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Dorin Andrica
Themistocles M. Rassias *Editors*

Differential and Integral Inequalities

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Differential and Integral Inequalities

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Preface

There are three reasons for the study of inequalities: practical, theoretical, and aesthetic

—Richard Bellman (1920–1984)

Differential and Integral Inequalities presents a number of papers written by some eminent mathematicians who have greatly contributed in the vast domain of inequalities. These inequalities play an instrumental role both as a separate domain of research and as a tool for the understanding and solution of a number of other problems deriving from real and complex analysis, operator theory, functional analysis, differential geometry, metric geometry, and related subjects.

The chapters of this book focus mainly on Cauchy–Schwarz inequality, Fejer inequalities, Hardy–Sobolev inequalities, Taylor-type representations, weighted inequalities for Riemann–Stieltjes integral, Poincaré-type inequalities, inequalities connected with generating functions, error estimates of approximations for the complex-valued integral transforms, operator inequalities, Hermite–Hadamard-type integral inequalities, multiple Hardy–Littlewood integral operator norm inequalities, Levin–Steckin inequality, Lyapunov-type inequalities, inequalities in statistics and information measures, norm inequalities for generalized fractional integral operators, variational inequalities, integral equations and inequalities, exact bounds on the zeros of solutions of second-order differential inequalities, double-sided Taylor's approximations, Meir–Keeler sequential contractions, inequalities for harmonic-exponential convex functions, and multidimensional half-discrete Hardy–Hilbert-type inequalities with a general homogeneous kernel.

We would like to thank all the contributors of papers in this volume who, throughout this collective effort, made this publication possible. Last but not least, we would like to acknowledge the superb assistance of the Springer staff for the publication of this work.

Cluj-Napoca, Romania — Dorin Andrica
Athens, Greece — Themistocles M. Rassias

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Convexity Variants and Fejer Inequalities with General Weight

Shoshana Abramovich

Abstract Some recent results related to convexity, superquadracity and Fejer type inequalities, in particular with generalized weight functions are discussed in this presentation.

1 Introduction

The Hermite–Hadamard inequality says that for any convex function $f : I \to \mathbb{R}$, I an interval, and for $a, b \in I$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2} \tag{1}$$

holds, and the Fejer inequality reads

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx \leq \int_a^b f(t)\,p(t)dt \leq \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx, \tag{2}$$

when f is convex and $p : [a, b] \to \mathbb{R}$ is non-negative, integrable and symmetric around the midpoint $x = \frac{a+b}{2}$.

One of many developments related to these inequalities is to replace the notion of classical convexity by other variants and generalizations of convexity. An early paper is that by Dragomir et al. [12], see also [5, 6, 11, 15, 17] and [19].

In particular, in the book [16] by C. Niculescu and L.E. Persson, several generalizations, variants and applications are described and are placed into a more general context of convexity. On this subject see also the book by Pečarić et al. [17].

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The development presented here deals with and emphasizes the Fejer and other types of inequalities with weight function p not necessarily symmetric around the mid point $x = \frac{a+b}{2}$ of the interval $[a, b]$, for convex, superquadratic, N-quasiconvex functions, and functions φ for which $\varphi^{'}(x)/p(x)$ is increasing.

We start with quoting some definitions and lemmas that we mention in the theorems presented in the sequel:

Definition 1 A function $\varphi : [0, B) \to \mathbb{R}$ is superquadratic provided that for all $x \in [0, B)$ there exists a constant $C_\varphi(x) \in \mathbb{R}$ such that the inequality

$$\varphi(y) \geq \varphi(x) + C_\varphi(x)(y - x) + \varphi(|y - x|)$$

holds for all $y \in [0, B)$, (see [7, Definition 2.1], there $[0, \infty)$ instead $[0, B)$).

Lemma 1 ([7, Inequality 1.2]) *The inequality*

$$\int \varphi(f(s))\, d\mu(s) \geq \varphi\left(\int f d\mu\right) + \int \varphi\left(\left|f(s) - \int f d\mu\right|\right) d\mu(s)$$

holds for all probability measures μ and all non-negative, $\mu-$integrable functions f if and only if φ is superquadratic.

Lemma 2 ([7, Lemma 2.1]) *Let φ be a superquadratic function with $C_\varphi(x)$ as in Definition 1.*

(i) Then $\varphi(0) \leq 0$.
(ii) If $\varphi(0) = \varphi^{'}(0) = 0$, then $C_\varphi(x) = \varphi^{'}(x)$ whenever φ is differentiable on $[0, B)$.
(iii) If $\varphi \geq 0$, then φ is convex and $\varphi(0) = \varphi^{'}(0) = 0$.

Corollary 1 *Suppose that f is superquadratic. Let $0 \leq x_i < B$, $i = 1, \ldots, n$ and let $\overline{x} = \sum_{i=1}^{n} a_i x_i$, where $a_i \geq 0$, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} a_i = 1$. Then*

$$\sum_{i=1}^{n} a_i f(x_i) - f(\overline{x}) \geq \sum_{i=1}^{n} a_i f(|x_i - \overline{x}|). \tag{3}$$

If f is non-negative, it is also convex and the inequality refines Jensen's inequality. In particular, the functions $f(x) = x^r$, $x \geq 0$, $r \geq 2$ are superquadratic and convex, and equality holds in (3) when $r = 2$.

Lemma 3 ([7, Lemma 3.1]) *Suppose $\varphi : [0, B) \to \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If $\varphi^{'}$ is superadditive or $\varphi^{'}(x)/x$ is non-decreasing, then φ is superquadratic and $C_\varphi(x) = \varphi^{'}(x)$ with $C_\varphi(x)$ as in Definition 1.*

Definition 2 ([5]) Let $N \in \mathbb{N}$. A real-valued function ψ_N defined on an interval $[a, b)$ with $0 \leq a < b \leq \infty$ is called N-quasiconvex if it can be represented as the product of a convex function φ and the function $p(x) = x^N$. For $N = 0$, $\psi_0 = \varphi$ and for $N = 1$ the function $\psi_1(x) = x\varphi(x)$ is called 1-quasiconvex function.

This presentation is organized as follows: After this introductory section we discuss in Section 2 results proved in [6] and [14] about Fejer type inequalities where the weight function is not symmetric around $\frac{a+b}{2}$. Section 3 deals with Hermite–Hadamard and Fejer type inequalities for N-quasiconvex functions that appear in [5] which can also be seen as Fejer type inequalities for convex functions with non-symmetric weight functions. Section 4 discusses the monotonicity of some functions related to the Fejer inequality proved in [6, 8], and [13]. Section 5 quotes from [1–3, 9, 10] and [18] which deal with normalized Jensen functional and their bounds.

2 Fejer Type Inequalities with Non-symmetric Weight Functions

Firstly, Fejer's inequality for special convex functions is presented by replacing the non-negative symmetric function $p = p(x)$ in (2) with monotone functions.

Theorem 1 ([6, Theorem 5]) *Let $\varphi : [a, b] \to \mathbb{R}$ be a differentiable and convex function. Let $p : [a, b] \to \mathbb{R}$ be a non-negative, integrable and monotone function.*

(a) Let $p'(x) \leq 0$, $a \leq x \leq b$ and $\varphi(a) \leq \varphi(b)$. Then

$$\int_a^b \varphi(t)\, p(t) dt \leq \frac{\varphi(a) + \varphi(b)}{2} \int_a^b p(x)\, dx. \tag{4}$$

(b) Let $p'(x) \geq 0$, $a \leq x \leq b$ and $\varphi(a) \leq \varphi\left(\frac{a+b}{2}\right)$. Then

$$\varphi\left(\frac{a+b}{2}\right) \int_a^b p(x)\, dx \leq \int_a^b \varphi(t)\, p(t) dt. \tag{5}$$

(c) If $p'(x) \geq 0$, $a \leq x \leq b$ and $\varphi(a) \geq \varphi(b)$, then (4) holds.
(d) If $p'(x) \leq 0$, $a \leq x \leq b$ and $\varphi(a) \geq \varphi\left(\frac{a+b}{2}\right)$, then (5) holds.

Remark 1 In particular cases (a) and (b) hold when φ is increasing and cases (c) and (d) hold when φ is decreasing.

Secondly, we quote three Hermite–Hadamard and Fejer type inequalities resulting from functions φ and weight function p where $\frac{\varphi'}{p}$ is non-decreasing. These theorems appear in [14] where the following is proved:

Theorem 2 ([14, Corollary 2.3]) *Let $\varphi : [0, b] \to \mathbb{R}$ be a differentiable function and let $p : [a, b] \to (0, \infty)$ be an integrable function. If $\frac{\varphi'}{p}$ is increasing, then*

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b \varphi(x)\,p(x)\,dx \le \frac{\varphi(a)+\varphi(b)}{2}. \tag{6}$$

Theorem 3 ([14, Theorem 2.6]) *Let $\varphi : [a,b] \to \mathbb{R}$ be a convex function and let $p : [a,b] \to (0,\infty)$ be an integrable function. If φ and p are monotonic in the same direction, then*

$$\int_a^b \varphi(x)\,p(x)\,dx \ge \varphi\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx.$$

Theorem 4 ([14, Theorem 2.11]) *Let $\varphi : [a,b] \to \mathbb{R}$ be a convex function and let $p : [a,b] \to (0,\infty)$ be a continuous function. If the function φ and p are monotonic in the opposite directions, then*

$$\frac{1}{\int_a^b p(x)\,dx}\int_a^b \varphi(x)\,p(x)\,dx \le \frac{\varphi(a)+\varphi(b)}{2} - \left(\frac{1}{2} - \frac{G}{b-a}\right)\delta_\varphi$$

where

$$G = \frac{1}{\int_a^b p(x)\,dx}\int_a^b \left|x - \frac{a+b}{2}\right| p(x)\,dx,$$

and

$$\delta_\varphi = \varphi(a) + \varphi(b) - 2\varphi\left(\frac{a+b}{2}\right).$$

In the next three corollaries we choose p that appears in the three Theorems 2, 3 and 4 to be $p(x) = x$. As it is given in these three theorems that $p(x) \ge 0$, we deal now with $x \ge 0$.

Corollary 2 *From Theorem 2 we get that for a differentiable function $\varphi : [a,b) \to \mathbb{R}$, $0 \le a < b$, for which $\frac{\varphi'(x)}{x}$ is increasing, inequality (6) is*

$$\frac{1}{b-a}\int_a^b \varphi(x)\,x dx \le \frac{\varphi(a)+\varphi(b)}{2}\frac{a+b}{2}. \tag{7}$$

Remark 2 Inequality (7) in Corollary 2 is satisfied for a large set of those superquadratic functions that satisfy Lemma 3.

From Theorem 3 we get when $p(x) = x \ge 0$:

Corollary 3 *Let $\varphi : [a,b] \to \mathbb{R}$, $0 \le a \le x \le b$ be a convex increasing function. Then,*

$$\frac{1}{b-a}\int_a^b \varphi(x)\,xdx \geq \varphi\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)$$

Corollary 4 follows from Theorem 4 for $p(x) = x \geq 0$ and it says that

Corollary 4 *If φ is a convex decreasing function on $[a, b]$, $0 \leq a < b$, then*

$$\frac{1}{b-a}\int_a^b \varphi(x)\,xdx$$
$$\leq \left(\frac{\varphi(a)+\varphi(b)}{4} + \frac{1}{2}\varphi\left(\frac{a+b}{2}\right)\right)\frac{a+b}{2}$$

holds.

3 On Fejer Type Inequalities for N-Quasiconvex Functions and More on Non-symmetric Weight Functions

We quote from [5] some refined Hermite–Hadamard and Fejer type inequalities for N-quasiconvex functions that are used in the theorems in the sequel, in particular in Section 4 for $N = 1$ and in Section 5. On this subject see also [1] and [4].

In Section 2 Fejer type inequalities where the weight function p is not symmetric are presented. Therefore we emphasize also here that the results about Hermite–Hadamard and Fejer type Inequalities can be seen as Hermite–Hadamard and Fejer type inequalities for the convex functions φ where the general weight function is $q(x) = x^N p(x)$, $N = 0, 1, \ldots, \,, x \geq 0$, and for the N-quasiconvex function $\psi_N(x) = x^N \varphi(x)$ the weight function is $q(x) = p(x)$, where $p(x)$ is symmetric around $x = \frac{a+b}{2}$.

This is the reason that the theorems and the examples in this section are presented in two forms, first as inequalities related to N-quasiconvex functions and then as inequalities related to convex functions.

The N-quasiconvex form of Fejer type inequalities with symmetric weight function p reads:

Theorem 5 ([5, Theorem1 and Corollary 1]) *Let $\varphi : [a, b] \to \mathbb{R}$, $a \geq 0$ be differentiable, convex and $\psi_N(x) = x^N \varphi(x)$, $N = 0, 1, 2, \ldots$. Let $p : [a, b] \to \mathbb{R}$ be non-negative, integrable and symmetric around $x = \frac{a+b}{2}$.*

Then,

$$\int_a^b \psi_N(x)\,p(x)\,dx$$
$$\geq \psi_N\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx$$

$$+\int_a^b \left(x-\frac{a+b}{2}\right)^2 \sum_{k=1}^N x^{k-1}\psi'_{N-k}\left(\frac{a+b}{2}\right) p(x)\,dx$$

and

$$\int_a^b \psi_N(x)\,p(x)\,dx$$
$$\le \frac{\psi_N(a)+\psi_N(b)}{2}\int_a^b p(x)\,dx$$
$$-\frac{1}{(b-a)}\sum_{k=1}^N \int_a^b b^{k-1}(x-a)(b-x)^2\,\psi'_{N-k}(x)\,p(x)\,dx$$
$$-\frac{1}{(b-a)}\sum_{k=1}^N \int_a^b a^{k-1}(x-a)^2(b-x)\,\psi'_{N-k}(x)\,p(x)\,dx.$$

In particular if $\varphi:[a,b]\to\mathbb{R}$, $a\ge 0$, *is a differentiable and convex function and* $\psi_1(x)=x\varphi(x)$*, then*

$$\psi_1\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx+\psi'_0\left(\frac{a+b}{2}\right)\int_a^b\left(x-\frac{a+b}{2}\right)^2 p(x)\,dx \tag{8}$$
$$\le \int_a^b \psi_1(x)\,p(x)\,dx$$
$$\le \frac{\psi_1(a)+\psi_1(b)}{2}\int_a^b p(x)\,dx-\int_a^b \psi'_0(x)(b-x)(x-a)\,p(x)\,dx,$$

where $\varphi=\psi_0$.

The convex form of Theorem 5 with the weight function $x^N p(x)$ reads:

Theorem 5* *Let* $\varphi:[a,b]\to\mathbb{R}$, $a\ge 0$ *be differentiable, convex and* $N=1,2,\ldots$. *Let* $p:[a,b]\to\mathbb{R}$ *be non-negative, integrable and symmetric around* $x=\frac{a+b}{2}$. *Then,*

$$\int_a^b \varphi(x)\left(x^N p(x)\right)dx$$
$$\ge \varphi\left(\frac{a+b}{2}\right)\left(\left(\frac{a+b}{2}\right)^N\int_a^b p(x)\,dx\right.$$
$$\left.+\int_a^b\left(x-\frac{a+b}{2}\right)^2\left(\left(\frac{\partial}{\partial\overline{x}}\left(\frac{x^N-\overline{x}^N}{x-\overline{x}}\varphi(\overline{x})\right)|_{\overline{x}=\frac{a+b}{2}}\right)p(x)\right)dx,\right.$$

and

$$\begin{aligned}
&\int_a^b \varphi(x)\left(x^N p(x)\right) dx \\
&\leq \frac{a^N \varphi(a) + b^N \varphi(b)}{2} \int_a^b p(x)\, dx \\
&-\frac{1}{(b-a)} \int_a^b \left[(x-a)(b-x)^2 \frac{\partial}{\partial x}\left(\frac{b^N - x^N}{b-x}\varphi(x)\right)\right. \\
&\left. + (x-a)^2 (b-x) \frac{\partial}{\partial x}\left(\frac{x^N - a^N}{x-a}\varphi(x)\right)\right] p(x)\, dx.
\end{aligned}$$

In particular if $\varphi : [a,b] \to \mathbb{R}$, $a \geq 0$, is a differentiable and convex function, then

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)\int_a^b p(x)\, dx + \varphi'\left(\frac{a+b}{2}\right)\int_a^b \left(x - \frac{a+b}{2}\right)^2 p(x)\, dx \\
&\leq \int_a^b \varphi(x)(xp(x))\, dx \\
&\leq \frac{a\varphi(a) + b\varphi(b)}{2}\int_a^b p(x)\, dx - \int_a^b \varphi'(x)(b-x)(x-a)\, p(x)\, dx.
\end{aligned}$$

The 1-quasiconvex form of (8) with the weight function $p(x) = 1$ reads:

Example 1 ([5, Example 1]) If $\varphi : [a,b] \to \mathbb{R}$, $a \geq 0$, is differentiable, convex, $\psi_1(x) = x\varphi(x)$ and $\psi_0 = \varphi$, then

$$\begin{aligned}
&\psi_1\left(\frac{a+b}{2}\right) + \frac{1}{12}\psi_0'\left(\frac{a+b}{2}\right)(b-a)^2 \\
&\leq \frac{1}{b-a}\int_a^b \psi_1(x)\, dx \\
&\leq \frac{\psi_1(a) + \psi_1(b)}{2} - \frac{1}{b-a}\int_a^b \psi_0'(x)(b-x)(x-a)\, dx.
\end{aligned}$$

This is a refinement of the Hermite–Hadamard inequality (1) when $\psi_0 = \varphi$ is increasing.

The convex form of Example 1 with the weight function $q(x) = x$ reads:

Example 1 ([5, Example 1])* If $\varphi : [a,b] \to \mathbb{R}$, $a \geq 0$, is differentiable and convex, then

$$\begin{aligned}
&\varphi\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right)+\frac{1}{12}\varphi^{'}\left(\frac{a+b}{2}\right)(b-a)^2\\
&\le \frac{1}{b-a}\int_a^b \varphi(x)\,x dx\\
&\le \frac{a\varphi(a)+b\varphi(b)}{2}-\frac{1}{b-a}\int_a^b \varphi^{'}(x)(b-x)(x-a)\,dx.
\end{aligned}$$

This is a Fejer type inequality when the weight function is $q(x)=x$.

It is proved also in [5] that:

Theorem 6 *Let* $\varphi : [a,b] \to \mathbb{R}$, $a \ge 0$, *be a differentiable, convex function and let* $N = 1, 2, 3, \ldots,$. *Then for* $\psi_1(x) = x\varphi(x)$ *and* $\varphi = \psi_0$ *we get that the inequalities*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \psi_1(x)\,dx\\
&\le \frac{b-a}{6}\psi_0(b)+\frac{b+2a}{3}\frac{1}{b-a}\int_a^b \psi_0(x)\,dx\\
&\le \frac{(b-a)(\psi_N(a)+\psi_N(b))}{6(b^N-a^N)}\\
&\quad +\frac{(b^{N+1}-a^{N+1})+2ab(b^{N-1}-a^{N-1})}{3(b^N-a^N)}\frac{1}{b-a}\int_a^b \psi_0(x)\,dx\\
&\le \frac{\psi_1(a)+\psi_1(b)}{6}+\frac{(b+a)}{3}\frac{1}{b-a}\int_a^b \psi_0(x)\,dx\\
&\le \frac{\psi_1(a)+\psi_1(b)}{2}-\frac{(b-a)(\psi_0(b)-\psi_0(a))}{6}
\end{aligned} \tag{9}$$

hold, which are Hermite–Hadamard refinements of (1) for ψ_1 *when* $\psi_0(b) - \psi_0(a) \ge 0$.

The convex form of Theorem 6 with a non-symmetric weight function is:

Theorem 6* *Let* $\varphi : [a,b] \to \mathbb{R}$, $a \ge 0$, *be a differentiable, convex function and let* $N = 1, 2, 3, \ldots,$. *Then we get that the inequalities*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b \varphi(x)\,x dx\\
&\le \frac{b-a}{6}\varphi(b)+\frac{b+2a}{3}\frac{1}{b-a}\int_a^b \varphi(x)\,dx\\
&\le \frac{(b-a)\left(a^N\varphi(a)+b^N\varphi(b)\right)}{6(b^N-a^N)}
\end{aligned}$$

$$+\frac{(b^{N+1}-a^{N+1})+2ab(b^{N-1}-a^{N-1})}{3(b^N-a^N)}\frac{1}{b-a}\int_a^b\varphi(x)\,dx$$
$$\leq\frac{a\varphi(a)+b\varphi(b)}{6}+\frac{(b+a)}{3}\frac{1}{b-a}\int_a^b\varphi(x)\,dx$$
$$\leq\frac{a\varphi(a)+b\varphi(b)}{2}-\frac{(b-a)(\varphi(b)-\varphi(a))}{6}$$

hold, which are Fejer type inequalities for the convex function φ *and the monotone weight function* $q(x)=x$.

4 Monotonicity of Some Functions Related to the Fejer Inequality

In this section we present in Theorems 9 and 10 some of the results obtained in [6]. It is shown there that when ψ is 1 -quasiconvex, that is, $\psi(x)=x\varphi(x)$, φ is convex and $\varphi^{'}\geq 0$, then

$$P(t)=\int_a^b\psi\left(tx+(1-t)\frac{a+b}{2}\right)p(x)\,dx \tag{10}$$

and

$$Q(t)=\frac{1}{2}\int_a^b\left[\psi\left(\frac{1+t}{2}a+\frac{1-t}{2}x\right)p\left(\frac{x+a}{2}\right)+\psi\left(\frac{1+t}{2}b+\frac{1-t}{2}x\right)p\left(\frac{x+b}{2}\right)\right]dx \tag{11}$$

are non-decreasing in t, $0\leq t\leq 1$ when $p=p(x)$ is non-negative, differentiable and symmetric around $x=\frac{a+b}{2}$. Replacing ψ with $\psi(x)=\varphi(x)x$, as in Section 3, $P(t)$ and $Q(t)$ in Theorems 9 and 10 stated in the end of this section can be represented by the convex function φ and the weight function $q(x)=xp(x)$. Therefore these theorems can be seen as variants and analogs of the monotonicity results in the following theorems F and G (proved in [13]) for the convex increasing φ and the general weight function $q(x)=xp(x)$.

In the proofs of Theorems 9 and 10, in which ψ is 1-quasiconvex function, similar techniques as those used in the following theorems F and G for Wright-convex functions and in Theorems 7 and 8 (proved in [8]), for superquadratic functions are employed.

Theorem F ([13, Theorem 2.5]) *Let* $f:[a,b)\to R$ *be a Wright-convex function and* $p:[a,b)\to R$ *be a non-negative, integrable and symmetric around* $x=\frac{a+b}{2}$, *then*

$$P(t) = \int_a^b f(tx + (1-t)\frac{a+b}{2})p(x)\,dx, \quad 0 \le t \le 1$$

is Wright-convex and increasing on $[0,1]$ *and*

$$f\left(\frac{a+b}{2}\right)\int_a^b p(x)\,dx = P(0) \le P(t) \le P(1) = \int_a^b f(x)\,p(x)\,dx.$$

Also, in the same paper it was proved that for :

$$Q(t) = \frac{\int_a^b \left[f\left(\frac{1+t}{2}a + \frac{1-t}{2}x\right)p\left(\frac{x+a}{2}\right) + f\left(\frac{1+t}{2}b + \frac{1-t}{2}x\right)p\left(\frac{x+b}{2}\right)\right]dx}{2}.$$

the following holds:

Theorem G ([13, Theorem 2.7]) *Let f and p be defined as in Theorem F. Then Q is Wright-convex and increasing on* $[0,1]$ *and*

$$\int_a^b f(x)\,p(x)\,dx = Q(0) \le Q(t) \le Q(1) = \frac{f(a)+f(b)}{2}\int_a^b p(x)\,dx.$$

Theorem 7 ([8, Theorem 1]) *Let f be a superquadratic integrable function on* $[0,b]$ *and let $p(x)$ be nonnegative, integrable and symmetric about $x = \frac{a+b}{2}$, $0 \le a < b$. Let $P(t)$ be*

$$P(t) = \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)p(x)\,dx, \quad t \in [0,1].$$

Then for $0 \le s \le t \le 1$, $t > 0$

$$P(s) \le P(t) - \int_a^b \frac{t+s}{2t} f\left((t-s)\left(\left|\frac{a+b}{2} - x\right|\right)\right)p(x)\,dx \quad (12)$$
$$- \int_a^b \frac{t-s}{2t} f\left((t+s)\left(\left|\frac{a+b}{2} - x\right|\right)\right)p(x)\,dx.$$

Corollary 5 ([8, Corollary 1]) *For $p(x) = 1$ we get that*

$$(b-a)\,f\left(\frac{a+b}{2}\right) + \int_a^b f\left(t\left|\frac{a+b}{2} - x\right|\right) \le \int_a^b f\left(tx + (1-t)\frac{a+b}{2}\right)dx$$
$$\le \int_a^b f(x)\,dx - \int_a^b \frac{1+t}{2} f\left((1-t)\left|x - \frac{a+b}{2}\right|\right)dx$$
$$- \int_a^b \frac{1-t}{2} f\left((1+t)\left|x - \frac{a+b}{2}\right|\right)dx$$

$$\le (b-a)\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b ((b-t)f(t-a)+(t-a)f(b-t))\,dt$$
$$-\int_a^b \frac{1+t}{2} f\left((1-t)\left|x-\frac{a+b}{2}\right|\right)dx-\int_a^b \frac{1-t}{2} f\left((1+t)\left|x-\frac{a+b}{2}\right|\right)dx.$$

Theorem 8 ([8, Theorem 2]) *Let $f(x)$ and $p(x)$ be defined as in Theorem 7. Let $Q(t)$ be*

$$Q(t)=\frac{\int_a^b \left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}x\right)p\left(\frac{x+a}{2}\right)+f\left(\frac{1+t}{2}b+\frac{1-t}{2}x\right)p\left(\frac{x+b}{2}\right)\right]dx}{2}.$$

Then, if $0\le s\le t\le 1$, we get that

$$Q(s)-Q(t)\le -\frac{1}{2}\int_a^b [\frac{(b-x)+\frac{t+s}{2}(x-a)}{b-x+t(x-a)} f\left(\frac{t-s}{2}(x-a)\right)$$
$$+\frac{\frac{t-s}{2}(x-a)}{b-x+t(x-a)} f\left((b-x)+\frac{t+s}{2}(x-a)\right)]p\left(\frac{x+a}{2}\right)dx$$
$$-\frac{1}{2}\int_a^b [\frac{(x-a)+\frac{t+s}{2}(b-x)}{x-a+t(b-x)} f\left(\frac{t-s}{2}(b-x)\right)$$
$$+\frac{\frac{t-s}{2}(b-x)}{x-a+t(b-x)} f\left((x-a)+\frac{t+s}{2}(b-x)\right)]p\left(\frac{x+b}{2}\right)dx$$
$$=-\int_a^b \frac{p(x)\left(1-\frac{t+s}{2}\right)|2x-a-b|}{(1-t)|2x-a-b|+t(b-a)} f\left(\frac{t-s}{2}(b-a-|a+b-2x|)\right)dx$$
$$-\int_a^b \frac{p(x)\frac{t+s}{2}(b-a)}{(1-t)|2x-a-b|+t(b-a)} f\left(\frac{t-s}{2}(b-a-|a+b-2x|)\right)dx$$
$$-\int_a^b \frac{p(x)\frac{t-s}{2}(b-a)f\left(\left(1-\frac{t+s}{2}\right)|2x-a-b|+\frac{t+s}{2}(b-a)\right)}{(1-t)|2x-a-b|+t(b-a)}dx$$
$$+\int_a^b \frac{p(x)\frac{t-s}{2}|a+b-2x|f\left(\left(1-\frac{t+s}{2}\right)|2x-a-b|+\frac{t+s}{2}(b-a)\right)}{(1-t)|2x-a-b|+t(b-a)}dx.$$

Corollary 6 ([8, Corollary 2]) *In the case that f is superquadratic and also positive, and therefore according to Lemma 2 is also convex, as in the case of x^p, $p\ge 2$, $x\ge 0$, Theorem 7 and Theorem 8 refine Theorem F and Theorem G, respectively, for convex functions.*

Example 2 For the special case that $s=0$ and $t=1$ we get

$$Q(0)-Q(1)$$

$$= \int_a^b f(x)\, p(x)dx - \int_a^b \frac{f(a)+f(b)}{2} p(x)dx$$
$$\leq - \int_a^b \frac{|2x-a-b|+(b-a)}{2(b-a)} f\left(\frac{b-a-|a+b-2x|}{2}\right) p(x)dx$$
$$- \int_a^b \frac{(b-a-|a+b-2x|)}{2(b-a)} f\left(\frac{|2x-a-b|+(b-a)}{2}\right) p(x)dx$$
$$= - \int_a^{\frac{a+b}{2}} \frac{(b-x)}{(b-a)} f(x-a)\, p(x)dx - \int_{\frac{a+b}{2}}^b \frac{x-a}{b-a} f(b-x)\, p(x)dx$$
$$- \int_a^{\frac{a+b}{2}} \frac{x-a}{b-a} f(b-x)\, p(x)dx - \int_{\frac{a+b}{2}}^b \frac{b-x}{b-a} f(x-a)\, p(x)dx$$
$$= - \int_a^b \left(\frac{x-a}{b-a} f(b-x) + \frac{b-x}{b-a} f(x-a)\right) p(x)dx.$$

The monotonicity result in Theorem 9 is a refinement of the result stated in Theorem F which says that $P(s) \leq P(t)$ for the convex function ψ with symmetric weight function p, because when φ is convex increasing, ψ is convex too. It can also be seen as an analog of Theorem F, this time with a non-symmetric weight function $q(x) = xp(x)$. It reads:

Theorem 9 ([6, Theorem 2]) *Let ψ be 1-quasiconvex function on $[a, b]$, $a \geq 0$, that is $\psi(x) = x\varphi(x)$. Let φ be a differentiable convex function satisfying $\varphi' \geq 0$. Let $p = p(x)$ be non-negative, integrable and symmetric around $x = \frac{a+b}{2}$, then, for $0 \leq s \leq t \leq 1$, $t > 0$,*

$$\begin{aligned} &P(s) \\ &\leq P(t) - \left(t^2 - s^2\right) \int_a^b \left(x - \frac{a+b}{2}\right)^2 \varphi'\left(sx + (1-s)\frac{a+b}{2}\right) p(x)\, dx \\ &\leq P(t), \end{aligned}$$

where P is defined in (10).

Next we present the following further refinement of the Fejer inequality (2) for 1-quasiconvex functions:

Corollary 7 ([6, Corollary 1]) *Assume that the conditions of Theorem 9 on ψ and p hold. Then*

$$\psi\left(\frac{a+b}{2}\right) \int_a^b p(x)\, dx + s^2 \varphi'\left(\frac{a+b}{2}\right) \int_a^b \left(x - \frac{a+b}{2}\right)^2 p(x)\, dx$$

$$
\begin{aligned}
&\leq \int_a^b \psi\left(sx+(1-s)\frac{a+b}{2}\right)p(x)\,dx \\
&\leq \int_a^b \psi(x)\,p(x)\,dx \\
&\quad -\left(1-s^2\right)\int_a^b\left(x-\frac{a+b}{2}\right)^2\varphi^{'}\left(sx+(1-s)\frac{a+b}{2}\right)p(x)\,dx \\
&\leq \frac{\psi(a)+\psi(b)}{2}\int_a^b p(x)\,dx-\int_a^b\varphi^{'}(x)(b-x)(x-a)\,p(x)\,dx \\
&\quad -\left(1-s^2\right)\int_a^b\left(x-\frac{a+b}{2}\right)^2\varphi^{'}\left(sx+(1-s)\frac{a+b}{2}\right)p(x)\,dx.
\end{aligned}
$$

The corresponding result for the function Q defined in (11) reads:

Theorem 10 ([6, Theorem 3]) *Let ψ and p be defined as in Theorem 9. If $0 \leq s \leq t \leq 1$, then a refinement of Theorem G is as follows.*

$$Q(s) \leq Q(t) - \Delta(s,t),$$

where

$$
\begin{aligned}
\Delta(s,t) = \int_a^{\frac{a+b}{2}} &\left(\varphi^{'}((1-s)x+sa)+\varphi^{'}((1-s)(a+b-x)+sb)\right) \\
&\times (t-s)(x-a)(a+b-2x+(t+s)(x-a))\,p(x)\,dx.
\end{aligned}
$$

Example 3 ([6, Example 2]) In the special case that $s=0$, $t=1$ we have that

$$
\begin{aligned}
Q(0) = \int_a^b \psi(x)\,p(x)\,dx &\leq Q(1)-\Delta(0,1) \\
&= \frac{\psi(a)+\psi(a)}{2}\int_a^b p(x)\,dx \\
&\quad -\int_a^{\frac{a+b}{2}}\left(\varphi^{'}(x)+\varphi^{'}(a+b-x)\right)(x-a)(b-x)\,p(x)\,dx \\
&= \frac{\psi(a)+\psi(a)}{2}\int_a^b p(x)\,dx-\int_a^b\varphi^{'}(x)(x-a)(b-x)\,p(x)\,dx,
\end{aligned}
$$

which is the same as the right-hand side of (8). Hence, Theorem 10 implies in particular a further refinement of the Fejer inequality (2).

5 Convexity, Superquadracity and Extended Normalized Jensen Functional

In this section we present Jensen type inequalities appeared in [1–3, 9, 10] and [18] related to the **Jensen functional**

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right)$$

We start with some theorems that appeared in these papers where we quote results about bounds of difference between specific Jensen functional and another Jensen functional for which the function involved is convex (Theorem 11), superquadratic (Theorem 12) and N-quasiconvex (Theorem 13).

In [9] similar results to those proved in Theorem 11 and in Theorem 12 are proved when f is a convex function and when f is a superquadratic function.

Theorem 11 ([10]) *Consider the normalized Jensen functional where* $f : C \longrightarrow \mathbb{R}$ *is a convex function on the convex set* C *in a real linear space,* $\mathbf{x} = (x_1, \ldots, x_n) \in C^n$, *and* $\mathbf{p} = (p_1, \ldots, p_n)$, $\mathbf{q} = (q_1, \ldots, q_n)$ *are non-negative n-tuples satisfying* $\sum_{i=1}^{n} p_i = 1$, $\sum_{i=1}^{n} q_i = 1$, $q_i > 0$, $i = 1, \ldots, n$. *Then*

$$M J_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq m J_n(f, \mathbf{x}, \mathbf{q}),$$

provided

$$m = \min_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right), \quad M = \max_{1 \leq i \leq n} \left(\frac{p_i}{q_i}\right).$$

Theorem 12 ([3, Theorem 3]) *Under the same conditions and definitions on* $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, m *and* M *as in Theorem 11, if* $f : [0, b) \to \mathbb{R}$, $0 < b \leq \infty$, *is a superquadratic function,* $\sum_{j=1}^{n} p_j x_j = \overline{x}_p$ *and* $\sum_{j=1}^{n} q_j x_j = \overline{x}_q$, $\mathbf{x} \in [0, b)^n$, *then the following inequalities hold:*

$$J_n(f, \mathbf{x}, \mathbf{p}) - m J_n(f, \mathbf{x}, \mathbf{q}) \geq m f\left(\left|\overline{x}_q - \overline{x}_p\right|\right) + \sum_{i=1}^{n} (p_i - m q_i) f\left(\left|x_i - \overline{x}_p\right|\right),$$

and

$$J_n(f, \mathbf{x}, \mathbf{p}) - M J_n(f, \mathbf{x}, \mathbf{q}) \leq -\sum_{i=1}^{n} (M q_i - p_i) f\left(\left|x_i - \overline{x}_q\right|\right) - f\left(\left|\overline{x}_q - \overline{x}_p\right|\right).$$

As in the former sections the following theorem can be seen as a theorem related to the convex function φ, where we replace p_i and q_i with $x_i p_i$ and $x_i q_i$, respectively, $i = 1, \ldots, n$ and $\sum_{i=1}^{n} p_i = \sum_{i=1}^{n} q_i = 1$.

Theorem 13 ([1, Theorem 18]) *Suppose that $\psi_N : [a, b) \to \mathbb{R}$, $0 \le a < b \le \infty$, is N-quasiconvex function, that is $\psi_N = x^N \varphi(x)$, $N = 1, 2, \ldots$, when φ is convex on $[a, b)$. Let $\mathbf{p}$, $\mathbf{q}$, $\mathbf{x}$, m, M, $\overline{x}_p$, $\overline{x}_q$ and x_i, $i = 1, \ldots, n$ be as in Theorem 12. Then,*

$$\begin{aligned}
&J_n(\psi_N, \mathbf{x}, \mathbf{p}) - m J_n(\psi_N, \mathbf{x}, \mathbf{q}) \\
&\ge \sum_{i=1}^{n} (p_i - m q_i)\left(x_i - \overline{x}_p\right)^2 \frac{\partial}{\partial \overline{x}_p}\left(\frac{x_i^N - \overline{x}_p^N}{x_i - \overline{x}_p}\varphi\left(\overline{x}_p\right)\right) \\
&+ m\left(\overline{x}_q - \overline{x}_p\right)^2\left(\frac{\overline{x}_q^N - \overline{x}_p^N}{\overline{x}_q - \overline{x}_p}\varphi\left(\overline{x}_p\right)\right),
\end{aligned}$$

and

$$\begin{aligned}
&J_n(\psi_N, \mathbf{x}, \mathbf{p}) - m J_n(\psi_N, \mathbf{x}, \mathbf{q}) \\
&\le \sum_{i=1}^{n} (p_i - M q_i)\left(x_i - \overline{x}_q\right)^2 \frac{\partial}{\partial \overline{x}_q}\left(\frac{x_i^N - \overline{x}_q^N}{x_i - \overline{x}_q}\varphi\left(\overline{x}_q\right)\right) \\
&- M\left(\overline{x}_q - \overline{x}_p\right)^2 \frac{\partial}{\partial \overline{x}_q}\left(\frac{\overline{x}_q^N - \overline{x}_p^N}{\overline{x}_q - \overline{x}_p}\varphi\left(\overline{x}_q\right)\right).
\end{aligned}$$

For $N = 1$ we get that

$$\begin{aligned}
&J_n(\psi_1, \mathbf{x}, \mathbf{p}) - m J_n(\psi_1, \mathbf{x}, \mathbf{q}) \\
&\ge \varphi'\left(\overline{x}_p\right)\left(J_n\left(x^2, \mathbf{x}, \mathbf{p}\right) - m J_n\left(x^2, \mathbf{x}, \mathbf{q}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&J_n(\psi_1, \mathbf{x}, \mathbf{p}) - M J_n(\psi_1, \mathbf{x}, \mathbf{q}) \\
&\le \varphi'\left(\overline{x}_q\right)\left(J_n\left(x^2, \mathbf{x}, \mathbf{p}\right) - M J_n\left(x^2, \mathbf{x}, \mathbf{q}\right)\right).
\end{aligned}$$

The following theorems deal with bounds of difference between specific Jensen functional and the sum of N other functionals when

$$0 \le p_{i,1} \le 1, 0 < q_i \le 1, \sum_{i=1}^{n} p_{i,1} = \sum_{i=1}^{n} q_i = 1.$$

For this purpose we denote:

$m_1 = \min\left(\frac{p_{i,1}}{q_i}\right)$, $i = 1, \ldots, n$, s_1 is equal to the number of i-th for which m_1 occurs,

$$p_{i,k} = \{ \begin{matrix} p_{i,k-1} - m_{k-1}q_i, & m_{k-1} \neq \frac{p_{i,k-1}}{q_i} \\ \frac{1}{s_{k-1}} m_{k-1}, & m_{k-1} = \frac{p_{i,k-1}}{q_i} \end{matrix}, \qquad k = 2, \ldots \tag{13}$$

$$m_{k-1} = \min_{1 \le i \le n}\left(\frac{p_{i,k-1}}{q_i}\right), \ k = 2, \ldots, ,$$

s_{k-1} is the number of cases for which m_{k-1} occurs.

Let also $x_{i,1} \in (a, b)$, $i = 1, \ldots, n$ be

$$x_{i,k} = \{ \begin{matrix} x_{i,k-1}, & m_{k-1} \neq \frac{p_{i,k-1}}{q_i} \\ \sum_{i=1}^{n} q_i x_{i,k-1}, & m_{k-1} = \frac{p_{i,k-1}}{q_i} \end{matrix}, \tag{14}$$

$i = 1, \ldots n$, $k = 2, \ldots, .$

With these notations the following theorem is obtained:

Theorem 14 ([2, Theorem 5]) *Suppose that* $f : [a, b) \to \mathbb{R}$, $a < b \le \infty$ *is a convex function. Then, for every integer* N,

$$J_n(f, \mathbf{x}_1, \mathbf{p}_1) - \sum_{k=1}^{N} m_k J_n(f, \mathbf{x}_k, \mathbf{q}) \ge 0, \tag{15}$$

where $\mathbf{p}_1 = (p_{1,1}, \ldots, p_{n,1})$, $\mathbf{q} = (q_1, \ldots, q_n)$, $\mathbf{x}_k = (x_{1,k}, \ldots, x_{n,k})$, $k = 1, \ldots, N$, $p_{i,k}$, m_k, $x_{i,k}$, *are as denoted in (13) and (14),* $\sum_{i=1}^{n} p_{i,1} = \sum_{i=1}^{n} q_i = 1$, *and* $p_{i,1} \ge 0, q_i > 0, \quad i = 1, \ldots, n$, $m_1 = \min_{1 \le i \le n}\left(\frac{p_{i,1}}{q_i}\right)$.

Corollary 8 *Under the conditions of Theorem 14, if*

$$p_{i,N} = q_i, \quad i = 1, \ldots, n \tag{16}$$

we get an equality in (15).

Replacing q_i, $i = 1, \ldots, n$ by $\frac{1}{n}$ in Theorem 14 and Corollary 8 we get Theorem 15 in [18, Theorem 1] and in [18, Theorem 2]:

Theorem 15 ([18, Theorem 1]) *Let* $f : I \to \mathbb{R}$, *(I is an interval) be convex, and let* $\mathbf{x}_1 = (x_{1,1}, \ldots, x_{n,1}) \subset I^n$, $\mathbf{p}_1 = (p_{1,1}, \ldots, p_{n,1}) \subset (0, 1)^n$ *be such that* $\sum_{i=1}^{n} p_{i,1} = 1$. *Then for every* $N \in \mathbb{N}$ *we have*

$$\sum_{i=1}^{n} p_{i,1} f(x_{i,1}) - f\left(\sum_{i=1}^{n} p_{i,1} x_{i,1}\right) \tag{17}$$

$$-\sum_{k=1}^{N} m_k \left(\sum_{i=1}^{n} \frac{1}{n} f\left(x_{i,k}\right) - f\left(\sum_{i=1}^{n} \frac{1}{n} x_{i,k} \right) \right) \geq 0,$$

where $m_k = \min_{1 \leq i \leq n} \left(\frac{p_{i,k}}{q_i} \right)$ *and* $q_i = \frac{1}{n}$, $i = 1, .., n$, $k = 1, \ldots, N$.

If $p_{i,N} = \frac{1}{n}$, *we get equality in (17).*

We extend now the left-hand side inequality in Theorem 11.

We denote

$$\begin{aligned} \mathbf{p}_1 &= \left(p_{1,1}, \ldots, p_{1,n}\right), \quad \mathbf{q} = (q_1, \ldots, q_n), \\ \mathbf{x}_k &= \left(x_{1,k}, \ldots, x_{n,k}\right), \quad k = 1, \ldots, N \\ p_{i,1} &\geq 0, \quad q_i > 0, \quad i = 1, \ldots, n, \quad \sum_{i=1}^{n} p_{i,1} = \sum_{i=1}^{n} q_i = 1, \\ M_1 &= Max \left(\frac{p_{i,1}}{q_i} \right) = \frac{p_{j,1}}{q_j}, \quad i = 1, \ldots, n, \end{aligned} \tag{18}$$

where j is a fixed specific integer for which M_1 holds.

We also denote

$$\begin{aligned} p_{i,1} &= p^*_{i,1}, \;\; x^*_{i,1} = x_{i.1}, \;\; i = 1, \ldots, n, \\ p^*_{i,k} &= p^*_{i,k-1} - M_{k-1} q_i, \;\; x^*_{i,k} = x^*_{i,k-1}, \text{ when:} \\ M_{k-1} &\neq \frac{p^*_{i,k-1}}{q_i}, \;\; k = 2, \ldots, N; \\ p^*_{i,k} &= p^*_{i,k-1} - M_{k-1} q_i, \;\; x^*_{i,k} = x^*_{i,k-1}, \text{ when:} \\ M_{k-1} &= \frac{p^*_{i,k-1}}{q_i}, \;\; i \neq j_k, \;\; k = 2, \ldots, N; \\ p^*_{j_k,k} &= M_{k-1}, \;\; x^*_{j,k} = \sum_{i=1}^{n} q_i x^*_{i,k-1}, \text{ when } M_{k-1} = \frac{p^*_{j_k,k-1}}{q_{j_{k-1}}}, \\ M_k &= \underset{1 \leq i \leq n}{Max} \left(\frac{p^*_{i,k}}{q_i} \right) = \frac{p^*_{j_k,k}}{q_{j_k}}, \qquad k = 1, \ldots, N, \end{aligned} \tag{19}$$

where j_k is a specific index for which M_k holds.

With the notations and conditions in (18) and (19) we get:

Theorem 16 *Let* $f : [a, b) \to \mathbb{R}$, $a \leq b \leq \infty$, *be a convex function, and let (18) and (19) hold. Then, for every integer N*

$$J_n\left(f, \mathbf{x}_1, \mathbf{p}_1\right) - \sum_{k=1}^{N} M_k J_n\left(f, \mathbf{x}_k, \mathbf{q}\right) \leq 0,$$

and

$$M_k = \frac{p_{j_1,1}}{q_{j_1}^k}, \qquad k = 1,\dots,N$$

hold, where j_1 *is a fixed specific integer for which* $M_1 = \frac{p_{j_1,1}}{q_{j_1}}$ *is satisfied.*

Theorem 17 extends Theorem 13 and Theorem 14 for 1-quasiconvex functions, and Theorem 18 extends Theorem 12 for superquadratic functions:

Theorem 17 ([2, Theorem 7]) *Let* $\psi_1 : [a,b) \to \mathbb{R}$, $0 \le a < b \le \infty$ *be a* 1*-quasiconvex function where* $\psi_1(x) = x\varphi(x)$, *and* φ *is a differentiable convex function. Let* $\overline{x}_{\mathbf{p}_k} = \sum_{i=1}^n p_{i,k}x_{i,k}$ *and* $\overline{x}_{\mathbf{q}_k} = \sum_{i=1}^n q_i x_{i,k}$, $k = 1,\dots,N$. *Then under the same notations and conditions as used in Theorem 14 for* $p_{i,k}$, $x_{i,k}$, m_k, $\mathbf{p}_1$, $\mathbf{q}$, $k = 1,\dots,N$, $i = 1,\dots n$ *we get:*

$$\begin{aligned} & J_n(\psi_1,\mathbf{x}_1,\mathbf{p}_1) - \sum_{k=1}^N m_k J_n(\psi_1,\mathbf{x}_k,\mathbf{q}) \qquad (20) \\ & \ge \varphi'(\overline{x}_{\mathbf{p}_1})\left(\sum_{i=1}^n p_{i,N+1}x_{i,N+1}^2 - (\overline{x}_{\mathbf{p}_N})^2\right) \\ & = \varphi'(\overline{x}_{\mathbf{p}_1})\left(J_n\left(x^2,\mathbf{x}_1,\mathbf{p}_1\right) - \sum_{k=1}^N m_k J_n\left(x^2,\mathbf{x}_k,\mathbf{q}\right)\right). \end{aligned}$$

If φ *is also increasing, then (20) refines Theorem 11 and Theorem 14.*

In particular, for $N = 1$ *we get that*

$$\begin{aligned} & J_n(\psi_1,\mathbf{x}_1,\mathbf{p}_1) - m_1 J_n(\psi_1,\mathbf{x}_1,\mathbf{q}) \qquad (21) \\ & \ge \varphi'(\overline{x}_{\mathbf{p}_1})\left(J_n\left(x^2,\mathbf{x}_1,\mathbf{p}_1\right) - m_1 J_n\left(x^2,\mathbf{x}_1,\mathbf{q}\right)\right). \end{aligned}$$

Inequality (21) appears also in Theorem 13.

Similarly, we get for superquadratic functions (see Definition 1) the following theorem which extends Theorem 12:

Theorem 18 ([2, Theorem 8]) *Let* $f : [0,b) \to \mathbb{R}$, $0 < b \le \infty$ *be a superquadratic function. Let* $p_{i,k}$, $x_{i,k}$, m_k *and* s_k, $k = 1,\dots,N$, $i = 1,\dots,n$ *satisfy (13) and (14). Let* $\overline{x}_{\mathbf{p}_j} = \sum_{i=1}^n p_{i,j}x_{i,j}$ *and* $\overline{x}_{\mathbf{q}_j} = \sum_{i=1}^n q_i x_{i,j}$, $j = 1,\dots,N$, $p_{i,1} \ge 0$, $q_i > 0$, $i = 1,\dots,n$, $\mathbf{x} = (x_i,\dots,x_n) \in [0,b)^n$. *Then*

$$J_n(f,\mathbf{x}_1,\mathbf{p}_1) - \sum_{k=1}^N m_k J_n(f,\mathbf{x}_k,\mathbf{q}) \ge \sum_{i=1}^n p_{i,N+1} f\left(\left|x_{i,N+1} - \overline{x}_{\mathbf{p}_1}\right|\right)$$

If f is also non-negative, then f is convex and (16) refines Theorem 12.

In particular for $N = 1$ we get that

$$J_n(f, \mathbf{x}_1, \mathbf{p}_1) - mJ_n(f, \mathbf{x}_1, \mathbf{q})$$
$$\geq mf\left(\left|\overline{x}_q - \overline{x}_{p_1}\right|\right) + \sum_{i=1}^{n}(p_i - mq_i) f\left(\left|x_i - \overline{x}_{p_1}\right|\right).$$

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Error Estimates of Approximations for the Complex Valued Integral Transforms

Andrea Aglić Aljinović

Abstract In this survey paper error estimates of approximations in complex domain for the Laplace and Mellin transform are given for functions f which vanish beyond a finite domain $[a, b] \subset [0, \infty\rangle$ and whose derivative belongs to $L_p[a, b]$. New inequalities involving integral transform of f, integral mean of f and exponential and logarithmic mean of the endpoints of the domain of f are presented. These estimates enable us to obtain two associated numerical quadrature rules for each transform and error bounds of their remainders.

1 Introduction

1.1 Laplace and Mellin Transform

The **Laplace transform** $\mathscr{L}(f)$ of Lebesgue integrable mapping $f : [a, b] \to \mathbb{R}$ is defined by

$$\mathscr{L}(f)(z) = \int_0^\infty f(t)\, e^{-zt} \mathrm{d}t \tag{1}$$

for every $z \in \mathbb{C}$ for which the integral on the right-hand side of (1) exists, i.e. $\left|\int_0^\infty f(t)\, e^{-zt} \mathrm{d}t\right| < \infty$.

The **Mellin transform** $\mathscr{M}(f)$ of Lebesgue integrable mapping $f : [0, \infty\rangle \to \mathbb{R}$ is defined by

$$\mathscr{M}(f)(z) = \int_0^\infty f(t)\, t^{z-1} \mathrm{d}t \tag{2}$$

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for every $z \in \mathbb{C}$ for which the integral on the right-hand side of (2) exists, i.e. $\left|\int_0^\infty f(t)\, t^{z-1} \mathrm{d}t\right| < \infty$.

If $f : [a, b] \to \mathbb{R}$ is Lebesgue integrable mapping which vanishes beyond a finite domain, where $[a, b] \subset [0, \infty\rangle$ instead of (1) and (2), we have the finite Laplace and finite Mellin transform

$$\mathscr{L}(f)(z) = \int_a^b f(t)\, e^{-zt} \mathrm{d}t \qquad \mathscr{M}(f)(z) = \int_a^b f(t)\, t^{z-1} \mathrm{d}t.$$

The Laplace and Mellin transform not only are widely used in various branches of mathematics (for instance, for solving boundary value problem or Laplace equation, for summation of infinite series) but also have significant applications in the field of physics and engineering, particularly in computer science (in image recognition because of its scale invariance property). More about the Laplace, Mellin, and other integral transforms can be found in [5].

1.2 Weighted Montgomery Identity for a Complex Valued Weight Function

Montgomery identity states (see [6]):

$$f(x) = \frac{1}{b-a} \int_a^b f(t)\, \mathrm{d}t + \int_a^b P(x,t)\, f'(t)\, \mathrm{d}t, \tag{3}$$

where $P(x,t)$ is the Peano kernel, defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, \ a \le t \le x, \\ \frac{t-b}{b-a} \ \ x < t \le b. \end{cases}$$

The **weighted Montgomery identity** states (given by Pečarić in [7])

$$f(x) - \frac{1}{\int_a^b w(t)\, \mathrm{d}t} \int_a^b f(t)\, w(t)\, \mathrm{d}t = \int_a^b P_w(x,t)\, f'(t)\, \mathrm{d}t \tag{4}$$

where $w : [a, b] \to \mathbb{R}$ is a weight function, i.e. integrable function such that $\int_a^b w(t)\, \mathrm{d}t \neq 0$, $W(x) = \int_a^x w(t)\, \mathrm{d}t$, $x \in [a, b]$ and $P_w(x,t)$ the weighted Peano kernel, defined by

$$P_w(x,t) = \begin{cases} \frac{W(t)}{W(b)}, & a \le t \le x, \\ \frac{W(t)}{W(b)} - 1, & x < t \le b. \end{cases} \tag{5}$$

Obviously, weighted Montgomery identity (4) for uniform normalized weight function $w(t) = \frac{1}{b-a}$, $t \in [a,b]$ reduces to the Montgomery identity (3).

It is easy to check that the weighted Montgomery identity holds also for a complex valued weight function $w : [a,b] \to \mathbb{C}$ such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$.

Let us check the last condition for the kernels $w(t) = e^{-zt}$ $t \in [a,b]$ and $w(t) = t^{z-1}$, $t \in [a,b]$ of the Laplace and Mellin transform. Since $\int_a^b e^{-zt}\mathrm{d}t = \frac{1}{z}\left(e^{-za} - e^{-zb}\right)$, by using notation $z = x + iy$ we have

$$
\begin{aligned}
e^{-za} &= e^{-zb} \\
e^{-xa}\left(\cos(-ya) + i\sin(-ya)\right) &= e^{-xb}\left(\cos(-yb) + i\sin(-yb)\right) \\
a &= b
\end{aligned}
$$

and obviously $\int_a^b w(t)\,\mathrm{d}t \neq 0$ holds for the kernel of the Laplace transform.

Also, it holds that $\frac{d}{dt}t^z = zt^{z-1}$ for $z \in \mathbb{C}$ and $\int_a^b t^{z-1}\mathrm{d}t = \frac{b^z - a^z}{z}$. Using notation $z = x + iy$ we have

$$
\begin{aligned}
b^z &= a^z \\
e^{z\ln a} &= e^{z\ln b} \\
e^{x\ln a}\left(\cos(y\ln a) + i\sin(y\ln a)\right) &= e^{x\ln b}\left(\cos(y\ln b) + i\sin(y\ln b)\right) \\
a &= b.
\end{aligned}
$$

For the kernel of the Mellin transform $w(t) = t^{z-1}$, $t \in [a,b]$ we can also conclude $\int_a^b w(t)\,\mathrm{d}t \neq 0$.

1.3 Difference Between Two Weighted Integral Means

By subtracting two weighted Montgomery identities (4), one for the interval $[a,b]$ and the other for $[c,d] \subseteq [a,b]$, the next result is obtained (see [2, 3]).

Lemma 1 *Let $f : [a,b] \to \mathbb{R}$ be an absolutely continuous function on $[a,b]$, $w : [a,b] \to \mathbb{C}$ and $u : [c,d] \to \mathbb{C}$ some weight functions, such that $\int_a^b w(t)\,\mathrm{d}t \neq 0$, $\int_c^d u(t)\,\mathrm{d}t \neq 0$ and*

$$
W(x) = \begin{cases} 0, & t < a, \\ \int_a^x w(t)\,\mathrm{d}t, & a \leq t \leq b, \\ \int_a^b w(t)\,\mathrm{d}t, & t > b, \end{cases} \quad U(x) = \begin{cases} 0, & t < c, \\ \int_c^x u(t)\,\mathrm{d}t & c \leq t \leq d, \\ \int_c^d u(t)\,\mathrm{d}t, & t > d, \end{cases}
$$

and $[c,d] \subseteq [a,b]$. *Then the next formula is valid*

$$\frac{1}{\int_a^b w(t)\,\mathrm{d}t}\int_a^b w(t)\,f(t)\,\mathrm{d}t - \frac{1}{\int_c^d u(t)\,\mathrm{d}t}\int_c^d u(t)\,f(t)\,\mathrm{d}t = \int_a^b K(t)\,f'(t)\,\mathrm{d}t \tag{6}$$

where

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{U(t)}{U(d)}, & t \in \langle c,d \rangle, \\ 1 - \frac{W(t)}{W(b)}, & t \in [d,b]. \end{cases} \tag{7}$$

Remark 1 The result of the previous lemma for real-valued weight functions has been proved in [4].

2 Error Estimates of Approximations in Complex Domain for the Laplace Transform

In this chapter error estimates of approximations complex domain for the Laplace transform are given for functions which vanish beyond a finite domain $[a,b] \subset [0,\infty\rangle$ and such that $f' \in L_p[a,b]$. New inequalities involving Laplace transform of f, integral mean of f and exponential mean of the endpoints of the domain of f are presented. In the next chapter these inequalities are used to obtain two associated numerical rules and error bounds of their remainders in each case. These results are published in [1].

Here and hereafter the symbol $L_p[a,b]$ $(p \geq 1)$ denotes the space of p-power integrable functions on the interval $[a,b]$ equipped with the norm

$$\|f\|_p = \left(\int_a^b |f(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$

and $L_\infty[a,b]$ denotes the space of essentially bounded functions on $[a,b]$ with the norm

$$\|f\|_\infty = ess \sup_{t\in[a,b]} |f(t)|.$$

Exponential mean $E(z,w)$ of z and w is given by

$$E(z,w) = \begin{cases} \frac{e^z - e^w}{z-w}, & \text{if } z \neq w, \\ e^w, & \text{if } z = w. \end{cases} \qquad z,w \in \mathbb{C} \tag{8}$$

Definition 1 We say (p, q) is a pair of conjugate exponents if $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$; or if $p = 1$ and $q = \infty$; or if $p = \infty$ and $q = 1$.

The next theorem was proved in [5]:

Theorem 1 *Let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous mapping on $[a, b]$. Then for all $x \neq 0$ we have the inequality*

$$\left| \mathscr{F}(g)(x) - E(-2\pi i x a, -2\pi i x b) \int_a^b g(s)\, \mathrm{d}s \right|$$
$$\leq \begin{cases} \frac{1}{3}(b-a)^2 \left\| g' \right\|_\infty, & \text{if } g' \in L_\infty[a,b], \\ \frac{2^{\frac{1}{q}}}{[(q+1)(q+2)]^{\frac{1}{q}}} (b-a)^{1+\frac{1}{q}} \left\| g' \right\|_p, & \text{if } g' \in L_p[a,b], \\ (b-a) \left\| g' \right\|_1 & \text{if } g' \in L_1[a,b]. \end{cases}$$

where $\mathscr{F}(g)(x)$ is Fourier transform

$$\mathscr{F}(g)(x) = \int_a^b g(t)\, e^{-2\pi i x t} \mathrm{d}t.$$

and $E(z, w)$ is given by (8).

Next, we apply identity for the difference of the two weighted integral means (6) with two special weight functions: uniform weight function and kernel of the Laplace transform. In such a way new generalizations of the previous results are obtained.

Theorem 2 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous, $f' \in L_p[a, b]$ and $c, d \in [a, b]$, $c < d$. Then for* $\operatorname{Re} z \geq 0$ *and $1 < p \leq \infty$ we have inequalities*

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\, \mathrm{d}t \right|$$
$$\leq e^{-a \operatorname{Re} z} (d-c) \left(\frac{(2^q + 1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$
$$\leq (d-c) \left(\frac{(2^q + 1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p,$$

while for $p = 1$ we have

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\, \mathrm{d}t \right|$$
$$\leq 2e^{-a \operatorname{Re} z} (d-c) \left\| f' \right\|_1 \leq 2(d-c) \left\| f' \right\|_1,$$

where $E(z, w)$ is exponential mean of z and w given by (8).

Proof If we apply identity (6) with $w(t) = e^{-zt}$, $t \in [a,b]$ and $u(t) = \frac{1}{d-c}$, $t \in [c,d]$, we have $W(t) = (t-a)E(-za,-zt)$, $t \in [a,b]$; $U(t) = \frac{t-c}{d-c}$, $t \in [c,d]$ and

$$\frac{1}{(b-a)E(-za,-zb)}\mathscr{L}(f)(z) - \frac{1}{d-c}\int_c^d f(t)\,\mathrm{d}t = \int_a^b K(t) f'(t)\,\mathrm{d}t.$$

Since $[c,d] \subseteq [a,b]$ we use (7) so

$$K(t) = \begin{cases} -\frac{W(t)}{W(b)}, & t \in [a,c], \\ -\frac{W(t)}{W(b)} + \frac{t-c}{d-c}, & t \in \langle c,d\rangle, \\ 1 - \frac{W(t)}{W(b)}, & t \in [d,b]. \end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t = \frac{d-c}{b-a}W(b)\int_a^b K(t) f'(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathscr{L}(f)(z) - E(-za,-zb)\int_c^d f(t)\,\mathrm{d}t\right| \le \left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q \left\|f'\right\|_p.$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\left\|\frac{d-c}{b-a}W(b)K(t)\right\|_q = \left(\int_a^c \left|\frac{d-c}{b-a}W(t)\right|^q \mathrm{d}t\right.$$
$$\left. + \int_c^d \left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|^q \mathrm{d}t + \int_d^b \left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|^q \mathrm{d}t\right)$$

and since $\operatorname{Re} z \ge 0$ we have $|W(t)| = \left|\int_a^t e^{-zs}\mathrm{d}s\right| \le \int_a^t \left|e^{-zs}\right|\mathrm{d}s = \int_a^t \left|e^{-s\operatorname{Re} z}\right|\mathrm{d}s \le (t-a)e^{-a\operatorname{Re} z}$ for $t \in [a,b]$, thus

$$\int_a^c \left|\frac{d-c}{b-a}W(t)\right|^q \mathrm{d}t \le \int_a^c \left(e^{-a\operatorname{Re} z}\frac{d-c}{b-a}(t-a)\right)^q \mathrm{d}t = e^{-aq\operatorname{Re} z}\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\int_c^d \left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|^q \mathrm{d}t \le \int_c^d \left(\left|\frac{d-c}{b-a}W(t)\right| + \left|\frac{t-c}{b-a}W(b)\right|\right)^q \mathrm{d}t$$

$$\le e^{-aq\,\mathrm{Re}\,z}\int_c^d\left(\frac{d-c}{b-a}(t-a)+t-c\right)^q\mathrm{d}t$$
$$=\left(\frac{e^{-a\,\mathrm{Re}\,z}}{b-a}\right)^q\int_c^d((b-a+d-c)\,t-c\,(b-a)-a\,(d-c))^q\,\mathrm{d}t.$$

If we denote

$$\lambda(t)=(b-a+d-c)\,t-c\,(b-a)-a\,(d-c)$$

we have $\lambda(c)=(d-c)(c-a)$ and $\lambda(d)=(d-c)(b+d-2a)$ so

$$\left(\frac{e^{-a\,\mathrm{Re}\,z}}{b-a}\right)^q\int_c^d((b-a+d-c)\,t-c\,(b-a)-a\,(d-c))^q\,\mathrm{d}t$$
$$=\frac{e^{-aqs\,\mathrm{Re}\,z}\left(\lambda(d)^{q+1}-\lambda(c)^{q+1}\right)}{(b-a)^q\,(q+1)\,(b-a+d-c)}$$
$$=\frac{e^{-aq\,\mathrm{Re}\,z}\,(d-c)^{q+1}\left((b+d-2a)^{q+1}-(c-a)^{q+1}\right)}{(b-a)^q\,(q+1)\,(b-a+d-c)}\le\frac{e^{-aq\,\mathrm{Re}\,z}2^q\,(d-c)^q\,(b-a)}{(q+1)}.$$

Also

$$\int_d^b\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q\mathrm{d}t=\int_d^b\left|\frac{d-c}{b-a}\int_t^b e^{-zs}\mathrm{d}s\right|^q\mathrm{d}t$$
$$\le e^{-aq\,\mathrm{Re}\,z}\int_d^b\left(\frac{d-c}{b-a}(b-t)\right)^q\mathrm{d}t=e^{-aq\,\mathrm{Re}\,z}\left(\frac{d-c}{b-a}\right)^q\frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q$$
$$\le e^{-a\,\mathrm{Re}\,z}\left(\left(\frac{d-c}{b-a}\right)^q\frac{(c-a)^{q+1}}{(q+1)}+\frac{2^q\,(d-c)^q\,(b-a)}{(q+1)}+\left(\frac{d-c}{b-a}\right)^q\frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}}$$
$$\le e^{-a\,\mathrm{Re}\,z}\left(\left(\frac{d-c}{b-a}\right)^q\frac{(b-a)^{q+1}}{(q+1)}+\frac{2^q\,(d-c)^q\,(b-a)}{(q+1)}\right)^{\frac{1}{q}}$$
$$=e^{-a\,\mathrm{Re}\,z}\,(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}$$

and since $e^{-a\operatorname{Re}z} \leq 1$ inequalities in case $1 < p \leq \infty$ are proved. For $p = 1$ we have

$$\left\| \frac{d-c}{b-a} W(b)\, K(t) \right\|_\infty = \max \left\{ \sup_{t\in[a,c]} \left| \frac{d-c}{b-a} W(t) \right|, \right.$$
$$\left. \sup_{t\in[c,d]} \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right|, \sup_{t\in[d,b]} \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right| \right\}$$

and

$$\sup_{t\in[a,c]} \left| \frac{d-c}{b-a} W(t) \right| \leq e^{-a\operatorname{Re}z} \frac{(d-c)(c-a)}{(b-a)},$$

$$\sup_{t\in[c,d]} \left| \frac{d-c}{b-a} W(t) - \frac{t-c}{b-a} W(b) \right| \leq \sup_{t\in[c,d]} \left\{ \left| \frac{d-c}{b-a} W(t) \right| + \left| \frac{t-c}{b-a} W(b) \right| \right\}$$
$$\leq e^{-a\operatorname{Re}z} \frac{d-c}{b-a} (d-a) + e^{-a\operatorname{Re}z} (d-c) = e^{-a\operatorname{Re}z} (d-c) \frac{b+d-2a}{b-a},$$

$$\sup_{t\in[d,b]} \left| \frac{d-c}{b-a} W(t) - \frac{d-c}{b-a} W(b) \right| \leq e^{-a\operatorname{Re}z} \frac{(d-c)(b-d)}{(b-a)}.$$

Thus

$$\left\| \frac{d-c}{b-a} W(b)\, K(t) \right\|_\infty \leq e^{-a\operatorname{Re}z} \frac{d-c}{b-a} \max\{(c-a), (b+d-2a), (b-d)\}$$
$$\leq e^{-a\operatorname{Re}z} 2(d-c)$$

and since $e^{-a\operatorname{Re}z} \leq 1$ the proof is completed.

Remark 2 The inequalities from the previous theorem hold for $\operatorname{Re}z \geq 0$. Similarly it can be proved that in case $\operatorname{Re}z < 0$ and $1 < p \leq \infty$ we have the inequality

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\, \mathrm{d}t \right|$$
$$\leq e^{-b\operatorname{Re}z} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p,$$

while for $\operatorname{Re} z < 0$ and $p = 1$ we have

$$\left| \frac{d-c}{b-a} \mathscr{L}(f)(z) - E(-za, -zb) \int_c^d f(t)\,\mathrm{d}t \right| \le e^{-b \operatorname{Re} z} 2(d-c) \left\| f' \right\|_1 .$$

Theorem 3 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous, $f' \in L_p[a, b]$ and $c, d \in [a, b]$, $c < d$. Then for $\operatorname{Re} z \ge 0$ and $1 < p \le \infty$, we have inequalities*

$$\begin{aligned} &\left| \frac{d-c}{b-a} E(-zc, -zd) \int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t \right| \\ &\le e^{-c \operatorname{Re} z} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p \\ &\le (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p , \end{aligned}$$

while for $p = 1$ we have

$$\begin{aligned} &\left| \frac{d-c}{b-a} E(-zc, -zd) \int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t \right| \\ &\le e^{-c \operatorname{Re} z} 2(d-c) \left\| f' \right\|_1 \\ &\le 2(d-c) \left\| f' \right\|_1 , \end{aligned}$$

where $E(z, w)$ is exponential mean of z and w given by (8).

Proof By applying identity (6) with $w(t) = \frac{1}{b-a}$, $t \in [a, b]$ and $u(t) = e^{-zt}$, $t \in [c, d]$ and proceeding in the similar manner as in the proof of the Theorem 2.

Remark 3 The inequalities from the previous theorem hold for $\operatorname{Re} z \ge 0$. Similarly it can be proved that in case $\operatorname{Re} z < 0$ and $1 < p \le \infty$ we have the inequality

$$\begin{aligned} &\left| \frac{d-c}{b-a} E(-zc, -zd) \int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t \right| \\ &\le e^{-d \operatorname{Re} z} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)} \right)^{\frac{1}{q}} \left\| f' \right\|_p , \end{aligned}$$

while for $\operatorname{Re} z < 0$ and $p = 1$ we have

$$\left| \frac{d-c}{b-a} E(-zc, -zd) \int_a^b f(t)\,\mathrm{d}t - \int_c^d e^{-zt} f(t)\,\mathrm{d}t \right| \le e^{-d \operatorname{Re} z} 2(d-c) \left\| f' \right\|_1 .$$

Corollary 1 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $f' \in L_p [a, b]$. Then for all* $\operatorname{Re} z \geq 0$ *and* $1 < p \leq \infty$*, we have the inequality*

$$\left| E(-za, -zb) \int_a^b f(t)\, \mathrm{d}t - \mathscr{L}(f)(z) \right| \leq (b-a)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left\| f' \right\|_p ,$$

while for $p = 1$ *we have*

$$\left| E(-za, -zb) \int_a^b f(t)\, \mathrm{d}t - \mathscr{L}(f)(z) \right| \leq 2(b-a) \left\| f' \right\|_1 .$$

Proof By applying Theorem 2 or 3 in the special case when $c = a$ and $d = b$.

Corollary 2 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous and $f' \in L_p [a, b]$. Then for all* $\operatorname{Re} z \geq 0$*, for any $c \in [a, b]$ and* $1 < p \leq \infty$*, we have the inequality*

$$\begin{aligned} &\left| \mathscr{L}(f)(z) - (b-a) E(-za, -zb) f(c) \right| \\ &\leq (b-a)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left\| f' \right\|_p , \end{aligned}$$

while for $p = 1$ *we have*

$$\left| \mathscr{L}(f)(z) - (b-a) E(-za, -zb) f(c) \right| \leq 2(b-a) \left\| f' \right\|_1 .$$

Proof By applying the proof of the Theorem 2 in the special case when $c = d$. Since f is absolutely continuous, it is continuous, thus as a limit case we have $\lim_{c \to d} \frac{1}{d-c} \int_c^d f(t)\, \mathrm{d}t = f(c)$.

3 Numerical Quadrature Rules for the Laplace Transform

In this section we use two previous corollaries to obtain two numerical quadrature rules.

Let $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a division of the interval $[a, b]$, $h_k := t_{k+1} - t_k,\ k = 0, 1, \ldots, n-1$ and $\nu(h) := \max_k \{h_k\}$. Define the sum

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} E(-zt_k, -zt_{k+1}) \int_{t_k}^{t_{k+1}} f(t)\, \mathrm{d}t \tag{9}$$

where $\operatorname{Re} z \geq 0$.

The following approximation theorem holds.

Theorem 4 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous function on $[a, b]$, $f' \in L_p [a, b]$. Then we have the quadrature rule*

$$\mathscr{L}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$, $\mathscr{E}(f, I_n, z)$ is given by (9) and for $1 < p \leq \infty$ the reminder $R(f, I_n, z)$ satisfies the estimate

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \|f'\|_p ,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2\nu(h) \|f'\|_1 .$$

Proof For $1 < p \leq \infty$ by applying the Corollary 1 with $a = t_k, b = t_{k+1}$ we have

$$\left|E(-zt_k, -zt_{k+1}) \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t - \int_{t_k}^{t_{k+1}} e^{-zt} f(t)\,\mathrm{d}t\right|$$
$$\leq (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} |f'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} .$$

Summing over k from 0 to $n-1$ and using generalized triangle inequality, we obtain

$$|R(f, I_n, z)| = |\mathscr{L}(f)(z) - \mathscr{E}(f, I_n, z)|$$
$$\leq \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} |f'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} .$$

Using the Hölder discrete inequality, we get

$$\left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} |f'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}$$
$$\leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left(\left(\int_{t_k}^{t_{k+1}} |f'(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}}$$
$$= \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \|f'\|_p$$

and the first inequality is proved. For $p = 1$ we have

$$|R(f, I_n, z)| \leq \sum_{k=0}^{n-1} 2h_k \left(\int_{t_k}^{k+1} \left| f'(t) \right| \mathrm{d}t \right)$$
$$\leq 2\nu(h) \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} \left| f'(t) \right| \mathrm{d}t \right) = 2\nu(h) \left\| f' \right\|_1$$

and the proof is completed.

Corollary 3 *Suppose that all assumptions of Theorem 4 hold. Additionally suppose*

$$\mathscr{E}(f, I_n, z) = \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} f(t)\,\mathrm{d}t$$
$$\cdot \sum_{k=0}^{n-1} E\left(-z\left(a + k \cdot \frac{b-a}{n}\right), -z\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right).$$

Then we have the quadrature rule

$$\mathscr{L}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p,$$

while for $p = 1$ *we have*

$$|R(g, I_n, z)| \leq \frac{2(b-a)}{n} \left\| f' \right\|_1.$$

Proof If we apply Theorem 4 with equidistant partition of $[a, b]$.

Now, define the sum

$$\mathscr{A}(f, I_n, z) = \sum_{k=0}^{n-1} (t_{k+1} - t_k) E(-zt_k, -zt_{k+1}) f\left(\frac{t_{k+1} + t_k}{2} \right) \tag{10}$$

where $\operatorname{Re} z \geq 0$.

The following approximation theorem also holds.

Theorem 5 *Assume* (p, q) *is a pair of conjugate exponents. Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous function on* $[a, b]$, $f' \in L_p [a, b]$. *Then we have the quadrature rule*

$$\mathscr{L}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$, $\mathscr{A}(f, I_n, z)$ *is given by (10) and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq 2\nu(h) \left\|f'\right\|_1.$$

Proof By applying the Corollary 2 with $a = t_k$, $b = t_{k+1}$, $c = \frac{t_{k+1}+t_k}{2}$ and then summing over k from 0 to $n - 1$, we obtain results similarly as in the proof of the Theorem 4.

Corollary 4 *Suppose that all assumptions of Theorem 5 hold. Additionally suppose*

$$\mathscr{A}(f, I_n, z) = \frac{b-a}{n} f\left(a + \frac{k(k+1)(b-a)}{2n}\right) \cdot \sum_{k=0}^{n-1} E\left(-z\left(a + k \cdot \frac{b-a}{n}\right), -z\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right).$$

Then we have the quadrature rule

$$\mathscr{L}(f)(z) = \mathscr{A}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 0$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p,$$

while for $p = 1$ *we have*

$$|R(g, I_n, z)| \leq \frac{2(b-a)}{n} \left\|f'\right\|_1.$$

Proof By applying Theorem 5 with equidistant partition of $[a, b]$.

Remark 4 For both numerical quadrature formulae in case $\operatorname{Re} z < 0$, for $1 < p \leq \infty$, the reminder $R(f, I_n, z)$ satisfies the estimate

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\| f' \right\|_p ,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} 2\nu(h) \left\| f' \right\|_1 .$$

For equidistant partition of $[a, b]$ and for $1 < p \leq \infty$ we have

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \frac{(b - a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p ,$$

while for $p = 1$

$$|R(f, I_n, z)| \leq e^{-b \operatorname{Re} z} \frac{2(b - a)}{n} \left\| f' \right\|_1 .$$

4 Error Estimates of Approximations in Complex Domain for the Mellin Transform

In this chapter error estimates of approximations complex domain for the Laplace transform are given for functions which vanish beyond a finite domain $[a, b] \subset [0, \infty\rangle$ and such that $f' \in L_p[a, b]$. New inequalities involving Laplace transform of f, integral mean of f, exponential and logarithmic means of the endpoints of the domain of f are presented. In the next section these inequalities are used to obtain two associated numerical rules and error bounds of their remainders in each case. These results are published in [3].

Logarithmic mean $L(a, b)$ is given by

$$L(a, b) = \begin{cases} \frac{a-b}{\ln a - \ln b}, & \text{if } a \neq b, \\ a, & \text{if } a = b, \end{cases} \quad a, b \in \mathbb{R}. \tag{11}$$

Theorem 6 *Assume (p, q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous, $[a, b] \subset \langle 0, \infty\rangle$, $f' \in L_p[a, b]$ and $[c,d] \subseteq [a, b]$. Then for* $\operatorname{Re} z \geq 1$ *and* $1 < p \leq \infty$ *the following inequality holds:*

$$\left|\frac{d-c}{b-a}\mathscr{M}\left(f\right)\left(z\right)-\frac{E\left(z\ln a, z\ln b\right)}{L\left(a,b\right)}\int_c^d f\left(t\right)\mathrm{d}t\right|$$
$$\leq b^{(\operatorname{Re} z)-1}\left(d-c\right)\left(\frac{\left(2^q+1\right)\left(b-a\right)}{\left(q+1\right)}\right)^{\frac{1}{q}}\left\|f'\right\|_p$$

while for $p=1$ *it holds*

$$\left|\frac{d-c}{b-a}\mathscr{M}\left(f\right)\left(z\right)-\frac{E\left(z\ln a, z\ln b\right)}{L\left(a,b\right)}\int_c^d f\left(t\right)\mathrm{d}t\right|\leq 2b^{(\operatorname{Re} z)-1}\left(d-c\right)\left\|f'\right\|_1.$$

Here $E\left(z,w\right)$ *is exponential mean given by (8) and* $L\left(a,b\right)$ *is logarithmic mean given by (11).*

Proof Taking $w\left(t\right)=t^{z-1}$, $t\in[a,b]$ and $u\left(t\right)=\frac{1}{d-c}$, $t\in[c,d]$, we have

$$W\left(t\right)=\int_a^t t^{z-1}\mathrm{d}t=\frac{t^z-a^z}{z}=\frac{e^{z\ln t}-e^{z\ln a}}{z}$$
$$=\frac{e^{z\ln t}-e^{z\ln a}}{z\ln t-z\ln a}\cdot\frac{\ln t-\ln a}{t-a}\left(t-a\right)=\frac{E\left(z\ln a, z\ln t\right)}{L\left(a,t\right)}\left(t-a\right)$$

for all $t\in[a,b]$ and $U\left(t\right)=\frac{t-c}{d-c}$ for all $t\in[c,d]$. Now, we apply identity (6) with these weight functions

$$\frac{L\left(a,b\right)}{\left(b-a\right)E\left(z\ln a, z\ln b\right)}\mathscr{M}\left(f\right)\left(z\right)-\frac{1}{d-c}\int_c^d f\left(t\right)\mathrm{d}t=\int_a^b K\left(t\right)f'\left(t\right)\mathrm{d}t.$$

Since $[c,d]\subseteq[a,b]$ we use (7) so

$$K\left(t\right)=\begin{cases}-\frac{W(t)}{W(b)}, & t\in\left[a,c\right],\\ -\frac{W(t)}{W(b)}+\frac{t-c}{d-c}, & t\in\left\langle c,d\right\rangle,\\ 1-\frac{W(t)}{W(b)}, & t\in\left[d,b\right].\end{cases}$$

Thus

$$\frac{d-c}{b-a}\mathscr{M}\left(f\right)\left(z\right)-\frac{E\left(z\ln a, z\ln b\right)}{L\left(a,b\right)}\int_c^d f\left(t\right)\mathrm{d}t=\frac{d-c}{b-a}W\left(b\right)\int_a^b K\left(t\right)f'\left(t\right)\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left|\frac{d-c}{b-a}\mathscr{M}(f)(z)-\frac{E(z\ln a,z\ln b)}{L(a,b)}\int_c^d f(t)\,\mathrm{d}t\right|\leq\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q\left\|f'\right\|_p.$$

Now, for $1<p\leq\infty$ (for $1\leq q<\infty$) we have

$$\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q=\left(\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q\mathrm{d}t\right.$$

$$\left.+\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q\mathrm{d}t+\int_d^b\left|\frac{d-c}{b-a}W(t)-\frac{d-c}{b-a}W(b)\right|^q\mathrm{d}t\right)^{\frac{1}{q}}.$$

Using notation $x=\operatorname{Re}z$, $y=s\operatorname{Im}z$, since $x\geq 1$ we have $|W(t)|=\left|\int_a^t e^{(x-1+iy)\ln s}\mathrm{d}s\right|\leq\int_a^t\left|e^{(x-1+iy)\ln s}\right|\mathrm{d}s=\int_a^t\left|e^{(x-1)\ln s}\right|\mathrm{d}s\leq(t-a)\,e^{(x-1)\ln b}$ $=(t-a)\,b^{(x-1)}$ for $t\in[a,b]$, thus

$$\int_a^c\left|\frac{d-c}{b-a}W(t)\right|^q\mathrm{d}t\leq\int_a^c\left(b^{(x-1)}\frac{d-c}{b-a}(t-a)\right)^q\mathrm{d}t=b^{q(x-1)}\left(\frac{d-c}{b-a}\right)^q\frac{(c-a)^{q+1}}{(q+1)},$$

$$\int_c^d\left|\frac{d-c}{b-a}W(t)-\frac{t-c}{b-a}W(b)\right|^q\mathrm{d}t\leq\int_c^d\left(\left|\frac{d-c}{b-a}W(t)\right|+\left|\frac{t-c}{b-a}W(b)\right|\right)^q\mathrm{d}t$$

$$\leq b^{q(x-1)}\int_c^d\left(\frac{d-c}{b-a}(t-a)+t-c\right)^q\mathrm{d}t$$

$$=\left(\frac{b^{q(x-1)}}{b-a}\right)^q\int_c^d((b-a+d-c)\,t-c\,(b-a)-a\,(d-c))^q\,\mathrm{d}t.$$

If we denote

$$\lambda(t)=(b-a+d-c)\,t-c\,(b-a)-a\,(d-c) \tag{12}$$

we have $\lambda(c)=(d-c)(c-a)\geq 0$ and $\lambda(d)=(d-c)(b+d-2a)\geq 0$ so

$$\left(\frac{b^{q(x-1)}}{b-a}\right)^q\int_c^d((b-a+d-c)\,t-c\,(b-a)-a\,(d-c))^q\,\mathrm{d}t$$

$$=\frac{b^{q(x-1)}\left(\lambda(d)^{q+1}-\lambda(c)^{q+1}\right)}{(b-a)^q\,(q+1)\,(b-a+d-c)}$$

$$=\frac{b^{q(x-1)}\,(d-c)^{q+1}\left((b+d-2a)^{q+1}-(c-a)^{q+1}\right)}{(b-a)^q\,(q+1)\,(b-a+d-c)}\leq\frac{b^{q(x-1)}2^q\,(d-c)^q\,(b-a)}{(q+1)}.$$

Also

$$\int_d^b \left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|^q \mathrm{d}t = \int_d^b \left|\frac{d-c}{b-a}\int_t^b e^{(x-1+iy)\ln s} ds\right|^q \mathrm{d}t$$
$$\leq b^{q(x-1)} \int_d^b \left(\frac{d-c}{b-a}(b-t)\right)^q \mathrm{d}t = b^{q(x-1)}\left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_q$$
$$\leq b^{(x-1)}\left(\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q(d-c)^q(b-a)}{(q+1)} + \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}}$$
$$\leq b^{(x-1)}\left(\left(\frac{d-c}{b-a}\right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q(d-c)^q(b-a)}{(q+1)}\right)^{\frac{1}{q}}$$
$$= b^{(x-1)}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}$$

and the first inequality is proved. For $p=1$ we have

$$\left\|\frac{d-c}{b-a}W(b)\,K(t)\right\|_\infty = \max\left\{\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right|,\right.$$
$$\left.\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right|,\ \sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right|\right\}$$

and

$$\sup_{t\in[a,c]}\left|\frac{d-c}{b-a}W(t)\right| \leq b^{(x-1)}\frac{(d-c)(c-a)}{(b-a)},$$

$$\sup_{t\in[c,d]}\left|\frac{d-c}{b-a}W(t) - \frac{t-c}{b-a}W(b)\right| \leq \sup_{t\in[c,d]}\left\{\left|\frac{d-c}{b-a}W(t)\right| + \left|\frac{t-c}{b-a}W(b)\right|\right\}$$
$$\leq b^{(x-1)}\frac{d-c}{b-a}(d-a) + b^{(x-1)}(d-c) = b^{(x-1)}(d-c)\frac{b+d-2a}{b-a},$$
$$\sup_{t\in[d,b]}\left|\frac{d-c}{b-a}W(t) - \frac{d-c}{b-a}W(b)\right| \leq b^{(x-1)}\frac{(d-c)(b-d)}{(b-a)}.$$

Thus

$$\left\| \frac{d-c}{b-a} W(b) K(t) \right\|_\infty \le b^{(x-1)} \frac{d-c}{b-a} \max\{(c-a), (b+d-2a), (b-d)\}$$
$$\le b^{(x-1)} 2(d-c)$$

and the proof is completed.

Remark 5 The inequalities from the previous theorem hold for $\operatorname{Re} z \ge 1$. Similarly it can be proved that in case $\operatorname{Re} z < 1$ and $1 < p \le \infty$ the following inequality holds:

$$\left| \frac{d-c}{b-a} \mathscr{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)} \int_c^d f(t)\,\mathrm{d}t \right|$$
$$\le a^{(\operatorname{Re} z)-1}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}} \left\|f'\right\|_p$$

while for $\operatorname{Re} z < 1$ and $p = 1$ it holds

$$\left| \frac{d-c}{b-a} \mathscr{M}(f)(z) - \frac{E(z\ln a, z\ln b)}{L(a,b)} \int_c^d f(t)\,\mathrm{d}t \right| \le 2a^{(\operatorname{Re} z)-1}(d-c)\left\|f'\right\|_1.$$

Remark 6 In case $a = 0$ and $\operatorname{Re} z \ge 1$ proceeding in the same way as in the previous proof and using the fact that $0^z = 0$ and thus $\frac{b^z - a^z}{z(b-a)} = \frac{b^{z-1}}{z}$ we obtain

$$\left| \frac{d-c}{b} \mathscr{M}(f)(z) - \frac{b^{z-1}}{z} \int_c^d f(t)\,\mathrm{d}t \right| \le b^{(\operatorname{Re} z)-1}(d-c)\left(\frac{(2^q+1)(b)}{(q+1)}\right)^{\frac{1}{q}} \left\|f'\right\|_p$$

and

$$\left| \frac{d-c}{b} \mathscr{M}(f)(z) - \frac{b^{z-1}}{z} \int_c^d f(t)\,\mathrm{d}t \right| \le 2b^{(\operatorname{Re} z)-1}(d-c)\left\|f'\right\|_1.$$

Theorem 7 *Assume (p,q) is a pair of conjugate exponents, that is $\frac{1}{p} + \frac{1}{q} = 1$. Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous, $[a,b] \subset \langle 0, \infty\rangle$, $f' \in L_p[a,b]$ and $c, d \in [a,b]$, $c < d$. Then for* $\operatorname{Re} z \ge 1$ *and $1 < p \le \infty$ the following inequality holds:*

$$\left| \frac{(d-c)E(z\ln c, z\ln d)}{(b-a)L(c,d)} \int_a^b f(t)\,\mathrm{d}t - \int_c^d t^{z-1} f(t)\,\mathrm{d}t \right|$$
$$\le d^{(\operatorname{Re} z)-1}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}} \left\|f'\right\|_p,$$

while for $p = 1$ *we have*

$$\left| \frac{(d-c)\, E\,(z \ln c, z \ln d)}{(b-a)\, L\,(c,d)} \int_a^b f\,(t)\,\mathrm{d}t - \int_c^d t^{z-1} f\,(t)\,\mathrm{d}t \right| \le d^{(\operatorname{Re} z)-1} 2\,(d-c) \left\| f' \right\|_1,$$

where $E\,(z, w)$ *is given by (8) and* $L\,(a, b)$ *is logarithmic mean given by (11).*

Proof If we apply identity (6) with $w\,(t) = \frac{1}{b-a}$, $t \in [a,b]$ and $u\,(t) = t^{z-1}$, $t \in [c,d]$, we have $W\,(t) = \frac{t-a}{b-a}$, $t \in [a,b]$; $U\,(t) = \frac{E(z \ln c, z \ln t)}{L(c,t)}\,(t-c)$, $t \in [c,d]$ and

$$\frac{1}{(b-a)} \int_a^b f\,(t)\,\mathrm{d}t - \frac{L\,(c,d)}{(d-c)\, E\,(z \ln c, z \ln d)} \int_c^d t^{z-1} f\,(t)\,\mathrm{d}t = \int_a^b K\,(t)\, f'\,(t)\,\mathrm{d}t.$$

Since $[c,d] \subseteq [a,b]$ we use (7) so

$$K\,(t) = \begin{cases} -\frac{t-a}{b-a}, & t \in [a,c]\,, \\[2ex] \frac{U(t)}{U(d)} - \frac{t-a}{b-a}, & t \in \langle c,d \rangle\,, \\[2ex] \frac{b-t}{b-a}, & t \in [d,b]\,. \end{cases}$$

Thus

$$\frac{(d-c)\, E\,(z \ln c, z \ln d)}{(b-a)\, L\,(c,d)} \int_a^b f\,(t)\,\mathrm{d}t - \int_c^d t^{z-1} f\,(t)\,\mathrm{d}t = U\,(d) \int_a^b K\,(t)\, f'\,(t)\,\mathrm{d}t$$

and by taking the modulus and applying Hölder inequality we obtain

$$\left| \frac{(d-c)\, E\,(z \ln c, z \ln d)}{(b-a)\, L\,(c,d)} \int_a^b f\,(t)\,\mathrm{d}t - \int_c^d t^{z-1} f\,(t)\,\mathrm{d}t \right| \le \left\| U\,(d)\, K\,(t) \right\|_q \left\| f' \right\|_p.$$

Now, for $1 < p \le \infty$ (for $1 \le q < \infty$) we have

$$\left\| U\,(d)\, K\,(t) \right\|_q = \left(\int_a^c \left| \frac{t-a}{b-a} U\,(d) \right|^q \mathrm{d}t \right.$$
$$\left. + \int_c^d \left| U\,(t) - \frac{t-a}{b-a} U\,(d) \right|^q \mathrm{d}t + \int_d^b \left| \frac{b-t}{b-a} U\,(d) \right|^q \mathrm{d}t \right)^{\frac{1}{q}}.$$

Using notation $x = \operatorname{Re} z$, $y = s \operatorname{Im} z$, we have $x \geq 1$. Since $|U(t)| = \left|\int_c^t e^{(x-1+iy)\ln s} \mathrm{d}s\right| \leq \int_c^t \left|e^{(x-1+iy)\ln s}\right| \mathrm{d}s = \int_c^t \left|e^{(x-1)\ln s}\right| \mathrm{d}s \leq (t-c)\, e^{(x-1)\ln d} = (t-c)\, d^{(x-1)}$ for $t \in [c,d]$, we have

$$\int_a^c \left|\frac{t-a}{b-a} U(d)\right|^q \mathrm{d}t \leq d^{(x-1)q} \int_a^c \left(\frac{t-a}{b-a}(d-c)\right)^q \mathrm{d}t = d^{(x-1)q} \left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)},$$

$$\begin{aligned}
&\int_c^d \left|U(t) - \frac{t-a}{b-a} U(d)\right|^q \mathrm{d}t \leq \int_c^d \left(|U(t)| + \left|\frac{t-a}{b-a} U(d)\right|\right)^q \mathrm{d}t \\
&\leq d^{(x-1)q} \int_c^d \left(t - c + \frac{d-c}{b-a}(t-a)\right)^q \mathrm{d}t \\
&\leq \frac{d^{(x-1)q}}{(b-a)^q} \int_c^d ((b-a+d-c)\, t - c(b-a) - a(d-c))^q \mathrm{d}t \\
&= d^{(x-1)q} \frac{\left(\lambda(d)^{q+1} - \lambda(c)^{q+1}\right)}{(b-a)^q (q+1)(b-a+d-c)} \\
&= d^{(x-1)q} \frac{(d-c)^{q+1} \left((b+d-2a)^{q+1} - (c-a)^{q+1}\right)}{(b-a)^q (q+1)(b-a+d-c)} \leq d^{(x-1)q} \frac{2^q (d-c)^q (b-a)}{(q+1)},
\end{aligned}$$

where $\lambda(t)$ is given by (12) and

$$\int_d^b \left|\frac{b-t}{b-a} U(d)\right|^q \mathrm{d}t \leq d^{(x-1)q} \int_d^b \left(\frac{b-t}{b-a}(d-c)\right)^q \mathrm{d}t = d^{(x-1)q} \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}.$$

Thus

$$\begin{aligned}
&\|U(d)\, K(t)\|_q \\
&\leq d^{(x-1)} \left(\left(\frac{d-c}{b-a}\right)^q \frac{(c-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)} + \left(\frac{d-c}{b-a}\right)^q \frac{(b-d)^{q+1}}{(q+1)}\right)^{\frac{1}{q}} \\
&\leq d^{(x-1)} \left(\left(\frac{d-c}{b-a}\right)^q \frac{(b-a)^{q+1}}{(q+1)} + \frac{2^q (d-c)^q (b-a)}{(q+1)}\right)^{\frac{1}{q}} \\
&= d^{(x-1)} (d-c) \left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}
\end{aligned}$$

and the first inequality is proved. For $p = 1$ we have

$$\|U(d)\, K(t)\|_\infty = \max\left\{\sup_{t\in[a,c]} \left|\frac{t-a}{b-a} U(d)\right|,\right.$$

$$\sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right|,\ \sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right|\right\}$$

and

$$\sup_{t\in[a,c]}\left|\frac{t-a}{b-a}U(d)\right|\le d^{(x-1)}\frac{(c-a)(d-c)}{(b-a)},$$

$$\sup_{t\in[c,d]}\left|U(t)-\frac{t-a}{b-a}U(d)\right|=\sup_{t\in[c,d]}\left\{|U(t)|+\left|\frac{t-a}{b-a}U(d)\right|\right\}$$
$$\le d^{(x-1)}\sup_{t\in[c,d]}\left|d-c+\frac{d-a}{b-a}(d-c)\right|=d^{(x-1)}(d-c)\frac{b+d-2a}{b-a},$$

$$\sup_{t\in[d,b]}\left|\frac{b-t}{b-a}U(d)\right|\le d^{(x-1)}\frac{(b-d)(d-c)}{(b-a)}.$$

Thus

$$\|U(d)K(t)\|_\infty\le d^{(x-1)}\frac{d-c}{b-a}\max\{(c-a),(b+d-2a),(b-d)\}\le d^{(x-1)}2(d-c)$$

and the proof is completed.

Remark 7 The inequalities from the previous theorem hold for $\operatorname{Re} z\ge 1$. Similarly it can be proved that in case $\operatorname{Re} z<1$ and $1<p\le\infty$ the following inequality holds:

$$\left|\frac{(d-c)E(z\ln c,z\ln d)}{(b-a)L(c,d)}\int_a^b f(t)\,\mathrm{d}t-\int_c^d t^{z-1}f(t)\,\mathrm{d}t\right|$$
$$\le c^{(\operatorname{Re} z)-1}(d-c)\left(\frac{(2^q+1)(b-a)}{(q+1)}\right)^{\frac{1}{q}}\left\|f'\right\|_p,$$

while for $\operatorname{Re} z<1$ and $p=1$ it holds

$$\left|\frac{(d-c)E(z\ln c,z\ln d)}{(b-a)L(c,d)}\int_a^b f(t)\,\mathrm{d}t-\int_c^d t^{z-1}f(t)\,\mathrm{d}t\right|\le c^{(\operatorname{Re} z)-1}2(d-c)\left\|f'\right\|_1.$$

Remark 8 In case $a=c=0$ and $\operatorname{Re} z\ge 1$ all the inequalities from the Theorem 7 holds with a term $\frac{d^{z-1}}{z}$ instead of $\frac{E(z\ln c,z\ln d)}{L(c,d)}$.

Corollary 5 *Assume* (p, q) *is a pair of conjugate exponents, that is* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous and* $[a, b] \subset \langle 0, \infty \rangle$, $f' \in L_p[a, b]$. *Then for all* $\operatorname{Re} z \geq 1$ *and* $1 < p \leq \infty$ *we have the inequality*

$$\left| \mathscr{M}(f)(z) - \frac{E(z \ln a, z \ln b)}{L(a, b)} \int_a^b f(t)\,\mathrm{d}t \right| \leq b^{(\operatorname{Re} z) - 1} (b - a)^{1 + \frac{1}{q}} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$

while for $p = 1$ *we have*

$$\left| \mathscr{M}(f)(z) - \frac{E(z \ln a, z \ln b)}{L(a, b)} \int_a^b f(t)\,\mathrm{d}t \right| \leq 2b^{(\operatorname{Re} z) - 1} (b - a) \left\| f' \right\|_1 .$$

Proof By applying the proof of the Theorem 6 or 7 in the special case when $c = a$ and $d = b$.

Corollary 6 *Assume* (p, q) *is a pair of conjugate exponents, that is* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $f : [a, b] \to \mathbb{R}$ *be absolutely continuous and* $[a, b] \subset \langle 0, \infty \rangle$, $f' \in L_p[a, b]$. *Then for all* $\operatorname{Re} z \geq 1$, *for any* $c \in [a, b]$ *and* $1 < p \leq \infty$ *we have the inequality*

$$\left| \mathscr{M}(f)(z) - (b - a) \frac{E(z \ln a, z \ln b)}{L(a, b)} f(c) \right| \leq b^{(\operatorname{Re} z) - 1} (b - a)^{1 + \frac{1}{q}} \left(\frac{2^q + 1}{q + 1} \right)^{\frac{1}{q}} \left\| f' \right\|_p$$

while for $p = 1$ *we have*

$$\left| \mathscr{M}(f)(z) - (b - a) \frac{E(z \ln a, z \ln b)}{L(a, b)} f(c) \right| \leq 2b^{(\operatorname{Re} z) - 1} (b - a) \left\| f' \right\|_1 .$$

Proof By applying the proof of the Theorem 6 in the special case when $c = d$. Since f is absolutely continuous, it is continuous, thus as a limit case we have $\lim_{c \to d} \frac{1}{d - c} \int_c^d f(t)\,\mathrm{d}t = f(c)$.

5 Numerical Quadrature Rules for the Mellin Transform

Since the exponents of the term $(b - a)^{1 + \frac{1}{q}}$ in the inequalities from the last two corollaries are greater than 1, these inequalities can be useful to obtain numerical quadrature formulae. Using Corollaries 5 and 6 we obtain the following two numerical rules.

Let $I_n : a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be a division of the interval $[a, b]$, $h_k := t_{k+1} - t_k$, $k = 0, 1, \ldots, n - 1$ and $\nu(h) := \max_k \{h_k\}$. Define the sum

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} \frac{E(z \ln t_k, z \ln t_{k+1})}{L(t_k, t_{k+1})} \int_{t_k}^{t_{k+1}} f(t)\,\mathrm{d}t \tag{13}$$

where $\operatorname{Re} z \geq 1$.

The following approximation theorem holds.

Theorem 8 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous function on $[a, b]$, $[a, b] \subset \langle 0, \infty \rangle$, $f' \in L_p [a, b]$. Then we have the quadrature rule*

$$\mathscr{M} (f) (z) = \mathscr{E} (f, I_n, z) + R (f, I_n, z)$$

where $\operatorname{Re} z \geq 1$, $\mathscr{E} (f, I_n, z)$ *is given by (13) and for* $1 < p \leq \infty$ *the reminder* $R (f, I_n, z)$ *satisfies the estimate*

$$|R (f, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\| f' \right\|_p , \tag{14}$$

while for $p = 1$

$$|R (f, I_n, z)| \leq 2 b^{(\operatorname{Re} z)-1} \nu (h) \left\| f' \right\|_1 . \tag{15}$$

Proof For $1 < p \leq \infty$ by applying the Corollary 5 with $a = t_k$, $b = t_{k+1}$ we have

$$\begin{aligned} &\left| \frac{E (z \ln t_k, z \ln t_{k+1})}{L (t_k, t_{k+1})} \int_{t_k}^{t_{k+1}} f (t) \, \mathrm{d}t - \int_{t_k}^{t_{k+1}} t^{z-1} f (t) \, \mathrm{d}t \right| \\ &\leq (t_{k+1})^{x-1} (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f' (t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq b^{x-1} (t_{k+1} - t_k)^{1+\frac{1}{q}} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f' (t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \end{aligned}$$

where $x = \operatorname{Re} z$. Summing over k from 0 to $n - 1$ and using generalized triangle inequality, we obtain

$$\begin{aligned} |R (f, I_n, z)| &= |\mathscr{M} (f) (z) - \mathscr{E} (f, I_n, z)| \\ &\leq b^{x-1} \left(\frac{2^q + 1}{q+1}\right)^{\frac{1}{q}} \sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f' (t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} . \end{aligned}$$

Using the Hölder discrete inequality, we get

$$\begin{aligned} &\sum_{k=0}^{n-1} (h_k)^{1+\frac{1}{q}} \left(\int_{t_k}^{t_{k+1}} \left|f' (t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq \left[\sum_{k=0}^{n-1} \left((h_k)^{1+\frac{1}{q}}\right)^q\right]^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} \left(\left(\int_{t_k}^{t_{k+1}} \left|f' (t)\right|^p \mathrm{d}t\right)^{\frac{1}{p}}\right)^p\right]^{\frac{1}{p}} = \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\| f' \right\|_p \end{aligned}$$

and the inequality (14) is proved. For $p = 1$ we have

$$|R(f, I_n, z)| \leq \sum_{k=0}^{n-1} 2b^{x-1} h_k \left(\int_{t_k}^{k+1} \left| f'(t) \right| \mathrm{d}t \right)$$
$$\leq 2b^{x-1} \nu(h) \sum_{k=0}^{n-1} \left(\int_{t_k}^{t_{k+1}} \left| f'(t) \right| \mathrm{d}t \right) = 2b^{x-1} \nu(h) \left\| f' \right\|_1$$

and the proof is completed.

Corollary 7 *Suppose that all assumptions of Theorem 8 hold. Additionally suppose*

$$\mathscr{E}(f, I_n, z) = \sum_{k=0}^{n-1} \int_{a+k\cdot\frac{b-a}{n}}^{a+(k+1)\cdot\frac{b-a}{n}} f(t)\,\mathrm{d}t \cdot \frac{E\left(z \ln\left(a + k \cdot \frac{b-a}{n}\right), z \ln\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}{L\left(\left(a + k \cdot \frac{b-a}{n}\right), \left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}. \tag{16}$$

Then we have the quadrature rule

$$\mathscr{M}(f)(z) = \mathscr{E}(f, I_n, z) + R(f, I_n, z)$$

where $\operatorname{Re} z \geq 1$ *and for* $1 < p \leq \infty$ *the reminder* $R(f, I_n, z)$ *satisfies the estimate*

$$|R(f, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p, \tag{17}$$

while for $p = 1$ *we have*

$$|R(g, I_n, z)| \leq b^{(\operatorname{Re} z)-1} \frac{2(b-a)}{n} \left\| f' \right\|_1. \tag{18}$$

Proof If we apply Theorem 8 with equidistant partition of $[a, b]$, $t_j = a + j \cdot \frac{b-a}{n}$, $j = 0, 1, \ldots, n$, we have (16) and $h_k = \frac{b-a}{n}$, $k = 0, 1, \ldots, n-1$. For $1 < p \leq \infty$ we obtain

$$|R(f, I_n, z)| \leq \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1} \right]^{\frac{1}{q}} \left\| f' \right\|_p = \left(\frac{2^q + 1}{q+1} \right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\| f' \right\|_p,$$

while for $p = 1$, $\nu(h) = \frac{b-a}{n}$ and the claim immediately follows.

Now, define the sum

$$\mathscr{A}\left(f, I_n, z\right) = \sum_{k=0}^{n-1} \left(t_{k+1} - t_k\right) \frac{E\left(z \ln t_k, z \ln t_{k+1}\right)}{L\left(t_k, t_{k+1}\right)} f\left(\frac{t_{k+1} + t_k}{2}\right) \tag{19}$$

where $\operatorname{Re} z \geq 1$.

Also the following approximation theorem holds.

Theorem 9 *Assume (p, q) is a pair of conjugate exponents. Let $f : [a, b] \to \mathbb{R}$ be absolutely continuous function on $[a, b]$, $[a, b] \subset \langle 0, \infty\rangle$, $f' \in L_p[a, b]$. Then we have the quadrature rule*

$$\mathscr{M}(f)(z) = \mathscr{A}\left(f, I_n, z\right) + R\left(f, I_n, z\right)$$

where $\operatorname{Re} z \geq 1$, $\mathscr{A}\left(f, I_n, z\right)$ is given by (19) and for $1 < p \leq \infty$ the reminder $R\left(f, I_n, z\right)$ satisfies the estimate

$$\left|R\left(f, I_n, z\right)\right| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p, \tag{20}$$

while for $p = 1$

$$\left|R\left(f, I_n, z\right)\right| \leq 2b^{(\operatorname{Re} z)-1} \nu(h) \left\|f'\right\|_1. \tag{21}$$

Proof By applying the Corollary 6 with $a = t_k$, $b = t_{k+1}$, $c = \frac{t_{k+1}+t_k}{2}$ and then summing over k from 0 to $n - 1$, we obtain results similarly as in the proof of the Theorem 8.

Corollary 8 *Suppose that all assumptions of Theorem 9 hold. Additionally suppose*

$$\mathscr{A}\left(f, I_n, z\right) = \sum_{k=0}^{n-1} \frac{b-a}{n} f\left(a + \frac{k(k+1)(b-a)}{2n}\right)$$
$$\cdot \frac{E\left(z \ln\left(a + k \cdot \frac{b-a}{n}\right), z \ln\left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}{L\left(\left(a + k \cdot \frac{b-a}{n}\right), \left(a + (k+1) \cdot \frac{b-a}{n}\right)\right)}.$$

Then we have the quadrature rule

$$\mathscr{M}(f)(z) = \mathscr{A}\left(f, I_n, z\right) + R\left(f, I_n, z\right)$$

where $\operatorname{Re} z \geq 1$ and for $1 < p \leq \infty$ the reminder $R\left(f, I_n, z\right)$ satisfies the estimate

$$\left|R\left(f, I_n, z\right)\right| \leq b^{(\operatorname{Re} z)-1} \left(\frac{2^q + 1}{q + 1}\right)^{\frac{1}{q}} \frac{(b-a)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p, \tag{22}$$

while for $p = 1$ we have

$$\left|R\left(g, I_n, z\right)\right| \leq b^{(\operatorname{Re} z)-1} \frac{2\left(b-a\right)}{n} \left\|f'\right\|_1 . \tag{23}$$

Proof By applying Theorem 9 with equidistant partition of $[a, b]$.

Remark 9 Both numerical quadrature formulae hold also in case $a = 0$ with the term $\frac{t_1^{z-1}}{z}$ instead of $\frac{E(z \ln a, z \ln t_1)}{L(a, t_1)}$ in the first approximation sum (13) and the second approximation sum (19).

Remark 10 For both numerical quadrature formulae in case $\operatorname{Re} z < 1$, for $1 < p \leq \infty$, the reminder $R\left(f, I_n, z\right)$ satisfies the estimate

$$\left|R\left(f, I_n, z\right)\right| \leq a^{(\operatorname{Re} z)-1} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \left[\sum_{k=0}^{n-1} h_k^{q+1}\right]^{\frac{1}{q}} \left\|f'\right\|_p ,$$

while for $p = 1$

$$\left|R\left(f, I_n, z\right)\right| \leq 2a^{(\operatorname{Re} z)-1} \nu\left(h\right) \left\|f'\right\|_1 .$$

For equidistant partition of $[a, b]$ and for $1 < p \leq \infty$ we have

$$\left|R\left(f, I_n, z\right)\right| \leq a^{(\operatorname{Re} z)-1} \left(\frac{2^q+1}{q+1}\right)^{\frac{1}{q}} \frac{\left(b-a\right)^{1+\frac{1}{q}}}{n} \left\|f'\right\|_p ,$$

while for $p = 1$

$$\left|R\left(f, I_n, z\right)\right| \leq a^{(\operatorname{Re} z)-1} \frac{2\left(b-a\right)}{n} \left\|f'\right\|_1 .$$

Remark 11 It is easy to see that in all these numerical rules estimate tends to zero as n tends to infinity.

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Convexity Revisited: Methods, Results, and Applications

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Abstract We present some new aspects involving strong convexity, the pointwise and uniform convergence on compact sets of sequences of convex functions, circular symmetric inequalities and bistochastic matrices with examples and applications, the convexity properties of the multivariate monomial, and Schur convexity.

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1 Introduction

A function $f : D \to \mathbb{R}$ defined on a nonempty subset D of a real linear space E is called convex, if the domain D of the function is convex and for every $x, y \in D$ and every $t \in [0, 1]$ one has

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

If the above inequality is strict whenever $x \neq y$ and $0 < t < 1$, f is called strictly convex. A function f such that $-f$ is convex is called concave.

The simplest example of a convex function is an affine function $f(x) = a^T x + b$. This function clearly is convex on the entire space $\mathbb{R}^n$, and the convexity inequality for it is equality. The affine function is also concave. One can easily prove that the function which is both convex and concave on the entire space is an affine function. Other examples of convex functions are given by the norms on the space $\mathbb{R}^n$, i.e. the real-valued functions which are nonnegative everywhere, positive outside of

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the origin, homogeneous and satisfy the triangle inequality. The most important examples of norms are the so-called l_p-norms, $1 \le p \le \infty$, defined by

$$\|x\|_p = \left(\sum_{k=1}^{n} |x_k|^p \right)^{1/p}.$$

Three members of the above family of norms are very well-known:
the Euclidean norm

$$\|x\| = \|x\|_2 = \sqrt{x^T x} = \sqrt{\left(\sum_{k=1}^{n} |x_k|^2 \right)},$$

the l_1-norm or the Cartesian norm

$$\|x\|_1 = \sum_{k=1}^{n} |x_k|,$$

and the l_∞-norm or the Tchebychev norm

$$\|x\|_\infty = \max_{1 \le k \le n} |x_k|.$$

Convex functions and their generalizations have been used in a variety of fields such as economics business administration, engineering, statistics, applied sciences, and numerical mathematics. Some new characterizations of convex functions in various contexts are presented in the paper [31]. There are generalized concepts of convex functions, such as: quasi-convex, midpoint-convex, strong convex, logarithmically convex, Schur convex, etc. Also many classes of convex functions are studied in the complex plane in connection with some geometric properties. There is a huge literature devoted to the study of convex functions in various contexts with numerous applications. We mention here only the monographs [35, 42], and [52].

The present chapter is organized into five sections. In Sect. 2 we introduce a new class of functions called (h_1, h_2)-convex functions. This class contains the class of strong convex functions and the class of strong concave functions. The new class of functions is used for improving some algebraic and geometric inequalities. Also, we have included the results of [6] showing the proofs of four fundamental results, two on convex functions and two in approximation theory. Strong-convexity and strong-concavity with respect to a function with applications to Korovkin type results are presented in Sects. 2.4–2.6. Section 3 contains some generalizations of a result given in [26] about the pointwise and uniform convergence on compact sets of sequences of convex functions. Some new results on circular symmetric inequalities and bistochastic matrices are given in Sect. 4. The main result is contained in Theorem 4.2 and examples and applications are presented in Sect. 4.2. The convexity properties of the multivariate monomial are studied in Sect. 5. If $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is a vector in $\mathbb{R}^n$, we denote by f_a the multivariate monomial with exponents equal to the entries of vector **a**. We determine conditions that should be satisfied by the parameter $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that the multivariate monomial fa is a convex, concave, logarithmically convex, logarithmically concave, quasi-convex, quasi-concave, subadditive, or super-additive function. Conditions for convexity of $f_{\mathbf{a}}$ may be found in Crouzeix [18]. The proof given in the third subsection is different from the Crouzeix's proof from [18]. The convexity and concavity necessary and sufficient conditions for the multivariate monomial may be stated simply as follows. The multivariate monomial is convex if and only if all the exponents are negative or one exponent is positive, the rest of exponents are negative and the sum of all exponents is greater or equal than one. The multivariate monomial is concave if and only if all the exponents are positive and the sum of all exponents is smaller or equal than one. Section 6 is devoted to the study of the class of n-Schur functions. A study of the n-Schur functions for values of n greater than 3 is made in the second subsection while a detailed study of the 3-Schur functions is given in the third subsection. In the fourth subsection is studied the class of the 5-Schur functions. In the fifth subsection we introduce two general classes of functions, that are connected with the class of the n-Schur functions.

