x, t) = (x - t)_+ - l \left[(. - t)_+ \right] (x)$$

where

$$l \left[f \right] (x) = \int_0^1 \alpha(s) f(s) ds + \left(x - \int_0^1 s\alpha(s) ds \right) \int_0^1 \alpha(s) f'(s) ds.$$

In this situation the line $l_t \left[(. - t)_+ \right]$ intersects the truncated power function at the points

$$x_1(t) = j(t) / \textstyle\int_t^1 \alpha(s) ds$$
$$x_2(t) = (t - j(t)) / \textstyle\int_0^t \alpha(s) ds$$

where

$$j(t) = \int_0^t \alpha(s)ds \int_t^1 s\alpha(s)ds - \int_0^t s\alpha(s)ds \int_t^1 \alpha(s)ds.$$

Unfortunately, these expressions are too complicated to hope to get a complete characterization of the best weight functions, i. e., those that lead to

$$\int_0^1 |K(x,t)|\, dx = 1/16.$$

4 Conclusion

With direct and simple proofs, we establish, under the concavity, convexity, or monotonicity of the weight function, the best constants for a class of (weighted and non-weighted) new Poincaré-type inequalities. Finally, an (unweighted) inequality of a similar type involving the second derivative is studied. A sharp constant is determined.

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