2 (h_1, h_2)-Convex Functions and Some Applications

Strong convexity is one of the most important concepts in optimization, especially for guaranteeing a linear convergence rate of many gradient descent type algorithms. In this section we shall present some useful results on strong convexity with applications to the improvement of some algebraic and geometrical inequalities. Also, we discuss some connections with the approximation theory by linear and positive operators. Let us first begin with a definition. A differentiable function $f : D \subset \mathbb{R}^n \to \mathbb{R}$, where D is convex with nonempty interior, is *strongly convex* with respect to the real number $\alpha > 0$ if

$$f(y) \geq f(x) + \nabla f(x)^T (y - x) + \frac{\alpha}{2}\|y - x\|^2 \tag{2.1}$$

for all $x, y \in D$. Here $\|\cdot\|$ denotes the standard Euclidean norm in $\mathbb{R}^n$. Strong convexity condition does not necessarily require the function to be differentiable, and the gradient is replaced by the sub-gradient when the function is non-smooth.

The following property is very useful in applications: The differentiable function f is strongly-convex with constant $\alpha > 0$ if and only if the function

$$g(x) = f(x) - \frac{\alpha}{2}\|x\|^2$$

is convex. It follows from the first-order condition for convexity of g, i.e., it is convex if and only if

$$g(y) \geq g(x) + \nabla g(x)^T (y - x) \text{ for all } x, y \in D.$$

These remarks motivate us to introduce a new class of functions in a general context. Let E be a real linear space, D be a convex subset of D and let $h_1 : D \to \mathbb{R} \cup \{-\infty\}$, $h_2 : D \to \mathbb{R} \cup \{+\infty\}$ be two functions with the property that $h_2 - h_1$is convex. We say that the function $f : D \to \mathbb{R}$ is (h_1, h_2)-*convex* if the functions $f - h_1$ and $h_2 - f$ are both convex.

A special case which is of considerable interest is the following. Let E be an inner product space, D be a convex subset of E, and -$\infty \le m < M \le +\infty$. A map $f : D \to \mathbb{R}$ belongs to the set $\mathscr{C}(m, M)$ if the functions

$$g(x) = f(x) - \frac{m}{2} \|x\|^2, \quad x \in D$$

and

$$h(x) = \frac{M}{2} \|x\|^2 - f(x), \quad x \in D$$

are both convex. If $m > 0$ and $M = +\infty$, then functions from $\mathscr{C}(m, M)$ are called strong convex. If $m = -\infty$ and $M < 0$, then functions from $\mathscr{C}(m, M)$ are called strong concave. If f is two times differentiable, D is open and convex, then f belongs to $\mathscr{C}(m, M)$ if and only if

$$m \|u\|^2 \le f''(x)(u, u) \le M \|u\|^2, \quad x \in D, \ u \in E$$

2.1 Improving Some Classical Inequalities

In this subsection we use the property that every strictly convex function in $C^2[a, b]$ is strongly convex and strongly concave on the interval $[a, b]$ with respect to some constants. We use this property to improve some classical algebraic and geometric inequalities.

Let $[a, b]$ be a fixed interval of the real line. For $x_1, \ldots, x_n \in [a, b]$ and $p_1, \ldots, p_n \ge 0$ with $p_1 + \ldots + p_n = 1$, let us denote

$$x = (x_1, \ldots, x_n), \quad p = (p_1, \ldots, p_n), \quad S_1(x, p) = 0 \tag{2.2}$$

and

$$S_n(x, p) = \sum_{\substack{i,j=1 \\ i<j}}^{n} p_i p_j (x_i - x_j)^2 \text{ for } n \ge 2. \tag{2.3}$$

For a function $f \in C^2[a, b]$ we consider the numbers

$$m_2(f) = \min_{t \in [a,b]} f''(t), \quad M_2(f) = \max_{t \in [a,b]} f''(t) \tag{2.4}$$

The following result is a refinement of the Jensen's inequality and it was proved in [11] and used to improve some algebraic inequalities. A nice application to the Shannon and Rényi's entropy is given in the paper [22].

Theorem 2.1 *If* $f \in C^2[a, b]$ *and* $x_1, \ldots, x_n \in [a, b]$, $p_1, \ldots, p_n \geq 0$, $p_1 + \ldots + p_n = 1$, *then the following inequalities hold:*

$$\frac{1}{2}m_2(f)S_n(x, p) \leq \sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{2}M_2(f)S_n(x, p) \qquad (2.5)$$

Proof Consider the mapping

$$g : [a, b] \to \mathbb{R}, g(x) = f(x) - \frac{1}{2}m_2(f)x^2.$$

Then g is twice differentiable on (a, b) and

$$g'(x) = f'(x) - m_2(f)x, x \in (a, b),$$
$$g''(x) = f''(x) - m_2(f), x \in (a, b),$$

which shows that the mapping g is convex on $[a, b]$.

Applying Jensen's discrete inequality for the convex mapping g, i.e.,

$$g\left(\sum_{i=1}^{p} p_i x_i\right) \leq \sum_{i=1}^{n} p_i g(x_i),$$

to obtain

$$f\left(\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{2}m_2(f)\left(\sum_{i=1}^{n} p_i x_i\right)^2 \leq \sum_{i=1}^{n} p_i \left[f(x_i) - \frac{1}{2}m_2(f)x_i^2\right]$$
$$= \sum_{i=1}^{n} p_i f(x_i) - \frac{1}{2}m_2(f)\sum_{i=1}^{n} p_i x_i^2,$$

which is equivalent to

$$\sum_{i=1}^{n} p_i f(x_i) - f\left(\sum_{i=1}^{n} p_i x_i\right) \geq \frac{1}{2}m_2(f)\left[\sum_{i=1}^{n} p_i x_i^2 - \left(\sum_{i=1}^{n} p_i x_i\right)^2\right]$$
$$= \frac{1}{4}m_2(f)\sum_{i,j=1}^{n} p_i p_j (x_i - x_j)^2$$

and the first inequality in (4) is proved.

The proof of the second inequality goes likewise for the mapping

$$h : [a, b] \to \mathbb{R}, h(x) = \frac{1}{2} M_2(f) x^2 - f(x)$$

which is convex on $[a, b]$. We omit the details. □

Note that in the proof of the above result we have used the property that every function $f \in C^2[a, b]$ is (h_1, h_2)-convex, where $h_1(x) = \frac{1}{2} m_2(f) x^2$ and $h_2(x) = \frac{1}{2} M_2(f) x^2$, where the coefficients $m_2(f)$ and $M_2(f)$ are defined in (2.4).

Now, consider the classical means:

1. *The weighted arithmetic mean* $A_n(w, a)$

$$A_n(w, a) := \frac{1}{W_n} \sum_{i=1}^{n} w_i a_i, \text{ where } W_n = \sum_{i=1}^{n} w_i.$$

2. *The weighted geometric mean* $G_n(w, a)$

$$G_n(w, a) := \left(\prod_{i=1}^{n} a_i^{w_i} \right)^{\frac{1}{W_n}}.$$

3. *The weighted harmonic mean* $H_n(w, a)$

$$H_n(w, a) = \frac{W_n}{\sum_{i=1}^{n} \frac{w_i}{a_i}}, \text{ where } a_i, w_i > 0 \ (i = 1, \ldots, n).$$

The following inequality is well known in the literature as the *arithmetic mean–geometric mean–harmonic mean* inequality

$$A_n(w, a) \geq G_n(w, a) \geq H_n(w, a). \tag{2.6}$$

The equality holds in (2.6) if and only if $a_1 = \ldots = a_n$.

In the following corollary we shall use strong convexity to find estimations of the ratios between arithmetic and geometric means and between geometric and harmonic means.

Corollary 2.1 *Let* $a_i, w_i > 0$ $(i = 1, \ldots, n)$. *If* $0 < m \leq a_i \leq M < \infty$ $(i = 1, \ldots, n)$, *then we have the inequalities:*

$$1 \leq \exp \left[\frac{1}{4M^2} \cdot \frac{1}{W_n^2} \sum_{i,j=1}^{n} w_i w_j (a_i - a_j)^2 \right]$$

$$\leq \frac{A_n(w,a)}{G_n(w,a)}$$

$$\leq \exp\left[\frac{1}{4m^2} \cdot \frac{1}{W_n^2} \sum_{i,j=1}^{n} w_i w_j (a_i - a_j)^2\right] \tag{2.7}$$

and

$$1 \leq \exp\left[\frac{1}{4}m^2 \cdot \frac{1}{W_n^2} \sum_{i,j=1}^{n} \frac{w_i w_j (a_i - a_j)^2}{a_i^2 a_j^2}\right]$$

$$\leq \frac{G_n(w,a)}{H_n(w,a)}$$

$$\leq \exp\left[\frac{1}{4}M^2 \cdot \frac{1}{W_n^2} \sum_{i,j=1}^{n} \frac{w_i w_j (a_i - a_j)^2}{a_i^2 a_j^2}\right]. \tag{2.8}$$

Equality holds in both (2.7) and (2.8) if and only if $a_1 = \ldots = a_n$.

Proof The proof follows by Theorem 2.1, choosing $f(x) = -\ln x$. For this mapping we have

$$f''(x) = \frac{1}{x^2} \in \left[\frac{1}{M^2}, \frac{1}{m^2}\right],$$

and if we assume that $p_i = \frac{w_i}{W_n}$, $x_i = a_i$, then, by Theorem 2.1, we deduce (2.7).

The inequality (2.8) follows by (2.7) applied for $\frac{1}{a_i}$ instead of a_i $(i = 1, \ldots, n)$. We omit the details. □

2.2 Improving Some Geometric Inequalities

In what follows we will use the inequalities (2.5) to improve the following geometric inequalities [1, 3]:

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq \frac{1}{8} \tag{2.9}$$

$$R \geq 2r \tag{2.10}$$

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}, \tag{2.11}$$

where A, B, C are the angles of a triangle ABC, R is the circumradius, and r is the inradius.

Considering the function $f : (0, \pi) \to \mathbb{R}$, $f(x) = \ln \sin x$, after a simple computation we have

$$M_2(f) = -1.$$

Applying the right-hand side inequality in (2.5) with

$$x_1 = \frac{A}{2},\ x_2 = \frac{B}{2},\ x_3 = \frac{C}{2},\ p_1 = p_2 = p_3 = \frac{1}{3}$$

one obtains

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \le \frac{1}{8}\exp\{-\frac{1}{24}[(A-B)^2 + (B-C)^2 + (C-A)^2]\}, \tag{2.12}$$

inequality which improves the inequality (2.9). Using the well-known relation

$$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = \frac{r}{4R},$$

from (2.12) we derive the inequality

$$R \ge 2r\exp\{\frac{1}{24}[(A-B)^2 + (B-C)^2 + (C-A)^2]\}, \tag{2.13}$$

which is a refinement of the Euler's inequality (2.10).

For the function $f : [0, \pi] \to \mathbb{R}$, $f(x) = \sin x$, we have

$$m_2(f) = -1, \quad M_2(f) = 0.$$

Using the inequalities (2.5) with $x_1 = A$, $x_2 = B$, $x_3 = C$, $p_1 = p_2 = p_3 = \frac{1}{3}$, we obtain the inequalities

$$\frac{3\sqrt{3}}{2} - \frac{1}{6}[(A-B)^2 + (B-C)^2 + (C-A)^2] \le \sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}. \tag{2.14}$$

The left side in (2.14) is a complementary inequality for (2.11).

Applying Theorem 2.1 for the function $f : \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$, $f(x) = \cos x$, with $x_1, x_2, x_3, p_1, p_2, p_3$ as in (2.14), and using the well-known relation

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R},$$

we obtain that for every acute angled triangle ABC the following inequality holds:

$$\frac{r}{R} \geq \frac{1}{2} - \frac{1}{6}[(A-B)^2 + (B-C)^2 + (C-A)^2]. \tag{2.15}$$

The inequality (2.15) is a complementary inequality to Euler's (2.10) written in the equivalent form $\frac{1}{2} \geq \frac{r}{R}$. Other improvements to Euler's inequality (2.10) are given in [9].

2.3 *Jensen's and Jessen's Inequality and Convexity Preserving in Approximation Theory*

It is well-known that a continuous real-valued function defined on a compact interval $[a, b]$ of the real axis can be uniformly approximated by polynomials. Many constructive examples of approximating sequences of polynomial operators are provided by the famous test function theorem of Korovkin which is a very important and now classic result in Approximation Theory.

An interesting approach for obtaining approximation results is based on convexity (see Popoviciu [47]). For instance, the behavior of the classical Bernstein polynomials on the class of convex functions was considered by many authors (see the bibliography compiled by Stark [54] and Gonska and Meier [25]). A result in this direction is the fact, observed first by Popoviciu [46], that Bernstein operator is a convexity-preserving operator, i.e. $B_n(f, \cdot)$ of a convex function $f \in C[a, b]$ is also a convex function. This property and the apparently unrelated well-known fact that Bernstein polynomials are approximating operators, i.e. for every $f \in C[a, b]$ we have $B_n(f; x) \to f(x)$ uniformly as $n \to \infty$, make the proof of Theorem 1 of Bojanic and Roulier [14] very simple. Thus it appears that the existence of a sequence of approximating and convexity-preserving positive linear polynomial operators which reproduces the affine functions (as Bernstein operators do) has a certain importance. The existence of a such sequence may be of interest in statistics. Indeed, if we consider the interpolation operators

$$L_n(f; x) = \sum_{k=0}^{m(n)} w_{nk}(x) f(x_{nk})$$

where $m(n)$ may be finite or not, and moreover for $k = 0, 1, \ldots,$ and $n = 1, 2, \ldots,$ we have

(a) $0 \leq w_{nk}(x) \leq 1, x \in [a, b];$

(b) $\displaystyle\sum_{k=0}^{m(n)} w_{nk}(x) = 1,$

then $L_n(f; x)$ is the mathematical expectation of a certain discrete univariate variable and the fact that mathematical expectation preserves the convexity of the data will probably be of importance to the statisticians. For example, the convexity-preserving property of Bernstein polynomials has been used in statistics by Wegmüller [58]. Keeping in mind that the class of convex functions is characterized by the well-known inequality of Jensen, the following question arises in a natural way:

What are the implications between the Jensen's inequality in $C[a, b]$, the existence of a sequence of approximating and convexity-preserving positive linear polynomial operators which reproduce the affine functions, and Korovkin's theorem?

The aim of this subsection is to show that the three above-mentioned basic results and a certain generalization of Jensen's inequality due to Jessen [27] considered in $C[a, b]$ are connected. This property emphasizes the role of convexity and of convexity-preserving operators in the theory of approximation by positive linear operators.

2.4 Statement of the Results

Let $C[a, b]$ be the linear space of all real-valued and continuous functions defined on $[a, b]$ and the functions $e_n \in C[a, b]$ given by

$$e_n(x) = x^n, \ n = 0, 1, 2, \ldots$$

We denote by B_n the n-th classical Bernstein operator on $[a, b]$, i.e.,

$$B_n(f; x) = \frac{1}{(b-a)^n} \sum_{k=0}^{n} \binom{n}{k} (x-a)^k (b-x)^{n-k} f\left(a + k\frac{b-a}{n}\right). \tag{2.16}$$

If $\|\cdot\|$ is the supremum norm in $C[a, b]$, it is well-known that

$$\lim_{n\to\infty} \|B_n f - f\| = 0,$$

for every function $f \in C[a, b]$, i.e. Bernstein polynomials are approximating operators, and B_n is a convexity-preserving operator (see, for instance, Popoviciu [47, pp. 126]).

Korovkin's theorem states that if $\{L_n\}$, $n = 1, 2, \ldots$ is a sequence of positive linear operators on $C[a, b]$ such that

$$\lim_{n\to\infty} \|L_n e_i - e_i\| = 0 \text{ for } i = 0, 1, 2,$$

then

$$\lim_{n\to\infty} \|L_n f - f\| = 0 \text{ for every } f \in C[a, b].$$

If the function $f \in C[a, b]$ is convex on $[a, b]$, then for $x_k \in [a, b]$, $p_k \geq 0$, $k = 1, 2, \ldots, m$, with $\sum_{k=1}^{m} p_k = 1$ we have

$$f\left(\sum_{k=1}^{m} p_k x_k\right) \leq \sum_{k=1}^{m} p_k f(x_k). \tag{2.17}$$

The inequality (2.17) is the well-known Jensen's inequality in $C[a, b]$ (see also Sect. 2.2). A generalization of Jensen's inequality involving isotonic linear functionals is due to Jessen [27]. A short proof of this generalization and other related result may be found in the paper of Beesack and Pećarić [13].

In fact, in this subsection we shall consider only the following quite particular form of Jessen's inequality: *If $A : C[a, b] \to \mathbb{R}$ is a linear, positive (and thus isotonic) functional with $A(e_0) = 1$, then for every convex function $f \in C[a, b]$*

$$f(A(e_1)) \leq A(f). \tag{2.18}$$

It is worth mentioning that this inequality and other similar inequalities appear in the works of Slater [53], Pećarić [44], Pećarić and Andrica [45]. In what follows by *Jessen's inequality* we shall understand the inequality (2.18).

Recall that the function $f : [a, b] \to \mathbb{R}$ is *midpoint convex* if

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{x_1 + x_2}{2},$$

for every $x_1, x_2 \in [a, b]$. The following result shows that midpoint convexity, convexity, and Jessen's convexity are equivalent for the functions in $C[a, b]$.

Lemma 2.1 *Let $f \in C[a, b]$. The following properties are equivalent:*

1^0. *f is midpoint convex;*
2^0. *f is convex;*
3^0. *f is Jessen convex, i.e. it satisfies the inequality (2.18) for every linear positive functional $A : C\,[a, b] \to \mathbb{R}$ with $A\,(e_0) = 1$.*

Proof The equivalence of 1^0 and 2^0 is a well-known property (see the monograph [43, Theorem 1.1.4]).

In order to prove the implication $2^0 \Rightarrow 3^0$ suppose that f is convex. It is well-known that $f'_+(y)$ exists for every $y \in (a, b)$ and for every $x \in [a, b]$

$$f(x) \geq f(y) + f'_+(y)(x - y) \tag{2.19}$$

(see Roberts and Varberg [52, p. 12]). If we substitute $y = A(e_1)$ in the above inequality, we obtain:

$$f(x) \geq f(A(e_1)) + f_+'(A(e_1))(A(e_1) - x), \quad x \in [a, b]$$

Applying the linear and positive functional A to the preceding inequality we obtain inequality (2.18).

In order to prove the implication $3^0 \Rightarrow 2^0$, suppose that 2^0 holds. For $x_1, x_2 \in [a, b]$ define the functional

$$A(f) = \frac{f(x_1) + f(x_2)}{2}$$

Note that

$$A(e_1) = \frac{e_1(x_1) + e_1(x_2)}{2} = \frac{x_1 + x_2}{2}$$

By 3^0 it follows that inequality (2.18) holds hence

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{f(x_1) + f(x_2)}{2}$$

Since a continuous midpoint convex function is convex it follows that the function f is convex. □

The next result is an improvement of the Jessen's inequality (2.18) when the function f is (h_1, h_2)-convex.

Theorem 2.2 *Consider the functions $h_1, h_2 \in C[a, b]$ with the property that $h_2 - h_1$ is convex, and let $f \in C[a, b]$ be a (h_1, h_2)-convex function. Then for every linear positive functional $A : C[a, b] \to \mathbb{R}$ with $A(e_0) = 1$, the following inequalities hold:*

$$A(h_2) - h_2(A(e_1)) \geq A(f) - f(A(e_1)) \geq A(h_1) - h_1(A(e_1)).$$

Proof Applying Jessen's inequality for the convex function $f - h_1$ we get the right inequality. Similarly, applying Jessen's inequality for the convex function $h_2 - f$ we obtain the left inequality. □

The following result is the general version of the quadratic mean-arithmetic mean inequality.

Lemma 2.2 *Consider the linear positive functional $A : C[a, b] \to \mathbb{R}$ with $A(e_0) = 1$. Then, for every function $f \in C[a, b]$, the following inequality holds:*

$$A(f^2) \geq A^2(f).$$

Proof Because $(f - A(f))^2 \geq 0$, it follows $A((f - A(f))^2) \geq 0$, that is

$$A(f^2) - 2A(f)A(f) + A^2(f) \geq 0,$$

hence $A(f^2) \geq A^2(f)$. □

We have seen that every function $f \in C^2[a, b]$ is (h_1, h_2)-convex, with $h_1(x) = \frac{m_2(f)}{2}x^2$ and $h_2(x) = \frac{M_2(f)}{2}x^2$, where the coefficients $m_2(f)$ and $M_2(f)$ are defined in (2.4). From Theorem 2.2 we get the inequalities

$$\frac{M_2(f)}{2}(A(e_2) - A^2(e_1)) \geq A(f) - f(A(e_1)) \geq \frac{m_2(f)}{2}(A(e_2) - A^2(e_1)),$$

where by Lemma 2.2 we have $A(e_2) - A^2(e_1) \geq 0$. The result in Theorem 2.1 is a special case obtained for the discrete functional $A(f) = \sum_{i=1}^{n} p_i f(x_i)$. We mention that the quantity $D_A = A(e_2) - A^2(e_1)$ is the dispersion of the functional A.

The main results of this subsection are proved in the paper [6] and connect the following properties:

(i) there is a sequence of approximating and convex-preserving positive linear polynomial operators which reproduce the affine functions;
(ii) Korovkin's theorem in the space $C[a, b]$;
(iii) Jessen's inequality for positive linear functionals on $C[a, b]$.

Proof of (ii) Using (iii) Let $\{L_n\}$, $n = 1, 2, \ldots$, be a sequence of positive linear operators on $C[a, b]$ with

$$\lim_{n\to\infty} \|L_n e_i - e_i\| = 0,\ i = 0, 1, 2.$$

Because

$$\lim_{n\to\infty} \|L_n e_0 - e_0\| = 0$$

we can assume that $L_n(e_0; x) > 0$ and $L_n(e_0; x) < k$ for every $x \in [a, b]$ and all positive integers $n \geq n_0$.

For a fixed $x \in [a, b]$ we consider the functionals

$$A_n : C[a, b] \to \mathbb{R}, \quad A_n(f) = L_n(f; x)/L_n(e_0; x), n \geq n_0.$$

It is obvious that A_n is linear positive and we have $A_n(e_0) = 1$.
If $f \in C^2[a, b]$ let us denote by

$$m_2 = \min_{t\in[a,b]} f''(t), \quad M_2 = \max_{t\in[a,b]} f''(t).$$

Now we can apply (iii) for the above-defined functionals A_n and for the convex functions

$$f_1 = f - \frac{1}{2}m_2 e_2, \quad f_2 = \frac{1}{2}M_2 e_2 - f.$$

We get immediately

$$\begin{aligned} &\frac{1}{2} \cdot \frac{m_2}{L_n(e_0; x)}[L_n(e_0; x)L_n(e_2; x) - L_n^2(e_1; x)] \\ &\leq L_n(f; x) - L_n(e_0; x) \cdot f\left(\frac{L_n(e_1; x)}{L_n(e_0; x)}\right) \\ &\leq \frac{1}{2} \cdot \frac{M_2}{L_n(e_0; x)}[L_n(e_0; x)L_n(e_2; x) - L_n^2(e_1; x)] \end{aligned}$$

for every $x \in [a, b]$ and for every $n \geq n_0$. By Lemma 2.2 for $f = e_1$ we obtain $L_n(e_0; x)L_n(e_2; x) \geq L_n^2(e_1; x)$, for every $n \geq n_0$. Therefore, from the above inequalities, we get

$$\begin{aligned} &|L_n(f; x) - L_n(e_0; x)f(L_n(e_1; x)/L_n(e_0; x))| \\ &\leq \frac{\|f''\|}{2L_n(e_0; x)}[L_n(e_0; x)L_n(e_2; x) - L_n^2(e_1; x)]. \end{aligned} \tag{2.20}$$

Using the triangle inequality, we have

$$\begin{aligned} |L_n(f; x) - f(x)| \leq &\left| f(x) - L_n(e_0; x)f\left(\frac{L_n(e_1; x)}{L_n(e_0; x)}\right)\right| \\ &+ \left| L_n(f; x) - L_n(e_0; x)f\left(\frac{L_n(e_1; x)}{L_n(e_0; x)}\right)\right|, \end{aligned}$$

and from (2.20) we conclude that

$$\begin{aligned} |L_n(f; x) - f(x)| \leq &\left| f(x) - f\left(\frac{L_n(e_1; x)}{L_n(e_0; x)}\right)\right| + \|f\||L_n(e_0; x) - 1| \\ &+ \frac{\|f''\|}{2L_n(e_0; x)}[L_n(e_0; x)L_n(e_2; x) - L_n^2(e_1; x)]. \end{aligned} \tag{2.21}$$

Because f is continuous on $[a, b]$ it is also uniformly continuous on $[a, b]$ so

$$|f(x) - f(L_n(e_1; x)/L_n(e_0; x))| \to 0$$

uniformly as $n \to \infty$.

On the other hand, using the fact that $\{e_0, e_1, e_2\}$ is a set of test functions, we have

$$[L_n(e_0; x)L_n(e_2; x) - L_n^2(e_1; x)]/L_n(e_0; x) \to 0$$

uniformly as $n \to \infty$. From these remarks and (2.21) we deduce that

$$\lim_{n\to\infty} \|L_n f - f\| = 0 \text{ for every } f \in C^2[a, b].$$

For every $f \in C[a, b]$ we have

$$|L_n(f; x)| \le \|f\|L_n(e_0; x) \le k\|f\|$$

so $\|L_n f\| \le k\|f\|$ and we obtain $\|L_n\| \le k$ for every $n \ge n_0$. It follows

$$\|L_n(f) - f\| \le \|L_n(f - g) - (f - g)\| + \|L_n(g) - g\| \le$$

$$\|L_n\|\|f - g\| + \|L_n(g) - g\| \le k\|f - g\| + \|L_n(g) - g\|, n \ge n_0,$$

therefore

$$\limsup \|L_n(f) - f\| \le k\|f - g\|,$$

for every $g \in C^2[a, b]$. Because $C^2[a, b]$ is a dense subspace in $C[a, b]$, the proof is complete.

Remark 2.1

(1) The construction of the functions f_1 and f_2 has been used by Lupaş [32] in Approximation Theory and to obtain the improvements of some inequalities based upon convex functions by Raşa [51], Andrica and Raşa [11] (see also the above Sects. 2.1 and 2.2). The same idea appeared in Andrica et al. [12].
(2) Using the same method of proof one may find a more general class than the class of approximating positive linear interpolation operators. This may be done by using Jensen–Steffensen' inequality (see Mitrinović [40, p. 109]) instead of Jensen's inequality which is equivalent to Jessen' s inequality by Lemma 2.1.

Proof of (i) Using (ii) Because $B_n e_k = e_k, k = 0, 1,$ and $B_n(e_2) = e_2 + \dfrac{e_1 - e_2}{n}$, we get by Korovkin's theorem that Bernstein operators are approximating operators. In the same time B_n is a convexity-preserving polynomial operator (see, for instance, Popoviciu [47, p. 126]) which reproduces the affine functions.

Proof of (iii) Using (i) Let $\{L_n\}$, $n = 1, 2, \ldots$, be a sequence of approximating and convexity-preserving positive linear polynomial operators which preserves the affine functions. The existence of a such sequence is guaranteed by (i).

Let f be a convex function of $C^2[a, b]$. Using the Taylor's formula we get

$$f(x) \ge f(t) + (x - t)f'(t)$$

for every $x, t \in [a, b]$. Applying the functional A with respect to x, for $t = A(e_1)$, it follows

$$A(f) \geq f(A(e_1)) \tag{2.22}$$

for every convex function $f \in C^2[a, b]$. For these functions we have similarly $L_n(f; x) \geq f(x)$. However, this inequality holds for an arbitrary convex function $f \in C[a, b]$. Indeed, if f is convex on $[a, b]$, then $L_m f \in C^2[a, b]$ is also a convex function and thus $L_n(L_m f; x) \geq f(x)$. Letting m tends to infinity we get that $L_n(f; x) \geq f(x)$, for every convex function $f \in [a, b]$.

Finally we complete the proof of Jessen's inequality (2.22) by using an idea of Andrica [2]. Let $f \in C[a, b]$ be a convex function. Using the last inequality and the fact that the operators L_n are approximating we find that for every $\varepsilon > 0$ there is a positive integer $N = N(\varepsilon)$ such that for all $n \geq N$ we have

$$0 \leq L_n(f; x) - f(x) < \varepsilon, \ x \in [a, b].$$

Thus

$$A(L_n f) \leq A(f) + \varepsilon, \ n = 1, 2, \ldots \tag{2.23}$$

Because $L_n f \in C^2[a, b]$ is also a convex function we find, using (2.22), that

$$L_n(A(e_1)) \leq A(L_n f), \ n = 1, 2, \ldots \tag{2.24}$$

Hence from (2.23) and (2.24) we have

$$L_n(A(e_1)) \leq A(f) + \varepsilon, \ n = 1, 2, \ldots \tag{2.25}$$

and because the operators L_n are approximating we get

$$f(A(e_1)) \leq A(f) + \varepsilon$$

for every $\varepsilon > 0$. Consequently, Jessen's inequality is proved.

Final Remark

From the above proof we see that we may replace (i) by the weaker assertion:

(i') *there is a sequence of approximating and convex-preserving positive linear operators* $\{L_n\}$ *which reproduce the affine functions and which verify the condition* $L_n f \in C^2[a, b]$, $n = 1, 2, \ldots$, *for every convex function* f.

Also, we may extend the result of the theorem to an infinite interval by replacing Bernstein operators with those of Szász-Mirakjan.

2.5 Strong-Convexity and Strong-Concavity with Respect to a Function

Let X be a real normed space endowed with the norm $\|\cdot\|$ and let X^* be the algebraic topological dual of X. Consider $D \subseteq X$ a convex set, and the continuous function $h : D \times D \to \mathbb{R}_+$ satisfying for every $x \in D$ the condition $h(x, x) = 0$.

Definition 2.1 The real function $f : D \to \mathbb{R}$ is called *h-strong convex* if there exist a function $g_f^{(1)} : D \to X^*$ and a real number $m > 0$, such that for every $x, y \in D$ the following inequality is verified:

$$f(x) \geq f(y) + g_f^{(1)}(y)(x - y) + mh(x, y). \tag{2.26}$$

Definition 2.2 The function $f : D \to D$ is *h-strong concave* if there exists a function $g_f^{(2)} : D \to X^*$ and a real number $M < 0$, such that for every $x, y \in D$ the following relation is true:

$$f(x) \leq f(y) + g_f^{(2)}(y)(x - y) + Mh(x, y). \tag{2.27}$$

Some particular situations of h-strong convexity for real functions defined on an interval $[a, b]$ are studied in the papers [11] and [8]. Other results concerning the refinement of some inequalities related to classical convex functions are obtained in [7] and [8].

Remark 2.2

(1) Recall that the subdifferential of f at the point $y \in D$ is defined by

$$\partial f(y) = \{a^* \in X^* : \; f(x) \geq f(y) + a^*(y)(x - y) \text{ for every } x \in D\}. \tag{2.28}$$

If $\partial f(y) \neq \emptyset$ for every $y \in D$, then (2.26) is satisfied by considering

$$g_f^{(1)}(y) = a^*(y) \in X^*.$$

Thus every function $f : D \to \mathbb{R}$ with $\partial f(y) \neq \emptyset$, $y \in D$ is a 0-strong convex function (see [45] for more details).

(2) Let $f : D \to \mathbb{R}$ be a Lipschitz function, i.e. f satisfies for every $x, y \in D$ the inequality

$$|f(x) - f(y)| \leq k\|x - y\|.$$

That is, the following relations are verified:

$$-k\|x-y\|+f(y)\leq f(x)\leq f(y)+k\|x-y\|. \tag{2.29}$$

It follows that f is h-strong convex and concave function, where

$$h(x,y)=\|x-y\|,$$

$$g_f^{(1)}(y)=g_f^{(2)}(y)=0\in X^*,\quad m=-k,\quad M=k,$$

the Lipschitz constant.

Denote by $C(D)=C(D,\mathbb{R})$ the real vector space of all continuous functions defined on D. Let us remark that if the function $f:D\to\mathbb{R}$ is h-strong convex and h-strong concave on D, then it is continuous on D. This assertion follows from the inequalities

$$mh(x,y)+g_f^{(1)}(y)(x-y)\leq f(x)-f(y)\leq g_f^{(2)}(y)(x-y)+Mh(x,y) \tag{2.30}$$

by using the continuity of the functionals $g_f^{(1)}(y), g_f^{(2)}(y)\in X^*$, the continuity of h and the hypothesis $h(x,x)=0$.

Let us denote by $C_h(D)$ the set of all h-strong convex and concave functions on D. Taking into account the above remark one obtains $C_h(D)\subset C(D)$ as a linear subspace of $C(D)$.

Consider X a pre-Hilbert space endowed with the inner product $\langle\cdot,\cdot\rangle$. If $D\subset X$ is a convex, compact subset of X having nonempty interior, let us denote by $C^2(D)$ the space of all C^2-differentiable real functions $f:D\to\mathbb{R}$. It is clear that $C^2(D)\subset C(D)$.

Theorem 2.3 *With the above notations, the following inclusion holds:*

$$C^2(D)\subset C_h(D), \tag{2.31}$$

where $h:D\times D\to\mathbb{R}$ is given by

$$h(x,y)=\|x-y\|^2.$$

That is, every C^2-differentiable function on D is h-strong convex and h-strong concave on D.

Proof From the well-known Taylor formula one obtains:

$$f(x)=f(y)+(df)(x-y)+\frac{1}{2}(d^2f)_\xi(x-y,x-y) \tag{2.32}$$

where

$$\xi = \xi_{x,y} \in [x, y] = \{tx + (1-t)y : t \in [0, 1]\} \subset D.$$

Let us denote

$$k_f^{(2)} = \max\{\|(d^2 f)_z\| : z \in D\},$$

where $d^2 f : D \to L(X, L(X; \mathbb{R})) \simeq L^2(X, X; \mathbb{R})$ represents the second differential of f. Using the Schwarz inequality it follows:

$$\begin{aligned}
|(d^2 f)(x-y, x-y)| &= |\langle (d^2 f)_\xi (x-y), x-y\rangle| \\
&\le \|(d^2 f)_\xi (x-y)\| \cdot \|x-y\| \\
&\le \|(d^2 f)_\xi\| \cdot \|x-y\|^2 \\
&\le k_f^{(2)} \cdot \|x-y\|^2.
\end{aligned}$$

From (2.32) one obtains

$$\begin{aligned}
-\frac{1}{2} k_f^{(2)} \|x-y\|^2 + (df)_y(x-y) + f(y) &\le f(x) \\
&\le f(y) + (df)_y(x-y) + \frac{1}{2} k_f^{(2)} \|x-y\|^2.
\end{aligned}$$

Therefore the inequalities (2.30) are verified with

$$g_f^{(1)} = g_f^{(2)} = df, \quad m = -\frac{1}{2} k_f^{(2)}, \quad M = \frac{1}{2} k_f^{(2)}.$$

□

2.6 Markov Operators

Consider X a real normed space and let $D \subset X$ be a convex and compact subset of X. Let $C(D)$ be the Banach space of all continuous real functions on D, endowed with the maximum norm.

Definition 2.3 A *Markov operator* on $C(D)$ is a positive linear operator $L : C(D) \to C(D)$ such that $L(e_0) = e_0$, where $e_0(x) = 1$, $x \in D$, i.e. L preserves the constant functions.

It is known (see [10, Lemma 1]) that if L is a Markov operator acting on $C(D)$, then $\|L\| = 1$.

If $(L_n)_{n\geq 1}$ is a sequence of Markov operators acting on $C(D)$, let

$$\alpha_n(x) = L_n(h(\cdot, x); x) \tag{2.33}$$

for all $x \in D$ and $n = 1, 2, \ldots$, where $C_h(D)$ is the subspace of h-strong convex and h-strong concave functions on D.

For $f \in C_h(D)$ consider the functions $G^{(1)}_{f,y}, G^{(2)}_{f,y} : D \to \mathbb{R}$ given by

$$G^{(j)}_{f,y}(x) = g^{(j)}_f(y)(x), \ \ j = 1, 2,$$

where $g^{(1)}_f, g^{(2)}_f$ are the mappings satisfying the inequalities (2.30).

The main result in this section was proved in the paper [5] (see also [4]) and it is contained in the following Korovkin type theorem.

Theorem 2.4 *Let $(L_n)_{n\geq 1}$ be a sequence of Markov operators acting on $C(D)$. Suppose that the following conditions are satisfied:*

(i) $\alpha_n(x) \to 0$*, uniformly with respect to $x \in D$;*
(ii) $L_n(G^{(j)}_{f,y}) \to G^{(j)}_{f,y}$ *in the uniform norm of $C(D)$, for any $f \in C_h(D)$ and for every point $y \in D$;*
(iii) $C_h(D)$ *is a dense subspace of $C(D)$.*

Then $(L_n(f))_{n\geq 1}$ converges uniformly to f for all $f \in C(D)$.

Proof Let $f \in C_h(D)$. The inequalities (2.30) can be rewritten in the form:

$$mh(x, y) + g^{(1)}_f(y)(x) - g^{(1)}_f(y)(y) \leq f(x) - f(y)$$
$$\leq g^{(2)}_f(y)(x) - g^{(2)}_f(y)(y) + Mh(x, y).$$

Applying to these inequalities the operator L_n with respect to x and taking into account the positivity of L_n, one obtains:

$$mL_n(h(\cdot, y); x) + L_n(G^{(1)}_{f,y}; x) - g^{(1)}_f(y)(y) \leq L_n(f; x) - f(y)$$
$$\leq L_n(G^{(2)}_{f,y}; x) - g^{(2)}_f(y)(y) + ML_n(h(\cdot, y); x).$$

Consider $\beta^{(j)}_n(x, y) = L_n(G^{(j)}_{f,y}; x) - G^{(j)}_{f,y}(x)$, and from the above inequalities it follows:

$$mL_n(h(\cdot, y); x) + g^{(1)}_f(y)(x) - g^{(1)}_f(y)(y) + \beta^{(1)}_n(x, y) \leq L_n(f; x) - f(y)$$
$$\leq g^{(2)}_f(y)(x) - g^{(2)}_f(y)(y) + \beta^{(2)}_n(x, y) + ML_n(h(\cdot, y); x).$$

Considering $y = x$ one obtains:

$$m\alpha_n(x) + \beta_n^{(1)}(x, x) \le L_n(f, x) - f(x) \le \beta_n^{(2)}(x, x) + M\alpha_n(x). \tag{2.34}$$

From (2.34) it follows that $L_n(f; \cdot) \to f$, uniformly for every $f \in C_h(D)$. Using the hypothesis (iii) and the mentioned fact that

$$\|L_n\| = 1, \; n = 1, 2, \ldots,$$

the conclusion is obtained via the well-known Banach–Steinhaus theorem. □

As an application we consider the following situation. Let X be a Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and let $D \subset X$ be a convex, compact subset of X having nonempty interior. Consider the functions $e_a, e : D \to \mathbb{R}$, where

$$e_a(x) = \langle a, x \rangle, \; e(x) = \langle x, x \rangle = \|x\|^2, \; a \in X.$$

Corollary 2.2 *Let $(L_n)_{n \ge 1}$ be a sequence of Markov operators acting on $C(D)$. Suppose that $L_n(e_a) = e_a$, i.e. $(L_n)_{n \ge 1}$ preserves the functions contained in the family $\{e_a\}_{a \in X}$ and $L_n(e) \to e$ in the uniform form of $C(D)$. Then $(L_n(f))_{n \ge 1}$ converges uniformly to f, for all $f \in C(D)$.*

Proof Let $f \in C^2(D)$. According to Theorem 2.3 it follows that $f \in C_h(D)$, where

$$\begin{aligned} h(x, y) &= \|x - y\|^2 = \langle x - y, x - y \rangle \\ &= \langle x, x \rangle - 2\langle x, y \rangle + \langle y, y \rangle \\ &= e(x) - 2e_y(x) + \langle y, y \rangle. \end{aligned}$$

Let us verify the conditions (i)–(iii) in Theorem 2.4. First, observe that

$$\begin{aligned} \alpha_n(x) &= L_n(h(\cdot, x); x) \\ &= L_n(e, x) - 2e_x(x) + e(x) \\ &= L_n(e; x) - e(x) \to 0, \end{aligned}$$

uniformly with respect to $x \in D$.
From the proof of Theorem 2.3 we have $g_f^{(1)} = g_f^{(2)} = df$. Thus

$$G_{f,y}^{(j)}(x) = (df)_y(x), \; y \in D.$$

Taking into account the well-known Riesz representation theorem one obtains

$$G_{f,y}^{(j)}(x) = (df)(x) = \langle (\nabla f)_y, x \rangle = e_{(\nabla f)_y}(x),$$

where $(\nabla f)_y \in X$ is the gradient of f at the point $y \in D$. It follows that

$$L_n(G^{(j)}_{f,y}; x) = G^{(j)}_{f,y}$$

for every $f \in C^2(D)$ and for every $y \in D$.

Because $C^2(D)$ is a dense subspace of $C(D)$ it follows that $C_h(D)$ has also this property. Therefore we can apply Theorem 2.4 and the desired conclusion is obtained. □

In the case $X = \mathbb{R}^n$, endowed with the usual Euclidean inner product, from Corollary 2.2 one obtains a result contained in Corollary 2 of the paper [7]. For $n = 2$ it follows the result contained in [32] and [57].

2.7 A Density Result Involving the Subspace $C_h(D)$

It is natural to find some reasonable sufficient conditions on the function $h : D \times D \to \mathbb{R}$ in order that the hypothesis (iii) in the above Theorem 2.4 is satisfied. In what follows we consider that the continuous function $h : D \times D \to \mathbb{R}$ satisfies the following conditions:

(a) h is symmetric, i.e. $h(x, y) = h(y, x)$ for every $x, y \in D$;
(b) h is positively-nondegenerate, i.e. $h(x, x) = 0$ for every $x \in D$; $h(x, y) > 0$ if $x \neq y$.

Let us denote by $C_{[h]}(D)$ the vector space of all real-valued functions defined on D satisfying the condition:

$$|f(x) - f(y)| \leq K_f h(x, y) \tag{2.35}$$

for every $x, y \in D$, where $K_f \geq 0$ is a real number depending on f.

Our main result is the following:

Theorem 2.5 *Suppose that the continuous mapping $h : D \times D \to \mathbb{R}$ satisfies the conditions (a) and (b) and for every $z \in D$ the function $h_z : D \to \mathbb{R}$, $x \mapsto h(z, x)$, belongs to $C_{[h]}(D)$. Then the subspace $C_h(D)$ is dense in $C(D)$ with respect to the uniform norm.*

Proof Let us note firstly that the following inclusions hold:

$$C_{[h]}(D) \subseteq C_h(D) \subset C(D) \tag{2.36}$$

because from (2.35) one obtains

$$-K_f h(x, y) \leq f(x) - f(y) \leq K_f h(x, y)$$

therefore f is a h-strong convex and concave function on D, where for every $y \in D$,

$$g_f^{(1)}(y) = g_f^{(2)}(y) = 0 \in X^*, \; m = -K_f, \; M = K_f,$$

the constant given by (2.35). The continuity of f follows from the continuity of h with respect to the norm topology on D. The assertion of the theorem will follow from inclusions (2.36) and from the well-known Stone–Weierstrass theorem if we shall show that $C_{[h]}(D)$ is a subalgebra of $C(D)$ containing the constant functions and separating the points of D.

It is clear that $C_{[h]}(D)$ is a linear subspace of $C(D)$. If $f_1, f_2 \in C_{[h]}(D)$, then

$$\begin{aligned} |(f_1 f_2)(x) - (f_1 f_2)(y)| &\le |f_1(x)||f_2(x) - f_2(y)| + |f_1(x) - f_1(y)| \\ &\le (\|f_1\| K_{f_2} + \|f_2\| K_{f_1}) g(x, y), \end{aligned}$$

for all $x, y \in D$, where K_{f_1}, K_{f_2} are the constants given by the inequalities (2.35) and $\|f_1\|, \|f_2\|$ are the uniform norms. Therefore $f_1 f_2 \in C_{[h]}(D)$.

As the constant functions are obviously in $C_{[h]}(D)$ to finish the proof we have only to show that the algebra $C_{[h]}(D)$ separates the points of D. For $y, z \in D$, $y \neq z$, let us consider the function $h_z : D \to \mathbb{R}$, $x \mapsto h(z, x)$. From the hypothesis of the theorem one obtains $h_z \in C_{[h]}(D)$.
Moreover, we have $h_z(z) = h(z, z) = 0$, $h_z(y) = h(z, y) > 0$, therefore $h_z(z) \neq h_z(y)$. □

Corollary 2.3 *If the continuous function $h : D \times D \to \mathbb{R}$ is a distance on D, then the subspace $C_h(D)$ is dense in $C(D)$ with respect to the uniform norm.*

Proof In this situation the conditions (a), (b) are satisfied and one obtains $C_{[h]}(D) = Lip_{[h]}(D)$, where $Lip_{[h]}(D)$ denotes the vector space of all real valued Lipschitz functions defined on D with respect to the distance h. Let us show that for every $z \in D$, the function $h_z : D \to \mathbb{R}$, $x \mapsto h(z, x)$, belongs to $Lip_{[h]}(D)$. If $x, y \in D$ one obtains

$$|h_z(x) - h_z(y)| = |h(z, x) - h(z, y)| \le h(x, y)$$

therefore $h_z \in Lip_{[h]}(D)$ and $K_{h_z} = 1$. □

3 Pointwise and Uniform Convergence on Compact Sets of Sequences of Convex Functions

In Hiriart–Urruty [26, p. 105] the following theorem was proved:

Theorem 3.1 *Let the convex functions $f_n : \mathbb{R}^k \to \mathbb{R}$ converge pointwise for $n \to \infty$ to $f : \mathbb{R}^k \to \mathbb{R}$. Then f is convex and for each compact set K, the convergence of f_n to f is uniform on K.*

In this section we shall prove and generalize the above result in several directions. We shall prove that Theorem 3.1 holds in Banach spaces of arbitrary dimension. Also we can relax the hypothesis that all f_n are convex with the hypothesis that all f_n are locally convex on an open set D. We investigate the validity of Theorem 3.1 in case the open convex set D is replaced with a compact set.

Another result from [26] that asserts that if $(f_n)_{n\geq 1}$is a pointwise convergent sequence of convex Fréchet differentiable functions defined on the m-dimensional Euclidean space that converges to a Fréchet differentiable function f, then the sequence $(\nabla f_n)_{n\geq 1}$ converges uniformly on every compact to ∇f, is generalized.

Theorem 3.2 *Let* (X, d), (Y, ρ) *be two metric spaces,* $f, f_n : X \to Y$, $n \geq 1$, $\eta : [0, \infty) \to [0, \infty)$ *be an increasing function such that*

$$\eta(0) = \lim_{t\downarrow 0} \eta(t) = 0.$$

Suppose that:

(i) X *is compact.*
(ii) $\lim_{n\to\infty} f_n(x) = f(x), \ x \in X.$
(iii) $\rho(f_n(x), f_n(y)) \leq \eta(d(x, y))$, *for every* $x, y \in X$, $n \geq 1$.

Then the sequence $(f_n)_{n\geq 1}$ *converges uniformly to* f *on* X.

Proof It suffices to consider the case X is infinite. Since X is compact then X is separable. Let $M = \{x_n \mid n \geq 1\}$ be a countable subset of X which is dense in X. For every $r \geq 1$ let $M_r = \{x_1, x_2, \ldots, x_r\}$. Consider $x \in X$ and let $r \geq 1$ and $k \in \{1, 2, \ldots, r\}$ be such that

$$d(x, x_k) = \min_{1\leq j\leq r} d(x, x_j) = d(x, M_r).$$

Note that

$$\begin{aligned}\rho(f_n(x), f(x)) &\leq \rho(f_n(x), f_n(x_k)) + \rho(f_n(x_k), f(x_k)) + \rho(f(x_k), f(x))\\ &\leq \eta(d(x, x_k)) + \max_{1\leq j\leq r} [\rho(f_n(x_j), f(x_j))] + \eta(d(x, x_k))\\ &= 2\eta(d(x, M_r)) + \max_{1\leq j\leq r} [\rho(f_n(x_j), f(x_j))]\end{aligned}$$

hence

$$\sup_{x\in X}[\rho(f_n(x), f(x))] \leq 2\eta\left(\sup_{x\in X} d(x, M_r)\right) + \max_{1\leq j\leq r} [\rho(f_n(x_j), f(x_j))].$$

Letting $n \to \infty$ in the above inequality we obtain

$$\limsup_{n\to\infty}\left[\sup_{x\in X}[\rho(f_n(x), f(x))]\right] \le 2\eta\left(\sup_{x\in X}[d(x, M_r)]\right).$$

Denote

$$a_r = \sup_{x\in X}[d(x, M_r)],\ r \ge 1.$$

Note that the sequence $(a_r)_{r\ge1}$ is decreasing. For every $r \ge 1$ let $y_r \in X$ be such that $a_r = d(y_r, M_r)$. Since X is compact there exists a convergent subsequence $(y_{r_p})_{p\ge1}$ of $(y_r)_{r\ge1}$. Let

$$y = \lim_{p\to\infty} y_{r_p}.$$

Consider the following inequality

$$0 \le a_{r_p} = d(y_{r_p}, M_{r_p}) \le d(y_{r_p}, y) + d(y, M_{r_p})$$

and let $p \to \infty$. We obtain

$$\lim_{p\to\infty} a_{r_p} = 0.$$

Since $(a_r)_{r\ge1}$ is decreasing it follows that

$$\lim_{r\to\infty} a_r = 0.$$

Consequently the sequence $(f_n)_{n\ge1}$ converges uniformly to f on X. □

Lemma 3.1 ([52, p. 93]) *Let E be a linear normed space, D be a subset of E, and $f : D \to \mathbb{R}$ be a convex function. If f is bounded from above in a neighborhood of a point of D, then f is continuous on D.*

If E is a linear normed space, $a \in E$ and $r > 0$, we shall denote with $B(a, r)$ the open ball with center at $x = a$ of radius r. By $B[a, r]$ will be denoted the closed ball of center $x = a$ and radius r.

Lemma 3.2 *Let E be a Banach space, D be an open convex subset of E, and $f : D \to \mathbb{R}$ be a convex lower semicontinuous function. Then f is continuous.*

Proof f is lower semicontinuous if and only if for every $c \in \mathbb{R}$ the set

$$\{x \in D \mid f(x) > c\}$$

is open. Suppose by absurd that f is not continuous. From Lemma 3.1 it follows that for all $c \in \mathbb{R}$ the set $\{x \in D \mid f(x) > c\}$ is a dense open subset of D. If $x_1 \in A_1$, then there exists $r_1 > 0$ such that $B(x_1, r_1) \subset B[x_1, r_1] \subset A_1$. Let

$x_2 \in B(x_1, r_1) \cap A_1$. Since $B(x_1, r_1) \cap A_1$ is an open set it follows that there exists $r_2 > 0$ such that $B(x_2, r_2) \subset B[x_2, r_2] \subset A_2 \cap B(x_1, r_1)$. If we iterate the above argument we obtain a sequence $(x_n)_{n\geq 1}$ with $x_n \in A_n$ for all $n \geq 1$ such that $B(x_n, r_n) \subset B[x_n, r_n] \subset B(x_{n-1}, r_{n-1})$. Since E is a Banach space it follows that

$$\bigcap_{i=1}^{\infty} B[x_i, r_i] \neq \emptyset.$$

Let

$$x_0 \in \bigcap_{i=1}^{\infty} B[x_i, r_i] \subset \bigcap_{i=1}^{\infty} A_i.$$

We note that $f(x_0) > n$ for every $n \geq 1$, which is a contradiction. Consequently f is a continuous function. □

Lemma 3.3 *Let E be a linear normed space, $R > 0$, $x_0 \in E$, $f : B(x_0, R) \to \mathbb{R}$ be a convex function with the property*

$$m \leq f(x) \leq M \text{ for every } x \in B(x_0, R).$$

Then for every $r \in (0, R)$ the following inequality holds:

$$|f(x) - f(y)| \leq \frac{M-m}{R-r}\|x-y\|, \; x, y \in B(x_0, r).$$

Proof Let $r \in (0, R)$, $x, y \in B(x_0, r)$, $x \neq y$, $z = y + \dfrac{R-r}{\|x-y\|}(y-x)$. Note that we have $z \in B(x_0, R)$, hence $f(z) \leq M$. Let

$$a = \frac{\|y-x\|}{R-r+\|y-x\|}, \; b = \frac{R-r}{R-r+\|y-x\|}.$$

Note that we have $a, b \geq 0$, $a + b = 1$ and $y = az + bx$. We obtain

$$\begin{aligned} f(y) - f(x) &\leq af(z) + bf(x) - f(x) = a(f(z) - f(x)) \\ &\leq a(M-m) \leq \frac{M-m}{R-r}\|y-x\|. \end{aligned}$$

By changing x with y and y with x in the preceding inequality we obtain the inequality from the statement. □

Let E be a linear normed space and let D be an open subset of E. A function $f : D \to \mathbb{R}$ is called *locally convex* if for every $a \in D$ there exists $r > 0$ such that the restriction of f to $B(a, r)$ is convex.

The following result can be found in [30, Corollary 2.4].

Lemma 3.4 ([30]) *Let E be a linear normed space and D be an open subset of E. Then every locally convex function $f : D \to \mathbb{R}$ is convex on each convex subset of D.*

Theorem 3.3 *Let E be a Banach space, D be an open subset of E. Consider the functions $f_n, f : D \to \mathbb{R}$, $n \geq 1$. Suppose that:*

(i) f and f_n, $n \geq 1$ are locally convex and continuous;
(ii) $\lim_{n\to\infty} f_n(x) = f(x)$ for every $x \in D$.

Then the sequence $(f_n)_{n\geq 1}$ converges uniformly to f on every compact subset K of D.

Proof We shall prove the following assertion:

Statement A For every $x_0 \in D$ there exists $r > 0$ such that $B(x_0, r) \subset D$ and $(f_n)_{n\geq 1}$ converges uniformly to f on every compact K in $B[x_0, r]$.

Proof of Statement A Let

$$g(x) = \sup_{n\geq 1} f_n(x), \quad x \in D.$$

Note that g is locally convex and lower semicontinuous. By Lemmas 3.2 and 3.4, g is continuous.

Let $x_0 \in D$. Then there exists $R > 0$ such that $B[x_0, R]$ is included in D and g is bounded on $B[x_0, R]$. Let $M > 0$ be such that $|g(x)| \leq M$ for every $x \in B[x_0, R]$. Hence $f_n(x) \leq M$ for every $x \in B[x_0, R]$. Since $(f_n(x_0))_{n\geq 1}$ is convergent it follows that there exists $m_1 \in \mathbb{R}$ such that $f_n(x_0) \geq m_1$ for every $n \geq 1$.

Let $x \in B[x_0, R]$ and $x' = 2x_0 - x$. Note that $x' \in B[x_0, R]$ and

$$f_n(x) \geq 2f_n(x_0) - f_n(x') \geq 2m_1 - M.$$

Let $m = 2m_1 - M$. We have $m \leq f_n(x) \leq M$ for every $x \in B[x_0, R]$. From Lemma 3.5 it follows that f_n is $\dfrac{M-m}{R-r}$ Lipschitz on $B[x_0, r]$, $n \geq 1$.

Let K be a compact subset of $B[x_0, r]$. Since $(f_n)_{n\geq 1}$ is pointwise convergent on K, from Theorem 3.2 it follows that $(f_n)_{n\geq 1}$ is uniformly convergent on K. Thus Statement A is proved.

Let K be a compact subset of D. By Statement A it follows that for every $x_0 \in K$ there exists $r(x_0) > 0$ such that $(f_n)_{n\geq 1}$ is uniformly convergent on every compact subset of $B[x_0, r(x_0)]$. Since $K \subset \bigcup_{x_0 \in K} B(x_0, r(x_0))$ it follows that there exists a finite covering $B[x_i, r(x_i)]$ of K, $i = 1, 2, \dots, m$, with balls with centers in K. Since $(f_n)_{n\geq 1}$ is uniformly convergent on every compact $K \cap B[x_i, r(x_i)]$, $i = 1, 2, \dots, m$, it follows that $(f_n)_{n\geq 1}$ is uniformly convergent to f on K.

□

Lemma 3.5 *Let $f : [a, b] \to \mathbb{R}$ be a convex function. Then the following inequality holds:*

$$|f(x)-f(y)| \le 2\max\left(f(a) - f\left(\frac{a+b}{2}\right), f(b) - f\left(\frac{a+b}{2}\right)\right), \; x, y \in [a, b].$$

Proof Let $M = \max(f(a), f(b))$. Note that

$$2f\left(\frac{a+b}{2}\right) - M \le f(x) \le M \text{ for every } x \in [a, b].$$

The above inequalities can be proved as follows. Let $x \in [a, b]$, $x' = a + b - x$. Note that $x' \in [a, b]$ and

$$2f\left(\frac{a+b}{2}\right) - M = 2f\left(\frac{x+x'}{2}\right) - M \le f(x) + f(x') - M \le f(x) \le M.$$

If $x, y \in [a, b]$, then

$$\begin{aligned} |f(x) - f(y)| &\le M - \left(2f\left(\frac{a+b}{2}\right) - M\right) = 2\left(M - f\left(\frac{a+b}{2}\right)\right) \\ &= 2\max\left(f(a) - f\left(\frac{a+b}{2}\right), f(b) - f\left(\frac{a+b}{2}\right)\right). \end{aligned}$$

□

Theorem 3.4 *Let $f, f_n : [0, 1] \to \mathbb{R}$, $n \ge 1$ be convex functions. Suppose that f is continuous and $(f_n)_{n\ge1}$ converges pointwise to f on $[0, 1]$. Then $(f_n)_{n\ge1}$ converges uniformly on $[0, 1]$ to f.*

Proof Since f is uniformly continuous on $[0, 1]$ there exists $\eta : [0, \infty) \to [0, \infty)$ increasing such that

$$\eta(0) = \eta(0 + 0) = 0 \text{ and } |f(x) - f(y)| \le \eta(|x - y|), \; x, y \in [0, \infty).$$

Let $\Delta : 0 = x_0 < x_1 < \ldots < x_m = 1$ be a division of the interval $[0, 1]$. If $x \in [0, 1]$ consider $i \in \{1, 2, \ldots, m\}$ such that $x \in [x_{i-1}, x_i]$. Note that

$$\begin{aligned} |f_n(x) - f(x)| &\le |f_n(x) - f_n(x_i)| + |f_n(x_i) - f(x_i)| + |f(x_i) - f(x)| \\ &\le 2\max\left(f_n(x_i) - f_n\left(\frac{x_i + x_{i-1}}{2}\right), f_n(x_{i-1}) - f_n\left(\frac{x_i + x_{i-1}}{2}\right)\right) \\ &+ \max_{1\le j\le m} |f_n(x_j) - f(x_j)| + \eta(|x_i - x|). \end{aligned}$$

Let $\|\Delta\| = \max\limits_{1\le i\le m}(x_i - x_{i-1})$ be the norm of division Δ. Then

$$\sup_{x\in[0,1]} |f_n(x) - f(x)|$$

$$\le 2\max\left(f_n(x_i) - f_n\left(\frac{x_i + x_{i-1}}{2}\right), f_n(x_{i-1}) - f_n\left(\frac{x_i + x_{i-1}}{2}\right)\right) + \eta(\|\Delta\|).$$

Letting $n \to \infty$ we obtain that

$$\limsup_{n\to\infty}\left[\sup_{x\in[0,1]} |f_n(x) - f(x)|\right]$$

$$\le 2\max\left(f(x_i) - f\left(\frac{x_i + x_{i-1}}{2}\right), f(x_{i-1}) - f\left(\frac{x_i + x_{i-1}}{2}\right)\right) + \eta(\|\Delta\|)$$

$$\le 2\eta(\|\Delta\|) + \eta(\|\Delta\|) = 3\eta(\|\Delta\|).$$

Now, letting $\|\Delta\| \to 0$ we obtain that $(f_n)_{n\ge1}$ is uniformly convergent to f on the interval [0, 1]. □

Proposition 3.1 *Let $m \ge 1$,*

$$B = \{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m \mid x_1^2 + x_2^2 + \ldots + x_m^2 \le 1\}$$

and $f : B \to \mathbb{R}$, $f(x) = 0$ for $x \in B$. Consider the functions $f_n : B \to \mathbb{R}$, $n \ge 1$.

$$f_n(x_1, x_2, \ldots, x_m) = mn^2 \cdot \max\left(x_1 + \frac{x_2 + x_3 + \ldots + x_m}{n} - 1, 0\right),$$

where $(x_1, x_2, \ldots, x_m) \in B$. Then $(f_n)_{n\ge1}$ is pointwise convergent to f but $(f_n)_{n\ge1}$ is not uniformly convergent to f on B.

Proof Note that if $x = (x_1, x_2, \ldots, x_m) \in B$ and $x_1 < 1$ then there exists $n(x_1) \ge 1$ such that $f_n(x) = 0$ for all $n \ge n(x_1)$. Since $f_n(1, 0, 0, \ldots, 0) = 0$ for every $n \ge 1$ it follows that $(f_n)_{n\ge1}$ converges pointwise to f on B.

Consider the sequence of vectors $x_n = (x_{n1}, x_{n2}, \ldots, x_{nm})$,

$$x_{n1} = \frac{n}{\sqrt{n^2 + m - 1}},\ x_{n2} = x_{n3} = \ldots = x_{nm} = \frac{1}{\sqrt{n^2 + m - 1}},\ n \ge 1.$$

Note that $x_n \in B$ for all $n \geq 1$. More precisely $\|x_n\| = 1$ for all $n \geq 1$. We consider on $\mathbb{R}^m$ the Euclidean norm. Note that for all $n \geq 1$, we have

$$f_n(x_n) = mn^2 \cdot \max\left(\frac{n}{\sqrt{n^2+m-1}} + \frac{m-1}{n\sqrt{n^2+m-1}} - 1, 0\right)$$

$$= mn^2 \cdot \max\left(\frac{\sqrt{n^2+m-1}}{n} - 1, 0\right) = mn^2 \cdot \frac{\sqrt{n^2+m-1}-n}{n} = mn \cdot \frac{m-1}{\sqrt{n^2+m-1}+n}.$$

Since $\lim_{n\to\infty} f_n(x_n) = \dfrac{m(m-1)}{2}$ it follows that $(f_n)_{n\geq 1}$ is not uniformly convergent to f on B. □

Theorem 3.5 *Let D be an open set of $\mathbb{R}$ and $f, f_n : D \to \mathbb{R}$, $n \geq 1$ be convex, differentiable functions. If $(f_n)_{n\geq 1}$ converges pointwise to f on D, then $(f_n')_{n\geq 1}$ converges pointwise to f on D.*

Proof Let $a \in D$, $r > 0$ be such that $(a-r, a+r) \subset D$. For every $n \geq 1$ consider the function $u_n : (-r, r) \to \mathbb{R}$

$$u_n(h) = \begin{cases} \dfrac{f_n(a+h) - f_n(a)}{h}, & h \in (-r, 0) \cup (0, r) \\ f_n'(a), & h = 0 \end{cases}$$

Since f_n are convex it follows that u_n are increasing, hence

$$u_n(-h) \leq u_n(0) \leq u_n(h),\ \ h \in (0, r),\ \ n \geq 1.$$

Thus

$$|u_n(h) - u_n(0)| \leq |u_n(h) - u_n(-h)|,\ \ h \in (0, r),\ \ n \geq 1.$$

Let $u(h) = \lim_{n\to\infty} u_n(h)$, $h \in (-r, r)$. Then for $h \in (-r, 0) \cup (0, r)$ we have

$$\begin{aligned} |f_n'(a) - f'(a)| &\leq |f_n'(a) - u_n(h)| + |u_n(h) - u(h)| + |u(h) - f'(a)| \\ &= |u_n(0) - u_n(h)| + |u_n(h) - u(h)| + |u(h) - u(0)| \\ &\leq |u_n(h) - u_n(-h)| + |u_n(h) - u(h)| + |u(h) - u(0)|. \end{aligned}$$

Letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty} |f_n'(a) - f'(a)| \leq |u(h) - u(-h)| + |u(h) - u(0)|.$$

Letting $h \to 0$ we obtain that $(f_n')_{n\geq 1}$ converges pointwise to f on D. □

In Kosmol [30] the following two theorems were proved.

Theorem 3.6 ([30]) *Let (X, d), (Y, ρ) be two metric spaces, $f_n : X \to Y$, $n \geq 1$ be continuous functions. Suppose that there exists $f : X \to Y$ such that $(f_n)_{n\geq 1}$ is pointwise convergent to f. Then the following assertions are equivalent:*

(1) $(f_n)_{n\geq 1}$ is an equicontinuous family of functions.
(2) $(f_n)_{n\geq 1}$ converges uniformly on every compact set of X to f.

Theorem 3.7 ([30]) *Let D be an open subset of $\mathbb{R}^m$ and let $f_n : D \to \mathbb{R}^m$, $n \geq 1$ be a sequence of continuous monotone operators that is pointwise convergent on D to a continuous operator $f : D \to \mathbb{R}^m$.*

Then $(f_n)_{n\geq 1}$ is equicontinuous on D.

Here by a monotone operator on D we understand a function $g : D \to \mathbb{R}^m$ such that

$$\langle g(x) - g(y), x - y\rangle \geq 0 \text{ for every } x, y \in D.$$

From Theorems 3.6 and 3.7 we obtain

Theorem 3.8 *Let D be an open subset of $\mathbb{R}^m$ and let $f_n : D \to \mathbb{R}^m$, $n \geq 1$ be a sequence of continuous monotone operators that is pointwise convergent on D to the continuous operator $f : D \to \mathbb{R}^m$. Then $(f_n)_{n\geq 1}$ is uniformly convergent to f on every compact subset K of D.*

In [26, Corollary 6.2.8], the following result is proved.

Theorem 3.9 *Let $f_n, f : \mathbb{R}^m \to \mathbb{R}$ be Fréchet differentiable convex functions. If $(f_n)_{n\geq 1}$ converges pointwise to f, then $(\nabla f_n)_{n\geq 1}$ converges to ∇f uniformly on every compact set of $\mathbb{R}^m$.*

The next result is a generalization of the preceding theorem.

Theorem 3.10 *Let E be a Banach space, D be an open subset of E, and $f, f_n : D \to \mathbb{R}$, $n \geq 1$ be locally convex functions that are Fréchet differentiable. Suppose that $(f_n)_{n\geq 1}$ is pointwise convergent to f on D. Then the following assertions hold:*

(1) For every $x \in D$ and every compact K of E we have

$$\lim_{n\to\infty} \sup_{h\in K} |f_n'(x)(h) - f'(x)(h)| = 0.$$

(2) If E is finite dimensional, then $(f_n')_{n\geq 1}$ converges uniformly on every compact subset K of D to f'.

Proof In order to prove assertion (1) let $x \in D$ and let $r > 0$ be such that $B(x, r) \subset D$. Let $h \in B(0, r)$. Consider the functions

$$u(t) = f(x + th), \quad t \in [-1, 1],$$
$$u_n(t) = f_n(x + th), \quad t \in [-1, 1], \; n \geq 1.$$

Note that all the functions u_n are convex and $(u_n)_{n\geq 1}$ converges pointwise to u on $[-1, 1]$. From Theorem 3.5 it follows that

$$\lim_{n\to\infty} u_n'(0) = u'(0)$$

hence

$$\lim_{n\to\infty} f_n'(x)(h) = f'(x)(h).$$

Consider the functions

$$\varphi_n(h) = f_n'(x)(h),\ n \geq 1$$

and

$$\varphi(h) = f'(x)(h),\ h \in B(0, r).$$

Note that φ_n, $n \geq 1$ are convex and $(\varphi_n)_{n\geq 1}$ converges pointwise to φ. From Theorem 3.3 it follows that for every compact set K of E we have

$$\lim_{n\to\infty} \sup_{h\in K} |f_n'(x)(h) - f'(x)(h)| = 0.$$

In order to prove the second assertion let E be a finite dimensional space. Since $(f_n)_{n\geq 1}$ is pointwise convergent to f it follows from assertion (1) that $(f_n'(x))_{n\geq 1}$ converges uniformly on $B[0, 1]$ to $f'(x)$. This can be written as follows:

$$\lim_{n\to\infty} \sup_{h\in B[0,1]} |f_n'(x)(h) - f'(x)(h)| = \lim_{n\to\infty} \|f_n'(x) - f'(x)\| = 0.$$

Since $f'(x)$ and $f_n'(x)$, $n \geq 1$ are continuous monotone operators, from Theorem 3.9 it follows that $(f_n')_{n\geq 1}$ converges uniformly to f' on every compact subset of D. □

4 A Generalization of Schur Convexity and Applications

If A is a square matrix with complex entries, we shall denote by f_A the characteristic polynomial of A, that is $f_A(t) = \det(tI_n - A)$. An $n \times n$ real matrix $A = \left(a_{ij}\right)$ is called bistochastic if all its entries are nonnegative and the following equalities hold (for details we refer to the reference [38]):

$$\sum_{j=1}^{n} a_{ij} = 1, \quad \text{for every } i \in \{1, 2, \ldots, n\}$$

$$\sum_{i=1}^{n} a_{ij} = 1, \quad \text{for every } j \in \{1, 2, \ldots, n\}$$

The set of bistochastic matrices will be denoted with $BSM_n(\mathbb{R})$.

We will use the following notations. For the matrices $A = (a_{ij})$, $B = (b_{ij}) \in M_n(\mathbb{R})$, we write

1. $A \le B$ if and only if for all i, $j = 1, \ldots, n$, we have $a_{ij} \le b_{ij}$;
2. $A < B$ if and only if for all i, $j = 1, \ldots, n$, we have $a_{ij} < b_{ij}$;

Let E be real linear space and $A = (a_{ij})$ be an $n \times n$ real matrix. We define the linear application $\overline{A} : E^n \to E^n$ as follows:

$$\overline{A}(x_1, x_2, \ldots, x_n) = \left(\sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j, \right), \quad (x_1, x_2, \ldots, x_n) \in E^n$$

Let $\mathscr{A}$ be a subset of $BSM_n(\mathbb{R})$. A subset D of E^n will be called $\mathscr{A}$ invariant if $\overline{A}(D) \subset D$ for every $A \in \mathscr{A}$. A subset D of E^n will be called circular symmetric invariant if for every $(x_1, x_2, \ldots, x_n) \in D$ we have $(x_2, x_3, \ldots, x_n, x_1) \in D$. A function $f : D \to \mathbb{R}$ where D is circular symmetric invariant is called circular symmetric if

$$f(x_1, x_2, \ldots, x_n) = f(x_2, x_3, \ldots, x_n, x_1), \text{for every } (x_1, x_2, \ldots, x_n) \in D;$$

Let D be a convex subset of E^n that is $\mathscr{A}$ invariant. A function $f : D \to \mathbb{R}$ is called $\mathscr{A}$-Schur convex if $f\left(\overline{A}x\right) \le f(x)$ for every $x \in D$ and for every $A \in \mathscr{A}$. In case $\mathscr{A} = BSM_n(\mathbb{R})$, then the set of $\mathscr{A}$-Schur convex functions is equal to the set of Schur convex functions. A function $f : D \to \mathbb{R}$ is called $\mathscr{A}$-Schur concave If $-f$ is $\mathscr{A}$-Schur convex.

Remark 4.1 Let $\mathscr{A}$ be a subset of $BSM_n(\mathbb{R})$, D be a convex subset of E^n that is $\mathscr{A}$ invariant and $f : D \to \mathbb{R}$ be $\mathscr{A}$-Schur convex. Then the following assertions hold:

1. The function f is $\check{\mathscr{A}}$-Schur convex where with $\check{\mathscr{A}}$ we denoted the multiplicative monoid generated by $\mathscr{A}$.
2. The set of $\mathscr{A}$-Schur convex functions is a convex cone of $BSM_n(\mathbb{R})$.
3. Every Schur convex function is a $\mathscr{A}$-Schur convex function.
4. If I is an interval of the real axis and $\phi : I \to \mathbb{R}$ is an increasing function, then $\phi \circ f$ is $\mathscr{A}$-Schur convex.

Remark 4.2 Suppose $\mathscr{A} \subset \mathscr{B}$ are subsets of $BSM_n(\mathbb{R})$ such that D is a convex subset of E^n that is $\mathscr{B}$-invariant. If $f : D \to \mathbb{R}$ is $\mathscr{B}$-Schur convex, then f is $\mathscr{A}$-Schur convex.

Remark 4.3 Let $n \geq 3$ be a natural number and $e_1, e_2, \ldots, e_n$ be the canonical base of $\mathbb{R}^n$. Consider the matrices

$$T_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \tag{4.1}$$

$$U_n = \begin{pmatrix} 0\ 1\ 0\ 0\ 0 \ldots 0 \\ 0\ 0\ 1\ 0\ 0 \ldots 0 \\ 0\ 0\ 0\ 1\ 0 \ldots 0 \\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \\ 0\ 0\ 0\ 0\ 0 \ldots 1 \\ 1\ 0\ 0\ 0\ 0 \ldots 0 \end{pmatrix} \tag{4.2}$$

$$T_n = diag(T_2, I_{n-2}) \in BSM_n(\mathbb{R}). \tag{4.3}$$

Then the following assertions hold:

1. $U_n = e_1 e_2^T + e_2 e_3^T + \cdots + e_{n-1} e_n^T + e_n e_1^T \in BSM_n(\mathbb{R})$.
2. If $U_n \in \mathscr{A}$ and D is a convex subset of E^n that is circular symmetric invariant and $f : D \to \mathbb{R}$ is $\mathscr{A}$-Schur convex, then f is circular symmetric.
3. If $T_n \in \mathscr{A}$, then and D is a convex subset of E^n such that D is $\mathscr{A}$ invariant and $f : D \to \mathbb{R}$ is $\mathscr{A}$-Schur convex, then

$$f\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n\right) \leq f(x_1, x_2, \ldots, x_n), \quad \text{for every } (x_1, x_2, \ldots, x_n) \in D.$$

4.1 The Strong Mixing Variables Method

We shall apply the concept of $\mathscr{A}$ -Schur convex functions for proving a generalization of the following theorem from Andrica and Mare [8].

Theorem 4.1 *Let $n \geq 2$ be a natural number, I be an interval of the real axis, and $f : I^n \to \mathbb{R}$ be a symmetric continuous function. If*

$$f\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, x_4, \ldots, x_n\right) \leq f\left(x_1, x_2, \ldots, x_n\right), \quad (x_1, x_2, \ldots, x_n) \in I^n$$

then

$$f\left(\overline{x}, \overline{x}, \ldots, \overline{x}, \right) \leq f\left(x_1, x_2, \ldots, x_n\right), \quad (x_1, x_2, \ldots, x_n) \in I^n$$

where

$$\overline{x} = \frac{x_1 + x_2 + \ldots + x_n}{n} \tag{4.4}$$

We shall prove that the above result holds for a more general class of functions, that is for the class of circular symmetric functions. We refer to the papers [33] and [34] for the theory of symmetric functions in this context (see also [37, 38] and [40]). The above theorem is called also the Strong Mixing Variables Method. See also [15, 29, 48], Cvetkovski [19], and Venkatachala [56]. An application of the Strong Mixing Variables Method to an estimation of graph entropy can be found in Eliasi [23].

Our generalization of Theorem 4.1 will be based on the following result:

Proposition 4.1 *Consider $A \in BSM_n(\mathbb{R})$. Then the following assertions hold:*

1° *The matrix A is of rank 1 if and only if $A = \frac{1}{n}ee^T$, where $e \in M_{n,1}(\mathbb{R})$ is the vector with all entries equal to 1 and e^T denotes the transpose of e;*

2° *If there is a positive integer r with $A^r > O_n$, then the relation*

$$\lim_{p \to \infty} A^p = \frac{1}{n} ee^T$$

holds.

Proof 1° Because A is of rank 1, we have $A = uv^T$, for some vectors $u, v \in M_{n,1}(\mathbb{R})$. From the property that A is bistochastic, it follows $Ae = e$ and $A^T e = e$. One obtains the relations

$$uv^T e = e \quad \text{and} \quad vu^T e = e,$$

hence we get

$$e^T uv^T e = e^T e = n$$

$$u = \frac{1}{e^T v} e \quad \text{and} \quad v = \frac{1}{e^T u} e.$$

Finally, we obtain

$$A = uv^T = \frac{1}{e^T v} \cdot \frac{1}{e^T u} ee^T = \frac{1}{n} ee^T,$$

and the conclusion follows.

2° Applying the Perron–Frobenius Theorem, it follows $\lim\limits_{p \to \infty} A^p = B$, for some matrix $B \in BSM_n(\mathbb{R})$ of rank 1. From assertion 1°, we obtain $B = \frac{1}{n} ee^T$, and we are done.

□

Theorem 4.2 *Let E be a real Hausdorff linear topological space, D be a closed convex subset of E^n, $\mathscr{A}$ be a subset of $BSM_n(\mathbb{R})$ and $A \in \check{\mathscr{A}}$. Suppose that the following conditions hold:*

1° $A^p > 0$ *for some natural number* $p \geq 1$.
2° *The set D is $\mathscr{A}$-invariant.*

If $f : D \to \mathbb{R}$ is a continuous function that is $\mathscr{A}$-Schur convex, then the following inequality holds:

$$f(\overline{x}, \overline{x}, \dots, \overline{x},) \leq f(x_1, x_2, \dots, x_n), \quad (x_1, x_2, \dots, x_n) \in D$$

where $\overline{x}$ is defined in (4).

Proof Note that for every $x \in D$, $x = (x_1, x_2, \dots, x_n)$, we have:

$$f\left(\overline{A}^k x\right) \leq f\left(\overline{A}^{k-1} x\right) \leq \dots \leq f\left(\overline{A} x\right) \leq f(x)$$

By Proposition 4.1 we have $\lim_{k\to\infty} A^k = \frac{1}{n}ee^T$. Letting $k \to \infty$ in the above sequence of inequalities we obtain

$$f(\overline{x}, \overline{x}, \dots, \overline{x},) = f\left(\frac{1}{n}ee^T x\right) \leq f(x_1, x_2, \dots, x_n).$$

□

Lemma 4.1 *Let $A \in BSM_n(\mathbb{R})$ be a bistochastic matrix satisfying the following properties:*

(1) The eigenvalues of A are pairwise distinct;
(2) If λ is an eigenvalue of A such that $|\lambda| = 1$, then $\lambda = 1$.

Then $A^p > O_n$, for some positive integer p.

Proof Because A is bistochastic, it follows that every eigenvalue λ of A satisfies the inequality $|\lambda| \leq 1$. Using the assumptions from the hypothesis, we can write the spectrum of A as

$$\sigma(A) = \{\lambda_1, \lambda_2, \dots, \lambda_n\},$$

where $\lambda_1 = 1$ and $|\lambda_i| < 1$ for $i = 2, \dots, n$. Because the matrix A has distinct eigenvalues, it follows A is diagonalizable. That is $A = SDS^{-1}$ for some invertible matrix $S \in M_n(\mathbb{C})$ and some matrix $D \in M_n(\mathbb{C})$ with $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$.

We have

$$\lim_{p\to\infty} A^p = \lim_{p\to\infty} (SDS^{-1})^p = S(\lim_{p\to\infty} D^p)S^{-1} = Se_1e_1^T S^{-1} = B,$$

where e_1 denotes the n-dimensional vector with the first entry equal to one and the rest of the entries equal to zero. Therefore, the limit matrix B has the rank 1. From Proposition 4.1, it follows $B = \frac{1}{n}ee^T > O_n$, i.e. $A^p > O_n$ for some positive integer p. □

Lemma 4.2 *Let $n \geq 3$ be a natural number, U_n and T_n be the matrices defined by (4.2) and (4.3). Consider the matrix*

$$A_n = U_n T_n = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 \\ . & . & . & . & . & \dots & . \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then the following assertions hold:

(1) $A_n \in BSM_n(\mathbb{R})$ and $f_{A_n}(t) = \det(tI_n - A_n) = t^n - \frac{1}{2}t^{n-1} - \frac{1}{2}t$;
(2) The polynomial f_{A_n} has distinct roots;
(3) The roots of f_{A_n} satisfy $|t| \leq 1$;
(4) If for a root of f_{A_n} we have $|t| = 1$, then $t = 1$;
(5) $A_n^p > O_n$, for some positive integer p.

Proof

(1) It is clear that A_n is a bistochastic matrix. A simple computation with determinants shows that

$$f_{A_n}(t) = \begin{vmatrix} t-\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \dots & 0 \\ 0 & t & -1 & 0 & \dots & 0 \\ 0 & 0 & t & -1 & \dots & 0 \\ . & . & . & . & . & \dots . \\ 0 & 0 & 0 & 0 & \dots & -1 \\ -\frac{1}{2} & -\frac{1}{2} & 0 & 0 & \dots & t \end{vmatrix} = (t - \frac{1}{2})a_n + \frac{1}{2}b_n,$$

where

$$a_n = \begin{vmatrix} t & -1 & 0 & \dots & 0 & 0 \\ 0 & t & -1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ 0 & 0 & 0 & \dots & t & -1 \\ -\frac{1}{2} & 0 & 0 & \dots & 0 & t \end{vmatrix} = \begin{vmatrix} t-1 & -1 & 0 & \dots & 0 & 0 \\ t-1 & t & -1 & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ t-1 & 0 & 0 & \dots & t & -1 \\ t-\frac{1}{2} & 0 & 0 & \dots & 0 & t \end{vmatrix},$$

We obtain the recursive formula $a_n = t^{n-2}(t-1) + a_{n-1}$ and immediately

$$a_n = t^{n-1} - \frac{1}{2}.$$

Also, we have

$$b_n = \begin{vmatrix} 0 & -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & t & -1 & 0 & \dots & 0 & 0 \\ 0 & 0 & t & -1 & \dots & 0 & 0 \\ . & . & . & . & \dots. & . & . \\ 0 & 0 & 0 & 0 & \dots & t & -1 \\ -\frac{1}{2} & 0 & 0 & 0 & \dots & 0 & t \end{vmatrix} = -\frac{1}{2}.$$

It follows

$$f_{A_n}(t) = (t - \frac{1}{2})a_n + \frac{1}{2}b_n = (t - \frac{1}{2})(t^{n-1} - \frac{1}{2}) - \frac{1}{4} = t^n - \frac{1}{2}t^{n-1} - \frac{1}{2}t.$$

(2) Considering the polynomial $g(t) = t^{n-1} - \frac{1}{2}t^{n-2} - \frac{1}{2}$, we note that the system $g(t) = 0$, $g'(t) = 0$ has no solution.

(3) Because $A_n \in BSM_n(\mathbb{R})$ it follows that every eigenvalue λ of A_n satisfies $|\lambda| \leq 1$.

(4) Assume $f_{A_n}(t) = 0$ and $|t| = 1$. From $t^n - \frac{1}{2}t^{n-1} - \frac{1}{2}t = 0$, we obtain

$$|t - \frac{1}{2}| = |t^n - \frac{1}{2}t^{n-1}| = |\frac{1}{2}t| = \frac{1}{2}.$$

On the other hand, we have

$$\frac{1}{4} = |t - \frac{1}{2}|^2 = |t|^2 - \frac{1}{2}(t + \bar{t}) + \frac{1}{4},$$

hence $t + \bar{t} = 2$. Because $\bar{t} = \frac{1}{t}$, from the last relation it follows $t^2 + 1 = 2t$, that is $t = 1$.

(5) Because A_n has distinct eigenvalues, it follows that A_n is diagonalizable. Therefore, $A = SDS^{-1}$ for some invertible matrix $S \in M_n(\mathbb{C})$ and some matrix $D \in M_n(\mathbb{C})$ of the form $D = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, where $\lambda_1 = 1$ and $|\lambda_j| < 1$ for $j = 2, 3, \dots, n$. Applying the result from Lemma 4.1. we have $A^p > O_n$, for some positive integer p.

□

Theorem 4.3 *Let $n \geq 3$ be a natural number, E be a real Hausdorff topological linear space, and D be a closed convex subset of E^n that is circular symmetric invariant. Consider $2 \leq r \leq n - 1$, $A = (a_{ij}) \in BSM_r(\mathbb{R})$ and $f : D \to \mathbb{R}$ a continuous function with the following properties:*

(1) There exists a natural number $p \geq 1$ such that the following inequality holds:

$$a_{12}^{(p)} a_{21}^{(p)} \left(\prod_{i=1}^{r} a_{ii}^{(p)} \right) > 0, \tag{4.5}$$

Here we have denoted by $a_{ij}^{(p)}$ the entries of the matrix A^p.

(2) The function f is circular symmetric, i.e.

$$f(x_1, x_2, \ldots, x_n) = f(x_2, x_3, \ldots, x_n, x_1),$$

for every $(x_1, x_2, \ldots, x_n) \in D$;

(3) The following inequality holds:

$$f(\sum_{j=1}^{r} a_{1j}x_j, \sum_{j=1}^{r} a_{2j}x_j, \ldots, \sum_{j=1}^{r} a_{rj}x_j, x_{r+1}, \ldots, x_n) \leq f(x_1, x_2, \ldots, x_n),$$

for every $(x_1, x_2, \ldots, x_n) \in D$.

Then, for every $(x_1, x_2, \ldots, x_n) \in D$ the following inequality holds:

$$f(\overline{x}, \overline{x}, \ldots, \overline{x}) \leq f(x_1, x_2, \ldots, x_n), \tag{4.6}$$

where $\overline{x}$ is defined by (4.4).

Proof Let $B = \operatorname{diag}(A, I_{n-r})$ and $T_r = \operatorname{diag}(T_2, I_{r-2})$, where the matrix T_2 is defined in (4.1). From (4.5) it follows that $A^p \geq cT_r$, for some positive integer p and some real number $c \in (0, 1)$. Note that properties (2) and (3) are equivalent to $f(\overline{U}_n x) = f(x)$ and $f(\overline{B}x) \leq f(x)$, for every $x \in D$. This is equivalent with f is $\mathscr{A}$-Schur convex where $\mathscr{A} = \{U_n, B\}$. From $A^p \geq cT_r$ it follows that $B^p \geq cT_n$ hence $U_n B^p \geq cU_n T_n$ By Lemma 4.2 it follows that $(U_n T_n)^q > 0$ for some natural number $q \geq 1$. Hence $(U_n B^p)^q \geq c^q\,(U_n T_n)^q > 0$. Since $U_n B^p \in \check{\mathscr{A}}$ it follows by Theorem 4.2 that f is $\check{\mathscr{A}}$-Schur convex, hence inequality (4.6) holds.

Applying Lemma 4.2, it follows $(U_n \widetilde{A}^p)^q > O_n$, for some positive integer q, therefore $\lim_{m\to\infty} (U_n \widetilde{A}^p)^m = \frac{1}{n} ee^T$. Because the inequality $f(U_n \widetilde{A}^p x) \leq f(x)$, for every $x \in D^n$, and the continuity of function f, we obtain $f(\frac{1}{n} ee^T x) \leq f(x)$, for every $x \in D^n$, and the conclusion follows. □

Remark 4.4 The condition (1) in the above theorem is satisfied if $A^p > O_n$ for some positive integer p.

Corollary 4.1 *Let $n \geq 3$ be a natural number, E be a Hausdorff topological vector space and D be a convex subset of E^n. Consider a continuous function $f : D \to \mathbb{R}$ satisfying the following properties:*

(1) For every $(x_1, x_2, \ldots, x_n) \in D$, the following inequality holds:

$$f\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n\right) \leq f(x_1, x_2, \ldots, x_n); \tag{4.7}$$

(2) D is circular symmetric invariant and f is circular symmetric.

Then for every $(x_1, x_2, \ldots, x_n) \in D$ inequality (4.6) holds.

Proof Take in Theorem 4.3, $r = 2$, $A = T_r$. □

Corollary 4.2 *Let $n \geq 3$ and $2 \leq r \leq n-1$ be positive integers, E be a Hausdorff topological vector space and D be a convex subset of E^n. Consider a continuous function $f : D \to \mathbb{R}$ satisfying the following properties:*

(1) For every $(x_1, x_2, \ldots, x_n) \in D$, the following inequality holds:

$$f\left(\frac{x_1 + x_2 + \ldots + x_r}{r}, \ldots, \frac{x_1 + x_2 + \ldots + x_r}{r}, x_{r+1}, \ldots, x_n\right) \leq f(x_1, x_2, \ldots, x_n)$$

(2) D is circular symmetric invariant and f is circular symmetric.

Then for every $(x_1, x_2, \ldots, x_n) \in D$ inequality (4.6) holds.

Theorem 4.4 *Let $n \geq 3$ be a positive integer, E, F be two Hausdorff topological linear spaces, D_1 be a convex subset of E, D_2 be a convex subset of F, and let $\phi : D_1 \to D_2$ be a bijective homeomorphism. Consider a continuous function $f : D_1^n \to \mathbb{R}$ satisfying the following properties:*

(1) For every $(x_1, x_2, \ldots, x_n) \in D_1^n$, the inequality holds

$$f(\phi^{-1}\left(\frac{\phi(x_1) + \phi(x_2)}{2}\right), \phi^{-1}\left(\frac{\phi(x_1) + \phi(x_2)}{2}\right), x_3, \ldots, x_n) \leq f(x_1, x_2, \ldots, x_n);$$

(2) f is circular symmetric.

Then for every $(x_1, x_2, \ldots, x_n) \in D$ inequality (4.6) holds with

$$\overline{x} = \phi^{-1}\left(\frac{\phi(x_1) + \phi(x_2) + \ldots + \phi(x_n)}{2}\right). \tag{4.8}$$

Proof Consider the function

$$g(y_1, y_2, \ldots, y_n) = f\left(\phi^{-1}(y_1), \phi^{-1}(y_2), \ldots, \phi^{-1}(y_n)\right), \quad (y_1, y_2, \ldots, y_n) \in D_2^n$$

Let $x_i = \phi^{-1}(y_1)$, $i = 1, 2, \ldots, n$. Note that

$$g\left(\frac{y_1 + y_2}{2}, \frac{y_1 + y_2}{2}, y_3, \ldots, y_n\right)$$

$$= g\left(\frac{\phi(x_1) + \phi(x_2)}{2}, \frac{\phi(x_1) + \phi(x_2)}{2}, \phi(x_3), \ldots, \phi(x_n)\right)$$

$$= f\left(\phi^{-1}\left(\frac{\phi(x_1)+\phi(x_2)}{2}\right), \phi^{-1}\left(\frac{\phi(x_1)+\phi(x_2)}{2}\right), x_3, \ldots, x_n\right)$$

$$\leq f(x_1, x_2, \ldots, x_n) = g(y_1, y_2, \ldots, y_n).$$

By Corollary 4.2 we obtain that for every $(y_1, y_2, \ldots, y_n) \in D_2^n$ the following inequality holds:

$$g(\overline{y}, \overline{y}, \ldots, \overline{y}) \leq g(y_1, y_2, \ldots, y_n), \tag{4.9}$$

where

$$\overline{y} = \frac{y_1 + y_2 + \ldots + y_n}{n}$$

Inequality (4.9) is equivalent with inequality (4.6) with $\overline{x}$ defined by (4.8). □

Corollary 4.3 *Let $n \geq 3$ be a positive integer. Consider the continuous function $f : (0, \infty)^n \to \mathbb{R}$ satisfying the following properties:*

(1) For every $x_1, x_2, \ldots, x_n \in (0, \infty)$, the following inequality holds:

$$f\left(\sqrt{x_1x_2}, \sqrt{x_1x_2}, x_3, \ldots, x_n\right) \leq f(x_1, x_2, \ldots, x_n);$$

(2) f is circular symmetric.

Then for every $x_1, x_2, \ldots, x_n \in (0, \infty)$ the following inequality holds:

$$f(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x}) \leq f(x_1, x_2, \ldots, x_n),$$

where $\widetilde{x} = \sqrt[n]{x_1x_2 \cdots x_n}$.

Proof We apply the preceding theorem for $\phi(t) = \ln t$, $t \in (0, \infty)$. □

Corollary 4.4 *Let $n \geq 3$ be a positive integer and $p \in \mathbb{R} - \{0\}$. Consider a continuous function $f : (0, \infty)^n \to \mathbb{R}$ satisfying the following properties:*

(1) For every $x_1, x_2, \ldots, x_n \in (0, \infty)$, the following inequality holds:

$$f\left(\left(\frac{x_1^p + x_2^p}{2}\right)^{1/p}, \left(\frac{x_1^p + x_2^p}{2}\right)^{1/p}, x_3, \ldots, x_n\right) \leq f(x_1, x_2, \ldots, x_n);$$

(2) f is circular symmetric.

Then for every $x_1, x_2, \ldots, x_n \in (0, \infty)$ the following inequality holds:

$$f(\widetilde{x}, \widetilde{x}, \ldots, \widetilde{x}) \leq f(x_1, x_2, \ldots, x_n),$$

where

$$\widetilde{x} = \left(\frac{x_1^p + x_2^p + \ldots + x_n^p}{n} \right)^{1/p}$$

Proof We apply Theorem 4.4 for $\phi(t) = t^p$, $t \in (0, \infty)$. □

4.2 Examples and Applications

In the following is given an example of a function circular symmetric which is not symmetric but it satisfies the desired inequality.

Example 4.1 Let $(E, <, >)$ be an inner product space and let $f : E^4 \to \mathbb{R}$ be the function defined by

$$f(x_1, x_2, x_3, x_4) = [(\langle x_1, x_3\rangle - \langle x_2, x_4\rangle)^2 - (\langle x_2, x_3\rangle - \langle x_1, x_4\rangle)^2]^2 \cdot [(\langle x_1, x_3\rangle - \langle x_2, x_4\rangle)^2 - (\langle x_1, x_2\rangle - \langle x_3, x_4\rangle)^2]^2.$$

The function f is not symmetric, it is circular symmetric and it satisfies the inequality

$$f\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, x_4\right) \le f(x_1, x_2, x_3, x_4),$$

for every $x_1, x_2, x_3, x_4 \in E$.

Proposition 4.2 *Let* $\phi : (0, \infty) \to \mathbb{R}$ *be a convex function. Consider the function:*

$$f(x_1, x_2, \ldots, x_n) = \frac{\phi\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} \phi(x_i)}{\sum_{i=1}^{n} x_i}, \quad (x_1, x_2, \ldots, x_n) \in (0, \infty)^n.$$

Then the inequality (4.6) holds where $\overline{x}$ *is defined by (4.4).*

Proof Note that for every $(x_1, x_2, \ldots, x_n) \in (0, \infty)^n$ the following inequality holds:

$$f\left(\frac{x_1 + x_2}{2}, \frac{x_1 + x_2}{2}, x_3, \ldots, x_n\right) \le f(x_1, x_2, \ldots, x_n)$$

By Corollary 4.1 it follows that the inequality (4.6) holds. □

The following corollary is useful in the estimation of graphs entry (see the paper of Eliasi [23]).

Corollary 4.5 *The following inequality holds:*

$$0 \leq \ln\left(\sum_{i=1}^{n} x_i\right) - \frac{\sum\limits_{i=1}^{n} x_i \ln x_i}{\sum\limits_{i=1}^{n} x_i} \leq \ln n, \quad (x_1, x_2, \ldots, x_n) \in (0, \infty)^n . \tag{4.10}$$

Proof Let $\phi(t) = t \ln t,\ \ t > 0$. Note that $\phi''(t) = \frac{1}{t},\ \ t > 0,$ hence ϕ is convex. Consider the function

$$f(x_1, x_2, \ldots, x_n) = \ln\left(\sum_{i=1}^{n} x_i\right) - \frac{\sum\limits_{i=1}^{n} x_i \ln x_i}{\sum\limits_{i=1}^{n} x_i}$$

Note that

$$f(\overline{x}, \overline{x}, \ldots, \overline{x}) = \ln(n\overline{x}) - \frac{n\overline{x}\ln(\overline{x})}{n\overline{x}} = \ln n$$

By Proposition 4.2 the inequality (4.10) holds. □

Lemma 4.3 *Let $p \in [2, \infty)$. Then for every $x \in [1, \infty)$ the following inequality holds:*

$$p(x+1)^{p-1}(x-1) \leq 2^{p-1}(x^p - 1). \tag{4.11}$$

Proof Considering the function

$$h(x) = 2^{p-1}(x^p - 1) - p(x+1)^{p-1}(x-1),\ x \in [1, \infty),$$

we have

$$h'(x) = p[2^{p-1}x^{p-1} - (x+1)^{p-2}(px + 2 - p)].$$

Because $h'(1) = 0$, in order to prove the desired inequality it suffices to show that $h'(x) \geq 0$, $x \in [1, \infty)$. Applying the AM-GM inequality, it follows

$$(x+1)^{p-2}(px+2-p) \leq \left(\frac{(p-2)(x+1) + px + 2 - p}{p-1}\right)^{p-1} = (2x)^{p-1},\ x \in [1, \infty)$$

One can easily see that the preceding inequality is equivalent to $h'(x) \geq 0$, for every $x \in [1, \infty)$. □

Lemma 4.4 *Let $p \in [2, \infty)$. Then the following inequalities hold:*

(1) $p \cosh^{p-1} t \cdot \sinh t \le \sinh(pt), t \in \mathbb{R}_+$;
(2) $p(\cosh^p t - 1) \le \cosh(pt) - 1, t \in \mathbb{R}$.

Proof Dividing by $x^{\frac{p}{2}}$, from the inequality (4.11) we obtain

$$p\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^{p-1}\left(\sqrt{x}-\frac{1}{\sqrt{x}}\right) \le 2^{p-1}\left(x^{\frac{p}{2}}-\frac{1}{x^{\frac{p}{2}}}\right), \ x \in [1,\infty).$$

The inequality (1) is obtained from the preceding inequality with the notation

$$\sqrt{x} = e^t.$$

In order to prove (2), we use the inequality (1) and observe that for every $t \in \mathbb{R}_+$, we have

$$p(\cosh^p t - 1) = p\int_0^t p\cosh^{p-1} s \cdot \sinh s ds \le p\int_0^t \sinh(ps)ds = \cosh(pt) - 1.$$

Because the function cosh is even, the desired inequality follows. □

Lemma 4.5 *Let $p \in [2, \infty)$. Then for every $x, y \in (0, \infty)$, the following inequality holds:*

$$(p+1)2^{p+1}(x^{2p}+y^{2p}) - 8p(x^p+y^p)(x+y)^p + 2^{p+2}(3p-1)x^p y^p \ge 0. \quad (4.12)$$

Proof With $z = \frac{x}{y}$ the inequality (4.12) is equivalent to

$$(p+1)2^{p+1}\left(z^p+\frac{1}{z^p}\right) - 8p\left(\sqrt{z}^p+\frac{1}{\sqrt{z}^p}\right)\left(\sqrt{z}+\frac{1}{\sqrt{z}}\right) + 2^{p+2}(3p-1) \ge 0,$$

for every $z \in (0, \infty)$. Denote $\sqrt{z} = e^t$, $t \in \mathbb{R}$, and write the inequality above in the equivalent form

$$(p+1)2^{p+1}(2\cosh 2t) - 8p(2\cosh pt)(2\cosh t)^p + 2^{p+2}(3p-1) \ge 0, \ t \in \mathbb{R}.$$

The last inequality is equivalent to

$$(p+1)2^{p+2}(2\cosh 2pt) - 2^{p+4}p(2\cosh pt)(\cosh t)^p + 2^{p+2}(3p-1) \ge 0, \ t \in \mathbb{R}.$$

Using the well-known formula $\cosh 2t = 2\cosh t^2 - 1$, $t \in \mathbb{R}$, we obtain

$$(p+1)\cosh^2(pt) - 2p\cosh pt(\cosh t)^p \cosh^p t + p - 1 \ge 0, \ t \in \mathbb{R}.$$

From Lemma 4.4 we have $\cosh(pt) \geq 1 - p + p \cosh t^p$, hence we can write

$$(p+1)\cosh^2(pt) - 2p\cosh pt(\cosh t)^p \cosh^p t + p - 1$$

$$= (p+1)\left[\left(\cosh(pt) - \frac{p}{p+1}\cosh^p t\right)^2 - \frac{p^2}{(p+1)^2}\cosh^{2p} t + \frac{p^2-1}{(p+1)^2}\right]$$

$$\geq (p+1)\left[\left(1 - p + p\cosh^p t - \frac{p}{p+1}\cosh^p t\right)^2 - \frac{p^2}{(p+1)^2}\cosh^{2p} t + \frac{p^2-1}{(p+1)^2}\right]$$

$$\geq (p+1)\left[\left(\frac{1 - p^2(\cosh^p t - 1)}{p+1}\right)^2 - \frac{p^2}{(p+1)^2}\cosh^{2p} t + \frac{p^2-1}{(p+1)^2}\right]$$

$$= \frac{1}{p+1}(p^4 - p^2)(\cosh^p t - 1)^2 \geq 0,\ t \in \mathbb{R},$$

and we are done. □

The following result is an interpolation to the power mean inequality.

Theorem 4.5 *Let $p \in [2, \infty)$. For every $x = (x_1, \ldots, x_n) \in (0, \infty)^n$, the following inequality holds:*

$$A_n(x)^p \leq \frac{(p+1)n^2}{(p+1)n^2 + 4p - 4} A_n(x^p) + \frac{4p-4}{(p+1)n^2 + 4p - 4} H_n(x^p) \leq A_n(x^p),$$

where $x^p = (x_1^p, \ldots, x_n^p) \in (0, \infty)^n$, and $A_n(x)$ and $H_n(x)$ are the arithmetic mean and harmonic mean of the entries of vector x, respectively, that is

$$A_n(x) = \frac{x_1 + x_2 + \ldots + x_n}{n}$$
$$H_n(x) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}.$$

Proof Denote

$$u(x, y) = (p+1)2^{p+1}(x^{2p} + y^{2p}) - 8p(x^p + y^p)(x+y)^p + 2^{p+2}(3p-1)x^p y^p.$$

For $n = 1$, the inequality from the statement is obvious.
For $n = 2$, using the preceding Lemma 4.5, we have

$$\frac{4p+4}{8p} A_2(x^p) + \frac{4p+4}{8p} H_2(x^p) - A_2(x)^p = \frac{u(x_1, x_2)}{2^{p-3} p(x_1^p + x_2^p)} \geq 0.$$

For $n \geq 3$, consider the function $f : (0, \infty)^n \to \mathbb{R}$, defined by

$$f(x) = \frac{(p+1)n^2}{(p+1)n^2 + 4p - 4} A_n(x^p) + \frac{4p-4}{(p+1)n^2 + 4p - 4} H_n(x^p) - A_n(x)^p.$$

If $x = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$ and $B = \sum_{i=3}^{n} \frac{1}{x_i^p}$, then

$$H_n(x_1^p, x_2^p, \ldots, x_n^p) - H_n\left(\left(\frac{x_1+x_2}{2}\right)^p, \left(\frac{x_1+x_2}{2}\right)^p, x_3^p, \ldots, x_n^p\right)$$

$$= \frac{n}{\frac{1}{x_1^p} + \frac{1}{x_2^p} + B} - \frac{n}{\frac{2}{(\frac{x_1+x_2}{2})^p} + B} = \frac{\frac{2^{p+1}}{(x_1+x_2)^p} - \frac{1}{x_1^p} - \frac{1}{x_2^p}}{\left(\frac{1}{x_1^p} + \frac{1}{x_2^p} + B\right)\left(\frac{2^{p+1}}{(x_1+x_2)^p} + B\right)}$$

$$\geq \frac{\frac{2^{p+1}}{(x_1+x_2)^p} - \frac{1}{x_1^p} - \frac{1}{x_2^p}}{(\frac{1}{x_1^p} + \frac{1}{x_2^p})\frac{2^{p+1}}{(x_1+x_2)^p}},$$

therefore, in our case

$$f(x_1, x_2, \ldots, x_n) - f\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right)$$

$$= \frac{(p+1)n^2}{(p+1)n^2 + 4p - 4}\left[A_n(x^p) - A_n\left(\left(\frac{x_1+x_2}{2}\right)^p, \left(\frac{x_1+x_2}{2}\right)^p, x_3^p, \ldots, x_n^p\right)\right]$$

$$+ \frac{4p-4}{(p+1)n^2 + 4p - 4}\left[H_n(x^p) - H_n\left(\left(\frac{x_1+x_2}{2}\right)^p, \left(\frac{x_1+x_2}{2}\right)^p, x_3^p, \ldots, x_n^p\right)\right]$$

$$-\left[A_n(x)^p - A_n\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \ldots, x_n\right)\right]$$

$$\geq \frac{(p+1)n}{(p+1)n^2 + 4p - 4} \cdot \frac{2^{p-1}(x_1^p + x_2^p) - (x_1+x_2)^p}{2^{p-1}}$$

$$+ \frac{(4p-4)n}{(p+1)n^2 + 4p - 4} \cdot \frac{\frac{2^{p+1}}{(x_1+x_2)^p} - \frac{1}{x_1^p} - \frac{1}{x_2^p}}{\left(\frac{1}{x_1^p} + \frac{1}{x_2^p}\right)\frac{2^{p+1}}{(x_1+x_2)^p}}$$

$$= \frac{(p+1)n}{(p+1)n^2 + 4p - 4} \cdot \frac{u(x_1, x_2)}{2^{p+1}(x_1^p + x_2^p)} \geq 0.$$

Because the function f is circular symmetric, applying Corollary 4.2, we obtain that for every $x = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$, the inequality (4.6) holds. □

Corollary 4.6 *For every real numbers $x, y, z \geq 0$, the following inequality holds:*

$$(x + y)^2(y + z)^2(z + x)^2 \geq 4(x^2 + yz)(y^2 + zx)(z^2 + xy) + 32x^2y^2z^2.$$

Proof To prove the above inequality, we consider the function $f : [0, \infty)^3 \to \mathbb{R}$, defined by

$$f(x, y, z) = (x+y)^2(y+z)^2(z+x)^2 - 4(x^2+yz)(y^2+zx)(z^2+xy) - 32x^2y^2z^2.$$

Clearly, the function f is circular symmetric and, after a simple computation, we have

$$f(x, y, z) - f(\sqrt{xy}, \sqrt{xy}, z)$$

$$= (\sqrt{x} - \sqrt{y})^2[(\sqrt{x} + \sqrt{y})^2(4xyz^2 + (x - z)^2(y - z)^2 + 4xyz(xy + z^2))] \geq 0.$$

From Corollary 4.3 it follows that for every real numbers $x, y, z \geq 0$, the inequality

$$f(x, y, z) \geq f(\sqrt[3]{xyz}, \sqrt[3]{xyz}, \sqrt[3]{xyz})$$

holds. Since $f(\sqrt[3]{xyz}, \sqrt[3]{xyz}, \sqrt[3]{xyz}) = 0$ the inequality from the statement holds. □

Theorem 4.6 *The following inequality holds:*

$$(x + y + z)^6 \geq 27\left(x^2 + y^2 + z^2\right)(xy + yz + zx)^2, \qquad x, y, z \geq 0.$$

Proof Consider the function

$$f(x, y, z) = \frac{\left(x^2 + y^2 + z^2\right)(xy + yz + zx)^2}{(x + y + z)^6}, \qquad (x, y, z) \in (0, \infty)^3.$$

Denoting $x + y = s$ and $xy = p$, we obtain

$$f(x, y, z) - f\left(\frac{x + y}{2}, \frac{x + y}{2}, z\right) = \frac{\left(x^2 + y^2 + z^2\right)(xy + yz + zx)^2}{(x + y + z)^6} -$$

$$-\frac{\left(\frac{s^2}{2} + z^2\right)\left(sz + \frac{s^2}{4}\right)^2}{(x + y + z)^6} =$$

$$= \frac{4p - s^2}{32(s + z)^6}\left[-16p^2 + p\left(8z^2 - 32sz + 4s^2\right) + 16sz^3 - 14s^2z^2 + 8s^3z + s^4\right].$$

Consider the function $g : \left[0, \frac{s^2}{4}\right] \to \mathbb{R}$,

$$g(t) = -16t^2 + t\left(8z^2 - 32sz + 4s^2\right) + 16sz^3 - 14s^2z^2 + 8s^3z + s^4, \quad t \in \left[0, \frac{s^2}{4}\right].$$

Note that g is concave, hence

$$g(t) \geq \min\left(g(0), g\left(\frac{s^2}{4}\right)\right), \quad t \in \left[0, \frac{s^2}{4}\right].$$

Note that

$$g(0) = s\left(16z^3 - 14sz^2 + 8s^2z + s^3\right) = 2sz\left(8z^2 - 7sz + 4s^2\right) + s^4 \geq 0,$$

and

$$g\left(\frac{s^2}{4}\right) = 16sz^3 - 12s^2z^2 + s^4.$$

If $h(t) = 16t^3 - 12st^2 + s^4, \; t \in [0, \infty)$, then observe that $h'(t) \leq 0$ for $t \in \left[0, \frac{s}{2}\right]$, and $h'(t) \geq 0$ for $t \in [\frac{s}{2}, \infty)$. Therefore, we have $h(t) \geq h\left(\frac{s}{2}\right) = 0, \quad t \in [0, \infty)$. Thus $g\left(\frac{s^2}{4}\right) = sh(z) \geq 0$. Consequently, $g(t) \geq 0$ for every $t \in \left[0, \frac{s^2}{4}\right]$. Finally we obtain

$$f(x, y, z) \leq f\left(\frac{x+y}{2}, \frac{x+y}{2}, z\right), \quad (x, y, z) \in (0, \infty)^3.$$

By Corollary 4.1 we obtain

$$f(x, y, z) \leq f(a, a, a) = f(1, 1, 1) = \frac{1}{27}$$

where $\frac{x+y+z}{3} = a$, and the inequality in the statement is proved. □

5 Convexity Properties of the Multivariate Monomial

Let $n \geq 2, \; \mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Suppose that

$$a_i \neq 0 \text{ for every } i \in \{1, 2, \ldots, n\}. \tag{5.1}$$

The multivariate monomial in n variables of exponent $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ is defined as follows

$$f_{\mathbf{a}}(x_1, x_2, \ldots, x_n) = x_1^{a_1} x_2^{a_2} \ldots . x_n^{a_n}, \quad x_1, x_2, \ldots, x_n \in (0, \infty). \tag{5.2}$$

Without loss of generality we may suppose that

$$a_1 \geq a_2 \geq \ldots \geq a_n. \tag{5.3}$$

The multivariate monomial is connected with the Cobb–Douglas production function. In 1928, Cobb and Douglas [17] introduced a famous two-factor production function, nowadays called Cobb–Douglas production function, in order to describe the distribution of the national income by help of production functions. The production function is widely used to represent the technological relationship between the amounts of two or more inputs (particularly physical capital and labor) and the amount of output that can be produced by those inputs. The Cobb-Douglas function was developed and tested against statistical evidence by Charles Cobb and Paul Douglas during 1927–1947. In its most standard form for production of a single good with two factors, the function is

$$Y = AL^{\beta}K^{\alpha}$$

where:

- Y = total production (the real value of all goods produced in a year or 365.25 days)
- L = labor input (the total number of person-hours worked in a year or 365.25 days)
- K = capital input (the real value of all machinery, equipment, and buildings)
- A = total factor productivity and your usual depreciation by utility in day after
- α and β are the output elasticities of capital and labor, respectively. These values are constants determined by available technology.

In its generalized form, the Cobb–Douglas function models more than two goods. The generalized Cobb–Douglas function (see Chen [16]) may be written as

$$u_{\mathbf{a}}(x_1, x_2, \ldots, x_n) = Ax_1^{a_1} x_2^{a_2} \ldots . x_n^{a_n}, \quad x_1, x_2, \ldots, x_n \in [0, \infty)$$

where:

- A is an efficiency parameter
- n is the total number of goods
- $x_1, x_2, \ldots, x_n$ are the (nonnegative) quantities of good consumed, produced, etc.
- a_i is an elasticity parameter for good i

The Cobb–Douglas function is often used as a utility function. If the consumer has a finite wealth, the utility maximization takes the form:

$$\max_x u_a(x) = \max_x \prod_{i=1}^{n} x_i^{a_i} \quad \text{subject to the constraint} \quad \sum_{i=1}^{n} p_i x_i = w$$

where w is the wealth of the consumer and p_i are the prices of the goods. Other inequalities related to rearrangements of powers and symmetric polynomials are given in the paper [28].

In the following subsection we shall compute a determinant of a matrix which is necessary for determination of conditions that should be satisfied by the parameter $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ such that the multivariate monomial $f_{\mathbf{a}}$ be a convex, concave, logarithmically convex, logarithmically concave, quasi-convex, quasi-concave, subadditive or superadditive function. Conditions for convexity of $f_{\mathbf{a}}$ may be found in Crouzeix [18]. The proof given in the third subsection is different from the Crouzeix's proof from [18]. The convexity and concavity necessary and sufficient conditions for the multivariate monomial may be stated simply as follows. The multivariate monomial is convex if and only if all the exponents are negative or one exponent is positive, the rest of exponents are negative and the sum of all exponents is greater or equal than one.

The multivariate monomial is concave if and only if all the exponents are positive and the sum of all exponents is smaller or equal than one.

In the third section we shall suppose that all the exponents are non-null and are ranked in decreasing order.

5.1 *Computation of a Determinant*

The following lemma is known as the matrix determinant lemma.

Lemma 5.1 *Let* $\mathbf{A}$ *be a square matrix of dimension* n, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then*

$$\det\left(\mathbf{A} + \mathbf{u}\mathbf{v}^T\right) = \det(\mathbf{A}) + \mathbf{v}^T adj(\mathbf{A})\,\mathbf{u}$$

where $adj(\mathbf{A})$ *is the adjugate of the matrix* $\mathbf{A}$.

Lemma 5.2 *Let* $\mathbf{B} = (b_{ij})$ *be a square matrix of dimension* n *and consider* $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. *Suppose that* $b_{ij} = a_i(a_i - 1)$ *if* $i = j$ *and* $b_{ij} = a_i a_j$ *if* $i \neq j$. *Then*

$$\det(\mathbf{B}) = (-1)^n a_1 a_2 \ldots a_n \left(1 - \sum_{i=1}^{n} a_i\right).$$

Proof Note that $\mathbf{B} = \mathbf{a}\mathbf{a}^T - \mathrm{diag}(\mathbf{a})$. Denote $\mathbf{C} = -\mathrm{diag}(\mathbf{a})$. By the preceding lemma we obtain

$$\det(\mathbf{B}) = \det\left(\mathbf{C} + \mathbf{a}\mathbf{a}^T\right) = \det(\mathbf{C}) + \mathbf{a}^T \operatorname{adj}(\mathbf{C})\,\mathbf{a}$$

If $a_i \neq 0$ for all i, then the matrix $\mathbf{C}$ is invertible and we have

$$\operatorname{adj}(\mathbf{C}) = (\det \mathbf{C})\,\mathbf{C}^{-1}.$$

Note that

$$\det(\mathbf{B}) = \det(\mathbf{C}) + \det(\mathbf{C})\left[\mathbf{a}^T \mathbf{C}^{-1}\mathbf{a}\right] = \det(\mathbf{C})\left[1 + \mathbf{a}^T \mathbf{C}^{-1}\,\mathbf{a}\right].$$

Since $\mathbf{a}^T \mathbf{C}^{-1}\mathbf{a} = -\sum_{i=1}^{n} a_i$, it follows that

$$\det(\mathbf{B}) = \det(\mathbf{C})\left(1 - \sum_{i=1}^{n} a_i\right) = (-1)^n\, a_1 a_2 \ldots a_n \left(1 - \sum_{i=1}^{n} a_i\right)$$

□

5.2 Main Results

Let $n \geq 2$ be a positive integer and let $\mathbf{a} = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. In the following we shall suppose that the entries of vector $\mathbf{a}$ satisfy conditions (5.1) and (5.3) and that $f_{\mathbf{a}}$ is the multivariate monomial defined in (5.2).

Theorem 5.1 *The following two assertions are equivalent:*

1° $f_{\mathbf{a}}$ *is a convex function;*

2° $a_1 < 0$ *or* ($a_1 > 0 > a_2$ *and* $\sum_{i=1}^{n} a_i \geq 1$).

Proof One can easily see that the Hessian matrix of $f_{\mathbf{a}}$ is

$$Hf_{\mathbf{a}}(x) = \begin{bmatrix} \frac{a_1(a_1-1)f_a(x)}{x_1^2} & \frac{a_1a_2 f_a(x)}{x_1x_2} & \cdots & \frac{a_1a_n f_a(x)}{x_1x_n} \\ \frac{a_2a_1 f_a(x)}{x_2x_1} & \frac{a_2(a_2-1)f_a(x)}{x_2^2} & \cdots & \frac{a_2a_n f_a(x)}{x_2x_n} \\ \vdots & \vdots & & \vdots \\ \frac{a_na_1 f_a(x)}{a_na_1} & \frac{a_na_2 f_a(x)}{a_na_2} & \cdots & \frac{a_n(a_n-1)f_a(x)}{a_n^2} \end{bmatrix}.$$

Denote by $\mathscr{K}$ the family of all nonempty subsets of the set $\{1, 2, \ldots, n\}$. For every $K \in \mathscr{K}$ consider the submatrix of $Hf_{\mathbf{a}}(x)$ given by

$$H_K f_{\mathbf{a}}(x) = \left(\frac{\partial^2 f_a(x)}{\partial x_i \partial x_j}\right)_{i,j \in K}.$$

Denote $\Delta_k(x) = \det[H_K f_{\mathbf{a}}(x)]$.

For $x \in (0, \infty)^n$ and $K = \{i_1, i_2, \dots, i_k\} \in \mathscr{K}$ consider the diagonal matrix:

$$C_K(x) = \begin{bmatrix} \frac{1}{x_{i_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{x_{i_2}} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{x_{i_k}} \end{bmatrix}.$$

If

$$a_{ij} = \begin{cases} \dfrac{a_i(a_i - 1)}{x_i^2} & \text{if } i = j \\ \dfrac{a_i a_j}{x_i x_j} & \text{if } i \neq j \end{cases}$$

then consider the matrix $A_K(x) = (a_{ij}(x))_{i,j \in K}$.
Let $B = (b_{ij})$ be the matrix defined in the statement of Lemma 5.2. For every $K \in \mathscr{K}$ denote $B_K = (b_{ij})_{i,j \in K}$. Note that $A_K(x) = C_K(x) B_K C_K(x)$. If $k = |K|$, then we have

$$\begin{aligned} \Delta_K(x) &= [f_{\mathbf{a}}(x)]^k \det[A_K(x)] \\ &= [f_{\mathbf{a}}(x)]^k \det[C_K(x) B_K C_K(x)] \\ &= [f_{\mathbf{a}}(x)]^k \det(B_k)[\det(C_K(x))]^2 \\ &= [f_{\mathbf{a}}(x)]^k \det(B_K) \left(\prod_{i \in K} x_i \right)^{-2} \\ &= (-1)^k [f_{\mathbf{a}}(x)]^k \cdot \prod_{i \in K} a_i \left(1 - \sum_{i \in K} a_i \right). \end{aligned}$$

In the following we shall prove the implication $2^\circ \Rightarrow 1^\circ$. Suppose that conditions from assertion 2° hold.
If $1 \notin K$, then $a_i < 0$ for every $i \in K$ and

$$(-1)^k \left(\prod_{i \in K} a_i \right) \left(1 - \sum_{i \in K} a_i \right) > 0.$$

From the above inequality it follows that $\Delta_K(x) \geq 0$.
If $1 \in K$, then we have

$$(-1)^k \prod_{i \in K} a_i = \prod_{i \in K} (-a_i) < 0.$$

Since

$$1-\sum_{i\in K}a_i \le 1-\sum_{i=1}^{n}a_i \le 0$$

it follows that $\Delta_K(x) \ge 0$. Consequently, f_a is a convex function.

Now, we shall prove the implication $1^\circ \Rightarrow 2^\circ$. Suppose that $f_{\mathbf{a}}$ is a convex function. Then

$$\frac{\partial^2 f_{\mathbf{a}}}{\partial x_i^2} = \frac{a_i(a_i-1)f_{\mathbf{a}}(x)}{x_i^2} \ge 0$$

for every $x \in (0,\infty)^n$ and $i \in \{1,2,\ldots,n\}$. Hence, we obtain $a_i(a_i-1) \ge 0$ for every $i \in \{1,2,\ldots,n\}$. Thus $a_i \in (-\infty,0) \cup [1,\infty)$ for every $i \in \{1,2,\ldots,n\}$. If $a_1 < 0$, then the condition from assertion 2° is satisfied.

We shall study now the case $a_1 > 0$.

If $a_1 \ge a_2 > 0$, then from $a_1, a_2 \in (-\infty,0) \cup [1,\infty)$ it follows that $a_1 \ge a_2 \ge 1$. Consider the function

$$g(x_1,x_2) = f_{\mathbf{a}}(x_1,x_2,1,1,\ldots,1) = x_1^{a_1}x_2^{a_2},\ x_1,x_2 > 0.$$

The function g is convex since $f_{\mathbf{a}}$ is convex. The Hessian matrix of g is

$$Hg(x_1,x_2) = \begin{bmatrix} \dfrac{a_1(a_1-1)g(x_1,x_2)}{x_1^2} & \dfrac{a_1a_2g(x_1,x_2)}{a_1a_2} \\ \dfrac{a_1a_2g(x_1,x_2)}{a_1a_2} & \dfrac{a_2(a_2-1)g(x_1,x_2)}{x_2^2} \end{bmatrix}.$$

Note that

$$\det[Hg(x_1,x_2)] = \frac{a_1a_2g^2(x_1,x_2)}{x_1^2x_2^2}(1-a_1-a_2) < 0.$$

The last inequality shows that g is not convex. It follows that the case $a_1 \ge a_2 > 0$ is not possible. We shall consider now the case $a_1 > 0 > a_2$. Since $f_{\mathbf{a}}$ is convex it follows

$$0 \le \det[Hf_{\mathbf{a}}(x)] = \frac{(-1)^n a_1a_2\ldots a_n[f_{\mathbf{a}}(x)]^n\left(1-\sum_{i=1}^{n}a_i\right)}{x_1^2x_2^2\ldots x_n^2}$$

$$= \frac{[f_{\mathbf{a}}(x)]^n \cdot a_1(-a_2)(-a_3)\ldots(-a_n)\left(\sum_{i=1}^{n}a_i - 1\right)}{x_1^2x_2^2\ldots x_n^2},$$

hence

$$\sum_{i=1}^{n} a_i \geq 1.$$

□

Theorem 5.2 *Let $n \geq 2$ be a positive integer and $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Then the following assertions are equivalent:*

$1°$ *$f_{\mathbf{a}}$ is a concave function.*

$2°$ $1 \geq a_1 \geq a_2 \geq \ldots \geq a_n > 0$ *and* $\sum_{i=1}^{n} a_i \leq 1$.

Proof Keeping the notations in the proof of the preceding theorem we have

$$\Delta_K(x) = (-1)^k [f_{\mathbf{a}}(x)]^k \left(\prod_{i \in K} a_i \right) \left(1 - \sum_{i \in K} a_i \right)$$

for every $K \in \mathscr{K}$. We denoted $|K| = k$.

Suppose that the conditions from assertion $2°$ hold. Then for every $K \in \mathscr{K}$ we have

$$\begin{aligned}(-1)^k \Delta_K(x) &= [f_{\mathbf{a}}(x)]^k \left(\prod_{i \in K} a_i \right) \left(1 - \sum_{i \in K} a_i \right) \\ &\geq [f_{\mathbf{a}}(x)]^k \left(\prod_{i=1}^{n} a_i \right) \left(1 - \sum_{i=1}^{n} a_i \right) \geq 0.\end{aligned}$$

Therefore, the function $f_{\mathbf{a}}$ is concave. Thus we have proved the implication $2° \Rightarrow 1°$.

Suppose now that $f_{\mathbf{a}}$ is concave. Then

$$\frac{\partial^2 f_{\mathbf{a}}}{\partial x_i^2} \leq 0 \text{ for every } i \in \{1, 2, \ldots, n\},$$

hence we have

$$a_i(a_i - 1) \leq 0 \text{ for every } i \in \{1, 2, \ldots, n\}.$$

Thus $a_i \in (0, 1]$ for every $i \in \{1, 2, \ldots, n\}$. Because the function $f_{\mathbf{a}}$ is concave it follows that $(-1)^{|K|} \Delta_K(x) \geq 0$ for every $K \in \mathscr{K}$, $x \in (0, \infty)^n$. If $K = \{1, 2, \ldots, n\}$, then the above inequality is equivalent to

$$(-1)^n \Delta_K(x) = [f_{\mathbf{a}}(x)]^n \left(\prod_{i=1}^{n} a_i\right)\left(1 - \sum_{i=1}^{n} a_i\right) \geq 0,$$

hence

$$\sum_{i=1}^{n} a_i \leq 1.$$

Thus the implication $1^\circ \Rightarrow 2^\circ$ is proved. □

Theorem 5.3 *Let E be a linear space, D a convex set of E, $n \geq 2$ a positive integer, $u_i : D \to \mathbb{R}$, $i = 1, 2, \ldots, n$, and*

$$g_{\mathbf{a}}(x) = [u_1(x)]^{a_1}[u_2(x)]^{a_2} \ldots [u_n(x)]^{a_n}, \; x \in D.$$

Then the following assertions hold:

1° *If $a_1 < 0$ and all u_i are concave, then the function $g_{\mathbf{a}}$ is convex.*

2° *If $a_1 > 0 > a_2$, $\sum_{i=1}^{n} a_i \geq 1$, u_1 is convex and $u_2, u_3, \ldots, u_n$ are concave, then the function $g_{\mathbf{a}}$ is convex.*

3° *If $a_i \in (0, 1]$ for every $i \in \{1, 2, \ldots, n\}$, $\sum_{i=1}^{n} a_i \leq 1$, and all u_i are concave, then the function $g_{\mathbf{a}}$ is concave.*

Proof

1° The inequality $a_1 < 0$ implies $a_i < 0$ for every $i \in \{1, 2, \ldots, n\}$, hence $f_{\mathbf{a}}$ is decreasing in each variable. Note also that $f_{\mathbf{a}}$ is convex. Since

$$g_{\mathbf{a}}(x) = f_{\mathbf{a}}(u_1(x), u_2(x), \ldots, u_n(x)), \; x \in D,$$

it follows that $g_{\mathbf{a}}$ is convex.

2° Conditions $a_1 > 0 > a_2$ and $\sum_{i=1}^{n} a_i \geq 1$ imply that $f_{\mathbf{a}}$ is convex. Note that $f_{\mathbf{a}}$ is increasing in the first variable and decreasing in the rest of variables. It follows that $g_{\mathbf{a}}$ is convex.

3° Conditions $a_i \in (0, 1]$, $i \in \{1, 2, \ldots, n\}$, and $\sum_{i=1}^{n} a_i \leq 1$ imply that $f_{\mathbf{a}}$ is concave. Note that $f_{\mathbf{a}}$ is increasing in all variables. Hence $g_{\mathbf{a}}$ is concave.

□

Proposition 5.1 *The following two assertions hold:*

1° *$f_{\mathbf{a}}$ is logarithmically convex if and only if $a_1 < 0$.*
2° *$f_{\mathbf{a}}$ is logarithmically concave if and only if $a_n > 0$.*

Proof The above assertions follow at once from the identity

$$\ln[f_{\mathbf{a}}(x_1, x_2, \ldots, x_n)] = \sum_{i=1}^{n} a_i \ln x_i.$$

□

Theorem 5.4 *The following assertions are equivalent:*

1° $f_{\mathbf{a}}$ *is quasi-convex.*

2° $a_1 < 0$ *or* $(a_1 > 0 > a_2$ *and* $\sum_{i=1}^{n} a_i \geq 0)$.

Proof We shall prove first the implication 1° ⇒ 2°. Suppose that $f_{\mathbf{a}}$ is quasi-convex. We shall consider the following three cases.

Case 1 $a_1 \geq a_2 > 0$. From the inequality

$$f_{\mathbf{a}}\left(\frac{x_1 + y_1}{2}, \frac{x_2 + y_2}{2}, 1, 1, \ldots, 1\right) \leq \max[f_{\mathbf{a}}(x_1, x_2, 1, 1, \ldots, 1), f_{\mathbf{a}}(y_1, y_2, 1, 1, \ldots, 1)]$$

we obtain

$$(x_1 + y_1)^{a_1}(x_2 + y_2)^{a_2} \leq 2^{a_1+a_2} \cdot \max(x_1^{a_1} x_2^{a_2}, y_1^{a_1} y_2^{a_2}),\ x_i, y_i > 0,\ i = 1, 2.$$

If we let $x_1 \to 0$ and $y_2 \to 0$ in the preceding inequality, we obtain

$$y_1^{a_1} x_2^{a_2} \leq 0.$$

This contradicts with $y_1 > 0$ and $x_2 > 0$.

Case 2 $a_1 > 0 > a_2$. We have to check in this case that

$$s = \sum_{i=1}^{n} a_i \geq 0.$$

Since $f_{\mathbf{a}}$ is quasi-convex it follows that

$$f_{\mathbf{a}}\left(\frac{x + y}{2}\right) \leq \max(f_{\mathbf{a}}(x), f_{\mathbf{a}}(y)),\ x, y \in (0, \infty)^n,$$

hence

$$(x_1 + y_1)^{a_1}(x_2 + y_2)^{a_2} \ldots (x_n + y_n)^{a_n} \leq 2^s \cdot \max(x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}, y_1^{a_1} y_2^{a_2} \ldots y_n^{a_n}).$$

If we let $y_1 \to 0$ in the preceding inequality, we obtain

$$x_1^{a_1}(x_2+y_2)^{a_2}(x_3+y_3)^{a_3}\dots(x_n+y_n)^{a_n} \le 2^s x_1^{a_1}x_2^{a_2}\dots x_n^{a_n}.$$

Letting $y_2 \to 0$, $y_3 \to 0$, …, $y_n \to 0$ in the inequality above, we obtain

$$f_{\mathbf{a}}(x_1, x_2, \dots, x_n) \le 2^s f_{\mathbf{a}}(x_1, x_2, \dots, x_n)$$

hence $2^s \ge 1$. Thus $s \ge 0$.

Case 3 $a_1 < 0$. In this case the condition from assertion 2° is verified.

In the following we shall prove the implication 2° ⇒ 1°.

Suppose that assertion 2° holds. If $a_1 < 0$, then $f_{\mathbf{a}}$ is convex (cf. Theorem 3.1), whence $f_{\mathbf{a}}$ is quasi-convex.

Suppose now that $a_1 > 0 > a_2$. We shall study two cases.

Case 1 $s = \sum_{i=1}^{n} a_i > 0$. Let $t \ge \frac{1}{s}$ and observe that

$$\sum_{i=1}^{n} ta_i \ge \sum_{i=1}^{n} \frac{a_i}{s} = 1.$$

By Theorem 5.1, the function $f_{\mathbf{a}}^t = f_{t\mathbf{a}}$ is convex. Hence $f_{\mathbf{a}}^t$ is quasi-convex. Let $u(z) = z^{1/t}$, $z > 0$ and observe that u is increasing and we have $f_{\mathbf{a}} = u \circ f_{\mathbf{a}}^t$. Therefore, the function $f_{\mathbf{a}}$ is quasi-convex.

Case 2 $s = \sum_{i=1}^{n} a_i = 0$. Consider the sequence of vectors $(\mathbf{a}_k)_{k\ge1}$, where

$$\mathbf{a}_k = (a_{k1}, a_{k2}, \dots, a_{kn}),\ a_{ki} = a_i + \frac{1}{k},\ i \in \{1, 2, \dots, n\},\ k \ge 1.$$

Note that if $k \ge \frac{1}{|a_2|}$, then $a_{k2} < 0 < a_{k1}$, and

$$\sum_{i=1}^{n} a_{ki} = s_k = s + \frac{n}{k} > 0.$$

By the preceding case the functions f_{a_k} are quasi-convex for every $k \ge \frac{1}{|a_2|}$. Since $f_{\mathbf{a}}(x) = \lim_{k\to\infty} f_{\mathbf{a}_k}(x)$ for every $x \in (0, \infty)^n$, it follows that $f_{\mathbf{a}}$ is quasi-convex.

□

Theorem 5.5 *The following assertions are equivalent:*

1° $f_{\mathbf{a}}$ *is quasi-concave.*

2° $a_n > 0$ *or* ($a_{n-1} > 0 > a_n$ *and* $\sum_{i=1}^{n} a_i \leq 0$).

Proof The proof of the above statement follows at once from the result below: f is quasi-concave if and only if $1/f$ is quasi-convex. □

Lemma 5.3 *Let* $f, g : (0, \infty)^n \to \mathbb{R}_+$ *be two functions with the following properties:*

$$f(x + y) \leq f(x) + f(y),\ x, y \in (0, \infty)^n$$
$$g(x + y) \leq \min(g(x), g(y)),\ x, y \in (0, \infty)^n.$$

Then the function $h = f \cdot g$ *is sub-additive.*

Proof For every $x, y \in (0, \infty)^n$, we have

$$h(x+y) = f(x+y)g(x+y) \leq (f(x)+f(y))g(x+y) \leq f(x)g(x)+f(y)g(y) = h(x)+h(y).$$

□

Theorem 5.6 *The following assertions are equivalent:*

1° $f_{\mathbf{a}}$ *is sub-additive.*

2° $a_2 < 0$ *and* $\sum_{i=1}^{n} a_i \leq 1$.

Proof In order to prove the implication 1° ⇒ 2°, suppose that $f_{\mathbf{a}}$ is sub-additive and consider $s = \sum_{i=1}^{n} a_i$. It follows

$$2^s = f_{\mathbf{a}}(2, 2, \ldots, 2) \leq 2 f_a(1, 1, \ldots, 1) = 2,$$

hence $s \leq 1$. Suppose that $a_2 > 0$. Letting $y_1 \to 0$ in the inequality

$$(x_1 + y_1)^{a_1}(x_2 + y_2)^{a_2} \ldots (x_n + y_n)^{a_n} \leq x_1^{a_2} x_2^{a_2} \ldots x_n^{a_n} + y_1^{a_1} y_2^{a_2} \ldots y_n^{a_n}$$

we obtain

$$x_1^{a_1}(x_2 + y_2)^{a_2} \ldots (x_n + y_n)^{a_n} \leq x_1^{a_2} x_2^{a_2} \ldots x_n^{a_n}.$$

Letting $y_2 \to \infty$ in the preceding inequality we obtain a contradiction. Hence $a_2 < 0$.

In the following we shall prove the implication $2^\circ \Rightarrow 1^\circ$.

Case 1 $a_1 < 0$. Then $f_{\mathbf{a}}(x+y) \le f_{\mathbf{a}}(x) \le f_{\mathbf{a}}(x) + f_{\mathbf{a}}(y)$, hence $f_{\mathbf{a}}$ is sub-additive.
Case 2 $a_1 \in (0, 1)$. Consider the functions:

$$g(x_1, x_2, \dots, x_n) = x_1^{a_1},\ \ h(x_1, x_2, \dots, x_n) = x_2^{a_2} x_3^{a_3} \dots x_n^{a_n}.$$

Note that $f_{\mathbf{a}} = g \cdot h$, $g > 0$, is sub-additive and

$$h(x+y) \le \min(h(x), h(y)),\ \ x, y \in (0, \infty)^n \tag{5.4}$$

By the preceding lemma $f_{\mathbf{a}}$ is sub-additive.
Case 3 $a_1 \in [1, \infty)$. Let $t = \dfrac{a_1 - 1}{-a_2 - a_3 - \dots - a_n}$ and note that $t \in (0, 1]$ and we have

$$a_1 + t(a_2 + a_3 + \dots + a_n) = 1.$$

Consider the functions:

$$g(x_1, x_2, \dots, x_n) = x_1^{a_1} x_2^{ta_2} \dots x_n^{ta_n}$$
$$h(x_1, x_2, \dots, x_n) = x_2^{(1-t)a_2} x_3^{(1-t)a^3} \dots x_n^{(1-t)a_n}.$$

Note that $f_{\mathbf{a}} = g \cdot h$ and h satisfies the condition (5.4). From Theorem 5.1 it follows that g is convex. The function g is 1-homogeneous, hence g is sub-additive. By the preceding lemma $f_{\mathbf{a}}$ is sub-additive.

□

Theorem 5.7 *The following two assertions are equivalent:*

1° $f_{\mathbf{a}}$ *is super-additive.*

2° $a_n > 0$ *and* $\sum_{i=1}^{n} a_i \ge 1$.

Proof In order to prove the implication $1^\circ \Rightarrow 2^\circ$ suppose that $f_{\mathbf{a}}$ is super-additive. Then $f_{\mathbf{a}}(2x) \ge 2 f_{\mathbf{a}}(x)$ for every $x \in (0, \infty)^n$. If we replace $x_1 = x_2 = \dots = x_n = 1$ in the preceding inequality we obtain $2^s \ge 2$, where $s = \sum_{i=1}^{n} a_i$. Hence $s \ge 1$. Suppose now that $a_n < 0$ and let $y_n \to 0$ in the inequality

$$f_{\mathbf{a}}(x_1 + y_1, x_2 + y_2, \dots, x_n + y_n) \ge f_{\mathbf{a}}(x_1, x_2, \dots, x_n) + f_{\mathbf{a}}(y_1, y_2, \dots, y_n).$$

We obtain a contradiction. Hence $a_n > 0$.

In the following we shall prove the implication $2^\circ \Rightarrow 1^\circ$.

If $s = \sum_{i=1}^{n} a_i$, then $g_{\mathbf{a}} = (f_{\mathbf{a}})^{1/s} = f_{\mathbf{a}/s}$. Note that $a_i/s \in (0, 1)$ for every $i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^{n} \frac{a_i}{s} = 1$. By Theorem 5.2 it follows that $g_{\mathbf{a}}$ is concave. Note that

$$\begin{aligned} f_{\mathbf{a}}(x+y) &= f_{\mathbf{a}}\left(2 \cdot \frac{x+y}{2}\right) = 2^s f_{\mathbf{a}}\left(\frac{x+y}{2}\right) \\ &= 2^s \cdot g_{\mathbf{a}}^s\left(\frac{x+y}{2}\right) \geq 2^s \cdot \left(\frac{g_{\mathbf{a}}(x) + g_{\mathbf{a}}(y)}{2}\right)^s \\ &= (g_{\mathbf{a}}(x) + \mathbf{a}_a(y))^s \geq g_{\mathbf{a}}^s(x) + g_{\mathbf{a}}^s(y) \\ &= f_{\mathbf{a}}(x) + f_{\mathbf{a}}(y), \end{aligned}$$

that is $f_{\mathbf{a}}$ is super-additive. □

6 On the Class of n-Schur Functions

The following inequality is known as the Schur inequality.

Theorem 6.1 *Let x, y, z be nonnegative real numbers. Then for every $r > 0$ the following inequality holds:*

$$x^r (x-y)(x-z) + y^r (y-z)(y-x) + z^r (z-x)(z-y) \geq 0 \tag{6.1}$$

Equality holds if and only if $x = y = z$ or if two of x, y, z are equal and the third is zero.

In case the exponent r is an even number, then inequality (6.1) holds for every x, y, z real numbers.

One of the reasons for which Schur's inequality is studied is related to its applications to geometric programming. Geometric programming is a part of nonlinear programming where both the objective function and constraints are polynomials with positive coefficients (posynomials), that is

$$P(x_1, x_2, \ldots, x_n) = \sum_{|\alpha| \leq m} a_\alpha x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a n-dimensional vector with components natural numbers, $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$, and all coefficients a_α are nonnegative numbers.

Expanding terms in (6.1) we get

$$\sum x^{r+2} + xyz\left(\sum x^{r-1}\right) \geq \sum x^{r+1}y + \sum x^{r+1}z,$$

therefore Schur's inequality is equivalent to an inequality between two posynomials.

Starting from Schur's inequality one can introduce the class of 3-Schur functions.

Definition 6.1 Let D be a subset of $\mathbb{R}$ containing at least two elements and $f : D \to \mathbb{R}$ be a map. Denote by $S_3(f, x, y, z)$ the sum

$$f(x)(x - y)(x - z) + f(y)(y - z)(y - x) + f(z)(z - x)(z - y).$$

We shall say that a function $f : D \to \mathbb{R}$ belongs to the class $\mathscr{S}_3(D)$ of Schur functions if the following inequality holds:

$$S_3(f, x, y, z) \geq 0 \text{ for every } x, y, z \in D. \tag{6.2}$$

One can easily see that all the functions from $\mathscr{S}_3(D)$ are nonnegative. A more general definition of a Schur class of functions is given below.

Let D be a subset of the real axis $\mathbb{R}$ which has more than two elements and let $D_0 = \{x \in D : \text{there exist } x_1, x_2 \in D \text{ such that } x_1 < x < x_2\}$. For every map $f : D \to \mathbb{R}$ and a positive integer $n \geq 2$, denote

$$S_n(f, x_1, x_2, \ldots, x_n) = \sum f(x_1)(x_1 - x_2)(x_1 - x_3)\ldots(x_1 - x_n)$$

The above sum has n terms and its terms are obtained by circular permutations. For every positive integer $n \geq 2$ we shall say that f is an *n-Schur function* if $S_n(f, x_1, x_2, \ldots, x_n) \geq 0$ for every $x_1, x_2, \ldots, x_n \in D$. For a positive integer $n \geq 2$ denote by $\mathscr{S}_n(D)$ the set of all n-Schur functions, that is

$$\mathscr{S}_n(D) = \{f : D \to \mathbb{R} : S_n(f, x_1, x_2, \ldots, x_n) \geq 0 \text{ for every } x_1, x_2, \ldots, x_n \in D\}$$

One interesting problem connected with the class of n-Schur functions was proposed at the International Mathematical Olympiad in 1971.

Problem 6.1 (IMO 1971 [20]) Prove that the following statement is true for $n = 3$ and for $n = 5$, and false for all other $n > 2$: For any real numbers $a_1, a_2, \ldots, a_n$,

$$(a_1 - a_2)(a_1 - a_3)\ldots(a_1 - a_n) + (a_2 - a_1)(a_2 - a_3)\ldots(a_2 - a_n) + \ldots$$
$$+ (a_n - a_1)(a_n - a_2)\ldots(a_n - a_{n-1}) \geq 0.$$

One can easily see that a reformulation of the above problem is the following: The Problem 6.1 reformulation is the following: The constant function $f(t) = 1, t \in \mathbb{R}$, belongs to $\mathscr{S}_n(\mathbb{R})$ if and only if $n = 3$ or $n = 5$.

A different approach to solve this problem is given in the paper [7].

Our goal is to give characterizations of the functions from $\mathscr{S}_n(D)$. A study of n-Schur functions is made in the second subsection while a detailed study of 3-Schur functions is made in the third section. In the fourth section is studied the class of 5-Schur functions. The fifth section is devoted to the definition of two general classes of functions, that are connected with the class of n-Schur functions.

6.1 A Study of n-Schur Functions

Theorem 6.2 *The following relation holds:*

$$\mathscr{S}_2(D) = \{f : D \to \mathbb{R} : f \text{ is monotone increasing on } D\}.$$

Proof Observe that for every $x, y \in D$ we have the relation $0 \leq S_2(f, x, y) = (f(x) - f(y))(x - y)$. □

Theorem 6.3 *Denote* $a = \inf D$, $b = \sup D$ *and consider the maps* $f_0, f_{c,d} : D \to \mathbb{R}$, $f_0(x) = 0$ *and*

$$f_{c,d}(x) = \begin{cases} 0 & \text{if } x \in D - \{c\} \\ d & \text{if } x = c. \end{cases}$$

Suppose $n \geq 4$ *is an even number and* $f \in \mathscr{S}_n(D)$. *Then the following assertions hold:*

1° *If* $a \notin D$, *then* $f(x) \geq 0$ *for every* $x \in D$.
2° *If* $b \notin D$, *then* $f(x) \leq 0$ *for every* $x \in D$.
3° $f(x) = 0$ *for every* $x \in D_0$.
4° *If* $a \notin D$ *and* $b \notin D$ *then* $\mathscr{S}_n(D) = \{f_0\}$.
5° *If* $a \notin D$ *and* $b \in D$, *then* $\mathscr{S}_n(D) = \left\{f_{b,d} : d \geq 0\right\}$.
6° *If* $a \in D$ *and* $b \notin D$, *then* $\mathscr{S}_n(D) = \left\{f_{a,d} : d \leq 0\right\}$.
7° *If* $a \in D$ *and* $b \in D$, *then*

$$\mathscr{S}_n(D) = \{g : D \to \mathbb{R} : g(a) \leq 0,\ g(b) \geq 0,\ g(x) = 0 \text{ for every } x \in D - \{a, b\}\}$$

Proof To prove 1° suppose that $a \notin D$ and there exists $x_0 \in D$ such that $f(x_0) < 0$. Take $y \in D$, such that $y < x_0$. One can easily see that

$$0 \leq S_n(f, x_0, y, y, \ldots, y) = f(x_0)(x_0 - y)^{n-1} < 0. \tag{6.3}$$

The contradiction we have obtained shows that $f(x) \geq 0$ for every $x \in D$.

To prove 2° suppose that $b \notin D$ and there exists $x_0 \in D$ such that $f(x_0) > 0$.Take $y \in D$, such that $y > x_0$. One can easily see that (6.1) holds. The contradiction we have obtained shows that $f(x) \leq 0$ for every $x \in D$.

Assertion 3° follows at once from assertions 1° and 2°. If $a \notin D$ and $b \notin D$ then $D = D_0$ hence $\mathscr{S}_n(D) = \mathscr{S}_n(D_0) = \{f_0\}$. We have proved thus 4°. The rest of the assertions follows at once from the preceding assertions. □

Lemma 6.1 *Let $f : D \to \mathbb{R}$, $k \geq 2$, $n \geq k + 2$. For every $t \in D$ consider the map*

$$g_t(x) = f(x)(x - t)^{n-k}, \quad x \in D.$$

If $x_{k+1} = x_{k+2} = \ldots = x_{k+1} = x_n = t$, then

$$S_n\left(f, x_1, x_{2,} \ldots x_k, x_{k+1}, \ldots, x_n\right) = S_n\left(f, x_1, x_{2,} \ldots x_k, t, t, \ldots, t\right) = S_k\left(g_t, x_1, x_{2,} \ldots x_k\right).$$

Proof Obvious. □

Theorem 6.4 *Let $n \geq 3$ be an odd number and $f \in \mathscr{S}_n(D)$. Suppose that D has more than 3 elements. Then the following assertions hold:*

1° $f(x) \geq 0$ *for every* $x \in D$.
2° *If* $n \geq 7$, *then* $f(x) = 0$ *for every* $x \in D_0$.
3° *If* $\inf D = -\infty$, *then* f *is increasing.*
4° *If* $\sup D = +\infty$, *then* f *is decreasing.*
5° *If* $\inf D = -\infty$ *and* $\sup D = +\infty$, *then* f *is a nonnegative constant map.*

Proof If we take $x_1 = x, x_2 = y, x_3 = x_4 = \ldots = x_n = z$, then

$$0 \leq S_n(f, x_1, x_2, x_3, \ldots, x_n) = (x - y)\left[f(x)(x - z)^{n-2} - f(y)(y - z)^{n-2}\right] = g(x, y, z).$$

By $0 \leq g(x, y, y) = f(x)(x - y)^{n-1}$ for every $x, y \in D$ it follows that $f(x) \geq 0$ for every $x \in D$.Thus assertion 1° was proved. To prove 2° for every $t \in D$ consider the map

$$h_t(x) = f(x)(x - t)^3 \quad x \in D$$

By the preceding lemma we have

$$S_n\left(f, x_1, x_{2,} \ldots x_{n-3}, t, t, t\right) = S_{n-3}(h_t, x_1, x_2, x_3, \ldots, x_{n-3})$$

hence $h_t \in \mathscr{S}_{n-3}(D)$. Note that $n - 3 \geq 4$ and $n - 3$ is an even number. By Theorem 6.3., assertion 3°, it follows that $f(x) = 0$ for every $x \in D_0$.

To prove 4° suppose that $\sup D = +\infty$. Let $(z_k)_{k\geq 1}$ be a sequence from D such that $\lim_{k\to\infty} z_k = \infty$. By

$$0 \leq \lim_{k\to\infty} \frac{g(x, y, z_k)}{z_k^{n-2}} = (x - y)(f(y) - f(x))$$

it follows that f is a monotone decreasing map.

To prove 3° suppose that $\inf D = -\infty$. Let $(z_k)_{k\geq 1}$ be a sequence from D such that $\lim_{k\to\infty} z_k = -\infty$. By

$$0 \geq \lim_{k\to\infty} \frac{g(x, y, z_k)}{z_k^{n-2}} = \lim_{t\to\infty} \frac{g(x, y, -t)}{(-t)^{n-2}}$$

$$= -\lim_{t\to\infty} \frac{(x-y)\left[f(x)(x+t)^{n-2} - f(y)(y+t)^{n-2}\right]}{t^{n-2}} = -(x-y)(f(x) - f(y))$$

it follows that f is a monotone increasing map. Assertion 5° follows at once from the preceding assertions. □

6.2 Literature Review of the Godunova–Levin-3–Schur Functions

Definition 6.2 Let D be a subset of $\mathbb{R}$ containing at least two elements and $f : D \to \mathbb{R}$ be a map. Denote by $S_3(f, x, y, z)$ the sum

$$f(x)(x-y)(x-z) + f(y)(y-z)(y-x) + f(z)(z-x)(z-y). \tag{6.4}$$

We shall say that a function $f : D \to \mathbb{R}$ belongs to the class $\mathscr{S}_3(D)$ of Schur functions if the following inequality holds:

$$S_3(f, x, y, z) \geq 0 \quad \text{for every} \quad x, y, z \in D. \tag{6.5}$$

One can easily see that all the functions in $\mathscr{S}_3(D)$ are nonnegative.

In [59] Wright has generalized the Schur's inequality, showing that the inequality (6.2) holds if the function $f(x) = x^r$ is replaced with a nonnegative convex function or with a nonnegative monotone function. Consequently, nonnegative convex functions and nonnegative monotone functions belong to the class of 3-Schur functions defined on some interval D.

In 1985 Godunova and Levin [24] introduced the following class of functions: If D is an interval of $\mathbb{R}$ a function $f : D \to \mathbb{R}$ is said to belong to the class $Q(D)$ if it is nonnegative and for all $x, y \in D$ and $t \in (0, 1)$, the following inequality holds:

$$f((1-t)x + ty) \leq \frac{f(x)}{1-t} + \frac{f(y)}{t}. \tag{6.6}$$

Of course, one can extend the definition of the Godunova and Levin class of functions $Q(D)$ in the case D is a subset of $\mathbb{R}$ containing at least two elements. Therefore we shall say that a function $f : D \to \mathbb{R}$ is said to belong to the class

$Q(D)$ if it is nonnegative and for all $x, y \in D$ and $t \in (0,1)$, such that $(1-t)x + ty \in D$ the inequality (6.6) holds.

In the paper [24] Godunova and Levin have shown that $\mathscr{S}_3(D)$, the class of all 3-Schur functions defined on D coincides with the Godunova–Levin class of functions $Q(D)$. For the sake of completeness we shall include a proof of the above statement.

Theorem 6.5 *Let $f : D \to \mathbb{R}$ be a map. Then the following assertions are equivalent:*

1° *f is a 3-Schur map on D.*
2° $f(\alpha x + \beta y) \leq \dfrac{f(x)}{\alpha} + \dfrac{f(y)}{\beta}$, *for every $x, y \in D$ and $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, such that $\alpha x + \beta y \in D$.*

Proof Let S_3 be the map defined in (6.1), $x, y, z \in D$, $x < z < y$, $\alpha, \beta \in (0,1)$, $\alpha + \beta = 1$, $z = \alpha x + \beta y$. Then one can easily see that the following equality holds:

$$S_3(f, x, y, z) = \alpha\beta (x-y)^2 \left[\frac{f(x)}{\alpha} + \frac{f(y)}{\beta} - f(z)\right]. \tag{6.7}$$

Then the equivalence of the assertions from the statement of the above theorem follows at once from identity (6.7). □

In the following we shall denote by D a subset of $\mathbb{R}$ containing at least two elements. We shall denote with $\mathscr{S}_3(D)$ the class of Godunova–Levin-Schur functions defined on D. The class of the Godunova–Levin-Schur functions was intensively studied in a series of papers [21, 41], [42, pp. 410–413], [49, 55], and [50].

In the paper [55] Varošanec has introduced a very general class of functions known as the class of h-convex functions. More precisely, let I be an interval of $\mathbb{R}$ and $h : (0,1) \to \mathbb{R}$ be a nonnegative function with the property that there exists $t_0 \in (0,1)$ such that $h(t_0) > 0$. A function $f : I \to \mathbb{R}$ is called a h-convex function if f is nonnegative and for all $x, y \in I$, $\alpha \in (0,1)$ we have

$$f(\alpha x + (1-\alpha)y) \leq h(\alpha) f(x) + h(1-\alpha) f(y). \tag{6.8}$$

If inequality in (6.6) is reversed, then f is said to be h-concave. Denote by $SX(h, I)$ the class of all h-convex functions. The notion of h-convex function is of course more general than the notion of Godunova–Levin–Schur function. The class of h-convex functions contains in case that special selections are made for the function h some important classes of functions. Obviously, if $h(\alpha) = \alpha, \alpha \in (0,1)$, then all nonnegative convex functions are h-convex functions. If $h(\alpha) = \frac{1}{\alpha}, \alpha \in (0,1)$, then $SX(h, I) = \mathscr{S}_3(I)$. If $h(\alpha) = 1, \alpha \in (0,1)$, then $SX(h, I)$ contains the class $P(I)$ of all $P-$functions defined on I. By a $P-$function we understand a nonnegative function $f\ I \to \mathbb{R}$ with the property that

$$f(\alpha x + (1-\alpha)y) \leq f(x) + f(y) \quad \text{for all } x, y \in I$$

The paper [55] contains many interesting properties of the h-convex functions.

In the paper [41], it is proved the following version of the famous Jensen inequality for convex functions.

Theorem 6.6 *Let D be an interval of $\mathbb{R}$, $n \geq 2$, $w_1, w_2, \ldots, w_n$ be real numbers and $f \in \mathscr{S}_3(D)$. If $v_n = w_1 + w_2 + \ldots + w_n$, then for every $x_1, x_2, \ldots, x_n \in I$ the following inequality holds*

$$f\left(\frac{1}{v_n}\sum_{i=1}^{n} w_i x_i\right) \leq v_n \sum_{i=1}^{n} \frac{f\ (x_i)}{w_i}$$

Let $I = [a_0, b_o]$ be an interval of the real line, $a, b \in I,\ a < b$ and $f\ I \to \mathbb{R}$ be a convex function. The following inequality is known as the Hadamard's inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f\ (x)\, dx \leq \frac{f\ (a) + f\ (b)}{2}$$

In [21] were proved two sharp integral inequalities of Hadamard type for the Godunova–Levin–Schur functions.

Theorem 6.7 ([21]) *Let I be an interval of $\mathbb{R}$, $a, b \in I,\ a < b$ and let $f \in \mathscr{S}_3(I)$ be a function integrable on $[a, b]$ .Then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{4}{b-a}\int_a^b f\ (x)\, dx \tag{6.9}$$

and

$$\frac{1}{b-a}\int_a^b p\ (x)\ f\ (x)\, dx \leq \frac{f\ (a) + f\ (b)}{2}$$

where

$$p\ (x) = \frac{(b-x)\ (x-a)}{(b-a)^2},\ x \in I.$$

The constant 4 *in (6.9) is the best possible.*

In [49] were proved the following properties of the Godunova–Levin–Schur functions:

Proposition 6.1 ([49]) *The following assertions hold:*

1° *If $f \in \mathscr{S}_3(D)$, then $f \geq 0$.*
2° *If $f \in \mathscr{S}_3(D)$ and for some $a, b \in D$, $a < b$, we have $f\ (a) = f\ (b) = 0$, then $f\ (x) = 0$ for every $x \in [a, b] \cap D$.*

3° *If there exists* $a, b \in D,\ a < b$, *such that* $\frac{a+b}{2} \in D$ *and*

$$f\left(\frac{a+b}{2}\right) > 2f(a) + 2f(b) \tag{6.10}$$

then $f \notin \mathscr{S}_3(D)$.

Proof Note that the map $S_3(f, x, y, z)$ is a symmetric map. In order to prove 1° consider $f \in \mathscr{S}_3(D)$ and take two distinct elements of D, $x, y \in D$. Then

$$0 \leq S_3(f, x, y, y) = f(x)(x-y)^2,$$

hence $f \geq 0$.

To prove 2° let $f \in \mathscr{S}_3(D)$, $a, b \in D$, $a < b$, $f(a) = f(b) = 0$. Then

$$0 \leq S_3(f, x, a, b) = f(x)(x-a)(x-b)$$

for every $x \in [a, b] \cap D$. Suppose that there exists $x_0 \in [a, b] \cap D$ such that $f(x_0) > 0$. This implies $(x_0 - a)(x_0 - b) < 0$, hence $S_3(f, x_0, a, b) < 0$. We have obtained a contradiction. It follows that $f = 0$ on $[a, b] \cap D$.

If f satisfies (6.10), then

$$S_3\left(f, \frac{a+b}{2}, a, b\right) = \frac{(a-b)^2}{4}\left(2f(a) + 2f(b) - f\left(\frac{a+b}{2}\right)\right) < 0 \tag{6.11}$$

hence $f \notin \mathscr{S}_3(D)$. Thus we proved assertion 3°. □

6.3 *The Class of 3-Schur Functions*

Proposition 6.2 ([50]) *Let* $f : D \to \mathbb{R}$ *be a map. Suppose that there exist two positive constants* m, M *such that:*

$$0 < m \leq f(x) \leq M \leq 4m \quad \textit{for every} \quad x \in D.$$

Then f *is a* 3*-Schur map on* D.

Proof Let $x, y \in I, \alpha, \beta \in (0, 1), \alpha + \beta = 1$ such that $\alpha x + \beta y \in D$. Then

$$f(\alpha x + \beta y) \leq M \leq 4m \leq \frac{m}{\alpha\beta} = \frac{m}{\alpha} + \frac{m}{\beta} \leq \frac{f(x)}{\alpha} + \frac{f(y)}{\beta}.$$

By Theorem 6.5 it follows that f is a 3-Schur map on D. □

Proposition 6.3 ([49]) *Let $f : D \to \mathbb{R}$, be a map. Suppose that there exist two positive constants m, M such that:*

$$0 < m \le f(x) \le M \quad \textit{for every} \quad x \in D.$$

For every $a \ge 0$ consider the map $f_a : D \to \mathbf{R}$, $f_a(x) = f(x) + a$, $x \in D$. Then for every $a \ge \max\left(\frac{M-4m}{3}, 0\right)$ the map f_a is a 3-Schur map on D.

Proof Note that $a \ge \max\left(\frac{M-4m}{3}, 0\right)$ implies that

$$0 < m + a \le f_a(x) \le M + a \le 4(m + a) \quad \text{for every } x \in D$$

By the preceding proposition f_a is a 3-Schur map on D. □

In the following we shall give a definition of a quasi-convex map which is a slight more general than the classical one.

Definition 6.3 A map $f : D \to \mathbb{R}$ is *quasi-convex* if

$$f(\alpha x + \beta y) \le \max(f(x), f(y))$$

for every $x, y \in D, \alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ such that $\alpha x + \beta y \in D$.

Recall that in the classical definition of a quasi-convex map one supposes that the set D is convex.

Corollary 6.1 *The following assertions hold:*

1° *Every nonnegative quasiconvex map is a 3-Schur map.*
2° *Every nonnegative map which is a sum of two nonnegative monotone maps is a 3-Schur map.*

Proof Consider $x, y \in D, \alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ such that $\alpha x + \beta y \in D$. If f is a nonnegative quasi-convex map, then

$$f(\alpha x + \beta y) \le \max(f(x), f(y)) \le \frac{f(x)}{\alpha} + \frac{f(y)}{\beta}.$$

By Theorem 6.5, it follows that f is a 3-Schur map on D.
Suppose that $f = u_1 + u_2$, $u_i \ge 0$, u_i monotone $i = 1, 2$. Then one can easily see that

$$u_i(\alpha x + \beta y) \le \max(u_i(x), u_i(y)) \le u_i(x) + u_i(y) \le \frac{u_i(x)}{\alpha} + \frac{u_i(y)}{\beta} \quad i = 1, 2.$$

By Theorem 6.5, f is a 3-Schur map on D. □

Theorem 6.8 *Let $f : D \to \mathbb{R}_+$ be a map with the property*

$$f(\alpha x + \beta y) \leq \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2$$

for every $x, y \in D$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ such that $\alpha x + \beta y \in D$. Then f is a 3-Schur map on D.

Proof The assertion of the theorem follows at once from the inequalities:

$$f(\alpha x + \beta y) \leq \left(\sqrt{f(x)} + \sqrt{f(y)}\right)^2 \leq \frac{f(x)}{\alpha} + \frac{f(y)}{\beta}$$

for every $x, y \in D, \alpha, \beta \in (0, 1), \alpha + \beta = 1$ such that $\alpha x + \beta y \in D$. □

Theorem 6.9 *Let $f : D \to \mathbb{R}_+$ be a map and M a positive constant. Suppose that the following inequality holds:*

$$f(\alpha x + \beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta} \leq M$$

for every $x, y \in D, \alpha, \beta \in (0, 1), \alpha + \beta = 1$ such that $\alpha x + \beta y \in D$. For every $a \geq 0$ consider the map $f_a : D \to \mathbb{R}$, $f_a(x) = f(x) + a$, $x \in D$. Then for every $a \geq \frac{M}{3}$ we have $f_a \in \mathcal{S}_3(D)$.

Proof Let $a \geq \dfrac{M}{3}$. Then for every $x, y \in D$, $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ such that $\alpha x + \beta y \in D$, we have

$$f_a(\alpha x + \beta y) - \frac{f_a(x)}{\alpha} - \frac{f_a(y)}{\beta}$$

$$= f(\alpha x + \beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta} + a\left(1 - \frac{1}{\alpha\beta}\right) \leq M + a(1 - 4) = M - 3a \leq 0.$$

By Theorem 6.5, it follows that f_a is a 3-Schur map on D. □

Theorem 6.10 *Let $f : D \to \mathbb{R}_+$ be a map. For every $a \geq 0$ consider the map $f_a : D \to \mathbb{R}$, $f_a(x) = f(x) + a$, $x \in D$. If $f \notin \mathcal{S}_3(D)$, then there exists $a_0 > 0$ such that for every $a \in (0, a_0)$ we have $f_a \notin \mathcal{S}_3(D)$.*

Proof Suppose that $f \notin \mathcal{S}_3(D)$. Then by Theorem 6.5, there exist $x_0, y_0 \in D$, $\alpha_0, \beta_0 \in (0, 1)$, $\alpha_0 + \beta_0 = 1$, such that $\alpha_0 x_0 + \beta_0 y_0 \in D$ and

$$A(f) = f(\alpha_0 x_0 + \beta_0 y_0) - \frac{f(x_0)}{\alpha_0} - \frac{f(y_0)}{\beta_0} > 0$$

Take

$$a_0 = \frac{A(f)}{\frac{1}{\alpha_0\beta_0} - 1}.$$

Note that if $a \in [0, a_0)$ then

$$\begin{aligned} A(f_a) &= f_a(\alpha_0 x_0 + \beta_0 y_0) - \frac{f_a(x_0)}{\alpha_0} - \frac{f_a(y_0)}{\beta_0} \\ &= A(f) + a\left(1 - \frac{1}{\alpha_0\beta_0}\right) > A(f) + a_0\left(1 - \frac{1}{\alpha_0\beta_0}\right) = 0 \end{aligned}$$

Consequently $f_a \notin \mathscr{S}_3(D)$. □

Proposition 6.4 *Let* $f(x) = \left(x^2 - 1\right)^2$, $x \in \mathbb{R}, \psi: [0, 1] \to \mathbb{R}$,

$$\psi(t) = \frac{(1-t)(1+t)^2}{1 + t + t^2 + t^3 + t^4},\ t \in [0, 1].$$

Denote

$$Q = \left\{(\alpha, \beta) \in \mathbf{R}^2 : \alpha, \beta \in (0, 1), \alpha + \beta = 1\right\}, \gamma_0 = \max_{(\alpha,\beta)\in Q} [\psi(\alpha) + \psi(\beta)].$$

For every $x, y \in \mathbf{R}$, $(\alpha, \beta) \in Q$ *define*

$$g(x, y, \alpha, \beta) = f(\alpha x + \beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta}.$$

Then the following inequalities hold:

$$\frac{36}{31} \le \gamma_0 < 2,\ g(x, y, \alpha, \beta) \le \gamma_0 + 1 \le 3.$$

Proof Note that f is decreasing on $(-\infty, -1]$ and increasing on $[-1, 0]$. Hence the restriction of f to $(-\infty, 0]$ is quasiconvex. Thus if $x, y \le 0$ and $(\alpha, \beta) \in Q$, then

$$\begin{aligned} g(x, y, \alpha, \beta) &= f(\alpha x + \beta y) - \frac{f(x)}{\alpha} - \frac{f(y)}{\beta} \\ &\le \max(f(x), f(y)) - \frac{f(x)}{\alpha} + \frac{f(y)}{\beta} \le 0 \end{aligned}$$

Since f is decreasing on $[0, 1]$ and increasing on $(1, \infty]$ it follows that the restriction of f to $[0, \infty)$ is quasiconvex. Thus if $x, y \ge 0$ and $(\alpha, \beta) \in Q$, then $g(x, y, \alpha, \beta) \le 0$.

Now we shall consider the case $x \geq 0$, $y \leq 0$. Let $z = -y$. We shall prove that

$$g\,(x, -z, \alpha, \beta) \leq 1 + \gamma_0 < 3.$$

Consider the maps:

$$\phi_1\,(t, x) = \frac{t^5 - 1}{t} x^4 + \frac{2\left(1 - t^3\right)}{t} x^2 - \frac{1}{t},\ t \in (0, 1)\,,\ x \in \mathbb{R},$$

$$\phi_2\,(x) = ax^4 + bx^2 + c,\ x \in \mathbb{R}.$$

Note that $a < 0$ and $b > 0$ implies that

$$\phi_2\,(x) \leq \phi_2\left(\sqrt{-\frac{b}{2a}}\right) = \frac{4ac - b^2}{4a} \text{ for every } x \in \mathbb{R}.$$

If in the preceding inequality we take

$$a = \frac{t^5 - 1}{t},\ b = \frac{2\left(1 - t^3\right)}{t},\ t \in (0, 1)\,,$$

we obtain

$$\phi_1\,(t, x) \leq \phi_2\left(\sqrt{-\frac{b}{2a}}\right) = \frac{4ac - b^2}{4a} = \frac{t^2\left(2 - t^2 - t^3\right)}{t^5 - 1} = \psi\,(t) - 1. \tag{6.12}$$

Let $x, z \geq 0$. Then

$$g\,(x, -z, \alpha, \beta) = f\,(\alpha x - \beta z) - \frac{f\,(x)}{\alpha} - \frac{f\,(-z)}{\beta}$$

$$= (\alpha x - \beta z)^4 - \frac{x^4}{\alpha} - \frac{z^4}{\beta} - 2\,(\alpha x - \beta z)^2 + \frac{2x^2}{\alpha} + \frac{2z^2}{\beta} + 1 - \frac{1}{\alpha} - \frac{1}{\beta}$$

$$= \left(\alpha^4 - \frac{1}{\alpha}\right) x^4 + \left(\frac{2}{\alpha} - 2\alpha^2\right) x^2 - \frac{1}{\alpha} + \left(\beta^4 - \frac{1}{\beta}\right) z^4 + \left(\frac{2}{\beta} - 2\beta^2\right) z^2 - \frac{1}{\beta}$$

$$+8\alpha^2\beta^2x^2z^2 - 4\alpha^3\beta x^3 z - 4\alpha\beta^3 x z^3 - 2\alpha^2\beta^2x^2z^2 + 4\alpha\beta xz + 1$$

$$= \phi_1\,(\alpha, x) + \phi_1\,(\beta, z) - 4\alpha\beta xz\,(\alpha x - \beta z)^2 - 2\alpha^2\beta^2x^2z^2 + 4\alpha\beta xz + 1$$

$$\leq \phi_1\,(\alpha, x) + \phi_1\,(\beta, z) + 3 \leq \psi\,(\alpha) - 1 + \psi\,(\beta) - 1 + 3$$

$$= \psi\,(\alpha) + \psi\,(\beta) + 1 \leq \gamma_0 + 1.$$

Thus we proved that $x \geq 0$, $y \leq 0$ implies $g(x, y, \alpha, \beta) \leq \gamma_0 + 1 \leq 3$. Analogously one can prove the same inequality for $x \leq 0$, $y \geq 0$. □

Proposition 6.5 *Let* $f(x) = \left(x^2 - 1\right)^2$, $x \in \mathbb{R}$. *For every* $a \geq 0$ *consider the map* $f_a : \mathbb{R} \to \mathbb{R}$,

$$f_a(x) = f(x) + a,\ x \in \mathbb{R}.$$

Then the following assertions hold:

1° *For every* $a \in \left[0, \frac{1}{3}\right)$, $f_a \notin \mathscr{S}_3(\mathbb{R})$.
2° *For every* $a \in [1, \infty)$, $f_a \in \mathscr{S}_3(\mathbb{R})$.
3° *For every* $a \in \mathbb{R}$, f_a *is not quasi-convex and is not the sum of two positive monotone functions.*

Proof To prove 1° take $\alpha_0 = \beta_0 = \frac{1}{2}$, $x_0 = -1$, $y_0 = 1$. Then

$$\alpha_0 x_0 + \beta_0 y_0 = 0,$$

$$A(f) = f(\alpha_0 x_0 + \beta_0 y_0) - \frac{f(x_0)}{\alpha_0} - \frac{f(y_0)}{\beta_0} = f(0) - 2f(-1) - 2f(1) = 1 > 0$$

and

$$a_0 = \frac{A(f)}{\frac{1}{\alpha_0 \beta_0} - 1} = \frac{1}{3}.$$

By Theorem 6.10, $f_a \notin \mathscr{S}_3(\mathbb{R})$ for every $a \in [0, \frac{1}{3})$.The assertion from 2° follows at once from Theorem 6.9 and Proposition 6.4.

Note that f_a is not monotone and there does not exist $u_0 \in \mathbb{R}$ such that f_a is decreasing on $(-\infty, u_0]$ and increasing on $[u_0, \infty)$. Therefore, the function f_a is not quasi-convex. □

Proposition 6.6 *Let* $\phi : \mathbb{R} \to \mathbb{R}$ *be a map with the property that* $f \circ \phi \in \mathscr{S}_3(\mathbb{R})$ *for every* $f \in \mathscr{S}_3(\mathbb{R})$. *Then* ϕ *is monotone.*

Proof Suppose contrary that ϕ is not a monotone map. Then there exist $x < y < z$ such that

$$\max(\phi(x), \phi(z)) < \phi(y).$$

Let $\lambda \in (\max(\phi(x), \phi(z)), \phi(y))$. For every $a, b > 0$ consider the map

$$f_{a,b}(t) = \begin{cases} a \text{ if } t \in (-\infty, \lambda] \\ b \text{ if } t \in (\lambda, +\infty). \end{cases}$$

Note that all maps $f_{a,b}$ are monotone, hence they are 3-Schur maps. Since $y \in (x, z)$ it follows that there exist $\alpha, \beta \in (0, 1)$, $\alpha + \beta = 1$ such that $y = \alpha x + \beta z$.
By hypothesis $f_{a,b} \circ \phi \in \mathscr{S}_3(\mathbb{R})$. Therefore

$$b = \left(f_{a,b} \circ \phi\right)(y) = f_{a,b}\left(\phi\left(\alpha x + \beta z\right)\right) \leq \frac{f_{a,b}\left(\phi\left(x\right)\right)}{\alpha} + \frac{f_{a,b}\left(\phi\left(z\right)\right)}{\beta} = \frac{a}{\alpha} + \frac{a}{\beta} = \frac{a}{\alpha\beta}.$$

By the preceding inequality it follows that $\alpha\beta \leq \frac{a}{b}$ for every $a, b > 0$. The contradiction we have obtained proves that the map ϕ must be monotone. □

Proposition 6.7 *Let* $f : \left[0, \frac{2}{3}\right] \to \mathbb{R}$, $f(x) = x - x^2$, $x \in \left[0, \frac{2}{3}\right]$. *Then* $f \in \mathscr{S}_3\left(\left[0, \frac{2}{3}\right]\right)$.

Proof Let $\alpha, \beta \in (0, 1), \alpha + \beta = 1$,

$$g(x, y, \alpha, \beta) = f\left(\alpha x + \beta y\right) - \frac{f\left(x\right)}{\alpha} - \frac{f\left(y\right)}{\beta},\ x, y \in \left[0, \frac{2}{3}\right].$$

Note that

$$\begin{aligned} -\alpha\beta g\left(x, y, \alpha, \beta\right) &= \left(\alpha^3\beta - \beta\right)x^2 + \left(\beta - \alpha^2\beta\right)x \\ &\quad + \left(\alpha\beta^3 - \alpha\right)y^2 + \left(\alpha - \alpha\beta^2\right)y + 2\alpha^2\beta^2 xy \end{aligned}$$

Observe that the coefficients of x^2 and y^2 from the right-hand side of the above identity, that is $\alpha^3\beta - \beta$ and $\alpha\beta^3 - \alpha$ are strictly negative. Moreover, the matrix

$$\begin{pmatrix} \alpha^3\beta - \beta & \alpha^2\beta^2 \\ \alpha^2\beta^2 & \alpha\beta^3 - \alpha \end{pmatrix}$$

is negative definite. Consequently the map $h_{\alpha,\beta}(x, y) = -\alpha\beta g(x, y, \alpha, \beta)$ is concave. Hence

$$-\alpha\beta g\left(x, y, \alpha, \beta\right) \geq \min\left(h_{\alpha,\beta}\left(0, 0\right), h_{\alpha,\beta}\left(0, \frac{2}{3}\right), h_{\alpha,\beta}\left(\frac{2}{3}, 0\right), h_{\alpha,\beta}\left(\frac{2}{3}, \frac{2}{3}\right)\right).$$

Since all the arguments of min are nonnegative it follows that $g(x, y, \alpha, \beta) \leq 0$. This implies that $f \in \mathscr{S}_3\left(\left[0, \frac{2}{3}\right]\right)$. □

Proposition 6.8 *Let D be a subset of $\mathbb{R}$ with more than three elements and $f : D \to \mathbf{R}$ be an increasing $3-$Schur map. Consider $x, y, z \in D$ distinct elements. Then $S_3(f, x, y, z) = 0$ if and only if one of the following situations occurs:*

1° *All x, y, z are equal, that is $x = y = z$.*
2° *Two of x, y, z are equal and the third is a zero of f.*
3° *All three x, y, z are zeros of f.*

Proof Without any loss of generality we may suppose that $x \geq y \geq z$. Denote

$$A(f, x, y, z) = (x - y)\left[f(x)(x - y) + (f(x) - f(y))(y - z)\right]$$

$$B(f, x, y, z) = f(z)(x - z)(y - z)$$

Note that $S_3(f, x, y, z) = A(f, x, y, z) + B(f, x, y, z)$. and $A(f, x, y, z) \geq 0$ and $B(f, x, y, z) \geq 0$. Consequently $S_3(f, x, y, z) = 0$ implies the relations $A(f, x, y, z) = B(f, x, y, z) = 0$.

We shall study two cases.

Case 1 $x = y$. In this case we have $A(f, x, y, z) = 0$. From $B(f, x, y, z) = 0$ it follows that $y = z$ or $f(z) = 0$. Thus case 1 reduces to situation 1° or 2°.

Case 2 $x > y$. From $A(f, x, y, z) = 0$ it follows that $f(x) = 0$ and

$$(f(x) - f(y))(y - z) = 0.$$

Hence $f(y) = 0$ or $y = z$. From $B(f, x, y, z) = 0$ it follows that $f(z) = 0$ or $y = z$.

□

Proposition 6.9 *Let I be an open interval of the real axis, $a \in I$, $f \in \mathscr{S}_3(I)$. If $f(a) = 0$, then f has one sided limits at every point of $I - \{a\}$.*

Proof Let $g(x) = (x - a)f(x)$, $x \in I$. Since $f \in \mathscr{S}_3(I)$ it follows that for every $x, y \in I$ we have

$$0 \leq S_3(f, x, y, a) = (x - y)(g(x) - g(y)).$$

Hence g is monotone increasing and has one sided limits at every point $x_0 \in I$. Thus the function

$$f(x) = \frac{g(x)}{x - a},\ x \in I - \{a\}$$

has one sided limits at every point in $I - \{a\}$. □

6.4 A Study of 5-Schur Functions

Theorem 6.11 *The following inequality*

$$S_5(f, x_1, x_2, \ldots, x_5) = \sum (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5) \geq 0$$

holds for every $x_1, x_2, x_3, x_4, x_5 \in \mathbb{R}$.

Proof Without any loss of generality we can suppose that $x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5$. Denote $u(x) = (x - x_3)(x - x_4)(x - x_5)$, $v(x) = (x - x_1)(x - x_2)(x - x_3)$, $x \in \mathbb{R}$.
Then

$$S_5(f, x_1, x_2, \ldots, x_5) = \sum (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)$$

$$= (x_1 - x_2)[(x_1 - x_3)(x_1 - x_4)(x_1 - x_5) - (x_2 - x_3)(x_2 - x_4)(x_2 - x_5)]$$

$$+ (x_4 - x_5)[(x_4 - x_1)(x_4 - x_2)(x_4 - x_3) - (x_5 - x_1)(x_5 - x_2)(x_5 - x_3)]$$

$$+ (x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5) = (x_1 - x_2)(u(x_1) - u(x_2))$$

$$+ (x_4 - x_5)(v(x_4) - v(x_5)) + (x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5).$$

Since

$$(x_1 - x_2)(u(x_1) - u(x_2)) \geq 0, \ (x_4 - x_5)(v(x_4) - v(x_5)) \geq 0$$

and

$$(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5) \geq 0$$

it follows that

$$\sum (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5) \geq 0.$$

□

Theorem 6.12 *If* $\inf D = -\infty$ *and* $\sup D = +\infty$, *then*

$$\mathscr{S}_5(D) = \{f : D \to \mathbb{R} : \exists\, c \geq 0 \text{ such that } f(x) = c \text{ for every } x \in D\}.$$

Proof The equality from the statement follows at once from the preceding theorem and Lemma 6.1. □

Proposition 6.10 *Let* $f \in \mathscr{S}_5(D)$, $D_0 = \{x \in D :$ *there exist* $x_1, x_2 \in D$ *such that* $x_1 < x < x_2\}$ *and* $a \in D_0$. *If* $f(a) = 0$, *then* $f(x) = 0$ *for every* $x \in D_0$.

Proof Let $g(x) = f(x)(x - a)$, $x \in D$. Note that $g \in \mathscr{S}_4(D)$ hence $g = 0$ on D_0. Consequently $f = 0$ on D_0. □

Theorem 6.13 *Let* $\mathbb{R}_+^*$ *be the set of positive numbers. Suppose that* $f \in \mathscr{S}_5(\mathbb{R}_+^*)$. *For every* $t > 0$ *consider the maps*

$$f_t(x) = f(x)(x - t)^3, \ g_t(x) = f(x)(x - t)^2, \ x \in \mathbb{R}_+^*.$$

Consider the map $h(x) = \frac{1}{x^3} f\left(\frac{1}{x}\right), \; x \in \mathbb{R}_+^*$.
Then the following assertions hold:

1° $h \in \mathcal{S}_5\left(\mathbb{R}_+^*\right)$.
2° $f \geq 0$ *on* $\mathbb{R}_+^*$ *and* f *is monotone decreasing.*
3° f_t *is a monotone increasing map for every* $t > 0$.
4° $g_t \in \mathcal{S}_3\left(\mathbb{R}_+^*\right)$ *for every* $t > 0$.
5° *If* $x_0 \in \mathbb{R}_+^*$ *and* $f(x_0) = 0$ *then* $f = 0$ *on* $\mathbb{R}_+^*$.
6° f *is continuous on* $\mathbb{R}_+^*$.

Proof Assertion 1° follows at once from the identity

$$S_5\left(f, \frac{1}{x_1}, \frac{1}{x_2}, \ldots, \frac{1}{x_5}\right) = \frac{1}{x_1 x_2 \ldots x_5} S_5(h, x_1, x_2, \ldots, x_5)$$

Assertion 2° follows at once from assertions 1° and 4° of Theorem 6.4.
From the identity $S_5(f, x, y, t, t, t) = S_2(f_t, x, y)$ it follows that $f_t \in \mathcal{S}_2\left(\mathbb{R}_+^*\right)$.
Assertion 3° follows now from Theorem 6.4.
Assertion 4° follows from the identity $S_5(f, x, y, z, t, t) = S_3(g_t, x, y, z)$.
To prove 5° note that $g_t(t) = g_t(x_0) = 0$ and $g_t \in \mathcal{S}_3\left(\mathbb{R}_+^*\right)$. If $t \neq x_0$, then by assertion 2° of Proposition 6.1 we have $g_t = 0$ on the interval with the endpoints t and x_0.
To prove 6° we shall use assertion 2^0 and 3^0 of the present theorem. By assertion 3^0 we have

$$f_t(x-0) \leq f_t(x) \leq f_t(x+0) \quad \text{for every } x, t > 0$$

hence

$$f(x-0)(x-t)^3 \leq f(x)(x-t)^3 \leq f(x+0)(x-t)^3 \quad \text{for every } x > t > 0.$$

By the preceding sequence of inequalities it follows that

$$f(x-0) \leq f(x) \leq f(x+0) \quad \text{for every } x > 0.$$

Taking into account that f is monotone decreasing it follows that f is continuous on $\mathbb{R}_+^*$. □

Theorem 6.14 *Let* $D \subset \mathbb{R}$ *be a bounded set,* $a = \inf D$, $b = \sup D$, *and* $f : D \to \mathbb{R}$. *For every* $\alpha, \beta, \gamma \in D$ *we define*

$$g_{\alpha,\beta,\gamma}(x) = f(x)(x-\alpha)(x-\beta)(x-\gamma), \; x \in D.$$

If for every $a < \gamma < \beta <' alpha < b$, *the function* $g_{\alpha,\beta,\gamma}$ *is increasing on* $[a, \gamma] \cap D$ *and on* $[\alpha, b] \cap D$, *then* $f \in \mathcal{S}_5(D)$.

Proof Choose $x_1 \geq x_2 \geq \ldots \geq x_5 > 0$. Note that

$$S_5(f, x_1, x_2, \ldots, x_5) = (x_1 - x_2)[f(x_1)(x_1 - x_3)(x_1 - x_4)(x_1 - x_5)$$
$$-f(x_2)(x_2 - x_3)(x_2 - x_4)(x_2 - x_5)]$$
$$+f(x_3)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)$$
$$+(x_4 - x_5)[f(x_4)(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)$$
$$-f(x_5)(x_5 - x_1)(x_5 - x_2)(x_5 - x_3)]$$
$$= (x_1 - x_2)\left[g_{x_3,x_4,x_5}(x_1) - g_{x_3,x_4,x_5}(x_2)\right]$$
$$+f(x_3)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5)$$
$$+(x_4 - x_5)\left[g_{x_1,x_2,x_3}(x_4) - g_{x_1,x_2,x_3}(x_5)\right].$$

Since

$$(x_1 - x_2)\left[g_{x_3,x_4,x_5}(x_1) - g_{x_3,x_4,x_5}(x_2)\right] \geq 0,$$
$$f(x_3)(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)(x_3 - x_5) \geq 0$$

and

$$(x_4 - x_5)\left[g_{x_1,x_2,x_3}(x_4) - g_{x_1,x_2,x_3}(x_5)\right] \geq 0$$

it follows that

$$S_5(f, x_1, x_2, \ldots, x_5) \geq 0$$

hence $f \in \mathscr{S}_5(D)$. □

Lemma 6.2 *Let $a \geq b \geq c > 0$, $\alpha \in [0, 3]$. Consider the function*

$$g_\alpha(x) = \frac{1}{x^\alpha}(x - a)(x - b)(x - c), \quad x \in (0, \infty).$$

Then g_α is increasing on the intervals $(0, c]$ and $[a, \infty)$.

Proof Denote $\sigma_1 = a + b + c$, $\sigma_2 = ab + bc + ac$, $\sigma_3 = abc$.

Note that for every $x \in (0, \infty)$ we have:

$$g_\alpha(x) = x^{3-\alpha} - \sigma_1 x^{2-\alpha} + \sigma_2 x^{1-\alpha} - \sigma_3 x^{-\alpha},$$

$$g'_\alpha(x) = (3-\alpha)x^{2-\alpha} - \sigma_1(2-\alpha)x^{1-\alpha} + \sigma_2(1-\alpha)x^{-\alpha} + \alpha\sigma_3 x^{-\alpha-1}.$$

Let $h_\alpha(x) = x^{\alpha+1} g_\alpha(x)$. Note that

$$h_\alpha(x) = -\alpha(x-a)(x-b)(x-c) + 3x^3 - 2\sigma_1 x^2 + \sigma_2 x.$$

In the following we shall study two cases.

Case 1 $x \in (0, c]$. Let $\phi(t) = t^{-1} h_0(t) = 3t^2 - 2\sigma_1 x + \sigma_2$, $t > 0$.
Since $\frac{\sigma_1}{3} \geq c$ it follows that ϕ is decreasing on $(0, c]$. Consequently

$$\phi(x) \geq \phi(c) = (c-a)(c-b) > 0,$$

hence $h_\alpha(x) \geq h_0(x) \geq 0$.Thus g_α is increasing on $(0, c]$.
Case 2 $x \in [a, \infty)$. Note that

$$h_3(x) = \sigma_1 x^2 - 2\sigma_2 x + 3\sigma_3,$$

$$h'_3(x) = 2\sigma_1 x - 2\sigma_2 \geq h'_3(a) = 2\left(a^2 - bc\right) > 0.$$

Hence $h_\alpha(x) \geq h_3(x) \geq h_3(a) = a(a-b)(a-c) \geq 0$.

Thus g_α is increasing on $[a, \infty)$. □

Corollary 6.2 *For* $\alpha \in [1, 2]$ *and* $f_\alpha(x) = x^{-\alpha}$, $x \in (0, \infty)$ *we have* $f_\alpha \in \mathscr{S}_5\left(\mathbb{R}_+^*\right)$.

Proof The assertion follows at once from Theorem 6.14. and Lemma 6.2. □

Proposition 6.11 *Let D be a subset of* $\mathbb{R}$ *such that* $|D| \geq 3$. *Denote*

$$c = \inf(D),\ d = \sup(D) \text{ and } D_0 = D - \{c, d\}.$$

If $f \in \mathscr{S}_5(D)$*, then* f *is continuous on* D_0.

Proof Let $x_0 \in D_0$. If $x_0 \in D_0$ is an isolated point of D, then f is continuous at $x = x_0$. For every $t \in D$ consider the function

$$f_t(x) = (x-t)^3 f(x),\ x \in D.$$

If $f \in \mathscr{S}_5(D)$ and $a, x_1, x_2 \in D$, then

$$0 \le S_5(f, x_1, x_2, a, a, a) = (x_1 - x_2)(f_a(x_1) - f_a(x_2)).$$

Hence f_a is increasing. If there exists a sequence $(x_n)_{n\ge 1}$ in D such that $x_n \uparrow x_0$, then the left limit of f_a at $x = x_0$ exists (that is $f_a(x_0 - 0)$ exists).

If $x_0 < a$ then $(x_0 - a)^3 < 0$ then

$$f_a(x_0 - 0) = (x_0 - a)^3 f(x_0 - 0) \le (x_0 - a)^3 f(x_0) = f_a(x_0).$$

Thus $f(x_0 - 0) \ge f(x_0)$.

If $x_0 > a$, then $(x_0 - a)^3 > 0$ hence from $f_a(x_0 - 0) \le f_a(x_0)$ it follows

$$f(x_0 - 0) \le f(x_0).$$

Consequently $f(x_0 - 0) = f(x_0)$. □

Theorem 6.15 *Let $D \subset \mathbb{R}$ be a bounded set,*

$$a = \inf D,\ b = \sup D,\ f : D \to \mathbb{R},$$
$$u(x) = f(x)(x-a)^3,\ x \in D,$$
$$v(x) = f(x)(x-b)^3,\ x \in D.$$

Then the following assertions are equivalent:

1° *u, v are increasing;*
2° *$f \in \mathscr{S}_5(D)$.*

Proof For the implication 2° $\Longrightarrow$ 1° note that $f \in \mathscr{S}_5(D)$ implies

$$S_5(f, x_1, x_2, t, t, t) = (x_1 - x_2)[f(x_1)(x_1 - t)^3 - f(x_2)(x_2 - t)^3] \ge 0,$$

hence for every $t \in D$, the function

$$f_t(x) = f(x)(x-t)^3,\ x \in D$$

is increasing. Thus u and v are increasing.
Implication 1° $\Longrightarrow$ 2°. Let $a < \gamma < \beta < \alpha < b, \alpha, \beta, \gamma \in D$ and

$$g(x) = f(x)(x-\alpha)(x-\beta)(x-\gamma),\ x \in D.$$

Observe that the function

$$g(x) = u(x)\frac{x-\alpha}{x-a}\cdot\frac{x-\beta}{x-a}\cdot\frac{x-\gamma}{x-a},\ x \in [a,b] \cap D$$

is increasing on $[\alpha, b] \cap D$ as a product of positive increasing functions. Note that

$$g(x) = -\left(-v(x) \cdot \frac{\alpha - x}{b - x} \cdot \frac{\beta - k}{b - x} \cdot \frac{\gamma - x}{b - x}\right),\ x \in [a, \gamma] \cap D$$

is increasing on $[a, \gamma] \cap D$. By Theorem 6.14 it follows that $f \in \mathscr{S}_5(D)$. ◻

Corollary 6.3 *Let $D \subset \mathbb{R}$ be a set with the following properties:*

(i) $a = \inf(D) \in \mathbb{R}$.
(ii) There exists $(x_n)_{n\geq 1}$ in D such that $\lim\limits_{n\to\infty} x_n = +\infty$.

Then the following two assertions are equivalent:

1° $f \in \mathscr{S}_5(D)$*;*
2° f *is monotone decreasing and*

$$u(x) = (x - a)^3 f(x),\ x \in D$$

is monotone increasing.

Proof Proof of the implication $1^\circ \Longrightarrow 2^\circ$. Suppose $f \in \mathscr{S}_5(D)$.
For every $t \in D \cup \{a\}$ and for every $b > a$, $b \in D$ consider the function

$$f_t(x) = (x - t)^3 f(x),\ x \in D \cap [a, b].$$

Note that f_t is monotone increasing for every $t \in D \cup \{a\}$.
Note that for every $t \in (0, \infty) \cap D$ the function

$$g_t(x) = \frac{1}{t^3} f_t(x),\ x \in D$$

is monotone increasing. Since

$$g(x) = \lim_{t\to\infty} g_t(x) = -f(x),\ x \in D$$

is monotone increasing it follows that f is monotone decreasing.
Proof of the implication $2^\circ \Longrightarrow 1^\circ$. Suppose that f satisfies conditions from assertion 2°. In order to prove that $f \in \mathscr{S}_5(D)$ it suffices to prove that $f \in \mathscr{S}_5(D \cap [a, b])$ for every $b \in D$, $b > a$.
Note that

$$f_b(x) = (x - b)^3 f(x),\ x \in D \cap [a, b]$$

is monotone increasing since

$$-f_b(x) = (b - x)^3 f(x),\ x \in D$$

is the product of two positive monotone decreasing functions. Since f_a is monotone increasing it follows that f is positive.
By the preceding theorem we obtain $f \in \mathscr{S}_5(D \cap [a, b])$. □

Theorem 6.16 *Let $p \in (0, \infty)$, $a \in \mathbb{R}$, $f_{a,p} : [0, 1] \to \mathbb{R}$,*

$$f_{a,p}(x) = x^p + a,\ x \in [0, 1].$$

Then the following assertions hold:

1° *If $p \in (1, \infty)$, then $a \geq \frac{1}{3}\left(\frac{p-1}{p+3}\right)^{p-1}$ if and only if $f_{a,p} \in \mathscr{S}_5([0, 1])$.*
2° *if $p = 1$, then $a \geq \frac{1}{3}$ if and only if $f_{a,1} \in \mathscr{S}_5([0, 1])$.*
3° *If $p \in (0, 1)$, then $f_{a,p} \notin \mathscr{S}_5([0, 1])$ for every $a \in \mathbb{R}$.*

Proof Proof of assertion 1°. Let $p \in (1, \infty)$ and $f_{a,p} \in \mathscr{S}_5([0, 1])$. Then

$$u(x) = x^3 f_{a,p}(x),\ x \in [0, 1], v(x) = (x-1)^3 f_{a,p}(x),\ x \in [0, 1]$$

are monotone increasing. Hence

$$0 \leq u'(x) = x^2[(p+3)x^p + 3a]$$

and

$$\begin{aligned} 0 \leq v'(x) &= (x-1)^2 [3 f_{a,p}(x) + (x-1) f'_{a,p}(x)] \\ &= (x-1)^2[3x^p + 3a + p(x-1)x^{p-1}]. \end{aligned}$$

From the first inequality we obtain

$$a \geq \sup_{x \in [0,1]} \left[-\frac{p+3}{3} x^p\right] = 0.$$

From the second inequality we obtain

$$a \geq \sup_{x \in [0,1]} \left[\frac{\varphi(x)}{3}\right] = \frac{1}{3}\left(\frac{p-1}{p+3}\right)^{p-1}$$

where $\varphi(x) = px^{p-1} - (p+3)x^p$, $x \in \mathbb{R}$.
Proof of assertion 2°. Let $p = 1$. From assertion 1° it follows that $f_{a,p} \in \mathscr{S}_5([0, 1])$ if and only if

$$a \geq \frac{1}{3} \lim_{p \downarrow 1} \left(\frac{p-1}{p+3}\right)^{p-1} = \frac{1}{3}.$$

Proof of assertion 3°. Suppose $p \in (0, 1)$. Since

$$v'(x) = (x-1)^2 \left[(3+p)x^p - px^{p-1} + 3a\right],\ x \in [0, 1]$$

it follows that

$$\lim_{x \downarrow 0} v'(x) = 3a - p \lim_{x \downarrow 0} x^{p-1} = -\infty.$$

Hence v is not increasing on [0, 1]. Consequently $f_{a,p} \notin \mathscr{S}_5([0, 1])$. □

6.5 The n − u-Schur Functions

In the following we shall introduce two classes of functions that generalize the class of n-Schur functions.

Let $D \subset \mathbb{R}$, $|D| \geq 3$, $n \geq 2$, $u : \mathbb{R} \to \mathbb{R}$, $f : D \to \mathbb{R}$. Consider the function

$$T_n(f, u, x_1, x_2, \ldots, x_n) = \sum f(x_1)u(x_1 - x_2)u(x_1 - x_3)\ldots u(x_1 - x_n),$$

$$x_1, x_2, \ldots, x_n \in D.$$

The class of functions $\mathscr{T}_{n,u}^{(1)}(D)$ is defined as the set of functions $f : D \to \mathbb{R}$ that satisfy the condition

$$T_n(f, u, x_1, x_2, \ldots, x_n) \geq 0 \text{ for every } x_1, x_2, \ldots, x_n \in D.$$

If $|D| \geq n$, the class of functions $\mathscr{T}_{n,u}^{(2)}(D)$ is defined as the set of all functions $f : D \to \mathbb{R}$ that satisfy the condition

$$T_n(f, u, x_1, x_2, \ldots, x_n) \geq 0$$

for every distinct $x_1, x_2, \ldots, x_n \in D$.

Note that if $u(t) = t$, $t \in \mathbb{R}$ then we have $\mathscr{T}_{n,u}^{(1)}(D) = \mathscr{S}_n(D)$.

In case $u(t) = \dfrac{1}{t}$, $t \in \mathbb{R} \setminus \{0\}$, then

$$T_n(f, u, x_1, x_2, \ldots, x_n) = f[x_1, x_2, \ldots, x_n],$$

where $f[x_1, x_2, \ldots, x_n]$ is the divided difference of f at points $x_1, x_2, \ldots, x_n$. In this case $\mathscr{T}_{n,u}^{(2)}(D)$ is the set of n-convex functions defined on D.

Theorem 6.17 *Let $D \subseteq \mathbb{R}$, $|D| \geq 3$ and $u : \mathbb{R} \to \mathbb{R}$ be a function with the following properties:*

(i) u *is increasing on* $(0, \infty)$*;*
(ii) $u(t) > 0$ *for every* $t \in (0, \infty)$*;*
(iii) $u(-t) = -u(t)$, $t \in \mathbb{R}$.

If $f : D \to \mathbb{R}$ *is quasi-convex, then* $f \in \mathscr{T}^{(1)}_{3,u}(D)$.

Proof Let $x, y, z \in D$, $x > y > z$. Since f is quasi-convex we have

$$f(y) \leq \max(f(x), f(z)) \leq f(x) + f(z)$$
$$\leq f(x)\frac{u(x-z)}{u(y-z)} + f(z)\frac{u(x-z)}{u(x-y)}.$$

We used the monotony of u on $(0, \infty)$.
From the preceding inequality it follows that $f \in \mathscr{T}^{(1)}_{3,u}(D)$. □

Theorem 6.18 *Let $D \subset \mathbb{R}$, $|D| \geq 3$, $f : D \to \mathbb{R}$, $u : \mathbb{R} \to \mathbb{R}$ and $v : (0, \infty) \to \mathbb{R}$, $v(t) = tu(t)$, $t \in (0, \infty)$. Suppose that u satisfies conditions (ii) and (iii). Then the following assertions hold:*

(1) If $f \in \mathscr{T}^{(2)}_{3,u}(D)$, f is bounded below on D and v is monotone decreasing, then f is convex.
(2) If $f : D \to \mathbb{R}$ is nonnegative, convex and v is monotone increasing, then $f \in \mathscr{T}^{(2)}_{3,u}(D)$.

Proof In order to prove assertion (1), let $x, y, z \in D$, $x > y > z$. If $t = \dfrac{x-y}{x-z}$, then $t \in (0, 1)$ and $y = (1-t)x + tz$. Suppose $f \in \mathscr{T}^{(2)}_{3,u}(D)$ and $f(t) \geq m$, for every $t \in D$.
Let $f_1(t) = f(t) + |m|$, $t \in D$. Note that $f_1(t) \geq 0$, $t \in D$. We obtain

$$f_1((1-t)x + tz) = f_1(y) \leq \frac{u(x-z)}{u(y-z)} f_1(x) + \frac{u(x-z)}{u(x-y)} f_1(z)$$
$$= \frac{v(x-z)}{v(y-z)} \cdot \frac{y-z}{x-z} f_1(x) + \frac{v(x-z)}{v(x-y)} \cdot \frac{x-y}{x-z} f_1(z)$$
$$\leq \frac{y-z}{x-z} f_1(x) + \frac{x-y}{x-z} f_1(z) = (1-t) f_1(x) + t f_1(z).$$

It follows that f_1 is a convex function. Thus f is a convex function.
In order to prove assertion (2) suppose that f is convex and v is increasing.
Let $x, y, z \in D$, $x > y > z$. Since v is monotone increasing it follows that

$$\frac{v(x-z)}{v(y-z)} \geq 1 \quad \text{and} \quad \frac{v(x-z)}{v(x-y)} \geq 1.$$

Since f is convex it follows that

$$\begin{aligned} f(y) &\le \frac{y-z}{x-z} f(x) + \frac{x-y}{x-z} f(z) \\ &\le \frac{y-z}{x-z} \cdot \frac{v(x-z)}{v(y-z)} f(x) + \frac{x-y}{x-z} \cdot \frac{v(x-z)}{v(x-y)} f(z) \\ &= \frac{u(x-z)}{u(y-z)} f(x) + \frac{u(x-z)}{u(x-y)} f(z). \end{aligned}$$

From the above inequalities it follows that $f \in \mathscr{T}_{3,u}^{(2)}(D)$. □

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Some New Methods for Generating Convex Functions

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Abstract We present some new methods for constructing convex functions. One of the methods is based on the composition of a convex function of several variables which is separately monotone with convex and concave functions. Using several well-known results on the composition of convex and quasi-convex functions we build new convex, quasi-convex, concave, and quasi-concave functions. The third section is dedicated to the study of convexity property of symmetric Archimedean functions. In the fourth section the asymmetric Archimedean function is considered. A classical example of such a function is the Bellman function. The fifth section is dedicated to the study of convexity/concavity of symmetric polynomials. In the sixth section a new proof of Chandler–Davis theorem is given. Starting from symmetric convex functions defined on finite dimensional spaces we build several convex functions of hermitian matrices. The seventh section is dedicated to a generalization of Muirhead's theorem and to some applications of it. The last section is dedicated to the construction of convex functions based on Taylor remainder series.

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1 Introduction

Convexity testing for an arbitrary function may be non-trivial in some cases. One widely used test for convexity is to check the function's Hessian. A continuous, twice-differentiable function is convex if its Hessian is positive semidefinite every-

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where in interior of the convex set. For details we refer to the monographs [53] and [65].

The aim of the present chapter is to build several convex and concave functions (some of them defined on convex subsets of linear spaces) starting from some elementary results of the theory of convex functions. Direct checking of the convexity of the functions resulted from our constructions by computing the Hessian matrix is in the great majority of the cases a very difficult task. One of the methods used for generating convex functions is based on the composition of a convex function of several variables which is separately monotone with convex and concave functions.

Let $n \geq 2$ be a natural number, I and J be two intervals of the real axis, and $\phi : I \to J$ be a bijective function. We suppose that $J = [0, \infty)$ or $J = (0, \infty)$. Consider the function $S_\phi : I^n \to I$

$$S_\phi (x_1, x_2, \ldots, x_n) = \phi^{-1} (\phi (x_1) + \phi (x_2) + \ldots + \phi (x_n)), \quad (x_1, x_2, \ldots, x_n) \in I^n$$

We shall call S_ϕ the symmetric Archimedean function generated by the function ϕ. The symmetric Archimedean functions occur in the study of Copula functions which describe the dependence structure between random variables with arbitrary marginal distribution functions. Copula Theory is a chapter of Probability Theory.

Let $D = \{(x_1, x_2, \ldots, x_n) \in I^n : \phi (x_1) > \phi (x_2) + \ldots + \phi (x_n)\}$ and define $A_\phi : D \to I$ as follows:

$$A_\phi (x_1, x_2, \ldots, x_n) = \phi^{-1} (\phi (x_1) - \phi (x_2) - \ldots - \phi (x_n)), (x_1, x_2, \ldots, x_n) \in D.$$

A_ϕ will be called the asymmetric Archimedean function generated by the function ϕ.

The structure of the chapter is as follows. The second section contains statements of several well-known results on the composition of convex and quasi-convex functions. These results help us to build convex, quasi-convex, concave, and quasi-concave functions. The third section is dedicated to the convexity study of symmetric Archimedean functions. Several corollaries are given. In the fourth section is considered the asymmetric Archimedean function. A classical example of such a function is the Bellman function

$$f_p (x_1, x_2, \ldots, x_n) = \left(x_1^p - x_2^p - \ldots - x_n^p\right)^{\frac{1}{p}}, \ (x_1, x_2, \ldots, x_n) \in D_p$$

where

$$D_p = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ \ i = 1, 2, \ldots, n, \quad x_1^p \geq x_2^p + \ldots + x_n^p\right\} \text{ if } p > 0$$

and

$$D_p = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_i > 0, \ \ i = 1, 2, \ldots, n, \quad x_1^p > x_2^p + \ldots + x_n^p\right\} \text{ if } p < 0$$

In 1957 Bellman [5] proved that f_p is concave if $p \in [1, \infty)$. Several references on Bellman function can be found in Losonczi and Pales [43]. In the fourth section are studied convexity properties of the asymmetric Archimedean function. The fifth section is dedicated to the study of convexity/concavity of symmetric polynomials. In the sixth section a new proof of Chandler–Davis theorem is given. Starting from symmetric convex functions defined on finite dimensional spaces we build several convex functions of hermitian matrices. The seventh section is dedicated to a generalization of Muirhead's theorem and to some applications of it. The eighth section is dedicated to the construction of convex functions based on Taylor remainder series.

2 Convexity of Composite Functions

In this section we shall recall several basic results on the convexity and quasi-convexity of composite functions. Let E be a linear space over $\mathbb{R}$ and D be a convex subset of E.

Theorem 2.1 *Let I be an interval of the real axis, $u : D \to I$, $g : I \to \mathbb{R}$, $f(x) = g(u(x))$, $x \in D$. Then the following assertions hold:*

1^0 If g is increasing and convex, u is convex, then f is convex.
2^0 If g is increasing and concave, u is concave, then f is concave.
3^0 If g is decreasing and convex, u is concave, then f is convex.
4^0 If g is decreasing and concave, u is convex, then f is concave.

Corollary 2.2 *Let $u : D \to \mathbb{R}$. Then the following assertions hold:*

1^0 If u is convex on D, then e^u is convex.
2^0 If $u > 0$ on D is concave, then $\ln(u)$ is concave and $\frac{1}{u}$ is convex.
3^0. If $u > 0$ on D is concave and $p \in (0, 1]$, then u^p is concave.
4^0. If $u > 0$ on D is convex and $p \in [1, \infty)$, then u^p is convex.

Theorem 2.3 *Let $n \geq 1$ be a natural number, I_k, $k = 1, 2, \ldots, n$ be intervals of the real axis, $J = \{1, 2, \ldots, n\}$, $J_1 \subset J$, $J_2 = J_1$. Consider the functions $g : I_1 \times I_2 \times \ldots \times I_n \to \mathbb{R}$, $u_k : D \to I_k$, $k = 1, 2, \ldots, n$, $f : D \to \mathbb{R}$,*

$$f(x) = g(u_1(x), u_2(x), \ldots, u_n(x)), \quad x \in D. \tag{2.1}$$

Suppose that the following conditions hold:

(i) g is monotone increasing in the i-th variable for every $i \in J_1$.
(ii) g is monotone decreasing in the i-th variable for every $i \in J_2$.

Then the following assertions hold:

1^0. *If* g *is convex,* u_i *is convex for every* $i \in J_1, u_i$ *is concave for every* $i \in J_2$,, *then* f *is convex.*

2^0. *If* g *is concave,* u_i *is convex for every* $i \in J_2, u_i$ *is concave for every* $i \in J_1$,, *then* f *is concave.*

Theorem 2.4 *Let* I *be an interval of the real axis,* $u : D \to I$, $g : I \to \mathbb{R}$, $f(x) = g(u(x))$, $x \in D$. *Then the following assertions hold:*

1^0 *If g is increasing and u is quasi-convex, then* f *is quasi-convex.*
2^0 *If g is increasing and u is quasi-concave, then* f *is quasi-concave.*
3^0 *If g is decreasing and u is quasi-concave, then* f *is quasi-convex.*
4^0 *If g is decreasing and u is quasi-convex, then* f *is quasi-concave.*

Theorem 2.5 *Let* $n \geq 1$ *be a natural number,* I_k , $k = 1, 2, \ldots, n$ *be intervals of the real axis,* $J = \{1, 2, \ldots, n\}$, $J_1 \subset J$, $J_2 = J_1$.*Consider the functions* $g : I_1 \times I_2 \times \ldots \times I_n \to \mathbb{R}$, $u_k : D \to I_k$, $k = 1, 2, \ldots, n$, $f : D \to \mathbb{R}$ *defined as in (2.1).*

Suppose that the following conditions hold:

(i) g *is monotone increasing in the* i*-th variable for every* $i \in J_1$.
(ii) g *is monotone decreasing in the* i*-th variable for every* $i \in J_2$.

Then the following assertions hold:

1^0. *If* u_i *is quasi-convex for every* $i \in J_1, u_i$ *is quasi-concave for every* $i \in J_2$,, *then* f *is quasi-convex.*

2^0. *If* u_i *is quasi-convex for every* $i \in J_2, u_i$ *is quasi-concave for every* $i \in J_1$,, *then* f *is quasi-concave.*

3 A Study of Convexity of Symmetric Archimedean Functions

Symmetric Archimedean functions appear in the theory of Archimedean copulas [78] and in the theory of Archimedean t-norms [35] and [79]. In probability theory and statistics, a copula is a multivariate probability distribution for which the marginal-probability distribution of each variable is uniform. Copulas are used to describe the dependence between random variables. Archimedean copulas is an important class of copulas—because of the ease with which they can be constructed and the nice properties they possess. In this section we shall investigate the convexity, quasi-convexity, concavity, and quasi-concavity of the symmetric Archimedean functions

Theorem 3.1 *Let* $n \geq 2$ *be a natural number,* I, J *be two intervals of the real axis, and* $\phi : I \to J$ *be a two times differentiable bijective function. Denote by* I_1

(resp. J_1) the interior of I (resp. J). Suppose that $J = [0, \infty)$ or $J = (0, \infty)$ and $\phi'(t) \neq 0$, $\phi''(t) \neq 0$ for every $t \in I_1$. Consider the functions

$$g_n : I^n \to I,\ g_n(x_1, x_2, \ldots, x_n) = \phi^{-1}(\phi(x_1) + \phi(x_2) + \ldots + \phi(x_n)),$$

$$x_1, x_2, \ldots, x_n \in I,$$

$$v : J_1 \to J_1,\ v(t) = \frac{\phi'^2(\phi^{-1}(t))}{\phi''^{-1}(t))},\ t \in J_1.$$

Then the following assertions hold:

1° *If $\phi''(t) > 0,\ \phi'(t) > 0$ for every $t \in I_1$,*

$$v(t + s) \geq v(t) + v(s),\ t, s \in J_1,$$

then g_n is convex on I^n.

2° *If $\phi''(t) > 0,\ \phi'(t) < 0$ for every $t \in I_1$,*

$$v(t + s) \geq v(t) + v(s),\ t, s \in J_1,$$

then g_n is concave on I^n.

3° *If $\phi''(t) < 0,\ \phi'(t) > 0$ for every $t \in I_1$,*

$$v(t + s) \leq v(t) + v(s),\ t, s \in J_1,$$

then g_n is concave on I^n.

4° *If $\phi''(t) < 0,\ \phi'(t) < 0$ for every $t \in I_1$,*

$$v(t + s) \leq v(t) + v(s),\ t, s \in J_1,$$

then g_n is convex on I^n.

Proof We shall prove the assertions in the case $n = 2$. Denote $g = g_2$. Thus

$$g(x, y) = \phi^{-1}(\phi(x) + \phi(y)),\ x, y \in I_1.$$

Taking partial differentials in the equality

$$\phi(g(x, y)) = \phi(x) + \phi(y)$$

we obtain:

$$\phi'(g(x, y))g'_x(x, y) = \phi'(x)$$
$$\phi'(g(x, y))g'_y(x, y) = \phi'(y)$$

hence since $\phi'(t) \neq 0$ for every $t \in I_1$ we have

$$g'_x(x, y) = \frac{\phi'(x)}{\phi'(g(x, y))}, \ g'_y(x, y) = \frac{\phi'(y)}{\phi'(g(x, y))}.$$

Further computations give us:

$$g''_{xx}(x, y) = \frac{\phi'^2(g(x, y))\phi''(x) - \phi'^2(x)\phi''(g(x, y))}{\phi'^3(g(x, y))}$$

$$g''_{xx}(x, y) = \frac{\phi''(x)[v(\phi(x) + \phi(y)) - v(\phi(x))]}{\phi'(g(x, y))v(\phi(x) + \phi(y))}$$

$$g''_{yy}(x, y) = \frac{\phi''(y)[v(\phi(x) + \phi(y)) - v(\phi(y))]}{\phi'(g(x, y))v(\phi(x) + \phi(y))}$$

$$g''_{xy}(x, y) = -\frac{\phi'(x)\phi'(y)}{\phi'(g(x, y))v(\phi(x) + \phi(y))}.$$

Denote by $H(x, y)$ the determinant of the Hessian matrix of g. Note that

$$H(x, y) = g''_{xx}(x, y)g''_{yy}(x, y) - [g''_{xy}(x, y)]^2$$

$$= \frac{\phi''(x)\phi''(y)[v(\phi(x) + \phi(y)) - v(\phi(x)) - v(\phi(y))]}{\phi'^2(g(x, y))v(\phi(x) + \phi(y))}.$$

From conditions on ϕ from assertion 1° it follows that

$$g''_{xx} \geq 0, \ g''_{yy} \geq 0, \ H \geq 0$$

on I_1^2, hence g is convex. One can easily check that the other assertions hold in the case $n = 2$.

We return to the study of the case $n \geq 2$. Note that

$$g_n(x_1, x_2, \ldots, x_n) = g_{n-1}(g_2(x_1, x_2), x_3, x_3, \ldots, x_n)$$

and g_n is separately increasing, that is it is increasing in each variable.

Suppose that g_2 is convex. Then from equation

$$g_3(x_1, x_2, x_3) = g_2(g_2(x_1, x_2), x_3) \tag{3.1}$$

it follows that g_3 is convex. By induction we obtain that g_n is convex for every $n \geq 2$.

Suppose now that g_2 is concave. Then from Eq. (3.1) and from Theorem 2.3 it follows that g_3 is concave. By induction we obtain that g_n is concave for every $n \geq 2$. □

Corollary 3.2 *Let $n \geq 1$, $J_p = [0, \infty)$ if $p \in (0, \infty)$ and $J_p = (0, \infty)$ if $p \in (-\infty, 0)$. For every $p \in \mathbb{R}^* = \mathbb{R} - \{0\}$ consider the functions:*

$$f_p : J_p^n \to J_p,\ f_p(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^{n} x_i^p\right)^{1/p}, \tag{3.2}$$
$$(x_1, x_2, \ldots, x_n) \in J_p^n$$

and

$$g : \mathbb{R}^n \to \mathbb{R},\ g(x_1, x_2, \ldots, x_n) = \ln(e^{x_1} + e^{x_2} + \ldots + e^{x_n}), \tag{3.3}$$
$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n.$$

Then the following assertions hold:

1° *If $p \in [1, \infty)$, then f_p is convex on J_p^n.*
2° *If $p \in (-\infty, 0) \cup (0, 1]$, then f_p is concave on J_p^n.*
3° *g is convex on $\mathbb{R}^n$.*

Proof For every $p \in \mathbb{R}^*$ consider the function

$$\phi_p : J_p \to J_p,\ \phi_p(t) = t^p,\ t \in J_p.$$

Note that ϕ_p is bijective,

$$\phi_p^{-1}(t) = t^{1/p},\ \phi_p^{-1}(t) = t^{1/p},\ \phi_p'(t) = pt^{p-1},$$
$$\phi_p''(t) = p(p-1)t^{p-2},\ t \in J_p.$$

For every $p \in \mathbb{R}^* - \{1\}$ we have

$$v_p(t) = \frac{\phi_p'^2(\phi_p^{-1}(t))}{\phi_p''(\phi_p^{-1}(t))} = \frac{\phi_p'^2(t^{1/p})}{\phi_p''(t^{1/p})} = \frac{p^2(t^{1/p})^{p-2}}{p(p-1)(t^{1/p})^{p-2}} = \frac{p}{p-1}t,\ t \in J_p.$$

Note that

$$v_p(t+s) = v_p(t) + v_p(s),\ s, t \in J_p,\ p \in \mathbb{R}^* - \{1\}.$$

If $p = 1$, then f_p is convex and concave.

If $p \in (1, \infty)$, then $\phi_p'(t) > 0$, $\phi_p''(t) > 0$, $t \in J_p$. By assertion 1° of Theorem 3.1 it follows that f_p is convex.

If $p \in (0, 1)$, then $\phi'_p(t) > 0$, $\phi''_p(t) < 0$, $t \in J_p$. By assertion 3° of Theorem 3.1 it follows that f_p is concave.

If $p \in (-\infty, 0)$, then $\phi'_p(t) < 0$, $\phi''_p(t) > 0$, $t \in J_p$. By assertion 2° of Theorem 3.1 it follows that f_p is concave.

Let $\phi : \mathbb{R} \to (0, \infty)$, $\phi(t) = e^t$, $t \in \mathbb{R}$. Note that ϕ is bijective and we have

$$\phi'(t) > 0,\ \phi''(t) > 0,$$

$$v(t) = \frac{\phi'^2(\phi^{-1}(t))}{\phi''^{-1}(t))} = t,\ t \in \mathbb{R}.$$

By assertion 1° of Theorem 3.1 it follows that g is convex on $\mathbb{R}^n$. □

Corollary 3.3 *Let $J_p = [0, \infty)$ if $p \in (0, \infty)$ and $J_p = (0, \infty)$ if $p \in (-\infty, 0)$. Suppose that E is a linear space and D is a convex subset of E. For every $p \in \mathbb{R}^*$ consider the functions $u_{k,p} : D \to J_p$, $k = 1, 2, \ldots, n$, and $f_p : J_p^n \to \mathbb{R}$,*

$$f_p(x_1, x_2, \ldots, x_n) = (u_{1,p}^p(x) + u_{2,p}^p(x) + \ldots + u_{n,p}^p(x))^{1/p},\ x \in D.$$

Let $u_i : D \to \mathbb{R}$, $i = 1, 2, \ldots, n$, and $g : D \to \mathbb{R}$,

$$g(x) = \ln(e^{u_1(x)} + e^{u_2(x)} + \ldots + e^{u_n(x)}),\ x \in D.$$

Then the following assertions hold:

1° *If $p \in [1, \infty)$, $u_{i,p}$ are convex for $i = 1, 2, \ldots, n$, then f_p is convex.*
2° *If $p \in (-\infty, 0) \cup (0, 1)$, $u_{i,p}$ are concave for every $i = 1, 2, \ldots, n$, then f_p is concave.*
3° *If u_i are convex for $i = 1, 2, \ldots, n$, then g is convex.*

Proof The validity of the assertions follows at once from the preceding corollary and from Theorem 2.3. □

Corollary 3.4 *For every p, $p_i \in (0, \infty)$, $i = 1, 2, \ldots, n$, consider the functions*

$$g_p(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^n |x_i|^{p_i}\right)^{1/p},\ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

$$h_p(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^n x_i^{p_i}\right)^{1/p},\ (x_1, x_2, \ldots, x_n) \in [0, \infty)^n.$$

For every p, $p_i \in (-\infty, 0)$ consider the function

$$w_p(x_1, x_2, \ldots, x_n) = \left(\sum_{i=1}^n x_i^{p_i}\right)^{1/p},\ (x_1, x_2, \ldots, x_n) \in (0, \infty)^n.$$

Let $p_1, p_2, \dots, p_n \in \mathbb{R}^*$. *Denote*

$$a = \min_{1 \le i \le n} (p_i), \ b = \max_{1 \le i \le n} (p_i).$$

Then the following assertions hold:

1° *If* $p_1, p_2, \dots, p_n \in [1, \infty)$, $p \in (0, a]$, *then* g_p *is convex.*
2° *If* $p_1, p_2, \dots, p_n \in (0, 1]$, $p \in [b, \infty)$, *then* h_p *is concave.*
3° *If* $p_1, p_2, \dots, p_n \in (-\infty, 0)$, $p \in (-\infty, a]$, *then* w_p *is concave.*

Proof Let f_p be the function defined by (3.2). Suppose that hypotheses of assertion 1° hold. Note that $a \in [1, \infty)$. By Corollary 3.2 it follows that f_a is convex. For every $i = 1, 2, \dots, n$ let

$$u_i : \mathbb{R}^n \to \mathbb{R}, \ u_i(x_1, x_2, \dots, x_n) = |x_i|^{p_i/a}, \ (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

Consider the function

$$u : \mathbb{R}^n \to \mathbb{R}^n, \ u(x) = (u_1(x), u_2(x), \dots, u_n(x)), \ x \in \mathbb{R}^n.$$

Note that

$$g_p(x_1, x_2, \dots, x_n) = \left(\sum_{i=1}^n |x_i|^{p_i} \right)^{1/p} = \left(\sum_{i=1}^n [u_i(x)]^a \right)^{1/p} = [f_a(u(x))]^{a/p},$$
$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n.$$

Since all u_i are convex and f_a is convex, by Theorem 3.2 it follows that $f_a \circ u$ is convex. Since $\dfrac{a}{p} \ge 1$ it follows that g_p is convex.

Suppose that hypotheses of assertion 2° hold. Note that $b \in (0, 1]$. By Corollary 3.2, function f_b is concave. For every $i = 1, 2, \dots, n$ let

$$v_i(x_1, x_2, \dots, x_n) = x_i^{p_i/b}, \ (x_1, x_2, \dots, x_n) \in [0, \infty)^n.$$

Consider the function

$$v(x) = (v_1(x), v_2(x), \dots, v_n(x)), \ x \in [0, \infty)^n.$$

Note that all v_i are concave and

$$h_p(x) = \left(\sum_{i=1}^n [v_i(x)]^b \right)^{1/p} = [f_b(v(x))]^{b/p}, \ x \in [0, \infty)^n.$$

By Theorem 3.2 the function $f_b \circ v$ is concave. Since $\frac{b}{p} \leq 1$ it follows that h_p is concave. Suppose that hypotheses of assertion 3° hold. By Corollary 3.2 the function f_a is concave. Since $\frac{p_i}{a} \leq 1$ it follows that u_i are concave on $(0, \infty)^n$. Note that

$$w_p(x) = [f_a(u(x))]^{a/p}, \ x \in (0, \infty)^n.$$

By Theorem 3.2 the function $f_a \circ u$ is concave on $(0, \infty)^n$. Since $\frac{a}{p} \leq 1$ it follows that w_p is concave. □

Corollary 3.5 *Let $p_i > 0$, $i = 1, 2, \ldots, n$. Then the function*

$$f : (0, \infty)^n \to \mathbb{R}, \ f(x_1, x_2, \ldots, x_n) = \ln(x_1^{-p_1} + x_2^{-p_2} + \ldots + x_n^{-p_n}),$$
$$(x_1, x_2, \ldots, x_n) \in (0, \infty)^n$$

is convex.

Proof Let g be the function defined by (3.3). For every $i = 1, 2, \ldots, n$ consider the function

$$u_i(x_1, x_2, \ldots, x_n) = -p_i \ln(x_i), \ (x_1, x_2, \ldots, x_n) \in (0, \infty)^n.$$

Note that

$$f(x) = g(u_1(x), u_2(x), \ldots, u_n(x)), \ x \in (0, \infty)^n$$

and all u_i are convex.

Since g is convex it follows from Corollary 3.3 that f is convex. □

Corollary 3.6 *Let $n \geq 1$ and $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial function with positive coefficients. Then the function*

$$f(x_1, x_2, \ldots, x_n) = \ln(P(e^{x_1}, e^{x_2}, \ldots, e^{x_n})), \ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

is convex.

Proof Let $\alpha_k = (\alpha_{k1}, \alpha_{k2}, \ldots, \alpha_{kn}) \in \mathbb{N}^n$, $a_k > 0$, $k = 1, 2, \ldots, m$ and

$$P(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{m} a_k x_1^{\alpha_{k1}} x_2^{\alpha_{k2}} \ldots x_n^{\alpha_{kn}}.$$

If

$$u_k(x_1, x_2, \ldots, x_n) = \alpha_{k1}x_1 + \alpha_{k2}x_2 + \ldots + \alpha_{kn}x_n + \ln(a_k),$$
$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

then

$$f(x_1, x_2, \ldots, x_n) = \ln[P(e^{x_1}, e^{x_2}, \ldots, e^{x_n})]$$

$$= \ln\left[\sum_{k=1}^{m} a_k \exp\left(\sum_{i=1}^{n} \alpha_{ki} x_i\right)\right] = \ln\left[\sum_{k=1}^{m} \exp(u_k(x_1, x_2, \ldots, x_n))\right].$$

If g is the function defined by (3.3), then

$$f(x) = g(u_1(x), u_2(x), \ldots, u_n(x)),\ x \in \mathbb{R}^n.$$

Since g is convex and all u_i are convex, by Theorem 2.3 it follows that f is also convex. □

Lemma 3.7 *Let I, J be two intervals of the real axis, $\phi : I \to J$ be a bijective, two times differentiable function. Suppose that $\phi'(t) \neq 0$, $\phi''(t) \neq 0$ for every $t \in I$. Consider the functions*

$$u : I \to \mathbb{R},\ u(t) = \frac{1}{\phi'(t)},\ t \in I$$

$$v(t) = \frac{\phi'^2(\phi^{-1}(t))}{\phi''^{-1}(t))},\ t \in J.$$

Then the following assertions hold:

1° *If u is convex, then $v \circ \phi$ is increasing.*
2° *If u is concave, then $v \circ \phi$ is decreasing.*

Proof Note that

$$u'(t) = -\frac{1}{v(\phi(t))},\ t \in I.$$

If u is convex, then u' is increasing, hence $v \circ \phi$ is increasing.
If u is concave, then u' is decreasing, hence $v \circ \phi$ is decreasing. □

Theorem 3.8 *Let I, I_1, J, J_1, ϕ be as in the statement of Theorem 3.1. Suppose that*

$$\phi'(t) \neq 0,\ \phi''(t) \neq 0,\ t \in I_1.$$

Consider the functions

$$g_n(x_1, x_2, \ldots, x_n) = \phi^{-1}(\phi(x_1) + \phi(x_2) + \ldots + \phi(x_n)),\ x_1, x_2, \ldots, x_n \in I \tag{3.4}$$

$$u : I_1 \to \mathbb{R},\ u(t) = \frac{1}{\phi'(t)},\ t \in I_1$$

$$v : J_1 \to J_1,\ v(t) = \frac{\phi'^2(\phi^{-1}(t))}{\phi''^{-1}(t))},\ t \in J_1.$$

Then the following assertions hold:

1° *If u is convex on I_1, then g_n is separately convex (i.e. g_n is convex in each variable).*
2° *If u is concave on I_1, then g_n is separately concave (i.e. g_n is concave in each variable).*

Proof Note that for $i = 1, 2, \ldots, n$ we have

$$\frac{\partial g_n}{\partial x_i}(x) = \frac{\phi'(x_i)}{\phi'(g_n(x))},$$

$$\frac{\partial g_n}{\partial x_i^2}(x) = \frac{\phi''(x_i)\phi'(g_n(x)) - \phi'^2(x_i)\phi''(g_n(x_i))}{\phi'^3(g_n(x))}$$

$$= \frac{\phi''(g_n(x))\phi''(x_i)[v(\phi(g_n(x))) - v(\phi(x_i))]}{\phi'^3(g_n(x))},$$

$$x = (x_1, x_2, \ldots, x_n) \in I_1^n.$$

Suppose that u is convex. By Lemma 3.7 it follows that $v \circ \phi$ is increasing.

If $\phi' > 0$ on I_1, then ϕ^{-1} is increasing hence v is increasing. Note that

$$\frac{\partial^2 g_n}{\partial x_i^2} \geq 0 \text{ on } I_1^n. \tag{3.5}$$

If $\phi' < 0$ on I_1, then ϕ^{-1} is decreasing, hence v is decreasing. Note that (3.5) holds again. We proved that g_n is separately convex on I_1^n. Since g_n is continuous it follows that g_n is separately convex on I^n.

Assertion 2° follows with a similar argument. □

Theorem 3.9 *Let $n \geq 2$ be a natural number, I, J be two intervals of the real axis. Suppose that $J = [0, \infty)$ or $J = (0, \infty)$. Let $\phi : I \to J$ be a bijective continuous function and $g_n : I^n \to I$ be the function defined by (3.4). Then the following assertions hold:*

1° *If ϕ is increasing and convex, then g_n is quasi-convex.*
2° *If ϕ is increasing and concave, then g_n is quasi-concave.*
3° *If ϕ is decreasing and convex, then g_n is quasi-concave.*
4° *If ϕ is decreasing and concave, then g_n is quasi-convex.*

Proof Let

$$\psi_n(x_1, x_2, \ldots, x_n) = \phi(x_1) + \phi(x_2) + \ldots + \phi(x_n),\ x_1, x_2, \ldots, x_n \in I.$$

Note that we have $g_n = \phi^{-1} \circ \psi_n$.

In order to prove assertion 1° suppose that ϕ is increasing and convex. Then ψ_n is convex and ϕ^{-1} is increasing. By Theorem 2.4 it follows that g_n is quasi-convex.

The other assertions can be proved by a similar argument. □

Theorem 3.10 *Let $n \geq 2$ be a natural number, I, J be two intervals of the real axis, and $\phi : I \to J$ be a bijective differentiable function. Consider the function $g_n : I^n \to I$ defined by (3.4). Suppose that $\phi'(t) \neq 0$ for every $t \in I$. Then the following assertions hold:*

1° *If $\phi' > 0$ on I and ϕ is convex, then g_n is Schur-convex.*
2° *If $\phi' < 0$ on I and ϕ is convex, then g_n is Schur-concave.*
3° *If $\phi' > 0$ on I and ϕ is concave, then g_n is Schur-concave.*
4° *If $\phi' < 0$ on I and ϕ is concave, then g_n is Schur-convex.*

Proof Note that g_n is symmetric. In order to decide the Schur-convexity or Schur-concavity of g_n we have to study the sign of

$$A(x) = (x_1 - x_2)\left(\frac{\partial g_n}{\partial x_1}(x) - \frac{\partial g_n}{\partial x_2}(x)\right),\ x = (x_1, x_2, \ldots, x_n) \in I^n.$$

Note that

$$A(x) = \frac{(x_1 - x_2)(\phi'(x_1) - \phi'(x_2))}{\phi'(g_n(x))},\ x = (x_1, x_2, \ldots, x_n) \in I^n.$$

All the assertions from the statement of the theorem follow at once by computing the sign of $A(x)$. □

4 A Study of Convexity of Asymmetric Archimedean Functions

Theorem 4.1 *Let I, J be two intervals of the real axis and $\phi : I \to J$ be a differentiable bijective function. Denote by I_1 (resp. J_1) the interior of I (resp. J). Suppose that $J = [0, \infty)$ or $J = (0, \infty)$ and $\phi'(t) \neq 0$, $\phi''(t) \neq 0$ for every $t \in I_1$. Let $v : J_1 \to J_1$,*

$$v(t) = \frac{\phi'^2(\phi^{-1}(t))}{\phi''^{-1}(t))},\ t \in J_1 \tag{4.1}$$

$$D = \{(x, y) \in I^2 : \ \phi(x) > \phi(y)\},$$

$$g : D \to I, \ g(x, y) = \phi^{-1}(\phi(x) - \phi(y)), \ (x, y) \in D.$$

Then the following assertions hold:

1° *If* $\phi''(t) > 0$, $\phi'(t) > 0$ *for every* $t \in I_1$ *and*

$$v(t + s) \geq v(t) + v(s), \ t, s \in J_1$$

then g *is concave on* D.

2° *If* $\phi''(t) > 0$, $\phi'(t) < 0$ *for every* $t \in I_1$ *and*

$$v(t + s) \geq v(t) + v(s), \ t, s \in J_1$$

then g *is convex on* D.

3° *If* $\phi''(t) < 0$, $\phi'(t) > 0$ *for every* $t \in I_1$ *and*

$$v(t + s) \leq v(t) + v(s), \ t, s \in J_1$$

then g *is convex on* D.

4° *If* $\phi''(t) < 0$, $\phi'(t) < 0$ *for every* $t \in I_1$ *and*

$$v(t + s) \leq v(t) + v(s), \ t, s \in J_1$$

then g *is concave on* D.

Proof Note that D is a convex subset of I^2,

$$\phi(g(x, y)) = \phi(x) - \phi(y), \ v(\phi(x)) = \frac{\phi'^2(x)}{\phi''(x)}$$

$$g'_x(x, y) = \frac{\phi'(x)}{\phi'(g(x, y))}, \ g'_y(x, y) = -\frac{\phi'(y)}{\phi'(g(x, y))}$$

$$g''_{xx}(x, y) = \frac{\phi''(x)(v(\phi(x) - \phi(y)) - v(\phi(x)))}{\phi'(g(x, y))v(\phi(x) - \phi(y))}$$

$$g''_{yy}(x, y) = -\frac{\phi''(y)(v(\phi(y)) + v(\phi(x) - \phi(y)))}{\phi'(g(x, y))v(\phi(x) - \phi(y))}$$

$$g''_{xy}(x, y) = \frac{\phi'(x)\phi'(y)}{\phi'(g(x, y))v(\phi(x) - \phi(y))}$$

Let H_g be the determinant of the Hessian matrix of g. Note that

$$H_g(x, y) = g''_{xx}(x, y)g''_{yy}(x, y) - [g''_{xy}(x, y)]^2$$

$$= \frac{\phi''(x)\phi''(y)[v(\phi(x)) - v(\phi(x) - \phi(y))][v(\phi(y)) + v(\phi(x) - \phi(y))] - \phi'^2(x)\phi'^2(y)}{\phi'^2(g(x, y))v^2(\phi(x) - \phi(y))}$$

$$= \frac{\phi''(x)\phi''(y)[v(\phi(x)) - v(\phi(y)) - v(\phi(x) - \phi(y))]}{\phi'^2(g(x, y))v(\phi(x) - \phi(y))}$$

A direct check shows that assertions 1°–4° hold. □

Theorem 4.2 *Let I, J, I_1, J_1, ϕ, v be defined as in the statement of Theorem 4.1. For every natural number $n \geq 2$ let*

$$D = \{(x_1, x_2, \ldots, x_n) \in I^n \mid \phi(x_1) > \phi(x_2) + \ldots + \phi(x_n)\}$$

and $g_n : D \to I$,

$$g_n(x_1, x_2, \ldots, x_n) = \phi^{-1}(\phi(x_1) - \phi(x_2) - \ldots - \phi(x_n)), \ (x_1, x_2, \ldots, x_n) \in D.$$

If U is a convex subset of D, then the following assertions hold:

1° *If $\phi''(t) > 0$, $\phi'(t) > 0$ for $t \in I_1$ and*

$$v(t + s) \geq v(t) + v(s), \ s, t \in J_1,$$

then D is convex and g_n is concave on D.

2° *If $\phi''(t) > 0$, $\phi'(t) < 0$ for $t \in I_1$ and*

$$v(t + s) \geq v(t) + v(s), \ s, t \in J_1,$$

then D is convex and g_n is convex on D.

3° *If $\phi''(t) < 0$, $\phi'(t) > 0$ for $t \in I_1$and*

$$v(t + s) \leq v(t) + v(s), \ s, t \in J_1,$$

then g_n is convex on D.

4° *If $\phi''(t) < 0$, $\phi'(t) < 0$ for $t \in I_1$and*

$$v(t + s) \leq v(t) + v(s), \ s, t \in J_1,$$

then g_n is concave on D.

Proof For every $n \geq 2$ let

$$u_n(x_1, x_2, \ldots, x_n) = \phi^{-1}(\phi(x_1) + \phi(x_2) + \ldots + \phi(x_n)),$$
$$(x_1, x_2, \ldots, x_n) \in I^n.$$

Note that

$$g_n(x_1, x_2, \ldots, x_n) = g_2(x_1, u_{n-1}(x_2, x_3, \ldots, x_n)) \tag{4.2}$$

Suppose that the hypotheses of assertion 1° hold. Note that

$$D = \{(x_1, x_2, \ldots, x_n) \in I^n \mid x_1 > \phi^{-1}(\phi(x_2) + \ldots + \phi(x_n))\}$$

$$= \{(x_1, x_2, \ldots, x_n) \in I^n \mid x_1 > u_{n-1}(x_1, x_2, \ldots, x_n)\}.$$

By Theorem 3.1, u_{n-1} is convex, hence D is convex. Note that g_2 is concave, is increasing in the first argument and decreasing in the second argument. By Theorem 2.3, g_n is concave.

Suppose now that the hypotheses of assertion 2° hold. Note that

$$D = \{(x_1, x_2, \ldots, x_n) \in I^n \mid x_1 < \phi^{-1}(\phi(x_2) + \ldots + \phi(x_n))\}$$

$$= \{(x_1, x_2, \ldots, x_n) \in I^n \mid x_1 < u_{n-1}(x_2, x_3, \ldots, x_n)\}.$$

By Theorem 3.1, u_{n-1} is concave, hence D is convex. By Theorem 4.1, g_2 is convex. By (4.2) and by Theorem 2.3, it follows that g_n is convex. Similar arguments apply for the proof of assertions 3° and 4°. □

Corollary 4.3 *Let*

$$D_p = \{(x, y) \in [0, \infty)^2 \mid x > y\} \text{ if } p \in (0, \infty) \text{ and}$$

$$D_p = \{(x, y) \in (0, \infty)^2 \mid x < y\} \text{ if } p \in (-\infty, 0).$$

For $p \in \mathbb{R}^* = \mathbb{R} - \{0\}$ *consider the function*

$$f_p(x, y) = (x^p - y^p)^{\frac{1}{p}},\ (x, y) \in D_p.$$

Then the following assertions hold:

1° *If* $p \in [1, \infty)$*, then* f_p *is concave on* D_p*.*
2° *If* $p \in (-\infty, 0) \cup (0, 1]$*, then* f_p *is convex on* D_p*.*

Proof Note that for every $p \in \mathbb{R}^*$ the set D_p is convex. If $p = 1$, then f_p is convex and concave. Suppose that $p \in (1, \infty)$ and let

$$\phi_p(t) = t^p,\ t \in [0, \infty) \tag{4.3}$$

$$v_p(t) = \frac{\phi_p'^2(\phi_p^{-1}(t))}{\phi_p''(\phi_p^{-1}(t))},\ t \in (0, \infty). \tag{4.4}$$

Note that

$$\phi_p'(t) > 0,\ \phi_p''(t) > 0,\ v_p(t+s) = v_p(t) + v_p(s),\ t, s \in (0, \infty).$$

By Theorem 4.1 it follows that f_p is concave on D_p.

Suppose that $p \in (0, 1)$. Note that

$$\phi_p'(t) > 0,\ \phi_p''(t) < 0,\ t \in (0, \infty)$$

and v_p is additive on $(0, \infty)$. By Theorem 4.1 it follows that f_p is convex on D_p.

Suppose that $p \in (-\infty, 0)$. Note that

$$\phi_p'(t) < 0,\ \phi_p''(t) > 0,\ t \in (0, \infty)$$

and v_p is additive on $(0, \infty)$. By Theorem 4.1 f_p is convex on D_p. □

Theorem 4.4 *Let $J_p = [0, \infty)$ if $p \in (0, \infty)$ and $J_p = (0, \infty)$ if $p \in (-\infty, 0)$. For every $p \in \mathbb{R}^*$ consider the set*

$$D_p = \{(x_1, x_2, \ldots, x_n) \in J_p^n \mid x_1^p > x_2^p + x_3^p + \ldots + x_n^p\}$$

and the function $f_p : D_p \to \mathbb{R}$,

$$f_p(x_1, x_2, \ldots, x_n) = (x_1^p - x_2^p - \ldots - x_n^p)^{\frac{1}{p}},\ (x_1, x_2, \ldots, x_n) \in D_p. \tag{4.5}$$

Then the following assertions hold:

1° *D_p is convex for $p \in (-\infty, 0) \cup [1, \infty)$.*
2° *If $p \in [1, \infty)$, then f_p is concave on D_p.*
3° *If $p \in (-\infty, 0)$, then f_p is convex on D_p.*
4° *If $p \in (0, 1)$, then f_p is convex on every convex subset of D_p.*
5° *If $p \in (0, 1)$, then D_p is convex if and only if $n = 2$.*

Proof Let ϕ_p and v_p be defined as in (4.3) and (4.4). Note that

$$\phi_p'(t) = pt^{p-1},\ \phi_p''(t) = p(p-1)t^{p-2},\ v_p(t) = \frac{t}{p-1},\ t \in J_p.$$

Consider the function

$$\psi_p(x_1, x_2, \ldots, x_{n-1}) = (x_1^p + x_2^p + \ldots + x_{n-1}^p)^{\frac{1}{p}},\ (x_1, x_2, \ldots, x_{n-1}) \in J_p^{n-1}.$$

If $p \in [1, \infty)$, then ψ_p is convex. Note that the function

$$u_p(x_1, x_2, \ldots, x_n) = \psi_p(x_2, x_3, \ldots, x_n) - x_1,\ (x_1, x_2, \ldots, x_n) \in D_p$$

is convex and $D_p = \{x \in J_p^n \mid u_p(x) < 0\}$. Hence D_p is convex.

If $p \in (-\infty, 0)$, then ψ_p and u_p are concave and

$$D_p = \{x \in J_p^n \mid u_p(x) > 0\}.$$

Consequently D_p is convex.

Suppose that $p \in (1, \infty)$. Note that

$$\phi_p'(t) > 0,\ \phi_p''(t) > 0,\ v_p(t+s) = v_p(t) + v_p(s),\ t, s \in (0, \infty).$$

From Theorem 4.2 it follows that f_p is concave on D_p.

If $p \in (-\infty, 0)$, then $\phi_p'(t) < 0, \phi_p''(t) > 0, t \in (0, \infty)$ and v_p is additive. From Theorem 4.2 it follows that f_p is convex.

If $p \in (0, 1)$, then $\phi_p' > 0$, $\phi_p'' < 0$, v_p is additive on $(0, \infty)$. If U is a convex subset of D_p, then from Theorem 4.2 it follows that f_p is convex on U.

In order to prove assertion 5° let $p \in (0, 1)$ and $a \in \left(1, 2^{\frac{1-p}{p}}\right)$. If $n = 2$ one can easily see that D_p is convex. If $n \geq 3$ let

$$x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n),\ z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n,$$

$$x_1 = a,\ x_2 = 0,\ x_3 = 1,\ y_1 = a,\ y_2 = 1,\ y_3 = 0,\ x_i = y_i = 0 \text{ for } i \in \{4, 5, ..n\}.$$

Note that $x \in D_p$ if and only if $y \in D_p$ which is equivalent with $a > 1$. Now, observe that $z = \frac{1}{2}(x + y)$ does not belong to D_p. Hence D_p is not convex. □

Theorem 4.5 *For every $p \in \mathbb{R}^*$ let J_p be defined as in the statement of the preceding Theorem. Let E be a linear space, D be a convex subset of E. For every $p \in \mathbb{R}^*$ consider*

$$u_{i,p} : D \to J_p,\ i = 1, 2, \dots, n,$$

$$U_p = \{x \in D \mid u_{1,p}^p(x) > u_{2,p}^p(x) + \ldots + u_{n,p}^p(x)\},$$

$$g_p(x) = (u_{1,p}^p(x) - u_{2,p}^p(x) - \ldots - u_{n,p}^p(x))^{\frac{1}{p}},\ x \in U_p,$$

$$h_p(x) = \ln[u_{1,p}^p(x) - u_{2,p}^p(x) - \ldots - u_{n,p}^p(x)],\ x \in U_p,$$

$$w_p = (u_{1,p}^p(x) - u_{2,p}^p(x) - \ldots - u_{n,p}^p(x))^a,\ x \in U_p.$$

Then the following assertions hold:

1° *If $p \in [1, \infty)$, $a \in (-\infty, 0)$, $u_{1,p}$ is concave and $u_{i,p}$ are convex for $i \in \{2, 3, \dots, n\}$, then g_p and h_p are concave on U_p and w_p is convex.*

2° *If $a, p \in (-\infty, 0)$, $ap \in [1, \infty)$, $u_{1,p}$ is convex and $u_{i,p}$ are concave for every $i \in \{2, 3, \dots, n\}$, then w_p is convex on U_p.*

Proof Let f_p be defined as in (4.5). Suppose that $p \in [1, \infty)$ and $a \in (-\infty, 0)$. Note that

$$g_p(x) = f_p(u_1(x), u_2(x), \dots, u_n(x)), \; x \in U_p.$$

By Theorem 4.4 it follows that f_p is concave. By Theorem 2.3 it follows that g_p is concave. By Corollary 2.2 it follows that h_p is concave and w_p is convex.

Suppose that the hypotheses of assertion 2° hold. Note that

$$w_p(x) = [f_p(u_1(x), u_2(x), \dots, u_n(x))]^{ap}, \; x \in U_p.$$

By Theorem 4.4 it follows that f_p is convex. Using Theorem 2.3 it follows that g_p is convex. By Corollary 2.2 it follows that w_p is convex on U_p. □

Corollary 4.6 *Let $n \geq 2$, $p \geq 1$, E be a linear normed space, D be a convex subset of E, and $v : E^n \to \mathbb{R}$ be a linear function. Consider the set*

$$U_p = \left\{ (x_1, x_2, \dots, x_n) \in D^n \mid v(x_1, x_2, \dots, x_n) > \left(\sum_{i=2}^{n} \|x_i\|^p \right)^{\frac{1}{p}} \right\}$$

and the function $g_p : U_p \to \mathbb{R}$,

$$g_p(x_1, x_2, \dots, x_n) = \left([v(x_1, x_2, \dots, x_n)]^p - \sum_{i=2}^{n} \|x_i\|^p \right)^{\frac{1}{p}},$$

$$(x_1, x_2, \dots, x_n) \in U_p.$$

Then g_p is concave on U_p.

Proof Let

$$u_1 = v, \; u_i(x_1, x_2, \dots, x_n) = \|x_i\|, \; i = 2, 3, \dots, n, \; (x_1, x_2, \dots, x_n) \in D^n.$$

Consider the function

$$u(x) = (u_1(x), u_2(x), \dots, u_n(x)), \; x \in D^n.$$

Note that u_1 is concave and $u_2, u_3, \dots, u_n$ are convex. If f_p is defined by (4.5), then

$$g_p(x) = f_p(u(x)), \; x \in U_p.$$

By Theorem 4.5 it follows that g_p is concave on U_p. □

Corollary 4.7 *For every $p \in \mathbb{R}^*$ let*

$$U_p = \left\{(z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \mid \mathrm{Re}(z_1) > (|z_2|^p + |z_3|^p + \ldots + |z_n|^p)^{\frac{1}{p}}\right\},$$

$$g_p(z_1, z_2, \ldots, z_n) = (|\mathrm{Re}(z_1)|^p - |z_2|^p - |z_3|^p - \ldots - |z_n|^p)^{\frac{1}{p}},$$

$$(z_1, z_2, \ldots, z_n) \in U_p.$$

If $p \in [1, \infty)$, then g_p is concave on U_p.

Proof Let

$$u_1(z_1, z_2, \ldots, z_n) = \mathrm{Re}(z_1),\ u_k(z_1, z_2, \ldots, z_n) = |z_k|,$$

$$k \in \{2, 3, \ldots, n\},\ (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n.$$

Let f_p be defined by (4.5). Note that

$$g_p(z) = f_p(u_1(z), u_2(z), \ldots, u_n(z)),\ z \in U_p$$

and u_1 is concave and $u_2, u_3, \ldots, u_n$ are convex. By the preceding Corollary 4.6 it follows that g_p is concave on U_p. □

Corollary 4.8 *For every $n \geq 2$, $p \in [1, \infty)$, $a < 0$ consider*

$$D_{p,n} = \{(x_1, x_2, \ldots, x_n) \in [0, \infty)^n \mid x_1^p > x_2^p + x_3^p + \ldots + x_n^p\}$$

$$U_{p,n} = \left\{(x_1, x_2, \ldots, x_n) \in [0, \infty)^n \mid \left(\sum_{i=1}^n x_i\right)^p > \sum_{i=1}^n x_i^p\right\}$$

$$g_{1,p,n}(x_1, x_2, \ldots, x_n) = \ln(x_1^p - x_2^p - \ldots - x_n^p),\ (x_1, x_2, \ldots, x_n) \in D_{p,n}$$

$$g_{2,p,n}(x_1, x_2, \ldots, x_n) = (x_1^p - x_2^p - \ldots - x_n^p)^a,\ (x_1, x_2, \ldots, x_n) \in D_{p,n}$$

$$g_{3,p,n}(x_1, x_2, \ldots, x_n) = \ln\left[\left(\sum_{i=1}^n x_i\right)^p - \left(\sum_{i=1}^n x_i^p\right)\right],\ (x_1, x_2, \ldots, x_n) \in U_{p,n}$$

$$g_{4,p,n}(x_1, x_2, \ldots, x_n) = \left[\left(\sum_{i=1}^n x_i\right)^p - \left(\sum_{i=1}^n x_i^p\right)\right]^a,\ (x_1, x_2, \ldots, x_n) \in U_{p,n}.$$

If $p \in [1, \infty)$ then the following assertions hold:

1° $D_{p,n}$ *and* $U_{p,n}$ *are convex sets.*
2° $g_{1,p,n}$ *and* $g_{3,p,n}$ *are concave.*
3° $g_{3,p,n}$ *and* $g_{4,p,n}$ *are convex.*

Proof For every $k \geq 1$ let $h_{k,p} : [0, \infty)^k \to \mathbb{R}$ be defined as follows:

$$h_{k,p}(x_1, x_2, \ldots, x_k) = (x_1^p + x_2^p + \ldots + x_k^p)^{\frac{1}{p}}, \ (x_1, x_2, \ldots, x_k) \in [0, \infty)^k.$$

By Corollary 3.2, $h_{k,p}$ is convex. Since

$$D_{p,n} = \{(x_1, x_2, \ldots, x_n) \in [0, \infty)^n \mid x_1 > h_{n-1,p}(x_2, x_3, \ldots, x_n)\}$$

$$U_{p,n} = \left\{(x_1, x_2, \ldots, x_n) \in [0, \infty)^n \mid \sum_{i=1}^{n} x_i > h_{n,p}(x_1, x_2, \ldots, x_n)\right\}$$

it follows that $D_{p,n}$ and $U_{p,n}$ are convex sets.

By Theorem 4.5 it follows that $g_{1,p}$ is concave and $g_{2,p}$ is convex. Since

$$g_{3,p,n}(x_1, x_2, \ldots, x_n) = g_{1,p}\left(\sum_{i=1}^{n} x_i, x_1, x_2, \ldots, x_n\right)$$

and

$$g_{4,p,n}(x_1, x_2, \ldots, x_n) = g_{2,p,n+1}\left(\sum_{i=1}^{n} x_i, x_1, x_2, \ldots, x_n\right)$$

it results that $g_{3,p,n}$ is concave and $g_{4,p,n}$ is convex. □

Theorem 4.9 *Let $n \geq 2$, $p_i > 0$, $i = 1, 2, \ldots, n$, $p > 0$ and $a = \min_{2 \leq i \leq n}(p_i)$. Consider the set*

$$D = \left\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \left(\sum_{i=2}^{n} |x_i|^{p_i}\right)^{\frac{1}{p_1}}\right\}$$

and the function $g_p : D \to \mathbb{R}$,

$$g_p(x_1, x_2, \ldots, x_n) = (|x_1|^{p_1} - |x_2|^{p_2} - \ldots - |x_n|^{p_n})^{\frac{1}{p}}, \ (x_1, x_2, \ldots, x_n) \in D.$$

If $a \geq 1$ and $p_1 \leq a \leq p$, then D is a convex set and g_p is concave on D.

Proof Let $h : \mathbb{R}^{n-1} \to \mathbb{R}$,

$$h(x_1, x_2, \dots, x_{n-1}) = \left(\sum_{i=1}^{n-1} |x_i|^{p_{i+1}} \right)^{\frac{1}{p_1}}, \ (x_1, x_2, \dots, x_{n-1}) \in \mathbb{R}^{n-1}.$$

By Corollary 3.4, h is convex. Since

$$D = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n \mid x_1 > h(x_2, x_3, \dots, x_n)\}$$

it follows that D is a convex set.

Let f_p be defined as in (4.5). Note that

$$g_p(x) = [f_a(u_1(x), u_2(x), \dots, u_n(x))]^{\frac{a}{p}}, \ x \in D.$$

By Theorem 4.4 it follows that f_a is concave. By Theorem 2.3, g_p is concave. □

Theorem 4.10 *Let* $p_i < 0$, $i = 1, 2, \dots, n$, $p < 0$, $a = \min_{2 \le i \le n} (p_i)$,

$$D = \left\{ (x_1, x_2, \dots, x_n) \in (0, \infty)^n \mid x_1 < \left(\sum_{i=2}^{n} x_i^{p_i} \right)^{\frac{1}{p_1}} \right\}$$

$$g_p(x_1, x_2, \dots, x_n) = (x_1^{p_1} - x_2^{p_2} - \dots - x_n^{p_n})^{\frac{1}{p}}, \ (x_1, x_2, \dots, x_n) \in D.$$

If $p_1 \le a \le p$, *then* D *is a convex set and* g_p *is convex on* D.

Proof Let $h : (0, \infty)^{n-1} \to \mathbb{R}$,

$$h(x_1, x_2, \dots, x_{n-1}) = \left(\sum_{i=1}^{n-1} x_i^{p_{i+1}} \right)^{\frac{1}{p_1}}, \ (x_1, x_2, \dots, x_{n-1}) \in (0, \infty)^{n-1}.$$

By Corollary 3.4, h is concave on $(0, \infty)^{n-1}$. Since

$$D = \{(x_1, x_2, \dots, x_n) \in (0, \infty)^n \mid x_1 < h(x_2, x_3, \dots, x_n)\}$$

it follows that D is a convex set.

Let $u_k : D \to \mathbb{R}$, $k = 1, 2, \dots, n$ be defined as follows:

$$u_1(x_1, x_2, \dots, x_n) = x_1^{p_1/a},$$

$$u_k(x_1, x_2, \dots, x_n) = x_k^{p_k/a}, \ k = 2, 3, \dots, n, \ (x_1, x_2, \dots, x_n) \in D.$$

Since $p_1/a \ge 1$ it follows that u_1 is convex.

Since $p_k/a \in (0, 1]$, $k = 2, 3, \ldots, n$ it follows that u_k, $k = 2, 3, \ldots, n$ are concave.

Let $u(x) = (u_1(x), u_2(x), \ldots, u_n(x))$, $x \in D$. Denote with f_p the function defined by (4.5). Note that

$$g_p(x) = [f_a(u(x))]^{a/p}, \; x \in D.$$

By Theorem 4.5, f_a is convex. By Theorem 2.3 it follows that g_p is convex on D. □

Theorem 4.11 *Let*

$$D = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > \ln(e^{x_2} + e^{x_3} + \ldots + e^{x_n})\}, \; a < 0,$$

$$f(x_1, x_2, \ldots, x_n) = \ln(e^{x_1} - e^{x_2} - \ldots - e^{x_n}), \; (x_1, x_2, \ldots, x_n) \in D,$$

$$g(x_1, x_2, \ldots, x_n) = (e^{x_1} - e^{x_2} - \ldots - e^{x_n})^a, \; (x_1, x_2, \ldots, x_n) \in D.$$

$$h(x_1, x_2, \ldots, x_n) = \ln(e^{x_1+x_2+\ldots+x_n} - e^{x_1} - e^{x_2} - \ldots - e^{x_n}), (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$$

Then D is a convex set, f is concave on D, g is convex on D and h is concave on $(0, \infty)^n$.

Proof Let

$$h(x_1, x_2, \ldots, x_{n-1}) = \ln(e^{x_1} + e^{x_2} + \ldots + e^{x_{n-1}}), \; (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{n-1}.$$

By Corollary 3.3 it follows that h is convex. Since

$$D = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_1 > h(x_2, x_3, \ldots, x_n)\}$$

it follows that D is a convex set.

Let $\phi(t) = e^t$, $t \in \mathbb{R}$. Note that

$$v(t) = \frac{\phi'^2(\phi^{-1}(t))}{\phi''^{-1}(t))} = t, \; \phi'(t) > 0, \; \phi''(t) > 0, \; t \in \mathbb{R}.$$

By Theorem 4.2 it follows that f is concave on D. Since $g = \exp(af)$ it follows from Corollary 2.2 that g is a convex function. The concavity of h follows from Theorem 2.3. and from the concavity of f. □

Theorem 4.12 *Let E be a linear space, D be a convex subset of E and $u_k : D \to \mathbb{R}$, $k = 1, 2, \ldots, n$. Suppose that u_1 is concave and $u_2, u_3, \ldots, u_n$ are convex. Let $a < 0$,*

$$U = \{x \in D \mid u_1(x) > \ln(e^{u_2(x)} + e^{u_3(x)} + \ldots + e^{u_n(x)})\}$$

and $f, g : U \to \mathbb{R}$,

$$f(x) = \ln(e^{u_1(x)} - e^{u_2(x)} - \ldots - e^{u_n(x)}),\ x \in U,$$

$$g(x) = (e^{u_1(x)} - e^{u_2(x)} - \ldots - e^{u_n(x)})^a,\ x \in U.$$

Then U *is a convex set,* f *is concave and* g *is convex.*

Proof Let $h(x) = \ln(e^{u_2(x)} + e^{u_3(x)} + \ldots + e^{u_n(x)}),\ x \in D$. By Corollary 3.3 it follows that h is convex on D. Since

$$U = \{x \in D \mid u_1(x) > h(x)\}$$

it follows that U is a convex set. By the preceding theorem and by Theorem 2.3 it follows that f is concave and g is convex. □

Theorem 4.13 *Let* $P : \mathbb{R}^n \to \mathbb{R}$ *be a function. Suppose that*

$$P = P_1 - P_2$$

where

$$P_1(x_1, x_2, \ldots, x_n) = d_0 x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$$

$$P_2(x_1, x_2, \ldots, x_n) = \sum_{r=1}^{m} d_r x_1^{b_{r1}} x_2^{b_{r2}} \ldots x_n^{b_{rn}},$$

$$a_i \geq 0,\ b_{ri} \geq 0,\ d_r > 0,\ i \in \{1, 2, \ldots, n\},\ r \in \{0, 1, \ldots, m\}.$$

Denote

$$c_i = \max_{1 \leq r \leq m} (b_{ri}),\ i \in \{1, 2, \ldots, n\},$$

$$D_1 = \{x \in (0, \infty)^n \mid P(x) > 0\}$$

$$D_2 = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid P(e^{x_1}, e^{x_2}, \ldots, e^{x_n}) > 0\}.$$

Consider the functions $g_i : D_i \to \mathbb{R}$, $i = 1, 2$,

$$g_1(x) = \ln[P(x)],\ x \in D_1$$

$$g_2(x_1, x_2, \ldots, x_n) = \ln[P(e^{x_1}, e^{x_2}, \ldots, e^{x_n})],\ (x_1, x_2, \ldots, x_n) \in D_2.$$

Then the following assertions hold:

1° *If* $a_i \geq c_i$ *for every* $i \in \{1, 2, \ldots, n\}$*, then* D_1 *is a convex set and* g_1 *is concave on* D_1.
2° *The set* D_2 *is convex and* g_2 *is concave on* D_2.

Proof Suppose that $a_i \geq c_i$ for every $i \in \{1, 2, \ldots, n\}$.
For $r \in \{1, 2, \ldots, m\}$ let

$$Q_r(x_1, x_2, \ldots, x_n) = x_1^{b_{r1}-a_1} x_2^{b_{r2}-a_2} \ldots x_n^{b_{rn}-a_n}.$$

Consider

$$Q(x) = \sum_{r=1}^{m} \frac{d_r}{d_0} Q_r(x).$$

Note that

$$Q(x) = \frac{P_2(x)}{P_1(x)},\ x \in (0, \infty)^n$$

and

$$D_1 = \{x \in (0, \infty)^n \mid P(x) < 0\} = \{x \in (0, \infty)^n \mid Q(x) < 1\}.$$

Since $b_{ri} - a_i \leq 0$ for every $i \in \{1, 2, \ldots, n\}$, $r \in \{1, 2, \ldots, m\}$ it follows that all functions Q_r are convex. Hence Q is convex and D_1 is a convex set. Note that

$$\begin{aligned} g_1(x) &= \ln[P(x)] = \ln[P_1(x)(1 - Q(x))] = \ln[P_1(x)] + \ln[1 - Q(x)] \\ &= \ln(d_0) + \sum_{i=1}^{n} a_i \ln(x_i) + \ln(1 - Q(x)), \end{aligned}$$

$$x = (x_1, x_2, \ldots, x_n) \in D_1.$$

Since all the terms in the right hand side of the preceding equation are concave it follows that g_1 is concave on D_1. Let

$$w(x_1, x_2, \ldots, x_n) = a_1x_1 + a_2x_2 + \ldots + a_nx_n + \ln(d_0),$$

$$u_r(x_1, x_2, \ldots, x_n) = b_{r1}x_1 + b_{r2}x_2 + \ldots + b_{rn}x_n + \ln(d_r),\ r \in \{1, 2, \ldots, m\},$$

$$v(\mathbf{x}) = \ln\left[\sum_{r=1}^{m} \exp(u_r(\mathbf{x}))\right] - w(\mathbf{x}),\ \mathbf{x} \in D_2.$$

Note that

$$P(e^{x_1}, e^{x_2}, \ldots, e^{x_n}) = \exp(u_1(x_1, x_2, \ldots, x_n)) - \sum_{r=1}^{m} \exp(u_r(x_1, x_2, \ldots, x_n)).$$

One can easily see that

$$D_2 = \{\mathbf{x} \in \mathbb{R}^n \mid P(e^{x_1}, e^{x_2}, \ldots, e^{x_n}) > 0\} = \{\mathbf{x} \in \mathbb{R}^n \mid v(\mathbf{x}) < 0\}.$$

Since v is convex it follows that D_2 is convex. Note that

$$g_2(\mathbf{x}) = \ln[\exp(w(\mathbf{x})) - \exp(v(\mathbf{x}) + w(\mathbf{x}))],\ \mathbf{x} \in D_2.$$

Since w is concave and $v + w$ is convex, from Theorem 4.12 it follows that g_2 is concave on D_2. □

Lemma 4.14 *Let $p \geq 1$, $0 \leq b \leq a$,*

$$f(x) = (x + a)^p - (x + b)^p,\ x \in [0, \infty).$$

Then f is increasing.

Proof Note that $f'(x) = p\left[(x + a)^{p-1} - (x + b)^{p-1}\right] \geq 0$, $x \in [0, \infty)$. □

Lemma 4.15 *Let $n \geq 2$, $p \geq 1$, $x_1, x_2, \ldots, x_n \geq 0$. If*

$$\sum_{i=1}^{n} x_i \geq 2 \max(x_1, x_2, \ldots, x_n)$$

then

$$\left(\sum_{i=1}^{n} x_i\right)^p \geq 2^{p-1} \left(\sum_{i=1}^{n} x_i^p\right). \tag{4.6}$$

Proof Without loss of generality we may suppose that

$$0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq x_1 + x_2 + \ldots + x_{n-1}.$$

Let $n = 2$. Then inequality (4.6) becomes

$$(x_1 + x_2)^p \geq 2^{p-1}(x_1^p + x_2^p) \text{ if } x_1 = x_2.$$

Thus inequality (4.6) holds in the case $n = 2$. Suppose now that $n \geq 3$. Denote

$$A = x_1 + x_2 + \ldots + x_{n-2}.$$

Let f be defined as in the statement of the preceding lemma. Take

$$x = A + x_{n-1},\ a = x_n,\ b = x_{n-1}.$$

Since $A + x_{n-1} \geq x_n$, from the preceding lemma we have that

$$(A + x_{n-1} + x_n)^p - (A + 2x_{n-1})^p = f(A + x_{n-1}) \geq f(x_n)$$

$$= (2x_n)^p - (x_n + x_{n-1})^p \tag{4.7}$$

In the following we propose to prove the inequality

$$\left(\sum_{i=1}^{n} x_i\right)^p - 2^{p-1}\left(\sum_{i=1}^{n} x_i^p\right)$$

$$\geq \left[\left(\sum_{i=1}^{n-2} x_i\right) + 2x_{n-1}\right]^p - 2^{p-1}\left[\left(\sum_{i=1}^{n-2} x_i^p\right) + 2x_{n-1}^p\right] \tag{4.8}$$

Using inequality (4.7) we obtain:

$$\left(\sum_{i=1}^{n} x_i\right)^p - 2^{p-1}\left(\sum_{i=1}^{n} x_i^p\right) - \left[\left(\sum_{i=1}^{n-2} x_i\right) + 2x_{n-1}\right]^p$$

$$+2^{p-1}\left[\left(\sum_{i=1}^{n-2} x_i^p\right) + 2x_{n-1}^p\right]$$

$$= (A + x_{n-1} + x_n)^p - (A + 2x_{n-1})^p - 2^{p-1}(x_n^p - x_{n-1}^p)$$

$$\geq (2x_n)^p - (x_n + x_{n-1})^p - 2^{p-1}(x_n^p - x_{n-1}^p)$$

$$= 2^{p-1}(x_{n-1}^p + x_n^p) - (x_{n-1} + x_n)^p \geq 0.$$

Thus we have proved inequality (4.8).
In the following we propose to prove the inequality

$$\left[\left(\sum_{i=1}^{n-2} x_i\right) + 2x_{n-1}\right]^p - 2^{p-1}\left[\left(\sum_{i=1}^{n-2} x_i^p\right) + 2x_{n-1}^p\right] \geq 0 \tag{4.9}$$

From the Bernoulli inequality it follows that

$$(A + 2x_{n-1})^p \geq (2x_{n-1})^p + pA(2x_{n-1})^{p-1} \tag{4.10}$$

From the above inequality we obtain

$$\left[\left(\sum_{i=1}^{n-2} x_i\right) + 2x_{n-1}\right]^p - 2^{p-1}\left[\left(\sum_{i=1}^{n-2} x_i^p\right) + 2x_{n-1}^p\right]$$

$$= (A + 2x_{n-1})^p - (2x_{n-1})^p - 2^{p-1}\left(\sum_{i=1}^{n-2} x_i^p\right)$$

$$\geq pA(2x_{n-1})^{p-1} - 2^{p-1}\left(\sum_{i=1}^{n-2} x_i^p\right) = 2^{p-1}\left[\sum_{i=1}^{n-2} x_i(px_{n-1}^{p-1} - x_i^{p-1})\right] \geq 0.$$

Thus we have proved inequality (4.9). Inequality (4.6) follows at once from inequalities (4.8) and (4.9). □

Theorem 4.16 *Let* $n \geq 3,\ p \in [1, \infty),\ a \in (-\infty, 0)$,

$$D = \left\{(x_1, x_2, \ldots, x_n) \in [0, \infty)^n \mid \sum_{i=1}^{n} x_i \geq 2\max(x_1, x_2, \ldots, x_n)\right\},$$

$$U = \left\{(x_1, x_2, \ldots, x_n) \in D \mid \left(\sum_{i=1}^{n} x_i\right)^p > 2^{p-1}\left(\sum_{i=1}^{n} x_i^p\right)\right\},$$

$$f(x_1, x_2, \ldots, x_n) = \left[\left(\sum_{i=1}^{n} x_i\right)^p - 2^{p-1}\left(\sum_{i=1}^{n} x_i^p\right)\right]^{1/p},\ (x_1, x_2, \ldots, x_n) \in D,$$

$$g(x_1, x_2, \ldots, x_n) = \ln\left[\left(\sum_{i=1}^{n} x_i\right)^p - 2^{p-1}\left(\sum_{i=1}^{n} x_i^p\right)\right],\ (x_1, x_2, \ldots, x_n) \in U,$$

$$h(x_1, x_2, \ldots, x_n) = \left[\left(\sum_{i=1}^{n} x_i\right)^p - 2^{p-1}\left(\sum_{i=1}^{n} x_i^p\right)\right]^a,\ (x_1, x_2, \ldots, x_n) \in U.$$

Then the following assertions hold:

1° *D and U are convex sets.*
2° *f is concave on D.*
2° *g is concave on U.*
4° *h is convex on U.*

Proof Let

$$v(x_1, x_2, \dots, x_n) = 2\max(x_1, x_2, \dots, x_n),\ (x_1, x_2, \dots, x_n) \in [0, \infty)^n,$$

$$w(x_1, x_2, \dots, x_n) = 2^{1-\frac{1}{p}} \left(\sum_{i=1}^{n} x_i^p\right)^{1/p},\ (x_1, x_2, \dots, x_n) \in [0, \infty)^n,$$

$$u(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} x_i.$$

Note that $v - u$ is a convex function and

$$D = \{x \in [0, \infty)^n \mid v(x) - u(x) \leq 0\}.$$

Hence D is a convex set. Note that $w - u$ is a convex function and

$$U = \{x \in D \mid w(x) - u(x) < 0\}.$$

Hence U is a convex set. By Lemma 4.15 the function f is well defined. Note that u is concave, w is convex,

$$f(x) = [u^p(x) - w^p(x)]^{1/p},\ x \in D.$$

By Corollary 4.3 and Theorem 2.3 it follows that f is concave on D. Since

$$g(x) = \ln[f(x)],\ x \in U,$$

by Corollary 2.2 it follows that g is concave on U. Since

$$h(x) = \exp(ag(x)),\ x \in U,$$

by Corollary 2.2 it follows that h is convex on U. □

Theorem 4.17 *Let $a < 0$, E be a linear space, D be a convex subset of E and $v, u_1, u_2, \dots, u_n : D \to [0, \infty)$ be such that $u_1, u_2, \dots, u_n$ are concave functions and v is a convex function. Consider the set*

$$U = \{x \in D \mid u_1(x)u_2(x)\dots u_n(x) > v^n(x)\}$$

and the functions

$$f(x) = \sqrt[n]{u_1(x)u_2(x)\dots u_n(x) - v^n(x)},\ x \in U$$

$$g(x) = \ln[u_1(x)u_2(x)\dots u_n(x) - v^n(x)],\ x \in U$$

$$h(x) = [u_1(x)u_2(x)\dots u_n(x) - v^n(x)]^a,\ x \in U.$$

Then the following assertions hold:

1° *U is convex set.*
2° *The functions f and g are concave on U.*
3° *If $a < 0$, then the function h is convex on U.*

Proof Consider the function

$$u(x) = \sqrt[n]{u_1(x)u_2(x)\dots u_n(x)},\ x \in U.$$

Note that u is concave and $v - u$ is convex. Since

$$U = \{x \in D \mid v(x) - u(x) < 0\}$$

it follows that U is a convex set. Note that

$$f(x) = [u^n(x) - v^n(x)]^{1/n},\ x \in U$$

$$g(x) = \ln[u^n(x) - v^n(x)],\ x \in U$$

$$h(x) = [u^n(x) - v^n(x)]^a,\ x \in U.$$

By Corollary 4.3 it follows that f is concave. Since

$$g(x) = n \cdot \ln[f(x)] \text{ and } h(x) = \exp(ag(x)),\ x \in U$$

it follows that g is concave and h is convex on U. □

Corollary 4.18 *Let $c_1, c_2 \in \mathbb{R}$, $a_i, b_i > 0$, $i = 1, 2, \dots, n$ be such that*

$$a_1a_2\dots a_n > c_1^n,\ b_1b_2\dots b_n > c_2^n.$$

Then the following inequality holds:

$$\sqrt[n]{(a_1 + b_1)(a_2 + b_2)\dots(a_n + b_n) - (c_1 + c_2)^n}$$

$$\geq \sqrt[n]{a_1a_2\dots a_n - c_1^n} + \sqrt[n]{b_1b_2\dots b_n - c_2^n}.$$

Proof Let $D = \{(x_1, x_2, \dots, x_n, y) \in \mathbb{R}_+^n \times \mathbb{R} \mid x_1x_2\dots x_n > y^n\}$.
Note that D is a convex set. Consider the function

$$f(x_1, x_2, \dots, x_n, y) = \sqrt[n]{x_1x_2\dots x_n - y^n},\ (x_1, x_2, \dots, x_n, y) \in D.$$

By Theorem 4.17 the function f is concave on D. Hence

$$f\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}, \ldots, \frac{a_n+b_n}{2}\right)$$
$$\geq \frac{1}{2}[f(a_1, a_2, \ldots, a_n, c_1) + f(b_1, b_2, \ldots, b_n, c_2)].$$

The inequality from the statement follows at once from the above inequality. □

Corollary 4.19 *Let* $z_i \in \mathbb{R}$, $x_i, y_i \in (0, \infty)$, $i = 1, 2$.
If $x_i y_i - z_i^2 > 0$ *for* $i = 1, 2$, *then the following inequality holds:*

$$\frac{8}{(x_1+x_2)(y_1+y_2)-(z_1+z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}.$$

Proof Let

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0,\ y > 0,\ xy - z^2 > 0\}$$

and

$$f(x, y, z) = \frac{1}{xy - z^2},\ (x, y, z) \in D.$$

Note that D is a convex set. By Theorem 4.17 it follows that f is convex on D. Hence

$$f\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right) \leq \frac{1}{2}[f(x_1, y_1, z_1) + f(x_2, y_2, z_2)].$$

The inequality from the statement of the corollary follows at once from the above inequality. □

The inequality from the statement was proposed at the International Mathematical Olympiad in 1969 (see [4] and [17]). A different proof is given in [3].

Theorem 4.20 *Let E be a linear space, D be a convex subset of E,* $a > 0$, $u, v, w : D \to (0, \infty)$,

$$U_1 = \{x \in D \mid u(x) > v(x)\},$$
$$U_2 = \{x \in D \mid u(x)v(x) > w^2(x)\},$$
$$f_1(x) = \frac{u(x)v(x)}{(u^a(x) + v^a(x))^{1/a}},\ x \in D,$$
$$f_2(x) = \frac{u(x)v(x)}{(u^a(x) - v^a(x))^{1/a}},\ x \in U_1,$$

$$f_3(x) = \frac{w(x)\sqrt{u(x)v(x)}}{\sqrt{u(x)v(x) - w^2(x)}}, \ x \in U_2,$$

$$f_4(x) = \frac{u(x)v(x)w^2(x)}{u(x)v(x) - w^2(x)}, \ x \in U_2.$$

Then the following assertions hold:

1° *If u, v are concave, then f_1 is concave.*
2° *If u is concave and v is convex, then U_1 is a convex set and f_2 is convex on U_1.*
3° *If u, v are concave and w is convex, then U_2 is a convex set and f_3 and f_4 are convex on U_2.*

Proof Let $p < 0$. Consider the functions

$$g_1(x, y) = (x^p + y^p)^{1/p}, \ (x, y) \in (0, \infty)^2,$$

$$g_2(x, y) = (x^p - y^p)^{1/p}, \ (x, y) \in (0, \infty)^2, \ x < y.$$

By Corollary 3.2, g_1 is concave. By Corollary 4.3, g_2 is convex.
Suppose that u, v are concave and $p = -a$. Note that

$$f_1(x) = g_1(u(x), v(x)), \ x \in (0, \infty)^2.$$

By Theorem 2.3, f_1 is concave.
Suppose that u is concave and v is convex and $p = -a$. Note that

$$f_2(x) = g_2(v(x), u(x)), \ x \in U_1.$$

By Theorem 2.3, f is convex.
Suppose u, v are concave, w is convex and $p = -a$. Note that $\sqrt{uv}$ is concave,

$$f_3(x) = g_2\left(w(x), \sqrt{u(x)v(x)}\right), \ x \in U_2,$$

$$f_4(x) = f_3^2(x), \ x \in U_2.$$

By Theorem 2.3, f_3 and f_4 are convex on U_2. □

5 Convexity of Symmetric Functions

An n-variable function f is called symmetric if it does not change by any permutation of its variables. A class of symmetric functions that have many applications in various domains is the class of symmetric polynomials. The symmetric n-variable polynomials form a ring that plays an important role in mathematics and mathematical physics [68, 69]. Symmetric polynomials are widely used in many fields such as algebra [1, 7, 14, 64, 84], linear algebra [18–20], algebraic geometry [54], representation theory [26], combinatorics [2, 11, 25, 77], statistics [31, 46], mechanics [26], physics [8, 44], discrete mathematics [74], geometry [28], information theory [32] and many others. For the basic properties of symmetric polynomials, see [45] and for references on recent studies on symmetric polynomials, we refer to [21].

In this section we shall study the convexity and concavity of symmetric functions which are defined as multivariate monomials or ratios or powers of elementary symmetric polynomials. We prove some "power" generalizations of Marcus–Lopes inequality [48] concerning elementary symmetric polynomials. Our results extend some recent results obtained by Sra [76] and Lachaume [36].

In the second part of this section we study the convexity and concavity of the functions of the type $f(x) = [\sigma_r(x)]^a \ [\sigma_s(x)]^b$, $g(x) = [\sigma_k(x^p)]^a$, and $h(x) = \ln(\sigma_k(x^p))$, where $x \in (0,\infty)^n$.

For $n \geq 2$ and for every $k \in \{1, 2, \ldots, n\}$, $(x_1, x_2, \ldots, x_n) \in (0,\infty)^n$ we denote by $\sigma_{k,n}(x_1, x_2, \ldots, x_n)$ the $k-$th elementary symmetric polynomial.

$$\sigma_{k,n}(x_1, x_2, \ldots, x_n) = \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq n} x_{i_1} x_{i_2} \ldots x_{i_k}$$

We set $\sigma_{0,n}(x_1, x_2, \ldots, x_n) = 1$. In the case no confusion may arise we shall write σ_k instead $\sigma_{k,n}$.

If $p \in \mathbb{R}$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in (0,\infty)^n$, we shall denote by $\mathbf{x}^p$ the vector $\left(x_1^p, x_2^p, \ldots, x_n^p\right)$.

A well-known result concerning the concavity of σ_k is the Marcus–Lopes inequality [48]

$$\frac{\sigma_k(\mathbf{x}+\mathbf{y})}{\sigma_{k-1}(\mathbf{x}+\mathbf{y})} \geq \frac{\sigma_k(\mathbf{x})}{\sigma_{k-1}(\mathbf{x})} + \frac{\sigma_k(\mathbf{y})}{\sigma_{k-1}(\mathbf{y})}, \quad 1 \leq k \leq n, \quad \mathbf{x}, \mathbf{y} \in (0,\infty)^n$$

The above inequality can be used to prove the following inequalities

$$[\sigma_k(\mathbf{x}+\mathbf{y})]^{\frac{1}{k}} \geq [\sigma_k(\mathbf{x})]^{\frac{1}{k}} + [\sigma_k(\mathbf{y})]^{\frac{1}{k}}, \quad 1 \leq k \leq n, \quad \mathbf{x}, \mathbf{y} \in (0,\infty)^n$$

and

$$\left[\frac{\sigma_k(\mathbf{x}+\mathbf{y})}{\sigma_{k-r}(\mathbf{x}+\mathbf{y})}\right]^{\frac{1}{r}} \geq \left[\frac{\sigma_k(\mathbf{x})}{\sigma_{k-r}(\mathbf{x})}\right]^{\frac{1}{r}} + \left[\frac{\sigma_k(\mathbf{y})}{\sigma_{k-r}(\mathbf{y})}\right]^{\frac{1}{r}}, \mathbf{x} \in (0,\infty)^n, 1 \leq r \leq k \leq n.$$

The above inequalities can be reformulated as follows:

Theorem 5.1 ([48]) *If* $1 \le r \le k \le n$*, then the functions*

$$f(\mathbf{x}) = \frac{\sigma_k(\mathbf{x})}{\sigma_{k-1}(\mathbf{x})}, \ \mathbf{x} \in (0, \infty)^n,$$

$$g(\mathbf{x}) = [\sigma_k(\mathbf{x})]^{\frac{1}{k}}, \mathbf{x} \in (0, \infty)^n,$$

$$h(x) = \left[\frac{\sigma_k(\mathbf{x})}{\sigma_{k-r}(\mathbf{x})}\right]^{\frac{1}{r}}, \mathbf{x} \in (0, \infty)^n$$

are concave.

The above theorem may be used to prove concavity and convexity of several symmetric functions constructed with the help of elementary symmetric polynomials. Recent results concerning the concavity of such functions can be found in Sra [76] and Lachaume [36]. The next three theorems present some important contributions from the above two mentioned papers.

Theorem 5.2 (Sra [76]) *If* $1 \le r \le k \le n$ *and* $p \in [0, 1]$, *then the functions*

$$f(\mathbf{x}) = \left[\frac{\sigma_k(\mathbf{x}^p)}{\sigma_{k-1}(\mathbf{x}^p)}\right]^{\frac{1}{p}}, \ \mathbf{x} \in (0, \infty)^n$$

$$g(\mathbf{x}) = \left[\frac{\sigma_k(\mathbf{x}^p)}{\sigma_{k-r}(\mathbf{x}^p)}\right]^{\frac{1}{rp}}, \mathbf{x} \in (0, \infty)^n$$

$$h(\mathbf{x}) = \left[\sigma_k\left(\mathbf{x}^p\right)\right]^{\frac{1}{kp}}, \quad \mathbf{x} \in (0, \infty)^n$$

are concave.

Theorem 5.3 (Lachaume [36]) *Let* $n \ge 2$*,* $a_i \ge 0$*,* $i \in \{0, 1, 2\}$,

$$f(\mathbf{x}) = \sqrt{a_0 + a_1\sigma_1(\mathbf{x}) + a_2\sigma_2(\mathbf{x})}, \quad \mathbf{x} \in (0, \infty)^n$$

If $na_1^2 - 2(n-1)a_0a_2 \ge 0$*, then* f *is concave.*

Theorem 5.4 (Lachaume [36]) *Let* $2 \le k \le n$, $a \ge 0$, $b \ge 0$,

$$f(\mathbf{x}) = (a\sigma_{k-1}(\mathbf{x}) + b\sigma_k(\mathbf{x}))^{\frac{1}{k}}, \quad \mathbf{x} \in (0, \infty)^n$$

Then f *is concave.*

Theorem 5.5 *Let* $n \ge 2$*,* $r, s \in \{1, 2, \ldots, n\}$*,* $r < s$*,* $a \in \mathbb{R}$*,*

$$f(x) = \left[\frac{\sigma_s(x)}{\sigma_r(x)}\right]^a, \ x \in (0, \infty)^n$$

$$g(x) = \ln[\sigma_s(x)] - \ln[\sigma_r(x)], \ x \in (0, \infty)^n.$$

Then the following assertions hold:

1° *If* $a \in \left[0, \dfrac{1}{s-r}\right]$*, then* f *is concave.*
2° *If* $a \in (-\infty, 0]$*, then* f *is convex.*
3° g *is concave.*

Proof Let

$$h(x) = \left[\frac{\sigma_s(x)}{\sigma_r(x)}\right]^{\frac{1}{s-r}}, \; x \in (0, \infty)^n.$$

For every $t \in \{r+1, r+2, \dots, s\}$ let

$$g_t(x) = \frac{\sigma_t(x)}{\sigma_{t-1}(x)}, \; x \in (0, \infty)^n.$$

By the Marcus–Lopes theorem (Theorem 5.1) all the functions g_t are concave. Hence $h = (g_{r+1}g_{r+2}\dots g_s)^{\frac{1}{s-r}}$ is concave. Suppose $a \in \left[0, \dfrac{1}{s-r}\right]$. Since h is concave, $f = h^{a(s-r)}$ and $a(s-r) \in [0, 1]$ it follows that f is concave. Since

$$g(x) = (s-r)\ln[h(x)], \; x \in (0, \infty)^n$$

it follows that g is concave. Suppose $a \in (-\infty, 0]$. Then ag is convex. Since $f = e^{ag}$ it follows that f is convex. □

Theorem 5.6 *Let* $n \geq 2$*,* $r, s \in \{1, 2, \dots, n\}$*,* $r < s$*,* $a, b \in \mathbb{R}$*,*

$$f(x) = [\sigma_r(x)]^a[\sigma_s(x)]^b, \; x \in (0, \infty)^n.$$

Then f *is convex if and only if* $b \leq 0$ *and* $a + b \leq 0$*.*

Proof Suppose $b \leq 0$ and $a + b \leq 0$. Then

$$\ln[f(x)] = a\ln[\sigma_r(x)] + b\ln[\sigma_s(x)]$$

$$= b(\ln[\sigma_s(x)] - \ln[\sigma_r(x)]) + (a+b)\ln[\sigma_r(x)], \; x \in (0, \infty)^n.$$

By the preceding theorem $\ln(f)$ is convex, hence $f = e^{\ln(f)}$ is convex.
Suppose now that f is convex. Let $x_{s+1} = x_{s+2} = \dots = x_n = 0$ in the equation that defined f. Then

$$g(x_1, x_2, \dots, x_s) = f(x_1, x_2, \dots, x_s, 0, 0, \dots, 0)$$

$$= [\sigma_r(x_1, x_2, \dots, x_s, 0, 0, \dots, 0)]^a(x_1x_2\dots x_s)^b.$$

Let $x_1 = x, x_2 = y, x_3 = x_4 = \ldots = x_s = 1$ in the above equation. Then

$$g(x, y, 1, 1, \ldots, 1) = [c(xy + \alpha x + \alpha y + \beta)]^a (xy)^b,\ (x, y) \in (0, \infty)^2$$

for some $c > 0, \alpha > 0, \beta > 0$.
Since f is convex it follows that the function

$$h(x, y) = (xy + \alpha x + \alpha y + \beta)^a (xy)^b,\ (x, y) \in (0, \infty)^2$$

is convex. Since h is symmetric it follows that h is Schur convex.
Let

$$u(x, y) = axy(y + \alpha) + by(xy + \alpha x + \alpha y + \beta),\ (x, y) \in (0, \infty)^2.$$

Note that

$$h'_x(x, y) = (xy + \alpha x + \alpha y + \beta)^{a-1}(xy)^{b-1}u(x, y)$$

$$h'_y(x, y) = (xy + \alpha x + \alpha y + \beta)^{a-1}(xy)^{b-1}u(y, x)$$

$$u(x, y) - u(y, x) = (y - x)[(a + b)xy + b(\alpha x + \alpha y + \beta)]$$

$$h_x(x, y) - h_y(x, y) = (xy + \alpha x + \alpha y + \beta)^{a-1}(xy)^{b-1}(u(x, y) - u(y, x)).$$

Since h is Schur convex it follows that

$$(x - y)(h'_x(x, y) - h'_y(x, y)) \geq 0 \text{ for every } (x, y) \in (0, \infty)^2$$

hence

$$(a + b)xy + \alpha b(x + y) + b\beta \geq 0 \text{ for every } (x, y) \in (0, \infty)^2.$$

If we put $x = y$ in the preceding inequality we obtain

$$(a + b)x^2 + 2\alpha bx + b\beta \leq 0 \text{ for every } x \in (0, \infty)$$

hence $b \leq 0$ and $a + b \leq 0$. □

Theorem 5.7 *Let $n \geq 2$, $r, s \in \{1, 2, \ldots, n\}$, $r < s$, $a, b \in \mathbb{R}$,*

$$f(x) = [\sigma_r(x)]^a [\sigma_s(x)]^b,\ x \in (0, \infty)^n.$$

Then f is concave if and only if $a + b \in [0, 1]$, $b \in [0, 1]$ and $ar + bs \in [0, 1]$.

Proof Suppose that f is concave. Let $x_{s+1} = x_{s+2} = \ldots = x_n = 0$ in the equation that defines f. Then

$$g(x_1, x_2, \ldots, x_s) = f(x_1, x_2, \ldots, x_s, 0, 0, \ldots, 0)$$

$$= [\sigma_r(x_1, x_2, \ldots, x_s, 0, 0, \ldots, 0)]^a (x_1 x_2 \ldots x_s)^b$$

is concave. Let $x_1 = t$, $x_2 = x_3 = \ldots = x_s = 1$ in the preceding equation. We obtain

$$g(t, 1, 1, \ldots, 1) = \left[C_{n-1}^{r-1}\left(t + \frac{n-r}{r}\right)\right]^a t^b.$$

Denote $\alpha = \dfrac{n-r}{r}$ and consider the function

$$h(t) = (t+\alpha)^a t^b,\ t \in (0, \infty).$$

Since g is concave it follows that h is concave. Note that

$$h''^{a-2} t^b + 2ab(t+\alpha)^{a-1} t^{b-1} + b(b-1)(t+\alpha)^a t^{b-2}$$

$$= (t+\alpha)^{a-2} t^{b-2} [a(a-1)t^2 + 2abt(t+\alpha) + b(b-1)(t+\alpha)^2].$$

Since $h'' \leq 0$ it follows that

$$(a+b)(a+b-1)t^2 + 2\alpha b(a+b-1)t + (b^2 - b)\alpha^2 \leq 0,\ t \in (0, \infty).$$

From the above inequality it follows that $b \in [0, 1]$, $a + b \in [0, 1]$.

If in the equation that defines f we put $x_1 = x_2 = \ldots = x_n = t$ we obtain

$$w(t) = f(t, t, \ldots, t) = ct^{ar+bs},\ t \in (0, \infty)$$

is concave. Hence $ar + bs \in [0, 1]$.
Suppose now that $a + b \in [0, 1]$, $b \in [0, 1]$ and $ar + bs \in [0, 1]$.
Consider the functions

$$u(x) = [\sigma_r(x)]^{\frac{1}{r}},\ v(x) = \left[\frac{\sigma_s(x)}{\sigma_r(x)}\right]^{\frac{1}{s-r}},\ x \in (0, \infty)^n.$$

Note that

$$f(x) = [u(x)]^{(a+b)r} \cdot [v(x)]^{b(s-r)},\ x \in (0, \infty)^n.$$

By Theorem 5.1 (Marcus–Lopes theorem), u and v are concave. Since

$$(a+b)r \geq 0,\ \ b(s-r) \geq 0$$

and

$$(a+b)r + b(s-r) = ar + bs \in [0, 1]$$

it follows that f is concave. □

Corollary 5.8 *Let* $r, s \in \{1, 2, \dots, n\}$*,* $n \geq 2$*,*

$$f(x) = \frac{\sigma_r(x)}{\sigma_s(x)},\ \ x \in (0, \infty)^n.$$

Then the following assertions hold:

1° f *is convex if and only if* $s \in \{r, r+1, \dots, n\}$*.*
2° f *is concave if and only if* $s \in \{r-1, r\}$*.*

Theorem 5.9 *Let* $n \geq 2$*,* $k \in \{1, 2, \dots, n\}$*,* $p \geq n+1$*,*

$$f(x) = \frac{\sigma_1(x^p)}{\sigma_k(x)},\ \ x \in (0, \infty)^n.$$

Then f *is convex.*

Proof Consider the functions

$$g(x, y) = x^p y^{-k},\ \ (x, y) \in (0, \infty)^2,$$

$$u(x) = [\sigma_1(x^p)]^{\frac{1}{p}},\ \ v(x) = [\sigma_k(x)]^{\frac{1}{k}},\ \ x \in (0, \infty)^n.$$

Note that u, g are convex, v is concave and

$$f(x) = g(u(x), v(x)),\ \ x \in (0, \infty)^n.$$

By Theorem 2.3 it follows that f is convex. □

Theorem 5.10 *Let* $2 \leq k \leq n$*,* $a, b \geq 0$*,* $p \in [0, 1]$*,*

$$f(x) = \sqrt[k]{a[\sigma_{k-1}(x)]^p + b[\sigma_k(x)]^p},\ \ x \in (0, \infty)^n.$$

Then f *is concave.*

Proof Let

$$g_1(x) = [\sigma_1(x)]^p,\ \ g_k(x) = a + b\left[\frac{\sigma_k(x)}{\sigma_{k-1}(x)}\right]^p,\ \ x \in (0, \infty)^n.$$

For every $r \in \{2, 3, \ldots, k-1\}$ let

$$g_r(x) = \left[\frac{\sigma_r(x)}{\sigma_{r-1}(x)}\right]^p, \; x \in (0, \infty)^n.$$

By Theorem 5.1 all g_r are concave. Since

$$f = [g_1 g_2 \ldots g_k]^{\frac{1}{k}}$$

it follows that f is concave. One can easily see that the above theorem is a generalization of Theorem 5.4. □

Theorem 5.11 *Let $n \geq 2$, $k \in \{1, 2, \ldots, n\}$, $p \in \mathbb{R}$,*

$$f(x) = \ln[\sigma_k(x^p)], \; x \in (0, \infty)^n.$$

Then the following assertions hold:

1° *f is convex if and only if $p \in (-\infty, 0]$.*
2° *If $k = n$, then f is concave if and only if $p \in [0, \infty)$.*
3° *$k \leq n-1$, then f is concave if and only if $p \in [0, 1]$.*

Proof Suppose f is convex. Then

$$g(t) = f(t, t, \ldots, t) = \ln(\alpha t^{kp}) = kp \ln(\alpha t), \; t \in (0, \infty)$$

is convex. Hence $p \in (-\infty, 0]$.
Suppose now that $p \in (-\infty, 0]$. If $K \subset \{1, 2, \ldots, n\}$ let

$$u_K(x_1, x_2, \ldots, x_n) = p \sum_{i \in K} \ln(x_i), \; (x_1, x_2, \ldots, x_n) \in (0, \infty)^n.$$

Note that all functions u_k are convex. From Corollary 3.2 it follows that

$$f(x) = \ln[\sigma_k(x^p)] = \ln\left(\sum_{|K|=k} \exp(u_K(x))\right), \; x \in (0, \infty)^n$$

is convex. Thus assertion 1° is proved.
If $k = n$, then

$$f(x) = \ln(x_1 x_2 \ldots x_n)^p = p \sum_{i=1}^{n} \ln(x_i).$$

Thus f is concave if and only if $p \in [0, \infty)$.

Suppose now that $k \leq n-1$ and f is concave. Then there exist $\alpha, \beta > 0$ such that

$$h(t) = f(t, 1, 1, \ldots, 1) = \ln(\alpha t^p + \beta),\ t \in (0, \infty)$$

is concave. Note that

$$h'(t) = \frac{\alpha p t^{p-1}}{\alpha t^p + \beta},\ h''p^{-2}\frac{(p-1)\beta - \alpha t^p}{(\alpha t^p + \beta)^2},\ t \in (0, \infty).$$

One can easily see that $h'' \leq 0$ if and only if $p \in [0, 1]$. We proved that f is concave implies $p \in [0, 1]$.
Suppose now that $p \in [0, 1]$. By Theorem 5.1 the function

$$v(x) = [\sigma_k(x)]^{\frac{1}{k}},\ x \in (0, \infty)$$

is concave. Since

$$f(x_1, x_2, \ldots, x_n) = k \ln[v(x_1^p, x_2^p, \ldots, x_n^p)],\ (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$$

from Theorem 2.3 it follows that f is concave. □

Theorem 5.12 *Let* $n \geq 2$*,*

$$A = \{(a, p) \in \mathbb{R}^2 \mid (ap-1)(p-1) \geq 0 \text{ and } ap(ap-1) \geq 0\},$$
$$B = \{(a, p) \in \mathbb{R}^2 \mid ap \in [0, 1] \text{ and } p \in (-\infty, 1]\}.$$

For every $(a, p) \in \mathbb{R}^2$ *consider the functions*

$$s_p(x_1, x_2, \ldots, x_n) = x_1^p + x_2^p + \ldots + x_n^p,\ (x_1, x_2, \ldots, x_n) \in (0, \infty)^n,$$
$$f(x) = [s_p(x)]^a,\ x \in (0, \infty)^n.$$

Then the following assertions hold:

1° f *is convex if and only if* $(a, p) \in A$*.*
2° f *is concave if and only if* $(a, p) \in B$*.*

Proof The proof of the theorem will be made in two steps: the case $n = 2$ and the case $n \geq 2$.

Suppose that $n = 2$. Let $u(x, y) = x^p + y^p$, $(x, y) \in (0, \infty)^2$. Then $f = u^a$. Note that

$$f'_x = au^{a-1}u'_x,\ f'_y = au^{a-1}u'_y,\ f''_{xx} = a(a-1)u^{a-2}u'^2_x + au^{a-1}u''_{xx}$$
$$f''_{xy} = a(a-1)u^{a-2}u'_x u'_y$$
$$f''_{yy} = a(a-1)u^{a-2}u'^2_y + au^{a-1}u''_{yy}.$$

Denote by H the determinant of the Hessian matrix of f. Then

$$H = f''_{xx} f''_{yy} - (f''_{xy})^2$$

$$= a^2 u^{2a-4}[(a-1)u'^2_x + uu''_{xx}][(a-1)u'^2_y + uu''_{yy}] - a^2 u^{2a-4}(a-1)^2 u'^2_x u'^2_y$$

$$= a^2 u^{2a-3}[(a-1)(u'^2_x u''_{yy} + u'^2_y u''_{xx}) + uu''_{xx} u''_{yy}].$$

Denote

$$H_1 = \frac{H}{a^2 u^{2a-3}}.$$

Then

$$H_1(x, y) = (a-1)[p^2 x^{2p-2} p(p-1) y^{p-2} + p^2 y^{2p-2} p(p-1) x^{p-2}]$$

$$+(x^p + y^p) p^2 (p-1)^2 x^{p-2} y^{p-2} = p^2 x^{p-2} y^{p-2} (x^p + y^p)(ap-1)(p-1)$$

$$f''_{xx}(x, y) = a(x^p + y^p)^{a-2}[(a-1)p^2 x^{2p-2} + (x^p + y^p) p(p-1) x^{p-2}]$$

$$= apx^{p-2}(x^p + y^p)^{a-2}[(ap-1)x^p + (p-1)y^p].$$

Note that $f''_{xx} \geq 0$ on $(0, \infty)^2$ if and only if $(a, p) \in A$. This also implies that $f''_{yy} \geq 0$ and $H \geq 0$. Hence assertion 1° is proved.

Note that $f''_{xx} \leq 0$ on $(0, \infty)^2$ if and only if $(a, p) \in B$. This also implies that $f''_{yy} \leq 0$ and $H \geq 0$. Hence assertion 2° is proved.

In the following we shall study the case $n \geq 2$. Let

$$v(x, y) = (x^p + y^p)^a, \ (x, y) \in (0, \infty)^2.$$

Note that $v(x, y)$ equals to

(a) the limit of $f(x, y, x_3, x_3, \ldots, x_n)$ when $p > 0$ and $x_i \to 0$ for $i \in \{3, 4, \ldots, n\}$;
(b) the limit of $f(x, y, x_3, x_4, \ldots, x_n)$ when $p < 0$ and $x_i \to +\infty$ for $i \in \{3, 4, \ldots, n\}$.

From the above remark it follows that if f is convex (resp. concave) then v is convex (resp. concave).

Hence if f is convex (resp. concave) then $(a, p) \in A$ (resp. $(a, p) \in B$).

Suppose now that $(a, p) \in A$. This is equivalent with the fact that (a, p) satisfies one of the following two conditions:

(i) $ap \in (-\infty, 0]$ and $p \in (-\infty, 1]$;
(ii) $ap \in [1, \infty)$ and $p \in [1, \infty)$.

If (i) holds then $g = (s_p)^{1/p}$ is concave hence $h = \ln(g)$ is concave. Since aph is convex and $f = e^{aph}$ it follows that f is convex.

If (ii) holds then g is convex hence $f = g^{ap}$ is convex. Thus assertion 1° is proved.

Suppose now that $(a, p) \in B$. It follows that g is concave hence $f = g^{ap}$ is concave. Thus assertion 2° is proved. □

Lemma 5.13 *Let* $2 \le k \le n$, $ap > 0$,

$$f(x) = [\sigma_k(x^p)]^a, \ x \in (0, \infty)^n.$$

Then f *is not convex.*

Proof If $p > 0$ let $x_i \to 0$, $i \in \{k+1, k+2, \dots, n\}$ in the equation that defines f. It follows that

$$f(x_1, x_n, \dots, x_k, 0, 0, \dots, 0) = [\sigma_k(x_1^p, x_2^p, \dots, x_k^p, 0, 0, \dots, 0)]^a$$
$$= (x_1 x_2 \dots x_k)^{ap}, \ (x_1, x_2, \dots, x_k) \in (0, \infty)^k.$$

Since $k \ge 2$, the multivariate monomial from the right-hand side of the above equation is not convex. Hence f is not convex.

If $p < 0$ let $x_i \to +\infty$, $i \in \{k+1, k+2, \dots, n\}$ in the equation that defines f. We obtain that

$$f(x_1, x_2, \dots, x_k, +\infty, +\infty, \dots, +\infty) = (x_1 x_2 \dots x_k)^{ap},$$
$$(x_1, x_2, \dots, x_k) \in (0, \infty)^k.$$

Since the multivariate monomial from the right-hand side of the above equation is not convex it follows that f is not convex. □

Lemma 5.14 *Let* $1 \le k \le n - 1$, $a, p \in \mathbb{R}$, $ap \ne 0$,

$$f(x) = [\sigma_k(x^p)]^a, \ x \in (0, \infty)^n.$$

If $(ap - 1)(p - 1) < 0$, *then* f *is neither convex nor concave.*

Proof Note that there exist $\alpha, \beta > 0$ such that

$$g(t) = f(t, 1, 1, \dots, 1) = (\alpha t^p + \beta)^a, \ t \in (0, \infty).$$

The first two derivatives of g are:

$$g'^{p} + \beta)^{a-1},$$

$$g''^{p} + \beta)^{a-2} t^{p-2}[\alpha(ap - 1)t^p + \beta(p - 1)], \ t \in (0, \infty).$$

One can easily see that $(ap-1)(p-1) < 0$ implies that g'' has no constant sign. Consequently f is neither convex nor concave. ◻

Theorem 5.15 *Let* $2 \le k \le n-1$,

$$A_1 = \{(a, p) \in \mathbb{R}^2 \mid a \in (-\infty, 0) \text{ and } p \in [0, 1]\},$$
$$A_2 = \{(a, p) \in \mathbb{R}^2 \mid a \in [0, \infty) \text{ and } p \in (-\infty, 0]\},\ A = A_1 \cup A_2,$$
$$f(x) = [\sigma_k(x^p)]^a,\ x \in (0, \infty)^n.$$

Then f *is convex if and only if* $(a, p) \in A$.

Proof Suppose $(a, p) \in A$. Let

$$g(x) = \ln[\sigma_k(x^p)],\ x \in (0, \infty)^n.$$

If $(a, p) \in A_1$ then by Theorem 5.11, g is concave. Hence ag and $f = e^{ag}$ are convex.

If $(a, p) \in A_2$ then by Theorem 5.11, g is convex. Hence ag and $f = e^{ag}$ are convex.

We proved that $(a, p) \in A$ implies f is convex. Suppose now that f is convex. Then there exists $a > 0$ such that

$$h(t) = f(t, t, \ldots, t) = \alpha t^{akp},\ t \in (0, \infty).$$

Since f is convex it follows that $akp \in (-\infty, 0] \cup [1, \infty)$. By Lemma 5.14 it follows that $ap \in (-\infty, 0]$.

We shall study two cases: $a \le 0$ and $a \ge 0$.

If $a \in (-\infty, 0]$, then $p \in [0, \infty)$. By Lemma 5.14 it follows that $p \le 1$. Hence $p \in [0, 1]$ and $(a, p) \in A_1$.

If $a \in [0, \infty)$, then $p \in (-\infty, 0]$, hence $(a, p) \in A_2$. Because f is convex it follows $(a, p) \in A$. ◻

Definition 5.16 Let A be a subset of $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$. A function $u : [0, \infty)^n \to [0, \infty)$ will be called A-*concave* if for every $p \in A$ the function

$$u_p(x) = [u(x^p)]^{\frac{1}{p}},\ x \in (0, \infty)^n$$

is concave.

Lemma 5.17 *Let* $n \ge 2$, $A = [-1, 0) \cup (0, \infty)$ *and*

$$f(x) = \frac{\sigma_n(x)}{\sigma_{n-1}(x)},\ x \in (0, \infty)^n.$$

Then f *is* A-*concave.*

Proof From Corollary 3.2 it follows that the function

$$g_p(x) = [\sigma_1(x^p)]^{\frac{1}{p}},\ x \in (0,\infty)^n$$

is concave for every $p \in (-\infty, 0) \cup (0, 1]$. Let $q = -p$. Then

$$g_p(x) = [\sigma_1(x^{-q})]^{-\frac{1}{q}} = \left[\frac{\sigma_n(x^q)}{\sigma_{n-1}(x^q)}\right]^{\frac{1}{q}} = [f(x^q)]^{\frac{1}{q}},\ x \in (0,\infty)^n.$$

Since $p \in (-\infty, 0) \cup (0, 1]$ is equivalent with $q \in A$ it follows that f is A-concave. □

Lemma 5.18 *Let $B = (-\infty, 0) \cup (0, 1]$ and $u, v : (0,\infty)^n \to (0,\infty)$ be two B-concave functions. Then the function*

$$w = \frac{uv}{u+v}$$

is B-concave.

Proof Note that:

$$w_p(x) = [w(x^p)]^{\frac{1}{p}} = \left[\frac{u(x^p)v(x^p)}{u(x^p)+v(x^p)}\right]^{\frac{1}{p}}$$

$$= \frac{[u(x^p)]^{\frac{1}{p}}[v(x^p)]^{\frac{1}{p}}}{\{[u(x^p)^{\frac{1}{p}}]^p + [v(x^p)^{\frac{1}{p}}]^p\}^{\frac{1}{p}}} = \frac{u_p(x)v_p(x)}{\{[u_p(x)]^p + [v_p(x)]^p\}^{\frac{1}{p}}}.$$

If $p \in B$ then u_p and v_p are concave. By Theorem 4.20 it follows that w_p is concave for every $p \in B$. Hence w is B-concave. □

Lemma 5.19 *Let*

$$B = (-\infty, 0) \cup (0, 1],\ u_1, u_2, \ldots, u_m : (0,\infty)^n \to (0,\infty).$$

If $u_1, u_2, \ldots, u_m$ are B-concave, then $u = u_1 + u_2 + \ldots + u_m$ is B-concave.

Proof For every $p \in \mathbb{R}^*$ let

$$v_{i,p}(x) = [u_i(x^p)]^{\frac{1}{p}},\ x \in (0,\infty)^n,\ i = 1, 2, \ldots, m.$$

Note that

$$w_p(x) = [u(x^p)]^{\frac{1}{p}} = \left(\sum_{i=1}^{m}[v_{i,p}(x)]^p\right)^{\frac{1}{p}},\ x \in (0,\infty)^n.$$

Since $v_{i,p}$ are concave for every $i \in \{1, 2, \ldots, m\}$ and $p \in B$, from Corollary 3.2 it follows that w_p is concave for every $p \in B$. Consequently u is B-concave. □

Lemma 5.20 *Let* $2 \le k \le n$ *and* $\phi : (0, \infty)^2 \to \mathbb{R}$,

$$\phi(x, y) = \frac{xy}{x + y}, \ (x, y) \in (0, \infty)^2.$$

If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$, $i \in \{1, 2, \ldots, n\}$ *we denote*

$$\mathbf{x}_i' = (x_1, x_2, \ldots, \widehat{x}_i, \ldots, x_n) \in (0, \infty)^{n-1}.$$

Here the notation $\widehat{x}_i$ *means that* x_i *is missing from the components of the vector.*
Then the following equality holds:

$$\frac{\sigma_{k,n}(\mathbf{x})}{\sigma_{k-1,n}(\mathbf{x})} = \frac{1}{k} \sum_{i=1}^{n} \phi\left(x_i, \frac{\sigma_{k-1,n-1}(\mathbf{x}_i')}{\sigma_{k-2,n-1}(\mathbf{x}_i')}\right), \ \mathbf{x} = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n.$$

Proof One can easily see that for every $i \in \{1, 2, \ldots, n\}$ we have

$$\sigma_{k,n}(\mathbf{x}) = x_i \sigma_{k-1,n-1}(\mathbf{x}_i') + \sigma_{k,n-1}(\mathbf{x}_i')$$

and

$$\sum_{i=1}^{n} \sigma_{k,n-1}(\mathbf{x}_i') = (n - k)\sigma_{k,n}(\mathbf{x}).$$

Note that

$$\frac{\sigma_{k,n}(\mathbf{x})}{\sigma_{k-1,n}(\mathbf{x})} = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_{k,n}(\mathbf{x})}{\sigma_{k-1,n}(\mathbf{x})} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i \sigma_{k-1,n-1}(\mathbf{x}_i') + \sigma_{k,n-1}(\mathbf{x}_i')}{\sigma_{k-1,n}(\mathbf{x})}$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{x_i \sigma_{k-1,n-1}(\mathbf{x}_i')}{\sigma_{k-1,n}(\mathbf{x})} + \frac{1}{n\sigma_{k-1,n}(\mathbf{x})} \sum_{i=1}^{n} \sigma_{k,n-1}(\mathbf{x}_i')$$

$$= \frac{1}{n} \sum_{i=1}^{n} \frac{x_i \sigma_{k-1,n-1}(x_i')}{\sigma_{k-1,n}(x)} + \frac{(n-k)\sigma_{k,n}(x)}{n\sigma_{k-1,n}(x)}.$$

From the above sequence of equalities we obtain

$$\frac{\sigma_{k,n}(\mathbf{x})}{\sigma_{k-1,n}(\mathbf{x})} - \frac{(n-k)\sigma_{k,n}(\mathbf{x})}{n\sigma_{k-1,n}(\mathbf{x})} = \frac{1}{n} \sum_{i=1}^{n} \frac{x_i \sigma_{k-1,n-1}(\mathbf{x}_i')}{\sigma_{k-1,n}(\mathbf{x})}$$

hence

$$\frac{\sigma_{k,n}(\mathbf{x})}{\sigma_{k-1,n}(\mathbf{x})} = \frac{1}{k}\sum_{i=1}^{n}\frac{x_i\sigma_{k-1,n-1}(\mathbf{x}_i')}{x_i\sigma_{k-2,n-1}(\mathbf{x}_i')+\sigma_{k-1,n-1}(\mathbf{x}_i')}$$

$$= \frac{1}{k}\sum_{i=1}^{n}\phi\left(x_i, \frac{\sigma_{k-1,n-1}(\mathbf{x}_i')}{\sigma_{k-2,n-1}(\mathbf{x}_i')}\right).$$

□

Theorem 5.21 *Let* $n \geq 2$, $1 \leq k \leq n-1$, $C = [-1, 0) \cup (0, 1]$,

$$f_{k,n}(\mathbf{x}) = \frac{\sigma_{k,n}(\mathbf{x})}{\sigma_{k-1,n}(\mathbf{x})},\ \mathbf{x} \in (0, \infty)^n.$$

Then $f_{k,n}$ *is C-concave. We supposed that* $\sigma_{0,n}(\mathbf{x}) \equiv 1$.

Proof We shall prove that if $k \geq 1$ then $f_{k,n}$ is C-concave for every $n \geq k$. If $k = 1$ then $f_{k,n}(\mathbf{x}) = \sigma_{1,n}(\mathbf{x})$, $\mathbf{x} \in (0, \infty)^n$. From Corollary 3.2 it follows that

$$w_p(\mathbf{x}) = [f_{k,n}(\mathbf{x}^p)]^{\frac{1}{p}} = [\sigma_1(\mathbf{x}^p)]^{\frac{1}{p}},\ \mathbf{x} \in (0, \infty)^n$$

is concave for every $p \in C$. This implies that $f_{k,p}$ is C-concave.
Suppose now that $f_{k-1,n}$ is C-concave. By Lemma 5.20 it follows that

$$f_{k,n+1}(\mathbf{x}) = \frac{1}{k}\sum_{i=1}^{n}\phi(x_i, f_{k-1,n}(\mathbf{x})).$$

By Lemma 5.18 all the functions

$$\psi_i(\mathbf{x}) = \phi(x_i, f_{k-1,n}(\mathbf{x})),\ \mathbf{x} \in (0, \infty)^n$$

are C-concave. By Lemma 5.19 it follows that $f_{k+1,n}$ is C-concave. □

Theorem 5.22 *let* $2 \leq k \leq n$, $p \in [-1, 0) \cup (0, 1]$,

$$f(\mathbf{x}) = [\sigma_k(\mathbf{x}^p)]^{1/kp},\ \mathbf{x} \in (0, \infty)^n.$$

Then f *is concave.*

Proof Let $\sigma_0(\mathbf{x}) \equiv 1$. For every $r \in \{1, 2, \ldots, n\}$ let

$$g_r(\mathbf{x}) = \left[\frac{\sigma_r(\mathbf{x}^p)}{\sigma_{r-1}(\mathbf{x}^p)}\right]^{\frac{1}{p}},\ \mathbf{x} \in (0, \infty)^n.$$

By the preceding theorem, g_r is concave for every $r \in \{1, 2, \ldots, n\}$. Since

$$f = (g_1 g_2 \ldots g_k)^{\frac{1}{k}}$$

it follows that f is concave. □

In the following we shall recall two definitions.

Definition 5.23 Let E be a linear space over $\mathbb{R}$. A subset C of E is called a *convex cone* if the following conditions are verified:

(i) If $x \in C$ then $tx \in C$ for every $t \in [0, \infty)$.
(ii) If $x, y \in C$ then $x + y \in C$.

Definition 5.24 Let E be a linear space over $\mathbb{R}$, C be a convex cone in E and $f : C \to \mathbb{R}$. The function f is called *positive homogeneous* of degree $a \in \mathbb{R}$ if

$$f(tx) = t^a f(x) \text{ for every } t \in (0, \infty) \text{ and } x \in C.$$

Theorem 5.25 *Let E be a real linear space, C be a convex cone in E and $f : C \to [0, \infty)$ be a positive homogeneous function of degree one. Then the following assertions hold:*

1° *If f is quasi-convex, then f is convex.*
2° *If f is quasi-concave, then f is concave.*

Proof Let $x, y \in C$ and $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$ be two sequences of positive numbers such that

$$\lim_{n\to\infty} a_n = f(x) \text{ and } \lim_{n\to\infty} b_n = f(y).$$

Suppose f is quasi-convex. Then

$$f(x+y) = (a_n + b_n) f\left(\frac{x+y}{a_n + b_n}\right) = f\left(\frac{a_n}{a_n + b_n} \cdot \frac{x}{a_n} + \frac{b_n}{a_n + b_n} \cdot \frac{y}{b_n}\right)$$

$$\leq (a_n + b_n) \max\left(f\left(\frac{x}{a_n}\right), f\left(\frac{y}{b_n}\right)\right)$$

$$= (a_n + b_n) \max\left(\frac{f(x)}{a_n}, \frac{f(y)}{b_n}\right), \ n \geq 1.$$

If we let $n \to \infty$ we obtain

$$f(x+y) \leq f(x) + f(y).$$

Since f is positive homogeneous of degree one it follows that f is convex.

Suppose now that f is quasi-concave. Then

$$\begin{aligned} f(x+y) &= (a_n+b_n) f\left(\frac{a_n}{a_n+b_n}\cdot\frac{x}{a_n}+\frac{b_n}{a_n+b_n}\cdot\frac{y}{b_n}\right) \\ &\geq (a_n+b_n)\min\left(\frac{f(x)}{a_n},\frac{f(y)}{b_n}\right). \end{aligned}$$

If we let $n\to\infty$ we obtain

$$f(x+y)\geq f(x)+f(y).$$

Since f is positive homogeneous of degree one it follows that f is concave. □

Corollary 5.26 *Let E be a real linear space, let C be a convex cone in E, $a\in\mathbb{R}^*=\mathbb{R}\setminus\{0\}$ and let $u_i: C\to\mathbb{R}_+$, $i=1,2,\ldots,n$, be positive homogeneous functions of degree a. Consider the function:*

$$f(x)=\left(\sum_{i=1}^n u_i(x)\right)^{\frac{1}{a}},\ x\in C.$$

Then the following assertions hold:

1° *If all u_i are convex and $a\in[1,\infty)$, then f is convex.*
2° *If all u_i are concave and $a\in(0,1]$, then f is concave.*
3° *If all u_i are convex and $a\in(-\infty,0)$, then f is concave.*

Proof Note that f is positive homogeneous of degree one. Let

$$\phi(t)=t^a,\ t\in(0,\infty),\ u=\sum_{i=1}^n u_i.$$

Suppose that all u_i are convex and $a\in[1,\infty)$. Then u is convex, hence u is quasi-convex. Since ϕ is increasing and $f=\phi(u)$ it follows that f is quasi-convex. By the preceding theorem it follows that f is convex.

Suppose now that all u_i are concave and $a\in(0,1]$. Then u is concave, hence u is quasi-concave. Since ϕ is increasing and $f=\phi(u)$ it follows that f is quasi-concave. By the preceding theorem f is concave.

Suppose now that all u_i are convex and $a\in(-\infty,0)$. Then u is convex, hence quasi-convex. Since ϕ is decreasing it follows that $f=\phi(u)$ is quasi-concave. By the preceding theorem f is concave. □

Remark If in the preceding corollary we take $E=\mathbb{R}^n$, $C=[0,\infty)^n$,

$$u_i=(x_1,x_2,\ldots,x_n)=x_i^p,\ (x_1,\ldots,x_n)\in C,$$

then the validity of assertions 1° and 2° of Corollary 3.2 follows.

Theorem 5.27 *Let $n \geq 2$, $1 \leq k \leq n$, $a, p \in \mathbb{R}$,*

$$f(\mathbf{x}) = [\sigma_k(\mathbf{x}^p)]^a, \ \mathbf{x} \in (0, \infty)^n.$$

Then f is concave if and only if $ap \in \left[0, \frac{1}{k}\right]$ and $p \in (-\infty, 1]$.

Proof Suppose that f is concave. Then

$$g(t) = f(t, t, \ldots, t) = ct^{akp}, \ t \in (0, \infty)$$

is concave. Hence $akp \in [0, 1]$. By Lemma 5.14 it follows that $p \in (-\infty, 1]$.

Suppose now that $ap \in \left[0, \frac{1}{k}\right]$ and $p \in (-\infty, 1]$. Let

$$h(\mathbf{x}) = [\sigma_k(\mathbf{x}^p)]^{\frac{1}{kp}}, \ \mathbf{x} \in (0, \infty)^n, \ \phi(t) = t^{\frac{1}{kp}}, \ t \in (0, \infty).$$

If $p \in [0, 1]$ then by Theorem 5.22, h is concave, hence $f = h^{akp}$ is concave.

If $p \in (-\infty, 0]$ then $u(\mathbf{x}) = \sigma_k(\mathbf{x}^p)$, $\mathbf{x} \in (0, \infty)^n$ is convex, hence quasi-convex. Since ϕ is decreasing it follows that $h = \phi(u)$ is quasi-concave. Note that h is positive homogeneous of degree one. From Theorem 5.25 it follows that h is concave. Since $akp \in [0, 1]$ it follows that $f = h^{akp}$ is concave. □

6 A Generalization of Davis Theorem and Applications

In this section we shall give a generalization of Davis theorem and we shall make various applications of this theorem. Let $n \geq 2$.

Recall that a square matrix with complex entries $A = (a_{ij})$ is called Hermitian if $A = A^*$. Here by A^* we denoted the adjoint of A. We have $A^* = \bar{A}^T = (\bar{a}_{ji})$.

All eigenvalues of a Hermitian matrix are real. A Hermitian matrix is called positive definite if all its eigenvalues are positive. A square matrix Q is called unitary if $QQ^* = Q^*Q = I_n$. If Q is a real matrix, then Q is unitary if and only if Q is an orthogonal matrix, that is $QQ^T = Q^TQ = I_n$.

Denote by $H_n(\mathbb{C})$ the set of $n \times n$ hermitian matrices, $P_n(\mathbb{C})$ the set of $n \times n$ Hermitian matrices with positive eigenvalues, $U_n(\mathbb{C})$ the set of $n \times n$ unitary matrices, $S_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices, $P_n(\mathbb{R})$ the set of $n \times n$ positive definite matrices and by $O_n(\mathbb{R})$ the set of $n \times n$ orthogonal matrices. In the following we shall give several definitions.

A subset K of $H_n(\mathbb{C})$ is called unitary invariant if for every $A \in K$ we have $UAU^* \in K$ for every $U \in U_n(\mathbb{C})$.

A subset K of $S_n(\mathbb{R})$ is called orthogonally invariant if for every $A \in K$ we have $UAU^T \in K$ for every $U \in O_n(\mathbb{R})$.

Let K be a unitary invariant subset of $H_n(\mathbb{C})$. A function $F : K \to \mathbb{R}$ is called a unitary invariant if $F(UAU^*) = F(A)$ for every $A \in K$ and $U \in U_n(\mathbb{C})$.

Let K be an orthogonally invariant subset of $S_n(\mathbb{R})$. A function $F : K \to \mathbb{R}$ is called a unitary invariant if $F\left(UAU^T\right) = F(A)$ for every $A \in K$ and $U \in O_n(\mathbb{R})$.

Unitary invariant and orthogonally invariant functions are called also spectral functions because they are functions of eigenvalues. They depend only on the spectrum of the operator A. All the results we present in this paper have parallel versions both for unitarily invariant functions on the space of Hermitian matrices and for orthogonally invariant functions defined on the space of real symmetric matrices. The proofs are essentially identical.

A spectral function is symmetric since it is invariant to permutation matrices (which are a special case of unitary matrices). There are many papers studying spectral functions (see, for example, Borwein and Lewis [9], Borwein and Vanderwerff [10], Lewis [38–40], Lewis and Overton[41], Lewis and Sendov [42], Meyer [51], Niculescu and Persson [58], and Seeger [72]).

Spectrally defined functions arise in various areas of applied mathematics: optimality criteria in experimental design theory (Pazman [59], Hoang and Seeger [29]), barrier functions in matrix optimization (Nesterov and Nemirovskii [57], Lewis [38]), matrix updates in quasi-Newton methods (Fletcher [22], Wolkowicz [83]), potential energy densities for isotropic elastic materials (Section 2.3 of Curnier et al. [13]), etc.

Spectral functions may be encountered also in quantum mechanics (Kemble [33] and Schiff [67]), nonlinear elasticity (Lehmich et al. [37], Spector [75], Šilhavý [73], and Gao et al. [23]). Nowadays they are an inseparable part of optimization (Lewis and Overton [41]) and matrix analysis (Horn and Johnson [30]).

Spectral functions like log det(A), $\lambda_{\max}(A)$ the largest eigenvalue of the matrix argument A, $\lambda_{\min}(A)$ the smallest eigenvalue of the matrix argument A, tr(A^p) the trace of power p of matrix A or the constraint that A must be positive definite, etc. appear in semidefinite programming (Vandenberghe et al. [82]),

Remarkably, many properties of the function f defined by $f(x) = F(\text{diag}(x))$ are inherited by the spectral function F. For example, this holds for differentiability and convexity (Lewis [38]), various types of generalized differentiability (Lewis [39]), analyticity (Tsing et al. [81]), various second-order properties (Torki [80]), etc.

For every $A \in H_n(\mathbb{C})$ let $\lambda_1(A) \geq \lambda_2(A) \geq \ldots \geq \lambda_n(A)$ be the eigenvalues of A arranged in descending order. Let $\Lambda : H_n(\mathbb{C}) \to \mathbb{R}$ be the function defined by

$$\Lambda(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)) \quad A \in H_n(\mathbb{C})$$

A subset C of $\mathbb{R}^n$ is called permutation invariant if $\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right) \in C$ for every $(x_1, x_2, \ldots, x_n) \in C$ and for every permutation $\sigma \in S_n$.A function $f : C \to \mathbb{R}$ is called symmetric if $f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right) = f(x_1, x_2, \ldots, x_n)$ for every $(x_1, x_2, \ldots, x_n) \in C$ and for every permutation $\sigma \in S_n$.

Let K be a unitary invariant subset of $H_n(\mathbb{C})$ and $F : K \to \mathbb{R}$ be a spectral function. Let $C = \{x \in \mathbb{R}^n : \text{diag}(x) \in K\}$. Then C is a permutation invariant subset of $\mathbb{R}^n$ and the function $f(x) = F(\mathrm{diag}(x))\ x \in C$ is permutation invariant. Spectral decomposition theorem shows that

$$F(A) = f(\Lambda(A)) = f(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)), \qquad A \in K$$

Here we denoted by diag(x) the diagonal matrix having the entries of vector x on the diagonal.

In the paper of Davis [15] was proved the following result:

Theorem 6.1 *Let K be a unitary invariant subset of $H_n(\mathbb{C})$, $F : K \to \mathbb{R}$ be a spectral function, $C = \{x \in \mathbb{R}^n : diag(x) \in K\}$ and $f(x) = F(diag(x))\ x \in C$.*

Then F is convex if and only if its associated symmetric function f is convex.

In the monograph of Niculescu and Persson [58] the following remark was made: The Davis's result remains valid in the framework of log-convex functions, log-concave functions, concave functions, quasi-convex functions, quasi-concave functions. The authors provide a proof of Davis theorem that is for the case the function implied is convex. Unfortunately the case when the function is quasi-convex is not studied.

In the following we shall state and prove a generalization of Davis theorem. We shall recall some definitions from Hardy–Littlewood–Polya theory of majorization. If $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, we shall denote by $x_{[1]} \geq x_{[2]} \geq \ldots \geq x_{[n]}$ the components of x in descending order. Thus $x_{[k]}$ is the $k-$th largest component of x. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ we say that $\mathbf{x}$ is *majorized* by $\mathbf{y}$ if

$$\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]} \quad \text{for } k = 1, 2, \ldots, n$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]}$$

If $\mathbf{x}$ is majorized by $\mathbf{y}$ we shall write $\mathbf{x} \prec_{HLP} \mathbf{y}$. The basic result relating majorization and convexity is the Hardy–Littlewood–Polya inequality of majorization:

Theorem 6.2 *If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\mathbf{x} \prec_{HLP} \mathbf{y}$ then*

$$\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)$$

for every continuous convex function whose domain includes the components of $\mathbf{x}$ and $\mathbf{y}$. Conversely if the above inequality holds for every continuous convex function whose domain includes the components of x and y then $\mathbf{x} \prec_{HLP} \mathbf{y}$.

Theorem 6.3 *Let $x, y \in \mathbb{R}^n$. Then the following assertions are equivalent:*

(a) $\mathbf{x} \prec_{HLP} \mathbf{y}$
(b) $\mathbf{x} = \mathbf{P}\mathbf{y}$ *for a suitable doubly stochastic matrix* $\mathbf{P} \in M_n(\mathbb{R})$.

For a proof of the following theorem, see the monograph of Niculescu and Persson [58].

Theorem 6.4 *Let C be a permutation invariant subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}$ be a symmetric quasi-convex function. Then f is Schur convex, that is $x, y \in C$ and $x \prec_{HLP} y$ implies $f(x) \leq f(y)$.*

Schur [71] has proved that the diagonal elements $a_{11}, a_{22}, \ldots, a_{nn}$ of a Hermitian $n \times n$ matrix $A = (a_{ij})$ are majorized by the eigenvalues $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$ that is

$$(a_{11}, a_{22}, \ldots, a_{nn}) \prec_{HLP} (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$$

The following theorem is a slightly modified version of Schur's theorem that will be used for the proof of a generalization of Davis theorem.

Theorem 6.5 *Let $u_j \in \mathbb{C}^n$, $j = 1, 2, \ldots, n$ be an orthonormal basis in $\mathbb{C}^n$, that is $u_j^T \bar{u}_k = \delta_{jk}$ for $j, k \in\in \{1, 2, \ldots, n\}$. If $A \in H_n(\mathbb{C})$ then*

$$\left(u_1^T A\bar{u}_1, u_2^T A\bar{u}_2, \ldots, u_n^T A\bar{u}_n\right) \prec_{HLP} (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$$

Let E be a linear space over $\mathbb{R}$ and C be a convex subset of E. For every $(p, q) \in [0, 1]^2$ consider the set $\mathscr{A}(p, q, C)$ of quasi-convex functions $f : C \to \mathbb{R}$ with the following property

$$f((1-p)x + py) \leq (1-q)[(1-p)f(x) + pf(y)] + q\max(f(x), f(y))$$

for every $x, y \in C$

If M is a subset of $[0, 1]^2$ denote by $\mathscr{B}(M, C)$ the set of quasi-convex functions $f : C \to \mathbb{R}$ with the property that $f \in \mathscr{A}(p, q, C)$ for every $(p, q) \in M$.
Note that

1^0. *If $M = [0, 1] \times \{0\}$, then $\mathscr{B}(M, C)$ is the set of convex functions on C.*
2^0. *If $M = [0, 1] \times \{1\}$, then $\mathscr{B}(M, C)$ is the set of quasi-convex functions on C.*
3^0. *If $M = \left\{\left(\frac{1}{2}, 0\right)\right\}$, then $\mathscr{B}(M, C)$ is the set of mid-convex functions on C*

The following theorem is a generalization of Davis theorem. It includes only the nontrivial implication from the original theorem.

Theorem 6.6 *Let C be a permutation invariant convex subset of $\mathbb{R}^n$, $f : C \to \mathbb{R}$ be a symmetric quasi-convex function and M be a subset of $[0, 1]^2$. Consider the set*

$K = \{A \in H_n(\mathbb{C}) : \Lambda(A) \in C\}$, *and the function* $\widetilde{f} : K \to \mathbb{R}$,

$$\widetilde{f}(A) = f(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)) \qquad A \in K$$

If $f \in \mathscr{B}(M, C)$*, then* $\widetilde{f} \in \mathscr{B}(M, K)$.

Proof Let $(p, q) \in M$, $f \in \mathscr{A}(p, q, C)$, $A, B \in K$, $D = (1-p)A + pB$. Denote by $u_j \in \mathbb{C}^n$, $j = 1, 2, \ldots, n$ the eigenvectors of the eigenvalues of the matrix D, that is $Du_j =. \lambda_j(D)u_j$, $j = 1, 2, \ldots, n$. By applying Theorems 6.4. and 6.5 we obtain:

$$\widetilde{f}(D) = f(\lambda_1(D), \lambda_2(D), \ldots, \lambda_n(D)) = f\left(u_1^T D\bar{u}_1, u_2^T D\bar{u}_2, \ldots, u_n^T D\bar{u}_n\right) =$$

$$= f\left((1-p)u_1^T A\bar{u}_1 + pu_1^T B\bar{u}_1, (1-p)u_2^T A\bar{u}_2 + pu_2^T B\bar{u}_2, \ldots, (1-p)u_n^T A\bar{u}_n + pu_n^T B\bar{u}_n\right) \leq$$

$$\leq (1-q)\left[(1-p) f\left(u_1^T A\bar{u}_1, u_2^T A\bar{u}_2, \ldots, u_n^T A\bar{u}_n\right) + pf\left(u_1^T B\bar{u}_1, u_2^T B\bar{u}_2, \ldots, u_n^T B\bar{u}_n\right)\right] +$$

$$+ q \max\left(f\left(u_1^T A\bar{u}_1, u_2^T A\bar{u}_2, \ldots, u_n^T A\bar{u}_n\right),\ f\left(u_1^T B\bar{u}_1, u_2^T B\bar{u}_2, \ldots, u_n^T B\bar{u}_n\right)\right) \leq$$

$$\leq (1-q)\left[(1-p) f(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)) + pf(\lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B))\right] +$$

$$+ q \max((f(\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)),\ f(\lambda_1(B), \lambda_2(B), \ldots, \lambda_n(B)))) =$$

$$= (1-q)\left[(1-p)\widetilde{f}(A) + p\widetilde{f}(B)\right] + q \max\left(\widetilde{f}(A),\ \widetilde{f}(B)\right)$$

Thus $\widetilde{f} \in \mathscr{A}(p, q, C)$. Consequently the conclusion of the theorem follows. □

Examples

1. If $C = \mathbb{R}^n$ and $f_1(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \ldots + x_n$, $(x_1, x_2, \ldots, x_n) \in C$ then $\widetilde{f_1}(A) = \lambda_1(A) + \lambda_2(A) + \ldots + \lambda_n(A) = \mathrm{Tr}(A)$, $A \in S_n(\mathbb{R})$
2. If $C = \mathbb{R}^n$ and $f_2(x_1, x_2, \ldots, x_n) = x_1 x_2 \ldots x_n$, $(x_1, x_2, \ldots, x_n) \in C$ then $\widetilde{f_2}(A) = \lambda_1(A)\lambda_2(A)\ldots\lambda_n(A) = \det(A)$, $A \in S_n(\mathbb{R})$
3. If $C = (0, \infty)^n$ and $f_3(x_1, x_2, \ldots, x_n) = \frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}$, $(x_1, x_2, \ldots, x_n) \in C$ then $\widetilde{f_3}(A) = \frac{1}{\lambda_1(A)} + \frac{1}{\lambda_2(A)} + \ldots + \frac{1}{\lambda_n(A)} = \lambda_1\left(A^{-1}\right) + \lambda_2\left(A^{-1}\right) + \ldots + \lambda_n\left(A^{-1}\right) = \mathrm{Tr}\left(A^{-1}\right)$, $A \in S_n(\mathbb{R})$

4. If $1 \leq k \leq n$ and $C = \mathbb{R}^n$ and $f_4(x_1, x_2, \dots, x_n) = \sigma_k(x_1, x_2, \dots, x_n) =$ the $k-$th elementary symmetric polynomial then $\widetilde{f_4}(A) = \widetilde{\sigma}_k(A)$ =the sum of principal minors of A of order k.

If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ are two vectors from $\mathbb{R}^n$ we shall write $x \leq y$ if $x_i \leq y_i$ for every $i \in \{1, 2, \dots, n\}$. Let C be a subset of $\mathbb{R}^n$ A function $h : C \to \mathbb{R}$ is called *monotonic increasing* if $x, y \in C$ and $x \leq y$ implies $h(x) \leq h(y)$. Note that h is monotone increasing if and only if it is separately monotone increasing.

Theorem 6.7 *Let $n \geq 2$, $1 \leq k \leq n$, C be a convex permutation invariant subset of $\mathbb{R}^n$, $P_k : \mathbb{R}^n \to \mathbb{R}^k$, $P_k(x_1, x_2, \dots, x_n) = (x_1, x_2, \dots, x_k)$, $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $C_k = P_k(\mathbb{R}^n)$.*

If $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, we denote by $x_{[1]} \geq x_{[2]} \geq \dots \geq x_{[n]}$ the components of x in descending order. Consider the functions $h_i : C_k \to \mathbb{R}$, $i = 1, 2$ and $f, g : C \to \mathbb{R}$ defined as follows:

$$f(x_1, x_2, \dots, x_n) = h_1\left(x_{[1]}, x_{[2]}, \dots, x_{[k]}\right), \qquad (x_1, x_2, \dots, x_n) \in C \tag{6.1}$$

$$g(x_1, x_2, \dots, x_n) = h_2\left(x_{[n]}, x_{[n-1]}, \dots, x_{[n-k+1]}\right), \qquad (x_1, x_2, \dots, x_n) \in C \tag{6.2}$$

Suppose that the following conditions hold:

1^0. h_1 and h_2 are symmetric and monotone increasing.
2^0. h_1 is convex and h_2 is concave.
Then f is convex and g is concave.

Proof For every permutation $\sigma \in S_n$ and $i = 1, 2$ let

$$u_{i,\sigma}(x_1, x_2, \dots, x_n) = h_i\left(x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(k)}\right), \quad (x_1, x_2, \dots, x_n) \in C.$$

Note that for every permutation $\sigma \in S_n$, $u_{1,\sigma}$ is convex and $u_{2,\sigma}$ is concave. One can easily see that

$$f(x) = \sup_{\sigma \in S_n} \left[u_{1,\sigma}(x)\right], \qquad x \in C,$$

$$g(x) = \inf_{\sigma \in S_n} \left[u_{2,\sigma}(x)\right], \qquad x \in C,$$

and f and g are symmetric. Since the maximum of convex functions is a convex function it follows that f is convex. Since the minimum of concave functions is a concave function it follows that g is concave. Note that (6.1) and (6.2) hold. □

Corollary 6.8 *For every matrix A let $\lambda_1(A) \geq \lambda_2(A) \geq \dots \geq \lambda_n(A)$ be the eigenvalues of A arranged in descending order. For every $1 \leq k \leq n$ consider the functions:*

$$f_{1,k}(A) = \sum_{r=1}^{k} \lambda_r(A), \qquad A \in H_n(\mathbb{C})$$

$$f_{2,k}(A) = \sum_{r=0}^{k-1} \lambda_{n-r}(A), \qquad A \in H_n(\mathbb{C})$$

$$f_3(A) = \lambda_1(A), \qquad A \in H_n(\mathbb{C})$$

$$f_4(A) = \lambda_n(A), \qquad A \in H_n(\mathbb{C})$$

$$f_{5,k}(A) = \sqrt[k]{\lambda_n(A)\lambda_{n-1}(A)\ldots\lambda_{n-k+1}(A)}, \qquad A \in P_n(\mathbb{C})$$

$$f_6(A) = \sqrt[n]{\det(A)}, \qquad A \in P_n(\mathbb{C})$$

Then $f_{1,k}$, f_3 are convex and $f_{2,k}$, f_4, $f_{5,k}$ and f_6 are concave.

Proof Let $h_k(x_1, x_2, \ldots, x_k) = x_1 + x_2 + \ldots + x_k$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^k$. Note that h_k is convex and concave. By Theorem 6.7. it follows that

$$g_{1,k}(x_1, x_2, \ldots, x_n) = h_k\left(x_{[1]}, x_{[2]}, \ldots, x_{[k]}\right), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

is convex and

$$g_{2,k}(x_1, x_2, \ldots, x_n) = h_k\left(x_{[n]}, x_{[n-1]}, \ldots, x_{[n-k+1]}\right), \qquad (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$$

is concave. Note that

$$f_{1,k}(A) = \widetilde{g}_{1,k}(A), \qquad A \in H_n(\mathbb{C})$$

$$f_{2,k}(A) = \widetilde{g}_{2,k}(A), \qquad A \in H_n(\mathbb{C})$$

By the Davis theorem it follows that $f_{1,k}$ is convex and $f_{2,k}$ is concave. Since $f_3 = f_{1,1}$ and $f_4 = f_{2,1}$ it follows that f_3 is convex and f_4 is concave. Let $u_k(x_1, x_2, \ldots, x_n) = \sqrt[k]{x_1 x_2 \ldots x_k}$, $(x_1, x_2, \ldots, x_n) \in (0, \infty)^n$. By applying the preceding theorem and the Davis theorem we obtain that f_5 is concave. Note that $f_6 = \widetilde{u}_n$. By Davis theorem it follows that f_6 is concave. □

Lemma 6.9 *Let E be a linear normed space, D be a convex open subset of E, and f be a two times differentiable function. For every $a \in D$ and $h \in E$ let $\varepsilon(a, h) > 0$, $J(a, h) = \{t \geq 0 : a + th \in D\} = [0, \varepsilon(a, h))$, $u_{a,h}(t) = f(a + th)$, $t \in J(a, h)$. Then f is convex if and only if $u''_{a,h}(0) \geq 0$ for every $a \in D$ and $h \in E$.*

Proof The conclusion of the lemma follows at once from the equality $u''_{a,h}(0) = f''(a)(h, h)$, $a \in D$ and $h \in E$. □

Lemma 6.10 *Let B, C be two $n \times n$ positive semidefinite matrices, $A \in P_n(\mathbb{R})$ and $w(t) = Tr\left(C\left(A^{-1}B + tI_n\right)^2 A^{-1}\right)$, $t \in \mathbb{R}$. Then the following assertions hold:*

1^0. $w(t) \geq 0$ for every $t \in \mathbb{R}$.
2^0. $\left[Tr\left(CA^{-1}BA^{-1}\right)\right]^2 \leq Tr\left(CA^{-1}\right)Tr\left(C\left(A^{-1}B\right)^2 A^{-1}\right)$

Proof Let $Q(t) = B + tA$, $t \in \mathbb{R}$. There exists a positive semidefinite matrix E such that $C = EE$ and exists a $D \in P_n(\mathbb{R})$ such that $A^{-1} = DD$. Denote $S(t) = EDDQ(t)D$, $t \in \mathbb{R}$. Then

$$w(t) = \mathrm{Tr}\left(C\left[A^{-1}(B + tA)\right]^2 A^{-1}\right) = \mathrm{Tr}\left(CA^{-1}Q(t)A^{-1}Q(t)A^{-1}\right) =$$

$$= \mathrm{Tr}\left(EEDDQ(t)DDQ(t)DD\right) = \mathrm{Tr}\left(ES(t)DQ(t)DD\right) =$$

$$= \mathrm{Tr}\left(S(t)DQ(t)DDE\right) = \mathrm{Tr}\left(S(t)S(t)^T\right) \geq 0$$

Thus assertion 1^0 is proved. Note that

$$w(t) = t^2\mathrm{Tr}\left(CA^{-1}\right) + 2t \cdot \mathrm{Tr}\left(CA^{-1}BA^{-1}\right) + \mathrm{Tr}\left(C\left(A^{-1}B\right)^2 A^{-1}\right)$$

Since the quadratic polynomial in t has constant sign it follows that its discriminant is negative. Hence the inequality from assertion 2^0 holds. □

Theorem 6.11 *Let $C \in P_n(\mathbb{R})$. Then the function*

$$f(A) = \frac{1}{Tr\left(CA^{-1}\right)}, \quad A \in P_n(\mathbb{R})$$

is concave.

Proof Let $B \in H_n(\mathbb{R})$, $A \in P_n(\mathbb{R})$. Then there exists $\varepsilon = \varepsilon(A, B) > 0$ such that $A + tB$ is invertible for every $t \in (-\epsilon, \epsilon)$. Let $v(t) = (A + tB)^{-1}$ and $u(t) = \mathrm{Tr}(Cv(t))$, $t \in (-\epsilon, \epsilon)$ .Note that $u(t) > 0$ for every $t \in (-\epsilon, \epsilon)$ and

$$\left(\frac{1}{u}\right)'' = \frac{2u\,u' - u\,u''}{u^3}.$$

Hence the function $\frac{1}{u}$ is concave if and only if $u\,u'' - 2u\,u' \geq 0$

Note that $v(0) = A^{-1}$. If we differentiate with respect to t the identity $(A + tB)v(t) = I_n$ we obtain $Bv(t) + (A + tB)v'(t) = 0$, hence $v'(t) = -v(t)Bv(t)$ .If we differentiate again the last identity we obtain

$$v''(t) = -v'(t)Bv(t) - v(t)Bv'(t) = 2v(t)Bv(t)Bv(t)$$

Note that

$$u''(t) = \mathrm{Tr}\left(Cv''(t)\right) = 2\mathrm{Tr}\left(Cv(t)\, Bv(t)\, Bv(t)\right)$$

By the preceding lemma we obtain

$$\begin{aligned} u(t)\ u''(t) - 2u(t)\ u'(t) =& 2\mathrm{Tr}\left(Cv(t)\right)\mathrm{Tr}\left(Cv(t)\, Bv(t)\, Bv(t)\right) \\ & - 2\left[\mathrm{Tr}\left(Cv(t)\, Bv(t)\right)\right]^2 \geq 0 \end{aligned}$$

By Lemma 6.9 it follows that f is concave. □

The Bergstrom Inequality [6]
For every $n \times n$ matrix A and $k \in \{1, 2, \ldots, n\}$ denote by A_k the matrix obtained from A by deleting the k-th row and the k-th column of A. In the paper of Bergstrom [6] the following inequality was proved:

$$\frac{\det(A+B)}{\det(A_k+B_k)} \geq \frac{\det(A)}{\det(A_k)} + \frac{\det(B)}{\det(B_k)}, \quad A, B \in P_n(\mathbb{R})$$

One can easily see that Bergstrom inequality is equivalent with the concavity of the function

$$f(A) = \frac{\det(A)}{\det(A_k)}, \quad A \in P_n(\mathbb{R})$$

A generalization of the Bergstrom inequality can be obtained from Theorem 6.11.

Theorem 6.12 *Let $c_1, c_2, \ldots, c_n$ be nonnegative numbers such that $\sum_{i=1}^{n} c_i > 0$. Then the function*

$$f(A) = \frac{\det(A)}{\sum_{i=1}^{n} c_i \det(A_i)}, \quad A \in P_n(\mathbb{R})$$

is concave.

Proof It suffices to prove the concavity of f in the case all $c_i > 0$. Note that if $C = \mathrm{diag}(c_1, c_2, \ldots, c_n)$ then

$$\mathrm{Tr}\left(CA^{-1}\right) = \sum_{i=1}^{n} c_i \frac{\det(A_i)}{\det(A)}$$

hence by Theorem 6.11.

$$f(A) = \frac{1}{\operatorname{Tr}\left(CA^{-1}\right)} = \frac{\det(A)}{\sum_{i=1}^{n} c_i \det(A_i)}, \quad A \in P_n(\mathbb{R})$$

is concave. □

Theorem 6.13 *Let $n \geq 2$, $1 \leq k \leq \tilde{n}$, $a \in \mathbb{R}$. Then the function*

$$f(A) = [\widetilde{\sigma}_k(A)]^a, \quad A \in P_n(\mathbb{R})$$

is concave if and only if $ap \in \left[0, \frac{1}{k}\right]$ and $p \in (-\infty, 1]$.

Proof Let $g(x) = [\sigma_k(x^p)]^a$, $x \in (0, \infty)^n$. Note that by Theorem 5.27. g is concave if and only if $ap \in \left[0, \frac{1}{k}\right]$ and $p \in (-\infty, 1]$. By the Davis theorem $f = \widetilde{g}$ is concave if and only if $ap \in \left[0, \frac{1}{k}\right]$ and $p \in (-\infty, 1]$. □

In the paper of Sra [76], Corollary 3.4., the following result is stated:

Theorem 6.14 *If $1 \leq k \leq n$ and $p \in (-1, 0)$ then the function*

$$f_{k,p}(A) = \frac{1}{\widetilde{\sigma}_k(A)}, \quad A \in P_n(\mathbb{R})$$

is concave.

Taking into account Theorem 6.13. one can easily see that $f_{k,p}$ is concave if and only if $k = 1$. Consequently Sra's result holds only in the presence of supplementary hypotheses. One complete statement is that the condition $1 \leq k \leq n$ and $p \in \left[-\frac{1}{k}, 0\right]$ implies that $f_{k,p}$ is concave.

Theorem 6.15 *Let $n \geq 2$, $1 \leq k \leq n$.Consider the functions*

$$f_{1,k}(A) = \frac{\widetilde{\sigma}_{k+1}(A)}{\widetilde{\sigma}_k(A)}, \quad A \in P_n(\mathbb{R}), \quad k \leq n-1$$

$$f_{2,k}(A) = [\widetilde{\sigma}_k(A)]^{1/k}, \quad A \in P_n(\mathbb{R})$$

$$f_{3,k}(A) = \ln[\widetilde{\sigma}_k(A)], \quad A \in P_n(\mathbb{R}), \quad k \leq n-1$$

$$f_{4,k}(A) = \ln\left[\widetilde{\sigma}_k\left(A^{-1}\right)\right], \quad A \in P_n(\mathbb{R})$$

Then $f_{1,k}$, $f_{2,k}$, $f_{3,k}$ are concave and $f_{4,k}$ is convex.

Proof Concavity of $f_{1,k}$ and $f_{2,k}$ follows from Theorem 5.1. and from Davis theorem. Concavity of $f_{3,k}$ and convexity of $f_{4,k}$ follow from Theorem 5.11. and from Davis theorem. Convexity of $f_{4,k}$ was proved with variational techniques in the paper of Muir [55]. □

Corollary 6.16 *Let $n \geq 3$ and $p \in \mathbb{R}$. Consider the functions:*

$$f_1(A) = [\det(A)]^p, \qquad A \in P_n(\mathbb{R})$$

$$f_2(A) = \ln[\det(A)], \qquad A \in P_n(\mathbb{R})$$

$$f_3(A) = \sqrt[3]{[Tr(A)]^3 + 2Tr\left(A^3\right) - 3Tr\left(A^2\right)Tr(A)}, \qquad A \in P_n(\mathbb{R}).$$

Then the following assertions hold:

1^0. If $p \in \left[0, \frac{1}{n}\right]$, then f_1 is concave.
2^0. f_2 and f_3 are concave.
3^0. If $p \in (-\infty, 0]$, then f_1 is convex.

Proof Note that $f_1(A) = \left\{[\widetilde{\sigma}_n(A)]^{1/n}\right\}^{pn}$, $A \in P_n(\mathbb{R})$ .If $p \in \left[0, \frac{1}{n}\right]$, then $pn \in [0, 1]$ hence f_1 is concave. Note also that $f_2(A) = \ln[\widetilde{\sigma}_n(A)]$ and $f_3(A) = \sqrt[3]{6}\sqrt[3]{\widetilde{\sigma}_3(A)}$ $A \in P_n(\mathbb{R})$. By the preceding theorem the functions f_2 and f_3 are concave. Suppose now that $p \in (-\infty, 0]$. Then the function g defined below is convex.

$$g(A) = \ln(f_1(A)) = p\ln[\det(A)] = pf_2(A), \qquad A \in P_n(\mathbb{R})$$

Since $f_1 = e^g$ it follows that f_1 is convex. □

Theorem 6.17 *Let $n \geq 2$ and $p \in \mathbb{R}^* = \mathbb{R} - \{0\}$. Consider the functions*

$$f_{1,p}(A) = \left[Tr\left(A^p\right)\right]^{1/p}, \qquad A \in P_n(\mathbb{R})$$

$$f_{2,p}(A) = \ln\left[Tr\left(A^p\right)\right], \qquad A \in P_n(\mathbb{R})$$

$$f_{3,p}(A) = \left\{[Tr(A)]^p - Tr\left(A^p\right)\right\}^{1/p}, \qquad A \in P_n(\mathbb{R})$$

Then the following assertions hold:

1^0. If $p \in [1, \infty)$, then $f_{1,p}$ is convex.
2^0. $p \in (-\infty, 0) \cup (0, 1]$ then $f_{1,p}$ is concave.
3^0. If $p \in (-\infty, 0)$, then $f_{2,p}$ is convex.
4^0. If $p \in (0, 1]$, then $f_{2,p}$ is concave.
5^0. If $p \in [1, \infty)$, then $f_{3,p}$ is concave.

Proof Let $g_p(x) = [\sigma_1(x^p)]^{1/p}$, $x \in (0, \infty)^n$. By Corollary 3.2. g_p is convex for $p \in [1, \infty)$ and g_p is concave for $p \in (-\infty, 0) \cup (0, 1]$ The validity of assertions 1^0. and 2^0. follow from the equality $f_{1,p} = \tilde{g}_p$ and from the Davis theorem. Let $h_{k,p}(x) = \ln[\sigma_k(x^p)]$, $x \in (0, \infty)^n$, $k \in \{1, 2, \ldots, n\}$. By Theorem 5.11 it follows that $h_{k,p}$ is convex for $p \in (-\infty, 0)$ and $h_{k,p}$ is concave for $k = n$ and $p \in [0, \infty)$ or for $k \leq n-1$ and $p \in [0, 1]$. The validity of assertions 3^0. and 4^0. follows from the equality $f_{2,p} = \tilde{h}_{k,p}$ and from the Davis theorem.

Let $u_p(x) = [\sigma_1(x)]^p - \sigma_1(x^p)$, $x \in (0, \infty)^n$. By Corollary 4.8 it follows that u_p is concave for $p \in [1, \infty)$. By the Davis theorem $f_{3,p} = \tilde{u}_p$ is concave. □

Theorem 6.18 *Let $n \geq 2$ and $p, q \in \mathbb{R}^* = \mathbb{R} - \{0\}$. Consider the functions:*

$$f_{1,p}(A) = \frac{Tr(A^p)}{\det(A)}, \quad A \in P_n(\mathbb{R})$$

$$f_{2,p,q}(A) = \frac{Tr(A^p)}{[Tr(A)]^q}, \quad A \in P_n(\mathbb{R})$$

$$f_{3,p,q}(A) = \frac{Tr(A^p)}{Tr(A^q)}, \quad A \in P_n(\mathbb{R}).$$

Then the following assertions hold:

1^0. If $p \in [n+1, \infty)$, then $f_{1,p}$ is convex.
2^0. If $p \geq q+1$ and $q \geq 0$, then $f_{2,p,q}$ is convex.
3^0. If $p \geq q+1$ and $q \in (0, 1]$, then $f_{3,p,q}$ is convex.

Proof Let $u(A) = [\mathrm{Tr}(A^p)]^{1/p}$, $v(A) = [\det(A)]^{1/n}$, $A \in P_n(\mathbb{R})$. If $p \in [n+1, \infty)$, then the function $h(x, y) = x^p y^{-n}$, $(x, y) \in (0, \infty)^2$, is convex. By Theorem 6.17. u is convex. By Corollary 6.16. v is concave. Hence $f_{1,p}(A) = h(u(A), v(A))$, $A \in P_n(\mathbb{R})$ is convex. If $p \geq q+1$ and $q \geq 0$, then $g(x, y) = x^p y^{-q}$, $(x, y) \in (0, \infty)^2$ is convex. Since $f_{2,p,q}(A) = g(u(A), \mathrm{Tr}(A))$, $A \in P_n(\mathbb{R})$ it follows that $f_{2,p,q}$ is convex. If $p \geq q+1$ and $q \in (0, 1]$, then $w(A) = [\mathrm{Tr}(A^q)]^{1/q}$, $A \in P_n(\mathbb{R})$ is concave hence $f_{3,p,q}(A) = g(u(A), w(A))$, $A \in P_n(\mathbb{R})$ is convex. □

Theorem 6.19 *Let $n \geq 2$ and $p, q \in \mathbb{R}$. Consider the function*

$$f(A) = [Tr(A)]^p [\det(A)]^q, \quad A \in P_n(\mathbb{R})$$

Then the following assertions hold:

1^0. If $p+q \leq 0$ and $q \leq 0$, then f is convex.
2^0. If $p+q \in [0, 1]$, $q \in [0, 1]$, $p+nq \in [0, 1]$, then f is concave.

Proof $g(x) = [\sigma_1(x)]^p [\sigma_n(x)]^q$, $x \in (0, \infty)^n$ .If $p+q \leq 0$ and $q \leq 0$, then by Theorem 5.6. it follows that g is convex. By Davis theorem $f = \tilde{g}$ is convex. If $p+q \in [0, 1]$, $q \in [0, 1]$, $p+nq \in [0, 1]$, then by Theorem 5.7 it follows that g is concave. By Davis theorem $f = \tilde{g}$ is concave. □

Theorem 6.20 *Let $n \geq 2$, $D = \{A \in H_n(\mathbb{C}) : \exp(Tr(A)) > Tr(\exp(A))\}$,*

$$f_1(A) = \ln\left[Tr(\exp(A))\right], \quad A \in H_n(\mathbb{C})$$

$$f_2(A) = \ln\left[\exp(Tr(A)) - Tr(\exp(A))\right], \quad A \in D.$$

Then the following assertions hold:

1^0. f_1 is convex.
2^0. D is convex.
3^0. f_2 is concave.

Proof Let $u(x_1, x_2, \ldots x_n) = \ln\left(\sum_{i=1}^{n} e^{x_i}\right)$, $(x_1, x_2, \ldots x_n) \in \mathbb{R}^n$. By Corollary 3.2 it follows that u is convex. By Davis theorem $f_1 = \tilde{u}$ is convex. Let $v(A) = f_1(A) - \text{Tr}(A)$, $A \in H_n(\mathbb{C})$. Note that v is convex and $D = \{A \in H_n(\mathbb{C}) : v(A) < 0\}$. Hence D is convex. Let

$$U = \left\{(x_1, x_2, \ldots x_n) \in \mathbb{R}^n : \exp\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} e^{x_i} > 0\right\}$$

$$w(x_1, x_2, \ldots x_n) = \ln\left[\exp\left(\sum_{i=1}^{n} x_i\right) - \sum_{i=1}^{n} e^{x_i}\right], \quad (x_1, x_2, \ldots x_n) \in U$$

By Theorem 4.12 the function w is concave. By the Davis theorem $f_2 = \tilde{w}$ is concave on D. □

Corollary 6.21 *Let $n \geq 2$. Then the following inequalities hold:*

1^0. $[\det(A + B)]^{1/n} \geq [\det(A)]^{1/n} + [\det(B)]^{1/n}$, $A, B \in P_n(\mathbb{R})$;
2^0. $\det(A + B) \geq \det(A) + \det(B)$, $A, B \in P_n(\mathbb{R})$;
3^0. $Tr\left(e^{A+B}\right) \geq Tr\left(e^{A}\right) \cdot Tr\left(e^{B}\right)$, $A, B \in S_n(\mathbb{R})$;
4^0.

$$\frac{k^{n+1}}{\det(A_1 + A_2 + \ldots + A_k)} \leq \frac{1}{\det(A_1)} + \frac{1}{\det(A_2)} + \ldots + \frac{1}{\det(A_k)}, \tag{6.3}$$

for every $A_1, A_2, \ldots, A_k \in P_n(\mathbb{R})$

Proof The first inequality follows from the concavity of the function $f(A) = [\det(A)]^{1/n}$, $A \in P_n(\mathbb{R})$. The second inequality follows at once from the first inequality.

From Theorem 6.20 it follows that the function $g(A) = \ln\left[\text{Tr}(\exp(A))\right]$, where $A \in H_n(\mathbb{C})$, is convex. Hence

$$\ln\left[\text{Tr}\left(\exp\left(\frac{A+B}{2}\right)\right)\right] = g\left(\frac{A+B}{2}\right) \leq \frac{g(A) + g(B)}{2} = \ln\left(\sqrt{g(A)\, g(B)}\right).$$

Thus

$$\mathrm{Tr}\left(\exp\left(A+B\right)\right) \leq \left[\mathrm{Tr}\left(\exp\left(\frac{A+B}{2}\right)\right)\right]^2 \leq \mathrm{Tr}\left(e^A\right)\cdot\mathrm{Tr}\left(e^B\right), \ \ A, B \in S_n\left(\mathbb{R}\right).$$

By Corollary 6.16. the function

$$h\left(A\right) = \frac{1}{\det\left(A\right)}, \quad A \in P_n\left(\mathbb{R}\right)$$

is convex. By Jensen inequality applied to function h we obtain inequality (6.3). In the case $k = n = 2$ the inequality (6.3) becomes:

$$\frac{8}{\det\left(A_1 + A_2\right)} \leq \frac{1}{\det\left(A_1\right)} + \frac{1}{\det\left(A_2\right)}, \quad A_1, A_2 \in P_2\left(\mathbb{R}\right).$$

If

$$A_j = \begin{pmatrix} x_j & y_j \\ y_j & z_j \end{pmatrix}, \quad j = 1, 2,$$

then the above inequality becomes

$$\frac{8}{\left(x_1 + x_2\right)\left(z_1 + z_2\right) - \left(y_1 + y_2\right)^2} = \frac{8}{\det\left(A_1 + A_2\right)} \leq$$

$$\leq \frac{1}{\det\left(A_1\right)} + \frac{1}{\det\left(A_2\right)} = \frac{1}{x_1 z_1 - y_1^2} + \frac{1}{x_2 z_2 - y_2^2},$$

where $x_j z_j - y_j^2 > 0, x_j > 0, z_j > 0,\ j = 1, 2$. The above inequality is the inequality from the statement of Corollary 4.19. □

Proposition 6.22 *Let* $n \geq 2,\ f : H_n\left(\mathbb{C}\right) \to \mathbb{R},\ f\left(A\right) = \left[Tr\left(A\right)\right]^2 - 4\det\left(A\right),\quad A \in H_n\left(\mathbb{C}\right)$. *Then* f *is convex if and only if* $n = 2$.

Proof Let $B\left(t\right) = \mathrm{diag}(t, t, \ldots, t),\ t \in \mathbb{R}$. If f is convex, then $g\left(t\right) = f\left(B\left(t\right)\right) = n^2 t^2 - 4t^n,\ t \in \mathbb{R}$, is convex. Note that

$$g''\left(t\right) = 2n^2 - 4n\left(n-1\right)t^{n-2} = 4n\left(n-1\right)\left[\frac{n}{2\left(n-1\right)} - t^{n-2}\right], \ \ t \in \mathbb{R}.$$

If $n \geq 3$, then g'' change its sign on $\mathbb{R}$.This shows that $n = 2$. Note that $h\left(x, y\right) = \left(x+y\right)^2 - 4xy = \left(x-y\right)^2,\ x, y \in \mathbb{R}$ is convex and if $n = 2$ we have $f = \tilde{h}$. By the Davis theorem it follows that f is convex. □

7 A Generalization of Muirhead Theorem and Applications

The Muirhead's inequality, also known as the "bunching" method, generalizes the inequality between arithmetic and geometric means. It was named after its author Robert Franklin Muirhead, a Scottish mathematician who lived between 1860 and 1941. The inequality is between two homogeneous symmetric polynomials of several variables. It is often useful in proofs involving inequalities between homogeneous symmetric polynomials. For example, Newton's inequality for elementary symmetric functions can be proved with Muirhead's inequality.

Theorem 7.1 (Muirhead) *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$, $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ *and* $\mathbf{a} \leq_{HLP} \mathbf{b}$. *Then the following inequality holds*

$$\sum_{\sigma \in S_n} x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \ldots x_{\sigma(n)}^{a_n} \leq \sum_{\sigma \in S_n} x_{\sigma(1)}^{b_1} x_{\sigma(2)}^{b_2} \ldots x_{\sigma(n)}^{b_n} \tag{7.1}$$

In his original paper Muirhead [56] has considered the case where the exponents a_i and b_i are positive integers. The extension of Muirhead result to real exponents was done by Hardy, Littlewood, and Polya in the book [27]. Proofs of Muirhead's inequality can be found in the monographs of Niculescu and Persson [58], Marshall Olkin and Arnold [49], Garling [24], and Manfrino et al. [47].

A more complete statement of Muirhead theorem is the following (see the references [27] and [47]):

Theorem 7.2 (Schulman [70]) *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_n),, \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$. *Then inequality (7.1) holds for every* $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$ *if and only if* $\mathbf{a} \leq_{HLP} \mathbf{b}$. *Equality takes place only when* $\mathbf{a} = \mathbf{b}$ *or when all the* x_i*-s are equal.*

There are several generalizations of the Muirhead's inequality. Marshall and Proschan proved in [50] an inequality for the expectation of permutation-invariant and convex functions of permutation-invariant random variables. Muirhead's theorem can be obtained as a special case of their result. A different generalization of Muirhead's theorem was given by Proschan and Sethuraman [63]. In this generalization the multivariate monomials were replaced with the product of log-convex functions. Their result states:

Theorem 7.3 *For every* $i \in \{1, 2, \ldots, n\}$ *let* $\psi_i : \mathbb{R} \to (0, \infty)$ *be log-convex functions. If* $\mathbf{a} = (a_1, a_2, \ldots, a_n),, \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ *and* $\mathbf{a} \leq_{HLP} \mathbf{b}$, *then*

$$\sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} \psi_{\sigma(i)}(a_i) \right) \leq \sum_{\sigma \in S_n} \left(\prod_{i=1}^{n} \psi_{\sigma(i)}(b_i) \right)$$

In case that in the above theorem we take $\psi_i(z) = x_i^z$, $z \in \mathbb{R}$ we obtain Muirhead's inequality.

A continuous analog of Muirhead's inequality was proved by Ryff [66]. Muirhead's majorization inequality was extended by Rado to the case of arbitrary permutation groups (that is subgroups of the permutations group S_n). A generalization of Muirhead's inequality considering such subgroups of permutations was given by Daykin [16]. In Kimelfeld [34] was defined the function

$$F(z) = \sum_{g \in G} \exp\left(\xi\left(gz\right)\right),$$

where G is a finite group of linear transformations acting on a real linear space E and ξ is a linear functional on E. With the help of this function was formulated a generalization of Muirhead's theorem. In the paper of Schulman [70] Muirhead's inequality was further generalized to compact groups and their linear representations over the reals. Several discussions about Muirhead's inequality can be found in the paper of Pecaric, Proschan, and Tong [61] and in the monograph of Marshall, Olkin, Arnold [49].

In this section we shall give a different generalization of Muirhead theorem and we shall present various applications of the main result. Let $n \geq 2$. Let E be a real linear space and $A = \left(a_{ij}\right)$ be an $n \times n$ matrix with real entries. We shall define the linear application $\tilde{A} : E^n \to E^n$,

$$\tilde{A}\left(x_1, x_2, \ldots, x_n\right) = \left(\sum_{i=1}^{n} a_{1j} x_j, \sum_{i=1}^{n} a_{2j} x_j, \ldots, \sum_{i=1}^{n} a_{nj} x_j\right), \qquad \left(x_1, x_2, \ldots, x_n\right) \in E^n$$

In the following we shall introduce a majorization preorder on the vectors of E^n. Let E be a real linear space and $\mathbf{x} = \left(x_1, x_2, \ldots, x_n\right) \in E^n$, $\mathbf{y} = \left(y_1, y_2, \ldots, y_n\right) \in E^n$. We shall say that $\mathbf{x}$ *majorize* $\mathbf{y}$ and we shall write $\mathbf{x} \geq_{\mathrm{HLP}} \mathbf{y}$ or $\mathbf{y} \leq_{\mathrm{HLP}} \mathbf{x}$ if there exists a double stochastic matrix $A = \left(a_{ij}\right)$ of order n such that $\mathbf{y} = \tilde{A}\mathbf{x}$.

Theorem 7.4 *Let E be a real linear space, D be a convex subset of E^n and $f : D \to \mathbb{R}$ be a quasi-convex function. Suppose that D is permutation invariant that is if $\left(x_1, x_2, \ldots, x_n\right) \in D$ then $\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right) \in D$ for every permutation $\sigma \in S_n$ and f is symmetric. Then for every $\mathbf{x}, \mathbf{y} \in D$, $\mathbf{x} \geq_{HLP} \mathbf{y}$ we have $f\left(\mathbf{x}\right) \geq f\left(\mathbf{y}\right)$.*

Proof Let $\mathbf{x}, \mathbf{y} \in D$ be such that $\mathbf{x} \geq_{\mathrm{HLP}} \mathbf{y}$. Then there exists a double stochastic matrix $A = \left(a_{ij}\right)$ of order n such that $\mathbf{y} = \tilde{A}\mathbf{x}$. By Birkhoff theorem there exist m permutation matrices P_j, $j = 1, 2, \ldots, m$ such that $A = \sum_{j=1}^{m} \lambda_j P_j$ where $\lambda_j \geq 0$, $j = 1, 2, \ldots, m$ and $\sum_{j=1}^{m} \lambda_j = 1$. Note that

$$f\left(\mathbf{y}\right) = f\left(\tilde{A}\mathbf{x}\right) = f\left(\sum_{j=1}^{m} \lambda_j \tilde{P}_j \mathbf{x}\right) \leq \max_{1 \leq j \leq m} f\left(\tilde{P}_j \mathbf{x}\right) = f\left(\mathbf{x}\right)$$

□

The following two theorems will be called Generalized Muirhead theorems.

Theorem 7.5 *Let E be a real linear space, D be a convex subset of E and $f : D \to \mathbb{R}$ be a convex function. Denote*

$$\Delta = \left\{ (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n : \sum_{i=1}^{n} a_i = 1,\ a_i \geq 0,\ i = 1, 2, \ldots, n \right\}$$

Then the following assertions hold:

1^0. If $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \Delta$ with $\mathbf{a} \geq_{HLP} \mathbf{b}$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ the following inequality holds:

$$\sum_{\sigma \in S_n} f\left(\sum_{i=1}^{n} a_i x_{\sigma(i)} \right) \geq \sum_{\sigma \in S_n} f\left(\sum_{i=1}^{n} b_i x_{\sigma(i)} \right) \tag{7.2}$$

2^0. If $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in [0, \infty)^n$, D is a convex cone and $\mathbf{a} \geq_{HLP} \mathbf{b}$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ inequality (7.2) holds.

3^0 If $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, $D = E$ and $\mathbf{a} \geq_{HLP} \mathbf{b}$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ inequality (7.2) holds.

Proof For every $\mathbf{x}=(x_1, x_2, \ldots, x_n) \in D^n$ consider the function

$$F_{\mathbf{x}}(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} f\left(\sum_{i=1}^{n} a_i x_{\sigma(i)} \right), \qquad (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$$

Note that for every $(x_1, x_2, \ldots, x_n) \in D^n$ the function $F_{\mathbf{x}}$ is convex and symmetric on Δ. By Theorem 7.4., if $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \Delta$, $\mathbf{a} \geq_{\mathrm{HLP}} \mathbf{b}$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ the inequality (7.2) holds.

If D is a convex cone, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ the function $F_{\mathbf{x}}$ is convex and symmetric on $[0, \infty)^n$. Applying Theorem 7.4., if $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in [0, \infty)^n$, with $\mathbf{a} \geq_{\mathrm{HLP}} \mathbf{b}$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ the inequality (7.2) holds.

If $D = E$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ the function $F_{\mathbf{x}}$ is convex and symmetric on $\mathbb{R}^n$. By Theorem 7.4. if $\mathbf{a} = (a_1, a_2, \ldots, a_n)$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, and $\mathbf{a} \geq_{\mathrm{HLP}} \mathbf{b}$, then for every $(x_1, x_2, \ldots, x_n) \in D^n$ the inequality (7.2) holds. □

Theorem 7.6 *Let D_1 be a subset of $(0, \infty)^m$, $f : D_1 \to \mathbb{R}$. Consider the set*

$$D_2 = \left\{ (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : \left(e^{x_1}, e^{x_2}, \ldots, e^{x_m} \right) \in D_1 \right\}.$$

Suppose that U *is a convex subset of* D_2*, and* $g : U \to \mathbb{R}$*,*

$$g(x_1, x_2, \ldots, x_m) = f\left(e^{x_1}, e^{x_2}, \ldots, e^{x_m}\right), \quad (x_1, x_2, \ldots, x_m) \in U$$

is a convex function. If $\mathbf{a} = (a_1, a_2, \ldots, a_n),$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ *and* $\mathbf{a} \leq_{HLP} \mathbf{b}$ *then for every* $\mathbf{x}_i = \left(x_{i,1}, x_{i,2}, \ldots, x_{i,n}\right) \in (0, \infty)^n$, $i = 1, 2, \ldots, m$, *the following inequality holds*

$$\sum_{\sigma \in S_n} f\left(x_{\sigma(1),1}^{a_1} x_{\sigma(2),1}^{a_2} \cdots x_{\sigma(n),1}^{a_n}, x_{\sigma(1),2}^{a_1} x_{\sigma(2),2}^{a_2} \cdots x_{\sigma(n),2}^{a_n}, \ldots, x_{\sigma(1),m}^{a_1} x_{\sigma(2),m}^{a_2} \cdots x_{\sigma(n),m}^{a_n},\right) \leq$$

$$\leq \sum_{\sigma \in S_n} f\left(x_{\sigma(1),1}^{b_1} x_{\sigma(2),1}^{b_2} \cdots x_{\sigma(n),1}^{b_n}, x_{\sigma(1),2}^{b_1} x_{\sigma(2),2}^{b_2} \cdots x_{\sigma(n),2}^{b_n}, \ldots, x_{\sigma(1),m}^{b_1} x_{\sigma(2),m}^{b_2} \cdots x_{\sigma(n),m}^{b_n}\right).$$

Proof Since the function g is convex, by the Generalized Muirhead theorem (Theorem 7.5), we have

$$\sum_{\sigma \in S_n} g\left(a_1 \mathbf{y}_{\sigma(1)} + a_2 \mathbf{y}_{\sigma(2)} + \ldots + a_n \mathbf{y}_{\sigma(n)}\right) \leq \sum_{\sigma \in S_n} g\left(b_1 \mathbf{y}_{\sigma(1)} + b_2 \mathbf{y}_{\sigma(2)} + \ldots + b_n \mathbf{y}_{\sigma(n)}\right)$$

for $\mathbf{y}_i = \left(y_{i,1}, y_{i,2}, \ldots, y_{i,n}\right) \in \mathbb{R}^n$, $i = 1, 2, \ldots, m$. This is equivalent to

$$\sum_{\sigma \in S_n} g\left(\sum_{j=1}^n a_j y_{\sigma(j),1}, \sum_{j=1}^n a_j y_{\sigma(j),2}, \ldots, \sum_{j=1}^n a_j y_{\sigma(j),m}\right) \leq$$
$$\leq \sum_{\sigma \in S_n} g\left(\sum_{j=1}^n b_j y_{\sigma(j),1}, \sum_{j=1}^n b_j y_{\sigma(j),2}, \ldots, \sum_{j=1}^n b_j y_{\sigma(j),m}\right),$$

hence

$$\sum_{\sigma \in S_n} f\left(\exp\left(\sum_{j=1}^n a_j y_{\sigma(j),1}\right), \exp\left(\sum_{j=1}^n a_j y_{\sigma(j),2}\right), \ldots, \exp\left(\sum_{j=1}^n a_j y_{\sigma(j),m}\right)\right) \leq$$

$$\leq \sum_{\sigma \in S_n} f\left(\exp\left(\sum_{j=1}^n b_j y_{\sigma(j),1}\right), \exp\left(\sum_{j=1}^n b_j y_{\sigma(j),2}\right), \ldots, \exp\left(\sum_{j=1}^n b_j y_{\sigma(j),m}\right)\right).$$

Denote $x_{i,j} = \exp\left(y_{i,j}\right)$. Taking into account that

$$\exp\left(\sum_{j=1}^{n} a_j y_{\sigma(j),i}\right) = \mathop{\Pi}_{j=1}^{n} \left[\exp\left(y_{\sigma(j),i}\right)\right]^{a_j} = \mathop{\Pi}_{j=1}^{n} x_{\sigma(j),i}^{a_j}$$

we obtain the inequality from the statement. □

Theorem 7.7 *Let $m \geq 2, n \geq 2$, $\mathbf{a} = (a_1, a_2, \ldots, a_n),$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, $x_{ij} \in (0, \infty)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$. Then the following assertions hold:*

1^0. If $\mathbf{a} \leq_{HLP} \mathbf{b}$, then

$$\prod_{\sigma \in S_n} \left(\sum_{j=1}^{m} x_{\sigma(1),j}^{a_1} x_{\sigma(2),j}^{a_2} \ldots x_{\sigma(n),j}^{a_n}\right) \leq \prod_{\sigma \in S_n} \left(\sum_{j=1}^{m} x_{\sigma(1),j}^{b_1} x_{\sigma(2),j}^{b_2} \ldots x_{\sigma(n),j}^{b_n}\right)$$

2^0. If $\mathbf{a} \leq_{HLP} \mathbf{b}$ and for every $\sigma \in S_n$ we have

$$x_{\sigma(1),1}^{a_1} x_{\sigma(2),1}^{a_2} \ldots x_{\sigma(n),1}^{a_n} > \sum_{j=2}^{m} x_{\sigma(1),j}^{a_1} x_{\sigma(2),j}^{a_2} \ldots x_{\sigma(n),j}^{a_n}$$

and

$$x_{\sigma(1),1}^{b_1} x_{\sigma(2),1}^{b_2} \ldots x_{\sigma(n),1}^{b_n} > \left(\sum_{j=2}^{m} x_{\sigma(1),j}^{b_1} x_{\sigma(2),j}^{b_2} \ldots x_{\sigma(n),j}^{b_n}\right)$$

then the following inequality holds

$$\prod_{\sigma \in S_n} \left(x_{\sigma(1),1}^{a_1} x_{\sigma(2),1}^{a_2} \ldots x_{\sigma(n),1}^{a_n} - \sum_{j=2}^{m} x_{\sigma(1),j}^{a_1} x_{\sigma(2),j}^{a_2} \ldots x_{\sigma(n),j}^{a_n}\right) \geq$$

$$\geq \prod_{\sigma \in S_n} \left(x_{\sigma(1),1}^{b_1} x_{\sigma(2),1}^{b_2} \ldots x_{\sigma(n),1}^{b_n} - \left(\sum_{j=2}^{m} x_{\sigma(1),j}^{b_1} x_{\sigma(2),j}^{b_2} \ldots x_{\sigma(n),j}^{b_n}\right)\right).$$

Proof Let $f_1(x_1, x_2, \ldots, x_m) = \ln(x_1 + x_2 + \ldots + x_m)$, $(x_1, x_2, \ldots, x_m) \in (0, \infty)^m$. Note that

$$g_1(x_1, x_2, \ldots, x_m) = f_1\left(e^{x_1}, e^{x_2}, \ldots, e^{x_m}\right) =$$

$$= \ln\left(e^{x_1} + e^{x_2} + \ldots + e^{x_m}\right), \qquad (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$$

is a convex function. By Theorem 7.6. it follows the first assertion holds.

Let $D_1 = \left\{(x_1, x_2, \ldots, x_m) \in (0, \infty)^m : x_1 > x_2 + \ldots + x_m\right\}$, and

$$f_2(x_1, x_2, \ldots, x_m) = \ln(x_1 - x_2 + \ldots + x_m), \ (x_1, x_2, \ldots, x_m) \in D_1.$$

Let

$$D_2 = \left\{(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m : e^{x_1} > e^{x_2} + e^{x_3} + \ldots + e^{x_m}\right\}$$

$$g_2(x_1, x_2, \ldots, x_m) = f_2\left(e^{x_1}, e^{x_2}, \ldots, e^{x_m}\right) =$$

$$= \ln\left(e^{x_1} - e^{x_2} - \ldots - e^{x_m}\right), \qquad (x_1, x_2, \ldots, x_m) \in D_2$$

Note that g_2 is concave. By Theorem 7.6. it follows the second assertion holds. □

Theorem 7.8 *Let* $n \geq 2$, $\mathbf{a} = (a_1, a_2, \ldots, a_n),$, $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, $x_i, y_i \in (0, \infty)$, $i = 1, 2, \ldots, n$. *If* $\mathbf{a} \preceq_{HLP} \mathbf{b}$ *then the following inequalities hold:*

$$\prod_{\sigma \in S_n} \left(x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \ldots x_{\sigma(n)}^{a_n} + y_{\sigma(1)}^{a_1} y_{\sigma(2)}^{a_2} \ldots y_{\sigma(n)}^{a_n}\right) \leq$$

$$\leq \prod_{\sigma \in S_n} \left(x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \ldots x_{\sigma(n)}^{a_n} + y_{\sigma(1)}^{a_1} y_{\sigma(2)}^{a_2} \ldots y_{\sigma(n)}^{a_n}\right)$$

$$\sqrt[n]{\prod_{i=1}^{n} (x_i + y_i)} \geq \sqrt[n]{x_1 x_2 \ldots x_n} + \sqrt[n]{y_1 y_2 \ldots y_n}.$$

Proof The first inequality is a special case of the first inequality from the preceding theorem. In the case we replace $y_1 = y_2 = \ldots = y_n = 1$ in the first inequality we obtain an inequality from [12]. The second inequality (known as the Huygens inequality) follows from the first inequality in the special case $\mathbf{a} = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n}\right)$, $\mathbf{b} = (1, 0, 0, \ldots, 0)$. □

Corollary 7.9 *Let* $x_i, y_i \in (0, \infty)$, $i = 1, 2, 3$. *Then the following inequality holds*

$$\left[\left(x_1^3 + y_1^3\right)\left(x_2^3 + y_2^3\right)\left(x_3^3 + y_3^3\right)\right]^2 \geq$$

$$\left(x_1^2 x_2 + y_1^2 y_2\right)\left(x_1^2 x_3 + y_1^2 y_3\right)\left(x_2^2 x_1 + y_2^2 y_1\right)\left(x_2^2 x_3 + y_2^2 y_3\right)$$
$$\left(x_3^2 x_1 + y_3^2 y_1\right)\left(x_3^2 x_2 + y_3^2 y_2\right) \geq$$

$$\geq (x_1 x_2 x_3 + y_1 y_2 y_3)^6$$

Proof Let $\mathbf{a} = (3, 0, 0)$, $\mathbf{b} = (2, 1, 0)$, $\mathbf{c} = (1, 1, 1)$. Note that $\mathbf{a} \succeq_{\text{HLP}} \mathbf{b} \succeq_{\text{HLP}} \mathbf{c}$ and apply the first inequality from the statement of the preceding theorem. □

Theorem 7.10 *Let* $n \geq 2$, $\mathbf{a} = (a_1, a_2, \ldots, a_n), \mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$, $a_i, b_i, x_i, y_i \in (0, \infty)$, $i = 1, 2, \ldots, n$. *If* $\mathbf{a} \leq_{HLP} \mathbf{b}$ *and* $x_i \geq y_i$, $i = 1, 2, \ldots, n$, *then the following inequalities hold*

$$\prod_{\sigma \in S_n} \left(x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \ldots x_{\sigma(n)}^{a_n} - y_{\sigma(1)}^{a_1} y_{\sigma(2)}^{a_2} \ldots y_{\sigma(n)}^{a_n} \right) \geq$$

$$\geq \prod_{\sigma \in S_n} \left(x_{\sigma(1)}^{a_1} x_{\sigma(2)}^{a_2} \ldots x_{\sigma(n)}^{a_n} - y_{\sigma(1)}^{a_1} y_{\sigma(2)}^{a_2} \ldots y_{\sigma(n)}^{a_n} \right)$$

$$\sqrt[n]{\prod_{i=1}^{n} (x_i - y_i)} \geq \sqrt[n]{x_1 x_2 \ldots x_n} - \sqrt[n]{y_1 y_2 \ldots y_n}$$

Proof The proof is similar with the proof of the preceding theorem. □

Corollary 7.11 *Let* $x_i > y_i > 0$, $i = 1, 2, 3$. *Then the following inequality holds*

$$\left[\left(x_1^3 - y_1^3 \right) \left(x_2^3 - y_2^3 \right) \left(x_3^3 - y_3^3 \right) \right]^2 \geq$$

$$\left(x_1^2 x_2 - y_1^2 y_2 \right) \left(x_1^2 x_3 - y_1^2 y_3 \right) \left(x_2^2 x_1 - y_2^2 y_1 \right) \left(x_2^2 x_3 - y_2^2 y_3 \right)$$
$$\left(x_3^2 x_1 - y_3^2 y_1 \right) \left(x_3^2 x_2 - y_3^2 y_2 \right) \geq$$
$$(x_1 x_2 x_3 - y_1 y_2 y_3)^6$$

Let E be a real linear space, D be a convex set of E, $n \in N$, $n \geq 1$, $k \in \{1, 2, \ldots, n\}$, $f : D \to R$, $x = (x_1, \ldots, x_n) \in D^n$. We define the *mixed mean* of order k, n:

$$\overline{f}_{k,n}(x) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{x_{i_1} + \cdots + x_{i_k}}{k} \right)$$

The following theorem is a refinement of Jensen's inequality without weights. For a proof we refer to the papers of Mitrinović and Pečarić [52], Pečaric and Volenec [62] (see also Pečarić [60]).

Theorem 7.12 *Let E be a real linear space, D be a convex set of E, $f : D \to \mathbb{R}$ be a convex function. Then for every $\mathbf{x} = (x_1, \ldots, x_n) \in D^n$ the following inequalities hold:*

$$f\left(\frac{x_1 + \ldots + x_n}{n}\right) = \overline{f}_{n,n}(\mathbf{x}) \le \overline{f}_{n-1,n}(\mathbf{x}) \le \cdots \le \overline{f}_{1,n}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} f(x_i).$$

In the following we shall give a proof of Theorem 7.12. by using the Generalized Muirhead's inequality (Theorem 7.5).

Proof of Theorem 7.12. Consider the n-dimensional vectors

$$\mathbf{a}_1 = \left(a_{1,1}, a_{1,2}, \ldots, a_{1,n}\right) = (1, 0, 0, \ldots, 0)$$

$$\mathbf{a}_2 = \left(a_{2,1}, a_{2,2}, \ldots, a_{2,n}\right) = \left(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0\right)$$

$$\ldots\ldots\ldots\ldots$$

$$\mathbf{a}_n = \left(a_{n,1}, a_{n,2}, \ldots, a_{n,n}\right) = \left(\frac{1}{n}, \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$$

Note that for every $k \in \{1, 2, \ldots, n\}$ we have

$$\overline{f}_{k,n}(\mathbf{x}) = \frac{1}{n!}\sum_{\sigma \in S_n} f\left(a_{k,1}x_{\sigma(1)} + a_{k,2}x_{\sigma(2)} + \ldots + a_{k,n}x_{\sigma(n)}\right)$$

and

$$\mathbf{a}_1 \ge_{\mathrm{HLP}} \mathbf{a}_2 \ge_{\mathrm{HLP}} \ldots . \ge_{\mathrm{HLP}} \mathbf{a}_n$$

By the Theorem 7.5 (the generalized Muirhead's theorem) the sequence of inequalities from the statement of Theorem 7.12. holds. □

The following result is a corollary of the preceding theorem.

Theorem 7.13 *Let $n \ge 2$. For every $k \in \{1, 2, \ldots, n\}$ denote by σ_k the $k-$th elementary symmetric polynomial in n variables. If $\mathbf{x} = (x_1, \ldots, x_n) \in (0, \infty)^n$ and $p \in \mathbb{R}$, we shall write*

$$\sigma_k\left(\mathbf{x}^p\right) = \sigma_k\left(x_1^p, x_2^p, \ldots, x_n^p\right)$$

Then for every $\mathbf{x} = (x_1, \ldots, x_n) \in (0, \infty)^n$ *the following inequalities hold:*

1^0.

$$\frac{\sigma_1(\mathbf{x})}{\binom{n}{1}} \geq \frac{\sigma_2\left(\mathbf{x}^{1/2}\right)}{\binom{n}{2}} \geq \ldots \geq \frac{\sigma_k\left(\mathbf{x}^{1/k}\right)}{\binom{n}{k}} \geq \ldots \geq \frac{\sigma_n\left(\mathbf{x}^{1/n}\right)}{\binom{n}{n}}$$

2^0

$$\sigma_k\left(\mathbf{x}^{k+1}\right)\binom{n}{k+1} \geq \sigma_{k+1}\left(\mathbf{x}^k\right)\binom{n}{k} \qquad k = 1, 2, \ldots, n-1$$

Proof Let $f(t) = e^t, t \in \mathbb{R}$. If $\mathbf{y} = (y_1, y_2, \ldots y_n) \in (0, \infty)^n$, then

$$\overline{f}_{k,n}(\mathbf{y}) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} f\left(\frac{y_{i_1} + \cdots + y_{i_k}}{k}\right)$$

Denote $f(y_i) = x_i, i \in \{1, 2, \ldots, n\}$. Then

$$\overline{f}_{k,n}(\mathbf{y}) = \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \left[f\left(y_{i_1}\right)\right]^{1/k} \left[f\left(y_{i_2}\right)\right]^{1/k} \ldots \left[f\left(y_{i_k}\right)\right]^{1/k} =$$

$$= \frac{1}{\binom{n}{k}} \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1}^{1/k} x_{i_2}^{1/k} \ldots x_{i_k}^{1/k} = \frac{\sigma_k\left(\mathbf{x}^{1/k}\right)}{\binom{n}{k}}$$

Since f is convex, by Theorem 7.12. it follows that

$$\overline{f}_{1,n}(\mathbf{y}) \geq \overline{f}_{2,n}(\mathbf{y}) \geq \ldots \geq \overline{f}_{n,n}(\mathbf{y})$$

Thus the first inequality was proved. The second inequality follows from the first inequality performing a substitution, □

Corollary 7.14 *Let* E *be a linear normed space,* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n \in E$ *and* $p \geq 1$. *Then the following sequence of inequalities holds:*

$$\frac{\sum_{i=1}^{n} \|\mathbf{x}_i\|^p}{\binom{n}{1}} \geq \frac{\sum_{i<j}^{n} \left\|\mathbf{x}_i + \mathbf{x}_j\right\|^p}{2^p\binom{n}{2}} \geq \frac{\sum_{i<j<k}^{n} \left\|\mathbf{x}_i + \mathbf{x}_j + \mathbf{x}_k\right\|^p}{3^p\binom{n}{3}} \geq \cdots$$

$$\geq \frac{\|\mathbf{x}_1 + \mathbf{x}_2 + \ldots + \mathbf{x}_n\|^p}{n^p\binom{n}{n}}$$

Proof Note that $f(\mathbf{x}) = \|\mathbf{x}\|^p$, $\mathbf{x} \in E$ is a convex function. The inequality from the statement follows by applying Theorem 7.12. to function f. □

Corollary 7.15 *Let $x_i, y_i \in [0,\infty)$, $i = 1,2,\ldots,n$. Then the following inequalities hold:*

$$\frac{\sum_{i=1}^{n}\sqrt{x_i y_i}}{\binom{n}{1}} \le \frac{\sum_{i<j}^{n}\sqrt{\left(x_i + x_j\right)\left(y_i + y_j\right)}}{\binom{n}{2}} \le$$

$$\le \frac{\sum_{i<j<k}^{n}\sqrt{\left(x_i + x_j + x_k\right)\left(y_i + y_j + y_k\right)}}{3\binom{n}{3}} \le \ldots . \le$$

$$\le \frac{\sqrt{(x_1 + x_2 + \ldots + x_n)(y_1 + y_2 + \ldots + y_n)}}{n\binom{n}{n}}$$

and

$$\left[\prod_{i=1}^{n}(x_i + y_i)\right]^{1/\binom{n}{1}} \ge \left[\prod_{i<j}\left(\sqrt{x_i x_j} + \sqrt{y_i y_j}\right)\right]^{1/\binom{n}{2}} \ge$$

$$\ge \left[\prod_{i<j<k}\left(\sqrt[3]{x_i x_j x_k} + \sqrt{y_i y_j y_k}\right)\right]^{1/\binom{n}{3}} \ge \ldots \ge$$

$$\ge \left(\sqrt[n]{x_1 x_2 \ldots x_n} + \sqrt[n]{y_1 y_2 \ldots y_n}\right)^{1/\binom{n}{n}}.$$

Proof In order to prove the first sequence of inequalities we apply Theorem 7.12. to the concave function $f(x,y) = \sqrt{xy}$, $x, y \in (0,\infty)^2$. Consider the function $g(x,y) = \ln\left(e^x + e^y\right)$, $(x,y) \in \mathbb{R}^2$. By Corollary 3.3 g is convex. If we apply Theorem 7.12. to function g we obtain:

$$\frac{\sum_{i=1}^{n}\ln\left(e^{a_i} + e^{b_i}\right)}{\binom{n}{1}} \ge \frac{\sum_{i=1}^{n}\ln\left(\exp\left(\frac{a_i + a_i}{2}\right) + \exp\left(\frac{b_i + b_j}{2}\right)\right)}{\binom{n}{2}} \ge \ldots \ge$$

$$\ge \frac{\ln\left[\exp\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right) + \exp\left(\frac{b_1 + b_2 + \ldots + b_n}{n}\right)\right]}{\binom{n}{n}}$$

for $a_i, b_i \in \mathbb{R}$, $i = 1,2,\ldots,n$. If we put $x_i = \exp(a_i)$, $y_i = \exp(b_i)$, $i = 1,2,\ldots,n$ in the last sequence of inequalities we obtain the second sequence of inequalities from the statement. □

Corollary 7.16 *Let $x_i, y_i \in (0, \infty)$, $i = 1, 2, \ldots, n$. Then the following assertions hold:*

1^0. If $p \in [0, 1]$, then

$$\left[\prod_{i=1}^{n}\left(x_i^p + y_i^p\right)\right]^{1/\binom{n}{1}} \leq \frac{1}{2^p}\left\{\prod_{i<j}\left[\left(x_i + x_j\right)^p + \left(y_i + y_j\right)^p\right]\right\}^{1/\binom{n}{2}} \leq$$

$$\leq \frac{1}{3^p}\left\{\prod_{i<j<k}\left[\left(x_i + x_j + x_k\right)^p + \left(y_i + y_j + y_k\right)^p\right]\right\}^{1/\binom{n}{3}} \leq$$

$$\leq \ldots . \leq \frac{1}{n^p}\left[\left(\sum_{i=1}^{n} x_i\right)^p + \left(\sum_{i=1}^{n} y_i\right)^p\right]^{1/\binom{n}{n}} .$$

2^0. If $p \in (-\infty, 0)$, then the above sequence of inequalities holds with the reversed sign of inequalities.

Proof For every $p \in \mathbb{R}$ consider the function

$$f_p(x, y) = \ln\left(x^p + y^p\right), \quad (x, y) \in (0, \infty)^2 .$$

Note that f_p is concave if $p \in [0, 1]$ and f_p is convex if $p \in (-\infty, 0)$. Inequalities from the statement follow at once from Theorem 7.12. □

Lemma 7.17 *Let $a_i, \alpha_i \in (0, \infty)$, $i = 1, 2, \ldots, n$, $x_0 \in \mathbb{R}$, and consider the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(x) = \left(\frac{\sum_{i=1}^{n} \alpha_i a_i^x}{\sum_{i=1}^{n} \alpha_i a_i^{x_0}}\right)^{1/(x-x_0)} \qquad \text{if } x \in \mathbb{R} - \{x_0\}$$

with

$$f(x_0) = \frac{\sum_{i=1}^{n} \alpha_i a_i^{x_0} \ln a_i}{\sum_{i=1}^{n} \alpha_i a_i^{x_0}} .$$

Then f is increasing on $\mathbb{R}$.

Proof Let $g(x) = \ln\left(\sum_{i=1}^{n} \alpha_i a_i^x\right)$, $x \in \mathbb{R}$. Note that

$$g'(x) = \frac{\sum_{i=1}^{n} \alpha_i a_i^x \ln a_i}{\sum_{i=1}^{n} \alpha_i a_i^x}, \quad x \in \mathbb{R}$$

and

$$g''(x) = \frac{\left(\sum_{i=1}^{n} \alpha_i a_i^x \ln^2 a_i\right)\left(\sum_{i=1}^{n} \alpha_i a_i^x\right) - \left(\sum_{i=1}^{n} \alpha_i a_i^x \ln a_i\right)^2}{\left(\sum_{i=1}^{n} \alpha_i a_i^x\right)^2}, \quad x \in \mathbb{R}.$$

By the Cauchy–Schwarz inequality it follows that $g'' \geq 0$. Let

$$h(x) = \frac{g(x) - (x_0)}{x - x_0}, \quad x \in \mathbb{R} - \{x_0\}$$

and

$$h(x_0) = g'(x_0).$$

Since g is convex it follows that h is increasing, hence $f = e^h$ is increasing. □

Theorem 7.18 *Let* $a_i \in [1, \infty)$, $i = 1, 2, \ldots, n$, $a = a_1 + 2a_2 + 3a_3 + \ldots + na_n$. *Then for every* $\mathbf{x} \in (0, \infty)^n$ *the following inequalities hold:*

$$[\sigma_n(\mathbf{x})]^{a/n} \leq \left(\frac{\sigma_1(\mathbf{x})}{\binom{n}{1}}\right)^{a_1} \left(\frac{\sigma_2(\mathbf{x})}{\binom{n}{2}}\right)^{a_2} \cdots \left(\frac{\sigma_n(\mathbf{x})}{\binom{n}{n}}\right)^{a_n} \leq \frac{\sigma_1(\mathbf{x}^a)}{n}.$$

Proof From the first sequence of inequalities in Theorem 7.12. we obtain

$$\frac{\sigma_1(\mathbf{x})}{\binom{n}{1}} \geq \frac{\sigma_k\left(\mathbf{x}^{1/k}\right)}{\binom{n}{k}} \geq \frac{\sigma_n\left(\mathbf{x}^{1/n}\right)}{\binom{n}{n}},$$

hence by a substitution we get

$$\frac{\sigma_1\left(\mathbf{x}^k\right)}{\binom{n}{1}} \geq \frac{\sigma_k(\mathbf{x})}{\binom{n}{k}} \geq \frac{\sigma_n\left(\mathbf{x}^{k/n}\right)}{\binom{n}{n}} = [\sigma_n(\mathbf{x})]^{k/n}.$$

By the preceding lemma we have

$$\left[\frac{\sigma_1\left(\mathbf{x}^{ka_k}\right)}{n}\right]^{1/ka_k} \leq \left[\frac{\sigma_1\left(\mathbf{x}^{a}\right)}{n}\right]^{1/a},$$

hence

$$\frac{\sigma_1\left(\mathbf{x}^{ka_k}\right)}{n} \leq \left[\frac{\sigma_1\left(\mathbf{x}^{a}\right)}{n}\right]^{ka_k/a}$$

From the preceding inequalities we obtain

$$\left[\frac{\sigma_1\left(\mathbf{x}^{a}\right)}{n}\right]^{ka_k/a} \geq \frac{\sigma_1\left(\mathbf{x}^{ka_k}\right)}{n} \geq \left[\frac{\sigma_1\left(\mathbf{x}^{k}\right)}{n}\right]^{a_k} \geq$$

$$\geq \left[\frac{\sigma_k\left(\mathbf{x}\right)}{\binom{n}{k}}\right]^{a_k} \geq \left[\sigma_n\left(\mathbf{x}\right)\right]^{ka_k/n}.$$

If we multiply the preceding inequalities for $k = 1, 2, \ldots, n$, then we get the inequality in the statement. □

Theorem 7.19 *Let $n \geq 3$, $m \geq 2$, $J = \{1, 2, \ldots, n\}$. For every $k \in J$ and every $\mathbf{x} \in (0, \infty)^n$ let $p(k, \mathbf{x}) = \frac{\sigma_k(\mathbf{x})}{\binom{n}{k}}$. If $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in J^m$, $\mathbf{a} \geq_{HLP} \mathbf{b}$, then for every $\mathbf{x} \in (0, \infty)^n$ the following inequality holds*

$$p(a_1, \mathbf{x})\, p(a_2, \mathbf{x}) \ldots p(a_m, \mathbf{x}) \leq p(b_1, \mathbf{x})\, p(b_2, \mathbf{x}) \ldots p(b_m, \mathbf{x}).$$

Proof For $\mathbf{x} \in (0, \infty)^n$ let $f_{\mathbf{x}}(k) = \ln[p(k, \mathbf{x})]$, $k \in J$. By Newton inequalities we have that $f_{\mathbf{x}}$ is concave, that is $2 f_{\mathbf{x}}(k) \geq f_{\mathbf{x}}(k-1) + f_{\mathbf{x}}(k+1)$, $k \in \{2, 3, \ldots, n-1\}$. Let $\tilde{f}_{\mathbf{x}} : [1, n] \to \mathbb{R}$ be the piecewise affine continuation of $f_{\mathbf{x}}$, that is

$$\tilde{f}_{\mathbf{x}}(t) = (t-k)\, f_{\mathbf{x}}(k+1) + (k+1-t)\, f_{\mathbf{x}}(k), \quad t \in [k, k+1], k \in \{1, 2, \ldots, n-1\}$$

Note that $\tilde{f}$ is concave. If

$$F_{\mathbf{x}}(c_1, c_2, \ldots, c_m) = \tilde{f}_{\mathbf{x}}(c_1) + \tilde{f}_{\mathbf{x}}(c_2) + \ldots + \tilde{f}_{\mathbf{x}}(c_m), (c_1, c_2, \ldots, c_m) \in [1, n]^m,$$

then $F_{\mathbf{x}}$ is symmetric and concave, hence $F_{\mathbf{x}}$ is Schur concave. Therefore, the inequality in the statement holds. □

Theorem 7.20 *Let* $m, n \geq 2, k \in \{1, 2, \ldots, n\}$. *For every* $\mathbf{x}= (x_1, \ldots, x_n) \in (0, \infty)^n$, *let*

$$q_{k,\mathbf{x}}(t) = \sigma_k\left(\mathbf{x}^t\right) = \sum_{|K|=k}\prod_{i\in K} x_i^t, \quad t \in \mathbb{R}.$$

If $\mathbf{a} = (a_1, a_2, \ldots, a_m),, \mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$ *and* $\mathbf{a} \leq_{HLP} \mathbf{b}$, *then for every* $\mathbf{x} \in (0, \infty)^n$ *the following inequality holds:*

$$q_{k,\mathbf{x}}(a_1)\, q_{k,\mathbf{x}}(a_2) \ldots q_{l,\mathbf{x}}(a_m) \leq q_{k,\mathbf{x}}(b_1)\, q_{k,\mathbf{x}}(b_2) \ldots q_{k,\mathbf{x}}(b_m).$$

Proof For every $K \subset \{1, 2, \ldots, n\}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in (0, \infty)^n$ let

$$u_K(x_1, \ldots, x_n) = \sum_{i\in K} \ln(x_i).$$

Note that

$$v_{k,\mathbf{x}}(t) = \ln\left[q_{k,\mathbf{x}}(t)\right] = \ln\left[\sum_{|K|=k} \exp\left(t u_K(\mathbf{x})\right)\right], \quad t \in \mathbb{R}$$

is convex. For every $k \in \{1, 2, \ldots, n\}$ and $\mathbf{x} \in (0, \infty)^n$, let

$$f_{k,\mathbf{x}}(c_1, c_2, \ldots, c_m) = \ln\left[q_{k,\mathbf{x}}(c_1)\right] + \ln\left[q_{k,\mathbf{x}}(c_2)\right] + \ldots + \ln\left[q_{k,\mathbf{x}}(c_m)\right],$$

where $(c_1, c_2, \ldots, c_m) \in \mathbb{R}^m$. Note that for every $k \in \{1, 2, \ldots, n\}$ and $\mathbf{x} \in (0, \infty)^n$ the function $f_{k,\mathbf{x}}$ is convex and symmetric, hence $f_{k,\mathbf{x}}$ is Schur convex. Thus $f_{k,\mathbf{x}}(\mathbf{a}) \leq f_{k,\mathbf{x}}(\mathbf{b})$. This is equivalent with the inequality from the statement. □

Corollary 7.21 *Let* $n \geq 2$. *Then for every* $(x_1, \ldots, x_n) \in (0, \infty)^n$ *the following inequalities hold:*

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n x_i^9\right) \geq \left(\sum_{i=1}^n x_i^3\right)\left(\sum_{i=1}^n x_i^4\right)\left(\sum_{i=1}^n x_i^5\right)$$

$$\left(\sum_{i<j}^n x_i x_j\right)\left(\sum_{i<j}^n x_i^2 x_j^2\right)\left(\sum_{i<j}^n x_i^9 x_j^9\right) \geq \left(\sum_{i<j}^n x_i^3 x_j^3\right)\left(\sum_{i<j}^n x_i^4 x_j^4\right)\left(\sum_{i<j}^n x_i^5 x_j^5\right)$$

Proof Note that $(9, 2, 1) \geq_{\text{HLP}} (5, 4, 3)$ and then apply the result in preceding the theorem. □

8 Construction of Convex Functions Based on Taylor Remainder

In this section we shall prove the convexity or the concavity of some functions with the help of Taylor remainder of power function and exponential function.

If $f : I \to \mathbb{R}$ is a k times differentiable function, then it can be written as

$$f(x) = T_k f(x) + R_k f(x), \quad x \in I.$$

Here $T_k f(x)$ is the Taylor polynomial of f and $R_k f(x)$ is the corresponding Taylor remainder. If f is the power function or the exponential function we shall prove that the function $g(x) = f^{-1}(R_k f(x))$ is concave. Two conjectures related with concavity of functions built with the help of Taylor remaiders are formulated.

Let I be an interval of the real axis $x_0, x_1, \ldots, x_n \in I$ and let $f : I \to \mathbb{R}$ be a function. Denote by $f[x_0, x_1, \ldots, x_n]$ the divided difference of f at the distinct points $x_0, x_1, \ldots, x_n$. In case f is n-times differentiable then $f[x_0, x_1, \ldots, x_n]$ is defined for all $x_0, x_1, \ldots, x_n \in I$ not necessarily distinct.

The Hermite–Genocchi formula is presented in the following theorem.

Theorem 8.1 *Let I be an interval of the real axis $x_0, x_1, \ldots, x_n \in I$ and $f : I \to \mathbb{R}$ be a function of class C^n. Denote by Δ_n the n-dimensional standard simplex, that is*

$$\Delta_n = \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} : t_i \geq 0 \, for \; i = 0, 1, 2, \ldots, n \; and \sum_{i=0}^{n} t_i = 1 \right\}.$$

Then

$$f[x_0, x_1, \ldots, x_n] = \int_{\Delta_n} f^{(n)}(t_0 x_0 + t_1 x_1 + \ldots + t_n x_n)\, dt_0 dt_1 \ldots dt_n.$$

Theorem 8.2 *Let $n \geq 1$ be a natural number, I be an open interval of the real axis, $a \in I$ and $f : I \to \mathbb{R}$ be a function of class C^n. Consider the function $g : I \to \mathbb{R}$:*

$$g(x) = \frac{f(x) - f(a) - f'(a)\frac{x-a}{1!} - \ldots - f^{(n-1)}(a)\frac{(x-a)^{n-1}}{(n-1)!}}{(x-a)^n}, \quad x \in I - \{a\} \tag{8.1}$$

and

$$g(a) = \frac{f^{(n)}(a)}{n!}.$$

If $f^{(n)}$ is a convex function, then g is a convex function.

Proof If $x_0, x_1, \ldots, x_n \in I$ let $f[x_0, x_1, \ldots, x_n]$ be the divided difference of f at the points $x_0, x_1, \ldots, x_n$. Let

$$h(x_0, x_1, \ldots, x_n) = f[x_0, x_1, \ldots, x_n], \quad (x_0, x_1, \ldots, x_n) \in I^{n+1} \tag{8.2}$$

Note that

$$g(x) = h(x, a, a, \ldots, a), \quad x \in I - \{a\}$$

The number of a in the argument of h is equal to n. By the Hermite–Genocchi formula we obtain:

$$g(x) = \int_{\Delta_n} f^{(n)}(t_0 x + (t_1 + t_2 + \ldots + t_n) a)\, dt_0 dt_1 \ldots dt_n \tag{8.3}$$

Since $f^{(n)}$ is convex it follows that g is convex. □

Corollary 8.3 *Let $n \geq 1$ be a natural number, $c \in (0,1)$, $I = (c, \infty)$ and $f : I \to \mathbb{R}$ be a function of class C^n. If $f^{(n)}$ is convex, then the function $u : I \to \mathbb{R}$*

$$u(x) = \frac{f(x+1) - f(1) - f'(1)\frac{x}{1!} - f''(1)\frac{x^2}{2!} - - \ldots - f^{(n-1)}(1)\frac{x^{n-1}}{(n-1)!}}{x^n}, \ x \in I$$

is convex.

Proof Consider the function g defined by (8.1) and substitute $x \to x+1$ and $a = 1$. Note that

$$u(x) = g(x+1) \quad x \in I$$

Since $f^{(n)}$ is convex, by Theorem 8.2., it follows that the function g is convex, hence u is convex. □

Corollary 8.4 *Let $n \geq 1$ be a natural number and $p \in \mathbb{R}$. For every positive integer k define the generalized binomial coefficient as follows:*

$$\binom{n}{k} = \frac{p(p-1)(p-2)\ldots(p-k+1)}{k!}.$$

If $k = 0$ we shall put $\binom{n}{k} = 1$. Consider the functions

$$g(x) = \frac{(1+x)^p - \binom{p}{0} - \binom{p}{1}x - \binom{p}{2}x^2 - \ldots - \binom{p}{n-1}x^{n-1}}{x^n}, \quad x \in (0, \infty) \tag{8.4}$$

$$h(x) = \frac{(1+x)^{-p} - \binom{p-1}{0} + \binom{p}{1}x - \binom{p+1}{2}x^2 + \ldots + (-1)^{n-1}\binom{p+n-2}{n-1}x^{n-1}}{x^n},$$

$$x \in (0, \infty).$$

Then the following assertions hold:

1^0. *If* $p \geq n+1$, *then* g *is convex.*
2^0. *If* $p \in (0, \infty)$, *then* $(-1)^n h$ *is convex.*

Proof Suppose that $p \geq n+1$ and let $f_1(x) = x^p$, $x \in (0, \infty)$ .Note that $f_1^{(n+2)}(x) = p(p-1)\ldots(p-n-1)x^{p-n-2} \geq 0$, $x \in (0, \infty)$. By Corollary 8.3., g is a convex function.

Suppose that $p \in [0, \infty)$ and let $f_2(x) = x^{-p}$, $x \in (0, \infty)$. Note that $(-1)^n f_2^{(n+2)}(x) = p(p+1)\ldots(p+n+1)x^{-p-n-2} \geq 0$, $x \in (0, \infty)$. By Corollary 8.3., $(-1)^n h$ is a convex function. □

Corollary 8.5 *Let* $n \geq 1$ *be a natural number. Consider the functions* $g, h : \mathbb{R} \to \mathbb{R}$

$$g(x) = \frac{e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^{n-1}}{(n-1)!}}{x^n}, \quad x \in \mathbb{R} - \{0\}$$

with

$$g(0) = \frac{1}{n!}$$

and

$$h(x) = \frac{1 - e^{-x} - \frac{x}{1!} + \frac{x^2}{2!} - \ldots + (-1)^{n-1}\frac{x^{n-1}}{(n-1)!}}{x^n}, \quad x \in \mathbb{R} - \{0\}$$

with

$$h(0) = \frac{1}{n!}.$$

Then g *and* $(-1)^{n+1} h$ *are convex.*

Proof If we apply Theorem 8.2 for $a = 0$ and $f(x) = e^x$, $x \in (0, \infty)$, then we obtain that g is convex. In order to prove that $(-1)^{n+1} h$ is convex consider the function $f(x) = (-1)^{n+1}\left(1 - e^{-x}\right)$, $x \in \mathbb{R}$, and then apply Theorem 8.2. □

Theorem 8.6 *Let* (X, Σ, μ) *be a space with measure,* E *be a real linear space,* D *be a convex subset of* E, *and* $u : D \times X \to \mathbb{R}$. *Suppose that* $u(\cdot, y)$ *is convex for every* $y \in X$ *and* $\exp(u(x, \cdot))$ *is integrable for every* $x \in D$. *Then the function*

$$f(x) = \ln\left[\int_X \exp(u(x, y))\, d\mu(y)\right], \quad x \in D$$

is convex.

Proof Let $a, b \geq 0, a + b = 1$. Then for every $x_1, x_2 \in D$, we have:

$$f(ax_1 + bx_2) = \ln \left[\int_X \exp(u(ax_1 + bx_2, y)) \, d\mu(y) \right] \leq$$

$$\leq \ln \left[\int_X \exp(au(x_1, y) + bu(x_2, y)) \, d\mu(y) \right] =$$

$$= \ln \left[\int_X \exp(au(x_1, y)) \exp(bu(x_2, y)) \, d\mu(y) \right] \leq$$

$$\leq \ln \left[\int_X \left[\exp(au(x_1, y))\right]^{1/a} d\mu(y) \right]^a \ln \left[\int_X \left[\exp(bu(x_2, y))\right]^{1/b} d\mu(y) \right]^b =$$

$$= a \ln \left[\int_X \exp(u(x_1, y)) \, d\mu(x) \right] + b \ln \left[\int_X \exp(u(x_2, y)) \, d\mu(x) \right] =$$

$$= af(x_1) + bf(x_2).$$

Consequently, f is convex. □

Corollary 8.7 *Let $n \geq 1$ be a natural number, I be an interval of the real axis, $f : I \to \mathbb{R}$ be an n times differentiable function such that $f^{(n)} > 0$ on I. If $u = \ln\left(f^{(n)}\right)$ is convex, then the function*

$$g(x_0, x_1, \ldots, x_n) = \ln(f[x_0, x_1, \ldots, x_n]), \quad (x_0, x_1, \ldots, x_n) \in I^{n+1}$$

is convex.

Proof By the Genocchi–Hadamard theorem we have:

$$g(x_0, x_1, \ldots, x_n) = \ln \left(\int_{\Delta_n} f^{(n)}(t_0 x_0 + t_1 x_1 + \ldots + t_n x_n) \, dt_0 dt_1 \ldots dt_n \right) =$$

$$= \ln \left(\int_{\Delta_n} \exp(u(t_0 x_0 + t_1 x_1 + \ldots + t_n x_n)) \, dt_0 dt_1 \ldots dt_n \right)$$

By the preceding theorem it follows that g is convex on I^{n+1}. □

Corollary 8.8 *Let $n \geq 1$ be a natural number. Then the function $g : (0, \infty) \to \mathbb{R}$ defined by*

$$g(x) = \ln \left(\frac{e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^{n-1}}{(n-1)!}}{x^n} \right), \quad x \in (0, \infty)$$

is convex.

Proof Let $f(x) = e^x, x \in (0, \infty)$. Note that $\ln\left(f^{(n)}(x)\right) = x,\ x \in (0, \infty)$ is convex. Since

$$g(x) = \ln(f[x, 0, 0, \ldots, 0])$$

by the preceding Corollary 8.7 it follows that g is convex. □

Corollary 8.9 *Let $n \geq 1$ be a natural number and $p \in \mathbb{R}$. Consider the function:*

$$g(x) = \ln\left(\frac{(1+x)^p - \binom{p}{0} - \binom{p}{1}x - \binom{p}{2}x^2 - \ldots - \binom{p}{n-1}x^{n-1}}{x^n}\right), \quad x \in (0, \infty)$$

Then the following assertions hold:

1^0. *If $p(p-1)(p-2)\ldots(p-n+1) > 0$ and $p < n$, then the function g is convex.*
2^0. *If $p(p-1)(p-2)\ldots(p-n-1) \leq 0$, then g is concave.*

Proof In order to prove 1^0 suppose that $p(p-1)(p-2)\ldots(p-n+1) > 0$ and $p < n$ Consider the function $f(x) = x^p, x \in (0, \infty)$. Note that

$$f^{(n)}(x) = p(p-1)(p-2)\ldots(p-n+1)\, x^{p-n}, x \in (0, \infty)$$

$f^{(n)} > 0$ and $\ln\left(f^{(n)}\right)$ is convex on $(0, \infty)$. By Corollary 8.7. it follows that g is convex.

In order to prove 2^0 suppose that $p(p-1)(p-2)\ldots(p-n-1) \leq 0$. Note that

$$f^{(n+2)}(x) = p(p-1)(p-2)\ldots(p-n-1)\, x^{p-n-2}, x \in (0, \infty)$$

and $f^{(n)}$ is concave on $(0, \infty)$. By Corollary 8.3 the function defined by (8.4) is concave. Hence the function g is concave. □

Lemma 8.10 *For every natural number $k \geq 0$, consider the functions:*

$$v_k(x) = e^x - 1 - \frac{x}{1!} - \frac{x^2}{2!} - \ldots - \frac{x^k}{k!}, \quad x \in (0, \infty) \tag{8.1}$$

$$f_k(x) = k!\,(k-x)\, v_k(x) - x^{k+1}, x \in (0, \infty) \tag{8.2}$$

Then $f_k(x) \leq 0$ for every $x \in (0, \infty)$.

Proof Note that for every $x \in (0, \infty)$ we have successively

$$f_k(x) = k!\,(k-x)\left(\frac{x^{k+1}}{(k+1)!} + \frac{x^{k+2}}{(k+2)!} + \dots.\right) - x^{k+1} =$$

$$= (k-x)\left[\frac{x^{k+1}}{k+1} + \frac{x^{k+2}}{(k+1)(k+2)} + \dots.\right] - x^{k+1} =$$

$$= \left(\frac{k}{k+1} - 1\right)x^{k+1} + \left(\frac{k}{(k+1)(k+2)} - \frac{1}{k+1}\right)x^{k+2} +$$

$$+ \left(\frac{k}{(k+1)(k+2)(k+3)} - \frac{1}{(k+1)(k+2)}\right)x^{k+3} + \dots. =$$

$$= -\frac{x^{k+1}}{k+1} - \frac{2x^{k+2}}{(k+1)(k+2)} - \frac{3x^{k+2}}{(k+1)(k+2)(k+3)} - \dots \le 0$$

□

Theorem 8.11 *For every $k \ge 0$ let v_k be the function defined in (8.1). Then the function $g_k(x) = \ln[v_k(x)]$, $x \in (0, \infty)$ is concave.*

Proof A direct computation of the second derivative shows that g_1 and g_2 are concave. If $k \ge 2$, then

$$v_k' = v_{k-1},\ v_k'' = v_{k-2},\ g_k' = \frac{v_k'}{v_k}$$

$$g_k'' = \frac{v_k v_k'' - \left(v_k'\right)^2}{v_k^2} = \frac{v_k v_{k-2} - v_{k-1}^2}{v_k^2}$$

Let

$$w_k(x) = v_k(x)\,v_k''(x) - \left[v_k'(x)\right]^2, \quad x \in (0, \infty).$$

Note that g_k is concave if and only if $w_k \le 0$. From the above lemma we obtain:

$$w_k(x) = v_k(x)\,v_{k-2}(x) - [v_{k-1}(x)]^2 =$$

$$= v_k(x)\left[v_k(x) + \frac{x^{k-1}}{(k-1)!} + \frac{x^k}{k!}\right] - \left(v_k(x) + \frac{x^k}{k!}\right)^2 =$$

$$= v_k(x)\left(\frac{x^{k-1}}{(k-1)!} - \frac{x^k}{k!}\right) - \frac{x^{2k}}{(k!)^2} =$$

$$= \frac{x^{k-1}}{(k!)^2}\left[k!\,(k-x)\,v_k\,(x) - x^{k+1}\right] \leq 0 \ \text{ if } x \in (0,\infty)\,.$$

This proves that g_k is concave. □

Lemma 8.12 *For $a, b \in [0,1]$ consider the functions $u\,(x) = e^x - a - bx$ and $f\,(x) = \ln\left[u\,(x)\right]$, where $x \in (0,\infty)$. If $b \leq 1 - \sqrt{1-a}$, then f is concave.*

Proof Note that f is concave if and only if $v = uu'' - \left(u^{'}\right)^2 \leq 0$ By $b \leq 1 - \sqrt{1-a}$ it follows that $-a + 2b \leq b^2$, hence

$$v\,(x) = \left(e^x - a - bx\right)e^x - \left(e^x - b\right)^2 == e^x\left(b^2 - bx\right) - b^2 = be^x\left(b - x - be^{-x}\right).$$

Since $e^{-x} \geq 1 - x$ it follows that

$$v\,(x) \leq be^x\left[b - x - b\,(1-x)\right] = b\,(b-1)\,xe^x \leq 0.$$

Consequently, f is concave. □

Theorem 8.13 *Let $n \geq 2$ be a natural number,*

$$\sigma_2\,(x_1, x_2, \ldots, x_n) = \sum_{1\leq i<j\leq n} x_i x_j\ , \quad (x_1, x_2, \ldots, x_n) \in (0,\infty)^n\,.$$

Then the function

$$f\,(x_1, x_2, \ldots, x_n) = \ln\left[\exp\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} e^{x_i}\right) + n - 1 - \sigma_2\,(x_1, x_2, \ldots, x_n)\right],$$

where $(x_1, x_2, \ldots, x_n) \in (0,\infty)^n$ is separately concave.

Proof Since f is symmetric it suffices to prove the concavity of it in the first variable. Fix $\left(x_2, x_3, \ldots, x_n\right) \in (0,\infty)^{n-1}$ and let

$$s = x_2 + x_3 + \ldots + x_n,$$

$$q_1 = \sum_{2\leq i<j\leq n} x_i x_j,$$

$$q_2 = \sum_{i=2}^{n}\left(e^{x_i} - 1\right),$$

$$a = \frac{q_1 + q_2}{e^s - 1},\ \ b = \frac{s}{e^s - 1},$$

$$\psi(t) = \ln\left(e^t - a - bt\right), \quad t \in (0, \infty),$$

and

$$\alpha = \sum_{i=2}^{n} \phi(x_i).$$

Note that we have the relations

$$q_1 + q_2 = q_1 + \sum_{i=2}^{n} \left(x_i + \frac{x_i^2}{2} + \phi(x_i) \right) = s + \frac{s^2}{2} + \alpha,$$

hence $a, b \in [0, 1]$. One can easily see that

$$f(x_1, x_2, \ldots, x_n) = \ln\left[e^{x_1}\left(e^s - 1\right) - q_2 - sx_1 - q_1\right] =$$

$$\ln\left(e^s - 1\right) + \ln\left[e^{x_1} - \frac{q_1 + q_2}{e^s - 1} - \frac{s}{e^s - 1} x_1\right] = \ln\left(e^s - 1\right) + \ln\left[e^{x_1} - a - bx_1\right]$$
$$= \psi(x_1).$$

We shall prove that $-a + 2b \leq b^2$, hence $b \leq 1 - \sqrt{1 - a}$. By Lemma 8.12. we obtain that ψ is concave, hence f will be separately concave. Note that

$$-a+2b-b^2 = -\frac{s + \frac{s^2}{2} + \alpha}{e^s - 1} + \frac{2s}{e^s - 1} - \left(\frac{s}{e^s - 1} \right)^2 = \frac{\left(s - \frac{s^2}{2} - \alpha\right)(e^s - 1) - s^2}{(e^s - 1)^2}$$

$$= \frac{\left(s - \frac{s^2}{2}\right)(e^s - 1) - s^2 - \alpha(e^s - 1)}{(e^s - 1)^2} = \frac{\frac{s}{2}\left[(2 - s)e^s - 2 - s\right] - \alpha(e^s - 1)}{(e^s - 1)^2}.$$

Let $v(t) = (2 - t)e^t - t - 2$, $t \in [0, \infty)$. Note that $v(0) = 0$, and v is decreasing. Hence $v(t) \leq 0$ if $t \in [0, \infty)$. Thus

$$-a + 2b - b^2 = \frac{\frac{s}{2} v(s) - \alpha(e^s - 1)}{(e^s - 1)^2} \leq 0$$

□

For every $a \in \mathbb{R}$ denote by $[a]$ the greatest integer smaller or equal than a.

Theorem 8.14 *For every* $p \in (1, \infty)$, $k \in \{0, 1, 2, \ldots, [p] + 1\}$, *consider the function*

$$f_{k,p}(x) = \left(\sum_{i=0}^{k} \binom{p}{i} x^i \right)^{1/p}, \quad x \in (0, \infty).$$

Then the following assertions hold:

1^0. *If* $k \in \{0, 1, 2, \ldots, [p]\}$, *then* $f_{k,p}$ *is concave.*
2^0. *If* $k = [p] + 1$, *then* $f_{k,p}$ *is convex.*

Proof Let

$$u_{k,p}(x) = \sum_{i=0}^{k} \binom{p}{i} x^i, \quad x \in [0, \infty).$$

Note that

$$f_{k,p} = \left(u_{k,p}\right)^{1/p}, \quad f_{k,p}' = \frac{1}{p} \left(u_{k,p}\right)^{(1/p)-1} u_{k,p}'$$

and

$$f_{k,p}'' = \frac{1}{p^2} \left(u_{k,p}\right)^{(1/p)-2} \left[p u_{k,p} u_{k,p}'' - (p-1) \left(u_{k,p}'\right)^2 \right].$$

One can easily see that $f_{k,p}$ is concave if and only if

$$v_{k,p} = (p-1) \left(u_{k,p}'\right)^2 - p u_{k,p} u_{k,p}'' \geq 0.$$

Note that

$$u_{k,p}' = p u_{k-1,p-1}, \quad u_{k,p}'' = p(p-1) u_{k-2,p-2},$$

hence

$$v_{k,p} = p^2 (p-1) \left[\left(u_{k-1,p-1}\right)^2 - u_{k,p} u_{k-2,p-2} \right].$$

Thus

$$\left(u_{k-1,p-1}(x)\right)^2 - u_{k,p}(x) u_{k-2,p-2}(x) =$$

$$= \left(\binom{p-1}{0} + \binom{p-1}{1} x + \ldots + \binom{p-1}{k-1} x^{k-1} \right)^2 -$$

$$-\left(\binom{p}{0}+\binom{p}{1}x+\ldots+\binom{p}{k-1}x^{k}\right)\left(\binom{p-2}{0}+\binom{p-2}{1}x+\ldots\right.$$
$$\left.+\binom{p-2}{k-2}x^{k-2}\right)=$$
$$=c_0+c_1x+\ldots+c_{2k-2}x^{2k-2}.$$

One notes that

$$c_{2k-r-1}=\frac{r}{p-1}\binom{p-1}{k}\binom{p-1}{k-r},\quad r\in\{1,2,\ldots,k\}$$

$$c_r=0,\quad r\in\{0,1,2\ldots.,k-2\}.$$

If $k\in\{0,1,2,\ldots,[p]\}$, then we have $v_{k,p}\geq 0$, hence $f_{k,p}$ is concave. If $k=[p]+1$, then $v_{k,p}\leq 0$, hence $f_{k,p}$ is convex. □

Conjecture A *For every natural number* $p\geq 2$ *and* $k\in\left\{0,1,2,\ldots,\left[\frac{p}{2}\right]-1\right\}$ *consider the function*

$$g_{k,p}(x)=\left[(1+x)^p-\sum_{i=0}^{k}\binom{p}{i}\left(x^i+x^{p-i}\right)\right]^{1/p},\quad x\in(0,\infty)$$

Then $g_{k,p}$ *is concave.*

Note that

$$g_{k,p}(x)=\left(\sum_{i=k+1}^{p-k-1}\binom{p}{i}x^i\right)^{1/p},\quad x\in(0,\infty)$$

The following lemma contains a classical result from matrix theory.

Lemma 8.15 *Let* A *be an* $n\times n$ *matrix with complex entries. Denote by* A^* *the adjugate of* A*. If* $x,y\in\mathbb{R}^n$ *and* $A=I_n+xy^T$*, then the following equalities hold:*

$$\det(A)=1+x^Ty$$

$$A^*=\left(1+x^Ty\right)I_n-xy^T$$

$$A^{-1}=I_n-\frac{1}{1+x^Ty}xy^T\quad \textit{if}\ 1+x^Ty\neq 0$$

Lemma 8.16 *Let $x, y, u, v \in \mathbb{R}^n$. Then the following equality holds:*

$$\det\left(I_n + xy^T + uv^T\right) = \left(1 + x^T y\right)\left(1 + u^T v\right) - \left(x^T v\right)\left(y^T u\right)$$

Proof Let $A = I_n + xy^T$. Then

$$\det\left(I_n + xy^T + uv^T\right) = \det\left(A + uv^T\right) = \det(A) + v^T A^* u =$$

$$= \det(A) + v^T\left[\left(1 + x^T y\right) I_n - xy^T\right] u =$$
$$= \left(1 + x^T y\right)\left(1 + u^T v\right) - \left(x^T v\right)\left(y^T u\right).$$

□

Theorem 8.17 *Let $n \geq 2$ be a natural number. Then the function*

$$f(x_1, x_2, \ldots, x_n) = \ln\left[\exp\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} e^{x_i}\right) + n - 1\right],$$
$$(x_1, x_2, \ldots, x_n) \in (0, \infty)^n$$

is concave.

Proof Let $\phi : (0, \infty) \to \mathbb{R}$ be a two times differentiable function and

$$u(x_1, x_2, \ldots, x_n) = \phi\left(\sum_{i=1}^{n} x_i\right) - \left(\sum_{i=1}^{n} \phi(x_i)\right), \ (x_1, x_2, \ldots, x_n) \in (0, \infty)^n$$

Suppose that $u > 0$ on $(0, \infty)^n$ and $\phi'' > 0$ on $(0, \infty)$. Note that

$$\frac{\partial u}{\partial x_i}(x) = \phi'\left(\sum_{r=1}^{n} x_r\right) - \phi'(x_i)$$

$$\frac{\partial^2 u}{\partial x_i^2}(x) = \phi''\left(\sum_{r=1}^{n} x_r\right) - \phi''(x_i)$$

$$\frac{\partial^2 u}{\partial x_i \partial x_j}(x) = \phi''\left(\sum_{r=1}^{n} x_r\right).$$

Let $A(x) = \left(a_{ij}(x)\right)_{1\leq i,j\leq n}$ be the Jacobian matrix of $g = \ln(u)$. Denote $s = \sum_{r=1}^{n} x_r$. Then

$$a_{ij}(x) = \left(\phi''(s) - \phi''(x_i)\right)\delta_{ij}u(x) - \left(\phi'(s) - \phi'(x_i)\right)\left(\phi'(s) - \phi'\left(x_j\right)\right),$$

where we have denoted by δ_{ij} the Kronecker symbol. Note that g is concave if and only if the matrix

$$\left(\frac{a_{ij}(x)}{u^2(x)}\right)$$

is negative semidefinite. Let

$$b_{ij}(x) = \delta_{ij} - \frac{\phi''(s)}{\phi''(x_i)} + \frac{\left(\phi'(s) - \phi'(x_i)\right)\left(\phi'(s) - \phi'\left(x_j\right)\right)}{u(x)\phi''(x_i)}.$$

Note that g is concave if and only if the matrix $B(x) = \left(b_{ij}(x)\right)_{1\leq i,j\leq n}$ is positive semidefinite. For every $m \in \{1, 2, \dots, n\}$ consider the matrix

$$B_m(x) = \left(b_{ij}(x)\right)_{1\leq i,j\leq m}$$

Note that g is concave if and only for every $m \in \{1, 2, \dots, n\}$ we have $\det(B_m(x)) \geq 0$

Let $m \in \{1, 2, \dots, n\}$. Consider the m-dimensional vectors

$$\alpha = \begin{pmatrix} \frac{\phi''(s)}{\phi''(x_1)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\phi''(s)}{\phi''(x_m)} \end{pmatrix}, \quad \beta = \begin{pmatrix} -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \frac{\phi'(s)-\phi'(x_1)}{u(x)\phi''(x_1)} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\phi'(s)-\phi'(x_m)}{u(x)\phi''(x_m)} \end{pmatrix},$$

$$\delta = \begin{pmatrix} \phi'(s) - \phi'(x_1) \\ \cdot \\ \cdot \\ \cdot \\ \phi'(s) - \phi'(x_m) \end{pmatrix}.$$

By the preceding lemma we obtain

$$\det(B_m(x)) = \det\left(I_m + \alpha\beta^T + \gamma\delta^T\right) = \left(1 + \alpha^T\beta\right)\left(1 + \gamma^T\delta\right) - \left(\alpha^T\delta\right)\left(\beta^T\gamma\right) =$$

$$1-\sum_{i=1}^{m}\frac{\phi''(s)}{\phi''(x_i)}+\sum_{i=1}^{m}\frac{\left(\phi'(s)-\phi'(x_i)\right)^2}{u(x)\,\phi''(x_i)}-$$

$$\left(\sum_{i=1}^{m}\frac{\phi''(s)}{\phi''(x_i)}\right)\left(\sum_{i=1}^{m}\frac{\left(\phi'(s)-\phi'(x_i)\right)^2}{u(x)\,\phi''(x_i)}\right)+$$

$$\left(\sum_{i=1}^{m}\frac{\phi''(s)\left(\phi'(s)-\phi'(x_i)\right)}{\phi''(x_i)}\right)\left(\sum_{i=1}^{m}\frac{\phi'(s)-\phi'(x_i)}{u(x)\,\phi''(x_i)}\right).$$

Note that

$$u(x)\det(B_m(x))=u(x)\left(1-\sum_{i=1}^{m}\frac{\phi''(s)}{\phi''(x_i)}\right)+$$

$$+\sum_{i=1}^{m}\frac{\left(\phi'(s)-\phi'(x_i)\right)^2}{\phi''(x_i)}-\left(\sum_{i=1}^{m}\frac{\phi''(s)}{\phi''(x_i)}\right)\left(\sum_{i=1}^{m}\frac{\left(\phi'(s)-\phi'(x_i)\right)^2}{\phi''(x_i)}\right)+$$

$$+\phi''(s)\left(\sum_{i=1}^{m}\frac{\phi'(s)-\phi'(x_i)}{\phi''(x_i)}\right)^2=$$

$$=\left(1-\sum_{i=1}^{m}\frac{\phi''(s)}{\phi''(x_i)}\right)\left(u(x)+\sum_{i=1}^{m}\frac{\left(\phi'(s)-\phi'(x_i)\right)^2}{\phi''(x_i)}\right)+$$

$$+\phi''(s)\left(\sum_{i=1}^{m}\frac{\phi'(s)-\phi'(x_i)}{\phi''(x_i)}\right)^2.$$

Let $p_m=\sum\limits_{i=1}^{m}e^{-x_i}$, $q_r=\sum\limits_{i=1}^{r}e^{x_i}$, $r\in\{1,2,\ldots,m\}$, $c_m=q_n-q_m$. If $\phi(t)=e^t-1$, where $t\in(0,\infty)$, then

$$u(x)\det(B_m(x))=\left(1-e^s p_m\right)\left[e^s-q_n+n-1+\sum_{i=1}^{m}\frac{(e^s-e^{x_i})^2}{e^{x_i}}\right]+$$

$$+e^s\left(\sum_{i=1}^{m}\frac{e^s-e^{x_i}}{e^{x_i}}\right)^2=$$

$$= \left(1 - e^s p_m\right)\left(e^s - q_n + n - 1 + e^{2s} p_m - 2me^s + q_m\right) + e^s \left(p_m e^s - m\right)^2 =$$

$$= e^s \left[(m-1)^2 - p_m (n - 1 - c_m) + e^{-s} (n - 1 - c_m)\right].$$

Note that

$$c_m - n + 1 \geq n - m - n + 1 = 1 - m,$$

$$0 \leq p_m - e^{-s} \leq m - 1,$$

$$\left(p_m - e^{-s}\right)(c_m - n + 1) \geq \left(p_m - e^{-s}\right)(1 - m),$$

hence

$$e^{-s} u(x) \det(B_m(x)) = (m-1)^2 + \left(p_m - e^{-s}\right)(c_m - n + 1) \geq$$

$$\geq (m-1)^2 + \left(p_m - e^{-s}\right)(1 - m) = (m-1)\left(m - 1 - p_m + e^{-s}\right) \geq 0.$$

We proved that $\det(B_m(x)) \geq 0$ for every $m \in \{1, 2, \ldots, n\}$. Thus the function $g = f$ is concave. □

Corollary 8.18 *Let $a_i, b_i \in [1, \infty)$, $i = 1, 2, \ldots, n$. Then the following inequality holds:*

$$\left[a_1^2 a_2^2 \ldots a_n^2 - \left(\sum_{i=1}^{n} a_i^2\right) + n - 1\right] \cdot \left[b_1^2 b_2^2 \ldots b_n^2 - \left(\sum_{i=1}^{n} b_i^2\right) + n - 1\right] \leq$$

$$\leq \left(a_1 a_2 \ldots a_n b_1 b_2 \ldots b_n - \left(\sum_{i=1}^{n} a_i b_i\right) + n - 1\right)^2.$$

Theorem 8.19 *Let $n \geq 2$ be a natural number. For $i, k \geq 0$ natural numbers consider the functions*

$$q_{i,n}(x_1, x_2, \ldots, x_n) = \frac{\left(\sum\limits_{r=1}^{n} x_r\right)^i - \left(\sum\limits_{r=1}^{n} x_r^i\right)}{i!},$$

$$u_{k,n}(x_1, x_2, \ldots, x_n) = \exp\left(\sum_{r=1}^{n} x_r\right) - \left(\sum_{r=1}^{n} e^{x_r}\right) - \sum_{i=0}^{k} q_{i,n}(x_1, x_2, \ldots, x_n),$$

where $(x_1, x_2, \ldots, x_n) \in (0, \infty)^n$. Then for every natural number $k \geq 0$ the function $f_{k,n} = \ln\left(u_{k,n}\right)$ is Schur concave on $(0, \infty)^n$.

Proof Note that we have $q_{0,n}(x_1, x_2, \ldots, x_n) = 1 - n$, $q_{1,n}(x_1, x_2, \ldots, x_n) = 0$, $q_{2,n}(x_1, x_2, \ldots, x_n) = \sum_{1 \le i < j \le n} x_i x_j$. If $k = 0$, then

$$f_{0,n}(x_1, x_2, \ldots, x_n) = \ln\left(\exp\left(\sum_{r=1}^{n} x_r\right) - \left(\sum_{r=1}^{n} e^{x_r}\right) + n - 1\right),$$

$$(x_1, x_2, \ldots, x_n) . \in (0, \infty)^n$$

Note that

$$(x_1 - x_2)\left(\frac{\partial f_{0,n}}{\partial x_1}(\mathbf{x}) - \frac{\partial f_{0,n}}{\partial x_2}(\mathbf{x})\right) =$$

$$= \frac{(x_1 - x_2)}{u_{0,n}(\mathbf{x})}\left(\frac{\partial u_{0,n}}{\partial x_1}(\mathbf{x}) - \frac{\partial u_{0,n}}{\partial x_2}(\mathbf{x})\right) =$$

$$= -\frac{(x_1 - x_2)}{u_{0,n}(\mathbf{x})}\left(e^{x_1} - e^{x_2}\right) \le 0$$

hence $f_{0,n}$ is Schur concave. If $k \ge 1$ is a natural number, then

$$\frac{\partial u_{k,n}}{\partial x_j}(\mathbf{x}) = \exp\left(\sum_{r=1}^{n} x_r\right) - e^{x_j} - \sum_{i=0}^{k-1} \frac{\left(\sum_{r=1}^{n} x_r\right)^i - x_j^i}{i!}$$

hence if we denote

$$w(t) = e^t - \sum_{i=0}^{k-1} \frac{t^i}{i!}, \quad t \in (0, \infty)$$

we have

$$(x_1 - x_2)\left(\frac{\partial f_{k,n}}{\partial x_1}(\mathbf{x}) - \frac{\partial f_{k,n}}{\partial x_2}(\mathbf{x})\right) = -\frac{(x_1 - x_2)}{u_{k,n}(\mathbf{x})}\left[w(x_1) - w(x_2)\right] \le 0.$$

Consequently, $f_{k,n}$ is Schur concave on $(0, \infty)^n$. □

We conclude this section with the following conjecture.

Conjecture B *Let $n \ge 2$ be a natural number. Then for every natural number $k \ge 0$ the function*

$$f_{k,n}(x_1, x_2, \ldots, x_n) = \ln\left(\exp\left(\sum_{r=1}^{n} x_r\right) - \left(\sum_{r=1}^{n} e^{x_r}\right) - \sum_{i=0}^{k} q_{i,n}(x_1, x_2, \ldots, x_n)\right),$$

where $(x_1, x_2, \ldots, x_n) \in (0, \infty)^n$ *is concave. Here the functions* $q_{i,n}$ *are defined in the statement of Theorem 8.19.*

Recall that until now we proved the following results:

(i) $f_{2,n}$ is separately concave (cf. Theorem 8.13);
(ii) $f_{0,n}$ is concave (cf. Theorem 8.17);
(iii) $f_{k,n}$ is Schur concave (cf. Theorem 8.19).

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Harmonic Exponential Convex Functions and Inequalities

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Abstract In this chapter, we intend to introduce and study a new class of harmonic exponential h-convex functions. We show that this class includes several new and previously known classes of harmonic convex functions. We derive several Hermite–Hadamard type integral inequalities. Numerous special cases are also discussed.

1 Introduction

The significance and importance of the convexity theory can be imagined through its applications in different fields of pure and applied sciences. Ideas explaining the convexity theory lead to the developments of new, novel, and powerful techniques to solve linear and nonlinear problems. It has been shown that convexity theory provides us the most natural, direct, simple, and efficient framework for unified treatment of unrelated problems. In recent years, convex sets and convex functions have been generalized and extended in various directions using innovative techniques and ideas, for example, see [19, 21]. An important aspect of convexity theory is its close relationship with theory of inequalities. Many inequalities are direct consequences of the applications of convex functions. There are two types of the inequalities, namely variational inequalities and integral inequalities. Variational inequalities are closely related to the optimization theory. In fact, it is worth mentioning that the minimum $u \in K$ of a differentiable convex functions can be characterized by an inequality of the type

$$\langle f'(u), v - u \rangle \geq 0, \quad \forall v \in K,$$

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which is known as the variational inequality, introduced and considered by Stampacchia [41]. Variational inequalities can be viewed as natural extension of the variational principle, the origin of which can be traced back to Euler, Lagrange, and Newton. It is remarkable that the applications of variational inequalities and techniques have played a crucial role in the developments of various fields of pure and applied sciences such as Nash equilibria, dynamical systems, transportation, structural analysis, and sensibility analysis. For the applications, formulation, numerical methods, sensitivity analysis, neural network, and other aspects of variational inequalities and related area, see [6, 14, 19, 22–27, 29, 30, 41] and the references therein.

Integral inequalities play important part in estimating the upper and lower bounds of the integral of the functions. It has been shown that by Hermite and Hadamard that a function is convex function if, and only if, it satisfies the inequality of the type

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \forall a, b \in I = [a, b].$$

which is known as Hermite–Hadamard type inequality. For the applications of Hermite–Hadamard type inequalities and their generalizations, see [1–4, 8, 9, 11, 12, 18, 20, 21, 28, 31, 33, 35, 37–39] and the references therein.

In recent years, various extensions and generalizations of the convex sets and convex functions have been introduced and studied. Motivated and inspired by the research activities in the convexity theory, Varosanec [42] introduced the notion of h-convex functions. It has been shown that under some suitable assumptions this class enjoys some nice properties which the classical convex functions have. It is worth mentioning that the class of h-convex functions generalizes not only the class of classical convex functions but also several other classes of convex functions such as Breckner type of s-convex functions [7], Godunova-Levin functions [15], P-functions [13], and Godunova–Levin–Dragomir type of functions [10]. In recent years, a considerable number of research articles have been devoted to the study of h-convex functions. For more details about this fascinating class of h-convex functions, see [42]. İscan [16] introduced the notion of harmonically convex functions. Motivated by this, Noor et al. [34] extended the class of harmonically convex functions and hconvex functions. They introduced the notion of harmonically h-convex functions, which generalizes different classes of harmonically convex functions.

Inspired and motivated by the ongoing research in the convex analysis, Awan et al. [5] introduced the concept of exponential convex functions and derived some integral inequalities. In this chapter, we consider and investigate the harmonic exponential convex functions involving an arbitrary non-negative function h. It is shown that harmonic exponential h-convex functions are more general and unifying ones. For different and appropriate choice of the arbitrary function, one can obtain a wide class of new and known classes of convex and harmonic convex functions. We derive a wide class of integral inequalities via harmonic exponential h-convex functions. Several new and special cases will also be discussed in detail. It is expected the ideas and technique of this paper may be a starting point for exploring the applications of the harmonic exponential convex functions in various branches of pure and applied sciences.

2 Preliminaries

In this section, we discuss previously known concepts and results.

Definition 1 ([40]) A set $\mathscr{H} \subset \mathbb{R}_+$ is said to be harmonic convex, if

$$\frac{xy}{tx+(1-t)y} \in \mathscr{H}, \quad \forall x, y \in \mathscr{H}, t \in [0, 1]. \tag{1}$$

We now define the class of harmonic h-convex function.

Definition 2 ([34]) Let $h : (0, 1) \subseteq J \to \mathbb{R}$ be a real function. A function $f : \mathscr{H} \to \mathbb{R}$ is said to be harmonic h-convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq h(1-t)f(x) + h(t)f(y), \quad \forall x, y \in \mathscr{H}, t \in (0, 1). \tag{2}$$

Note that, if $t = \frac{1}{2}$, then we have Jensen's type of harmonic h-convex function

$$f\left(\frac{2xy}{x+y}\right) \leq h\left(\frac{1}{2}\right)[f(x) + f(y)], \quad \forall x, y \in \mathscr{H}.$$

Now we discuss some special cases of Definition 2.

I. If $h(t) = t$ in (2), then Definition 2 reduces to the definition of harmonic convex functions.

Definition 3 ([16]) A function $f : \mathscr{H} \to \mathbb{R}$ is said to be harmonic convex function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)f(x) + tf(y), \quad \forall x, y \in \mathscr{H}, t \in [0, 1].$$

Noor and Noor [29] have shown that the optimality conditions of differentiable harmonic convex functions can be characterized by a class of variational inequalities, which is called the harmonic variational inequalities. Harmonic variational inequality is an interesting problem for future research. This field is new one and needs further efforts.

II. If $h(t) = t^s$ in (2), then Definition 2 reduces to the definition of Breckner type of harmonic s-convex function.

Definition 4 ([34]) A function $f : \mathscr{H} \to \mathbb{R}$ is said to be Breckner type of harmonic s-convex function, where $s \in (0, 1]$, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq (1-t)^s f(x) + t^s f(y), \quad \forall x, y \in \mathscr{H}, t \in [0, 1].$$

III. If $h(t) = 1$ in (2), then Definition 2 reduces to the definition of harmonic P-function.

Definition 5 ([34]) A function $f : \mathscr{H} \to \mathbb{R}$ is said to be harmonic P-function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq f(x) + f(y), \quad \forall x, y \in \mathscr{H}, t \in [0, 1].$$

IV. If $h(t) = \frac{1}{t}$ in (2), then we have

Definition 6 ([34]) A function $f : \mathscr{H} \subset \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonic Godunova–Levin function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{1-t} f(x) + \frac{1}{t} f(y), \quad \forall x, y \in \mathscr{I}_h, t \in (0, 1).$$

V. If $h(t) = \frac{1}{t^s}$ in (2), then Definition 2 reduces to the definition of Godunova–Levin type of harmonic s-convex function.

Definition 7 ([32]) A function $f : \mathscr{H} \to \mathbb{R}$ is said to be Godunova–Levin type of harmonic s-convex function, where $s \in [0, 1]$, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \leq \frac{1}{(1-t)^s} f(x) + \frac{1}{t^s} f(y), \quad \forall x, y \in \mathscr{H}, t \in (0, 1).$$

We now recall some special functions, which will be helpful in our coming results.

Definition 8 ([17]) The gamma function $\Gamma : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$\Gamma(x) = \int_0^\infty e^{-x} t^{x-1} \mathrm{d}t,$$

Definition 9 ([17]) The beta function B is a function of two variables defined as

$$\mathrm{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, \mathrm{d}t = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The hypergeometric function is defined as:

$${}_2F_1[a, b; c; z] = \frac{1}{\mathrm{B}(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} \mathrm{d}t,$$

where $|z| < 1, c > b > 0$.

3 Harmonically Exponential *h*-Convex Functions

We now introduce a new class of convex functions, which is called "exponentially h-convex functions."

Definition 10 Let $h : (0, 1) \to \mathbb{R}$ be a real function. A function $f : \mathscr{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically exponential h-convex function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le h(1-t)e^{\alpha x} f(x) + h(t)e^{\alpha y} f(y),$$

for all $x, y \in \mathscr{H}$, $t \in (0, 1)$ and $\alpha \in \mathbb{R}$. If the above inequality holds in the reversed sense, then f is said to be harmonically exponential h-concave function.

It is worth to mention here that if $\alpha = 0$, then the class of harmonically exponential h-convex functions reduces to the class of classical h-convex function.

We now discuss some special cases of Definition 10

I. If we suppose $h(t) = t$ in Definition 10, then, we have a new definition of harmonically exponential convex functions.

Definition 11 A function $f : \mathscr{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically exponential convex function, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le (1-t)e^{\alpha x} f(x) + te^{\alpha y} f(y),$$

for all $x, y \in \mathscr{H}$, $t \in [0, 1]$ and $\alpha \in \mathbb{R}$.

II. If we suppose $h(t) = t^s$ in Definition 10, then, we have a new definition of harmonically exponential s-convex functions of Breckner type.

Definition 12 A function $f : \mathscr{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically exponential s-convex function of Breckner type, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le (1-t)^s e^{\alpha x} f(x) + t^s e^{\alpha y} f(y),$$

for all $x, y \in \mathscr{H}$, $t \in [0, 1]$, $s \in (0, 1)$ and $\alpha \in \mathbb{R}$.

III. If we suppose $h(t) = t^{-s}$ in Definition 10, then, we have a new definition of harmonically exponential s-convex functions of Godunova–Levin-Dragomir type.

Definition 13 A function $f : \mathscr{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically exponential s-convex function of Godunova–Levin-Dragomir type, if

$$f\left(\frac{xy}{tx + (1-t)y}\right) \le (1-t)^{-s} e^{\alpha x} f(x) + t^{-s} e^{\alpha y} f(y),$$

for all $x, y \in \mathscr{H}$, $t \in (0, 1)$, $s \in [0, 1]$ and $\alpha \in \mathbb{R}$.

IV. If we suppose $h(t) = t^{-1}$ in Definition 10, then, we have a new definition of harmonically exponential Godunova–Levin functions.

Definition 14 A function $f : \mathscr{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically exponential Godunova–Levin function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq (1-t)^{-1}e^{\alpha x}f(x) + t^{-1}e^{\alpha y}f(y),$$

for all $x, y \in \mathscr{H}$, $t \in (0,1)$ and $\alpha \in \mathbb{R}$.

V. If we suppose $h(t) = 1$ in Definition 10, then, we have a new definition of harmonically exponential P-functions.

Definition 15 A function $f : \mathscr{H} \subseteq \mathbb{R}_+ \to \mathbb{R}$ is said to be harmonically exponential P-function, if

$$f\left(\frac{xy}{tx+(1-t)y}\right) \leq e^{\alpha x}f(x) + e^{\alpha y}f(y),$$

for all $x, y \in \mathscr{H}$, $t \in (0,1)$ and $\alpha \in \mathbb{R}$.

4 Integral Inequalities

In this section, we derive some new Hermite–Hadamard like inequalities via harmonically exponential h-convex functions.

Theorem 1 (Hermite–Hadamard like inequality) *Let $f : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be an integrable harmonically exponential h-convex function, then, for $h\left(\frac{1}{2}\right) \neq 0$, we have*

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c}\int_c^d \frac{e^{\alpha x}f(x)}{x^2}\mathrm{d}x \leq [e^{\alpha c}f(c) + e^{\alpha d}f(d)]\int_0^1 h(t)\mathrm{d}t.$$

Proof Let f be a harmonically exponentially h-convex function. Then

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{2xy}{x+y}\right) \leq e^{\alpha x}f(x) + e^{\alpha y}f(y).$$

Changing the variables, we get

$$\frac{1}{h\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) \leq e^{\alpha\left(\frac{cd}{(1-t)c+td}\right)} f\left(\frac{cd}{(1-t)c+td}\right) + e^{\alpha\left(\frac{cd}{tc+(1-t)d}\right)} f\left(\frac{cd}{tc+(1-t)d}\right).$$

Integrating with respect to t on [0, 1], we have

$$\frac{1}{2h\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c}\int_{c}^{d} \frac{e^{\alpha x} f(x)}{x^2}\mathrm{d}x. \tag{3}$$

Also, we have

$$f\left(\frac{cd}{tc+(1-t)d}\right) \leq h(1-t)e^{\alpha c} f(c) + h(t)e^{\alpha d} f(d).$$

Integrating with respect to t on [0, 1], we have

$$\frac{cd}{d-c}\int_{c}^{d} \frac{e^{\alpha x} f(x)}{x^2}\mathrm{d}x \leq [e^{\alpha c} f(c) + e^{\alpha d} f(d)] \int_{0}^{1} h(t)\mathrm{d}t. \tag{4}$$

Combining inequalities (3) and (4) completes the proof. □

Remark 1 Note that, if $\alpha = 0$ in Theorem 1, then we have Hermite–Hadamard like inequality for harmonically h-convex functions obtained by Noor et al. [34].

We now discuss some more new special cases of Theorem 1.

I. If $h(t) = t$ in Theorem 1, then we have Hermite–Hadamard like inequality for harmonically exponential convex function. The result reads as:

Corollary 1 *Let $f : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be an integrable harmonically exponential convex function, then*

$$f\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c}\int_{c}^{d} \frac{e^{\alpha x} f(x)}{x^2}\mathrm{d}x \leq \frac{e^{\alpha c} f(c) + e^{\alpha d} f(d)}{2}.$$

If we assume $\alpha = 0$ in Corollary 1, then we have classical Hermite–Hadamard inequality obtained via harmonic convex functions [16].

II. If $h(t) = t^s$ in Theorem 1, then we have Hermite–Hadamard like inequality for harmonically exponential s-convex function of Breckner type. The result reads as:

Corollary 2 *Let $f : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be an integrable harmonically exponential s-convex function of Breckner type, then*

$$\frac{1}{2^{1-s}} f\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c}\int_{c}^{d} \frac{e^{\alpha x} f(x)}{x^2}\mathrm{d}x \leq \frac{e^{\alpha c} f(c) + e^{\alpha d} f(d)}{1+s}.$$

If we assume $\alpha = 0$ in Corollary 2, then we have classical Hermite–Hadamard inequality obtained via harmonic s-convex functions of Breckner type [34].

III. If $h(t) = t^{-s}$ in Theorem 1, then we have Hermite–Hadamard like inequality for harmonically exponential s-convex function of Godunova–Levin-Dragomir type. The result reads as:

Corollary 3 *Let $f : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be an integrable harmonically exponential s-convex function of Godunova–Levin-Dragomir type, then*

$$\frac{1}{2^{1+s}} f\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c} \int_c^d \frac{e^{\alpha x} f(x)}{x^2} \mathrm{d}x \leq \frac{e^{\alpha c} f(c) + e^{\alpha d} f(d)}{1-s}.$$

If we assume $\alpha = 0$ in Corollary 3, then we have classical Hermite–Hadamard inequality obtained via harmonic s-convex functions of Godunova–Levin type [32].

IV. If $h(t) = 1$ in Theorem 1, then we have Hermite–Hadamard like inequality for harmonically exponential P-functions. The result reads as:

Corollary 4 *Let $f : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be an integrable harmonically exponential P-function, then*

$$\frac{1}{2} f\left(\frac{2cd}{c+d}\right) \leq \frac{cd}{d-c} \int_c^d \frac{e^{\alpha x} f(x)}{x^2} \mathrm{d}x \leq e^{\alpha c} f(c) + e^{\alpha d} f(d).$$

If we assume $\alpha = 0$ in Corollary 4, then we have classical Hermite–Hadamard inequality obtained via harmonic P-functions [34].

We now derive Hermite–Hadamard like inequality via product of two harmonically exponential h-convex functions.

Theorem 2 *Let $f, g : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be two integrable harmonically exponential hconvex functions, then for $h^2\left(\frac{1}{2}\right) \neq 0$, we have*

$$\frac{1}{2h^2\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)$$

$$-e^{\alpha(c+d)}\left[M(c,d;e)\int_0^1 h(t)h(1-t)\mathrm{d}t + \frac{1}{2}N(c,d;e)\int_0^1 [h^2(t)+h^2(1-t)]\mathrm{d}t\right]$$

$$\leq \frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x \leq M(c,d;e)\int_0^1 h^2(t)\mathrm{d}t + N(c,d;e)\int_0^1 h(t)h(1-t)\mathrm{d}t,$$

where

$$M(c,d;e) := e^{\alpha c} f(c)g(c) + e^{\alpha d} f(d)g(d), \tag{5}$$

and

$$N(c,d;e) := e^{\alpha(c+d)} \left[f(d)g(c) + f(c)g(d)\right], \tag{6}$$

respectively.

Proof Since it is given that f and g are harmonically exponential h-convex functions, then

$$\begin{aligned}&\frac{1}{h^2\left(\frac{1}{2}\right)} f\left(\frac{2xy}{x+y}\right) g\left(\frac{2xy}{x+y}\right)\\ &\le \left\{\left(e^{\alpha x} f(x) + e^{\alpha y} f(y)\right)\right\}\left\{\left(e^{\alpha x} g(x) + e^{\alpha y} g(y)\right)\right\}.\end{aligned}$$

This implies

$$\begin{aligned}&\frac{1}{h^2\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)\\ &\le \left\{\left(e^{\alpha\left(\frac{cd}{tc+(1-t)d}\right)} f\left(\frac{cd}{tc+(1-t)d}\right) + e^{\alpha\left(\frac{cd}{(1-t)c+td}\right)} f\left(\frac{cd}{(1-t)c+td}\right)\right)\right\}\\ &\quad\times\left\{\left(e^{\alpha\left(\frac{cd}{tc+(1-t)d}\right)} g\left(\frac{cd}{tc+(1-t)d}\right) + e^{\alpha\left(\frac{cd}{(1-t)c+td}\right)} g\left(\frac{cd}{(1-t)c+td}\right)\right)\right\}.\end{aligned}$$

Integrating with respect to t on [0, 1], we have

$$\begin{aligned}&\frac{1}{h^2\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)\\ &\le \frac{2cd}{d-c}\int_c^d e^{2\alpha x}\frac{f(x)g(x)}{x^2}\mathrm{d}x\\ &+e^{\alpha(c+d)}\int_0^1 \left[\left\{h(1-t)e^{\alpha c} f(c) + h(t)e^{\alpha d} f(d)\right\}\right.\\ &\quad\times\left\{h(t)e^{\alpha c} g(c) + h(1-t)e^{\alpha d} g(d)\right\} + \left\{h(t)e^{\alpha c} f(c) + h(1-t)e^{\alpha d} f(d)\right\}\\ &\quad\left.\times\left\{h(1-t)e^{\alpha c} g(c) + h(t)e^{\alpha d} g(d)\right\}\right]\mathrm{d}t.\end{aligned}$$

This implies

$$\frac{1}{2h^2\left(\frac{1}{2}\right)} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)$$
$$-e^{\alpha(c+d)}\left[M(c,d;e)\int_0^1 h(t)h(1-t)\mathrm{d}t + \frac{1}{2}N(c,d;e)\int_0^1 [h^2(t)+h^2(1-t)]\mathrm{d}t\right]$$
$$\leq \frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x. \tag{7}$$

Also it is given that f and g are harmonically exponential h-convex functions, we have

$$f\left(\frac{cd}{(1-t)c+td}\right) g\left(\frac{cd}{(1-t)c+td}\right)$$
$$\leq \left[h(t)e^{\alpha c} f(c) + h(1-t)e^{\alpha d} f(d)\right]\left[h(t)e^{\alpha c} f(c) + h(1-t)e^{\alpha d} f(d)\right].$$

Integrating with respect to t on [0, 1], we have

$$\frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x \leq M(c,d;e)\int_0^1 h^2(t)\mathrm{d}t + N(c,d;e)\int_0^1 h(t)h(1-t)\mathrm{d}t. \tag{8}$$

Combining inequalities (7) and (8) completes the proofcqed □

It is worth to mention here that if we take $\alpha = 0$ in Theorem 2, then we have a result for classical harmonically h-convex functions.

We now discuss some new special cases of Theorem 2.

I. If we take $h(t) = t$, in Theorem 2, then we have result for harmonically exponential convex functions.

Corollary 5 *Let $f, g : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be two integrable harmonically exponential hconvex functions, then we have*

$$2f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)$$
$$-e^{\alpha(c+d)}\left[\frac{1}{6}M(c,d;e) + \frac{1}{3}N(c,d;e)\right]$$
$$\leq \frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x \leq \frac{1}{3}M(c,d;e) + \frac{1}{6}N(c,d;e),$$

where $M(c,d;e)$ and $N(c,d;e)$ are given by (5) and (6) respectively.

II. If we take $h(t) = t^s$, in Theorem 2, then we have result for harmonically exponential s-convex functions of Breckner type.

Corollary 6 *Let $f, g : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be two integrable harmonically exponential s-convex functions of Breckner type, then we have*

$$\frac{1}{2^{1-2s}} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)$$
$$-e^{\alpha(c+d)}\left[B(s+1, s+1)M(c, d; e) + \frac{B(2s+1, 2s+1)}{2}N(c, d; e)\right]$$
$$\leq \frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x \leq \frac{1}{1+2s}M(c, d; e) + B(s+1, s+1)N(c, d; e),$$

where $M(c, d; e)$ and $N(c, d; e)$ are given by (5) and (6) respectively.

III. If we take $h(t) = t^{-s}$, in Theorem 2, then we have result for harmonically exponential s-convex functions of Godunova–Levin type.

Corollary 7 *Let $f, g : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be two integrable harmonically exponential s-convex functions of Godunova–Levin type, then we have*

$$\frac{1}{2^{1+2s}} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right)$$
$$-e^{\alpha(c+d)}\left[B(1-s, 1-s)M(c, d; e) + \frac{B(1-2s, 1-2s)}{2}N(c, d; e)\right]$$
$$\leq \frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x \leq \frac{1}{1-2s}M(c, d; e) + B(1-s, 1-s)N(c, d; e),$$

where $M(c, d; e)$ and $N(c, d; e)$ are given by (5) and (6) respectively.

IV. If we take $h(t) = 1$, in Theorem 2, then we have result for harmonically exponential P-convex functions.

Corollary 8 *Let $f, g : \mathscr{I} \subset \mathbb{R}_+ \to \mathbb{R}$ be two integrable harmonically exponential P-convex functions, then we have*

$$\frac{1}{2} f\left(\frac{2cd}{c+d}\right) g\left(\frac{2cd}{c+d}\right) - e^{\alpha(c+d)}\left[M(c, d; e) + N(c, d; e)\right]$$
$$\leq \frac{cd}{d-c}\int_c^d \frac{e^{2\alpha x} f(x)g(x)}{x^2}\mathrm{d}x \leq M(c, d; e) + N(c, d; e),$$

where $M(c, d; e)$ and $N(c, d; e)$ are given by (5) and (6) respectively.

5 Differentiable Harmonic *h*-Convex Functions

In this section, we derive some new Hermite–Hadamard like inequalities via differentiable harmonically exponential h-convex functions. For this, we need three following auxiliary results. For the sake of completeness and to convey the main idea, we include the proofs of these auxiliary results.

Lemma 1 ([16]) *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$, $c < d$ and $f' \in L[c, d]$, then*

$$\frac{f(c) + f(d)}{2} - \frac{cd}{d-c} \int_c^d \frac{f(x)}{x^2} \, \mathrm{d}x = \frac{cd(d-c)}{2} \int_0^1 \frac{1-2t}{A_t^2} f'\left(\frac{cd}{A_t}\right) \mathrm{d}t, \tag{9}$$

where $A_t = (1-t)c + td$.

Proof It suffices to show that

$$\int_0^1 \frac{1-2t}{((1-t)c+td)^2} f'\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t$$
$$= \frac{f(c)+f(d)}{cd(d-c)} - \frac{2}{cd(d-c)} \int_0^1 f\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t$$
$$= \frac{f(c)+f(d)}{cd(d-c)} - \frac{2}{cd(d-c)} \frac{cd}{d-c} \int_c^d \frac{f(x)}{x^2} \mathrm{d}x.$$

Multiplying both sides of above equation by $\frac{cd(d-c)}{2}$ completes the proofcqed □

Lemma 2 ([36]) *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$, $c < d$ and $f' \in L[c, d]$, then*

$$\frac{cd}{d-c} \int_c^d \frac{f(x)}{x^2} \, \mathrm{d}x - f\left(\frac{2cd}{c+d}\right) = cd(d-c) \left[\int_0^{1/2} \frac{t}{A_t^2} f'\left(\frac{cd}{A_t}\right) \mathrm{d}t + \int_{1/2}^1 \frac{t-1}{A_t^2} f'\left(\frac{cd}{A_t}\right) \mathrm{d}t \right], \tag{10}$$

where $A_t = (1-t)c + td$.

Proof Let

$$L_1 + L_2 = \int_0^{1/2} \frac{t}{((1-t)c+td)^2} f'\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t + \int_{1/2}^{1} \frac{t-1}{((1-t)c+td)^2} f'\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t. \tag{11}$$

Now

$$\begin{aligned} L_1 &= \int_0^{1/2} \frac{t}{((1-t)c+td)^2} f'\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t \\ &= -\frac{1}{2} f\left(\frac{2cd}{c+d}\right) + \frac{1}{cd(d-c)} \int_0^{1/2} f\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t. \end{aligned} \tag{12}$$

Similarly

$$\begin{aligned} L_2 &= \int_{1/2}^{1} \frac{t-1}{((1-t)c+td)^2} f'\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t \\ &= -\frac{1}{2} f\left(\frac{2cd}{c+d}\right) + \frac{1}{cd(d-c)} \int_{1/2}^{1} f\left(\frac{cd}{(1-t)c+td}\right) \mathrm{d}t. \end{aligned} \tag{13}$$

Summation of (11), (12), and (13) completes the proof. □

Lemma 3 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$. If $f' \in L[c, d]$, then*

$$\begin{aligned} &\frac{1}{2}\left[\frac{f(c)+f(d)}{2} + f\left(\frac{2cd}{c+d}\right)\right] - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x \\ &= cd(d-c) \\ &\quad \times \left[\int_0^1 \left(\frac{1}{2}-t\right)\left(\frac{1}{(1-t)c+(1+t)d}\right)^2 f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\mathrm{d}t \right. \\ &\quad \left. + \int_0^1 \left(t-\frac{1}{2}\right)\left(\frac{1}{(1+t)c+(1-t)d}\right)^2 f'\left(\frac{2cd}{(1+t)c+(1-t)d}\right)\mathrm{d}t\right]. \end{aligned}$$

Proof Let

$$V_1 = \int_0^1 \left(\frac{1}{2} - t\right)\left(\frac{1}{(1-t)c+(1+t)d}\right)^2 f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\mathrm{d}t$$

$$= \frac{1}{4cd(d-c)} f(c) + \frac{1}{4cd(d-c)} f\left(\frac{2cd}{c+d}\right) - \frac{cd}{cd(d-c)^2}\int_c^{\frac{2cd}{c+d}} \frac{f(x)}{x^2}\mathrm{d}x. \quad (14)$$

Similarly

$$V_2 = \int_0^1 \left(t - \frac{1}{2}\right)\left(\frac{1}{(1+t)c+(1-t)d}\right)^2 f'\left(\frac{2cd}{(1+t)c+(1-t)d}\right)\mathrm{d}t$$

$$= \frac{1}{4cd(d-c)} f(d) + \frac{1}{4cd(d-c)} f\left(\frac{2cd}{c+d}\right) - \frac{cd}{cd(d-c)^2}\int_{\frac{2cd}{c+d}}^{d} \frac{f(x)}{x^2}\mathrm{d}x. \quad (15)$$

Combining (14) and (15) and then multiplying by $cd(d-c)$ completes the proof. □

Theorem 3 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ such that $f' \in L[c,d]$, where $c, d \in \mathscr{I}^0$ with $c < d$. If $|f'|^q$, $q > 1$ is harmonically exponential h-convex, then*

$$\left|\frac{f(c)+f(d)}{2} - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right|$$
$$\leq \frac{cd(d-c)}{2}\mu^{1-\frac{1}{q}}(c,d)\left(|e^{\alpha c} f'(c)|^q \int_0^1 |1-2t|h(t)A_t^{-2}\mathrm{d}t\right.$$
$$\left. + |e^{\alpha d} f'(d)|^q \int_0^1 |1-2t|h(1-t)A_t^{-2}\mathrm{d}t\right)^{1/q}, \quad (16)$$

where

$$\mu(c,d) = c^{-2}\left[{}_2F_1\left(2,2;3;1-\frac{d}{c}\right) - {}_2F_1\left(2,1;2;1-\frac{d}{c}\right)\right.$$
$$\left. + \frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)\right],$$

and $A_t = (1-t)c + td$.

Proof Using Lemma 1, the power mean inequality and the harmonic h-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\frac{f(c)+f(d)}{2}-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right| \\
&\leq \frac{cd(d-c)}{2}\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|\mathrm{d}t \\
&\leq \frac{cd(d-c)}{2}\left(\int_0^1 \frac{|1-2t|}{A_t^2}\mathrm{d}t\right)^{1-1/q}\left(\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|^q \mathrm{d}t\right)^{1/q} \\
&\leq \frac{cd(d-c)}{2}\mu^{1-1/q}(c,d) \\
&\quad\times\left(\int_0^1 \frac{|1-2t|}{A_t^2}\left(h(t)|e^{\alpha c}f'(c)|^q+h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q} \\
&= \frac{cd(d-c)}{2}\mu^{1-1/q}(c,d)\left(|e^{\alpha c}f'(c)|^q\int_0^1 |1-2t|h(t)A_t^{-2}\mathrm{d}t\right. \\
&\quad\left.+|e^{\alpha d}f'(d)|^q\int_0^1 |1-2t|h(1-t)A_t^{-2}\mathrm{d}t\right)^{1/q},
\end{aligned}$$

where

$$\begin{aligned}
\mu(c,d)=\int_0^1 \frac{|1-2t|}{A_t^2}\mathrm{d}t \;&= c^{-2}\left[{}_2F_1\left(2,2;3;1-\frac{d}{c}\right)-{}_2F_1\left(2,1;2;1-\frac{d}{c}\right)\right. \\
&\quad\left.+\frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)\right].
\end{aligned}$$

This completes the proof. □

We now discuss some new special cases of Theorem 3.

I. If $h(t)=t^s$ and the function f is Breckner type of harmonically exponential s-convex, then inequality (16) becomes

$$\begin{aligned}
&\left|\frac{f(c)+f(d)}{2}-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right| \\
&\leq \frac{cd(d-c)}{2}\mu^{1-1/q}(c,d)\left[\mu_1(s;c,d)|e^{\alpha c}f'(c)|^q+\mu_2(s;c,d)|e^{\alpha d}f'(d)|^q\right]^{1/q},
\end{aligned}$$

where

$$\begin{aligned}\mu_1(s;c,d) &= c^{-2}\Bigg[\frac{2}{s+2}\,{}_2F_1\left(2,s+2;s+3;1-\uplus\frac{d}{c}\right)\\ &\quad -\frac{1}{s+1}\,{}_2F_1\left(2,s+1;s+2;1-\frac{d}{c}\right)\\ &\quad +\frac{1}{2^s(s+1)(s+2)}\,{}_2F_1\left(2,s+1;s+3;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)\Bigg],\end{aligned}$$

and

$$\begin{aligned}\mu_2(s;c,d) &= c^{-2}\Bigg[\frac{2}{(s+1)(s+2)}\,{}_2F_1\left(2,2;s+3;1-\frac{d}{c}\right)\\ &\quad -\frac{1}{s+1}\,{}_2F_1\left(2,1;s+2;1-\frac{d}{c}\right)\\ &\quad +\frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)\Bigg].\end{aligned}$$

II. If $h(t)=t^{-s}$ and the function f is Breckner type of harmonically exponential s-Godunova–Levin function, then inequality (16) becomes

$$\begin{aligned}&\left|\frac{f(c)+f(d)}{2}-\frac{cd}{d-c}\int_c^d\frac{f(x)}{x^2}\,\mathrm{d}x\right|\\ &\le\frac{cd(d-c)}{2}\mu^{1-1/q}(c,d)\left[\nu_1(s;c,d)|e^{\alpha c}f'(c)|^q+\nu_2(s;c,d)|e^{\alpha d}f'(d)|^q\right]^{1/q},\end{aligned}$$

where

$$\begin{aligned}\nu_1(s;c,d) &= c^{-2}\Bigg[\frac{2}{2-s}\,{}_2F_1\left(2,2-s;3-s;1-\frac{d}{c}\right)\\ &\quad -\frac{1}{1-s}\,{}_2F_1\left(2,1-s;2-s;1-\frac{d}{c}\right)\\ &\quad +\frac{2^s}{(1-s)(2-s)}\,{}_2F_1\left(2,1-s;3-s;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)\Bigg],\end{aligned}$$

and

$$\begin{aligned}\nu_2(s;c,d) &= c^{-2}\Bigg[\frac{1}{(1-s)(2-s)}\,{}_2F_1\left(2,2;3-s;1-\frac{d}{c}\right)\\ &\quad -\frac{1}{1-s}\,{}_2F_1\left(2,1;2-s;1-\frac{d}{c}\right)\\ &\quad +\frac{1}{2}\,{}_2F_1\left(2,1;3;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)\Bigg].\end{aligned}$$

Theorem 4 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ such that $f' \in L[c, d]$, where $c, d \in \mathscr{I}^0$ with $c < d$. If the function $|f'|^q$, $q > 1$ is harmonic h-convex, then we have*

$$\left|\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x - f\left(\frac{2cd}{c+d}\right)\right|$$
$$\leq cd(d-c)\left[\psi_1^{1-1/q}(c,d)\left(\int_0^{1/2}\frac{t}{A_t^2}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right.$$
$$\left. + \psi_2^{1-1/q}(c,d)\left(\int_{1/2}^{1}\frac{1-t}{A_t^2}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right], \quad (17)$$

where

$$\psi_1(c,d) = \frac{1}{8c^2}\,{}_2F_1\left[2, 2; 3; \frac{1}{2}\left(1 - \frac{d}{c}\right)\right],$$

$$\psi_2(c,d) = \frac{1}{2(c+d)^2}\,{}_2F_1\left[2, 1; 3; \frac{c-d}{c+d}\right],$$

and $A_t = (1-t)c + td$.

Proof From Lemma 2, the power mean inequality and the harmonic h-convexity of $|f'|^q$, with $q > 1$, we have

$$\left|\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x - f\left(\frac{2cd}{c+d}\right)\right|$$
$$\leq cd(d-c)\left[\int_0^{1/2}\frac{t}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|\mathrm{d}t + \int_{1/2}^{1}\frac{|t-1|}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|\mathrm{d}t\right]$$
$$\leq cd(d-c)\left[\left(\int_0^{1/2}\frac{t}{A_t^2}\mathrm{d}t\right)^{1-1/q}\left(\int_0^{1/2}\frac{t}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|^q\mathrm{d}t\right)^{1/q}\right.$$
$$\left. + \left(\int_{1/2}^{1}\frac{1-t}{A_t^2}\mathrm{d}t\right)^{1-1/q}\left(\int_{1/2}^{1}\frac{1-t}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|^q\mathrm{d}t\right)^{1/q}\right]$$
$$\leq cd(d-c)\left[\psi_1(c,d)^{1-1/q}\left(\int_0^{1/2}\frac{t}{A_t^2}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right.$$
$$\left. + \psi_2(c,d)^{1-1/q}\left(\int_{1/2}^{1}\frac{1-t}{A_t^2}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right],$$

where

$$\psi_1(c,d) = \int_0^{1/2} \frac{t}{A_t^2}\mathrm{d}t = \frac{1}{8c^2}\,{}_2F_1\left(2,2;3;\frac{1}{2}\left(1-\frac{d}{c}\right)\right)$$

and

$$\psi_2(c,d) = \int_{1/2}^{1} \frac{1-t}{A_t^2}\mathrm{d}t = \frac{1}{2(c+d)^2}\,{}_2F_1\left(2,1;3;\frac{c-d}{c+d}\right).$$

This completes the proof. □

Theorem 5 *Let* $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$ *such that* $f' \in L[c,d]$, *where* $c, d \in \mathscr{I}^0$ *with* $c < d$. *If the function* $|f'|^q$, $q > 1$ *is harmonic* h*-convex, then*

$$\left|\frac{f(c)+f(d)}{2} - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right|$$
$$\leq \frac{cd(d-c)}{2(p+1)^{1/p}}\left(|e^{\alpha c}f'(c)|^q \int_0^1 h(t)A_t^{-2q}\mathrm{d}t\right.$$
$$\left. + |e^{\alpha d}f'(d)|^q \int_0^1 h(1-t)A_t^{-2q}\mathrm{d}t\right)^{1/q}, \tag{18}$$

where $A_t = (1-t)c + td$, $\frac{1}{p} + \frac{1}{q} = 1$.

Proof Using Lemma 1, Hölder's inequality, and the harmonic h-convexity of $|f'|^q$, we have

$$\left|\frac{f(c)+f(d)}{2} - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right|$$
$$\leq \frac{cd(d-c)}{2}\int_0^1 \frac{|1-2t|}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|\mathrm{d}t$$
$$\leq \frac{cd(d-c)}{2}K_1^{1/p}\left(\int_0^1 \frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}$$
$$\leq \frac{cd(d-c)}{2(p+1)^{1/p}}\left(|e^{\alpha c}f'(c)|^q \int_0^1 h(t)A_t^{-2q}\mathrm{d}t + |e^{\alpha d}f'(d)|^q \int_0^1 h(1-t)A_t^{-2q}\mathrm{d}t\right)^{1/q},$$

where

$$K_1 = \int_0^1 |1-2t|\mathrm{d}t = \frac{1}{p+1}.$$

This completes the proof. □

We discuss some special cases of Theorem 5.

I. If $h(t) = t^s$ and the function f is harmonic s-convex and inequality (18) reduces to

$$\begin{aligned}&\left|\frac{f(c)+f(d)}{2}-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right|\\&\leq \frac{d(d-c)}{2c}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{s+1}\right)^{1/q}\\&\times\left({}_2F_1\left(2q, s+1; s+2; 1-\frac{d}{c}\right)|e^{\alpha c}f'(c)|^q\right.\\&\qquad\left.+{}_2F_1\left(2q, 1; s+2; 1-\frac{d}{c}\right)|e^{\alpha d}f'(d)|^q\right)^{1/q},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

II. If $h(t) = t^{-s}$ and the function f is harmonic s-Godunova–Levin function, then the inequality (18) reduces to the following new result.

$$\begin{aligned}&\left|\frac{f(c)+f(d)}{2}-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x\right|\\&\leq \frac{d(d-c)}{2c}\left(\frac{1}{p+1}\right)^{1/p}\left(\frac{1}{1-s}\right)^{1/q}\left({}_2F_1\left(2q, 1-s; 2-s; 1-\frac{d}{c}\right)|e^{\alpha c}f'(c)|^q\right.\\&\qquad\left.+{}_2F_1\left(2q, 1; 2-s; 1-\frac{d}{c}\right)|e^{\alpha d}f'(d)|^q\right)^{1/q},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 6 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$ such that $f' \in L[c,d]$, where $c, d \in \mathscr{I}^0$ with $c < d$. If $|f'|^q$ with $q > 1$ is harmonic h-convex function, then*

$$\begin{aligned}&\left|\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x - f\left(\frac{2cd}{c+d}\right)\right|\\&\leq \frac{cd(d-c)}{2}\left(\frac{1}{2(p+1)}\right)^{1/p}\\&\times\left[\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q+h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right.\\&\left.+\left(\int_{1/2}^{1}\frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q+h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right],\end{aligned}$$

where $A_t = (1-t)c + td$ and $\frac{1}{p}+\frac{1}{q}=1$.

Proof From Lemma 2, Hölder's inequality, and the harmonic h-convexity of $|f'|^q$, we get

$$\begin{aligned}
&\left|\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\,\mathrm{d}x - f\left(\frac{2cd}{c+d}\right)\right| \\
&\le cd(d-c)\left[\int_0^{1/2}\frac{t}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|\mathrm{d}t + \int_{1/2}^1 \frac{t-1}{A_t^2}\left|f'\left(\frac{cd}{A_t}\right)\right|\mathrm{d}t\right] \\
&\le cd(d-c)\left[\left(\int_0^{1/2} t^p\mathrm{d}t\right)^{1/p}\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left|f'\left(\frac{cd}{A_t}\right)\right|^q\mathrm{d}t\right)^{1/q}\right. \\
&\qquad\left. + \left(\int_{1/2}^1 |t-1|^p\mathrm{d}t\right)^{1/p}\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left|f'\left(\frac{cd}{A_t}\right)\right|^q\mathrm{d}t\right)^{1/q}\right]
\end{aligned}$$

$$\begin{aligned}
&\le cd(d-c)\left[\left(\frac{1}{2^{p+1}(p+1)}\right)^{1/p}\right. \\
&\quad\times\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q} \\
&\quad+\left(\frac{1}{2^{p+1}(p+1)}\right)^{1/p} \\
&\quad\left.\times\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right] \\
&= \frac{cd(d-c)}{2}\left(\frac{1}{2(p+1)}\right)^{1/p} \\
&\quad\times\left[\left(\int_0^{1/2}\frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right. \\
&\qquad\left.+\left(\int_{1/2}^1\frac{1}{A_t^{2q}}\left(h(t)|e^{\alpha c}f'(c)|^q + h(1-t)|e^{\alpha d}f'(d)|^q\right)\mathrm{d}t\right)^{1/q}\right].
\end{aligned}$$

This completes the proof. □

Now using Lemma 3 as an auxiliary result, we derive our coming results.

Theorem 7 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$ and $f' \in L[c, d]$. If $|f'|$ is harmonic h-convex function, then*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2}+f\left(\frac{2cd}{c+d}\right)\right]-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq cd(d-c)\left[\{|e^{\alpha c}f'(c)|+|e^{\alpha d}f'(d)|\}\left\{\int_0^1\left|t-\frac{1}{2}\right|(\nu_1+\nu_2)(\nu_3+\nu_4)\,\mathrm{d}t\right\}\right],$$

where

$$\nu_1 = h\left(\frac{1+t}{2}\right) \tag{19}$$

$$\nu_2 = h\left(\frac{1-t}{2}\right) \tag{20}$$

$$\nu_3 = \frac{1}{((1-t)c+(1+t)d)^2}, \tag{21}$$

and

$$\nu_4 = \frac{1}{((1+t)c+(1-t)d)^2}. \tag{22}$$

Proof Using Lemma 3, the harmonic h-convexity of $|f'|$, we have

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2}+f\left(\frac{2cd}{c+d}\right)\right]-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$=\left|cd(d-c)\left[\int_0^1\left(\frac{1}{2}-t\right)\left(\frac{1}{(1-t)c+(1+t)d}\right)^2 f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\mathrm{d}t\right.\right.$$

$$\left.\left.+\int_0^1\left(t-\frac{1}{2}\right)\left(\frac{1}{(1+t)c+(1-t)d}\right)^2 f'\left(\frac{2cd}{(1+t)c+(1-t)d}\right)\right]\right|$$

$$\le cd(d-c)$$
$$\times\left[\int_0^1 \left|t-\frac{1}{2}\right|\left(\frac{1}{(1-t)c+(1+t)d}\right)^2 \left|f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\right|\mathrm{d}t\right.$$
$$\left.+\int_0^1 \left|t-\frac{1}{2}\right|\left(\frac{1}{(1+t)c+(1-t)d}\right)^2 \left|f'\left(\frac{2cd}{(1+t)c+(1-t)d}\right)\right|\mathrm{d}t\right]$$

$$\le cd(d-c)$$
$$\times\left[\int_0^1 \left|t-\frac{1}{2}\right|\left(\frac{1}{(1-t)c+(1+t)d}\right)^2\right.$$
$$\times\left\{h\left(\frac{1+t}{2}\right)|e^{\alpha c}f'(c)|+h\left(\frac{1-t}{2}\right)|e^{\alpha d}f'(d)|\right\}\mathrm{d}t$$
$$+\int_0^1 \left|t-\frac{1}{2}\right|\left(\frac{1}{(1-t)c+(1+t)d}\right)^2$$
$$\left.\times\left\{h\left(\frac{1-t}{2}\right)|e^{\alpha c}f'(c)|+h\left(\frac{1+t}{2}\right)|e^{\alpha d}f'(d)|\right\}\mathrm{d}t\right]$$
$$= cd(d-c)$$
$$\times\left[\{|e^{\alpha c}f'(c)|+|e^{\alpha d}f'(d)|\}\left\{\int_0^1 \left|t-\frac{1}{2}\right|\left(h\left(\frac{1+t}{2}\right)+h\left(\frac{1-t}{2}\right)\right)\right.\right.$$
$$\left.\times\left(\frac{1}{((1-t)c+(1+t)d)^2}+\frac{1}{((1+t)c+(1-t)d)^2}\right)\mathrm{d}t\right\}.$$

This completes the proof. □

We now discuss some special cases of Theorem 7. It is worth to mention here that all these special cases also appear to be new in the literature.

I. If $h(t)=t$ in Theorem 7, then we have result for harmonic convex functions.

Corollary 9 *Let $f:\mathscr{I}\subseteq\mathbb{R}_+\to\mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c,d\in\mathscr{I}$ with $c<d$ and $f'\in L[c,d]$. If $|f'|$ is harmonic convex function, then*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2}+f\left(\frac{2cd}{c+d}\right)\right]-\frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq \frac{cd(d-c)}{2(c+d)^2}\left[|e^{\alpha c} f'(c)| + |e^{\alpha d} f'(d)|\right](\varphi_1 + \varphi_2),$$

where

$$\varphi_1 = {}_2F_1\left(2, 2; 3; \frac{c-d}{c+d}\right) - {}_2F_1\left(2, 1; 2; \frac{c-d}{c+d}\right) + \frac{1}{4}\,{}_2F_1\left(2, 1; 3; \frac{c-d}{c+d}\right)$$

and

$$\varphi_2 = {}_2F_1\left(2, 2; 3; \frac{d-c}{c+d}\right) - {}_2F_1\left(2, 1; 2; \frac{d-c}{c+d}\right) + \frac{1}{4}\,{}_2F_1\left(2, 1; 3; \frac{d-c}{c+d}\right).$$

II. If $h(t) = t^s$ in Theorem 7, then we have result for harmonic s-convex functions.

Corollary 10 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$ and $f' \in L[c, d]$. If $|f'|$ is harmonic s-convex function, then for $s \in (0, 1)$, we have*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2} + f\left(\frac{2cd}{c+d}\right)\right] - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq cd(d-c)\left[|e^{\alpha c} f'(c)| + |e^{\alpha d} f'(d)|\right](I_1 + I_2),$$

where

$$I_1 = \int_0^{1/2}\left(\frac{1}{2} - t\right)\left(\nu_1^* + \nu_2^*\right)\left(\nu_3^* + \nu_4^*\right)\mathrm{d}t = J_1 + J_2 + J_3 + J_4,$$

with

$$J_1 = \int_0^{1/2}\left(\frac{1}{2} - t\right)\nu_1^*\nu_3^*\mathrm{d}t$$

$$J_2 = \int_0^{1/2}\left(\frac{1}{2} - t\right)\nu_1^*\nu_4^*\mathrm{d}t$$

$$J_3 = \int_0^{1/2}\left(\frac{1}{2} - t\right)\nu_2^*\nu_3^*\mathrm{d}t$$
$$= \frac{1}{2^{s+3}d^2}\left[\frac{2}{s+2}\,{}_2F_1\left(2, s+2; s+3; \frac{d-c}{2d}\right)\right.$$

$$-\frac{1}{s+1}\,{}_2F_1\left(2, s+1; s+2; \frac{d-c}{2d}\right)$$
$$+\frac{1}{2^{s+1}(s+1)(s+2)}\,{}_2F_1\left(2, s+1; s+3; \frac{d-c}{4d}\right)\bigg],$$

$$\begin{aligned}
J_4 &= \int_0^{1/2}\left(\frac{1}{2}-t\right)\nu_2^*\nu_4^*\mathrm{d}t \\
&= \frac{1}{2^{s+3}c^2}\bigg[\frac{2}{s+2}\,{}_2F_1\left(2, s+2; s+3; \frac{c-d}{2c}\right) \\
&\quad -\frac{1}{s+1}\,{}_2F_1\left(2, s+1; s+2; \frac{c-d}{2c}\right) \\
&\quad +\frac{1}{2^{s+1}(s+1)(s+2)}\,{}_2F_1\left(2, s+1; s+3; \frac{c-d}{4c}\right)\bigg],
\end{aligned}$$

and

$$I_2 = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left(\nu_1^*+\nu_2^*\right)\left(\nu_3^*+\nu_4^*\right)\mathrm{d}t = K_1+K_2+K_3+K_4$$

with

$$K_1 = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_1^*\nu_3^*\mathrm{d}t$$

$$K_2 = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_1^*\nu_4^*\mathrm{d}t$$

$$\begin{aligned}
K_3 &= \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_2^*\nu_3^*\mathrm{d}t \\
&= \frac{1}{2^{2s+4}d^2(s+1)(s+2)}\,{}_2F_1\left(2, 2; s+3; \frac{d-c}{4d}\right),
\end{aligned}$$

$$\begin{aligned}
K_4 &= \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_2^*\nu_4^*\mathrm{d}t \\
&= \frac{1}{2^{2s+4}c^2(s+1)(s+2)}\,{}_2F_1\left(2, 2; s+3; \frac{c-d}{4c}\right).
\end{aligned}$$

III. If $h(t) = t^{-s}$ in Theorem 7, then we have result for harmonic s-Godunova–Levin convex functions.

Corollary 11 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$ and $f' \in L[c, d]$. If $|f'|$ is harmonic s-Godunova–Levin convex function, then for $s \in [0, 1]$, we have*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2}+f\left(\frac{2cd}{c+d}\right)\right]-\frac{cd}{d-c}\int\limits_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq cd(d-c)\left[|e^{\alpha c}f'(c)|+|e^{\alpha d}f'(d)|\right]\left(I_1^*+I_2^*\right),$$

where

$$I_1^* = \int_0^{1/2}\left(\frac{1}{2}-t\right)\left(\nu_1^\sharp+\nu_2^\sharp\right)\left(\nu_3^\sharp+\nu_4^\sharp\right)\mathrm{d}t = J_1^*+J_2^*+J_3^*+J_4^*,$$

with

$$J_1^* = \int_0^{1/2}\left(\frac{1}{2}-t\right)\nu_1^\sharp\nu_3^\sharp\mathrm{d}t$$

$$J_2^* = \int_0^{1/2}\left(\frac{1}{2}-t\right)\nu_1^\sharp\nu_4^\sharp\mathrm{d}t$$

$$\begin{aligned}J_3^* &= \int_0^{1/2}\left(\frac{1}{2}-t\right)\nu_2^\sharp\nu_3^\sharp\mathrm{d}t\\ &= \frac{1}{2^{-s+3}d^2}\left[\frac{2}{-s+2}\,{}_2F_1\left(2,-s+2;-s+3;\frac{d-c}{2d}\right)\right.\\ &\quad -\frac{1}{-s+1}\,{}_2F_1\left(2,-s+1;-s+2;\frac{d-c}{2d}\right)\\ &\quad \left.+\frac{1}{2^{-s+1}(-s+1)(-s+2)}\,{}_2F_1\left(2,-s+1;-s+3;\frac{d-c}{4d}\right)\right]\end{aligned}$$

$$\begin{aligned}J_4^* &= \int_0^{1/2}\left(\frac{1}{2}-t\right)\nu_2^\sharp\nu_4^\sharp\mathrm{d}t\\ &= \frac{1}{2^{s+3}c^2}\left[\frac{2}{s+2}\,{}_2F_1\left(2,s+2;s+3;\frac{c-d}{2c}\right)\right.\end{aligned}$$

$$-\frac{1}{s+1}\,{}_2F_1\left(2, s+1; s+2; \frac{c-d}{2c}\right)$$
$$+\frac{1}{2^{s+1}(s+1)(s+2)}\,{}_2F_1\left(2, s+1; s+3; \frac{c-d}{4c}\right)\Bigg],$$

and

$$I_2^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\left(\nu_1^\sharp+\nu_2^\sharp\right)\left(\nu_3^\sharp+\nu_4^\sharp\right)\mathrm{d}t = K_1^*+K_2^*+K_3^*+K_4^*,$$

with

$$K_1^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_1^\sharp\nu_3^\sharp\mathrm{d}t$$

$$K_2^* = \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_1^\sharp\nu_4^\sharp\mathrm{d}t$$

$$\begin{aligned} K_3^* &= \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_2^\sharp\nu_3^\sharp\mathrm{d}t \\ &= \frac{1}{2^{-2s+4}d^2(-s+1)(-s+2)}\,{}_2F_1\left(2, 2; -s+3; \frac{d-c}{4d}\right) \end{aligned}$$

$$\begin{aligned} K_4^* &= \int_{1/2}^{1}\left(t-\frac{1}{2}\right)\nu_2^\sharp\nu_4^\sharp\mathrm{d}t \\ &= \frac{1}{2^{-2s+4}c^2(-s+1)(-s+2)}\,{}_2F_1\left(2, 2; -s+3; \frac{c-d}{4c}\right). \end{aligned}$$

Theorem 8 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$ and $f' \in L[c, d]$. If $|f'|^q$ is harmonic h-convex function where $q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, then*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2}+f\left(\frac{2cd}{c+d}\right)\right]-\frac{cd}{d-c}\int\limits_c^d\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq cd(d-c)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1 \nu_3^q\left\{\nu_1|e^{\alpha c}f'(c)|^q+\nu_2|e^{\alpha d}f'(d)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \nu_4^q\left\{\nu_2|e^{\alpha c}f'(c)|^q+\nu_1|e^{\alpha d}f'(d)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right],$$

where ν_1, ν_2 are given by (19), (20) and

$$\nu_3^q = \frac{1}{((1-t)c+(1+t)d)^{2q}}, \tag{23}$$

and

$$\nu_4^q = \frac{1}{((1+t)c+(1-t)d)^{2q}}. \tag{24}$$

Proof Using Lemma 3, Holder's inequality, and the fact that $|f'|^q$ is harmonic h-convex function, we have

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2}+f\left(\frac{2cd}{c+d}\right)\right]-\frac{cd}{d-c}\int_c^d\frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$=\left|cd(d-c)\left[\int_0^1\left(\frac{1}{2}-t\right)\left(\frac{1}{(1-t)c+(1+t)d}\right)^2 f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\mathrm{d}t\right.\right.$$

$$\left.\left.+\int_0^1\left(t-\frac{1}{2}\right)\left(\frac{1}{(1+t)c+(1-t)d}\right)^2 f'\left(\frac{2cd}{(1+t)c+(1-t)d}\right)\right]\right|$$

$$\le cd(d-c)\left[\left(\int_0^1\left|t-\frac{1}{2}\right|^p\mathrm{d}t\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1\frac{1}{((1-t)c+(1+t)d)^{2q}}\left|f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\right|\mathrm{d}t\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1\left|t-\frac{1}{2}\right|^p \mathrm{d}t\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1 \frac{1}{((1-t)c+(1+t)d)^{2q}}\left|f'\left(\frac{2cd}{(1-t)c+(1+t)d}\right)\right|\mathrm{d}t\right)^{\frac{1}{q}}\Bigg]$$

$$\leq cd(d-c)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}$$

$$\times\Bigg[\Bigg(\int_0^1\left(\frac{1}{((1-t)c+(1+t)d)^{2q}}\right)$$

$$\times\left[h\left(\frac{1+t}{2}\right)|e^{\alpha c}f'(c)|^q+h\left(\frac{1-t}{2}\right)|e^{\alpha d}f'(d)|^q\right]\mathrm{d}t\Bigg)^{\frac{1}{q}}$$

$$+\Bigg(\int_0^1\left(\frac{1}{((1-t)c+(1+t)d)^{2q}}\right)$$

$$\times\left[h\left(\frac{1-t}{2}\right)|e^{\alpha c}f'(c)|^q+h\left(\frac{1+t}{2}\right)|e^{\alpha d}f'(d)|^q\right]\mathrm{d}t\Bigg)^{\frac{1}{q}}\Bigg]$$

$$=cd(d-c)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}$$

$$\times\left[\left(\int_0^1 \nu_3^q\left\{\nu_1|e^{\alpha c}f'(c)|^q+\nu_2|e^{\alpha d}f'(d)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \nu_4^q\left\{\nu_2|e^{\alpha c}f'(c)|^q+\nu_1|e^{\alpha d}f'(d)|^q\right\}\mathrm{d}t\right)^{\frac{1}{q}}\right].$$

This completes the proof. □

Now we discuss some special cases of Theorem 8.

I. If $h(t) = t$ in Theorem 8, then we have result for harmonic convex functions.

Corollary 12 *Let* $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$*, where* $c, d \in \mathscr{I}$ *with* $c < d$ *and* $f' \in L[c, d]$*. If* $|f'|^q$ *is harmonic convex function where* $q > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$*, then*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2} + f\left(\frac{2cd}{c+d}\right)\right] - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq cd(d-c)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left(\phi_1^{1/q} + \phi_2^{1/q}\right),$$

where

$$\begin{aligned}
\phi_1 &= \int_0^1 \nu_3^q \left\{\nu_1 |e^{\alpha c} f'(c)|^q + \nu_2 |e^{\alpha d} f'(d)|^q\right\} \mathrm{d}t \\
&= \frac{1}{2(c+d)^{2q}} \left\{ |e^{\alpha c} f'(c)|^q \left[{}_2F_1\left(2q, 1; 2; \frac{c-d}{c+d}\right) + \frac{1}{2}\,{}_2F_1\left(2q, 2; 3; \frac{c-d}{c+d}\right)\right]\right. \\
&\quad \left. + \frac{1}{2}|e^{\alpha d} f'(d)|^q\,{}_2F_1\left(2q, 1; 3; \frac{c-d}{c+d}\right)\right\},
\end{aligned}$$

and

$$\begin{aligned}
\phi_2 &= \int_0^1 \nu_4^q \left\{\nu_2 |e^{\alpha c} f'(c)|^q + \nu_1 |e^{\alpha d} f'(d)|^q\right\} \mathrm{d}t \\
&= \frac{1}{2(c+d)^{2q}} \left\{ \frac{1}{2}|e^{\alpha c} f'(c)|^q\,{}_2F_1\left(2q, 1; 3; \frac{d-c}{c+d}\right)\right. \\
&\quad \left. + |e^{\alpha d} f'(d)|^q \left[{}_2F_1\left(2q, 1; 2; \frac{d-c}{c+d}\right) + \frac{1}{2}\,{}_2F_1\left(2q, 2; 3; \frac{d-c}{c+d}\right)\right]\right\}.
\end{aligned}$$

II. If $h(t) = t^s$ in Theorem 8, then we have result for harmonic s-convex functions.

Corollary 13 *Let* $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ *be a differentiable function on* $\mathscr{I}^0$*, where* $c, d \in \mathscr{I}$ *with* $c < d$ *and* $f' \in L[c, d]$*. If* $|f'|^q$ *is harmonic* s*-convex function, where* $q > 1$*,* $\frac{1}{p} + \frac{1}{q} = 1$*,* $s \in (0, 1)$*, then*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2} + f\left(\frac{2cd}{c+d}\right)\right] - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$

$$\leq cd(d-c)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left(\eta_1^{1/q} + \eta_2^{1/q}\right),$$

where

$$\eta_1 = \int_0^1 \nu_3^q \left\{ \nu_1 |e^{\alpha c} f'(c)|^q + \nu_2 |e^{\alpha d} f'(d)|^q \right\} \mathrm{d}t = |e^{\alpha c} f'(c)|^q M_1 + |e^{\alpha d} f'(d)|^q M_2$$

with

$$M_1 = \int_0^1 \nu_1 \nu_3^q \mathrm{d}t = \frac{1}{2^s (c+d)^{2q}} \int_0^1 (1+t)^s \left(1 - \frac{c-d}{c+d} t\right)^{-2q} \mathrm{d}t$$

$$\begin{aligned} M_2 &= \int_0^1 \nu_2 \nu_3^q \mathrm{d}t \\ &= \frac{1}{2^s (c+d)^{2q}} \int_0^1 (1-t)^s \left(1 - \frac{c-d}{c+d} t\right)^{-2q} \mathrm{d}t \\ &= \frac{1}{2^s (s+1)(c+d)^{2q}} {}_2F_1\left(2q, 1; s+2; \frac{c-d}{c+d}\right), \end{aligned}$$

and

$$\eta_2 = \int_0^1 \nu_4^q \left\{ \nu_2 |e^{\alpha c} f'(c)|^q + \nu_1 |e^{\alpha d} f'(d)|^q \right\} \mathrm{d}t = |e^{\alpha c} f'(c)|^q N_1 + |e^{\alpha d} f'(d)|^q N_2,$$

with

$$\begin{aligned} N_1 &= \int_0^1 \nu_2 \nu_4^q \mathrm{d}t \\ &= \frac{1}{2^s (c+d)^{2q}} \int_0^1 (1-t)^s \left(1 - \frac{d-c}{c+d} t\right)^{-2q} \mathrm{d}t \\ &= \frac{1}{2^s (s+1)(c+d)^{2q}} {}_2F_1\left(2q, 1; s+2; \frac{d-c}{c+d}\right) \end{aligned}$$

$$\begin{aligned} N_2 &= \int_0^1 \nu_1 \nu_4^q \mathrm{d}t \\ &= \frac{1}{2^s (c+d)^{2q}} \int_0^1 (1+t)^s \left(1 - \frac{d-c}{c+d} t\right)^{-2q} \mathrm{d}t. \end{aligned}$$

III. If $h(t) = t^{-s}$ in Theorem 8, then we have result for harmonic s-Godunova–Levin convex functions.

Corollary 14 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$ and $f' \in L[c, d]$. If $|f'|^q$ is harmonic s-Godunova–Levin convex function where $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $s \in [0, 1]$, then*

$$\left|\frac{1}{2}\left[\frac{f(c)+f(d)}{2} + f\left(\frac{2cd}{c+d}\right)\right] - \frac{cd}{d-c}\int_c^d \frac{f(x)}{x^2}\mathrm{d}x\right|$$
$$\leq cd(d-c)\left(\frac{1}{2^p(1+p)}\right)^{\frac{1}{p}}\left(\psi_1^{1/q} + \psi_2^{1/q}\right),$$

where

$$\psi_1 = \int_0^1 \nu_3^q \left\{\nu_1|e^{\alpha c} f'(c)|^q + \nu_2|e^{\alpha d} f'(d)|^q\right\} \mathrm{d}t = |e^{\alpha c} f'(c)|^q M_1^* + |e^{\alpha d} f'(d)|^q M_2^*,$$

with

$$M_1^* = \int_0^1 \nu_1 \nu_3^q \mathrm{d}t = \frac{1}{2^{-s}(c+d)^{2q}} \int_0^1 (1+t)^{-s}\left(1 - \frac{c-d}{c+d}t\right)^{-2q} \mathrm{d}t$$

$$\begin{aligned} M_2^* &= \int_0^1 \nu_2 \nu_3^q \mathrm{d}t \\ &= \frac{1}{2^{-s}(c+d)^{2q}} \int_0^1 (1-t)^{-s}\left(1 - \frac{c-d}{c+d}t\right)^{-2q} \mathrm{d}t \\ &= \frac{1}{2^{-s}(-s+1)(c+d)^{2q}}\, {}_2F_1\left(2q, 1; -s+2; \frac{c-d}{c+d}\right), \end{aligned}$$

and

$$\psi_2 = \int_0^1 \nu_4^q \left\{\nu_2|e^{\alpha c} f'(c)|^q + \nu_1|e^{\alpha d} f'(d)|^q\right\} \mathrm{d}t = |e^{\alpha c} f'(c)|^q N_1^* + |e^{\alpha d} f'(d)|^q N_2^*,$$

with

$$\begin{aligned} N_1^* &= \int_0^1 \nu_2 \nu_4^q \mathrm{d}t \\ &= \frac{1}{2^{-s}(c+d)^{2q}} \int_0^1 (1-t)^{-s}\left(1 - \frac{d-c}{c+d}t\right)^{-2q} \mathrm{d}t \\ &= \frac{1}{2^{-s}(-s+1)(c+d)^{2q}}\, {}_2F_1\left(2q, 1; -s+2; \frac{d-c}{c+d}\right) \end{aligned}$$

$$N_2^* = \int_0^1 \nu_1 \nu_4^q \mathrm{d}t = \frac{1}{2^{-s}(c+d)^{2q}} \int_0^1 (1+t)^{-s} \left(1 - \frac{d-c}{c+d}t\right)^{-2q} \mathrm{d}t.$$

IV. If $h(t) = 1$ in Theorem 8, then we have result for harmonic P-functions.

Corollary 15 *Let $f : \mathscr{I} \subseteq \mathbb{R}_+ \to \mathbb{R}$ be a differentiable function on $\mathscr{I}^0$, where $c, d \in \mathscr{I}$ with $c < d$ and $f' \in L[c, d]$. If $|f'|^q$ is harmonic P-function where, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\left| \frac{1}{2} \left[\frac{f(c) + f(d)}{2} + f\left(\frac{2cd}{c+d}\right) \right] - \frac{cd}{d-c} \int_c^d \frac{f(x)}{x^2} \mathrm{d}x \right|$$

$$\leq \frac{cd(d-c)}{(c+d)^2} \left(\frac{1}{2^p(1+p)} \right)^{\frac{1}{p}}$$

$$\times \left(|e^{\alpha c} f'(c)|^q + |e^{\alpha d} f'(d)|^q \right)^{1/q} \left[{}_2F_1\left(2q, 1; 2; \frac{c-d}{c+d}\right) + {}_2F_1\left(2q, 1; 2; \frac{d-c}{c+d}\right) \right]^{1/q}.$$

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On the Hardy–Sobolev Inequalities

Athanase Cotsiolis and Nikos Labropoulos

Abstract In this paper, we present a short survey on the Hardy–Sobolev inequalities, considering the classical case and the fractional as well, by collecting some important known results in the area and some new results where the concept of symmetry plays an important role.

1 Introduction

Hardy–Sobolev inequalities are among the most important functional inequalities in analysis because of their very interesting autonomous existence and also because of their strong connection with the solvability of a large number of nonlinear partial differential equations. As it is known, Hardy-type and Sobolev-type inequalities constitute essential tools in Analysis, in the study of partial differential equations, and in the Calculus of variations. In addition, we can find various applications in Geometry, in Mathematical Physics, and in Astrophysics.

In this paper, we present a short survey on Hardy–Sobolev inequalities by collecting some important known results in the area and some new results where the concept of symmetry plays an important role.

The paper is organized as follows: In Sects. 1.1–1.4 we give a short survey, concluding some important results, concerning firstly the Sobolev and secondly the Hardy inequalities. In Sect. 2, we present the Hardy–Sobolev inequalities (old and new results). Section 3 is devoted to the concept of symmetry of the view of the analysis to the sharp Hardy–Sobolev inequality.

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1.1 Sobolev Inequalities

We consider the Euclidean space $\mathbb{R}^n$, $n \geq 3$. The Sobolev embedding $W^{1,2}(\mathbb{R}^n) \hookrightarrow L^{2^*}(\mathbb{R}^n)$, where $2^* = \frac{2n}{n-2}$, proved by Sobolev in 1938 [60], asserts that for every $u \in W^{1,2}(\mathbb{R}^n)$ exists a positive constant C_n such that

$$\left(\int_{\mathbb{R}^n} |u|^{2^*}\,dx\right)^{\frac{1}{2^*}} \leqslant C_n \left(\int_{\mathbb{R}^n} |\nabla u|^2\,dx\right)^{\frac{1}{2}}. \tag{1}$$

Also, by Sobolev embedding theorem arises that for any $p \in [1, n)$ exists a positive constant $C_{n,p}$ such that for every $u \in W^{1,p}(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |u|^{p^*}\,dx\right)^{\frac{1}{p^*}} \leqslant C_{n,p} \left(\int_{\mathbb{R}^n} |\nabla u|^p\,dx\right)^{\frac{1}{p}}, \tag{2}$$

where $p^* = \frac{np}{n-p}$, is the critical exponent for this inequality in the sense that it cannot become lower nor higher and the inequality be in effect, and $|\cdot|$ denotes the standard Euclidean norm on $\mathbb{R}^n$.

$W^{1,p}(\mathbb{R}^n)$ is the classical Sobolev space, that is

$$W^{1,p}(\mathbb{R}^n) = \{u \in L^p(\mathbb{R}^n) : \ \nabla u \in L^p(\mathbb{R}^n)\},$$

which is defined for any integer $n \geq 1$ and for all real numbers $p \geq 1$. Here, $L^p(\mathbb{R}^n)$ is the usual Lebesgue space of order p, and ∇ stands for the gradient operator, acting on the distribution space $\mathscr{D}'(\mathbb{R}^n)$, where $\mathscr{D}(\mathbb{R}^n) = C_0^\infty(\mathbb{R}^n)$.

Recall that the best constant in front of the gradient term in inequality (2) is defined to be

$$\mathscr{C}_{n,p}^{-1} = \inf_{\substack{u \in L^{p*}(\mathbb{R}^n)\setminus\{0\} \\ \nabla u \in L^p(\mathbb{R}^n)}} \frac{\int_{\mathbb{R}^n} |\nabla u|^p dx}{\left(\int_{\mathbb{R}^n} |u|^{p^*} dx\right)^{\frac{p}{p^*}}}, \tag{3}$$

and it has been proven that

$$\mathscr{C}_{n,1} = \frac{1}{n}\left(\frac{n}{\omega_{n-1}}\right)^{\frac{1}{n}}, \tag{4}$$

$$\mathscr{C}_{n,p} = \frac{1}{n}\left(\frac{n\,(p-1)}{n-p}\right)^{1-\frac{1}{p}} \left(\frac{\Gamma\,(n+1)}{\Gamma\left(\frac{n}{p}\right)\Gamma\left(n+1-\frac{n}{p}\right)\omega_{n-1}}\right)^{\frac{1}{p}}, \tag{5}$$

where ω_{n-1} is the area of the unit sphere in $\mathbb{R}^n$ and Γ is the gamma function.

In this case, i.e. when $1 < p < n$, the value of $\mathscr{C}_{n,1}$ was explicitly computed independently by Aubin [1] and Talenti [64] and is attained by the functions

$$\varphi(x) = c(\lambda + |x - x_0|^{\frac{p}{p-1}})^{1-\frac{n}{p}}, \tag{6}$$

where $c \in \mathbb{R}$, $\lambda > 0$ and $x_0 \in \mathbb{R}^n$ are fixed constants.

In particular by (5) for $p = 2$ arises that

$$\mathscr{C}_{n,2} = \frac{1}{n\,(n-2)\,\pi} \left(\frac{\Gamma\,(n)}{\Gamma\left(\frac{n}{2}\right)} \right)^{\frac{2}{n}} \tag{7}$$

and the extremal functions are

$$\varphi(x) = c\left(\lambda + |x - x_0|^2\right)^{-\frac{n-2}{2}}. \tag{8}$$

In the special case where $p = 1$, the sharp form of the Sobolev inequality in $\mathbb{R}^n$ is of the form

$$\left(\int\limits_{\mathbb{R}^n} |f|^{\frac{n}{n-1}}\, dx \right)^{\frac{n-1}{n}} \leqslant \mathscr{C}_{n,1} \int\limits_{\mathbb{R}^n} |\nabla f|\, dx, \tag{9}$$

where the constant $\mathscr{C}_{n,1}$ is defined by (4).

Federer, Fleming, and Rishel in [27] and [29] by using the Coarea Formula have proved that the above inequality arises from the usual isoperimetric inequality, (see [27, 29] and [26]). The exact value of $\mathscr{C}_{n,1}$ was computed by Federer and Fleming [27] and by Maz'ya [52], and the extremal functions in this case are the characteristic functions of the balls of $\mathbb{R}^n$.

Concerning the values of the best constants, it is worth mentioning that Aubin [2] proved that the constant $\mathscr{C}_{n,1}$ is obtained as a limit of $\mathscr{C}_{n,p}$ as p tends to 1^+, and that Talenti's proof is also based on the Coarea Formula and follows implicitly that sharp Sobolev inequality for $p = 1$ reduces to the isoperimetric inequality.

By completing this brief survey on Sobolev inequalities, i.e. where one wants to control the size of a function in terms of the size of its gradient, we consider it useful to point out that the inequalities of this type first appeared long before the time of Sobolev. In particular, Steklov in 1896 [61] proved that the inequality

$$\int\limits_0^a u^2\,(x)dx \leqslant \left(\frac{1}{\pi}\right)^2 \int\limits_0^a \left|u'\,(x)\right|^2 dx \tag{10}$$

holds for all functions which are continuously differentiable on $[0, a]$ and they have zero mean there. The inequality (10) was among the earliest inequalities with sharp constant that appeared in Mathematical Physics. The fact that the constant in (10) is sharp was emphasized by Steklov in [63].

The next year (1897), Steklov published the article [62], in which the following analogue of the inequality (10) was proved:

$$\int_{\Omega} u^2 dx \leqslant C \int_{\Omega} |\nabla u|^2 \, dx. \tag{11}$$

Here, ∇ stands for the gradient operator and the integral on the right-hand side is called the Dirichlet integral. The assumptions made by Steklov are as follows: Ω is a bounded three-dimensional domain whose boundary is piecewise smooth and u is a real C^1-function on $\bar{\Omega}$ vanishing on $\partial\Omega$. Again, the inequality (11) was obtained by Steklov with the sharp constant equal to λ_1^{-1}, where λ_1 is the smallest eigenvalue of the Dirichlet Laplacian in Ω.

For a complete study on the best constants in Sobolev inequalities, see in the books [2, 19, 41, 53, 59] and for a short survey in [47].

1.2 Hardy Inequalities

The classical Hardy inequality was established by Hardy in the 1920s and in the continuous form it informs us that:

If $1 < p < \infty$ and f is a non-negative p-integrable function on $(0, \infty)$, then f is integrable over the interval $(0, x)$ for each positive x and

$$\int_0^\infty \left(\frac{1}{x} \int_0^x f(t) \, dt \right)^p dx \leqslant \left(\frac{p}{p-1} \right)^p \int_0^\infty f^p(x) \, dx. \tag{12}$$

The constant $\left(\frac{p}{p-1}\right)^p$ in (12) is sharp, i.e. it cannot be replaced with a smaller number such that (12) remains true for all relevant functions, respectively, and equality holds only if $f = 0$.

The inequality (12) was established by Hardy in [37] and was first highlighted in the famous book [39] of Hardy, Littlewood, and Polya or in the original article of Hardy [38], which also showed that the constant is not attained, i.e. the variational problem has no minimizer. A proof of the above inequality was given by Landau, in a letter to Hardy, which was officially published in [48]. For a short but very informative presentation of the prehistory of Hardy's inequality see in [46].

Coming back to the inequality (12), if we set $u(x) = \int_0^x f(t) \, dt$, we obtain the inequality

$$\int_0^\infty \left(\frac{u(x)}{x}\right)^p dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^\infty \left(u'(x)\right)^p dx, \tag{13}$$

which is the most popular form of the classical Hardy inequality in contemporary literature.

The following Hardy inequality is the classical generalization of Hardy inequality (13) to higher dimensions and asserts that for $n > 1$, $1 \le p < \infty$ with $p \neq n$ and for all $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\}$, it holds (see [39] or [57])

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^p} dx \leqslant \left(\frac{p}{|n-p|}\right)^p \int_{\mathbb{R}^n} |\nabla u(x))^p\, dx. \tag{14}$$

The constant $\left(\frac{p}{|n-p|}\right)^p$ is sharp and is not attained in the corresponding Sobolev spaces, which is $W^{1,p}(\mathbb{R}^n)$ when $1 \le p < n$ and $W^{1,p}(\mathbb{R}^n \backslash \{0\})$ when $n < p < \infty$. If $p = 1$, equality holds for any symmetric decreasing function. For $p = 2$ and $n > 2$, this inequality is also called "the uncertainty principle."

We note here that in the one-dimensional case, it was proved by Hardy in 1925 that for all p-integrable, $p > 1$ on $(0, 1)$, functions u, it holds

$$\int_0^1 \frac{|u(x)|^p}{d_{(0,1)}^p(x)} dx \leqslant \left(\frac{p}{p-1}\right)^p \int_0^1 \left|u'(x)\right|^p dx, \tag{15}$$

where $d_{(0,1)}(x) = \min(x, 1-x)$ (see in [37, 38] and [9]). In addition, Hardy showed that the constant is not attained, i.e. the variational problem has no minimizer. Furthermore, inequality (15) confirms that in the one-dimensional case no geometrical assumption is required on the domain.

In regard to Hardy inequalities for domains in $\mathbb{R}^n$, $n \geqslant 2$ the situation is far more complicated than in the one-dimensional case and in general the best constant in (14) depends on the domain.

In aim to establish it in domains it is necessary first to establish it in the half space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0, \infty)$. If we denote by $x = \left(x', x_n\right)$ a point in $\mathbb{R}^n$, where $x' = (x_1, \ldots, x_{n-1})$, the Hardy inequality in the half space $\mathbb{R}^n_+$ asserts that if $p > 1$, then for all $u \in C_0^\infty\left(\mathbb{R}^n_+\right)$

$$\int_{\mathbb{R}^n_+} \frac{|u|^p}{x_n^p} dx \leqslant \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}^n_+} |\nabla u|^p\, dx, \tag{16}$$

where the constant $\left(\frac{p}{p-1}\right)^p$ is sharp and is not attained in $W_0^{1,2}(\mathbb{R}^n_+)$ [56, 57].

As a direct generalization of inequality (14) on domains in $\mathbb{R}^n$, $n \geq 2$ we can take the following: Let Ω be a domain in $\mathbb{R}^n$, $n \geq 2$ with nonempty boundary and $1 \leq p < \infty$. Given Ω, let $d_\Omega(x)$ be the distance from x to the boundary $\partial\Omega$, that is $d_\Omega(x) = \min\{|x-y| : y \notin \Omega\}$. Then, the Hardy inequality in higher dimensions should be of the type

$$\int_\Omega \frac{|u|^p}{d_\Omega^p} dx \leqslant \mu \int_\Omega |\nabla u|^p \, dx, \tag{17}$$

which means that there exists a positive constant μ such that the inequality (17) is valid for all u belonging to some suitable space. And if that is so, does it valid unconditionally on Ω, or are some prerequisites necessary, and if so, which ones? Maz'ya has shown in 1960 that the validity of the Hardy inequality depends on measure theoretical conditions on the domain [52, 53]. Additionally, Hardy-type inequalities in $\mathbb{R}^n$, $n \geqslant 2$, appeared by Něcas in 1962 [56] in the context of Lipschitz domains. However, in regard to Hardy inequalities for domains Ω in $\mathbb{R}^n$, $n \geqslant 2$, the best constant in (17) depends on the domain Ω and no universal Hardy constant exists.

Regarding the study on the best constants in Hardy inequalities, see in the books [4, 35, 57], in the paper [5] and the references of it. For a short survey in [13].

1.3 Fractional Sobolev Inequalities

We recall the definition of the Fourier transform of a distribution. Consider the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decaying C^∞ functions in $\mathbb{R}^n$ and $\mathcal{S}'(\mathbb{R}^n)$ be the set of all tempered distributions, that is the topological dual of it (see in [17] for details).

For any $u \in \mathcal{S}(\mathbb{R}^n)$, we denote by

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x) dx, \quad \xi \in \mathbb{R}^n \tag{18}$$

the Fourier transform of u.

Let Ω be a general, possibly nonsmooth, open set in $\mathbb{R}^n$ and a fix fractional exponent $\sigma \in (0, 1)$. Then, for any $p \in [1, +\infty)$, we define the fractional Sobolev space $W^{\sigma,p}(\Omega)$ as follows:

$$W^{\sigma,p}(\Omega) = \left\{ u \in L^p(\Omega) : \frac{|u(x) - u(y)|}{|x-y|^{\frac{n}{p}+\sigma}} \in L^p(\Omega \times \Omega) \right\}$$

endowed with the natural norm

$$\|u\|_{W^{\sigma,p}(\Omega)} = \left(\int_{\Omega} |u|^p \, dx + \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\sigma}} dx dy, \right)^{\frac{1}{p}},$$

where the term

$$[u]_{W^{\sigma,p}(\Omega)} = \left(\iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\sigma}} dx dy \right)^{\frac{1}{p}} \tag{19}$$

is the so-called Gagliardo (semi)norm of u.

We note at this point that although we have defined the space $W^{\sigma,p}(\Omega)$ for any $\sigma \in (0, 1)$, this definition, with an appropriate procedure, can be extended for each $\sigma > 0$ (see in [17] for details).

As in the classic case with σ being an integer, for any $\sigma > 0$, the space $C_0^\infty(\mathbb{R}^n)$ of smooth functions with compact support is dense in $W^{\sigma,p}(\mathbb{R}^n)$ (see [17]).

We define also the space $W_0^{\sigma,p}(\Omega)$ denote the closure of $C_0^\infty(\Omega)$ in the norm $\|u\|_{W^{\sigma,p}(\Omega)}$, defined above. Note, that $W_0^{\sigma,p}(\mathbb{R}^n) = W^{\sigma,p}(\mathbb{R}^n)$, but in general, for $\Omega \subset \mathbb{R}^n$, $W_0^{\sigma,p}(\Omega) \neq W^{\sigma,p}(\Omega)$. The spaces $W^{\sigma,2}(\mathbb{R}^n)$ and $W_0^{\sigma,2}(\mathbb{R}^n)$ as Hilbert spaces are usually denoted by $H^\sigma(\mathbb{R}^n)$ and $H_0^\sigma(\mathbb{R}^n)$, respectively.

We introduce now the Dirichlet fractional Laplacian which will be denoted by $(-\Delta)^\sigma$, $\sigma \in (0, 1)$. Let $u \in \mathscr{J}(\mathbb{R}^n)$. (If $x \in \Omega$, we extend the function u in all of the $\mathbb{R}^n$ by setting $u(x) = 0$ for any $x \notin \Omega$). Then, the fractional Laplacian $(-\Delta)^\sigma$ is defined via Fourier transform by

$$\left((-\Delta)^\sigma u\right)^\wedge(\xi) = |\xi|^{2\sigma} \hat{u}(\xi), \; u \in C_0^\infty\left(\mathbb{R}^n\right). \tag{20}$$

We can easily verify that $\|\nabla u\|_2 = \|(-\Delta)^{\frac{1}{2}} u\|_2$. Using this notation, for any $\sigma > 0$ we can define, again, the Sobolev spaces $H^\sigma(\mathbb{R}^n)$ by

$$H^\sigma\left(\mathbb{R}^n\right) = \left\{ u \in L^2\left(\mathbb{R}^n\right) : \left\| (-\Delta)^{\frac{\sigma}{2}} u \right\|_2 < \infty \right\}.$$

The following sharp fractional Sobolev inequality consists the direct generalization of (1) for functions which belong to the space $H^\sigma(\mathbb{R}^n)$ and it asserts that if $0 < \sigma < \frac{n}{2}$, then for all $u \in H^\sigma(\mathbb{R}^n)$,

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{2n}{n-2\sigma}} dx \right)^{\frac{n-2\sigma}{n}} \leqslant \mathscr{S}_{n,\sigma} \int_{\mathbb{R}^n} \left| (-\Delta)^{\frac{\sigma}{2}} u(x) \right|^2 dx, \tag{21}$$

where

$$\mathscr{S}_{n,\sigma} = \frac{1}{(4\pi)^{\sigma}} \frac{\Gamma\left(\frac{n-2\sigma}{2}\right)}{\Gamma\left(\frac{n+2\sigma}{2}\right)} \left(\frac{\Gamma(n)}{\Gamma\left(\frac{n}{2}\right)}\right)^{\frac{2\sigma}{n}}, \tag{22}$$

and, the equality in (21) holds if and only if u is an Aubin-Talenti type function, i.e.:

$$u(x) = c\left(\mu^2 + (x - x_0)^2\right)^{-\frac{n-2\sigma}{2}}, \ x \in \mathbb{R}^n, \tag{23}$$

where $c \in \mathbb{R}$, $\mu > 0$ and $x_0 \in \mathbb{R}^n$ are fixed constants.

The best constant $\mathscr{S}_{n,\sigma}$ and the extremal functions were computed by Cotsiolis and Tavoularis in [15]. Note that for $\sigma = 1$ we obtain the inequality (1) and for $\sigma = \frac{1}{2}$ the best value for $\mathscr{S}_{n,\frac{1}{2}}$ is calculated by Lieb and Loss (see [49]).

Coming back to the operator $(-\Delta)^{\sigma}$, $\sigma \in (0, 1)$, we can define it as follows (see in [32, 55] and [17]):

$$\left\|(-\Delta)^{\frac{\sigma}{2}} u\right\|_2^2 = \int\limits_{\mathbb{R}^n} |\xi|^{2\sigma} \left|\hat{u}(\xi)\right|^2 d\xi = \mathscr{S}_{n,\sigma} \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2\sigma}} dxdy. \tag{24}$$

Under the above last considerations the fractional Sobolev inequality of order $\sigma \in (0, 1)$ (with the additional assumption $\sigma < \frac{1}{2}$ if $n = 1$) is exactly the inequality (21). Thus, its best constant $\mathscr{S}_{n,\sigma}(\mathbb{R}^n)$ remains the same as computed in [15].

Here, $H^{\sigma}(\mathbb{R}^n)$ denotes the space of all real valued functions u on $\mathbb{R}^n$ such that

$$[u]^2_{H^{\sigma}(\mathbb{R}^n)} = \iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{(u(x) - u(y))^2}{|x - y|^{n+2\sigma}} dxdy < +\infty.$$

Frank, Jin, and Xiong in [33], as well as Musina and Nazarov in [55] studied the fractional Sobolev inequality on the half-space. In this case we need to define the appropriate Sobolev space. In particular, we define the Sobolev space $W_0^{\sigma,p}(\mathbb{R}^n_+)$ as the completion of $C_0^{\infty}(\mathbb{R}^n_+)$ with respect to the

$$[u]_{W^{\sigma,p}(\mathbb{R}^n_+)} = \iint\limits_{\mathbb{R}^n_+ \times \mathbb{R}^n_+} \frac{|u(x) - u(y)|^p}{|x - y|^{n+p\sigma}} dx'dy', \tag{25}$$

and set $W_0^{\sigma,2}(\mathbb{R}^n_+) = H_0^{\sigma}(\mathbb{R}^n_+)$.

1.4 Fractional Hardy Inequalities

We recall the Hardy-type inequality (see in [32])

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2\sigma}} dx \le \mathscr{C}_{n,\sigma}^{-1} \int_{\mathbb{R}^n} |\xi|^{2\sigma} |\hat{u}(\xi)|^2 d\xi, \ \text{for all} \in C_0^\infty(\mathbb{R}^n), \tag{26}$$

valid for $0 < 2\sigma < n$. Here, $\hat{u}(\xi)$ is the Fourier transform of u defined by (18).

The sharp constant in (26)

$$\mathscr{C}_{n,\sigma} = 2^{2\sigma} \frac{\Gamma^2\left(\frac{n+2\sigma}{4}\right)}{\Gamma^2\left(\frac{n-2\sigma}{4}\right)} \tag{27}$$

has been found independently by Herbst [43] and Yafaev [65].

For $n \ge 1$ and $0 < \sigma < 1$ we consider the homogeneous Sobolev spaces $W_0^{\sigma,p}(\mathbb{R}^n)$ and $W_0^{\sigma,p}(\mathbb{R}^n \setminus \{0\})$ defined as the completion with respect to $[u]_{W^{\sigma,p}(\mathbb{R}^n)}$ (defined by (19)) of $C_0^\infty(\mathbb{R}^n)$ for $1 \le p < \frac{n}{\sigma}$ and of $C_0^\infty(\mathbb{R}^n \setminus \{0\})$ for $p > \frac{n}{\sigma}$, respectively.

The fractional analog of Hardy inequality (14) is obtained if the term $\int_{\mathbb{R}^n} |\nabla u(x))^p\, dx$ on the right-hand side of it is replaced by $[u]_{W^{\sigma,p}(\mathbb{R}^n)}$, for some $0 < \sigma < 1$. So, in this case we have the sharp fractional Hardy inequality on $\mathbb{R}^n$, $n \ge 1$

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{p\sigma}} dx \leqslant \mathscr{C}_{n,p,\sigma}^{-1} \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\sigma}} dx dy. \tag{28}$$

The best constant $\mathscr{C}_{n,p,\sigma}$ was computed in [30].

If $p = 1$, equality holds if and only if u is proportional to a symmetric-decreasing function. It is worth mentioning the very important contribution of Maz'ya and Shaposhnikova [54], and of Bourgain et al. [7], since in these works the bases for calculating the optimum constant $\mathscr{C}_{n,p,\sigma}$ were set.

The fractional Hardy inequality in the half-space $\mathbb{R}_+^n$ states that for $0 < \sigma < 1$ and $1 < p < \infty$ with $p\sigma \neq 1$ there is a positive constant $\mathscr{D}_{n,p,\sigma}$ such that

$$\int_{\mathbb{R}_+^n} \frac{|u(x)|^p}{x_n^{p\sigma}} dx \leqslant \mathscr{D}_{n,p,\sigma}^{-1} \iint_{\mathbb{R}_+^n \times \mathbb{R}_+^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\sigma}} dx dy, \tag{29}$$

for all $u \in C_0^\infty\left(\overline{\mathbb{R}_+^n}\right)$ if $p\sigma < 1$ and for all $u \in C_0^\infty\left(\mathbb{R}_+^n\right)$ if $p\sigma > 1$.

The sharp value of the constant $\mathscr{D}_{N,p,\sigma}$ for $p = 2$ and $2\sigma = \alpha$ (this last substitution has been made by the authors) is calculated in [6] to be equal to

$$\mathscr{D}_{n,2,\sigma} = \mathscr{D}_{n,\alpha} = \frac{\pi^{\frac{n-1}{2}} \Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{n+\alpha}{2}\right)} \frac{\mathrm{B}\left(\frac{1+\alpha}{2}, \frac{2-\alpha}{2}\right) - 2^{\alpha}}{\alpha 2^{\alpha-1}}, \tag{30}$$

where Γ, B are the gamma and the beta functions, respectively.

For arbitrary p, the best constant $\mathscr{D}_{n,p,\sigma}$ and the extremal functions are calculated in [31].

Dyda in [20] investigated the following integral inequality

$$\int\limits_{\Omega} \frac{|u(x)|^p}{d_{\Omega}^{a}(x)} dx \leqslant C(\Omega) \iint\limits_{\Omega\times\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+\alpha}} dxdy, \tag{31}$$

where $\alpha, p > 0$, $\Omega \subset \mathbb{R}^n, n \geq 1$ is a Lipschitz domain or its complement or a complement of a point, and $d_{\Omega}(x) = \min\{|x-y| : y \notin \Omega\}$. In this paper, the author gives sufficient conditions on Ω for the validity of (31) for same $C(\Omega) > 0$. In addition, counterexamples for cases where (31) is not hold are given. Also, another case of the fractional Hardy inequality with a remainder term studied in [21].

Bogdan and Dyda in [6] conjectured that if Ω is a convex, open subset of $\mathbb{R}^n$ and $\alpha \in (0, 2)$ then the largest number $\mathscr{C}(\Omega)$ such that

$$\int\limits_{\Omega} \frac{u^2(x)}{d_{\Omega}^{a}(x)} dx \leqslant \mathscr{C}(\Omega) \iint\limits_{\Omega\times\Omega} \frac{(u(x) - u(y))^2}{|x-y|^{n+\alpha}} dxdy, \tag{32}$$

is equal to $\mathscr{D}_{n,\alpha}$, defined by (30).

Loss and Sloane in [50] proved a sharp Hardy inequality for fractional integrals for functions that are supported in a general domain. The constant is the same as the one for the half-space and hence their result settles the conjecture of Bogdan and Dyda. In the same paper, a weaker form of (31) was established for $p > 1$. However the sharp constant remains the same as the sharp constant for the Hardy inequality for the half-space, i.e. $\mathscr{D}_{n,p,\sigma}$, computed in [31]. For $0 < p \leq 1$ the inequality remains valid (see [20]), however, the sharp constant is not known.

We mention here that by Dyda and Vähäkangas a general framework for fractional Hardy inequalities [22], a Maz'ya type characterization [23] are provided. Also, Ihnatsyeva, Lehrbäck, Tuominen, and Vähäkangas proved in [44] fractional order Hardy inequalities on open sets under a combined fatness and visibility condition on the boundary. Concluding this part we refer that Brasco and Cinti in [8] give a quick overview on Hardy inequality and prove a Hardy inequality on convex sets, for fractional Sobolev-Slobodeckiĭ spaces of order (s, p).

2 Hardy–Sobolev Type Inequalities

2.1 Hardy–Sobolev Inequalities

It is well known that for any $1 < p < n$ and for all $u \in W_0^{1,p}(\mathbb{R}^n)$, the following Sobolev and Hardy inequalities hold, respectively,

$$\left(\int\limits_{\mathbb{R}^n} |u|^{p^*}\, dx\right)^{\frac{1}{p^*}} \leqslant \mathscr{C}_{n,p} \left(\int\limits_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}}, \tag{33}$$

$$\int\limits_{\mathbb{R}^n} \frac{|u|^p}{|x|^p} dx \leqslant \left(\frac{p}{n-p}\right)^p \int\limits_{\mathbb{R}^n} |\nabla u|^p\, dx, \tag{34}$$

where $W_0^{1,p}(\mathbb{R}^n)$ is the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|^p_{W^{1,p}(\mathbb{R}^n)} = \int\limits_{\mathbb{R}^n} |\nabla u|^p\, dx + \int\limits_{\mathbb{R}^n} |u|^p\, dx.$$

Interpolating between these two inequalities, i.e. (33) and (34), we obtain the Hardy–Sobolev inequality which is a particular case of the family of functional inequalities obtained by Caffarelli et al. [11]. In this case the Hardy–Sobolev Inequality states as follows:

$$\left(\int\limits_{\mathbb{R}^n} |u|^{p^*(s)} |x|^{-s} dx\right)^{\frac{1}{p^*(s)}} \leqslant \left(\frac{p}{n-p}\right)^{\frac{p}{p^*(s)}} \mathscr{C}_{n,p}^{\frac{n(p-s)}{p(n-s)}} \left(\int\limits_{\mathbb{R}^n} |\nabla u|^p dx\right)^{\frac{1}{p}}, \tag{35}$$

for all $u \in W_0^{1,p}(\mathbb{R}^n)$, (i.e., see in [16]).

We can extend now the inequality (35) for any domain Ω in $\mathbb{R}^n$ (see [35], Theorem 15.1.1). Then for any $p \in (1, n)$, $s \in (0, p)$ and $p^*(s) = p\frac{n-s}{n-p}$, there exists a constant $C(p, s, \Omega) > 0$ such that

$$\left(\int\limits_{\Omega} |u|^{p^*(s)} |x|^{-s} dx\right)^{\frac{1}{p^*(s)}} \leqslant C(p, s, \Omega) \left(\int\limits_{\Omega} |\nabla u|^p dx\right)^{\frac{1}{p}} \text{ for all } u \in W_0^{1,p}(\Omega). \tag{36}$$

In addition, if Ω is bounded, then the inequality holds with $p^*(s)$ replaced by any q with $p \leq q \leq p^*(s)$.

Note that, for $s = 0$ (resp., $s = p$), (36) is just the Sobolev (resp., the Hardy) inequality. We therefore have to only consider the case where $0 < s < p$.

Regarding the best constants, it is well known that the cases $s = 0$ and $s = p$ have been studied extensively during the last few years. At this time we will give the meaning of the critical exponent. The exponent $p^*(s) = \frac{p(n-s)}{n-p}$ is critical in the following sense:

The Sobolev space $W_0^{1,p}(\Omega)$ is continuously embedded in the weighted Lebesgue space $\left(L^p(\Omega), |x|^{-s}\right)$ (i.e., the space of all functions $u : \Omega \to \mathbb{R}$ such that $|u|^p \cdot |x|^{-s} \in L^1(\Omega)$) if and only if $1 \le p \le p^*(s)$, and the embedding is compact if and only if $1 \le p < p^*(s)$. These results are derived directly from the generalizations of Theorems 3 and 4 (see Sect. 3 bellow) in arbitrary bounded domains in R^n.

We will now define the best Hardy–Sobolev constant $\mu_s(\Omega)$ to all cases, i.e. for any $s \in [0, p]$,

$$\mu_s(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^{p^*(s)}|x|^{-s}dx\right)^{\frac{p}{p^*(s)}}};\ u \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

By definition it follows that $\mu_s(\Omega) > 0$. At this point, in the spirit of Robert [58], we present some known results concerning the value of $\mu_s(\Omega)$ and the existence of extremals for it. In this case, i.e. when $s \in (0, p)$ the extremals functions are of the form (see Theorem 15.2.2 [35])

$$u^\alpha(x) = \left[\alpha(n-s)\left(\frac{n-p}{p-1}\right)^{p-1}\right]^{\frac{n-p}{p(p-s)}} \left(\alpha + |x|^{\frac{p-s}{p-1}}\right)^{-\frac{n-p}{p-s}}, \tag{37}$$

where α is any positive real number.

In this case, when $s \in (0, p)$, it is remarkable that the Hardy–Sobolev inequality inherits the singularity at 0 from the Hardy inequality and the critical exponent from the Sobolev inequality. Due to the singularity at 0, the situation will depend on the location of 0 with respect to Ω. However, concerning general open subsets of $\mathbb{R}^n$ so that $0 \in \Omega$, for all Ω open subsets of $\mathbb{R}^n$ holds

$$\mu_s(\Omega) = \mu_s(\mathbb{R}^n). \tag{38}$$

This last assertion can be proved by a direct generalization of the studied case by Robert (see [58]). In addition, if there is an extremal for $\mu_s(\Omega)$, then it is also an extremal for $\mu_s(\mathbb{R}^n)$. In particular, there is no extremal for $\mu_s(\Omega)$ if Ω is bounded.

When $0 \notin \bar{\Omega}$ and Ω is bounded, then $\left(L^{p^*(s)}(\Omega), |x|^{-s}\right) = L^{p^*(s)}(\Omega)$ and since $1 \le p^*(s) < \frac{np}{n-p}$ the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*(s)}(\Omega)$ is compact and thus, by the standard minimization methods we can prove the existence of extremals for $\mu_s(\Omega)$, however, we cannot find the exact value of it in general.

The remaining case $0 \in \partial\Omega$ is the most complicated, but particularly interesting. In this case, since 0 belongs to the boundary $\partial\Omega$ of Ω we can "roughly" say that, around 0, Ω looks like the half space $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}$ and not the whole space $\mathbb{R}^n$ as it happens in the case where $0 \in \Omega$. This ascertainment obliges us to compare the optimal constant $\mu_s(\Omega)$ with $\mu_s(\mathbb{R}^n_-)$ and not with $\mu_s(\mathbb{R}^n)$, which means that we need to take $\Omega \subset \mathbb{R}^n_-$. Indeed, for a such Ω, mimicking and generalizing the arguments used by Robert in [58] we can obtain that $\mu_s(\Omega) \leq \mu_s(\mathbb{R}^n_-)$, and by definition of $\mu_s(\Omega)$ arises that $\mu_s(\Omega) \geq \mu_s(\mathbb{R}^n_-)$. Thus, for $\Omega \subset \mathbb{R}^n_-$,

$$\mu_s(\Omega) = \mu_s(\mathbb{R}^n_-). \tag{39}$$

Moreover, if Ω is bounded, then there is no extremal for $\mu_s(\Omega)$. Assuming that Ω is a subset of half the space $\mathbb{R}^n_-$ immediately we put a hypothesis of convexity for Ω at 0. In particular, this hypothesis is satisfying for balls.

Remaining in the same case when $0 \in \partial\Omega$ but in the special case $p = 2$, we present some important results as well as some very useful comments by Robert (collected from [58]), without convexity assumptions on Ω, but strongly influenced by geometry.

Egnell in [24] proved that if $C = \{r\theta : r > 0, \theta \in D\}$ is the cone based at 0 induced by D, where D is a nonempty connected domain (not necessarily smooth at 0) of $\mathbb{S}^{n-1}$, the unit sphere in $\mathbb{R}^n$, then there are extremals for $\mu_s(C)$. Moreover, it is proved that there are extremals for $\mu_s(\mathbb{R}^n_-)$, but we do not know the value of $\mu_s(\mathbb{R}^n_-)$.

Ghoussoub and Kang in [34] proved that if Ω is a smooth bounded domain of $\mathbb{R}^n$ with $0 \in \partial\Omega$ and such that $\mu_s(\Omega) < \mu_s(\mathbb{R}^n_-)$, then there are extremals for $\mu_s(\Omega)$. Also, in the same paper, it is proved that for a such Ω, if the principal curvatures at 0 are all negative (i.e., Ω is locally concave at 0) and $n \geq 4$, then there are extremals for $\mu_s(\Omega)$.

Ghoussoub and Robert in [36] proved that if Ω is a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \partial\Omega$ and if the mean curvature of $\partial\Omega$ at 0 is negative and $n \geq 3$, then there are extremals for $\mu_s(\Omega)$. These last results clearly include the immediate preceding. Qualitatively, the last result tells us that there are extremals for $\mu_s(\Omega)$ when the domain is rather concave than convex at 0 in the sense that the negative principal directions dominate quantitatively the positive principal directions. This allows us to exhibit new examples either convex or concave for which the extremals exist. Note that this result does not tell anything about the value of the best constant.

Hashizume in [40] proved that the size of domain affects the attainability of $\mu_s(\Omega)$. More precisely it is proved that if $\partial\Omega$ has a smoothness which the Sobolev embeddings hold and if Ω is sufficiently small, i.e., such that

$$|\Omega| \left(\int_\Omega |x|^{-s} dx \right)^{-\frac{2}{2^*(s)}} \leq \mu_s(\Omega), \tag{40}$$

then $\mu_s(\Omega)$ is attained, where $|\Omega|$ is the n-dimensional Lebesgue measure of domain Ω. In addition, it is proved that there is a positive constant M which depends on only Ω such that $\mu_s(r\Omega)$ is never attained if $r > M$.

Remark 1 We note here that Filippas, Maz'ya, and Tertikas in [28] studied the following inequality (the so-called Hardy–Sobolev–Maz'ya inequality) (see in [53], Edition 1985, Corollary 3, p. 97)

$$\int\limits_{\mathbb{R}^n} |\nabla u|^2 dx - \left(\frac{k-2}{2}\right)^2 \int\limits_{\mathbb{R}^n} \frac{|u(x)|^2}{d^2} dx \geq C \left(\int\limits_{\mathbb{R}^n} |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{n}}, \tag{41}$$

for all $u \in C_0^\infty(\mathbb{R}^n \backslash K)$, where $K = \{x \in \mathbb{R}^n : x_1 = x_2 = \cdots = x_k = 0\}$, $1 \leq k \leq n-1$ and $d(x) = \mathrm{dist}(x, K)$.

The Hardy–Sobolev–Maz'ya inequality has been not only the motivation but also the basis for a particularly extensive study in functional inequalities and in the PDEs.

2.2 Fractional Hardy–Sobolev Inequalities

In this subsection some new representative results on Fractional Hardy–Sobolev inequalities are presented, which can be used as a basis for further study in the area.

Yang in [66] showed that the minimizing problem

$$\Lambda_{\sigma,\alpha} = \inf_{u \in H_0^\sigma(\mathbb{R}^n),\ u \not\equiv 0} \frac{\int_{\mathbb{R}^n} \left|(-\Delta)^{\frac{\sigma}{2}} u(x)\right|^2 dx}{\left(\int_{\mathbb{R}^n} \frac{|u(x)|^{2^*_{\sigma,\alpha}}}{|x|^\alpha} dx\right)^{\frac{2}{2^*_{\sigma,\alpha}}}} \tag{42}$$

is achieved by a positive, radially symmetric, and strictly decreasing function provided $0 < \sigma < \frac{n}{2}$, $0 < \alpha < 2\sigma$ and $2^*_{\sigma,\alpha} = \frac{n-\alpha}{n-2\sigma}$.

Marano and Mosconi in [51] established the existence of optimizers u in the space $W^{\sigma,p}(\mathbb{R}^n)$, with differentiability order $\sigma \in (0,1)$ for the Hardy–Sobolev inequality through concentration-compactness. In particular, the scale-invariant, nonlocal functional inequality

$$\left(\int\limits_{\mathbb{R}^n} \frac{|u(x)|^q}{|x|^\alpha} dx\right)^{\frac{1}{q}} \leqslant C \left(\iint\limits_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x-y|^{n+p\sigma}} dx dy\right)^{\frac{1}{p}} \tag{43}$$

for some constant $C > 0$. Here, $n > \alpha \geq 0$, $q > p \geq 1$, and $\sigma \in (0,1)$ are determined by scale invariance. The asymptotic behavior $u(x) \simeq |x|^{-\frac{n-p\sigma}{p-1}}$ as

$|x| \to +\infty$ and the summability information $u \in W_0^{\sigma,\gamma}(\mathbb{R}^n)$ for all $\gamma \in \left(\frac{n(p-1)}{n-\sigma}, p\right]$ are then obtained.

We refer here to Sobolev–Hardy inequality considered by Frank, Lieb, and Seiringer in [32]. Firstly, we need to define the form

$$h_s[u] = \int_{\mathbb{R}^n} |\xi|^{2s} \|\hat{u}(\xi)\|^2 d\xi - \mathscr{C}_{s,n} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2s}} dx, \tag{44}$$

where the constant $\mathscr{C}_{s,n}$ is as defined by (27). It should be noted here that the fractional exponent (see (20)) in the minuend of the second part of (44) is the same as the exponent of the weight the subtrahend in the same part of it.

We present now two theorems concerning the local and the global Sobolev–Hardy inequalities.

Theorem 1 (Local Sobolev–Hardy Inequality ([32], Theorem 2.3)) *Let* $0 < s < \min\{1, \frac{n}{2}\}$ *and* $1 \le q < 2^* = \frac{2n}{n-2s}$. *Then there exists a constant* $C_{q,n,s} > 0$ *such that for any domain* $\Omega \subset \mathbb{R}^n$ *with finite measure* $|\Omega|$ *one has*

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \, dx\right)^{\frac{2}{q}} \leqslant C_{q,n,s} \, |\Omega|^{2\left(\frac{1}{q}-\frac{1}{2^*}\right)} h_s[u], \quad u \in C_0^\infty(\Omega). \tag{45}$$

Note that the exponent q is strictly smaller than the critical exponent $2^* = \frac{2n}{n-2s}$. We refer that the analogue of Theorem 1 in the local case $s = 1$ is proved in [10].

Theorem 2 (Global Sobolev–Hardy Inequality ([32], Corollary 2.5)) *Let* $0 < s < \min\{1, \frac{n}{2}\}$ *and* $1 \le q < 2^* = \frac{2n}{n-2s}$. *Then there exists a constant* $C'_{q,n,s} > 0$ *such that*

$$\left(\int_{\mathbb{R}^n} |u(x)|^q \, dx\right)^{\frac{2}{q}} \leqslant C'_{q,n,s} \left(h_s[u] + \int_{\mathbb{R}^n} |u(x)|^2 \, dx\right), \quad u \in C_0^\infty\left(\mathbb{R}^n\right). \tag{46}$$

3 Hardy–Sobolev Inequalities: The Influence of Symmetries

In this part, we are interested in the case $s \in (0, p)$ with supercritical exponent, in particular with the critical of the supercritical one. For this purpose, it is necessary to incorporate the symmetry of Ω into the analysis. So, we consider the optimal Hardy–Sobolev inequalities on smooth bounded domains of the Euclidean space in the presence of symmetries. Our model domain is the solid torus because of its particular interest in terms both of the geometry and of the analysis, however we could also study the cases of the spheroid or the cylinder as well as the punctured ones.

The prevailing perception about the concept of symmetry is that this has to do with the shape of an object in the space or the shape of the boundary of a domain and generally with the sense of the beauty, of the perfection. This view has been expressed by the ancient Greeks and dominates as perception even in mathematical texts sometimes directly and other times indirectly. In this research we will try to make clear that the symmetry has a deeper and more meaningful sense. Specifically, in the case of Euclidean domains, the symmetry property is intrinsic characterizing the whole domain, and what we see traditionally called as symmetry is nothing but the effect of it in the appearance of the boundary of the domain. So, any classification, from a geometrical point, of the various domains based on the shape of their boundaries specifically is interesting (i.e., the curvature), but in terms of the analysis, especially of the PDEs does not offer something substantial as it referred in the boundary, namely in a set of zero measure, and not in the domain itself consistently disregarded the internal structure of the whole domain.

Let T be the solid ring torus in $\mathbb{R}^3$ with minor radius r and major radius R. This is the "doughnut-shaped" domain generated by rotating a disk of radius r about a co-planar axis at a distance R from the center of the disk, and it is represented by

$$T = \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \left(\sqrt{x_1^2 + x_2^2} - R \right)^2 + x_3^2 < r^2, \;\; R > r > 0 \right\}.$$

Before we explain the symmetry the torus presents we introduce some background material from the geometry. Consider a group G acting on a set X. The *orbit* of a point x in X is the set of elements of X to which x can be moved by the elements of G. (Just as gravity moves a planet around its orbit, the group action moves an element around its orbit.)

A brief explanation concerning the symmetry presented by the torus we are working as follows: Consider an arbitrary plane Π containing the axis $x_3'x_3$ which forms with the positive semi-axis Ox_1 angle θ, $\theta \in \mathbb{R}$, and the interval $I = [\theta, \theta + 2\pi)$, (or $I = (\theta, \theta + 2\pi]$). We also denote D the unit disk centered on the beginning of the axes and consider the transformation $\xi : T \setminus \{T \cap \Pi\} \to I \times D$, defined to be $\xi(x) = \xi(x_1, x_2, x_3) = (\omega, t_1, t_2)$, with $\omega \in I$ and such that for any function u defined on T we define the function $\phi = u \circ \xi^{-1}$, which does not depend on the variable ω, i.e. it holds that

$$\phi(t) = \phi(t_1, t_2) = (u \circ \xi^{-1})(\omega, t_1, t_2). \tag{47}$$

Because of the C-symmetry of the torus, each point P of it belongs to only one orbit. Thus, by rotating the torus around the $x_3'x_3$ axis all the points of this orbit will pass through the point P. Thus, if we know the value of a function u at the point P, then we know that at each point of this orbit the value of u remains the same as at P. On the other hand, we consider that the space is isotropic and then the orientation does not play a role. Thus, when we refer to points of the torus T, we may identify each $P \in T$ with its image on the disk $D_r = \{(\mu, \nu) \in \mathbb{R}^2 : \mu^2 + \nu^2 < r^2\}$, the intersection of an arbitrary plane Π containing the axis $x_3'x_3$ with the torus T. Therefore, when we refer to points of the torus T whether they are in its interior or on its boundary we can assume that they belong to the interior of the disk D_r or on its boundary. However, for calculational purposes of convenience only in the definition of the transformation ξ we identify the points of the torus T with the points of the unit disk D and not with the points of the disk D_r. So, in the following when we refer to points of the torus T, we can assume that they belong to the disk D. Due to the above, we need to use functions whose values do not depend on the orientation in the x_1x_2-plane and therefore must be of the form

$$u(x_1, x_2, x_3) = u(\sqrt{x_1^2 + x_2^2},\ 0,\ x_3) = u(0, \sqrt{x_1^2 + x_2^2},\ x_3). \tag{48}$$

Note that these functions play the same role for the torus as the radial functions do for the sphere or for the whole isotropic Euclidian space (for more details, see in [13] or [14] and in [41] for further consideration). Thus, according to the above analysis, the solid torus $\overline{T} = T \cup \partial T$ is invariant under the action of the subgroup $G = O(2) \times I$ of the isometry group $O(3)$ and so we must rely on spaces containing functions which are invariant under the action of G.

Let $W^{1,p}(T)$, $p \geq 1$, be the classical Sobolev space, and $W_0^{1,p}(T)$ be the closure of $C_0^\infty(T)$ in $W^{1,p}(T)$. Since the solid torus T is an open bounded domain in $\mathbb{R}^3$ and its boundary is smooth, in order to study the Hardy–Sobolev inequality it seems that the suitable functional space to be used is $W^{1,p}(T)$. We denote by, $W^{1,p}(T) = H_1^p(T)$, where $H_1^p(T)$ is the completion of $C^\infty(T)$ with respect to the norm

$$\|u\|^p_{H_1^p(T)} = \int_T |\nabla u|^p\, dx + \int_T |u|^p\, dx.$$

Therefore, it would be natural to work in $H_1^p(T)$. However, as referred above the solid torus $\overline{T} = T \cup \partial T$ is invariant under the action of the group $G = O(2) \times I$ and so we must rely on spaces containing functions which are invariant under the action of G. Thus, the largest and most suitable space of functions that must be used in the case of the solid torus is the one that contains the G-invariant functions, i.e. the Sobolev space

$$H^p_{1,G}(T) = \left\{u \in H_1^p(T) : u \circ g = u\,,\ \forall\, g \in G\right\}.$$

We refer here two lemmas which are the keys to incorporate the geometry into the analysis. (For the proofs see in [14]).

Lemma 1 *For any $x, \in T$, holds $|x| = r|t|$, where $|x|$ is the distance of the point $x \in T$ to the circle $C_R = \{x = (x_1, x_2, 0) \in T : x_1^2 + x_2^2 = R^2\}$.*

Lemma 2 *For any function $u \in H^p_{1,G}(T)$, the following equalities hold:*

$$\int_T |u|^p dx = 2\pi r^2 \int_D |\phi|^p \, (R + rt_1)\, dt, \tag{49}$$

$$\int_T |u|^p |x|^{-s} dx = 2\pi r^{2-s} \int_D |\phi|^p |t|^{-s} \, (R + rt_1)\, dt, \tag{50}$$

$$\int_T |\nabla u|^p dx = 2\pi r^{2-p} \int_D |\nabla\phi|^p \, (R + rt_1)\, dt, \tag{51}$$

where the function ϕ is as defined by (47), i.e. $\phi(t) = \phi(t_1, t_2) = (u \circ \xi^{-1})(\omega, t_1, t_2)$.

We need now to present two basic theorems (the proofs are presented in [14]).

Theorem 3 (Sobolev Embedding Theorem) *Let T be a solid torus, $1 \le p < 2$, $0 \le s \le p$ and $p^*(s) = \frac{p(2-s)}{2-p}$. Then, the embedding $H^{1,p}_G(T) \hookrightarrow \left(L^q(T), |x|^{-s}\right)$ is continuous for all $q \in [1, p^*(s)]$.*

Theorem 4 (Kondrakov Embedding Theorem) *Let T be a solid torus, $1 \le p < 2$, $0 \le s < p$ and $p^*(s) = \frac{p(2-s)}{2-p}$. Then, the embedding $H^{1,p}_G(T) \hookrightarrow \left(L^q(T), |x|^{-s}\right)$ is compact for all $q \in [1, p^*(s))$.*

The proofs (see in [14]) of the Theorems 3 and 4 based on the Theorems 1.1.1 and 1.1.2 of [45]. In fact, these two theorems regarding the Sobolev spaces constitute a generalization of the aforementioned theorems and for the domains a limination of them.

We are now in a position to explain why the exponent $p^*(s)$ is supercritical. As we have mentioned in the previous paragraph the Sobolev space $W^{1,p}_0(\Omega)$ is continuously embedded in the weighted Lebesgue space $\left(L^p(\Omega), |x|^{-s}\right)$ if and only if $1 \le p \le p^*(s)$, and the embedding is compact if and only if $1 \le p < p^*(s)$. Hence, the critical exponent for this embedding in the n-dimensional case is

$$p^*(s) = \frac{p(n-s)}{n-p}.$$

Thus, if $\Omega \in \mathbb{R}^3$ the critical exponent is

$$p^*(s) = \frac{p(3-s)}{3-p}.$$

As it was mentioned above, the symmetry presented by the torus allows us to take it into consideration, and due to Lemma 2 it becomes absolutely clear that eventually we will have to work in two dimensions. Hereafter, if $1 < p < 2$, $0 \le s \le p$ then the critical exponent of the Sobolev embedding $H^{1,p}_G(T) \hookrightarrow \left(L^q(T), |x|^{-s}\right)$ is

$$p^*(s) = \frac{p(2-s)}{2-p}$$

and since

$$p^*(s) = \frac{p(2-s)}{2-p} > \frac{p(3-s)}{3-p},$$

arises that it is a *supercritical* exponent and indeed the *critical of the supercritical* one.

Under the above considerations, we improved the best constant in the inequality (36) for $0 < s < p$ exploiting the symmetries that the domain Ω presents. More precisely, we established the Hardy–Sobolev inequality using as a model domain the solid torus T, and we calculated (see [14]) its best constant to be equal to

$$\begin{aligned}
[\mu_{s,G}(T)]^{-1} &= \frac{1}{[2\pi(R-r)]^{\frac{p-s}{p(2-s)}}} \left[\mu_s\left(\mathbb{R}^2\right)\right]^{-1} \\
&= \frac{1}{[2\pi(R-r)]^{\frac{p-s}{p(2-s)}}} \left(\frac{p}{2-p}\right)^{\frac{s}{p^*(s)}} \mathscr{C}_{2,p}^{\frac{2(p-s)}{p(2-s)}},
\end{aligned}$$

where $2\pi(R-r)$ is the length of the orbit with minimum radius $R-r$.

We note here that in [12] we calculated the best constant in the case of the solid torus T for $s = 0$. In particular, we proved that for any $p \in (1, 2)$ the best constant in the resulting supercritical Sobolev inequality is equal to

$$\mu_0(T) = \left(\frac{\mathscr{C}_{2,p}}{\sqrt{2\pi(R-r)}}\right)^{-1}.$$

We, also, proved in [13] that if $s = p$ then

$$\mu_p(T) = \left(\frac{p-1}{p}\right)^p,$$

i.e., the best constant in the Hardy inequality is the same as the standard Hardy best constant which appears in convex domains although the solid torus has no convex boundary but it has all kinds of curvature. In this research our results are obtained without any assumption concerning the "shape" (i.e., some convexity) of the boundary and both of the used techniques that exploit the symmetry presented by the solid torus and the clarity of the results confirm that:

The symmetry of a domain is an intrinsic property which determines its structure and characterizes both the interior and its boundary. Regarding the "shape" of the boundary, it is also determined by the symmetry of the domain and is not the one that can determine the behavior of the whole domain.

Our first result concerns the supercritical Hardy–Sobolev inequality in the case of the solid torus. (All results and proofs of this part are presented in detail in [14]). Firstly, we have to prove the following proposition.

Proposition 1 *Let T be the 3-dimensional solid torus, $1 < p < 2$, $0 \leq s < p$ and $p^*(s) = \frac{p(2-s)}{2-p}$. Then, for any $\varepsilon > 0$ there exists $B = B(p, \varepsilon) > 0$ such that for all $u \in H^p_{1,G}(T)$,*

$$\left(\int_T |u|^{p^*(s)} |x|^{-s} dx \right)^{\frac{1}{p^*(s)}} \leqslant \left[[\mu_{s,G}(T)]^{-1} + \varepsilon \right] \left(\int_T |\nabla u|^p dx \right)^{\frac{1}{p}} + B \left(\int_T |u|^p dx \right)^{\frac{1}{p}}. \tag{52}$$

In addition,

$$(\mu_{s,G}(T))^{-1} = \frac{1}{[2\pi(R-r)]^{\frac{p-s}{p(2-s)}}} \left(\frac{p}{2-p} \right)^{\frac{s}{p^*(s)}} \mathscr{C}_{2,p}^{\frac{2(p-s)}{p(2-s)}}$$

is the best constant for this inequality.

We address now the following question: If we set $\varepsilon = 0$ to (52) does this inequality remain valid? Concerning this question we note that the parameter ε that appeared in Proposition 1 controls in some sense the thinness of the cover that we use in each case through the related partition of unity. Thus, its existence is absolutely necessary because we do not know if the inequality is valid without this parameter. Although in some cases, Sobolev inequalities exist without ε (see [12, 18, 25, 42]), but in general we cannot make it disappear. For instance, regarding the classical Sobolev inequality, Aubin conjectured a positive answer. This conjecture was first proved for $p = 2$ by Hebey and Vaugon [42], and for any p by Druet [18] and by Aubin and Li [3]. On Riemannian manifolds without boundary in the presence of symmetries a positive answer is given by Faget [25]. In the case of the solid torus a positive answer is, also, given by Cotsiolis and Labropoulos [12]. Unfortunately, in our case it does not seem possible to use the same arguments as those used in the above cases. However, we can give a possible positive answer to the above question in the following sense: We cannot find $\varepsilon > 0$, arbitrarily small such that the inequality (52) to remain valid for all $u \in H^p_{1,G}(T)$. In particular, we can state the following theorem:

Theorem 5 *Let T be the 3-dimensional solid torus, $1 < p < 2$, $0 \leq s < p$ and $p^*(s) = \frac{p(2-s)}{2-p}$. Then, there exists $B = B(p) > 0$ such that for all $u \in H^p_{1,G}(T)$,*

$$\left(\int_T |u|^{p^*(s)} |x|^{-s} dx \right)^{\frac{p}{p^*(s)}} \leqslant [\mu_{s,G}(T)]^{-p} \int_T |\nabla u|^p dx + B \int_T |u|^p dx. \tag{53}$$

In addition, $(\mu_{s,G}(T))^{-p}$ is the best constant for this inequality.

The Hardy–Sobolev inequalities with supercritical exponent allow us to solve nonlinear elliptic problems with supercritical nonlinearity, i.e as the following

$$\text{(P)}\ \Delta_p u + a(x)u^{p-1} = f(x)\frac{u^{p^*(s)-1}}{|x|^s},\ u > 0\ \text{ on } T,\quad u = 0\ \text{ on } \partial T,$$
$$1 < p < 2,\quad 0 \leq s \leq p\ \text{ and }\ p^*(s) = \frac{p(2-s)}{2-p},$$

as well as some variants of it (i.e., see [14]).

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Two Points Taylor's Type Representations with Integral Remainders

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Abstract In this chapter we establish some two points Taylor's type representations with integral remainders and apply them for the logarithmic and exponential functions. Some inequalities for weighted arithmetic and geometric means are provided as well.

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1 Introduction

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder.

Theorem 1 *Let $I \subset \mathbb{R}$ be a closed interval, $c \in I$ and let n be a positive integer. If $f : I \longrightarrow \mathbb{C}$ is such that the n-derivative $f^{(n)}$ is absolutely continuous on I, then for each $y \in I$*

$$f(y) = T_n(f; c, y) + R_n(f; c, y), \tag{1.1}$$

where $T_n(f; c, y)$ is Taylor's polynomial, i.e.,

$$T_n(f; c, y) := \sum_{k=0}^{n} \frac{(y-c)^k}{k!} f^{(k)}(c). \tag{1.2}$$

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Note that $f^{(0)} := f$ *and* $0! := 1$ *and the remainder is given by*

$$R_n(f; c, y) := \frac{1}{n!} \int_c^y (y-t)^n f^{(n+1)}(t)\, dt. \tag{1.3}$$

A simple proof of this theorem can be achieved by mathematical induction using the integration by parts formula in the Lebesgue integral.

For related results, see [1–5, 11–14, 18, 19] and [22].

Let $a, b > 0$, then we have the equality:

$$\ln b - \ln a = \sum_{k=1}^{n} \frac{(-1)^{k-1} (b-a)^k}{k a^k} + (-1)^n \int_a^b \frac{(b-t)^n}{t^{n+1}} dt, \ n \geq 1. \tag{1.4}$$

Indeed, if we consider the function $f : (0, \infty) \longrightarrow \mathbb{R}$, $f(x) = \ln x$, then,

$$f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}, \ n \geq 1, \ x > 0,$$

$$T_n(\ln; a, x) = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1} (x-a)^k}{k a^k}, \ a > 0$$

and

$$R_n(\ln; a, x) = (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}} dt.$$

Now, using (1.1), we have the equality,

$$\ln x = \ln a + \sum_{k=1}^{n} \frac{(-1)^{k-1} (x-a)^k}{k a^k} + (-1)^n \int_a^x \frac{(x-t)^n}{t^{n+1}} dt.$$

Choosing in the last equality $x = b$, we get (1.4).

Consider the function $f : \mathbb{R} \longrightarrow (0, \infty)$, $f(y) = \exp y$. Then for any $c \in \mathbb{R}$ we have

$$T_n(\exp; c, y) = \sum_{k=0}^{n} \frac{(y-c)^k}{k!} \exp c$$

and

$$R_n(\exp; c, y) := \frac{1}{n!} \int_c^y (y-t)^n \exp t\, dt.$$

On applying Taylor's formula (1.1) we have

$$\exp y - \exp c - \sum_{k=1}^{n} \frac{(y-c)^k}{k!} \exp c = \frac{1}{n!} \int_c^y (y-t)^n \exp t dt \tag{1.5}$$

for any $c,\ y \in \mathbb{R}$.

If we take $y = \ln x,\ c = \ln a$ where $x,\ a > 0$, then we get

$$x - a - a \sum_{k=1}^{n} \frac{(\ln x - \ln a)^k}{k!} = \frac{1}{n!} \int_{\ln a}^{\ln x} (\ln x - t)^n \exp t dt.$$

By using the change of variable, $s = \exp t$, we have

$$\int_{\ln a}^{\ln x} (\ln x - t)^n \exp t dt = \int_a^x (\ln x - \ln s)^n ds$$

giving that

$$b - a - a \sum_{k=1}^{n} \frac{(\ln b - \ln a)^k}{k!} = \frac{1}{n!} \int_a^b (\ln b - \ln s)^n ds, \tag{1.6}$$

for any $b,\ a > 0$.

Now, if $n \geq 2$, then by (1.6) we have

$$\frac{b-a}{a} - \sum_{k=1}^{n} \frac{(\ln b - \ln a)^k}{k!} = \frac{1}{n!a} \int_a^b (\ln b - \ln s)^n ds,$$

namely

$$\ln b - \ln a = \frac{b-a}{a} - \sum_{k=2}^{n} \frac{(\ln b - \ln a)^k}{k!} - \frac{1}{n!a} \int_a^b (\ln b - \ln s)^n ds, \tag{1.7}$$

for any $b,\ a > 0$.

By taking in (1.4) and (1.7) $n = 2m + 1$, we get the following equalities of interest

$$\ln b - \ln a = \sum_{k=1}^{2m+1} \frac{(-1)^{k-1} (b-a)^k}{ka^k} - \int_a^b \frac{(b-t)^{2m+1}}{t^{2m+2}} dt,\ m \geq 0 \tag{1.8}$$

and

$$\ln b - \ln a = \frac{b-a}{a} - \sum_{k=2}^{2m+1} \frac{(\ln b - \ln a)^k}{k!} - \frac{1}{(2m+1)!a} \int_a^b (\ln b - \ln s)^{2m+1} ds, \ m \geq 1. \tag{1.9}$$

Since for any a, $b > 0$

$$\int_a^b \frac{(b-t)^{2m+1}}{t^{2m+2}} dt \geq 0 \text{ and } \int_a^b (\ln b - \ln s)^{2m+1} ds \geq 0,$$

then we have from (1.8) that

$$\ln b - \ln a \leq \frac{b-a}{a} + \sum_{k=2}^{2m+1} \frac{(-1)^{k-1} (b-a)^k}{ka^k}, \ m \geq 1 \tag{1.10}$$

and from (1.9) that

$$\ln b - \ln a \leq \frac{b-a}{a} - \sum_{k=2}^{2m+1} \frac{(\ln b - \ln a)^k}{k!}, \ m \geq 1. \tag{1.11}$$

The case $m = 1$ provides the following inequalities

$$\ln b - \ln a \leq \frac{b-a}{a} - \frac{(b-a)^2}{2a^2} + \frac{(b-a)^3}{3a^3} \tag{1.12}$$

and

$$\ln b - \ln a \leq \frac{b-a}{a} - \frac{(\ln b - \ln a)^2}{2} - \frac{(\ln b - \ln a)^3}{6} \tag{1.13}$$

for any a, $b > 0$.

Now, if $0 < a < b$, then by (1.7) we have

$$\ln b - \ln a \leq \frac{b-a}{a} - \sum_{k=2}^{n} \frac{(\ln b - \ln a)^k}{k!} \tag{1.14}$$

for any $n \geq 2$.

If $0 < a < b$ and $n = 2m$, then by (1.4) we have

$$\ln b - \ln a \geq \sum_{k=1}^{2m} \frac{(-1)^{k-1} (b-a)^k}{ka^k}, \; m \geq 1 \tag{1.15}$$

while in the case $n = 2m + 1$ we have

$$\ln b - \ln a \leq \sum_{k=1}^{2m+1} \frac{(-1)^{k-1} (b-a)^k}{ka^k}, m \geq 0. \tag{1.16}$$

In this chapter we establish some two points Taylor's type representations with integral remainders and apply them for the logarithmic and exponential functions. Some inequalities for weighted arithmetic and geometric means are provided as well.

2 Some Two Points Identities

The following identity can be stated:

Theorem 2 *Let $f : I \to \mathbb{C}$ be n-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(n)}$, with $n \geq 1$, be locally absolutely continuous on $\mathring{I}$. Then for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in \mathbb{R} \setminus \{0, 1\}$ we have the representation*

$$\begin{aligned} f(x) &= (1-\lambda) f(a) + \lambda f(b) \\ &+ \sum_{k=1}^{n} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a) (x-a)^k + (-1)^k \lambda f^{(k)}(b) (b-x)^k \right] \\ &+ S_{n,\lambda}(x, a, b), \end{aligned} \tag{2.1}$$

where the remainder $S_{n,\lambda}(x, a, b)$ is given by

$$\begin{aligned} &S_{n,\lambda}(x, a, b) \\ &:= \frac{1}{n!} \left[(1-\lambda)(x-a)^{n+1} \int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n \, ds \right. \\ &\left. + (-1)^{n+1} \lambda (b-x)^{n+1} \int_0^1 f^{(n+1)}((1-s)x + sb) s^n ds \right]. \end{aligned} \tag{2.2}$$

Proof Using Taylor's representation with the integral remainder (1.1) we can write the following two identities:

$$f(x)=\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^{k}+\frac{1}{n!}\int_{a}^{x}f^{(n+1)}(t)(x-t)^{n}dt \tag{2.3}$$

and

$$f(x)=\sum_{k=0}^{n}\frac{(-1)^{k}}{k!}f^{(k)}(b)(b-x)^{k}+\frac{(-1)^{n+1}}{n!}\int_{x}^{b}f^{(n+1)}(t)(t-x)^{n}dt \tag{2.4}$$

for any $x, a, b \in \mathring{I}$.

For any integrable function h on an interval and any distinct numbers c, d in that interval, we have, by the change of variable $t=(1-s)c+sd$, $s\in[0,1]$ that

$$\int_{c}^{d}h(t)\,dt=(d-c)\int_{0}^{1}h((1-s)c+sd)\,ds.$$

Therefore,

$$\begin{aligned}&\int_{a}^{x}f^{(n+1)}(t)(x-t)^{n}dt\\&=(x-a)\int_{0}^{1}f^{(n+1)}((1-s)a+sx)(x-(1-s)a-sx)^{n}ds\\&=(x-a)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)a+sx)(1-s)^{n}ds\end{aligned}$$

and

$$\begin{aligned}&\int_{x}^{b}f^{(n+1)}(t)(t-x)^{n}dt\\&=(b-x)\int_{0}^{1}f^{(n+1)}((1-s)x+sb)((1-s)x+sb-x)^{n}ds\\&=(b-x)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)x+sb)s^{n}ds.\end{aligned}$$

The identities (2.3) and (2.4) can then be written as

$$\begin{aligned}f(x)=&\sum_{k=0}^{n}\frac{1}{k!}f^{(k)}(a)(x-a)^{k} &(2.5)\\&+\frac{1}{n!}(x-a)^{n+1}\int_{0}^{1}f^{(n+1)}((1-s)a+sx)(1-s)^{n}ds\end{aligned}$$

and

$$f(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(b)(b-x)^k \tag{2.6}$$
$$+ (-1)^{n+1} \frac{(b-x)^{n+1}}{n!} \int_0^1 f^{(n+1)}((1-s)x + sb)\, s^n ds.$$

Now, if we multiply (2.5) with $1-\lambda$ and (2.6) with λ and add the resulting equalities, a simple calculation yields the desired identity (2.1). □

Remark 1 If we take in (2.1) $x = \frac{a+b}{2}$, with $a, b \in \mathring{I}$,, then we have for any $\lambda \in \mathbb{R} \setminus \{0, 1\}$ that

$$f\left(\frac{a+b}{2}\right) = (1-\lambda) f(a) + \lambda f(b) \tag{2.7}$$
$$+ \sum_{k=1}^{n} \frac{1}{2^k k!} \left[(1-\lambda) f^{(k)}(a) + (-1)^k \lambda f^{(k)}(b)\right](b-a)^k$$
$$+ \tilde{S}_{n,\lambda}(a,b),$$

where the remainder $\tilde{S}_{n,\lambda}(a,b)$ is given by

$$\tilde{S}_{n,\lambda}(a,b) \tag{2.8}$$
$$:= \frac{1}{2^{n+1} n!}(b-a)^{n+1}\left[(1-\lambda)\int_0^1 f^{(n+1)}\left((1-s)a + s\frac{a+b}{2}\right)(1-s)^n ds\right.$$
$$\left.+ (-1)^{n+1}\lambda \int_0^1 f^{(n+1)}\left((1-s)\frac{a+b}{2} + sb\right) s^n ds\right].$$

In particular, for $\lambda = \frac{1}{2}$ we have

$$f\left(\frac{a+b}{2}\right) = \frac{f(a) + f(b)}{2} \tag{2.9}$$
$$+ \sum_{k=1}^{n} \frac{1}{2^{k+1} k!} \left[f^{(k)}(a) + (-1)^k f^{(k)}(b)\right](b-a)^k$$
$$+ \tilde{S}_n(a,b),$$

where the remainder $\tilde{S}_n(a,b)$ is given by

$$\begin{aligned}
&\tilde{S}_n(a,b) \qquad (2.10)\\
&:= \frac{1}{2^{n+2}n!}(b-a)^{n+1}\left[\int_0^1 f^{(n+1)}\left((1-s)a + s\frac{a+b}{2}\right)(1-s)^n\,ds\right.\\
&\left.+(-1)^{n+1}\int_0^1 f^{(n+1)}\left((1-s)\frac{a+b}{2} + sb\right)s^n ds\right].
\end{aligned}$$

Corollary 1 *With the assumptions in Theorem 2 we have for each distinct* x, a, $b \in \mathring{I}$

$$\begin{aligned}
f(x) &= \frac{1}{b-a}\left[(b-x)f(a) + (x-a)f(b)\right] + \frac{(b-x)(x-a)}{b-a} \qquad (2.11)\\
&\times \sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k-1}f^{(k)}(a) + (-1)^k(b-x)^{k-1}f^{(k)}(b)\right\}\\
&+ L_n(x,a,b),
\end{aligned}$$

where

$$\begin{aligned}
L_n(x,a,b) := \frac{(b-x)(x-a)}{n!(b-a)}&\left[(x-a)^n\int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n\,ds\right.\\
&\left.+(-1)^{n+1}(b-x)^n\int_0^1 f^{(n+1)}((1-s)x + sb)s^n ds\right]
\end{aligned}$$

and

$$\begin{aligned}
f(x) &= \frac{1}{b-a}\left[(x-a)f(a) + (b-x)f(b)\right] \qquad (2.12)\\
&+ \frac{1}{b-a}\sum_{k=1}^{n}\frac{1}{k!}\left\{(x-a)^{k+1}f^{(k)}(a) + (-1)^k(b-x)^{k+1}f^{(k)}(b)\right\}\\
&+ P_n(x,a,b),
\end{aligned}$$

where

$$\begin{aligned}
P_n(x,a,b) := \frac{1}{n!(b-a)}&\left[(x-a)^{n+2}\int_0^1 f^{(n+1)}((1-s)a + sx)(1-s)^n\,ds\right.\\
&\left.+(-1)^{n+1}(b-x)^{n+2}\int_0^1 f^{(n+1)}((1-s)x + sb)s^n ds\right],
\end{aligned}$$

respectively.

The proof is obvious. Choose $\lambda = (x-a)/(b-a)$ and $\lambda = (b-x)/(b-a)$, respectively, in Theorem 2. The details are omitted.

Corollary 2 *With the assumption in Theorem 2 we have for each* $\lambda \in [0,1]$ *and any distinct* $a, b \in \mathring{I}$ *that*

$$f((1-\lambda)a+\lambda b) = (1-\lambda) f(a) + \lambda f(b) + \lambda(1-\lambda) \times \sum_{k=1}^{n} \frac{1}{k!}\left[\lambda^{k-1} f^{(k)}(a) + (-1)^k (1-\lambda)^{k-1} f^{(k)}(b)\right](b-a)^k + S_{n,\lambda}(a,b), \tag{2.13}$$

where the remainder $S_{n,\lambda}(a,b)$ *is given by*

$$S_{n,\lambda}(a,b) := \frac{1}{n!}(1-\lambda)\lambda(b-a)^{n+1}\left[\lambda^n \int_0^1 f^{(n+1)}((1-s\lambda)a+s\lambda b)(1-s)^n ds + (-1)^{n+1}(1-\lambda)^n \int_0^1 f^{(n+1)}((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)s^n ds\right]. \tag{2.14}$$

We also have

$$f((1-\lambda)b+\lambda a) = (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{n} \frac{1}{k!}\left[(1-\lambda)^{k+1} f^{(k)}(a) + (-1)^k \lambda^{k+1} f^{(k)}(b)\right](b-a)^k + P_{n,\lambda}(a,b), \tag{2.15}$$

where the remainder $P_{n,\lambda}(a,b)$ *is given by*

$$P_{n,\lambda}(a,b) := \frac{1}{n!}(b-a)^{n+1}\left[(1-\lambda)^{n+2}\int_0^1 f^{(n+1)}((1-s+\lambda s)a+(1-\lambda)sb)(1-s)^n ds + (-1)^{n+1}\lambda^{n+2}\int_0^1 f^{(n+1)}((1-s)\lambda a+(1-\lambda+\lambda s)b)s^n ds\right]. \tag{2.16}$$

The case $n=0$, namely when the function f is locally absolutely continuous on $\mathring{I}$ with the derivative f' existing almost everywhere in $\mathring{I}$ is important and produces the following simple identities for each distinct $x, a, b \in \mathring{I}$ and $\lambda \in \mathbb{R}\setminus\{0,1\}$

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + S_\lambda(x,a,b), \tag{2.17}$$

where the remainder $S_\lambda(x,a,b)$ is given by

$$S_\lambda(x,a,b) := (1-\lambda)(x-a)\int_0^1 f'((1-s)a+sx)\,ds - \lambda(b-x)\int_0^1 f'((1-s)x+sb)\,ds. \tag{2.18}$$

We then have for each distinct $x,\ a,\ b \in \mathring{I}$

$$f(x) = \frac{1}{b-a}\left[(b-x)f(a)+(x-a)f(b)\right] + L(x,a,b), \tag{2.19}$$

where

$$L(x,a,b) := \frac{(b-x)(x-a)}{b-a}\left[\int_0^1 f'((1-s)a+sx)\,ds - \int_0^1 f'((1-s)x+sb)\,ds\right] \tag{2.20}$$

and

$$f(x) = \frac{1}{b-a}\left[(x-a)f(a)+(b-x)f(b)\right] + P(x,a,b), \tag{2.21}$$

where

$$P(x,a,b) := \frac{1}{b-a}\left[(x-a)^2\int_0^1 f'((1-s)a+sx)\,ds - (b-x)^2\int_0^1 f'((1-s)x+sb)\,ds\right]. \tag{2.22}$$

We also have

$$f((1-\lambda)a+\lambda b) = (1-\lambda)f(a)+\lambda f(b)+S_\lambda(a,b), \tag{2.23}$$

where the remainder $S_\lambda(a,b)$ is given by

$$S_\lambda(a,b) := (1-\lambda)\lambda(b-a)\left[\int_0^1 f'((1-s\lambda)a+s\lambda b)\,ds - \int_0^1 f'((1-s-\lambda+s\lambda)a+(\lambda+s-s\lambda)b)\,ds\right] \tag{2.24}$$

and

$$f((1-\lambda)b+\lambda a) = (1-\lambda)f(a)+\lambda f(b)+P_\lambda(a,b), \tag{2.25}$$

where the remainder $P_\lambda(a,b)$ is given by

$$P_\lambda(a,b) := (b-a)\left[(1-\lambda)^2 \int_0^1 f'((1-s+\lambda s)a + (1-\lambda)sb)\,ds \right.$$
$$\left. -\lambda^2 \int_0^1 f'((1-s)\lambda a + (1-\lambda+\lambda s)b)\,ds\right]. \tag{2.26}$$

Moreover, if we take in (2.17) $x = \frac{a+b}{2}$ for each distinct $a, b \in \mathring{I}$ and $\lambda \in \mathbb{R}\setminus\{0,1\}$,, then we have

$$f\left(\frac{a+b}{2}\right) = (1-\lambda) f(a) + \lambda f(b) + S_\lambda(a,b), \tag{2.27}$$

where the remainder $S_\lambda(a,b)$ is given by

$$S_\lambda(a,b) := \frac{1}{2}(b-a)$$
$$\times\left[(1-\lambda)\int_0^1 f'\left((1-s)a + s\frac{a+b}{2}\right)ds - \lambda\int_0^1 f'\left((1-s)\frac{a+b}{2} + sb\right)ds\right]. \tag{2.28}$$

In particular, for $\lambda = \frac{1}{2}$ we have

$$f\left(\frac{a+b}{2}\right) = \frac{f(a)+f(b)}{2} + S(a,b), \tag{2.29}$$

where

$$S(a,b) := \frac{1}{4}(b-a)$$
$$\times\left[\int_0^1 f'\left((1-s)a + s\frac{a+b}{2}\right)ds - \int_0^1 f'\left((1-s)\frac{a+b}{2} + sb\right)ds\right]. \tag{2.30}$$

3 Examples for Logarithm and Exponential

Consider the function $f : (0,\infty) \longrightarrow \mathbb{R},\ f(x) = \ln x$, then,

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n},\ n \geq 1,\ x > 0. \tag{3.1}$$

Using the identity (2.1) for this function we get for any $x, a, b > 0$ and $\lambda \in \mathbb{R}\backslash \{0, 1\}$ that

$$\ln x = (1-\lambda)\ln a + \lambda \ln b + \sum_{k=1}^{n} \frac{1}{k}\left[(-1)^{k-1}(1-\lambda)\left(\frac{x}{a}-1\right)^k - \lambda\left(1-\frac{x}{b}\right)^k\right] + U_{n,\lambda}(x,a,b), \tag{3.2}$$

where the remainder $U_{n,\lambda}(x,a,b)$ is given by

$$U_{n,\lambda}(x,a,b) := \left[(-1)^n (1-\lambda)(x-a)^{n+1}\int_0^1 \frac{(1-s)^n}{((1-s)a+sx)^n}ds - \lambda (b-x)^{n+1}\int_0^1 \frac{s^n}{((1-s)x+sb)^n}ds\right]. \tag{3.3}$$

Using the identity (2.7) for the function $f:(0,\infty)\longrightarrow \mathbb{R}$, $f(x)=\ln x$, then

$$\ln\left(\frac{a+b}{2}\right) = (1-\lambda)\ln a + \lambda \ln b + \sum_{k=1}^{n} \frac{1}{2^k k}\left[(-1)^{k-1}\frac{1-\lambda}{a^k} - \frac{\lambda}{b^k}\right](b-a)^k + U_{n,\lambda}(a,b), \tag{3.4}$$

where the remainder $U_{n,\lambda}(a,b)$ is given by

$$U_{n,\lambda}(a,b) := \frac{1}{2^{n+1}}(b-a)^{n+1}\left[(1-\lambda)\int_0^1 \frac{(-1)^n (1-s)^n}{\left((1-s)a+s\frac{a+b}{2}\right)^{n+1}}ds - \lambda\int_0^1 \frac{s^n}{\left((1-s)\frac{a+b}{2}+sb\right)^{n+1}}ds\right]. \tag{3.5}$$

In particular, for $\lambda = \frac{1}{2}$, we have for all $a, b > 0$ that

$$\ln\left(\frac{a+b}{2}\right) = \frac{\ln a + \ln b}{2} + \sum_{k=1}^{n} \frac{1}{2^{k+1} k}\left[\frac{(-1)^{k-1}}{a^k} - \frac{1}{b^k}\right](b-a)^k + U_n(a,b), \tag{3.6}$$

where the remainder $U_{n,\lambda}(a,b)$ is given by

$$U_n(a,b) \tag{3.7}$$
$$:= \frac{1}{2^{n+2}}(b-a)^{n+1}$$
$$\times \left[\int_0^1 \frac{(-1)^n (1-s)^n}{\left((1-s)a + s\frac{a+b}{2}\right)^{n+1}} ds - \int_0^1 \frac{s^n}{\left((1-s)\frac{a+b}{2} + sb\right)^{n+1}} ds\right].$$

From (2.13) we have for any $a, b > 0$ and $\lambda \in [0,1]$ that

$$0 \le \ln\left(\frac{(1-\lambda)a + \lambda b}{a^{1-\lambda}b^{\lambda}}\right) \tag{3.8}$$
$$= \lambda(1-\lambda)\sum_{k=1}^{n}\frac{1}{k}\left[\frac{(-1)^{k-1}\lambda^{k-1}}{a^k} - \frac{(1-\lambda)^{k-1}}{b^k}\right](b-a)^k$$
$$+ U_{n,\lambda}(a,b),$$

where the remainder $U_{n,\lambda}(a,b)$ is given by

$$U_{n,\lambda}(a,b) \tag{3.9}$$
$$:= (1-\lambda)\lambda(b-a)^{n+1}\left[\lambda^n\int_0^1 \frac{(-1)^n(1-s)^n}{((1-s\lambda)a + s\lambda b)^{n+1}} ds\right.$$
$$\left.-(1-\lambda)^n\int_0^1 \frac{s^n}{((1-s-\lambda+s\lambda)a + (\lambda+s-s\lambda)b)^{n+1}} ds\right].$$

Consider the function $f : \mathbb{R} \longrightarrow (0,\infty)$, $f(y) = \exp y$, then,

$$f^{(n)}(y) = \exp y,\ n \ge 1,\ x > 0. \tag{3.10}$$

If we write the equality (2.1) for this function we get for any $y, c, d \in \mathbb{R}$ and $\lambda \in \mathbb{R}\setminus\{0,1\}$ that

$$\exp y = (1-\lambda)\exp c + \lambda \exp d \tag{3.11}$$
$$+ \sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)(y-c)^k \exp c + (-1)^k \lambda (d-y)^k \exp d\right]$$
$$+ R_{n,\lambda}(y,c,d),$$

where the remainder $R_{n,\lambda}(y,c,d)$ is given by

$$R_{n,\lambda}(y,c,d) \tag{3.12}$$
$$:= \frac{1}{n!}\left[(1-\lambda)(y-c)^{n+1}\int_0^1 (1-s)^n \exp((1-s)c+sy)\,ds\right.$$
$$\left.+(-1)^{n+1}\lambda(d-y)^{n+1}\int_0^1 s^n \exp((1-s)y+sd)\,ds\right].$$

Let $x, a, b > 0$. If we take in (3.11) and (3.12) $y = \ln x$, $c = \ln a$ and $d = \ln b$,, then we get for any $\lambda \in \mathbb{R}\setminus\{0,1\}$ that

$$x = (1-\lambda)a + \lambda b \tag{3.13}$$
$$+\sum_{k=1}^{n}\frac{1}{k!}\left[(1-\lambda)a(\ln x - \ln a)^k + (-1)^k\lambda b(\ln b - \ln x)^k\right]$$
$$+R_{n,\lambda}(x,a,b),$$

where the remainder $R_{n,\lambda}(x,a,b)$ is given by

$$R_{n,\lambda}(x,a,b) \tag{3.14}$$
$$:= \frac{1}{n!}\left[(1-\lambda)(\ln x - \ln a)^{n+1}\int_0^1 (1-s)^n a^{1-s}x^s\,ds\right.$$
$$\left.+(-1)^{n+1}\lambda(\ln b - \ln x)^{n+1}\int_0^1 s^n x^{1-s}b^s\,ds\right].$$

If we write the equality (2.13) for the function $f:\mathbb{R}\longrightarrow(0,\infty)$, $f(y) = \exp y$, we get for any $c, d \in \mathbb{R}$ and $\lambda \in \mathbb{R}\setminus\{0,1\}$ that

$$\exp((1-\lambda)c+\lambda d) = (1-\lambda)\exp c + \lambda\exp d$$
$$+\lambda(1-\lambda)\sum_{k=1}^{n}\frac{1}{k!}\left[\lambda^{k-1}\exp c + (-1)^k(1-\lambda)^{k-1}\exp d\right](d-c)^k + T_{n,\lambda}(c,d), \tag{3.15}$$

where

$$T_{n,\lambda}(c,d)$$
$$:= \frac{1}{n!}(d-c)^{n+1}\left[(1-\lambda)^{n+2}\int_0^1 \exp((1-s+\lambda s)c + (1-\lambda)sd)(1-s)^n\,ds\right.$$
$$\left.+(-1)^{n+1}\lambda^{n+2}\int_0^1 \exp((1-s)\lambda c + (1-\lambda+\lambda s)d)\,s^n\,ds\right]. \tag{3.16}$$

Let $x, a, b > 0$. If we take in (3.15) and (3.16) $y = \ln x$, $c = \ln a$ and $d = \ln b$,, then we get for any $\lambda \in \mathbb{R}\backslash \{0, 1\}$ that

$$a^{1-\lambda} b^{\lambda} = (1-\lambda) a + \lambda b$$
$$+ \lambda (1-\lambda) \sum_{k=1}^{n} \frac{1}{k!} \left[\lambda^{k-1} a + (-1)^k (1-\lambda)^{k-1} b \right] (\ln b - \ln a)^k + T_{n,\lambda} (a, b), \tag{3.17}$$

where

$$T_{n,\lambda} (a, b)$$
$$:= \frac{1}{n!} (\ln b - \ln a)^{n+1} \left[(1-\lambda)^{n+2} \int_0^1 a^{1-s+\lambda s} b^{(1-\lambda)s} (1-s)^n \, ds \right.$$
$$\left. + (-1)^{n+1} \lambda^{n+2} \int_0^1 a^{(1-s)\lambda} b^{1-\lambda+\lambda s} s^n ds \right]. \tag{3.18}$$

If $\lambda \in [0, 1]$ and $a, b > 0$, then we have from (3.17) that

$$0 \le (1-\lambda) a + \lambda b - a^{1-\lambda} b^{\lambda}$$
$$= \lambda (1-\lambda) \sum_{k=1}^{n} \frac{1}{k!} \left[(-1)^{k-1} (1-\lambda)^{k-1} b - \lambda^{k-1} a \right] (\ln b - \ln a)^k - T_{n,\lambda} (a, b). \tag{3.19}$$

4 Some Inequalities

We have the following inequality:

Theorem 3 *Let $f : I \to \mathbb{R}$ be $(2m+1)$-time differentiable function on the interior $\mathring{I}$ of the interval I and $f^{(2m+1)}$, with $m \ge 0$, be locally absolutely continuous on $\mathring{I}$. If $f^{(2m+2)}(t) \ge (\le)\, 0$ for almost every $t \in \mathring{I}$, then for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in [0, 1]$ we have*

$$f(x) \ge (\le)\, (1-\lambda) f(a) + \lambda f(b) \tag{4.1}$$
$$+ \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a) (x-a)^k + (-1)^k \lambda f^{(k)}(b) (b-x)^k \right].$$

Proof From Theorem 2 we have for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in [0, 1]$ that

$$f(x) = (1-\lambda) f(a) + \lambda f(b) + \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(1-\lambda) f^{(k)}(a) (x-a)^k + (-1)^k \lambda f^{(k)}(b) (b-x)^k \right] + S_{2m+1,\lambda}(x, a, b), \tag{4.2}$$

where the remainder $S_{2m+1,\lambda}(x, a, b)$ is given by

$$S_{2m+1,\lambda}(x, a, b) := \frac{1}{(2m+1)!} \left[(1-\lambda)(x-a)^{2m+2} \int_0^1 f^{(2m+2)}((1-s)a + sx)(1-s)^{2m+1} ds + \lambda (b-x)^{2m+2} \int_0^1 f^{(2m+2)}((1-s)x + sb) s^{2m+1} ds \right].$$

If $f^{(2m+2)}(t) \geq (\leq) 0$ for almost every $t \in \mathring{I}$,, then for each distinct $x, a, b \in \mathring{I}$ we have

$$\int_0^1 f^{(2m+2)}((1-s)a + sx)(1-s)^{2m+1} ds \geq (\leq) 0$$

and

$$\int_0^1 f^{(2m+2)}((1-s)x + sb) s^{2m+1} ds \geq (\leq) 0,$$

which implies that $S_{2m+1,\lambda}(x, a, b) \geq (\leq) 0$ for each distinct $x, a, b \in \mathring{I}$.

Using the identity (4.2) we deduce the desired result (4.1). □

Corollary 3 *With the assumptions of Theorem 3 for the function* $f : I \to \mathbb{R}$ *then for each distinct* $a, b \in \mathring{I}$ *and for any* $\lambda \in [0, 1]$ *we have*

$$f((1-\lambda)a + \lambda b) \geq (\leq) (1-\lambda) f(a) + \lambda f(b) + \lambda (1-\lambda) \sum_{k=1}^{2m+1} \frac{1}{k!} \left[\lambda^{k-1} f^{(k)}(a) + (-1)^k (1-\lambda)^{k-1} f^{(k)}(b) \right] (b-a)^k. \tag{4.3}$$

Remark 2 If the function $f : I \to \mathbb{R}$ is twice differentiable convex (concave) on $\mathring{I}$ then for each distinct $x, a, b \in \mathring{I}$ and for any $\lambda \in [0, 1]$ we have from (4.1) that

$$f(x) \geq (\leq) (1-\lambda) f(a) + \lambda f(b) + (1-\lambda) f'(a)(x-a) - \lambda f'(b)(b-x). \tag{4.4}$$

From (4.3) we have that

$$f((1-\lambda) a + \lambda b) \geq (\leq) (1-\lambda) f(a) + \lambda f(b) + \lambda (1-\lambda) \left[f'(a) - f'(b)\right](b-a)$$

that is equivalent to

$$\lambda (1-\lambda) \left[f'(b) - f'(a)\right](b-a) \geq (\leq) (1-\lambda) f(a) + \lambda f(b) - f((1-\lambda) a + \lambda b) \tag{4.5}$$

for any $a, b \in \mathring{I}$ and for any $\lambda \in [0, 1]$.

We get from (3.2) and (3.3) for any $x, a, b > 0$ and $\lambda \in \mathbb{R} \setminus \{0, 1\}$ that

$$\ln x = (1-\lambda) \ln a + \lambda \ln b + \sum_{k=1}^{2m+1} \frac{1}{k} \left[(-1)^{k-1} (1-\lambda) \left(\frac{x}{a} - 1\right)^k - \lambda \left(1 - \frac{x}{b}\right)^k\right] + U_{2m+1,\lambda}(x, a, b), \tag{4.6}$$

where the remainder $U_{2m+1,\lambda}(x, a, b)$ is given by

$$U_{2m+1,\lambda}(x, a, b) := -\left[(1-\lambda)(x-a)^{2m+2} \int_0^1 \frac{(1-s)^{2m+1}}{((1-s) a + sx)^{2m+1}} ds + \lambda (b-x)^{2m+2} \int_0^1 \frac{s^{2m+1}}{((1-s) x + sb)^{2m+1}} ds\right]. \tag{4.7}$$

If $x, a, b > 0$ and $\lambda \in [0, 1]$, then $U_{2m+1,\lambda}(x, a, b) \leq 0$ and by (4.6), we get

$$\ln x \leq (1-\lambda) \ln a + \lambda \ln b + \sum_{k=1}^{2m+1} \frac{1}{k} \left[(-1)^{k-1} (1-\lambda) \left(\frac{x}{a} - 1\right)^k - \lambda \left(1 - \frac{x}{b}\right)^k\right]. \tag{4.8}$$

From (3.8) we have for any $a, b > 0$, $m \geq 0$ and $\lambda \in [0, 1]$ that

$$0 \leq \ln\left(\frac{A_\lambda(a,b)}{G_\lambda(a,b)}\right) \leq \lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k}\left[\frac{(-1)^{k-1}\lambda^{k-1}}{a^k}-\frac{(1-\lambda)^{k-1}}{b^k}\right](b-a)^k, \tag{4.9}$$

where $A_\lambda(a,b) := (1-\lambda)a+\lambda b$ is the weighted arithmetic mean and $G_\lambda(a,b) := a^{1-\lambda}b^\lambda$ is the weighted geometric mean. For $\lambda = \frac{1}{2}$ we recapture the arithmetic mean $A(a,b)$ and geometric mean $G(a,b)$, respectively.

By taking the exponential in (4.9) we have

$$1 \leq \frac{A_\lambda(a,b)}{G_\lambda(a,b)} \leq \exp\left[\lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k}\left[\frac{(-1)^{k-1}\lambda^{k-1}}{a^k}-\frac{(1-\lambda)^{k-1}}{b^k}\right](b-a)^k\right], \tag{4.10}$$

for any $a, b > 0$, $m \geq 0$ and $\lambda \in [0, 1]$.

In particular, we have

$$1 \leq \frac{A(a,b)}{G(a,b)} \leq \exp\left[\frac{1}{4}\sum_{k=1}^{2m+1}\frac{1}{2^{k-1}k}\left[\frac{(-1)^{k-1}b^k-a^k}{a^k b^k}\right](b-a)^k\right], \tag{4.11}$$

for any $a, b > 0$ and $m \geq 0$.

If we take in (4.10) $m = 0$,, then we get

$$1 \leq \frac{A_\lambda(a,b)}{G_\lambda(a,b)} \leq \exp\left[\lambda(1-\lambda)\frac{(b-a)^2}{ab}\right] \tag{4.12}$$

for any $a, b > 0$ and $\lambda \in [0, 1]$.

We consider the *Kantorovich's constant* defined by

$$K(h) := \frac{(h+1)^2}{4h}, \; h > 0. \tag{4.13}$$

The function K is decreasing on $(0, 1)$ and increasing on $[1, \infty)$, $K(h) \geq 1$ for any $h > 0$ and $K(h) = K\left(\frac{1}{h}\right)$ for any $h > 0$.

Using Kantorovich's constant we can write the inequality (4.12) as

$$1 \leq \frac{(1-\nu)a+\nu b}{a^{1-\nu}b^\nu} \leq \exp\left[4\nu(1-\nu)\left(K\left(\frac{a}{b}\right)-1\right)\right] \tag{4.14}$$

for any $a, b > 0$ and $\lambda \in [0, 1]$. That has been obtained in [6].

In particular, we have [6]

$$1 \leq \frac{A(a,b)}{G(a,b)} \leq \exp\left(K\left(\frac{a}{b}\right) - 1\right) \tag{4.15}$$

for any $a, b > 0$.

Let $x, a, b > 0$ and $m \geq 0$. Then we get from (3.13) and (3.14) for any $\lambda \in \mathbb{R} \backslash \{0, 1\}$ that

$$\begin{aligned} x &= (1-\lambda) a + \lambda b \\ &+ \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(1-\lambda) a (\ln x - \ln a)^k + (-1)^k \lambda b (\ln b - \ln x)^k\right] \\ &+ R_{2m+1,\lambda}(x,a,b), \end{aligned} \tag{4.16}$$

where the remainder $R_{2m+1,\lambda}(x,a,b)$ is given by

$$\begin{aligned} &R_{2m+1,\lambda}(x,a,b) \\ &:= \frac{1}{(2m+1)!}\left[(1-\lambda)(\ln x - \ln a)^{2m+2} \int_0^1 (1-s)^{2m+1} a^{1-s} x^s ds\right. \\ &\left. +\lambda (\ln b - \ln x)^{2m+2} \int_0^1 s^{2m+1} x^{1-s} b^s ds\right]. \end{aligned} \tag{4.17}$$

If $x, a, b > 0$, $m \geq 0$ and $\lambda \in [0,1]$, then $R_{2m+1,\lambda}(x,a,b) \geq 0$ and by (4.16) we have

$$\begin{aligned} x &\geq (1-\lambda) a + \lambda b \\ &+ \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(1-\lambda) a (\ln x - \ln a)^k + (-1)^k \lambda b (\ln b - \ln x)^k\right]. \end{aligned} \tag{4.18}$$

If $\lambda \in [0,1]$ and $a, b > 0$, $m \geq 0$, then we have from (3.19) that

$$\begin{aligned} 0 &\leq (1-\lambda) a + \lambda b - a^{1-\lambda} b^{\lambda} \\ &= \lambda (1-\lambda) \sum_{k=1}^{2m+1} \frac{1}{k!} \left[(-1)^{k-1} (1-\lambda)^{k-1} b - \lambda^{k-1} a\right] (\ln b - \ln a)^k \\ &\qquad - T_{2m+1,\lambda}(a,b), \end{aligned} \tag{4.19}$$

where

$$T_{2m+1,\lambda}(a,b) := \frac{1}{n!}(\ln b - \ln a)^{2m+2}\left[(1-\lambda)^{2m+3}\int_0^1 a^{1-s+\lambda s}b^{(1-\lambda)s}(1-s)^{2m+1}ds + \lambda^{2m+3}\int_0^1 a^{(1-s)\lambda}b^{1-\lambda+\lambda s}s^{2m+1}ds\right]. \quad (4.20)$$

Since $T_{2m+1,\lambda}(a,b) \geq 0$ if $\lambda \in [0,1]$ and $a, b > 0$, $m \geq 0$, then from (4.19) we get

$$0 \leq A_\lambda(a,b) - G_\lambda(a,b) \leq \lambda(1-\lambda)\sum_{k=1}^{2m+1}\frac{1}{k!}\left[(-1)^{k-1}(1-\lambda)^{k-1}b - \lambda^{k-1}a\right](\ln b - \ln a)^k. \quad (4.21)$$

In particular, we have for any $a, b > 0$ and $m \geq 0$ that

$$0 \leq A(a,b) - G(a,b) \leq \frac{1}{4}\sum_{k=1}^{2m+1}\frac{1}{2^{k-1}k!}\left[(-1)^{k-1}b - a\right](\ln b - \ln a)^k. \quad (4.22)$$

If we take $m = 0$ in (4.21), then we get

$$0 \leq A_\lambda(a,b) - G_\lambda(a,b) \leq \lambda(1-\lambda)(b-a)(\ln b - \ln a), \quad (4.23)$$

for any $a, b > 0$ and $\lambda \in [0,1]$, that has been obtained in [6].

In particular, we have [6]

$$0 \leq A(a,b) - G(a,b) \leq \frac{1}{4}(b-a)(\ln b - \ln a), \quad (4.24)$$

for any $a, b > 0$.

For other recent inequalities between the weighted arithmetic mean and geometric mean, see [6–10, 15–17, 20, 21] and [23].

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Some Weighted Inequalities for Riemann–Stieltjes Integral When a Function Is Bounded

Silvestru Sever Dragomir

Abstract In this chapter we provide some simple ways to approximate the Riemann–Stieltjes integral of a product of two functions $\int_a^b f(t)\, g(t)\, dv(t)$ by the use of simpler quantities and under several assumptions for the functions involved, one of them satisfying the boundedness condition

$$\left| f(t) - \frac{\gamma + \Gamma}{2} \right| \le \frac{1}{2} |\Gamma - \gamma| \text{ for each } t \in [a, b],$$

where $f : [a, b] \to \mathbb{C}$. Applications for continuous functions of selfadjoint operators and functions of unitary operators on Hilbert spaces are also given.

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1 Introduction

One can approximate the *Stieltjes integral* $\int_a^b f(t)\, du(t)$ with the following simpler quantities:

$$\frac{1}{b-a} [u(b) - u(a)] \cdot \int_a^b f(t)\, dt \tag{1.1}$$

$$f(x) [u(b) - u(a)] \tag{1.2}$$

[13, 15, 24, 25] or with

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$$[u(b)-u(x)]f(b)+[u(x)-u(a)]f(a), \tag{1.3}$$

[26] where $x\in[a,b]$.

In order to provide *a priori* sharp bounds for the *approximation error*, consider the functionals:

$$D(f,u;a,b):=\int_a^b f(t)\,du(t)-\frac{1}{b-a}[u(b)-u(a)]\cdot\int_a^b f(t)\,dt,$$

$$\Theta(f,u;a,b,x):=\int_a^b f(t)\,du(t)-f(x)[u(b)-u(a)]$$

and

$$T(f,u;a,b,x):=\int_a^b f(t)\,du(t)-[u(b)-u(x)]f(b)-[u(x)-u(a)]f(a).$$

If the *integrand* f is *Riemann integrable* on $[a,b]$ and the *integrator* $u:[a,b]\to\mathbb{R}$ is $L-$*Lipschitzian*, i.e.,

$$|u(t)-u(s)|\le L|t-s| \qquad \text{for each } t,s\in[a,b], \tag{1.4}$$

then the Stieltjes integral $\int_a^b f(t)\,du(t)$ exists and, as pointed out in [24],

$$|D(f,u;a,b)|\le L\int_a^b\left|f(t)-\int_a^b\frac{1}{b-a}f(s)\,ds\right|dt. \tag{1.5}$$

The inequality (1.5) is sharp in the sense that the multiplicative constant $C=1$ in front of L cannot be replaced by a smaller quantity. Moreover, if there exist the constants $m,\ M\in\mathbb{R}$ such that $m\le f(t)\le M$ for a.e. $t\in[a,b]$, then [24]

$$|D(f,u;a,b)|\le\frac{1}{2}L(M-m)(b-a). \tag{1.6}$$

The constant $\frac{1}{2}$ is best possible in (1.6).

A different approach in the case of integrands of bounded variation was considered by the same authors in 2001, [25], where they showed that

$$|D(f,u;a,b)|\le\max_{t\in[a,b]}\left|f(t)-\frac{1}{b-a}\int_a^b f(s)\,ds\right|\bigvee_a^b(u), \tag{1.7}$$

provided that f is continuous and u is of bounded variation. Here $\bigvee_a^b (u)$ denotes the total variation of u on $[a, b]$. The inequality (1.7) is sharp.

If we assume that f is K-Lipschitzian, then [25]

$$|D(f, u; a, b)| \le \frac{1}{2} K (b-a) \bigvee_a^b (u), \tag{1.8}$$

with $\frac{1}{2}$ the best possible constant in (1.8).

For various bounds on the error functional $D(f, u; a, b)$ where f and u belong to different classes of function for which the Stieltjes integral exists, see [18–20], and [8] and the references therein.

For the functional $\theta(f, u; a, b, x)$ we have the bound [13]:

$$\begin{aligned} &|\theta(f, u; a, b, x)| \qquad (1.9)\\ &\le H\left[(x-a)^r \bigvee_a^x (f) + (b-x)^r \bigvee_x^b (f)\right] \\ &\le H \times \begin{cases} \left[(x-a)^r + (b-x)^r\right]\left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right]; \\ \left[(x-a)^{qr} + (b-x)^{qr}\right]^{\frac{1}{q}}\left[\left(\bigvee_a^x (f)\right)^p + \left(\bigvee_x^b (f)\right)^p\right]^{\frac{1}{p}} \\ \qquad \text{if } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^r \bigvee_a^b (f), \end{cases} \end{aligned}$$

provided f is of bounded variation and u is of *r-H-Hölder type*, i.e.,

$$|u(t) - u(s)| \le H |t-s|^r \qquad \text{for each } t, s \in [a, b], \tag{1.10}$$

with given $H > 0$ and $r \in (0, 1]$.

If f is of q-K-Hölder type and u is of bounded variation, then [15]

$$|\theta(f, u; a, b, x)| \le K\left[\frac{1}{2}(b-a) + \left|x - \frac{a+b}{2}\right|\right]^q \bigvee_a^b (u), \tag{1.11}$$

for any $x \in [a, b]$.

If u is monotonic nondecreasing and f of q-K-Hölder type, then the following refinement of (1.11) also holds [8]:

$$
\begin{aligned}
|\theta(f,u;a,b,x)| &\le K\Bigg[(b-x)^q u(b)-(x-a)^q u(a) \\
&\quad + q\left\{\int_a^x \frac{u(t)\,dt}{(x-t)^{1-q}}-\int_x^b \frac{u(t)\,dt}{(t-x)^{1-q}}\right\}\Bigg] \\
&\le K\left[(b-x)^q[u(b)-u(x)]+(x-a)^q[u(x)-u(a)]\right] \\
&\le K\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^q[u(b)-u(a)],
\end{aligned} \tag{1.12}
$$

for any $x\in[a,b]$.

If f is monotonic nondecreasing and u is of r-H-Hölder type, then [8]:

$$
\begin{aligned}
&|\theta(f,u;a,b,x)| \\
&\le H\Bigg[\left[(x-a)^r-(b-x)^r\right]f(x) \\
&\quad + r\left\{\int_a^x \frac{f(t)\,dt}{(b-t)^{1-r}}-\int_x^b \frac{f(t)\,dt}{(t-r)^{1-r}}\right\}\Bigg] \\
&\le H\left\{(b-x)^r[f(b)-f(x)]+(x-a)^r[f(x)-f(a)]\right\} \\
&\le H\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^r[f(b)-f(a)],
\end{aligned} \tag{1.13}
$$

for any $x\in[a,b]$.

The error functional $T(f,u;a,b,x)$ satisfies similar bounds, see [6, 8, 26] and [2] and the details are omitted. For other related results, see [3–5, 7, 11, 12, 17, 28, 29].

Motivated by the above results, in this chapter we provide some simple ways to approximate the Riemann–Stieltjes integral of a product of two functions $\int_a^b f(t)\,g(t)\,dv(t)$ by the use of simpler quantities and under several assumptions for the functions involved. Applications for continuous functions of selfadjoint operators and continuous functions of unitary operators on Hilbert spaces are also given.

2 General Results

We have the simple equality of interest for what follows:

Lemma 1 *Let $f,g,v:[a,b]\to\mathbb{C}$, $\lambda,\mu\in\mathbb{C}$ and $x\in[a,b]$. If $fg,g\in\mathcal{R}_{\mathbb{C}}(v,[a,x])\cap\mathcal{R}_{\mathbb{C}}(v,[x,b])$, then $fg,g\in\mathcal{R}_{\mathbb{C}}(v,[a,b])$ and*

$$\int_a^b f(t)\,g(t)\,dv(t) = \lambda \int_a^x g(t)\,dv(t) + \mu \int_x^b g(t)\,dv(t) \tag{2.1}$$
$$+ \int_a^x [f(t) - \lambda]\,g(t)\,dv(t) + \int_x^b [f(t) - \mu]\,g(t)\,dv(t)$$
$$= \mu \int_a^b g(t)\,dv(t) + (\lambda - \mu) \int_a^x g(t)\,dv(t)$$
$$+ \int_a^x [f(t) - \lambda]\,g(t)\,dv(t) + \int_x^b [f(t) - \mu]\,g(t)\,dv(t).$$

In particular, for $\mu = \lambda$, *we have*

$$\int_a^b f(t)\,g(t)\,dv(t) = \lambda \int_a^b g(t)\,dv(t) \tag{2.2}$$
$$+ \int_a^x [f(t) - \lambda]\,g(t)\,dv(t) + \int_x^b [f(t) - \lambda]\,g(t)\,dv(t)$$
$$= \lambda \int_a^b g(t)\,dv(t) + \int_a^b [f(t) - \lambda]\,g(t)\,dv(t).$$

Proof The integrability follows by Theorem 7.4 from [1] which says that if a function is Riemann–Stieltjes integrable on the intervals $[a, x]$, $[x, b]$ with $x \in [a, b]$, then it is integrable on the whole interval $[a, b]$.

Using the properties of the Riemann–Stieltjes integral, we have

$$\int_a^x [f(t) - \lambda]\,g(t)\,dv(t) + \int_x^b [f(t) - \mu]\,g(t)\,dv(t)$$
$$= \int_a^x f(t)\,g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) + \int_x^b f(t)\,g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t)$$
$$= \int_a^b f(t)\,g(t)\,dv(t) - \lambda \int_a^x g(t)\,dv(t) - \mu \int_x^b g(t)\,dv(t),$$

which is equivalent to the first equality in (2.1).

The rest is obvious. □

Corollary 1 *Assume that* $f, v : [a, b] \to \mathbb{C}$ *and* $x \in [a, b]$ *are such that* $f \in \mathcal{R}_{\mathbb{C}}(v, [a, x]) \cap \mathcal{R}_{\mathbb{C}}(v, [x, b])$. *Then for any* $\lambda, \mu \in \mathbb{C}$ *we have the equality*

$$\int_a^b f(t)\,dv(t) = \lambda [v(x) - v(a)] + \mu [v(b) - v(x)] \tag{2.3}$$
$$+ \int_a^x [f(t) - \lambda]\,dv(t) + \int_x^b [f(t) - \mu]\,dv(t).$$

In particular, for $\mu = \lambda$, *we have*

$$\int_a^b f(t)\,dv(t) = \lambda [v(b) - v(a)] + \int_a^x [f(t) - \lambda]\,dv(t) + \int_x^b [f(t) - \lambda]\,dv(t) = \lambda [v(b) - v(a)] + \int_a^b [f(t) - \lambda]\,dv(t). \tag{2.4}$$

The proof follows by Lemma 1 for $g(t) = 1$, $t \in [a, b]$.

Remark 1 We observe that, see [1, Theorem 7.27], if $f, g \in \mathcal{C}_{\mathbb{C}}[a, b]$, namely, are continuous on $[a, b]$ and $v \in \mathcal{BV}_{\mathbb{C}}[a, b]$, namely of bounded variation on $[a, b]$, then for any $x \in [a, b]$ the Riemann–Stieltjes integrals in Lemma 1 exist and the equalities (2.1) and (2.2) hold.

Now, for $\gamma, \Gamma \in \mathbb{C}$ and $[a, b]$ an interval of real numbers, define the sets of complex-valued functions

$$\bar{U}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \to \mathbb{C} | \operatorname{Re}\left[(\Gamma - f(t))\left(\overline{f(t)} - \overline{\gamma}\right)\right] \geq 0 \text{ for each } t \in [a, b] \right\}$$

and

$$\bar{\Delta}_{[a,b]}(\gamma, \Gamma) := \left\{ f : [a, b] \to \mathbb{C} | \left| f(t) - \frac{\gamma + \Gamma}{2} \right| \leq \frac{1}{2} |\Gamma - \gamma| \text{ for each } t \in [a, b] \right\}.$$

This family of functions is a particular case of the class introduced in [21]

$$\bar{\Delta}_{[a,b],g}(\gamma, \Gamma) := \left\{ f : [a, b] \to \mathbb{C} | \left| f(t) - \frac{\gamma + \Gamma}{2} g(t) \right| \leq \frac{1}{2} |\Gamma - \gamma| |g(t)| \text{ for each } t \in [a, b] \right\},$$

where $g : [a, b] \to \mathbb{C}$.

The following representation result may be stated.

Proposition 1 *For any* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, *we have that* $\bar{U}_{[a,b]}(\gamma, \Gamma)$ *and* $\bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ *are nonempty, convex, and closed sets and*

$$\bar{U}_{[a,b]}(\gamma, \Gamma) = \bar{\Delta}_{[a,b]}(\gamma, \Gamma). \tag{2.5}$$

Proof We observe that for any $z \in \mathbb{C}$ we have the equivalence

$$\left|z - \frac{\gamma + \Gamma}{2}\right| \le \frac{1}{2}|\Gamma - \gamma|$$

if and only if

$$\operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right] \ge 0.$$

This follows by the equality

$$\frac{1}{4}|\Gamma - \gamma|^2 - \left|z - \frac{\gamma + \Gamma}{2}\right|^2 = \operatorname{Re}\left[(\Gamma - z)(\bar{z} - \bar{\gamma})\right]$$

that holds for any $z \in \mathbb{C}$.

The equality (2.5) is thus a simple consequence of this fact. □

On making use of the complex numbers field properties we can also state that:

Corollary 2 *For any $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$, we have that*

$$\begin{aligned}\bar{U}_{[a,b]}(\gamma, \Gamma) = \{f : [a,b] \to \mathbb{C} \mid\ & (\operatorname{Re}\Gamma - \operatorname{Re} f(t))(\operatorname{Re} f(t) - \operatorname{Re}\gamma) \\ & + (\operatorname{Im}\Gamma - \operatorname{Im} f(t))(\operatorname{Im} f(t) - \operatorname{Im}\gamma) \ge 0 \textit{ for each } t \in [a,b]\}.\end{aligned} \tag{2.6}$$

Now, if we assume that $\operatorname{Re}(\Gamma) \ge \operatorname{Re}(\gamma)$ and $\operatorname{Im}(\Gamma) \ge \operatorname{Im}(\gamma)$, then we can define the following set of functions as well:

$$\begin{aligned}\bar{S}_{[a,b]}(\gamma, \Gamma) := \{f : [a,b] \to \mathbb{C} \mid\ & \operatorname{Re}(\Gamma) \ge \operatorname{Re} f(t) \ge \operatorname{Re}(\gamma) \\ & \text{and } \operatorname{Im}(\Gamma) \ge \operatorname{Im} f(t) \ge \operatorname{Im}(\gamma) \text{ for each } t \in [a,b]\}.\end{aligned} \tag{2.7}$$

One can easily observe that $\bar{S}_{[a,b]}(\gamma, \Gamma)$ is closed, convex and

$$\emptyset \neq \bar{S}_{[a,b]}(\gamma, \Gamma) \subseteq \bar{U}_{[a,b]}(\gamma, \Gamma). \tag{2.8}$$

We consider the following functional

$$P(f, g, v; \gamma, \Gamma, a, b) := \int_a^b f(t)\, g(t)\, dv(t) - \frac{\gamma + \Gamma}{2}\int_a^b g(t)\, dv(t) \tag{2.9}$$

for the complex-valued functions f, g, v defined on $[a,b]$ and such that the involved Riemann–Stieltjes integrals exist, and for $\gamma, \Gamma \in \mathbb{C}$.

Theorem 1 *Let* $f, g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$.

(i) *If* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$\begin{aligned} |P(f, g, v; \gamma, \Gamma, a, b)| &\leq \frac{1}{2}|\Gamma - \gamma| \int_a^b |g(t)|\, d\left(\bigvee_a^t (v)\right) \\ &\leq \frac{1}{2}|\Gamma - \gamma| \max_{t\in[a,b]} |g(t)| \bigvee_a^b (v). \end{aligned} \tag{2.10}$$

(ii) *If* $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, *namely,* v *is Lipschitzian with the constant* $L > 0$,

$$|v(t) - v(s)| \leq L|t - s| \text{ for any } t,\ s \in [a,b],$$

then we also have

$$\begin{aligned} |P(f, g, v; \gamma, \Gamma, a, b)| &\leq \frac{1}{2}|\Gamma - \gamma| L \int_a^b |g(t)|\, dt \\ &\leq \frac{1}{2}|\Gamma - \gamma| (b-a) \max_{t\in[a,b]} |g(t)|. \end{aligned} \tag{2.11}$$

(iii) *If* $v \in \mathcal{M}^{\nearrow}[a,b]$, *namely,* v *is monotonic increasing on* $[a,b]$, *then we have*

$$\begin{aligned} |P(f, g, v; \gamma, \Gamma, a, b)| &\leq \frac{1}{2}|\Gamma - \gamma| \int_a^b |g(t)|\, dv(t) \\ &\leq \frac{1}{2}|\Gamma - \gamma| [v(b) - v(a)] \max_{t\in[a,b]} |g(t)|. \end{aligned} \tag{2.12}$$

Proof

(i) It is well known that if $p \in \mathcal{R}(u, [a,b])$ where $u \in \mathcal{BV}_{\mathbb{C}}[a,b]$ then we have [1, p. 177]

$$\left|\int_a^b p(t)\, du(t)\right| \leq \int_a^b |p(t)|\, d\left(\bigvee_a^t (u)\right) \leq \sup_{t\in[a,b]} |p(t)| \bigvee_a^b (u). \tag{2.13}$$

By the equality (2.2) we have

$$\int_a^b f(t) g(t)\, dv(t) - \frac{\gamma + \Gamma}{2} \int_a^b g(t)\, dv(t) = \int_a^b \left[f(t) - \frac{\gamma + \Gamma}{2}\right] g(t)\, dv(t). \tag{2.14}$$

Since $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ then by (2.13) and (2.14) we have

$$\begin{aligned}\left|\int_a^b f(t)\, g(t)\, dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\, dv(t)\right| &\le \int_a^b \left|\left[f(t) - \frac{\gamma+\Gamma}{2}\right] g(t)\right| d\left(\bigvee_a^t (v)\right) \\ &= \int_a^b \left|f(t) - \frac{\gamma+\Gamma}{2}\right| |g(t)|\, d\left(\bigvee_a^t (v)\right) \\ &\le \frac{1}{2}|\Gamma-\gamma| \int_a^b |g(u)|\, d\left(\bigvee_a^t (v)\right)\end{aligned}$$

and the first inequality in (2.10) is proved. The second part is obvious.

(ii) It is well known that if $p \in \mathcal{R}(u, [a,b])$, where $u \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, then we have

$$\left|\int_a^b p(t)\, dv(t)\right| \le L \int_a^b |p(t)|\, dt. \tag{2.15}$$

By using (2.14) we then get (2.11).

(iii) It is well known that if $p \in \mathcal{R}(u, [a,b])$, where $u \in \mathcal{M}^{\nearrow}[a,b]$, then we have

$$\left|\int_a^b p(t)\, dv(t)\right| \le L \int_a^b |p(t)|\, dv(t). \tag{2.16}$$

By using (2.14) we then get (2.12).

□

Remark 2 We define the simpler functional for $g \equiv 1$ by

$$P(f, v; \gamma, \Gamma, a, b) := P(f, 1, v; \gamma, \Gamma, a, b) = \int_a^b f(t)\, dv(t) - \frac{\gamma+\Gamma}{2}[v(b) - v(a)].$$

Let $f \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. If $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then

$$|P(f, v; \gamma, \Gamma, a, b)| \le \frac{1}{2}|\Gamma-\gamma| \bigvee_a^b (v). \tag{2.17}$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, then

$$|P(f, v; \gamma, \Gamma, a, b)| \le \frac{1}{2} L |\Gamma-\gamma| (b-a). \tag{2.18}$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, then

$$\left|\int_a^b P(f,v;\gamma,\Gamma,a,b)\right| \le \frac{1}{2}|\Gamma-\gamma|[v(b)-v(a)]. \tag{2.19}$$

We observe that, if $f \in \mathcal{C}[a,b]$, namely f is real valued and continuous on $[a,b]$ and if we put $m := \min_{t\in[a,b]} f(t)$ and $M := \max_{t\in[a,b]} f(t)$ then by (2.17)–(2.19) we get

$$|P(f,v;m,M,a,b)| \le \frac{1}{2}(M-m)\bigvee_a^b(v)$$

if $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$,

$$|P(f,v;m,M,a,b)| \le \frac{1}{2}L(M-m)(b-a)$$

if $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$ and

$$|P(f,v;m,M,a,b)| \le \frac{1}{2}(M-m)[v(b)-v(a)]$$

if $v \in \mathcal{M}^{\nearrow}[a,b]$, that have been obtained in [21].

For other results of this type, see [16].

3 Quasi-Grüss Type Inequalities

We consider the functional

$$\begin{aligned} Q(f,g,v;\gamma,\Gamma,\delta,\Delta,a,b) &:= \int_a^b f(t)g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) \\ &\quad - \frac{\delta+\Delta}{2}\int_a^b f(t)\,dv(t) + \frac{\gamma+\Gamma}{2}\cdot\frac{\delta+\Delta}{2}[v(b)-v(a)] \end{aligned}$$

for the complex-valued functions f, g, v defined on $[a,b]$ and such that the involved Riemann–Stieltjes integrals exist, and for $\gamma, \Gamma, \delta, \Delta \in \mathbb{C}$.

We have the following quasi-Grüss type inequality:

Proposition 2 *Let $f, g \in \mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma, \Gamma, \delta, \Delta \in \mathbb{C}$, $\gamma \ne \Gamma$, $\delta \ne \Delta$ such that $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$ and $g \in \bar{\Delta}_{[a,b]}(\delta,\Delta)$. If $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, then*

$$|Q(f,g,v;\gamma,\Gamma,a,b)| \le \frac{1}{4}|\Gamma-\gamma|\,|\Delta-\delta| \bigvee_a^b (v). \tag{3.1}$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, *then*

$$|Q(f,g,v;\gamma,\Gamma,a,b)| \le \frac{1}{4}|\Gamma-\gamma|\,|\Delta-\delta|\,L(b-a).$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, *then*

$$|Q(f,g,v;\gamma,\Gamma,a,b)| \le \frac{1}{4}|\Gamma-\gamma|\,|\Delta-\delta|\,[v(b)-v(a)].$$

Proof If we replace in (2.10) g by $g - \frac{\delta+\Delta}{2}$, then we get

$$\begin{aligned}\Bigg|\int_a^b f(t)\,g(t)\,dv(t) &- \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t) \\ &- \frac{\delta+\Delta}{2}\int_a^b f(t)\,dv(t) + \frac{\gamma+\Gamma}{2}\cdot\frac{\delta+\Delta}{2}[v(b)-v(a)]\Bigg| \\ &\le \frac{1}{2}|\Gamma-\gamma|\int_a^b \left|g(t)-\frac{\delta+\Delta}{2}\right| d\left(\bigvee_a^t(v)\right).\end{aligned}$$

Since $g \in \bar{\Delta}_{[a,b]}(\delta,\Delta)$, then

$$\int_a^b \left|g(t)-\frac{\delta+\Delta}{2}\right| d\left(\bigvee_a^t(v)\right) \le \frac{1}{2}|\Delta-\delta|\int_a^b d\left(\bigvee_a^t(v)\right) = \frac{1}{2}|\Delta-\delta|\bigvee_a^b(v)$$

and the inequality (3.1) is proved.

The proofs of the other two statements follow in a similar way and we omit the details. □

Proposition 3 *Let* $f, g \in \mathcal{C}_{\mathbb{C}}[a,b]$, $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$ *and* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \ne \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. *If* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$|Q(f,g,v;g(a),g(b),a,b)| \le \frac{1}{4}|\Gamma-\gamma| \bigvee_a^b(g)\bigvee_a^b(v), \tag{3.2}$$

where

$$Q(f,g,v;g(a),g(b),a,b) = \int_a^b f(t)\,g(t)\,dv(t) - \frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t)$$

$$-\frac{g(a)+g(b)}{2}\int_a^b f(t)\,dv(t)+\frac{\gamma+\Gamma}{2}\cdot\frac{g(a)+g(b)}{2}[v(b)-v(a)].$$

If $v\in\mathcal{L}_{L,\mathbb{C}}[a,b]$, *then*

$$|Q(f,g,v;g(a),g(b),a,b)|\leq\frac{1}{4}|\Gamma-\gamma|\,L(b-a)\bigvee_a^b(g). \tag{3.3}$$

If $v\in\mathcal{M}^{\nearrow}[a,b]$, *then*

$$|Q(f,g,v;g(a),g(b),a,b)|\leq\frac{1}{4}|\Gamma-\gamma|\bigvee_a^b(g)[v(b)-v(a)]. \tag{3.4}$$

Proof If we replace in (2.10) g by $g-\frac{g(a)+g(b)}{2}$, then we get

$$\begin{aligned}&\left|\int_a^b f(t)g(t)\,dv(t)-\frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t)\right.\\&\left.-\frac{g(a)+g(b)}{2}\int_a^b f(t)\,dv(t)+\frac{\gamma+\Gamma}{2}\cdot\frac{g(a)+g(b)}{2}[v(b)-v(a)]\right|\\&\qquad\leq\frac{1}{2}|\Gamma-\gamma|\int_a^b\left|g(t)-\frac{g(a)+g(b)}{2}\right|d\left(\bigvee_a^t(v)\right).\end{aligned}$$

Since $g\in\mathcal{BV}_{\mathbb{C}}[a,b]$, hence

$$\begin{aligned}\left|g(t)-\frac{g(a)+g(b)}{2}\right|&=\left|\frac{g(t)-g(a)+g(t)-g(b)}{2}\right|\\&\leq\frac{1}{2}[|g(t)-g(a)|+|g(b)-g(t)|]\leq\frac{1}{2}\bigvee_a^b(g)\end{aligned}$$

for any $t\in[a,b]$.

Therefore

$$\int_a^b\left|g(t)-\frac{g(a)+g(b)}{2}\right|d\left(\bigvee_a^t(v)\right)\leq\frac{1}{2}\bigvee_a^b(g)\int_a^b d\left(\bigvee_a^t(v)\right)=\frac{1}{2}\bigvee_a^b(g)\bigvee_a^b(v)$$

and the inequality (3.2) is proved.

The proofs of the other statements follow in a similar way and we omit the details. □

Proposition 4 *Let* $f, g \in \mathcal{C}_{\mathbb{C}}[a,b]$ *and* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma,\Gamma)$. *If* $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, *then*

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right|$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|d\left(\bigvee_a^t(v)\right)$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|\bigvee_a^b(v), \tag{3.5}$$

where

$$Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)$$
$$=\int_a^b f(t)\,g(t)\,dv(t)-\frac{\gamma+\Gamma}{2}\int_a^b g(t)\,dv(t)$$
$$-\int_a^b f(t)\,dv(t)\,\frac{1}{b-a}\int_a^b g(t)\,dt+[v(b)-v(a)]\,\frac{\gamma+\Gamma}{2}\cdot\frac{1}{b-a}\int_a^b g(t)\,dt\Bigg|.$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, *then*

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right|$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\,L\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\,L\,(b-a)\max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dt. \tag{3.6}$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, *then*

$$\left|Q\left(f,g,v;\frac{1}{b-a}\int_a^b g(s)\,ds,\frac{1}{b-a}\int_a^b g(s)\,ds,a,b\right)\right|$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\int_a^b\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|dv(t)$$
$$\leq \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}\left|g(t)-\frac{1}{b-a}\int_a^b g(s)\,ds\right|[v(b)-v(a)]. \tag{3.7}$$

Proof The first inequality follows by Theorem 1 by replacing g with $g - \frac{1}{b-a}\int_a^b g(s)\,ds$. The second part follows by the fact that

$$\int_a^b \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| d\left(\bigvee_a^t (v)\right)$$
$$\leq \max_{t\in[a,b]} \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| \int_a^b d\left(\bigvee_a^t (v)\right)$$
$$= \max_{t\in[a,b]} \left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| \bigvee_a^b (v).$$

The proofs of the other statements follow in a similar way and we omit the details.

□

Remark 3 We observe that the quantity

$$\left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right|, \ t \in [a,b]$$

is the left-hand side in Ostrowski type inequalities for various classes of functions g. For a recent survey on these inequalities, see [23]. Therefore, if

$$\left| g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds \right| \leq M_{g,[a,b]}(t), \ t \in [a,b]$$

is such of inequality, then from (3.5) we get

$$\left| Q\left(f, g, v; \frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right|$$
$$\leq \frac{1}{2}|\Gamma - \gamma| \int_a^b M_{g,[a,b]}(t)\,d\left(\bigvee_a^t (v)\right) \quad (3.8)$$

if $v \in \mathcal{BV}_{\mathbb{C}}[a,b]$, from (3.6) we get

$$\left| Q\left(f, g, v; \frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right|$$
$$\leq \frac{1}{2}|\Gamma - \gamma| L \int_a^b M_{g,[a,b]}(t)\,dt \quad (3.9)$$

if $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$ and from (3.7) we get

$$\left| Q\left(f, g, v; \frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right| \le \frac{1}{2}|\Gamma-\gamma| \int_a^b M_{g,[a,b]}(t)\,dv(t), \qquad (3.10)$$

if $v \in \mathcal{M}^{\nearrow}[a,b]$.

For instance, if $g : [a,b] \to \mathbb{C}$ is of bounded variation, then we have, see [10] and [14]

$$\left| g(t) - \frac{1}{b-a}\int_a^b g(t)\,dt\right| \le \left[\frac{1}{2} + \frac{\left|t - \frac{a+b}{2}\right|}{b-a}\right] \bigvee_a^b (g) \qquad (3.11)$$

for any $t \in [a,b]$. The constant $\frac{1}{2}$ is the best possible one.

Observe that

$$\begin{aligned}
&\int_a^b \left[\frac{1}{2} + \frac{\left|t - \frac{a+b}{2}\right|}{b-a}\right] d\left(\bigvee_a^t (v)\right) \\
&= \frac{1}{2}\bigvee_a^b (v) + \frac{1}{b-a}\int_a^b \left|t - \frac{a+b}{2}\right| d\left(\bigvee_a^t (v)\right) \\
&= \frac{1}{2}\bigvee_a^b (v) + \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - t\right) d\left(\bigvee_a^t (v)\right) \\
&+ \frac{1}{b-a}\int_{\frac{a+b}{2}}^b \left(t - \frac{a+b}{2}\right) d\left(\bigvee_a^t (v)\right) \\
&= \frac{1}{2}\bigvee_a^b (v) + \frac{1}{b-a}\left[\left(\frac{a+b}{2} - t\right)\bigvee_a^t (v)\Big|_a^{\frac{a+b}{2}} + \int_a^{\frac{a+b}{2}} \bigvee_a^t (v)\,dt\right] \\
&+ \frac{1}{b-a}\left[\left(t - \frac{a+b}{2}\right)\bigvee_a^t (v)\Big|_{\frac{a+b}{2}}^b - \int_{\frac{a+b}{2}}^b \bigvee_a^t (v)\,dt\right] \\
&= \bigvee_a^b (v) + \frac{1}{b-a}\int_a^{\frac{a+b}{2}} \bigvee_a^t (v)\,dt - \frac{1}{b-a}\int_{\frac{a+b}{2}}^b \bigvee_a^t (v)\,dt \\
&= \bigvee_a^b (v) + \frac{1}{b-a}\left(\int_a^{\frac{a+b}{2}} \bigvee_a^t (v)\,dt - \int_{\frac{a+b}{2}}^b \bigvee_a^t (v)\,dt\right) \\
&= \bigvee_a^b (v) - \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right)\bigvee_a^t (v)\,dt.
\end{aligned}$$

Then by (3.8) we get

$$\left| Q\left(f, g, v; \frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right|$$
$$\leq \frac{1}{2}\left|\Gamma-\gamma\right| \bigvee_a^b (g) \left[\bigvee_a^b (v) - \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t (v)\,dt\right]$$
$$\leq \frac{1}{2}\left|\Gamma-\gamma\right| \bigvee_a^b (g)\bigvee_a^b (v) \tag{3.12}$$

if $v, g \in \mathcal{BV}_{\mathbb{C}}[a,b]$.

The last inequality in (3.12) follows by Chebyshev's inequality for monotonic functions that gives that

$$\frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)\bigvee_a^t (v)\,dt$$
$$\geq \frac{1}{b-a}\int_a^b \operatorname{sgn}\left(t-\frac{a+b}{2}\right)dt\,\frac{1}{b-a}\int_a^b \bigvee_a^t (v)\,dt = 0.$$

Observe also that

$$\int_a^b \left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]dt = \frac{3}{4}(b-a),$$

then by (3.9) we get

$$\left| Q\left(f, g, v; \frac{1}{b-a}\int_a^b g(s)\,ds, \frac{1}{b-a}\int_a^b g(s)\,ds, a, b\right)\right|$$
$$\leq \frac{3}{8}\left|\Gamma-\gamma\right| L(b-a)\bigvee_a^b (g) \tag{3.13}$$

if $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$ and $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$.

Finally, since

$$\int_a^b \left[\frac{1}{2}+\frac{\left|t-\frac{a+b}{2}\right|}{b-a}\right]dv(t) = v(b)-v(a)$$

$$- \frac{1}{b-a} \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) [v(t) - v(a)]\, dt,$$

then we get by (3.10) that

$$\left| Q\left(f, g, v; \frac{1}{b-a} \int_a^b g(s)\, ds, \frac{1}{b-a} \int_a^b g(s)\, ds, a, b\right) \right|$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \bigvee_a^b (g) \left[v(b) - v(a) - \frac{1}{b-a} \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2}\right) [v(t) - v(a)]\, dt \right]$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \bigvee_a^b (g) [v(b) - v(a)] \tag{3.14}$$

if $v \in \mathcal{M}^{\nearrow}[a, b]$ and $g \in \mathcal{BV}_{\mathbb{C}}[a, b]$.

4 Grüss Type Inequalities

Consider the *Grüss type functional*

$$G(f, g, v; a, b) := \int_a^b f(t)\, g(t)\, dv(t)$$
$$- \frac{1}{v(b) - v(a)} \int_a^b f(t)\, dv(t) \int_a^b g(t)\, dv(t) \tag{4.1}$$

for the complex-valued functions f, g, v defined on $[a, b]$ and such that the involved Riemann–Stieltjes integrals exist and $v(b) \neq v(a)$.

We have:

Proposition 5 *Let* $f, g \in \mathcal{C}_{\mathbb{C}}[a, b]$ *and* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$. *If* $v \in \mathcal{BV}_{\mathbb{C}}[a, b]$ *with* $v(b) \neq v(a)$, *then*

$$|G(f, g, v; a, b)|$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \int_a^b \left| g(t) - \frac{1}{v(b) - v(a)} \int_a^b g(s)\, dv(s) \right| d\left(\bigvee_a^t (v)\right)$$
$$\leq \frac{1}{2} |\Gamma - \gamma| \bigvee_a^b (v) \max_{t \in [a,b]} \left| g(t) - \frac{1}{v(b) - v(a)} \int_a^b g(s)\, dv(s) \right| dt. \tag{4.2}$$

If $v \in \mathcal{L}_{L,\mathbb{C}}[a,b]$, *then*

$$\begin{aligned}|G(f,g,v;a,b)| \\ &\le \frac{1}{2}|\Gamma-\gamma| L \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right| dt \\ \le \frac{1}{2}|\Gamma-\gamma| L(b-a) \max_{t\in[a,b]}\left| g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right|. \end{aligned} \tag{4.3}$$

If $v \in \mathcal{M}^{\nearrow}[a,b]$, *then*

$$\begin{aligned}|G(f,g,v;a,b)| \\ &\le \frac{1}{2}|\Gamma-\gamma| \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right| dv(t) \\ \le \frac{1}{2}|\Gamma-\gamma| [v(b)-v(a)] \max_{t\in[a,b]}\left| g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right|. \end{aligned} \tag{4.4}$$

Proof By Theorem 1, on replacing g with $g - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)$ we get

$$\begin{aligned}\Bigg|\int_a^b f(t)\left[g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right] dv(t) \\ -\frac{\gamma+\Gamma}{2}\int_a^b \left[g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right] dv(t)\Bigg| \\ \le \frac{1}{2}|\Gamma-\gamma| \int_a^b \left| g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right| d\left(\bigvee_a^t (v)\right). \end{aligned}$$

Since

$$\begin{aligned}\int_a^b f(t)\left[g(t) - \frac{1}{v(b)-v(a)}\int_a^b g(s)\,dv(s)\right] dv(t) \\ = \int_a^b f(t)g(t)\,dv(t) - \frac{1}{v(b)-v(a)}\int_a^b f(t)\,dv(t)\int_a^b g(t)\,dv(t)\end{aligned}$$

and

$$\int_a^b \left[g(t) - \frac{1}{v(b) - v(a)} \int_a^b g(s)\, dv(s) \right] dv(t) = 0,$$

hence the first inequality (4.2) is obtained. The second inequality is obvious.
The rest follow in a similar way and we omit the details. □

Remark 4 If g is of K-Lipschitzian and v is of bounded variation, then [15]

$$\left| g(t)[v(b) - v(a)] - \int_a^b g(s)\, dv(s) \right| \leq K \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] \bigvee_a^b (v),$$

for any $t \in [a, b]$.

By (4.2) we then have

$$\begin{aligned} &|G(f, g, v; a, b)| \\ &\leq \frac{1}{2} \frac{|\Gamma - \gamma|}{|v(b) - v(a)|} \int_a^b \left| g(t)[v(b) - v(a)] - \int_a^b g(s)\, dv(s) \right| d\left(\bigvee_a^t (v) \right) \\ &\quad \leq \frac{1}{2} \frac{|\Gamma - \gamma|}{|v(b) - v(a)|} K \bigvee_a^b (v) \int_a^b \left| \frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right| d\left(\bigvee_a^t (v) \right). \end{aligned}$$

Since, as above

$$\begin{aligned} &\int_a^b \left[\frac{1}{2}(b-a) + \left| t - \frac{a+b}{2} \right| \right] d\left(\bigvee_a^t (v) \right) \\ &\quad = (b-a) \bigvee_a^b (v) - \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2} \right) \bigvee_a^t (v)\, dt \leq (b-a) \bigvee_a^b (v), \end{aligned}$$

then we get the following upper bounds for the magnitude of $G(f, g, v; a, b)$

$$\begin{aligned} &|G(f, g, v; a, b)| \\ &\leq \frac{1}{2} \frac{|\Gamma - \gamma|}{|v(b) - v(a)|} K \bigvee_a^b (v) \left[(b-a) \bigvee_a^b (v) - \int_a^b \operatorname{sgn}\left(t - \frac{a+b}{2} \right) \bigvee_a^t (v)\, dt \right] \\ &\qquad \leq \frac{1}{2} K \frac{|\Gamma - \gamma|(b-a)}{|v(b) - v(a)|} \left(\bigvee_a^b (v) \right)^2. \qquad (4.5) \end{aligned}$$

Any other upper bounds for $|\theta(g, v; a, b, t)|$ with $t \in [a, b]$, see, for instance, the survey [9], will provide the corresponding bounds for $|G(f, g, v; a, b)|$. The details are left to the interested reader.

5 Applications for Selfadjoint Operators

We denote by $\mathcal{B}(H)$ the Banach algebra of all bounded linear operators on a complex Hilbert space $(H; \langle \cdot, \cdot \rangle)$. Let $A \in \mathcal{B}(H)$ be selfadjoint and let φ_λ be defined for all $\lambda \in \mathbb{R}$ as follows:

$$\varphi_\lambda(s) := \begin{cases} 1, & \text{for } -\infty < s \le \lambda, \\ \\ 0, & \text{for } \lambda < s < +\infty. \end{cases}$$

Then for every $\lambda \in \mathbb{R}$ the operator

$$E_\lambda := \varphi_\lambda(A) \tag{5.1}$$

is a projection which reduces A.

The properties of these projections are collected in the following fundamental result concerning the spectral representation of bounded selfadjoint operators in Hilbert spaces, see, for instance, [27, p. 256]:

Theorem 2 (Spectral Representation Theorem) *Let A be a bounded selfadjoint operator on the Hilbert space H and let $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ and $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. Then there exists a family of projections $\{E_\lambda\}_{\lambda \in \mathbb{R}}$, called the spectral family of A, with the following properties:*

(a) $E_\lambda \le E_{\lambda'}$ *for* $\lambda \le \lambda'$;
(b) $E_{a-0} = 0$, $E_b = I$ *and* $E_{\lambda+0} = E_\lambda$ *for all* $\lambda \in \mathbb{R}$;
(c) *We have the representation*

$$A = \int_{a-0}^{b} \lambda dE_\lambda.$$

More generally, for every continuous complex-valued function φ defined on $\mathbb{R}$ there exists a unique operator $\varphi(A) \in \mathcal{B}(H)$ such that for every $\varepsilon > 0$ there exists a $\delta > 0$ satisfying the inequality

$$\left\| \varphi(A) - \sum_{k=1}^{n} \varphi\left(\lambda_k'\right)\left[E_{\lambda_k} - E_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} \lambda_0 < a = \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = b, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(A) = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda, \tag{5.2}$$

where the integral is of Riemann–Stieltjes type.

Corollary 3 *With the assumptions of Theorem 2 for A, E_λ, and φ we have the representations*

$$\varphi(A)\, x = \int_{a-0}^{b} \varphi(\lambda)\, dE_\lambda x \ \text{ for all } x \in H$$

and

$$\langle \varphi(A)\, x, y\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, y\rangle \ \text{ for all } x,\ y \in H. \tag{5.3}$$

In particular,

$$\langle \varphi(A)\, x, x\rangle = \int_{a-0}^{b} \varphi(\lambda)\, d\langle E_\lambda x, x\rangle \ \text{ for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(A)\, x\|^2 = \int_{a-0}^{b} |\varphi(\lambda)|^2\, d\, \|E_\lambda x\|^2 \ \text{ for all } x \in H.$$

We need the following result that provides an upper bound for the total variation of the function $\mathbb{R} \ni \lambda \mapsto \langle E_\lambda x, y\rangle \in \mathbb{C}$ on an interval $[\alpha, \beta]$, see [22].

Lemma 2 *Let $\{E_\lambda\}_{\lambda\in\mathbb{R}}$ be the spectral family of the bounded selfadjoint operator A. Then for any x, $y \in H$ and $\alpha < \beta$ we have the inequality*

$$\left[\bigvee_{\alpha}^{\beta} \left(\left\langle E_{(\cdot)} x, y\right\rangle\right)\right]^2 \le \left\langle \left(E_\beta - E_\alpha\right) x, x\right\rangle \left\langle \left(E_\beta - E_\alpha\right) y, y\right\rangle, \tag{5.4}$$

where $\bigvee_{\alpha}^{\beta}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ *denotes the total variation of the function* $\left\langle E_{(\cdot)}x, y\right\rangle$ *on* $[\alpha, \beta]$.

Remark 5 For $\alpha = a - \varepsilon$ with $\varepsilon > 0$ and $\beta = b$ we get from (5.4) the inequality

$$\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \langle (I - E_{a-\varepsilon})\, x, x\rangle^{1/2} \langle (I - E_{a-\varepsilon})\, y, y\rangle^{1/2} \tag{5.5}$$

for any $x, y \in H$.

This implies, for any $x, y \in H$, that

$$\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \|x\| \, \|y\| , \tag{5.6}$$

where $\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)$ denotes the limit $\lim_{\varepsilon \to 0+}\left[\bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right)\right]$.

We can state the following result for functions of selfadjoint operators:

Theorem 3 *Let* A *be a bounded selfadjoint operator on the Hilbert space* H *and let* $a = \min\{\lambda \,|\lambda \in Sp(A)\} =: \min Sp(A)$ *and* $b = \max\{\lambda \,|\lambda \in Sp(A)\} =: \max Sp(A)$. *Also, assume that* $\{E_\lambda\}_{\lambda \in \mathbb{R}}$ *is the spectral family of the bounded selfadjoint operator* A *and* $f, g : I \to \mathbb{C}$ *are continuous on* I, $[a, b] \subset \mathring{I}$ *(the interior of* I*). If* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, *then*

$$\begin{aligned} &\left|\langle f(A)\, g(A)\, x, y\rangle - \frac{\gamma + \Gamma}{2}\langle g(A)\, x, y\rangle\right| \\ &\quad \leq \frac{1}{2}|\Gamma - \gamma| \max_{t \in [a,b]} |g(t)| \bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \leq \frac{1}{2}|\Gamma - \gamma| \max_{t \in [a,b]} |g(t)| \, \|x\| \, \|y\| \end{aligned} \tag{5.7}$$

for any $x, y \in H$.

Proof Using the inequality (2.10), we have

$$\begin{aligned} &\left|\int_{a-\varepsilon}^{b} f(t)\, g(t)\, d\langle E_t x, y\rangle - \frac{\gamma + \Gamma}{2}\int_{a-\varepsilon}^{b} g(t)\, d\langle E_t x, y\rangle\right| \\ &\leq \frac{1}{2}|\Gamma - \gamma| \max_{t \in [a-\varepsilon,b]} |g(t)| \bigvee_{a-\varepsilon}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \end{aligned}$$

for small $\varepsilon > 0$ and for any $x, y \in H$.

Taking the limit over $\varepsilon \to 0+$ and using the continuity of f, g and the Spectral Representation Theorem, we deduce the desired result (5.7). □

Corollary 4 *With the assumptions of Theorem 3 and if* $\gamma, \Gamma, \delta, \Delta \in \mathbb{C}$, $\gamma \neq \Gamma$, $\delta \neq \Delta$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$ *and* $g \in \bar{\Delta}_{[a,b]}(\delta, \Delta)$, *then*

$$\Bigg|\langle f(A)g(A)x, y\rangle - \frac{\gamma+\Gamma}{2}\langle g(A)x, y\rangle - \frac{\delta+\Delta}{2}\langle f(A)x, y\rangle + \frac{\gamma+\Gamma}{2}\frac{\delta+\Delta}{2}\langle x, y\rangle\Bigg| \le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{4}|\Gamma-\gamma||\Delta-\delta|\,\|x\|\,\|y\| \tag{5.8}$$

for any $x, y \in H$.

Corollary 5 *With the assumptions of Theorem 3 and if* $g \in \mathcal{BV}_{\mathbb{C}}[a,b]$ *and* γ, $\Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$, *then*

$$\Bigg|\langle f(A)g(A)x, y\rangle - \frac{\gamma+\Gamma}{2}\langle g(A)x, y\rangle - \frac{g(a)+g(b)}{2}\langle f(A)x, y\rangle + \frac{\gamma+\Gamma}{2}\frac{g(a)+g(b)}{2}\langle x, y\rangle\Bigg| \le \frac{1}{4}|\Gamma-\gamma|\bigvee_{a}^{b}(g)\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{4}|\Gamma-\gamma|\bigvee_{a}^{b}(g)\,\|x\|\,\|y\| \tag{5.9}$$

for any $x, y \in H$.

Corollary 6 *With the assumptions of Theorem 3 and if* $\gamma, \Gamma \in \mathbb{C}$, $\gamma \neq \Gamma$ *such that* $f \in \bar{\Delta}_{[a,b]}(\gamma, \Gamma)$,

$$\Bigg|\langle f(A)g(A)x, y\rangle - \frac{\gamma+\Gamma}{2}\langle g(A)x, y\rangle - \langle f(A)x, y\rangle\frac{1}{b-a}\int_a^b g(t)\,dt + \langle x, y\rangle\frac{\gamma+\Gamma}{2}\frac{1}{b-a}\int_a^b g(t)\,dt\Bigg| \le \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|\bigvee_{a-0}^{b}\left(\left\langle E_{(\cdot)}x, y\right\rangle\right) \le \frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}\left|g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right|\|x\|\,\|y\| \tag{5.10}$$

for any $x, y \in H$.

Finally, by the use of the inequality (4.2) in the form

$$\left|[v(b)-v(a)]\int_a^b f(t)\,g(t)\,dv(t)-\int_a^b f(t)\,dv(t)\int_a^b g(t)\,dv(t)\right|$$
$$\le \frac{1}{2}|\Gamma-\gamma|\bigvee_a^b(v)\max_{t\in[a,b]}\left|g(t)[v(b)-v(a)]-\int_a^b g(s)\,dv(s)\right|, \tag{5.11}$$

provided $v\in\mathcal{BV}_{\mathbb{C}}[a,b]$, $f,\ g\in\mathcal{C}_{\mathbb{C}}[a,b]$ and $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$ such that $f\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, we have

Corollary 7 *With the assumptions of Theorem 3 and if* $\gamma,\ \Gamma\in\mathbb{C}$, $\gamma\ne\Gamma$ *such that* $f\in\bar{\Delta}_{[a,b]}(\gamma,\Gamma)$, *then*

$$|\langle f(A)\,g(A)\,x,y\rangle\langle x,y\rangle-\langle f(A)\,x,y\rangle\langle g(A)\,x,y\rangle|$$
$$\le\frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}|g(t)\langle x,y\rangle-\langle g(A)\,x,y\rangle|\bigvee_{a-0}^b\left(\left\langle E_{(\cdot)}x,y\right\rangle\right)$$
$$\le\frac{1}{2}|\Gamma-\gamma|\max_{t\in[a,b]}|g(t)\langle x,y\rangle-\langle g(A)\,x,y\rangle|\,\|x\|\,\|y\| \tag{5.12}$$

for any $x,\ y\in H$.

6 Applications for Unitary Operators

A unitary operator is a bounded linear operator $U:H\to H$ on a Hilbert space H satisfying

$$U^*U=UU^*=1_H$$

where U^* is the adjoint of U, and $1_H:H\to H$ is the identity operator. This property is equivalent to the following:

(1) U preserves the inner product $\langle\cdot,\cdot\rangle$ of the Hilbert space, i.e., for all vectors x and y in the Hilbert space, $\langle Ux,Uy\rangle=\langle x,y\rangle$ and
(2) U is surjective.

The following result is well known [27, pp. 275–276]:

Theorem 4 (Spectral Representation Theorem) *Let U be a unitary operator on the Hilbert space H. Then there exists a family of projections* $\{P_\lambda\}_{\lambda\in[0,2\pi]}$, *called the spectral family of* U, *with the following properties:*

(a) $P_\lambda \le P_{\lambda'}$ *for* $\lambda \le \lambda'$;
(b) $P_0 = 0$, $P_{2\pi} = I$ *and* $P_{\lambda+0} = P_\lambda$ *for all* $\lambda \in [0, 2\pi)$*;*
(c) *We have the representation*

$$U = \int_0^{2\pi} \exp(i\lambda)\, dP_\lambda.$$

More generally, for every continuous complex-valued function φ *defined on the unit circle* $\mathcal{C}(0,1)$ *there exists a unique operator* $\varphi(U) \in \mathcal{B}(H)$ *such that for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *satisfying the inequality*

$$\left\| \varphi(U) - \sum_{k=1}^{n} \varphi\left(\exp\left(i\lambda_k'\right)\right)\left[P_{\lambda_k} - P_{\lambda_{k-1}}\right] \right\| \le \varepsilon$$

whenever

$$\begin{cases} 0 = \lambda_1 < \ldots < \lambda_{n-1} < \lambda_n = 2\pi, \\ \lambda_k - \lambda_{k-1} \le \delta \text{ for } 1 \le k \le n, \\ \lambda_k' \in [\lambda_{k-1}, \lambda_k] \text{ for } 1 \le k \le n \end{cases}$$

this means that

$$\varphi(U) = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda, \tag{6.1}$$

where the integral is of Riemann–Stieltjes type.

Corollary 8 *With the assumptions of Theorem 4 for* U, P_λ, *and* φ *we have the representations*

$$\varphi(U)x = \int_0^{2\pi} \varphi(\exp(i\lambda))\, dP_\lambda x \quad \text{for all } x \in H$$

and

$$\langle \varphi(U)x, y\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, y\rangle \quad \text{for all } x, y \in H. \tag{6.2}$$

In particular,

$$\langle \varphi(U)x, x\rangle = \int_0^{2\pi} \varphi(\exp(i\lambda))\, d\langle P_\lambda x, x\rangle \quad \text{for all } x \in H.$$

Moreover, we have the equality

$$\|\varphi(U)x\|^2 = \int_0^{2\pi} |\varphi(\exp(i\lambda))|^2 \, d\|P_\lambda x\|^2 \quad \text{for all } x \in H.$$

On making use of an argument similar to the one in [22, Theorem 6], we have:

Lemma 3 *Let $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ be the spectral family of the unitary operator U on the Hilbert space H. Then for any $x, y \in H$ and $0 \le \alpha < \beta \le 2\pi$ we have the inequality*

$$\bigvee_{\alpha}^{\beta} \left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \le \left\langle \left(P_\beta - P_\alpha\right)x, x\right\rangle^{1/2} \left\langle \left(P_\beta - P_\alpha\right)y, y\right\rangle^{1/2}, \tag{6.3}$$

where $\bigvee_{\alpha}^{\beta} \left(\left\langle P_{(\cdot)}x, y\right\rangle\right)$ denotes the total variation of the function $\left\langle P_{(\cdot)}x, y\right\rangle$ on $[\alpha, \beta]$.

In particular,

$$\bigvee_{0}^{2\pi} \left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \le \|x\| \|y\| \tag{6.4}$$

for any $x, y \in H$.

Theorem 5 *Let U be a unitary operator on the Hilbert space H and $\{P_\lambda\}_{\lambda\in[0,2\pi]}$ the spectral family of projections of U. Also, assume that $f, g : \mathcal{C}(0,1) \to \mathbb{C}$ are continuous on $\mathcal{C}(0,1)$. If $\phi, \Phi \in \mathbb{C}$, $\phi \ne \Phi$ are such that $f \circ \exp(i\cdot) \in \bar{\Delta}_{[0,2\pi]}(\phi, \Phi)$, then*

$$\begin{aligned} &\left| \langle f(U) g(U) x, y\rangle - \frac{\phi + \Phi}{2} \langle g(U) x, y\rangle \right| \\ &\qquad \le \frac{1}{2} |\Phi - \phi| \max_{t\in[0,2\pi]} |g(\exp(it))| \bigvee_{0}^{2\pi} \left(\left\langle P_{(\cdot)}x, y\right\rangle\right) \\ &\qquad\qquad \le \frac{1}{2} |\Phi - \phi| \max_{t\in[0,2\pi]} |g(\exp(it))| \|x\| \|y\| \end{aligned} \tag{6.5}$$

for any $x, y \in H$.

The proof follows by Theorem 1 and the Spectral Representation Theorem for unitary operators in a similar way with the proof of Theorem 3 and we omit the details.

Corollary 9 *With the assumptions of Theorem 5 and if* $\phi, \Phi, \psi, \Psi \in \mathbb{C}$, $\phi \neq \Phi$ *such that* $f \circ \exp(i\cdot) \in \bar{\Delta}_{[0,2\pi]}(\phi, \Phi)$, $g \circ \exp(i\cdot) \in \bar{\Delta}_{[0,2\pi]}(\psi, \Psi)$ *then*

$$\Big| \langle f(U) g(U) x, y\rangle - \frac{\phi + \Phi}{2} \langle g(U) x, y\rangle - \frac{\psi + \Psi}{2} \langle f(U) x, y\rangle + \frac{\phi + \Phi}{2} \frac{\psi + \Psi}{2} \langle x, y\rangle \Big|$$
$$\leq \frac{1}{4} |\Phi - \phi| |\Psi - \psi| \bigvee_0^{2\pi} \left(\left\langle P_{(\cdot)} x, y\right\rangle\right) \leq \frac{1}{4} |\Phi - \phi| |\Psi - \psi| \|x\| \|y\| \tag{6.6}$$

for any $x, y \in H$.

Corollary 10 *With the assumptions of Theorem 5 and if* $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$ *such that* $f \circ \exp(i\cdot) \in \bar{\Delta}_{[0,2\pi]}(\phi, \Phi)$, *then*

$$\Big| \langle f(U) g(U) x, y\rangle - \frac{\phi + \Phi}{2} \langle g(U) x, y\rangle - \langle f(U) x, y\rangle \frac{1}{2\pi} \int_0^{2\pi} g(\exp(it))\, dt + \langle x, y\rangle \frac{\phi + \Phi}{2} \frac{1}{2\pi} \int_0^{2\pi} g(\exp(it))\, dt \Big|$$
$$\leq \frac{1}{2} |\Phi - \phi| \max_{t \in [0,2\pi]} \left| g(\exp(it)) - \frac{1}{2\pi} \int_0^{2\pi} g(\exp(is))\, ds \right| \bigvee_0^{2\pi} \left(\left\langle P_{(\cdot)} x, y\right\rangle\right)$$
$$\leq \frac{1}{2} |\Phi - \phi| \max_{t \in [0,2\pi]} \left| g(\exp(it)) - \frac{1}{2\pi} \int_0^{2\pi} g(\exp(is))\, ds \right| \|x\| \|y\| \tag{6.7}$$

for any $x, y \in H$.

Corollary 11 *With the assumptions of Theorem 5 and if* $\phi, \Phi \in \mathbb{C}$, $\phi \neq \Phi$ *such that* $f \circ \exp(i\cdot) \in \bar{\Delta}_{[0,2\pi]}(\phi, \Phi)$, *then*

$$|\langle f(U) g(U) x, y\rangle \langle x, y\rangle - \langle f(U) x, y\rangle \langle g(U) x, y\rangle|$$
$$\leq \frac{1}{2} |\Phi - \phi| \max_{t \in [0,2\pi]} |g(\exp(it)) \langle x, y\rangle - \langle g(U) x, y\rangle| \bigvee_0^{2\pi} \left(\left\langle P_{(\cdot)} x, y\right\rangle\right)$$
$$\leq \frac{1}{2} |\Phi - \phi| \max_{t \in [0,2\pi]} |g(\exp(it)) \langle x, y\rangle - \langle g(U) x, y\rangle| \|x\| \|y\| \tag{6.8}$$

for any $x, y \in H$.

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Cauchy–Schwarz Inequality and Riccati Equation for Positive Semidefinite Matrices

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Abstract By the use of the matrix geometric mean #, the matrix Cauchy–Schwarz inequality is given as $Y^*X \leq X^*X \# U^*Y^*YU$ for $k \times n$ matrices X and Y, where $Y^*X = U|Y^*X|$ is a polar decomposition of Y^*X with unitary U. In this note, we generalize Riccati equation as follows: $X^*A^\dagger X = B$ for positive semidefinite matrices, where $A^\dagger$ is the Moore–Penrose generalized inverse of A. We consider when the matrix geometric mean $A \# B$ is a positive semidefinite solution of $XA^\dagger X = B$. For this, we discuss the case where the equality holds in the matrix Cauchy–Schwarz inequality.

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1 Introduction

One of the most important inequalities in functional analysis is the Cauchy–Schwarz inequality. It is originally an integral inequality, but is usually expressed as follows: Let H be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Then

$$|\langle x, y \rangle| \leq \langle x, x \rangle^{1/2} \langle y, y \rangle^{1/2} \quad \text{for } x, y \in H. \tag{1.1}$$

Matrix versions of the Cauchy–Schwarz inequality have been discussed by Marshall and Olkin [7], see also Bhatia and Davis [2] for operator versions.

Now we note that its right-hand side of (1.1) is the geometric mean of $\langle x, x \rangle$ and $\langle y, y \rangle$. From this viewpoint, Fujii [3] proposed a matrix Cauchy–Schwarz inequality by the use of the matrix geometric mean #, see [5, Lemma 2.6]. Let X and Y be $k \times n$

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matrices and $Y^*X = U|Y^*X|$ a polar decomposition of an $n \times n$ matrix Y^*X with unitary U. Then

$$|Y^*X| \leq X^*X\#U^*Y^*YU,$$

where the matrix geometric mean # is defined by

$$A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

for positive definite matrices A and B, see [6].

On the other hand, the original definition of it for operators is given by Ando [1] as follows: For $A, B \geq 0$, it is defined by

$$A\#B = \max\left\{X \geq 0; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\}.$$

Here a bounded linear operator A acting on a Hilbert space H is positive, denoted by $A \geq 0$, if $\langle Ax, x\rangle \geq 0$ for all $x \in H$. It is obvious that a matrix A is positive semidefinite if and only if $A \geq 0$, and A is positive definite if and only if $A > 0$, i.e., A is positive and invertible. It is known that if $A > 0$, then they coincide, that is,

$$\max\left\{X \geq 0; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\} = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$$

holds for any $B \geq 0$.

Another approach of geometric mean is the Riccati equation. For $A > 0$ and $B \geq 0$, $A\#B$ is the unique solution of the Riccati equation

$$XA^{-1}X = B.$$

This fact is easily checked by multiplying $A^{-1/2}$ on both sides. For importance of Riccati equation, we refer [8]. Throughout this paper, we restrict our attention to positive semidefinite matrices, by which we can consider the generalized inverse $X^\dagger$ in the sense of Moore–Penrose even if they are not invertible. Among others, we generalize the Riccati equation to

$$XA^\dagger X = B.$$

In this paper, we discuss order relations between $A\#B$ and $A^{1/2}((A^{1/2})^\dagger B(A^{1/2})^\dagger)^{1/2}A^{1/2}$ for positive semidefinite matrices A and B. As an application, we discuss the case where the equality holds in matrix Cauchy–Schwarz inequality. Finally we generalize some results in our previous paper [4] by the use of the generalized inverse $X^\dagger$.

2 A Generalization of Formula for Geometric Mean

Since $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ for invertible A, the geometric mean $A\#B$ for positive semidefinite matrices A and B might be expected the same formulae as for positive definite matrices, i.e.,

$$A\#B = A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2}.$$

As a matter of fact, the following result is mentioned by Fujimoto and Seo [5]. For convenience, we cite it as Theorem FS:

Theorem FS *Let A and B be positive semidefinite matrices. Then*

$$A\#B \le A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2},$$

If the kernel inclusion $\ker A \subset \ker B$ *is assumed, then the equality holds in above.*

We remark that the point of its proof is that A and B are expressed as $A = A_1 \oplus 0$ and $B = B_1 \oplus 0$ on $\operatorname{ran} A \oplus \ker A$, respectively, and $A^{\dagger} = (A_1)^{-1} \oplus 0$.

Now Theorem FS has an improvement in the following way. Below, let P_A be the projection onto $\operatorname{ran} A$, the range of A.

Theorem 2.1 *Let A and B be positive semidefinite matrices. Then*

$$A\#B \le A^{1/2}((A^{1/2})^{\dagger}B(A^{1/2})^{\dagger})^{1/2}A^{1/2},$$

In particular, the equality holds in above if and only if $P_A = AA^{\dagger}$ commutes with B.

To prove it, we cite the following lemma:

Lemma 2.2 *If* $\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \ge 0$*, then $X = AA^{\dagger}X = P_A X$ and $B \ge XA^{\dagger}X$.*

Proof The assumption implies that

$$\begin{pmatrix} (A^{1/2})^{\dagger} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} (A^{1/2})^{\dagger} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} P_A & (A^{1/2})^{\dagger}X \\ X^*(A^{1/2})^{\dagger} & B \end{pmatrix} \ge 0.$$

Moreover, since

$$\begin{aligned} 0 &\le \begin{pmatrix} 1 & -(A^{1/2})^{\dagger}X \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} P_A & (A^{1/2})^{\dagger}X \\ X^*(A^{1/2})^{\dagger} & B \end{pmatrix} \begin{pmatrix} 1 & -(A^{1/2})^{\dagger}X \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_A & 0 \\ 0 & B - X^*A^{\dagger}X \end{pmatrix}, \end{aligned}$$

we have $B \ge X^*A^{\dagger}X$.

Next we show that $X = P_A X$, which is equivalent to $\ker A \subseteq \ker X^*$. Suppose that $Ax = 0$. Putting $y = -\frac{1}{\|B\|} X^* x$, we have

$$\begin{aligned} 0 &\le \left(\begin{pmatrix} A & X \\ X^* & B \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right) \\ &= (Xy, x) + (X^*x, y) + (By, y) \\ &= -\frac{2}{\|B\|} \|X^*x\|^2 + \frac{1}{\|B\|^2} (BX^*x, X^*x) \\ &\le -\frac{\|X^*x\|^2}{\|B\|} \le 0. \end{aligned}$$

Hence we have $X^*x = 0$, that is, $\ker A \subseteq \ker X^*$ is shown.

Proof of Theorem 2.1 For the first half, it suffices to show that if $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \ge 0$, then

$$X \le A^{1/2}((A^{1/2})^\dagger B (A^{1/2})^\dagger)^{1/2} A^{1/2}$$

because of Ando's definition of the geometric mean. We here use the facts that $(A^{1/2})^\dagger = (A^\dagger)^{1/2}$, and that if $\begin{pmatrix} A & X \\ X & B \end{pmatrix} \ge 0$ for positive semidefinite X, then $X = AA^\dagger X = P_A X$ and $B \ge XA^\dagger X$ by Lemma 2.2.

Now, since $B \ge XA^\dagger X$, we have

$$(A^{1/2})^\dagger B (A^{1/2})^\dagger \ge [(A^{1/2})^\dagger X (A^{1/2})^\dagger]^2,$$

so that Löwner–Heinz inequality implies

$$[(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2} \ge (A^{1/2})^\dagger X (A^{1/2})^\dagger.$$

Hence it follows from $X = P_A X$ that

$$A^{1/2}[(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2} A^{1/2} \ge X.$$

Namely we have $Y = A^{1/2}[(A^{1/2})^\dagger B (A^{1/2})^\dagger]^{1/2} A^{1/2} \ge A \# B$.

Next suppose that $\ker A \subset \ker B$. Then we have $\operatorname{ran} B \subset \operatorname{ran} A$ and so

$$A^{1/2}(A^{1/2})^\dagger B (A^{1/2})^\dagger A^{1/2} = B.$$

Therefore, putting $C = (A^{1/2})^\dagger B (A^{1/2})^\dagger$, since

$$Y = A^{1/2}((A^{1/2})^\dagger B (A^{1/2})^\dagger)^{1/2} A^{1/2} = A^{1/2} C^{1/2} A^{1/2},$$

we have

$$\begin{pmatrix} A & Y \\ Y & B \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \begin{pmatrix} I & C^{1/2} \\ C^{1/2} & C \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \geq 0,$$

which implies that $Y \leq A\#B$ and thus $Y = A\#B$ by combining the result $Y \geq A\#B$ in the first paragraph.

Now we show the second half. Notation as in above. If $P_A = AA^\dagger (= A^{1/2}(A^{1/2})^\dagger)$ commutes with B, we have $P_A B P_A \leq B$. Therefore we have

$$\begin{pmatrix} A & Y \\ Y & B \end{pmatrix} \geq \begin{pmatrix} A & Y \\ Y & P_A B P_A \end{pmatrix} = \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \begin{pmatrix} I & C^{1/2} \\ C^{1/2} & C \end{pmatrix} \begin{pmatrix} A^{1/2} & 0 \\ 0 & A^{1/2} \end{pmatrix} \geq 0,$$

which implies that $Y \leq A\#B$ and hence $Y = A\#B$.

Conversely assume that the equality holds. Then $\begin{pmatrix} A & Y \\ Y & B \end{pmatrix} \geq 0$. Hence we have

$$B \geq YA^\dagger Y = A^{1/2} C A^{1/2} = P_A B P_A,$$

which means P_A commutes with B, cf. Lemma 2.2.

3 Solutions of a Generalized Riccati Equation

Noting that $A\#B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ for invertible A, the geometric mean $A\#B$ is the unique solution of the Riccati equation $XA^{-1}X = B$ if $A > 0$, see [8] for an early work. So we consider it for positive semidefinite matrices by the use of the Moore–Penrose generalized inverse, that is,

$$XA^\dagger X = B$$

for positive semidefinite matrices A, B.

Theorem 3.1 *Let A and B be positive semidefinite matrices satisfying the kernel inclusion* $\ker A \subset \ker B$. *Then* $A\#B$ *is a solution of a generalized Riccati equation*

$$XA^\dagger X = B.$$

Moreover, the uniqueness of its solution is ensured under the additional assumption $\ker A \subset \ker X$.

Proof We first note that $(A^{1/2})^\dagger = (A^\dagger)^{1/2}$ and $P_A = P_{A^\dagger}$. Putting $X_0 = A\#B$, either Theorem FS or 2.1 says that

$$X_0 = A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2}.$$

Therefore we have

$$\begin{aligned} X_0A^\dagger X_0 &= A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}P_A[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2} \\ &= A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]A^{1/2} \\ &= P_ABP_A = B \end{aligned}$$

Since ran $X_0 \subset$ ran $A^{1/2}$, X_0 is a solution of the equation.

The second part is proved as follows: If X is a solution of $XA^\dagger X = B$, then

$$(A^{1/2})^\dagger XA^\dagger X(A^{1/2})^\dagger = (A^{1/2})^\dagger B(A^{1/2})^\dagger,$$

so that

$$(A^{1/2})^\dagger X(A^{1/2})^\dagger = [(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}.$$

Hence we have

$$P_AXP_A = A^{1/2}[(A^{1/2})^\dagger B(A^{1/2})^\dagger]^{1/2}A^{1/2} = X_0.$$

Since $P_AXP_A = X$ by the assumption, $X = X_0$ is obtained.

As an application, we give a simple proof of the case where the equality holds in matrix Cauchy–Schwarz inequality, see [5, Lemma 2.5].

Corollary 3.2 *Let X and Y be $k \times n$ matrices and $Y^*X = U|Y^*X|$ a polar decomposition of an $n \times n$ matrix Y^*X with unitary U. If* ker $X \subset$ ker YU, *then*

$$|Y^*X| = X^*X\#U^*Y^*YU$$

if and only if $Y = XW$ for some $n \times n$ matrix W.

Proof Since ker $X^*X \subset$ ker U^*Y^*YU, the preceding theorem implies that $|Y^*X|$ is a solution of a generalized Riccati equation, i.e.,

$$U^*Y^*YU = |Y^*X|(X^*X)^\dagger|Y^*X| = U^*Y^*X(X^*X)^\dagger X^*YU,$$

or consequently

$$Y^*Y = Y^*X(X^*X)^\dagger X^*Y.$$

Noting that $X(X^*X)^\dagger X^*$ is the projection P_X, we have $Y^*Y = Y^*P_XY$ and hence

$$Y = P_XY = X(X^*X)^\dagger X^*Y$$

by $(Y - P_XY)^*(Y - P_XY) = 0$, so that $Y = XW$ for $W = (X^*X)^\dagger X^*Y$.

4 Geometric Mean in Operator Cauchy–Schwarz Inequality

The origin of Corollary 3.2 is the operator Cauchy–Schwarz inequality due to Fujii [3] as in below. Let $B(H)$ be the C^*-algebra of all bounded linear operators acting on a Hilbert space H.

OCS Inequality *If $X, Y \in B(H)$ and $Y^*X = U|Y^*X|$ is a polar decomposition of Y^*X with a partial isometry U, then*

$$|Y^*X| \leq X^*X\#U^*Y^*YU.$$

In his proof of it, the following well-known fact due to Ando [1] is used: For $A, B \geq 0$, the geometric mean $A\#B$ is given by

$$A\#B = \max\left\{X \geq 0; \begin{pmatrix} A & X \\ X & B \end{pmatrix} \geq 0\right\}$$

First of all, we discuss the case $Y^*X \geq 0$ in (OCS). That is,

$$Y^*X \leq X^*X\#Y^*Y$$

is shown: Noting that $Y^*X = X^*Y \geq 0$, we have

$$\begin{pmatrix} X^*X & X^*Y \\ Y^*X & Y^*Y \end{pmatrix} = \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} X & Y \\ 0 & 0 \end{pmatrix} \geq 0,$$

which means $Y^*X \leq X^*X\#Y^*Y$.

The proof for a general case is presented by applying the above: Noting that $(YU)^*X = |Y^*X| \geq 0$, it follows that

$$|Y^*X| = (YU)^*X \leq X^*X\#(YU)^*YU.$$

Incidentally, we can give a direct proof to the general case as follows:

$$\begin{pmatrix} X^*X & |Y^*X| \\ |Y^*X| & U^*Y^*YU \end{pmatrix} = \begin{pmatrix} X & YU \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} X & YU \\ 0 & 0 \end{pmatrix} \geq 0.$$

Related to matrix Cauchy–Schwarz inequality, the following result is obtained by Fujimoto–Seo [5]:

Let $\mathbb{A} = \begin{pmatrix} A & C \\ C^* & B \end{pmatrix}$ be positive definite matrix. Then $B \geq C^*A^{-1}C$ holds. Furthermore it is known by them:

Theorem 4.1 *Let $\mathbb{A}$ be as in above and $C = U|C|$ a polar decomposition of C with unitary U. Then*

$$|C| \leq U^*AU \;\#\; C^*A^{-1}C.$$

Proof It can be also proved as similar as in above: Since $|C| = U^*C = C^*U$, we have

$$\begin{pmatrix} U^*AU & |C| \\ |C| & C^*A^{-1}C \end{pmatrix} = \begin{pmatrix} A^{1/2}U & A^{-1/2}C \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} A^{1/2}U & A^{-1/2}C \\ 0 & 0 \end{pmatrix} \geq 0.$$

The preceding result is generalized a bit by the use of the Moore–Penrose generalized inverse, for which we note that $(A^{1/2})^\dagger = (A^\dagger)^{1/2}$ for $A \geq 0$:

Theorem 4.2 *Let $\mathbb{A}$ be of form as in above and positive semidefinite, and $C = U|C|$ a polar decomposition of C with unitary U. If* $\operatorname{ran} C \subseteq \operatorname{ran} A$, *then*

$$|C| \leq U^*AU \;\#\; C^*A^\dagger C.$$

Proof Let P_A be the projection onto the range of A. Since $P_AC = C$ and $C^*P_A = C^*$, we have $|C| = U^*P_AC = C^*P_AU$. Hence it follows that

$$\begin{pmatrix} U^*AU & |C| \\ |C| & C^*A^\dagger C \end{pmatrix} = \begin{pmatrix} A^{1/2}U & (A^\dagger)^{1/2}C \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} A^{1/2}U & (A^\dagger)^{1/2}C \\ 0 & 0 \end{pmatrix} \geq 0.$$

5 Solutions of Generalized Algebraic Riccati Equation

Following after [4], we discuss solutions of a generalized algebraic Riccati equation. Incidentally P_X means the projection onto the range of a matrix X.

Lemma 5.1 *Let A and B be positive semidefinite matrices and T an arbitrary matrix. Then W is a solution of a generalized Riccati equation*

$$W^*A^\dagger W = B + T^*AT$$

if and only if $X = W + AT$ is a solution of a generalized algebraic Riccati equation

$$X^*A^\dagger X - T^*P_AX - X^*P_AT = B.$$

Proof Put $X = W + AT$. Then it follows that

$$X^* A^\dagger X - T^* P_A X - X^* P_A T = W^* A^\dagger W - T^* AT,$$

so that we have the conclusion.

Theorem 5.2 *Let A and B be positive semidefinite matrices. Then W is a solution of a generalized Riccati equation*

$$W^* A^\dagger W = B \quad with \text{ ran } W \subseteq \text{ran } A$$

if and only if $W = A^{1/2} U B^{1/2}$ for some partial isometry U such that $U^* U \geq P_B$ and $UU^* \leq P_A$.

Proof Suppose that $W^* A^\dagger W = B$ and ran $W \subseteq$ ran A. Since $\|(A^{1/2})^\dagger W x\| = \|B^{1/2} x\|$ for all vectors x, there exists a partial isometry U such that $UB^{1/2} = (A^{1/2})^\dagger W$ with $U^* U = P_B$ and $UU^* \leq P_A$. Hence we have

$$A^{1/2} U B^{1/2} = P_A W = W.$$

The converse is easily checked: If $W = A^{1/2} U B^{1/2}$ for some partial isometry U such that $U^* U \geq P_B$ and $UU^* \leq P_A$, then ran $W \subseteq$ ran A and

$$W^* A^\dagger W = B^{1/2} U^* P_A U B^{1/2} = B^{1/2} U^* U B^{1/2} = B.$$

Corollary 5.3 *Notation as in above. Then X is a solution of a generalized algebraic Riccati equation*

$$X^* A^\dagger X - T^* X - X^* T = B$$

with ran $X \subseteq$ ran A *if and only if $X = A^{1/2} U (B + T^* AT)^{1/2} + AT$ for some partial isometry U such that $U^* U \geq P_{B+T^*AT}$ and $UU^* \leq P_A$.*

Proof By Lemma 5.1, X is a solution of a generalized algebraic Riccati equation $X^* A^\dagger X - T^* P_A X - X^* P_A T = B$ if and only if $W = X - AT$ is a solution of $W^* A^\dagger W = B + T^* AT$. Since ran $X \subseteq$ ran A if and only if ran $W \subseteq$ ran A, we have the conclusion by Theorem 5.2.

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Inequalities for Solutions of Linear Differential Equations in a Banach Space and Integro-Differential Equations

Michael Gil'

Abstract The chapter presents a survey of the recent results of the author on solution estimates for the linear differential equation $du(t)/dt = A(t)u(t)$ with a bounded operator $A(t)$ in a Banach space satisfying various conditions. These estimates give us sharp stability conditions as well as upper and lower bounds for the evolution operator. Applications to integro-differential equations are also discussed. In particular, we consider equations with differentiable in t operators, equations with the Lipschitz property, equations in the lattice normed spaces, and equations with the generalized Lipschitz property. In addition, we investigate integrally small perturbations of autonomous equations. In appropriate situations our stability conditions are formulated in terms of the commutators of the coefficients of the considered equations. A significant part of these results has been generalized in the available literature to equations with unbounded operators. Some results presented in the chapter are new.

AMS (MOS) Subject Classification 34G10, 34D05, 34D20

1 Introduction and Notations

Let $\mathscr{X}$ be a Banach space with a norm $\|.\|$ and the unit operator I. By $\mathscr{B}(\mathscr{X})$ we denote the algebra of bounded linear operators in $\mathscr{X}$. For a $C \in \mathscr{B}(\mathscr{X})$, $\sigma(C)$ is the spectrum, $\alpha(C) = \sup \Re\sigma(C)$, $\beta(C) = \inf \Re\sigma(C)$, $\|C\|$ is the operator norm, C^* is the adjoint operator.

By $\mathscr{H}$ we denote a separable complex Hilbert space with a scalar product $(.,.)$ and the norm $\|.\| = \sqrt{(.,.)}$.

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The present paper is a survey of the recent results of the author on solution estimates for the equation

$$\frac{du(t)}{dt} = A(t)u(t) \quad (t \geq 0), \tag{1}$$

with a bounded operator $A(t)$ in $\mathscr{X}$ satisfying the conditions pointed below. These estimates give us sharp stability conditions as well as upper and lower bounds for the evolution operator. Applications to integro-differential equations are also discussed.

A solution to (1) for a given $u_0 \in \mathscr{X}$ is a function $u : [0, \infty) \to \mathscr{X}$ having at each point $t > 0$ a strong derivative, at $t = 0$—the right strong derivative, and satisfying (1) for all $t > 0$ and the equality $u(0) = u_0$. The existence and uniqueness of solutions under considerations are obvious.

Equation (1) is said to be exponentially stable, if there are positive constants m_0 and ϵ, such that $\|u(t)\| \leq m_0 exp\ [-\epsilon t]\|u(0)\|$ $(t \geq 0)$ for any solution $u(t)$ of (1). Besides, the evolution operator $U(t, s)$ $(t \geq s \geq 0)$ of (1) is defined by the equality $U(t, s)u(s) = u(t)$.

The literature on stability of abstract differential equations is very rich, cf. [2, 3, 5, 7, 10, 32, 33], and the references therein, but the problem of stability analysis of such equations continues to attract the attention of many specialists despite its long history. It is still one of the most burning problems of the theory of differential equations, because of the absence of its complete solution. The basic method for the stability analysis of (1) is the direct Lyapunov method [7]. By that method many very strong results are obtained, but finding Lyapunov's functions is often connected with serious mathematical difficulties. Below we suggest various explicit stability conditions.

In particular, in Sect. 2 we derive a stability test for Eq. (1) in a Hilbert space assuming that $A(t)$ has "small" derivatives. Recall the Wintner inequalities

$$exp\ [\int_s^t \lambda(\Re A(t_1))dt_1] \leq \frac{\|U(t, s)v\|}{\|v\|} \leq exp\ [\int_s^t \Lambda(\Re A(t_1))dt_1]$$

$$(0 \leq s \leq t;\ v \in \mathscr{H}),$$

where $\Re A(t) = (A(t) + A^*(t))/2$,

$$\Lambda(\Re A(t)) := \sup \sigma(\Re A(t)) \text{ and } \lambda(\Re A(t)) := \inf \sigma(\Re A(t)),$$

cf. [7, Theorem III.4.7]. Note that if $A(t)$ is not dissipative, i.e. if $A(t) + A^*(t)$ is not negative definite, then the Wintner inequalities do not give us stability conditions even in the case of a constant operator. In Sect. 2 we suggest estimates for the evolution operator in $\mathscr{H}$, which give us stability conditions for equations with non-dissipative operators. The main result of this section has been generalized in [26] to equations with unbounded operators.

In Sects. 3 and 4 we consider the equation

$$du(t)/dt = (B + C(t))u(t), \tag{2}$$

where B is a constant bounded operator, and $C(t)$ is a function defined on $[0, \infty)$ whose values are bounded operators. Conditions for the exponential stability are derived in terms of the commutator $BC(t) - C(t)B$. The results of Sects. 3 and 4 have been generalized in [27, 28, 30] to equations with unbounded operators.

Section 5 is devoted to integrally small perturbations of equations with constant operators.

In Sect. 6 we extend the freezing method for ordinary differential equations [4, 18, 35] to equations in $\mathscr{X}$. Moreover, assuming that the norm of $A(t)$ satisfies the Lipschitz condition, we establish upper and lower bounds for the norm of $U(t, s)$. The main result of this section has been generalized in [29] to equations with unbounded operators.

Section 7 deals with the equations in a lattice normed space. Here the important role is played by the generalized (Kantorovich) norm.

In Sect. 8 we generalize some results from Sect. 6 assuming that $A(t)$ satisfies the so-called generalized Lipschitz condition.

In Sects. 9 and 10 we discuss applications of our results to integro-differential equations. Besides, our results supplement the well-known investigations of integro-differential equations [1, 6, 9, 11, 12, 31, 34, 37] and the references therein.

2 Equations with Differentiable in Time Coefficients

2.1 Statement of the Result

In this section we consider Eq. (1) a Hilbert space $\mathscr{H}$, assuming that $A(t)$ is bounded on $[0, \infty)$, has a measurable strong derivative bounded on $[0, \infty)$. In addition,

$$\sup_{t\geq 0} \alpha(A(t)) = \sup_{t\geq 0} \sup \Re\sigma(A(t)) < 0. \tag{3}$$

Then the integrals

$$Q(t) := 2\int_0^\infty e^{A^*(t)s} e^{A(t)s} ds \text{ and } q(t) := 2\int_0^\infty \|e^{A(t)s}\|^2 ds \tag{4}$$

converge.

Theorem 1 *Let the conditions (3) and*

$$\sup_{t\geq 0} q^2(t)\|A'(t)\| < 2 \tag{5}$$

hold. Then any solution $u(t)$ to Eq. (1) satisfies the inequality

$$(Q(t)u(t), u(t)) \le (Q(0)u(0), u(0)) \quad (t \ge 0). \tag{6}$$

Moreover, (1) is exponentially stable.

This theorem is proved in the next subsection.

2.2 Proof of Theorem 1

Recall that the Lyapunov equation

$$A_0^* Y + Y A_0 = E \tag{7}$$

with a constant bounded stable operator A_0 (i.e., $\alpha(A_0) < 0$) and a constant bounded operator E has a solution Y which is represented as

$$Y = -\int_0^\infty e^{A_0^* s} E e^{A_0 s} ds, \tag{8}$$

cf. [7, Section I.4.4]. Then due to (4), $Q(t)$ is a unique solution of the equation

$$A^*(t)Q(t) + Q(t)A(t) = -2I \quad (t \ge 0). \tag{9}$$

Clearly,

$$\|Q(t)\| \le q(t). \tag{10}$$

Lemma 1 *Let condition (3) hold and $A(t)$ be strongly differentiable. Then $Q(t)$ is strongly differentiable and $\|Q'(t)\| \le q^2(t)\|A'(t)\|$.*

For the proof see Lemma 2 from [22].

Lemma 2 *Let the condition*

$$\sup_{t \ge 0} \|Q'(t)\| < 2. \tag{11}$$

hold. Then inequality (6) is valid

Proof Multiplying Eq. (1) by $Q(t)$ and doing the scalar product, we can write

$$(Q(t)u'(t), u(t)) = (Q(t)A(t)u(t), u(t)).$$

Since

$$\frac{d}{dt}(Q(t)u(t), u(t)) = (Q(t)u'(t), u(t)) + (u(t), Q(t)u'(t)) + (Q'(t)u(t), u(t)),$$

in view of (9) it can be written

$$\frac{d}{dt}(Q(t)u(t), u(t)) = (Q(t)A(t)u(t), u(t)) + (u(t), Q(t)A(t)u(t)) + (Q'(t)u(t), u(t))$$

$$= ((Q(t)A(t) + A^*(t)Q(t))u(t), u(t)) + (Q'(t)u(t), u(t))$$

$$= -2(u(t), u(t)) + (Q'(t)u(t), u(t)).$$

Hence,

$$\frac{d}{dt}(Q(t)u(t), u(t)) \le (-2 + \|Q'(t)\|)(u(t), u(t)) < 0. \tag{12}$$

Solving this inequality we get the required result. □

Lemma 3 *Let conditions (3) and (5) hold. Then (1) is exponentially stable.*

Proof For a stable operator A_0 put $y(t) = e^{A_0 t}v$ ($v \in \mathscr{H}$). Then $y'(t) = A_0 y(t)$, and

$$\frac{d(y(t), y(t))}{dt} = ((A_0 + A_0^*)y(t), y(t)).$$

Hence denoting by $\lambda(\Re A_0)$ the smallest eigenvalue of $\Re A_0 := (A_0 + A_0^*)/2$ we have

$$\frac{d(y(t), y(t))}{dt} \ge 2\lambda(\Re A_0)(y(t), y(t)) \text{ and therefore } \|e^{A_0 t}v\| \ge e^{t\lambda(\Re A_0)}\|v\|.$$

Recall that A_0 is stable, so $\|e^{A_0 t}v\| \to 0$ and therefore $e^{t\lambda(\Re A_0)} \to 0$ ($t \to \infty$). Hence it follows that $\lambda(\Re A_0) < 0$. Put

$$Q_0 = 2\int_0^\infty e^{A_0^* s}e^{A_0 s}ds.$$

Then we get

$$(Q_0 h, h) = 2\int_0^\infty (e^{A_0^* s}e^{A_0 s}h, h)ds \ge 2\int_0^\infty e^{2\lambda(\Re A_0)s}ds\ \|h\|^2 = \frac{\|h\|^2}{|\lambda(\Re A_0)|}\ (h \in \mathscr{H}).$$

Hence,

$$(Q(t)u(t), u(t)) \geq \frac{\|u(t)\|^2}{|\lambda(\Re A(t))|}. \tag{13}$$

Now Lemma 3 implies

$$(u(t), u(t)) \leq 2|\lambda(\Re A(t))|(Q(0)u(0), u(0)) \quad (t \geq 0).$$

But $|\lambda(A(t))|$ is uniformly bounded on $[0, \infty$ and therefore all the solutions of (1) are uniformly bounded (i.e., (1) is Lyapunov stable). Furthermore, substitute into (1)

$$u(t) = u_\epsilon(t)e^{-\epsilon t} \quad (\epsilon > 0). \tag{14}$$

Then

$$\dot{u}_\epsilon(t) = (A(t) + \epsilon I)u_\epsilon(t). \tag{15}$$

Applying our above arguments to (15) can assert that Eq. (15) with small enough $\epsilon > 0$ is Lyapunov stable. So due to (14) Eq. (1) is exponentially stable, provided (3) and (11) hold. As claimed. □

Proof of Theorem 1 The required assertion follows from Lemmas 1 and 3. □

2.3 *Upper and Lower Bounds for Evolution Operators of Equations with Differentiable in Time Coefficients*

According to (12), under condition (6) we have

$$\frac{d}{dt}(Q(t)u(t), u(t)) = ((-2I + Q'(t))u(t), u(t)) \leq -(2 - \|Q'(t)\|)(u(t), u(t)).$$

But

$$\|Q(t)\|(u(t), u(t)) \geq (Q(t)u(t), u(t)).$$

Hence,

$$\frac{d}{dt}(Q(t)u(t), u(t)) = ((-2I + Q'(t))u(t), u(t)) \leq -\frac{(2 - \|Q'(t)\|)}{\|Q(t)\|}(Q(t)u(t), u(t)).$$

Solving this inequality and taking into account (10), we obtain

$$(Q(t)u(t), u(t)) \le (Q(s)u(s), u(s)) \exp\,[-\int_s^t \frac{(2 - \|Q'(t_1)\|)}{q(t_1)} dt_1].$$

Now (13) implies

$$\|u(t)\|^2 \le |\lambda(\Re A(t))| q(s) \|u(s)\|^2 \exp\,[-\int_s^t \frac{(2 - \|Q'(t_1)\|)}{q(t_1)} dt_1].$$

Making use of Lemma 1 we arrive at

Corollary 1 *Let conditions (3) and (5) hold. Then the evolution operator of (1) satisfies the inequality*

$$\|U(t,s)\|^2 \le |\lambda(\Re A(t))| q(s) \exp\,[-\int_s^t \frac{(2 - q^2(t_1)\|A'(t_1)\|)}{q(t_1)} dt_1].$$

Furthermore, due to (12)

$$\frac{d}{dt}(Q(t)u(t), u(t)) = ((-2I + Q'(t))u(t), u(t)) \ge -(2 + \|Q'(t)\|)(u(t), u(t)).$$

Hence by (13)

$$\frac{d}{dt}(Q(t)u(t), u(t)) = ((-2I + Q'(t))u(t), u(t)) \ge -(2 + \|Q'(t)\|)|\lambda(\Re A(t))|(Q(t)u(t), u(t)).$$

Solving this inequality, we obtain

$$(Q(t)u(t), u(t)) \ge (Q(s)u(s), u(s)) \exp\,[-\int_s^t (2 + \|Q'(t_1)\|)|\lambda(\Re A(t_1))dt_1].$$

Now (10) and (13) imply

$$q(t)\|u(t)\|^2 \ge \frac{\|u(s)\|^2}{|\lambda(\Re A(s))|} \exp\,[-\int_s^t (2 + \|Q'(t_1)\|)|\lambda(\Re A(t_1))|dt_1].$$

Making use of Lemma 1, we arrive at

Corollary 2 *Let conditions (3) and (5) hold. Then the evolution operator of (1) satisfies the inequality*

$$\|U(t,s)h\|^2 \ge \frac{\|h\|^2}{|\lambda(\Re A(s))|} \exp\,[-\int_s^t (2 + q^2(t_1)\|A'(t_1)\|)|\lambda(\Re A(t_1))|dt_1]$$

$$(h \in \mathscr{H}, 0 \le s \le t).$$

2.4 Coefficients with Hilbert–Schmidt Components

Assume that

$$\Im A_0 := (A_0 - A_0^*)/2i \text{ is a Hilbert–Schmidt operator,} \tag{16}$$

i.e., $N_2^2(\Im A_0) = \text{trace}\,(\Im A_0)^2 < \infty$, and put

$$g_I(A_0) := (2N_2^2(\Im A_0) - 2\sum_{k=1}^{\infty} |Im\ \lambda_k(A_0)|^2)^{1/2},$$

where $\lambda_k(A_0)$ $(k = 1, 2, \ldots)$ are the nonreal eigenvalues of A_0.

Lemma 4 *Let the conditions (16) and $\alpha(A_0) < 0$ hold. Then a solution Y of Eq. (7) is subject to the inequality*

$$\|Y\| \leq \|E\| \sum_{j,k=1}^{\infty} \frac{g_I^{j+k}(A_0)(k+j)!}{|2\alpha(A_0)|^{j+k+1}(j!\,k!)^{3/2}}.$$

Proof We need the estimate

$$\|e^{A_0 t}\| \leq \exp[\alpha(A_0)t] \sum_{k=0}^{\infty} \frac{g_I^k(A_0)t^k}{(k!)^{3/2}} \quad (t \geq 0)$$

proved in [17, Example 7.10.3]. Then due to (8),

$$\|Y\| \leq \|E\| \int_0^{\infty} \|e^{A_0 t}\|^2 dt \leq$$

$$\|E\| \int_0^{\infty} \exp[2\alpha(A_0)t](\sum_{k=0}^{\infty} \frac{g_I^k(A_0)t^k}{(k!)^{3/2}})^2 dt \leq \|E\| \int_0^{\infty} e^{2\alpha(A_0)t} \sum_{j,k=0}^{\infty} \frac{(g_I(A_0)t)^{k+j}}{(j!\,k!)^{3/2}} dt$$

$$= \|E\| \sum_{j,k=0}^{\infty} \frac{(k+j)!\,g_I^{j+k}(A_0)}{(2|\alpha(A_0)|)^{j+k+1}(j!\,k!)^{3/2}},$$

as claimed. □

Suppose that condition (3) holds and

$$\Im A(t) \quad (t \geq 0) \text{ is a Hilbert–Schmidt operator.}$$

Then due to the previous lemma

$$q(t) \le \mu(A(t)) := 2 \sum_{j,k=1}^{\infty} \frac{g_I^{j+k}(A(t))(k+j)!}{|2\alpha(A(t))|^{j+k+1}(j!\,k!)^{3/2}}.$$

Theorem 1 and the previous inequality imply

Corollary 3 *Let $\Im A_I(t)$ be a Hilbert-Schmidt operator. Let the conditions (3) and*

$$\sup_{t\ge 0} \mu^2(t)\|A'(t)\| < 2$$

hold. Then (1) is exponentially stable.

3 Estimates for Solutions of Differential Equations in a Banach Space via Commutators

3.1 Statement of the Result

We consider Eq. (2) where $B \in \mathscr{B}(\mathscr{X})$ is a constant operator and $C(t)$ is a function with values in $\mathscr{B}(\mathscr{X})$ uniformly bounded on $[0, \infty)$ and Riemann-integrable on each finite segment.

Equation (2) can be considered as Eq. (1) with a variable operator $A(t)$. This identification which is a common device in the theory of differential equations when passing from a given equation to an abstract evolution equation turns out to be useful also here. Observe that $A(t)$ in the considered case has a special form: it is the sum of operators B and $C(t)$. This fact allows us to use the information about the coefficients more completely than the theory of differential equations (1) containing an arbitrary operator $A(t)$.

Let $U(t, s)$ be evolution operator of (2) and $W(t, s)$ $(t \ge s \ge 0)$ be the evolution operator of the equation

$$\dot{y}(t) = C(t)y(t) \quad (t \ge 0). \tag{17}$$

In this section we suggest estimates for the sup-norm of solutions to (2) via the commutator $K(t) := [B, C(t)] = BC(t) - C(t)B$. These estimates give us sharp stability conditions, provided we have estimates for $W(t, s)$. Besides, we do not require that the operator $B + C(t)$ is dissipative.

Put

$$Y(t, s) = e^{B(t-s)}W(t, s),\ \kappa = \sup_{t\ge 0}\|K(t)\| \text{ and } \|U\|_C := \sup_{t\ge s\ge 0}\|U(t, s)\|.$$

It is assumed that there are positive constants c_0 and b_0, such that

$$\|W(t,s)\| \le c_0 \exp\left[-b_0(t-s)\right] \ \ (t \ge s \ge 0). \tag{18}$$

We say that a bounded operator *is stable* if its spectrum is in the open left half plane.

Theorem 2 *Let condition (18) hold and the operator $B - b_0 I$ be stable. In addition, let*

$$\zeta := \kappa c_0 \left(\int_0^\infty \|e^{(B-b_0 I)t}\| dt\right)^2 < 1. \tag{19}$$

Then

$$\|U - Y\|_C := \sup_{t \ge s \ge 0} \|U(t,s) - Y(t,s)\|_{C(s,\infty)} \le \frac{\zeta \|Y\|_C}{1-\zeta} \tag{20}$$

and Eq. (2) is exponentially stable.

The proof of this theorem is presented in the next section. Note that under consideration

$$\|Y\|_C \le c_0 \sup_t \|e^{(B-b_0 I)t}\|.$$

Theorem 2 is sharp: if B commutes with $C(t)$, then $U(t,s) = Y(t,s)$ $(t \ge s)$. In addition, from (20) it directly follows

$$\|U\|_C \le \frac{\|Y\|_C}{1-\zeta}. \tag{21}$$

3.2 *Proof of Theorem 2*

Let $[e^{Bt}, C(t)] := e^{tB}C(t) - C(t)e^{Bt}$.

Lemma 5 *One has*

$$[e^{B(t-s)}, C(t)] = \int_s^t e^{(v-s)B} K(t) e^{(t-v)B} dv \ \ (0 \le s \le t < \infty).$$

Proof We have

$$\int_s^t e^{(v-s)B} K(t) e^{(t-v)B} dv = \int_s^t e^{(v-s)B} (BC(t) - C(t)B) e^{(t-v)B} dv =$$

$$\int_s^t (\frac{d}{dv}e^{(v-s)B})C(t)e^{(t-v)B} + e^{vB}C(t)(\frac{d}{dv}e^{(t-v)B}))dv =$$

$$\int_s^t \frac{d}{dv}(e^{(v-s)B}C(t)e^{(t-v)B})dv = e^{(v-s)B}C(t)e^{(t-v)B}|_{v=s}^t = e^{B(t-s)}C(t) - Ce^{B(t-s)}.$$

We thus get the required result. □

The previous lemma is a generalization of Lemma 2.1 from [23].

Lemma 6 *With the notation* $F(t,s) := [e^{B(t-s)}, C(t)]W(t,s)$*, let*

$$\gamma(F) := \sup_s \int_s^\infty \|F(t,s)\|dt < 1. \tag{22}$$

Then

$$\|U\|_C \leq \frac{\|Y\|_C}{1-\gamma(F)} \tag{23}$$

and

$$\|U - Y\|_C \leq \frac{\gamma(F)\|Y\|_C}{1-\gamma(F)}. \tag{24}$$

Proof We have

$$\frac{\partial}{\partial t}Y(t,s) = \frac{\partial}{\partial t}(e^{B(t-s)}W(t,s)) = (\frac{\partial}{\partial t}e^{B(t-s)})W(t,s) + e^{B(t-s)}\frac{\partial}{\partial t}W(t,s)$$

$$= Be^{B(t-s)}W(t,s) + e^{B(t-s)}C(t)W(t,s) = (B + C(t))e^{B(t-s)}W(t,s) + F(t,s).$$

Thus,

$$\partial Y(t,s)/\partial t = (B + C(t))Y(t,s) + F(t,s). \tag{25}$$

Subtracting (2) from (25), for a fixed s, we get

$$\frac{\partial}{\partial t}(Y(t,s) - U(t,s)) = (B + C(t))(Y(t,s) - U(t,s)) + F(t,s). \tag{26}$$

Since $Y(s,s) = U(s,s) = I$, we can write

$$Y(t,s) - U(t,s) = \int_s^t U(t,t_1)F(t_1,s)dt_1.$$

Consequently,

$$\|Y(t,s) - U(t,s)\| \le \int_0^t \|U(t,t_1)\| \|F(t_1,s)\| dt_1, \tag{27}$$

and therefore

$$\|U(t,s)\| \le \|Y(t,s)\| + \int_s^t \|U(t,t_1)\| \|F(t_1,s)\| dt_1. \tag{28}$$

Hence, for any finite $t > 0$ we obtain

$$\sup_{0 \le s \le v \le t} \|U(v,s)\| \le \|Y\|_C + \sup_{0 \le s \le v \le t} \|U(v,s)\| \gamma(F). \tag{29}$$

Now (22) implies

$$\sup_{0 \le s \le v \le t} \|U(v,s)\| \le \|Y\|_C / (1 - \gamma(F)).$$

This proves (23). From (23), inequality (24) follows. This proves the lemma. □

Proof of Theorem 2 By Lemma 5,

$$\|F(t,s)\| \le \|[e^{B(t-s)}, C(t)]\| \|W(t,s)\| \le \kappa \|W(t,s)\| \int_s^t \|e^{B(v-s)}\| \|e^{B(t-v)}\| dv.$$

Hence,

$$\int_s^\infty \|F(t,s)\| ds \le \hat{\gamma}(s),$$

where

$$\hat{\gamma}(s) := \kappa \int_s^\infty \|W(t,s)\| \int_s^t \|e^{B(v-s)}\| \|e^{B(t-v)}\| dv \, dt.$$

So $\gamma(F) \le \sup_s \hat{\gamma}(s)$. From (18) it follows

$$\hat{\gamma}(s) \le \kappa c_0 \int_s^\infty e^{-b_0(t-s)} \int_s^t \|e^{B(t-v)}\| \|e^{B(v-s)}\| dv \, dt$$

$$= \kappa c_0 \int_s^\infty \|e^{B(v-s)}\| \int_v^\infty \|e^{B(t-v)}\| e^{-b_0(t-s)} \, dt \, \, dv$$

$$= \kappa c_0 \int_s^\infty \|e^{B(v-s)}\| \int_0^\infty \|e^{Bt_1}\| e^{-b_0(t_1+v-s)} dt_1 dv.$$

Thus

$$\gamma(F) \le \kappa c_0 \|e^{(B-b_0 I)t}\|^2_{L^1(0,\infty)}.$$

Here

$$\|e^{(B-b_0 I)t}\|_{L^1(0,\infty)} = \int_0^\infty \|e^{(B-b_0 I)t}\| dt.$$

Now Lemma 6 proves (20) and (21). Inequality (21) means that (2) is Lyapunov stable. Furthermore, substitute

$$u(t) = u_\epsilon(t) e^{-\epsilon t} \quad (\epsilon > 0) \tag{30}$$

into (2). Then

$$du_\epsilon(t)/dt = (B + C(t) + \epsilon I) u_\epsilon(t). \tag{31}$$

Applying our above arguments to (31) one can assert that Eq. (31) with small enough $\epsilon > 0$ is Lyapunov stable. So due to (30) Eq. (2) is exponentially stable. This proves the theorem. □

3.3 Auxiliary Results

To apply Theorem 2 to concrete equations we need some auxiliary results presented in this section. Introduce the products

$$\prod_{1\le k\le m}^{\leftarrow} (I + C(t_k^{(m)})\delta_k) :=$$

$$(I + C(t_m^{(m)})\delta_m)(I + C(t_{m-1}^{(m)})\delta_{m-1}) \cdots (I + C(t_1^{(m)})\delta_1),$$

where

$$s = t_1^{(m)} < t_2^{(m)} < \ldots < t_m^{(m)} = t \text{ and } \delta_k = t_k^{(m)} - t_{k-1}^{(m)} \quad (k = 1, \ldots, m).$$

That is, the arrow over the symbol of the product means that the indexes of the co-factors increase from right to left. The strong limit of these products as $\max_k \delta_k \to 0$ (if it exists) is called *the left multiplicative integral* and is denoted by $\int_{[s,t]}^{\leftarrow} (I + C(s_1)ds_1)$. As it is well-known,

$$W(t,s) = \int_{[s,t]}^{\leftarrow} (I + C(s_1)ds_1). \tag{32}$$

This equality is proved in [8]; in the finite dimensional case it is proved in [13, Chapter XV, Section 6] but for operators in a Banach space the proof is the same.

Furthermore, introduce the products

$$\prod_{1\le k\le m}^{\leftarrow} e^{C(t_k^{(m)})\delta_k} := e^{C(t_m^{(m)})\delta_m} e^{C(t_{m-1}^{(m)})\delta_{m-1}} \dots e^{C(t_1^{(m)})\delta_1}$$

$$(s = t_1^{(m)} < t_2^{(m)} < \dots < t_m^{(m)} = t).$$

The strong limit of these products as $\max_k \delta_k \to 0$ (if it exists) will be called *the left exponentially multiplicative integral* and denoted by $\int_{[s,t]}^{\leftarrow} e^{C(t_1)dt_1}$. As it is shown in [7, Section III.1],

$$W(t,s) = \int_{[s,t]}^{\leftarrow} e^{C(t_1)dt_1} \tag{33}$$

Lemma 7 *Let there be a real function $\phi(t)$ Riemann-integrable on each finite segment, such that*

$$\|e^{C(t)\delta}\| \le e^{\phi(t)\delta} \ (t \ge 0) \tag{34}$$

for all sufficiently small $\delta > 0$. Then

$$\|W(t,s)\| \le exp[\int_s^t \phi(s_1)ds_1] \ \ (t \ge s \ge 0). \tag{35}$$

Proof Condition (34) implies

$$\|\prod_{1\le k\le m}^{\leftarrow} e^{C(t_k^{(m)}))\delta_k}\| \le \prod_{k=1}^{m} e^{\phi(t_k^{(m)})\delta_k} = \exp\,[\sum_{k=1}^{m} \phi(t_k^{(m)})\delta_k].$$

The passage to the limit as $m \to \infty$ and representation (33) give the required estimate. □

Similarly, applying representation (32), we obtain the following result.

Lemma 8 *Let there be a real Riemann-integrable function $\phi(t)$, such that*

$$\|I + C(t)\delta\| \le 1 + \phi(t)\delta \ (t \ge 0) \tag{36}$$

for all sufficiently small $\delta > 0$. Then inequality (35) is valid.

Let $\mathscr{X} = \mathscr{H}$ be a Hilbert space. Recall that $\Lambda(\Re C(s)) = \sup\ \sigma(\Re C(s))$, where $\Re C(s) = \frac{1}{2}(C(s) + C^*(s))$ and the asterisk means the adjointness. Taking in (36) $\phi(t) = \Lambda(\Re C(t))$, we arrive at the Wintner inequality

$$\|W(t,s)\| \le \exp[\int_s^t \Lambda(\Re C(s_1))ds_1] \ \ (t \ge s \ge 0). \tag{37}$$

Theorem 2, and Lemmas 8 and 7 imply

Corollary 4 *Assume that one of conditions (34) or (36) holds and*

$$b_0 := - \inf_{t \ge s \ge 0} \frac{1}{(t-s)} \int_s^t \phi(t_1)dt_1 > 0. \tag{38}$$

Then $\|W(t,s)\| \le \exp[-b_0(t-s)]$ $(t \ge s \ge 0)$ *and Eq. (2) is exponentially stable, provided*

$$\kappa \|e^{(B-b_0 I)t}\|^2_{L^1(0,\infty)} < 1.$$

4 Stability Conditions for Equations in a Hilbert Space via Commutators

4.1 Stability Conditions

In this section we consider Eq. (2) in a Hilbert space $\mathscr{H}$. Besides, $B \in \mathscr{B}(\mathscr{H})$ is a constant operator and $C(t)$ is a function with values in $\mathscr{B}(\mathscr{H})$ uniformly bounded on $[0,\infty)$ and Riemann-integrable on each finite segment. Assume that

$$\alpha(B) < 0 \tag{39}$$

and put

$$X := 2\int_0^\infty e^{B^* t} e^{Bt} dt, \ \zeta(B) := 2\int_0^\infty \|e^{Bt}\| \int_0^t \|e^{Bs}\| \|e^{B(t-s)}\| ds\, dt$$

and

$$\psi(X, C(t)) := \begin{cases} \Lambda(\Re C(t))\|X\| & \text{if } \Lambda(\Re C(t)) > 0, \\ \Lambda(\Re C(t))\lambda(X) & \text{if } \Lambda(\Re C(t)) \le 0. \end{cases}$$

Recall that $\lambda(S) = \inf \sigma(S)$ and $\Lambda(S) = \sup \sigma(S)$ for a selfadjoint operator S. Below we suggest estimates for $\|X\|$ and $\lambda(X)$. Furthermore, $[B_1, B_2] = B_1 B_2 - B_2 B_1$ is the commutator of $B_1, B_2 \in \mathscr{B}(\mathscr{H})$ and $K(t) = [B, C(t)]$.

Theorem 3 *Let the conditions (39) and*

$$\sup_{t \ge 0} (\psi(X, C(t)) + \|K(t)\| \zeta(B)) < 1 \tag{40}$$

hold. Then Eq. (2) is exponentially stable.

This theorem is proved in the next section. If

$$\|e^{Bs}\| \le ce^{-\nu s} \quad (s \ge 0; c, \nu = const > 0), \tag{41}$$

then

$$(Xv, v) = 2\int_0^\infty \|e^{Bt}v\|^2 dt \le 2c^2 \int_0^\infty e^{-2\nu t} dt \|v\|^2 \quad (v \in \mathscr{H}).$$

So

$$\|X\| \le \frac{c^2}{\nu} \text{ and } \zeta(B) \le 2c^3 \int_0^\infty e^{-\nu t} \int_0^t e^{-\nu s} e^{-\nu(t-s)} ds\, dt$$
$$= 2c^3 \int_0^\infty e^{-2\nu t} t dt = \frac{c^3}{2\nu^2}. \tag{42}$$

Now let us estimate $\lambda(X)$. Due to the Wintner inequalities (see Sect. 1),

$$\|e^{Bt}v\| \ge e^{\lambda(\Re B)t}\|v\| \quad (v \in \mathscr{H}).$$

So in view of (39), $\lambda(\Re B)$ is negative. Consequently,

$$(Xv, v) = 2\int_0^\infty \|e^{Bt}v\|^2 dt \ge 2\int_0^\infty e^{2\lambda(\Re B)t}\|v\|^2 dt \ge \|v\|^2/|\lambda(\Re B)| \quad (v \in \mathscr{H}).$$

Thus

$$\lambda(X) \ge 1/|\lambda(\Re B)|. \tag{43}$$

If B is a normal operator: $BB^* = B^*B$, then $\|e^{Bt}\| = e^{\alpha(B)t}$ $(t \ge 0)$, and according to (42),

$$\|X\| \le \frac{1}{|\alpha(B)|}, \zeta(B) = \frac{1}{2|\alpha(B)|^2} \text{ and, in addition, } \lambda(\Re B) = \beta(B),$$

where $\beta(B) := \inf\ \Re\sigma(B)$. Consequently, $\psi(X, C(t)) = \psi_0(B, C(t))$, where

$$\psi_0(B, C(t)) = \begin{cases} \frac{\Lambda(\Re C(t))}{|\alpha(B)|} & \text{if } \Lambda(\Re C(t)) > 0, \\ \frac{\Lambda(\Re C(t))}{|\beta(B)|} & \text{if } \Lambda(\Re C(t)) \le 0. \end{cases}$$

So we arrive at

Corollary 5 *Let B be a normal operator, and the conditions (39) and*

$$\sup_{t\geq 0}\left(\psi_0(B, C(t)) + \frac{\|K(t)\|}{2|\alpha(B)|^2}\right) < 1 \quad (44)$$

hold. Then (2) is exponentially stable.

Theorem 3 and Corollary 5 are sharp: if $C(t)$ is a constant operator, then $\psi(B, C(t)) = \|K(t)\| = 0$, and (40) obviously holds. But condition (39) is necessary in this case.

4.2 Proof of Theorem 3

Under condition (39), the Lyapunov equation

$$XB + (XB)^* = -2I \quad (45)$$

has a unique solution $X \in \mathscr{B}(\mathscr{H})$ and it can be represented as in Subsect. 4.2, cf. [7, Theorem I.5.1] (see also Eq. (4.12) from Chapter I of [7]). For two selfadjoint operators S and S_1 the inequality $S < S_1$ ($S \leq S_1$) means $(Sh, h) < (S_1h, h)$ ($(Sh, h) \leq (S_1h, h)$) ($h \in \mathscr{H}$). In particular, the inequality $S < 0$ ($S > 0$) means that S is strongly negative (strongly positive) definite.

Lemma 9 *If condition (39) holds and X is a solution of (45), then*

$$\Re(XC(t)) = \frac{1}{2}(XC(t) + (XC(t))^*) \leq (\psi(X, C(t)) + \|K(t)\|\zeta(B))I.$$

Proof We can write (8)

$$\Re(XC(t)) = \frac{1}{2}(XC(t) + C^*(t)X) = \int_0^\infty (e^{B^*t_1}e^{Bt_1}C(t) + C^*(t)e^{B^*t_1}e^{Bt_1})dt_1.$$

But

$$e^{Bt_1}C(t) = C(t)e^{Bt_1} + [e^{Bt_1}, C(t)], C^*(t)e^{B^*t_1} = e^{B^*t_1}C^*(t) + [C^*(t), e^{B^*t_1}].$$

So $\Re(XC(t)) = J_1 + J_2$, where

$$J_1 = \int_0^\infty e^{B^*t_1}(C(t) + C^*(t))e^{Bt_1}dt_1$$

and

$$J_2 = \int_0^\infty (e^{B^* t_1}[e^{Bt_1}, C(t)] + (e^{B^* t_1}[e^{Bt_1}, C(t)])^*) dt_1.$$

We have

$$J_1 \le 2\Lambda(\Re C(t)) \int_0^\infty e^{B^* t_1} e^{Bt_1} dt_1 = \Lambda(\Re C(t)) X.$$

If $\Lambda(\Re C(t)) > 0$, then $J_1 \le \Lambda(\Re C(t)) \|X\| I$. If $\Lambda(\Re C(t)) < 0$, then

$$J_1 \le \Lambda(\Re C(t)) \lambda(X) I.$$

So $J_1 \le \psi(X, C(t)) I$. In addition, by Lemma 5,

$$\|J_2\| \le 2 \int_0^\infty \|e^{Bt_1}\| \|[e^{Bt_1}, C(t)]\| dt_1 \le 2 \int_0^\infty \|e^{Bt_1}\| \|K(t)\| \int_0^{t_1} \|e^{Bs}\| \|e^{B(t_1 - s)}\| ds \, dt_1$$

$$= \|K(t)\| \zeta(B).$$

This proves the lemma. □

Proof of Theorem 3 Due to the Lyapunov equation (45) and Lemma 9,

$$\Re X(B + C(t)) \le -(1 - \psi(X, C(t)) - \|K(t)\| \zeta(B)) I.$$

So (40) implies

$$\Re X(B + C(t)) < \sup_t (-1 + \psi(X, C(t)) + \|K(t)\| \zeta(B)) I < 0. \tag{46}$$

Applying the right-hand Wintner inequality (see Sect. 1) with the scalar product $(.,.)_X$ defined by $(h, g)_X = (Xh, g)$ $(h, g \in \mathscr{H})$, we can assert that Eq. (2) is exponentially stable, as claimed. □

4.3 Coefficients with Compact Hermitian Components

In this section we consider Eq. (2), assuming that

$$\Im B \text{ is a Hilbert–Schmidt operator,} \tag{47}$$

i.e., $N_2(\Im B) = (\text{trace } (\Im B)^2)^{1/2} < \infty$. Recall that

$$g_I(B) = [2N_2^2(\Im B) - 2\sum_{k=1}^\infty |\Im \lambda_k(B)|^2 \,]^{1/2} \le \sqrt{2} N_2(\Im B),$$

where $\lambda_k(B), k = 1, 2, \ldots,$ are nonreal eigenvalues of B, enumerated with their multiplicities. If B is a normal operator, then $g_I(B) = 0$, cf. [17, Section 7.7]. Again apply [17, Example 7.10.3]:

$$\|e^{Bt}\| \le e^{\alpha(B)t} \sum_{k=0}^{\infty} \frac{t^k g_I^k(B)}{(k!)^{3/2}} \quad (t \ge 0),$$

So

$$\|X\| \le 2\int_0^{\infty} \|e^{Bt}\|^2 dt \le 2\int_0^{\infty} e^{\alpha(B)t} \left(\sum_{k=0}^{\infty} \frac{t^k g_I^k(B)}{(k!)^{3/2}}\right)^2 dt = \mu(B),$$

where

$$\mu(B) = \sum_{j,k=0}^{\infty} \frac{g_I^{j+k}(B)(k+j)!}{2^{j+k}|\alpha(B)|^{j+k+1}(j!\,k!)^{3/2}}.$$

Put

$$\tilde{p}(B, t) = \sum_{k=0}^{\infty} \frac{t^k g_I^k(B)}{(k!)^{3/2}} \quad (t \ge 0).$$

Then $\|e^{Bt}\| \le e^{\alpha(B)t}\hat{p}(B, t)$ and

$$\zeta(B) \le \tilde{\zeta}(B) := 2\int_0^{\infty} e^{2\alpha(B)t}\tilde{p}(t, B)\int_0^t \tilde{p}(t-s, B)\tilde{p}(s, B)ds\,dt.$$

Moreover, $\psi(X, C(t)) \le \tilde{\psi}(B, C(t))$, where

$$\tilde{\psi}(B, C(t)) := \begin{cases} \mu(B)\Lambda(\Re C(t)) & \text{if } \Lambda(\Re C(t)) > 0, \\ \frac{\Lambda(\Re C(t))}{|\lambda(\Re B)|} & \text{if } \Lambda(\Re C(t)) \le 0. \end{cases}$$

Now Theorem 3 and (43) imply

Corollary 6 *If the conditions (39), (47) and*

$$\sup_{t\ge 0}\left(\tilde{\psi}(B, C(t)) + \|K(t)\|\tilde{\zeta}(B)\right) < 1$$

hold, then (2) is exponentially stable.

5 Integrally Small Perturbations of Autonomous Equations

5.1 Statement of the Result

Again consider Eq. (2). Throughout this section $B \in \mathscr{B}(\mathscr{X})$ is a stable operator and $C(t) : [0, \infty) \to \mathscr{B}(\mathscr{X})$ is Riemann integrable. Let $U(t) = U(t, 0)$ be the Cauchy operator to (2): that is $U(t)u(0) = u(t)$ for a solution $u(t)$ of (2). Put $A(t) = B + C(t)$,

$$J(t) := \int_0^t C(s)ds, \quad m(t) := \|BJ(t) - J(t)A(t)\|$$

and

$$r_J(t) := \inf_{h \in X; \|h\|=1} \|(J(t) - I)h\| \quad (t \geq 0).$$

Theorem 4 *Let the condition*

$$\inf_{t \geq 0} r_J(t) > 0 \tag{48}$$

hold. Then $\|U(t)\| \leq z(t), \ t \geq 0$, *where* $z(t)$ *is a solution of the equation*

$$z(t) = \frac{1}{r_J(t)}[\|e^{tB}\| + \int_0^t \|e^{(t-s)B}\|m(s)z(s)ds], \ t \geq 0. \tag{49}$$

In the next subsection we show that from Theorem 4 it follows

Theorem 5 *Let* B *be stable and*

$$\sup_{t \geq 0} (\|J(t)\| + \int_0^t \|e^{(t-s)B}\|m(s)ds) < 1. \tag{50}$$

Then Eq. (2) is exponentially stable.

5.2 Proofs of Theorems 4 and 5

We need the following simple result.

Lemma 10 *Let* $w(t), f(t)$ *and* $v(t)$ $(0 \leq t \leq a < \infty)$ *be functions whose values are bounded linear operators in* $\mathscr{X}$. *Assume that* $w(t)$ *is Riemann integrable and*

$f(t)$ and $v(t)$ have Riemann integrable derivatives on $[0, a]$. Then with the notation $j_w(t) = \int_0^t w(s)ds$, we have

$$\int_0^t f(s)w(s)v(s)ds = f(t)j_w(t)v(t) - \int_0^t [f'(s)j_w(s)v(s) + f(s)j_w(s)v'(s)]ds$$

$$(0 \le t \le a).$$

For the proof see Lemma 3 from [21].

Lemma 11 *One has*

$$(I - J(t))U(t) = e^{Bt} + \int_0^t e^{B(t-s)}[BJ(s) - J(s)(B + C(s))]U(s)ds.$$

Proof The equality

$$U(t) - e^{Bt} = \int_0^t e^{B(t-s)}C(s)U(s)ds, \tag{51}$$

can be checked by differentiation. Thanks to the previous lemma,

$$\int_0^t e^{(t-s)B}C(s)U(s)ds = T(0)J(t)U(t) - \int_0^t [(\partial e^{(t-s)B}/\partial s)J(s)U(s)$$

$$+e^{(t-s)B}J(s)U'(s)]ds.$$

But $\partial e^{(t-s)B}/\partial s = -Be^{(t-s)B}$. In addition, $U'(s) = A(s)U(s)$. Thus,

$$\int_0^t e^{(t-s)B}C(s)U(s)ds = J(t)U(t) + \int_0^t e^{(t-s)B}[BJ(s) - J(s)A(s)]U(s)ds.$$

Now (51) implies the required result. □

Proof of Theorem Thanks to the previous lemma,

$$\|(I - J(t))U(t)\| \le \|e^{tB}\| + \int_0^t \|e^{(t-s)B}\|m(s)\|U(s)\|ds.$$

Hence

$$r_J(t)\|U(t)\| \le \|e^{tB}\| + \int_0^t \|e^{(t-s)B}\|m(s)\|U(s)\|ds.$$

Then by the well-known (comparison) Lemma 3.2.1 from [7] we have the required result. □

Let

$$\eta_0 := \sup_{t\ge 0} \frac{1}{r_J(t)} \int_0^t \|e^{(t-s)B}\| m(s) ds < 1. \tag{52}$$

Then (49) implies

$$\sup_t z(t) \le \sup_{t\ge 0} \frac{\|e^{tB}\|}{r_J(t)} + \sup_t z(t)\eta_0.$$

Due to the previous lemma we get

Lemma 12 *Let conditions (48) and (52) hold. Then*

$$\sup_{t\ge 0} \|U(t)\| \le \sup_{t\ge 0} \frac{\|e^{tB}\|}{(1-\eta_0) r_J(t)}.$$

Proof of Theorem 5 Assume that

$$j(t) := \|J(t)\| \le q < 1 \ \ (q = const;\ t \ge 0),$$

then $r_J(t) \ge 1 - j(t)$. If

$$\eta_1 := \sup_{t\ge 0} \frac{1}{1-j(t)} \int_0^t \|e^{(t-s)B}\| m(s) ds < 1, \tag{53}$$

then $\eta_0 \le \eta_1 < 1$ and thanks to the previous lemma, Eq. (2) is stable. But condition (50) implies that

$$j(t) + \int_0^t \|e^{(t-s)B}\| m(s) ds < 1$$

or

$$\frac{1}{1-j(t)} \int_0^t \|e^{(t-s)B}\| m(s) ds < 1 \ (t \ge 0).$$

Thus (50) implies the inequality $\eta_1 < 1$, and therefore, from (50) condition (53) follows. This proves the stability. Substitute the equality

$$u(t) = y(t) e^{-\epsilon t} \tag{54}$$

into (1). Then we obtain the equation

$$\dot{y} = (A(t) + I\epsilon) y. \tag{55}$$

Denote the Cauchy operator of (55) by $U_\epsilon(t)$. Repeating our above arguments with $U_\epsilon(t)$ instead of $U(t)$, due to Lemma 12 we can assert that $U_\epsilon(t)$ is bounded, provided $\epsilon > 0$ is sufficiently small. Now (54) implies

$$\|U(t)\| \le e^{-\epsilon t} \sup_{t\ge 0} \|U_\epsilon(t)\|, \ t \ge 0.$$

This proves the theorem. □

5.3 A Particular Case of Theorem 5

To illustrate Theorem 5, consider the equation

$$\frac{du}{dt} = Bu + c(t)C_0 u, \tag{56}$$

where C_0 is a constant operator and $c(t)$ is a scalar real piece-wise continuous function bounded on $[0, \infty)$. So $C(t) = c(t)C_0$.

Without loss of generality assume that $\sup_t |c(t)| \le 1$. With the notation

$$i_c(t) = |\int_0^t c(s)ds|,$$

we obtain

$$m(t) = \|BJ(t) - J(t)A(t)\| \le i_c(t)\|BC_0 - C_0(B + c(t)C_0)\| \le$$

$$i_c(t)(\|BC_0 - C_0 B\| + |c(t)|\|C_0^2\|) \le$$

$$i_c(t)(\|BC_0 - C_0 B\| + \|C_0^2\|).$$

If $\|e^{tB}\| \le Me^{-\alpha t}$ $(M, \alpha, t > 0)$, then

$$\int_0^t \|e^{(t-s)B}\|ds \le M \int_0^t e^{-\alpha s} ds \le \frac{M}{\alpha} \ \ (t \ge 0).$$

Thus, denoting

$$\theta_0 = \sup_t |\int_0^t c(s)ds|,$$

due to Theorem 5, we arrive at the following result.

Corollary 7 *If the inequalities* $\sup_t |c(t)| \leq 1$, $\|e^{tB}\| \leq Me^{-\alpha t}$ $(M, \alpha, t > 0)$ *and*

$$\theta_0(\|C_0\| + \frac{M}{\alpha}(\|BC_0 - C_0 B\| + \|C_0^2\|)) < 1 \tag{57}$$

hold, then Eq. (56) is exponentially stable.

For example, let $c(t) = sin\ (\omega t)$ $(\omega > 0)$. Then $i_c(t) \leq \frac{2}{\omega}$ and

$$m(t) \leq \frac{2}{\omega}(\|BC_0 - C_0 B\| + \|C_0^2\|).$$

Thus (57) takes the form

$$\|C_0\| + \frac{M}{\alpha}(\|BC_0 - C_0 B\| + \|C_0^2\|) < \frac{\omega}{2}.$$

6 Equations with the Lipschitz Property

6.1 Stability Conditions

Again consider in $\mathscr{X}$ the equation

$$\dot{u}(t) = A(t)u(t) \quad (t \geq 0), \tag{58}$$

where $A(t)$ is a variable bounded stable operator (i. e. $\alpha(A(t)) < 0$), satisfying the conditions

$$\|A(t) - A(s)\| \leq q_0|t - s| \ \ (t, s \geq 0;\ \ q_0 = const > 0), \tag{59}$$

and

$$\|exp[A(s)t]\| \leq p(t) \ \ (t, s \geq 0), \tag{60}$$

where $p(t)$ is a piece-wise-continuous function independent on s uniformly bounded on $[0, \infty)$.

Theorem 6 *Let the conditions (59), (60) and*

$$\theta_0 := q_0 \int_0^\infty tp(t)dt < 1 \tag{61}$$

hold. Then a solution $u(t)$ *of (58) satisfies the inequality*

$$\sup_{t\geq 0} \|u(t)\| \leq \frac{\chi\|u(0)\|}{1 - \theta_0}$$

where

$$\chi := \sup_{t \geq 0} p(t) < \infty. \tag{62}$$

Proof Equation (58) can be rewritten in the form

$$du/dt - A(\tau)u = [A(t) - A(\tau)]ux$$

with an arbitrary fixed $\tau \geq 0$. This equation is equivalent to the following one:

$$u(t) = exp[A(\tau)t]u(0) + \int_0^t exp[A(\tau)(t - t_1)][A(t_1) - A(\tau)]u(t_1)dt_1.$$

So

$$\|u(t)\| \leq \|exp[A(\tau)t]\|\|u(0)\|$$

$$+ \int_0^t \|exp[A(\tau)(t - t_1)]\|\|A(t_1) - A(\tau)\|\|u(t_1)\|dt_1.$$

According to (59) and (60),

$$\|u(t)\| \leq p(t)\|u(0)\| + q_0 \int_0^t p(t - t_1)|t_1 - \tau|\|u(t_1)\|dt_1.$$

With $\tau = t$ this relation gives us

$$\|u(t)\| \leq p(t)\|u(0)\| + q_0 \int_0^t p(t - t_1)(t - t_1)\|u(t_1)\|dt_1. \tag{63}$$

Hence

$$\sup_{0 \leq t \leq T} \|u(t)\| \leq \chi\|u(0)\| + \sup_{0 \leq t \leq T} \|u(t)\|\theta_0$$

for any positive finite T. By the condition $\theta_0 < 1$ we arrive at the inequality

$$\sup_{0 \leq t \leq T} \|u(t)\| \leq \frac{\chi\|u(0)\|}{1 - \theta_0}.$$

Since the right-hand part of the latter inequality does not depend on T, we get the required inequality. □

By the substitution

$$u(t) = u_\epsilon(t)e^{-\epsilon t} \quad (\epsilon > 0)$$

with small enough $\epsilon > 0$ into (58) and taking into account Theorem 6, we arrive at

Corollary 8 *Under the hypothesis of Theorem 6 Eq. (58) is exponentially stable.*

6.2 An Upper Bound for Evolution Operators

Suppose that there are constants $a_\nu, \nu > 0$ independent of s, such that

$$\|exp[A(s)t]\| \le a_\nu e^{-\nu t} \quad (a_\nu, \nu = const > 0, t, s \ge 0). \tag{64}$$

That is, $p(t) = a_\nu e^{-\nu t}$. Then due to (63),

$$\|u(t)\| \le a_\nu e^{-\nu t}\|u(0)\| + q_0 \int_0^t a_\nu e^{-\nu(t-t_1)}(t-t_1)\|u(t_1)\|dt_1.$$

With $v(t) = \|u(t)\|e^{\nu t}$ this gives us

$$v(t) \le a_\nu v(0) + q_0 a_\nu \int_0^t (t-t_1)v(t_1)dt_1.$$

Due to the above-mentioned comparison principle this inequality yields $v(t) \le w(t)$, where $w(t)$ is the solution of

$$w(t) = a_\nu v(0) + a_\nu q_0 \int_0^t (t-t_1)w(t_1)dt_1.$$

Hence,

$$w''(t) = a_\nu q_0 w(t)$$

with $w(0) = v(0)a_\nu$ and $w'(0) = 0$. Solving this equality, we have

$$w(t) = v(0)a_\nu \cosh(\sqrt{a_\nu q_0}t) \quad (\cosh\ x = (e^x + e^{-x})/2).$$

But

$$\|u(t)\| = v(t)e^{-\nu t} \le w(t)e^{-\nu t}$$

and the therefore the inequality

$$\|u(t)\| \le \|u(0)\|a_\nu e^{-\nu t}\cosh(t\sqrt{a_\nu q_0}) \le \|u(0)\|a_\nu \exp[(-\nu+\sqrt{a_\nu q_0})t]$$

is true for any solution $u(t)$ of (58). Replacing in our reasonings zero by an arbitrary $t_0 \ge 0$, we get

$$\|u(t)\| \le \|u(t_0)\|a_\nu e^{-\nu(t-t_0)}\cosh(\sqrt{a_\nu q_0}(t-t_0))$$

$$\le \|u(t_0)\|a_\nu exp[(-\nu+\sqrt{a_\nu q_0})(t-t_0)] \quad (t \ge t_0).$$

We thus have proved

Theorem 7 *Let the conditions (59) and (64) hold. Then the evolution operator of (58) satisfies the inequalities*

$$\|U(t,t_0)\| \le a_\nu e^{-\nu(t-t_0)}\cosh(\sqrt{a_\nu q_0}(t-t_0))$$

$$\le a_\nu exp[(-\nu+\sqrt{a_\nu q_0})(t-t_0)] \quad (t \ge t_0 \ge 0).$$

Corollary 9 *Let the conditions (59), (64) and $\nu > \sqrt{a_\nu q_0}$ hold. Then (58) is exponentially stable.*

6.3 A Lower Bound for Evolution Operators

Assume that

$$\|\exp[-A(s)t]\| \le b_\mu e^{-\mu t} \quad (t,s \ge 0) \tag{65}$$

with constants $b_\mu > 0$ and $\mu > 0$. Put $V(t,t_0) = U^{-1}(t,t_0)$. Then

$$\frac{d}{dt}(V(t,t_0)U(t,t_0)) = \frac{d}{dt}I = 0$$

and therefore,

$$\frac{d}{dt}(V(t,t_0)U(t,t_0)) = (\frac{d}{dt}V(t,t_0))U(t,t_0) + V(t,t_0)\frac{d}{dt}U(t,t_0) =$$

$$= (\frac{d}{dt}V(t,t_0))U(t,t_0) + V(t,t_0)A(t)U(t,t_0) = 0.$$

Hence,

$$\frac{d}{dt}V(t,t_0) = -V(t,t_0)A(t)$$

and thus,

$$\frac{d}{dt}V^*(t,t_0) = -A^*(t)V^*(t,t_0).$$

So $V^*(t,t_0)$ is the evolution operator of the equation

$$dv(t)/dt = -A^*(t)v(t) \quad (t \geq 0). \tag{66}$$

Since the norms of adjoint operators coincide, condition (59) holds with $A^*(t)$ instead of $A(t)$. Applying Theorem 7 to Eq. (66) and taking into account (65), we obtain

$$\|U^{-1}(t,t_0)\| = \|V(t,t_0)\| \leq b_\mu \exp\left[(-\mu + \sqrt{b_\mu q_0})(t-t_0)\right] \ (t \geq t_0 \geq 0)$$

and consequently

$$\|U(t,t_0)h\| \geq \|h\|\frac{1}{b_\mu}\exp\left[(\mu - \sqrt{b_\mu q_0})(t-t_0)\right] \ (h \in \mathscr{X},\ t \geq t_0 \geq 0). \tag{67}$$

We thus have proved the following

Theorem 8 *Let conditions (59) and (65) hold. Then the evolution operator of (58) satisfies inequality (67).*

From this theorem it directly follows

Corollary 10 *Let the conditions (59), (65) and*

$$\mu - \sqrt{b_\mu q_0} > 0$$

hold. Then Eq. (58) is unstable.

6.4 Equations in a Hilbert Space

Consider Eq. (58) in a Hilbert space: $\mathscr{X} = \mathscr{H}$, and put

$$Z = \int_0^\infty e^{B^*t}e^{Bt}\,dt,$$

where $B \in \mathscr{B}(\mathscr{H})$ with the spectrum in the open left half-plane. So Z is a solution of the equation $ZB + B^*Z = -I$ (see Sect. 2). Put $w(t) = e^{Bt}h$ $(h \in \mathscr{H})$. We have

$$\frac{d}{dt}(Zw, w) = (Zw, w') + (Zw', w) = (Zw, Bw) + (ZBw, w) \leq -(w, w) \quad (w = w(t)).$$

Clearly, $(w, w) = (Z^{-1}Zw, w) \geq \lambda(Z^{-1})(Zw, w)$ where $\lambda(S)$ means the smallest eigenvalue of a selfadjoint operator S. But $\lambda(Z^{-1}) = 1/\|Z\|$ and therefore $\frac{d}{dt}(Zw, w) \leq -(Zw, w)/\|Z\|$. Consequently,

$$(Zw(t), w(t)) \leq \exp\left[-\frac{1}{\|Z\|}\right](Zw(0), w(0)).$$

Hence

$$\lambda(Z)(w(t), w(t)) \leq \|Z\| \exp\left[-\frac{t}{\|Z\|}\right](w(0), w(0)).$$

Or

$$\|e^{Bt}h\|^2 \leq \frac{\|Z\|}{\lambda(Z)} \exp\left[-\frac{t}{\|Z\|}\right]\|h\|^2.$$

So

$$\|e^{Bt}\| \leq \frac{\sqrt{\|Z\|}}{\sqrt{\lambda(Z)}} \exp\left[-\frac{t}{2\|Z\|}\right] \quad (t \geq 0). \tag{68}$$

Moreover, by the Parseval equality

$$\|Z\| \leq \psi(B) := \int_0^\infty \|e^{Bt}\|^2 dt = \frac{1}{2\pi}\int_{-\infty}^\infty \|(B - i\omega I)^{-1}\|^2 d\omega.$$

In addition, by the Wintner inequality $\|e^{Bt}h\| \geq e^{\lambda(\Re B)t}\|h\|$ and consequently,

$$(Zh, h) = \int_0^\infty \|e^{Bt}h\|^2 dt \geq \int_0^\infty e^{2\lambda(\Re B)t} dt \|h\|^2.$$

Hence $\lambda(Z) \geq 1/|2\lambda(\Lambda(\Re B))|$ and (68) yields

Lemma 13 *Let $B \in \mathscr{B}(\mathscr{H})$ be stable. Then*

$$\|e^{Bt}\| \leq \sqrt{2\psi(B)|\lambda(\Re B)|} exp\ [-t/(2\psi(B))]\ (t \geq 0). \tag{69}$$

About estimates for resolvents of and exponentials of nonselfadjoint operators see [24].

Furthermore, put

$$\psi_0 := \sup_{t\geq 0} \psi(A(t)) = \sup_{t\geq 0} \int_0^\infty \|e^{A(t)s}\|^2 ds = \sup_{t\geq 0} \frac{1}{2\pi} \int_{-\infty}^\infty \|(A(t) - i\omega I)^{-1}\|^2 d\omega$$

and

$$\chi_0 := \sup_{t\geq 0} |\lambda(\Re A(t))| = \sup_{t\geq 0} |\inf \sigma(\Re A(t))|.$$

Then Lemma 13 implies

$$\|e^{A(s)t}\| \leq \sqrt{2\psi_0\chi_0} \exp\,[-t/(2\psi_0)]. \tag{70}$$

Now we can immediately apply Theorem 7 and Corollary 9.

7 Equations in a Lattice Normed Space

7.1 Lattice Normed Spaces

Throughout this section $\mathscr{B}$ is a Banach lattice with a positive cone $\mathscr{B}^+$ and an order continuous norm $\|.\|_{\mathscr{B}}$ [36], $L(\mathscr{B})$ is the set of all bounded operators acting in $\mathscr{B}$. Let X be a linear space, and there be a mapping $M : X \to \mathscr{B}^+$ with the properties

$$M(x) > 0 \text{ iff } x \neq 0;\ M(\lambda x) = |\lambda| M(x)\ (\lambda \in \mathbf{C}) \text{ and } M(x+y) \leq M(x) + M(y) \tag{71}$$

for every $\lambda \in \mathbf{C}$ and $x, y \in X$. Such a mapping was introduced by L. Kantorovich (see [36, p. 334]) who called M the generalized norm. Since the words "the generalized norm" can confuse the reader, we will call a mapping M satisfying (71) *the normalizing mapping*, and X will be called the space with a normalizing mapping. Following [36], we will call $\mathscr{B}$ *the norming lattice*. Clearly, a space X with a normalizing mapping $M : X \to \mathscr{B}^+$ is a normed space with the norm

$$\|h\|_X = \|M(h)\|_{\mathscr{B}} \quad (h \in X). \tag{72}$$

The topology in space X in the sequel is defined by the norm (72), and X is a Banach space.

Again consider the equation

$$du/dt = A(t)u \quad (t \geq 0), \tag{73}$$

where $A(t)$ is a linear bounded operator in X. In the present section under some assumptions by a normalizing mapping, solution estimates for Eq. (73) are derived.

Note that a normalizing mapping enables us to use more complete information about the equation than a usual (number) norm.

7.2 Solution Estimates

Let us suppose that for small enough $\delta > 0$, there is a continuous operator-valued function $a(t) : R_+ = [0, \infty) \to L(\mathscr{B})$ such that

$$M((I_X + \delta A(t))h) \le (I_{\mathscr{B}} + a(t)\delta)M(h) \quad (h \in X;\ t \ge 0). \tag{74}$$

Here I_X and $I_{\mathscr{B}}$ are the unit operators in X and $\mathscr{B}$, respectively. We need the following linear equation in $\mathscr{B}$

$$\dot{z}(t) = a(t)z(t) \quad (t \ge 0). \tag{75}$$

Theorem 9 *Let condition (74) hold. Then any solution $u(t)$ of (73) subordinates the inequality*

$$M(u(t)) \le z(t) \quad (t \ge 0), \tag{76}$$

where $z(t)$ is a solution of Eq. (75) with the initial condition $z(0) = M(u(0))$.

Proof For some partitioning of a segment $[0, t]$: $0 = t_0^{(n)} < t_1^{(n)} < \ldots < t_n^{(n)} = t$ let us denote

$$U_{n,k} = (I_X + A(t_n^{(n)})\delta_n)(I_X + A(t_{n-1}^{(n)})\delta_{n-1}) \ldots (I_X + A(t_{k+1}^{(n)})\delta_{k+1}),$$

for $k < n$, and $U_{n,n} = I_X$. Here $\delta_k = \delta_k^{(n)} = t_k^{(n)} - t_{k-1}^{(n)}$ $(k = 1, \ldots, n)$.

According to (74) we easily get

$$M(U_{n,0}u_0) \le \prod_{1 \le k \le n}^{\leftarrow} (I_{\mathscr{B}} + a(t_k)\delta_k)M(u_0), \tag{77}$$

where the arrow over the symbol of the product means that the cofactor indices increase from right to left. On the other hand, the limit in the strong topology of the operators

$$\prod_{1 \le k \le n}^{\leftarrow} (I_{\mathscr{B}} + a(t_k)\delta_k) \text{ as } \max_k \delta_k^{(n)} \to 0,$$

is the Cauchy operator of Eq. (75), cf. [8]. Now the desired assertion follows from inequality (77). ☐

7.3 Systems of Equations in Banach Spaces

Let X be a direct sum of Banach spaces E_k ($k = 1, \ldots, n < \infty$) with norms $\|.\|_{E_k}$, and let $h = (h_k \in E_k)_{k=1}^n$ be an element of X. Define in X the normalizing mapping by the formula

$$M(h) = (\|h_k\|_{E_k})_{k=1}^n. \tag{78}$$

That is, $M(h)$ is the vector whose coordinates are $\|h_k\|_{E_k}$ $(k = 1, \ldots, n)$. Furthermore, let $A_{jk}(t)$ be bounded continuous linear operators acting from E_k into E_j $(j, k = 1, \ldots, n)$, and $A(t)$ be defined by the operator matrix $(A_{jk}(t))_{j,k=1}^n$. Then (73) takes the form

$$du_j/dt = \sum_{k=1}^n A_{jk}(t)u_k \quad (t \geq 0;\ j = 1, \ldots, n;\ u_j = u_j(t)). \tag{79}$$

Put

$$a_{jk}(t) = \|A_{jk}(t)\|_{E_k \to E_j} \ (j \neq k; t \geq 0)$$

and let for every small enough $\delta > 0$,

$$\|(I_{E_k} + \delta A_{kk}(t))h_k)\|_{E_k} \leq (1 + a_{kk}(t)\delta)\|h_k\|_{E_k} \ (h_k \in E_k), \tag{80}$$

where $a_{kk}(t)$ $(k = 1, \ldots, n)$ are continuous scalar-valued functions. We have

$$M_j((I_X + \delta A(t))h) = \|(I_{E_j} + A_{jj}(t)\delta)h_j + \delta \sum_{k \neq j} A_{jk}(t)h_k\|_{E_j}.$$

Here $M_j((I_X + \delta A(t))h)$ $(h \in X)$ is the coordinate of the vector $M((I_X + \delta A(t))h)$. Therefore,

$$M_j((I_X + \delta A(t))h) \leq \|(I_{E_j} + A_{jj}(t)\delta)h_j\|_{E_j} + \delta \sum_{k \neq j} \|A_{jk}(t)h_k\|_{E_j} \leq$$

$$(1 + a_{jj}(t)\delta)\|h_j\|_{E_j} + \delta \sum_{k \neq j} a_{jk}(t)\|h_k\|_{E_k}.$$

That inequality can be written in the vector form

$$M((I_X + \delta A(t))h) \leq (I_{C^n} + a(t)\delta)M(h) \ (h \in X)$$

with the matrix $a(t) = (a_{jk}(t))$. This relation yields condition (74). Now Theorem 9 immediately implies

Corollary 11 *Let conditions (80) be fulfilled. Then a solution $u(t)$ of system (79) satisfies inequality (76), where $z(t)$ is the solution of Eq. (75) with a variable $n \times n$-matrix $a(t) = (a_{jk}(t))$ and the initial condition $z(0) = (\|u_k(0)\|_{E_k})_{k=1}^n \in \mathbf{R}^n$.*

8 Equations with the Generalized Lipschitz Conditions

Consider Eq. (1) assuming that $A(t)$ is a variable operator in $\mathscr{X}$ uniformly bounded on $[0, \infty)$ and satisfying the generalized Lipschitz condition

$$\|A(t) - A(\tau)\| \le r(t - \tau) \quad (t, \tau \ge 0). \tag{81}$$

where $r(t)$ is a positive piece-wise continuous on $[0, \infty)$ function. In addition to (81) suppose that there is a positive Riemann integrable on finite real segments function $p(t)$ independent of s and uniformly bounded on $[0, \infty)$, such that

$$\|\exp[A(s)t]\| \le p(t) \quad (t, s \ge 0). \tag{82}$$

Now we are in a position to formulate the main result of the section

Theorem 10 *Let the conditions (81), (82) and*

$$\zeta_0 := \int_0^\infty r(s)p(s)ds < 1 \tag{83}$$

hold. Then any solution $u(t)$ of (1) satisfies the inequality

$$\sup_{t \ge 0} \|u(t)\| \le \frac{\chi \|u(0)\|}{1 - \zeta_0}, \tag{84}$$

where $\chi = \sup_t p(t)$. Moreover, Eq. (1) is exponentially stable.

Proof Rewrite (1) as

$$\frac{du(t)}{dt} = A(\tau)u(t) + [A(t) - A(\tau)]u(t)$$

with an arbitrary fixed $\tau \ge 0$. So (1) is equivalent to the equation

$$u(t) = \exp[A(\tau)t]u(0) + \int_0^t \exp[A(\tau)(t - s)][A(s) - A(\tau)]u(s)ds.$$

Hence,

$$\|u(t)\| \leq \|\exp[A(\tau)t]\|\|u(0)\| + \int_0^t \|\exp[A(\tau)(t-s)]\|\|A(s) - A(\tau)\|\|u(s)\|ds.$$

According to (81) and (82),

$$\|u(t)\| \leq p(t)\|u(0)\| + \int_0^t p(t-s)r(s-\tau)\|u(s)\|ds.$$

Taking $\tau = t$, we obtain

$$\|u(t)\| \leq p(t)\|u(0)\| + \int_0^t p(t-s)r(t-s)\|u(s)\|ds$$

and therefore,

$$\|u(t)\| \leq p(t)\|u(0)\| + \int_0^t p(t_1)r(t_1)\|u(t-t_1)\|dt_1.$$

Hence for any positive finite T,

$$\sup_{t\leq T}\|u(t)\| \leq \chi\|u(0)\| + \sup_{t\leq T}\|u(t)\| \int_0^T p(t_1)r(t_1)dt_1$$

$$\leq \chi\|u(0)\| + \sup_{t\leq T}\|u(t)\| \int_0^\infty p(t_1)r(t_1)dt_1 = \chi\|u(0)\| + \sup_{t\leq T}\|u(t)\|\zeta_0.$$

According to (83) we get

$$\sup_{t\leq T}\|u(t)\| \leq \chi\|u(0)\|(1-\zeta_0)^{-1}$$

Extending this result to all $T \geq 0$ we prove inequality (84).

Furthermore, by the substitution

$$u(t) = u_\epsilon(t)e^{-\epsilon t} \tag{85}$$

with an $\epsilon > 0$ into (1), we obtain the equation

$$du_\epsilon(t)/dt = (\epsilon I + A(t))u_\epsilon(t). \tag{86}$$

Taking ϵ small enough and applying (84) to Eq. (86) we can assert that $\|u_\epsilon(t)\| \leq const\|u(0)\|$. Hence due to (85) we prove exponential stability. □

9 Integro-Differential Equations with Differentiable in Time Coefficients

Our main object in this section is the equation

$$\frac{\partial u(t,x)}{\partial t} = c(t,x)u(t,x) + \int_0^1 k(t,x,s)u(t,s)ds \quad (t > 0;\ 0 \le x \le 1), \tag{87}$$

where $c(.,.) : [0,\infty) \times [0,1] \to \mathbf{R}$ and $k(.,.,.) : [0,\infty) \times [0,1]^2 \to \mathbf{R}$ are given functions, $u(.,.)$ is unknown.

We consider (87) in space $L^2(0,1)$ of scalar functions defined on $[0,1]$ with the traditional scalar product

$$(f,g) = \int_0^1 f(x)\overline{g}(x)dx \quad (f,g \in L^2(0,1)),$$

norm $\|.\|_{L^2(0,1)} = \sqrt{(.,.)}$ and initial condition

$$u(0,x) = u_0(x) \quad (0 \le x \le 1), \tag{88}$$

where $u_0 \in L^2(0,1)$ is given.

A solution of (87), (88) is a function $u(t,.)$, defined on $[0,\infty)$ with values in $L^2(0,1)$, absolutely continuous in t and satisfying (88) and (87), almost everywhere on $[0,\infty)$. The existence and uniqueness of solutions under consideration is obvious.

We will say that Eq. (87) is exponentially stable, if there are positive constants m_0 and ϵ, such that any its solution $u(t,.)$ satisfies the inequality $\|u(t,.)\|_{L^2} \le m_0 e^{-\epsilon t}\|u_0\|_{L^2}$ $(t \ge 0)$. It is assumed that, for almost all $x,s \in [0,1]$, $c(t,x)$ and $k(t,x,s)$ have bounded measurable derivatives in t, $c_t'(t,x)$ and $k_t'(t,x,s)$. In addition, the operators $A(t)$ and $A'(t)$ defined by

$$(A(t)w)(x) = c(t,x)w(x) + \int_0^1 k(t,x,s)w(s)ds$$

and

$$(A'(t)w(x)) = c_t'(t,x)w(x) + \int_0^1 k_t'(t,x,s)w(s)ds \quad (x \in [0,1];\ w \in L^2),$$

respectively, are assumed to be bounded uniformly in $t \in [0,\infty)$. In addition, it is assumed that

$$N_2(\Im A(t)) = (\int_0^1 \int_0^1 (k(t,x,s) - k(t,s,x))^2\, ds\, dx)^{1/2} < \infty \tag{89}$$

and

$$\sup_{t\geq 0} \alpha(A(t)) < 0. \tag{90}$$

Recall that $N_2(.)$ is the Hilbert–Schmidt norm. Note that in [22] estimates for $\alpha(A(t))$ are derived. To apply Corollary 3 take into account that $g_I^2(A) \leq 2N_2^2(\Im A(t))$ and therefore $\mu(t) \leq \hat{\mu}(t)$, where

$$\hat{\mu}(t) = \sum_{j,k=1}^{\infty} \frac{N_2^{j+k}(\Im A(t))(k+j)!}{2^{(j+k)/2}|\alpha(A(t))|^{j+k+1}(j!\,k!)^{3/2}}.$$

Then Corollary 3 implies

Corollary 12 *Let the conditions (89), (90) and*

$$\sup_{t\geq 0} \hat{\mu}^2(t)\|A'(t)\| < 2$$

hold. Then Eq. (87) is exponentially stable.

10 Integro-Differential Equations with Two Spatial Variables

Put $\Omega = [0,1] \times [0,1]$. Consider the equation

$$\frac{\partial u(t,x,y)}{\partial t}$$

$$= c(t,x,y)u(t,x,y) + \int_0^x \psi_1(x,s)u(t,s,y)ds + \int_0^1 \psi_2(t,y,s_1)u(t,x,s_1)ds_1$$

$$(0 \leq x, y \leq 1; t \geq 0), \tag{91}$$

where $c(\cdot,\cdot,\cdot) : [0,\infty) \times \Omega \to \mathbf{C}$ is piece-wise continuous, $\psi_1(\cdot,\cdot) : \Omega \to \mathbf{C}$, $\psi_2(\cdot,\cdot,\cdot) : [0,\infty) \times \Omega \to \mathbf{C}$ satisfy the conditions pointed below.

We consider Eq. (91) in the Hilbert space $\mathscr{H} = L^2(\Omega)$ of complex square integrable functions defined on Ω with the scalar product

$$(f,g) = \int_0^1 \int_0^1 f(x,y)\overline{g}(x,y)dx \quad (f,g \in L^2(\Omega)),$$

norm $\|.\|_{L^2(\Omega)} = \sqrt{(.,.)}$ and initial condition

$$u(0, x, y) = u_0(x, y) \quad (0 \leq x, y \leq 1), \tag{92}$$

where $u_0 \in L^2(\Omega)$ is given.

Define the operators B and $C(t)$ by

$$(Bw)(x, y) = \int_0^x \psi_1(x, s)w(s, y)ds$$

and

$$(C(t)w)(x, y) = c(t, x, y)w(x, y) + \int_0^1 \psi_2(t, y, s_1)w(x, s_1)ds_1$$

$$(x, y \in [0, 1]; \ w \in L^2(\Omega)),$$

respectively. Under consideration the commutator $K(t) = BC(t) - C(t)B$ is defined by

$$(K(t)w)(x, y) = \int_0^x m(t, x, y, s)w(s, y)ds,$$

where $m(t, x, y, s) = \psi_1(x, s)(c(t, s, y) - c(t, x, y))$. Assume that

$$N_2(B) = \left(\int_0^1 \int_0^x |\psi_1(x, s)|^2 \, ds \, dx \right)^{1/2} < \infty \tag{93}$$

and ψ_2 provides the boundedness of the operator $M(t)$ defined in $L^2(0, 1)$ by

$$(M(t)v)(y) = \int_0^1 \psi_2(t, y, s)v(s)ds \quad (y \in [0, 1]; \ v \in L^2(0, 1)).$$

Suppose

$$\Lambda_0 := \sup_t \Lambda(\Re C(t)) = \sup_t \sup \sigma(\Re C(t)) < 0. \tag{94}$$

Obviously,

$$\Lambda(\Re C(t)) \leq \sup_{x,y} \Re c(t, x, y) + \Lambda(\Re M(t)).$$

According to the Wintner inequalities (see Sect. 1) condition (18) is fulfilled with $b_0 = |\Lambda_0|$ and $c_0 = 1$.

Furthermore, under consideration B is quasi-nilpotent. Applying Corollary 7.4 from [24], we have

$$\|e^{(B-b_0)t}\| \leq e^{-b_0 t} \sum_{k=0}^{\infty} \frac{t^k N_2^k(B)}{(k!)^{3/2}} \quad (t \geq 0),$$

So $\|e^{(B-b_0 I)t}\|_{L^1(0,\infty)} \leq J(b_0, B)$, where

$$J(b_0, B) = \sum_{k=0}^{\infty} \frac{N_2^k(B)}{b_0^{k+1}(k!)^{1/2}}.$$

Making use of Theorem 2, we arrive at the following result.

Corollary 13 *Let the conditions (93), (94), and* $\sup_t \|K(t)\| J^2(b_0, B) < 1$. *hold. Then Eq. (91) is exponentially stable.*

Note that some estimates for the spectra of partial integral operators can be found in [19].

11 Bibliographical Comments

The material of Sect. 2 is adopted from the paper [22].

Section 3 is based on the paper [25].

The results of Sect. 4 are probably new.

Section 5 is a modification of the paper [20].

Section 6.1 is a generalization of the results from [18, Section 3.1] derived in the framework of the freezing method for ordinary differential equations (see also [15, 16]). The material of Sects. 6.2 and 6.3 is probably new.

The material of Sect. 7 is adopted from [14]. The results of Sect. 8 are probably new. Sections 9 and 10 are based on the papers [22] and [25].

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Best Constants for Poincaré-Type Inequalities in $W_1^n(0, 1)$

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Abstract For any positive integer n, let $\{T_k\}_{k=1}^n$ be a given set of linear functionals on $W_1^n(0, 1)$, which are unisolvent for polynomials of degree $n-1$. We determine the best possible constant $c(T_1, \ldots, T_n)$ in the following general higher-order Poincaré-type inequalities

$$\int_0^1 |f(x)|\,\mathrm{d}x \le c_n(T_1, \ldots, T_n) \int_0^1 \left|f^{(n)}(x)\right|\mathrm{d}x,$$

where $f \in W_1^n(0, 1)$ satisfying the conditions $T_k[f] = 0$, $k = 1, \ldots, n$. Our main result states that the minimal value c_n of the constants $c_n(T_1, \ldots, T_n)$ is just the L_∞-norm of the (properly normalized) perfect B-spline B_n of degree n on $[0, 1]$. We were also able to exhibit one particular set of extremal functionals for which this constant is achieved. Furthermore, comparison of the best constants in the previous inequality for some most frequently used functionals in practice is also given.

1 Introduction

The Poincaré-type inequalities are important and widely used in the study of many problems of partial differential equations and numerical analysis. Various extensions, analogues, variants, applications, and historical notes of the Poincaré inequality can be found in the excellent recent survey paper [7], and the references

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given there. For many practical purposes it is important to know the exact values of the best possible constants appearing in such inequalities (see, e.g., [7, 8, 12]).

The main contribution of this paper provides the best constant for the generalized Poincaré inequality

$$\int_0^1 |f(x)|\,\mathrm{d}x \le c_n(T_1,\ldots,T_n)\int_0^1 \left|f^{(n)}(x)\right|\mathrm{d}x, \tag{1}$$

where $f \in W_1^n(0,1)$ satisfying the conditions $T_k[f] = 0$, $k = 1,\ldots,n$. Throughout, $\{T_k\}_{k=1}^n$ denotes any set of linear functionals on $W_1^n(0,1)$, which are unisolvent with respect to π_{n-1}. Recall that the functionals T_k, $k = 1,\ldots,n$, are π_{n-1} unisolvent if for any $p \in \pi_{n-1}$, $T_k(p) = 0$, $k = 1,\ldots,n$, only when p is the identical zero polynomial. Here and in what follows, π_m denotes the class of all polynomials of degree less than or equal to m.

To state the result precisely let us first fix some more notation and terminology. By a normalized weight function α we mean a real valued nonnegative function on [0, 1] for which the Riemann integral exists and has the value one. To ease the notation, let us denote by $\mathscr{N}$ the class of normalized weight functions on [0, 1]. Throughout this paper, the usual L_1-norm on [0, 1] will be denoted by $\|\cdot\|_1$, and for any positive integer n, let $W_1^n(0,1)$ be the Sobolev space defined by

$$W_1^n(0,1) = \left\{f : f^{(n-1)} \text{ is absolutely continuous and } \left\|f^{(n)}(x)\right\|_1 < \infty\right\}.$$

Furthermore, let $\alpha_1,\ldots,\alpha_n$ be any n normalized weight functions on [0, 1]. They give rise to the linear functionals

$$T_k[f] = \int_0^1 \alpha_k(x) f(x)\mathrm{d}x, \quad k = 1,\ldots,n. \tag{2}$$

Let us start by considering inequality (1) in the special case when the functionals T_k, $k = 1,\ldots,n$, can be represented in weighted integrals as defined in (2). We will return to more general case when the functionals T_k, $k = 1,\ldots,n$ are not necessarily generated by densities in Sect. 3. With such notation at hand, we ask the following questions: For a given function belonging to $W_1^n(0,1)$, can we estimate $\|f\|_1$ in terms of $\left\|f^{(n)}\right\|_1$ provided that for any $k = 1,\ldots,n$,

$$T_k[f] = 0?$$

If so, what is the best choice of the extremal weight functions $\{\alpha_k\}_{k=1}^n$ in the sense that they provide the smallest constant $c(\alpha_1,\ldots,\alpha_n)$ in the following general higher-order Poincaré-type inequality

$$\|f\|_1 \le c(\alpha_1,\ldots,\alpha_n)\left\|f^{(n)}(x)\right\|_1? \tag{3}$$

Recently, in the case $n = 1$, one of us (see [5, Theorem 1]) has shown that the best constant in inequality (3) is equal to $1/2$. Moreover, they characterized all extremal weight functions α for which inequality (3) is satisfied with the best constant

$$c(\alpha) = \frac{1}{2}.$$

For $n > 1$, we did not give fully satisfactory answers, however, we were able to specify a class of weight functions $\alpha_1, \ldots, \alpha_n$ for which we find the exact value of the best possible constant in the following first order Poincaré inequality

$$\int_0^1 |f(x)| \,\mathrm{d}x \le c(\alpha_1, \ldots, \alpha_n) \int_0^1 \left|f'(x)\right| \mathrm{d}x, \tag{4}$$

in the case where *many* orthogonality conditions are satisfied by f,

$$\int_0^1 \alpha_k(x) f(x)\mathrm{d}x = 0, \quad k = 1, \ldots, n. \tag{5}$$

More precisely, it is shown (in Theorem 3) that in this more general context, the best possible value in inequality (4) is just $c(\alpha_1, \ldots, \alpha_n) = 1/(2n)$. The proof of this theorem is based on Theorem 1 and the Hobby–Rice theorem (see [9]), and it may be viewed as a natural generalization of Theorem 1 to the case where *many* orthogonality conditions are satisfied by f.

The corresponding questions for the most general inequality (1) are answered via a new approach in our main result (in Theorem 4). The minimal value c_n of the constants $c_n(T_1, \ldots, T_n)$ in inequality (1) is shown to be the L_∞-norm of the (properly normalized) perfect B-spline B_n of degree n on $[0, 1]$. Moreover we show that the best constant c_n is attained by exhibiting (in Corollary 1) one particular set of extremal functionals for which this constant is achieved. Furthermore, we discuss the comparison of the best constants in inequality (1) for some most frequently used functionals in practice.

2 Case of Weighted Integral Functionals

We begin by considering the (simplest) case when $n = 1$. To this end, we first recall the following Poincaré-type inequality. A proof can be found in Guessab [5].

Theorem 1 *Let α be a normalized weight function. Then, there exists a constant $c(\alpha)$ such that for every function $f \in W_1^1(0, 1)$ satisfying*

$$\int_0^1 \alpha(x) f(x)\mathrm{d}x = 0, \tag{6}$$

the following Poincaré inequality holds:

$$\int_0^1 |f(t| \,\mathrm{d}t \le c(\alpha) \int_0^1 \left|f'(t)\right| \mathrm{d}t. \tag{7}$$

The constant $c(\alpha)$ satisfies the inequality

$$c(\alpha) \ge \frac{1}{2},$$

and the best constant $c(\alpha) = 1/2$ is achieved if and only if the weight function α satisfies the additional condition

$$\int_0^{1/2} \alpha(t)\mathrm{d}t = \frac{1}{2}. \tag{8}$$

Moreover, if $c(\alpha) = \frac{1}{2}$, then the constant $\frac{1}{2}$ in (7) is optimal.

Proof In order to make the paper self-contained, we include a proof here for the best constant $c(\alpha) = 1/2$ appearing in (7). To see that the constant $1/2$ is optimal in the case where $c(\alpha) = 1/2$, take $f = f_\varepsilon$, where $\varepsilon > 0$, and f_ε is defined by

$$f_\varepsilon(x) = \begin{cases} -1 & \text{if } x \in \left[0, \frac{1}{2} - \varepsilon\right], \\ \frac{1}{\varepsilon}\left(x - \frac{1}{2}\right) & \text{if } x \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right], \\ 1 & \text{if } x \in \left[\frac{1}{2} + \varepsilon, 1\right]. \end{cases} \tag{9}$$

We clearly have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^1 f_\varepsilon(x)\alpha(x)\mathrm{d}x &= \lim_{\varepsilon \to 0} \left(-\int_0^{1/2-\varepsilon} \alpha(x)\mathrm{d}x + \int_{1/2+\varepsilon}^1 \alpha(x)\mathrm{d}x \right) \\ &\quad + \lim_{\varepsilon \to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)\mathrm{d}x \\ &= -\int_0^{1/2} \alpha(x)\mathrm{d}x + \int_{1/2}^1 \alpha(x)\mathrm{d}x + \lim_{\varepsilon \to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)\mathrm{d}x. \end{aligned}$$

Now since $c(\alpha) = \frac{1}{2}$, we have that $\int_0^{1/2} \alpha(x)dx = 1/2$, which yields

$$-\int_0^{1/2} \alpha(x)\mathrm{d}x + \int_{1/2}^1 \alpha(x)\mathrm{d}x = 0.$$

Hence, we get

$$\lim_{\varepsilon\to 0}\int_0^1 f_\varepsilon(x)\alpha(x)\mathrm{d}x = \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)\mathrm{d}x$$
$$= 0.$$

Observe that in the case where $\alpha \equiv 1$ condition $\int_0^1 f_\varepsilon(x)\alpha(x)\mathrm{d}x = 0$ is exactly fulfilled for arbitrary ε. As we have

$$\int_0^1 |f_\varepsilon(x)|\,\mathrm{d}x = 1-\varepsilon,$$

$$\int_0^1 \left|f_\varepsilon'(x)\right|\mathrm{d}x = 2,$$

it is easy to check that

$$\frac{\int_0^1 |f_\varepsilon(x)|\,\mathrm{d}x}{\int_0^1 \left|f_\varepsilon'(x)\right|\mathrm{d}x} = \frac{1-\varepsilon}{2} \to \frac{1}{2} \quad \text{when } \varepsilon\to 0.$$

Remark 1 A scaling argument shows that for a general interval $[a,b]$ the inequality (7) takes the form

$$\int_a^b |f(t)|\,\mathrm{d}t \le \frac{b-a}{2}\int_a^b \left|f'(t)\right|\mathrm{d}t.$$

We now return to our general Poincaré inequality (3).

Set $I_k := \left[\frac{k-1}{n}, \frac{k}{n}\right]$, $k = 1,\ldots,n$. Let us define the weight functions in the following way

$$\alpha_k^*(x) = \begin{cases} 0 \ \text{ on } \ I_j, \ \text{ if } j \neq k, \\ n \ \text{ on } \ I_k. \end{cases} \tag{10}$$

Clearly, it is easily verified that

$$\int_0^1 \alpha_k^*(x)\mathrm{d}x = \int_{I_k} \alpha_k^*(x)\mathrm{d}x = 1.$$

Thus, for any $k = 1,\ldots,n$, $\alpha_k^* \in \mathscr{N}$.

Our next result extends Theorem 1 when instead of (6) orthogonality conditions (5) are imposed. To establish this result, we shall use Theorem 1 and the

Hobby–Rice Theorem [6] (see [9] for a nice simple proof). This later, which is of importance in the study of L^1-approximation, plays a central role in this paper.

Theorem 2 (Hobby, Rice) *Let* $\alpha_i, i = 1, \ldots, n$, *be* n *real functions in* $L(\mathrm{d}\mu; [0, 1])$, *where* μ *is a finite, nonatomic, real measure on* $[0, 1]$. *Then there exist* $\{t_j\}_{j=1}^{m}$, $m \le n$, $t_0 = 0 < t_1 < \cdots < t_m < t_{m+1} = 1$, *such that*

$$\sum_{j=1}^{m+1} (-1)^j \int_{t_{j-1}}^{t_j} \alpha_i(x)\mathrm{d}\mu(x) = 0, \quad i = 1, \ldots, n.$$

Our first result can be stated as follows:

Theorem 3 *Let* α_k^*, $k = 1, \ldots, n$, *be the weight functions as defined in* (10). *Set*

$$T_k^*(f) = \int_0^1 \alpha_k^*(x) f(x)\mathrm{d}x, \quad k = 1, \ldots, n. \tag{11}$$

Then for every function $f \in W_1^1(0, 1)$ *satisfying*

$$T_k^*(f) = 0, \quad k = 1, \ldots, n, \tag{12}$$

the following Poincaré inequality holds:

$$\int_0^1 |f(x)|\, \mathrm{d}x \le \frac{1}{2n} \int_0^1 \left| f'(x) \right| \mathrm{d}x. \tag{13}$$

Moreover, for each choice of $\alpha_1, \ldots, \alpha_n$ *in* $\mathcal{N}$, *the constant* $\dfrac{1}{2n}$ *in* (13) *is optimal.*

Proof Let $f \in W_1^1(0, 1)$ and assume that f satisfies conditions (12). Then, it is seen from the definition of α_k^* that

$$\int_{I_k} \alpha_k^*(x) f(x)\mathrm{d}x = \int_0^1 \alpha_k^*(x) f(x)\mathrm{d}x = 0, \quad k = 1, \ldots, n.$$

Thus, we can apply Theorem 1 (see Remark 1) to the interval I_k and get

$$\int_{I_k} |f(x)|\, \mathrm{d}x \le \frac{|I_k|}{2} \int_{I_k} \left| f'(x) \right| \mathrm{d}x = \frac{1}{2n} \int_{I_k} \left| f'(x) \right| \mathrm{d}x. \tag{14}$$

Then summing (14) over k gives the desired result (13).

It remains to show that the constant $1/(2n)$ is optimal. Now, because of the Hobby–Rice Theorem, we know that for any given integrable functions $\alpha_1, \ldots, \alpha_n$ on $[0, 1]$ there exist points $0 < t_1 < \cdots < t_m < 1$, with $m \le n$, such that the sign function

$$\sigma(x) := \text{sign}\,[(x - t_1) \dots (x - t_m)]$$

is orthogonal on [0, 1] to each of $\alpha_{1,} \dots, \alpha_n$, i.e.,

$$\int_0^1 \alpha_k(x)\sigma(x)\mathrm{d}x = 0, \quad k = 1, \dots, n.$$

Next, let α_k^*, $k = 1, \dots, n$, be the weight functions as defined in (10), then for every sufficiently small $\varepsilon > 0$ we construct the function σ_ε, smoothing the jumps at $t_1, \dots, t_m$ in the way of definition of f_ε defined in (9). Then

$$\left| \int_0^1 \alpha_k^*(x)\sigma_\varepsilon(x)\mathrm{d}x \right| = O(\varepsilon), \quad k = 1, \dots, n,$$

$$\int_0^1 |\sigma_\varepsilon(x)|\mathrm{d}x = 1 - m\varepsilon,$$

$$\int_0^1 \left|\sigma_\varepsilon'(x)\right| \mathrm{d}x = 2m,$$

it is easy to check that, when $\varepsilon \to 0$, we have

$$\frac{\int_0^1 |\sigma_\varepsilon(x)|\, \mathrm{d}x}{\int_0^1 \left|\sigma_\varepsilon'(x)\right| \mathrm{d}x} = \frac{1 - m\varepsilon}{2m} \to \frac{1}{2m} \geq \frac{1}{2n}.$$

(The last inequality holds because $m \leq n$.) Thus, $1/(2n)$ is optimal and completes the proof of the theorem. □

3 Case of General Linear Functionals

We now turn to the main part of our contribution and consider the more general situation in which the functionals T_k, $k = 1, \dots, n$, are not necessarily generated by densities. To accomplish this, we use some well-known facts on B-splines and divided differences, see, e.g., [2]. We shall start with the simple observation any set of linear functionals $T_1, \dots, T_n$ satisfying

$$T_1\,[f] = \cdots = T_n\,[f] = 0 \quad (f \in W_1^n(0, 1)) \tag{15}$$

assure existence of a constant $c(T_1, \dots, T_n)$ such that

$$\int_0^1 |f(x)|\,\mathrm{d}x \le c(T_1,\ldots,T_n)\int_0^1 \left|f^{(n)}(x)\right|\mathrm{d}x, \tag{16}$$

must be unisolvent with respect to π_{n-1}. Note that the unisolvence condition is certainly satisfied by functionals T_k^*, $k = 1,\ldots,n$, defined by Eq. (12), as can be shown using the classical intermediate value theorem.

In order to formulate our main result we need the following important fact.

Lemma 1 *Assume that the functionals T_k, $k = 1,\ldots,n$, are unisolvent with respect to π_{n-1}. Then for every $f \in W_1^n(0,1)$ there exists a unique polynomial $p[f] \in \pi_{n-1}$ such that*

$$T_k\left[p[f]\right] = T_k\left[f\right], \quad k = 1,\ldots,n.$$

Proof The conditions above form a linear system of n linear equations in n unknowns with respect to the coefficients of $p[f]$. Now, the unisolvence of the set of functionals $\{T_k\}_{k=1}^n$ with respect to π_{n-1} ensures that the corresponding homogeneous system admits only the trivial zero solution, and this implies the existence and uniqueness of the solution $p[f]$ of the original system. □

We now briefly recall the relevant material from B-splines and divided differences defined on a set of distinct points $\{x_k\}_{k=0}^n$. They are called knots.

$$B(x_0,\ldots,x_{n+1},t) = (\cdot - t)_+^n\,[x_0,\ldots,x_{n+1}]$$

of the truncated power function $(x-t)_+^n$ is called a B-spline of degree n with knots $x_0,\ldots,x_{n+1}$. The B-spline has the properties (see [2, p. 30]):

$$B(x_0,\ldots,x_{n+1},t) > 0 \ \text{ on } \ (x_0,x_{n+1});$$

$$B(x_0,\ldots,x_{n+1},t) = 0 \ \text{ for } t \notin (x_0,x_{n+1});$$

$$B^{(j)}(x_0) = B^{(j)}(x_{n+1}) = 0, \quad j = 0,1,\ldots,n-1,$$

provided $x_0 < x_1 < \cdots < x_{n+1}$. The reader can find details in some book on splines, e.g., [2–4, 10, 11]. It can be easily derived from these properties $\frac{\mathrm{d}}{\mathrm{d}t}B(x_0,\ldots,x_{n+1},t)$ has a unique zero in (x_0,x_{n+1}). Therefore, $B(x_0,\ldots,x_{n+1},t)$ has a unique maximum in (x_0,x_{n+1}).

Consider the particular case when $x_0 = 0$, $x_{n+1} = 1$, and for $k = 1,\ldots,n$ the knots x_k coincide with the zeros $\eta_1^0 < \cdots < \eta_n^0$ of the Chebyshev polynomial of the second kind u_n^0 associated with the interval [0, 1], i.e., $u_n^0(x) = u_n(2x-1)$, where

$$u_n(x) := \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta.$$

Then the corresponding B-spline will be denoted by B_n. It is called perfect B-spline, since in this case its nth derivative maintains a constant absolute value

$$\left|B_n^{(n)}(t)\right| = \text{const}$$

for all points $t \in [0, 1]$ at which the derivative $B_n^{(n)}(t)$ is defined. We shall assume that B_n is normalized by the condition

$$B_n^{(n)}(t) = \text{sign } u_n^0(t), \quad t \in [0, 1] .$$

Perfect splines exhibit a number of interesting extremal properties, see e.g. [11]. Let us denote by $\|B_n\|$ the uniform norm of B_n on [0, 1]. Given $T_1, \ldots, T_n$, for each $f \in W_1^n$ we consider the polynomial $p[f]$ defined in Lemma 1. Since, as can be easily verified, the operator $p[f] : W_1^n(0, 1) \to \pi_{n-1}$ reproduces the polynomials from π_{n-1} Peano's theorem gives the representation

$$f(x) = \frac{1}{(n-1)!} \int_0^1 \left((x-t)_+^{n-1} - p\left[(\cdot - t)_+^{n-1}\right](x)\right) f^{(n)}(t)\mathrm{d}t$$

for all functions $f \in W_1^n(0, 1)$ for which $T_k[f] = 0$, $k = 1, \ldots, n$.

The above integral representation yields the estimate

$$\int_0^1 |f(x)|\,\mathrm{d}x \le c(T_1, \ldots, T_n) \int_0^1 \left|f^{(n)}(x)\right|\mathrm{d}x, \tag{17}$$

with

$$c(T_1, \ldots, T_n) := \frac{1}{(n-1)!} \max_{0 \le x \le 1} \int_0^1 \left|(x-t)_+^{n-1} - p\left[(\cdot - t)_+^{n-1}\right](x)\right| \mathrm{d}t. \tag{18}$$

So much more can be said about the best constant in the error bound given in (17).

Our main result is presented in the following theorem.

Theorem 4 *Let $\{T_k[f]\}_1^n$ be any set of linear functionals on $W_1^n(0, 1)$ which are unisolvent with respect to π_{n-1}. Then*

$$\int_0^1 |f(x)|\,\mathrm{d}x \le c(T_1, \ldots, T_n) \int_0^1 \left|f^{(n)}(x)\right|\mathrm{d}x \tag{19}$$

for every function $f \in W_1^n(0, 1)$ satisfying the conditions $T_k[f] = 0$, $k = 1, \ldots, n$. Furthermore, the constant $c(T_1, \ldots, T_n)$ defined in (18) is optimal.

Proof The first claim was proved. Next we show the lower bound. Note that $p\left[(\cdot - t)_+^{n-1}\right]$ is an algebraic polynomials from π_{n-1}. Therefore, for every fixed $t \in [0, 1]$,

$$c(T_1, \ldots, T_n) \geq \frac{1}{(n-1)!} \int_0^1 \left| (x-t)_+^{n-1} - q_t(x) \right| \mathrm{d}x,$$

where q_t is the polynomial of best L_1-approximation on [0, 1] to the function $(\cdot - t)_+^{n-1}$. It is a classical result (see, for example, the book of Achieser [1]) that if the difference $f(x) - p(x)$, where $p \in \pi_{n-1}$ interpolates f at the zeros $\eta_1^0, \ldots, \eta_n^0$ of the Chebyshev polynomial of the second kind u_n^0, changes sign only at the points $\eta_1^0, \ldots, \eta_n^0$, then p is the polynomial of best L_1-approximation to f on [0, 1] of degree $n-1$. In the case $f(\cdot) = (\cdot - t)_+^{n-1}$ we see that $f-p$ cannot have more than n zeros counting multiplicities. Indeed, otherwise $f^{(n-1)} - p^{(n-1)}$ would have at last two sign changes which is impossible since $f^{(n-1)}$ is a step function and $p^{(n-1)}$ is a constant. Thus, the only zeros of $f - p$ are the interpolation nodes $\eta_1^0, \ldots, \eta_n^0$, and they are simple zeros. This means that the polynomials $q_t \in \pi_{n-1}$ which interpolates $(\cdot - t)_+^{n-1}$ at $\eta_1^0, \ldots, \eta_n^0$, is a polynomial of best L^1-approximation of it. Then, as known from the theory, the sign of the difference, that is,

$$\operatorname{sign}((x-t)_+^{n-1} - q_t(x)) = \operatorname{sign} u_n^0(x)$$

must be orthogonal to all polynomials from π_{n-1}. This yields

$$\begin{aligned} \int_0^1 \left| (x-t)_+^{n-1} - q_t(x) \right| \mathrm{d}x &= \int_0^1 (x-t)_+^{n-1} \operatorname{sign} u_n^0(x) \, \mathrm{d}x \\ &:= \Phi(t), \end{aligned}$$

where Φ is the n-tuple integral of u_n^0 satisfying the conditions

$$\Phi(0) = \Phi'(0) = \cdots = \Phi^{(n-1)}(0) = 0.$$

By the orthogonality property of $\operatorname{sign} u_n^0(t)$, it satisfies also the conditions

$$\Phi(1) = \Phi'(1) = \cdots = \Phi^{(n-1)}(1) = 0.$$

Since $\Phi^{(n-1)} = (n-1)! \operatorname{sign} u_n^0$, the function $\Phi^{(n-1)}/(n-1)!$ is a spline of degree n with knots at $0, \eta_1^0, \ldots, \eta_n^0, 1$. Moreover, $\Phi^{(n-1)}/(n-1)!$ coincides with the perfect B-spline B_n. Therefore,

$$c(T_1, \ldots, T_n) \geq \|B_n\|,$$

which was to be shown.

To see that the constant $c(T_1, \ldots, T_n)$ is optimal for the given functionals $T_1, \ldots, T_n$, we consider the function

$$f_0(x) := \frac{1}{(n-1)!}\left\{(x-t)_+^{n-1} - p\left[(\cdot - t)_+^{n-1}\right](x)\right\}$$

for a point t at which

$$\frac{1}{(n-1)!}\int_0^1 \left|(x-t)_+^{n-1} - p\left[(\cdot - t)_+^{n-1}\right](x)\right| \mathrm{d}x$$

attained its maximal value (equal to $c(T_1, \dots, T_n)$ by definition). Observe that then

$$\int_0^1 |f_0(x)|\, \mathrm{d}x = c(T_1, \dots, T_n),$$

while the variation $V(f_0^{(n-1)})$ of $f_0^{(n-1)}$ on $[0,1]$ equals 1. The latter follows from the fact that $(n-1)$th derivative of $(x-t)_+^{n-1}/(n-1)!$ equals 0 on $[0,t]$ and 1 on $[t,1]$, and $p^{(n-1)}\left[(\cdot - t)_+^{n-1}\right](x)/(n-1)!$ is a constant. Then smoothing f_0 in a neighborhood $[t-\varepsilon, t+\varepsilon]$ of t we get a function f_ε from $W_1^n(0,1)$ such that

$$\int_0^1 |f_\varepsilon(x)|\, \mathrm{d}x = c(T_1, \dots, T_n) - \varepsilon,$$

$$V(f_\varepsilon^{(n-1)}) = \int_0^1 |f_\varepsilon^{(n)}(x)| \mathrm{d}x = 1.$$

Therefore, inequality (19) with a constant smaller than $c(T_1, \dots, T_n)$ cannot be hold for f_ε if ε is sufficiently small. The proof is complete. □

Our main result tells us that using any set of linear functionals $\{T_k\}_{k=1}^n$ we cannot get estimate with a constant better than $\|B_n\|$. In the following corollary, which is a simple consequence of Theorem 4, we obtain an example of a set of extremal functionals for which this best constant is achieved.

Corollary 1 *Let the functionals $T_1, \dots, T_n$ be defined by*

$$T_k[f] = f(\eta_k^0), \quad k = 1, \dots, n.$$

Then

$$\int_0^1 |f(x)|\, \mathrm{d}x \le \|B_n\| \int_0^1 \left|f^{(n)}(x)\right| \mathrm{d}x$$

for every function $f \in W_1^n(0,1)$ satisfying the conditions $T_k[f] = 0$, $k = 1, \dots, n$.

We conclude this section with the following open problem: *Are they other sets of functionals with this extremal property?*

As can be easily computed

$$\|B_1\| = \frac{1}{2}, \quad \|B_2\| = \frac{1}{16}.$$

Therefore this confirms and gives another proof of the exact value of the best possible constant $1/2$ derived in Theorem 1. On the other hand, by Corollary 1, we see that there exist another extremal functionals for which this constant is also achieved. Moreover, a direct calculation shows that the mean-value functionals

$$T_1\,[f] = \int_0^1 f(x)\mathrm{d}x \quad \text{and} \quad T_2\,[f] = \int_0^1 f'(x)\mathrm{d}x$$

are also best in this sense for the case $n = 2$. Based on this observation, it seems that we have a good reason to conjecture that the functionals

$$T_k\,[f] = \int_0^1 f^{(k)}(x)\mathrm{d}x, \quad k = 0, 1, \ldots, n-1,$$

are extremal for any n as well.

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Best Constants for Weighted Poincaré-Type Inequalities

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Abstract In this paper we will determine the best constant for a class of (weighted and non-weighted) new Poincaré-type inequalities. In particular, we obtain sharp inequalities under the concavity, convexity of the weight function. We also establish a family of sharp Poincaré inequalities involving the second derivative.

1 Introduction

If Ω is a sufficiently regular bounded convex domain in $\mathbb{R}^d$ and f a smooth function defined on Ω with vanishing mean value over Ω, then a well-known form of the Poincaré inequality states that there is a constant $c_{p,\Omega}$ independent of f such that

$$\|f\|_p \le c_{p,\Omega} \|\nabla f\|_p , \tag{1}$$

where $1 \le p < \infty$ and $\|.\|_p$ denotes the classical $L^p(\Omega)$-norm, see, e.g., [12]. Poincaré-type inequalities are a key tool in the study of many problems of partial differential equations and numerical analysis, see [3, 7, 10, 13]. In the estimate (1), the constant $c_{p,\Omega}$ is finite for any p and generally it is not explicitly known. For practical purposes it is important to know an explicit expression for this constant (see, e.g., [11]). The determination of analytical expression of the Poincaré constant $c_{p,\Omega}$ as function of p and Ω is a difficult task. Specific estimates related to $c_{p,\Omega}$ have been obtained only in very special cases. For example, for $p = 2$, by some elementary considerations, Payne and Weinberger [9] showed that in order to determine the best Poincaré constant $c_{2,\Omega}$ in (1), it is basically enough to consider *weighted* Poincaré inequalities in one dimension. In this way, the one-dimensional case is essential since the case of several dimensions can be reduced to it. The main

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idea is to decompose the original domain into smaller subdomains, keeping the same mean value of f on each subdomain, and then to apply sharp one-dimensional *weighted* Poincaré inequalities. Payne and Weinberger have exploited this method in [9] to prove that the optimal constant in (1) is d/π when the L^2-norm is used. For $p = 1$, this technique was also developed by Acosta and Durán in [1]. For general p, Chua and Wheede [2] have successfully used this method for estimating the constants $c_{p,\Omega}$.

Some simple generalizations of (1) are well known, see, e.g., [8, Theorems 8.11 and 8.12]. One involves replacing the condition $\int_\Omega f(\mathbf{x})d\mathbf{x} = 0$ by the α-weighted average of f over Ω, $\int_\Omega \alpha(\mathbf{x})f(\mathbf{x})d\mathbf{x} = 0$, where α is any weight function from $L_1(\Omega)$ satisfying $\int_\Omega \alpha(\mathbf{x})d\mathbf{x} = 1$. We do not know the exact value of the best possible constants appearing in such general inequalities. We must add at this place that determining exact constants in inequalities between norms of functions and their derivatives is a very difficult problem that usually requires delicate considerations. Each case when exact constants are found is a great achievement.

In this paper, continuing the previous line of research we will discuss just the 1-dimensional case of the above. To be specific, we are interested in finding the smallest constant which is admissible in the following Poincaré-type inequality:

$$\int_0^1 |f(t)|\, dt \le c(\alpha) \int_0^1 \left|f'(t)\right| dt, \tag{2}$$

where f is such that f is absolutely continuous, $f, f' \in L^1[0,1]$ and

$$\int_0^1 \alpha(t) f(t) dt = 0, \tag{3}$$

with α a weight function on [0, 1] whose integral over [0, 1] is one. Our first main result provides an explicit expression for the best constant in (2). We will show that $c(\alpha) = 1/2$ is the best possible value. Moreover, we shall characterize all weight functions α for which (2) holds with best possible value 1/2. In [1, Theorem 3.1], Acosta and Durán showed that if, in addition, α is *concave* on [0, 1] then the following weighted version of (2)

$$\int_0^1 \alpha(t)\, |f(t)|\, dt \le c(\alpha) \int_0^1 \alpha(t) \left|f'(t)\right| dt, \tag{4}$$

holds true with the constant $c(\alpha) = 1/2$. Moreover, they also showed that the constant 1/2 cannot be improved in the case when $\alpha \equiv 1$. We shall give a new proof of this result, and, under specified conditions, prove that inequality (4) continues to hold when concavity of the weight function α is replaced by convexity. It is also shown, under suitable conditions on the weighted function, that Poincaré inequality (4) still holds with the best constant $c(\alpha) = 1/2$ for this general class of functions. Sharp Poincaré inequalities involving the second derivative are also considered.

The present paper is organized as follows. In Sect. 2, we shall first determine the best constant in (2), and characterize all weight functions α for which (2) holds with best possible value. We then give a new proof of the Acosta and Durán's inequality (4) when the weight function is concave. Furthermore, under appropriate assumptions on the weight function like convexity or monotonicity we generalize it, showing that inequality (4) still holds with the best constant $c(\alpha) = 1/2$ for this class of functions. Finally, in Sect. 3 we show how our arguments can be used to establish new sharp Poincaré inequality involving the second derivative.

2 Sharp Inequalities for the First Derivative

For the sake of clarity of our presentation, we shall first consider inequalities which involve only the first derivative of a function and the function itself. We first discuss optimal weight functions corresponding to the best constant in (2) with respect to the choice to some large class of weight functions. We also establish several weighted Poincaré-type inequalities under some appropriate assumptions on the weight function α like concavity, convexity.

We now set down some of the notation which will be used throughout. Let $W^{1,1}[0,1]$ denote the space of absolutely continuous functions on [0, 1] such that f and $f' \in L^1\,[0,1]$. Consider a linear functional of the form

$$T_\alpha[f] := \int_0^1 \alpha(t) f(t)dt \tag{5}$$

where α is a weight on $[0,1]$. Throughout this paper, by weight function we mean a nonnegative *integrable* function on [0, 1]. We assume in addition that α is normalized in the sense that its integral is equal to one:

$$\int_0^1 \alpha(t)dt = 1. \tag{6}$$

To ease the notation let us denote by $\mathcal{N}$ the class of weight functions on [0, 1] satisfying (6). This condition simply means that $T_\alpha[f] = 1$ for the constant function f of value 1 on [0, 1]. In what follows, $(.)^0_+ : \mathbb{R} \to \mathbb{R}$ will denote the function defined by

$$(x)^0_+ = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0. \end{cases}$$

Applying integration by parts, we deduce

$$T_\alpha[f] = f(0) + \int_0^1 \left(\int_0^1 (s-t)^0_+ \alpha(s) ds \right) f'(t)dt.$$

Hence if we denote

$$K_\alpha(x,t) = (x-t)_+^0 - \int_0^1 (s-t)_+^0 \alpha(s)ds$$

then it follows from Taylor's formula

$$f(x) = f(0) + \int_0^1 (x-t)_+^0 f'(t)dt$$

that

$$f(x) - T_\alpha[f] = \textstyle\int_0^1 K_\alpha(x,t) f'(t)dt. \tag{7}$$

Thus, if we assume that $T_\alpha[f] = 0$, we may use the representation formula (7) to estimate L_1-norm of f in terms of the L_1-norm of f' as follows:

$$\begin{aligned} \textstyle\int_0^1 |f(x)|\, dx &= \textstyle\int_0^1 \left| \int_0^1 K_\alpha(x,t) f'(t)dt \right| dx \\ &\le \left(\sup_{0 \le t \le 1} \textstyle\int_0^1 |K_\alpha(x,t)|\, dx \right) \textstyle\int_0^1 \left| f'(t) \right| dt. \end{aligned} \tag{8}$$

Hence, denoting by

$$c(\alpha) := \sup_{0 \le t \le 1} \int_0^1 |K_\alpha(x,t)|\, dx \tag{9}$$

we get that for any function $f \in W^{1,1}[0,1]$ satisfying $T_\alpha[f] = 0$, the following Poincaré inequality holds:

$$\int_0^1 |f(t)|\, dt \le c(\alpha) \int_0^1 \left| f'(t) \right| dt. \tag{10}$$

An interesting problem is to know the dependence of the constant $c(\alpha)$, where $c(\alpha)$ is given by (9), on the weight function and, in particular, to find the best constant $c_{\min} = \min_{\alpha \in \mathcal{N}} c(\alpha)$. Therefore, there are two important questions that arise:

(1) What is the exact value of the best possible constant $c_{\min}$?
(2) What are all the normalized weight functions α for which the best constant $c_{\min}$ is achieved?
(3) What is the exact value of $c(\alpha)$ under the concavity or convexity of the weight function α ?

Our first result will be fundamental for the remainder of this paper. Indeed, it permits us to determine the exact value of the best possible constant $c_{\min}$ and also to establish a complete characterization of the weight functions for which inequality (10) is satisfied with the best possible value $c_{\min}$. More precisely, we have the following characterization:

Lemma 1 *For any weight function* $\alpha \in \mathcal{N}$,

$$c(\alpha) \geq 1/2.$$

The inequality $c(\alpha) = 1/2$ *holds if and only if* α *satisfies the additional condition*

$$\int_0^{1/2} \alpha(t)dt = 1/2. \tag{11}$$

Moreover, if $c(\alpha) = 1/2$*, then the constant* $\frac{1}{2}$ *in* (10) *is optimal.*

Proof Let α be a weight function belonging to $\mathcal{N}$. From the above considerations, we know that inequality (10) holds with the constant

$$c(\alpha) = \sup_{0 \leq t \leq 1} \int_0^1 |K_\alpha(x,t)|\, dx. \tag{12}$$

But, the integral appearing in (12) is easily calculated. Indeed, we have

$$\begin{aligned}
F(t) &:= \textstyle\int_0^1 |K_\alpha(x,t)|\, dx \\
&= \textstyle\int_0^1 \left| (x-t)_+^0 - \int_0^1 (s-t)_+^0 \alpha(s)ds \right| dx \\
&= \left(\textstyle\int_t^1 \alpha(s)ds \right) t + \left(1 - \int_t^1 \alpha(s)ds \right)(1-t) \\
&= \left(1 - \textstyle\int_0^t \alpha(s)ds \right) t + \left(\int_0^t \alpha(s)ds \right)(1-t) \\
&= \left(1 - 2\textstyle\int_0^t \alpha(s)ds \right) t + \int_0^t \alpha(s)ds.
\end{aligned}$$

Moreover, we clearly observe that

$$F(\frac{1}{2}) = \frac{1}{2}.$$

Therefore, by continuity of F on [0, 1] it becomes obvious that

$$c(\alpha) = \sup_{0 \leq t \leq 1} F(t) \geq \frac{1}{2}. \tag{13}$$

This proves the first statement of the lemma.

We now prove the second part of this lemma. To this end, a simple calculation shows that the derivative of F is given by

$$F'(t) = 1 - 2\int_0^t \alpha(s)ds + \alpha(t)(1-2t). \tag{14}$$

To establish the "if" part, let us assume that $c(\alpha) = 1/2$. Since $F(1/2) = 1/2$ then (13) implies $F'(\frac{1}{2}) = 0$. Hence, substituting $t = 1/2$ into Eq. (14) yields condition (11).

For the "only if" part, let us assume that $\int_0^{1/2} \alpha(x)dx = 1/2$. Then for every $\alpha \in \mathcal{N}$ that satisfies the last condition, we clearly have, by Eq. (14),

$$F'(t) > 0 \text{ on } [0, 1/2), \qquad \text{and} \qquad F'(t) < 0 \text{ on } (1/2, 1],$$

and thus $\sup_{0 \le t \le 1} F(t) = F(1/2)$. Remembering that $F(1/2) = 1/2$ we easily get $c(\alpha) = \sup_{0 \le t \le 1} F(t) = 1/2$, as required. To see that the constant $\frac{1}{2}$ is optimal in the case where $c(\alpha) = 1/2$, take $f = f_\varepsilon$, where $\varepsilon > 0$, and f_ε is defined by

$$f_\varepsilon(x) = \begin{cases} -1 & \text{if } x \in \left[0, \frac{1}{2} - \varepsilon\right] \\ \frac{1}{\varepsilon}\left(x - \frac{1}{2}\right) & \text{if } x \in \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right] \\ 1 & \text{if } x \in \left[\frac{1}{2} + \varepsilon, 1\right]. \end{cases} \tag{15}$$

We clearly have

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon \to 0}\left(-\int_0^{1/2-\varepsilon} \alpha(x)dx + \int_{1/2+\varepsilon}^1 \alpha(x)dx\right) \\ &\quad + \lim_{\varepsilon \to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= -\int_0^{1/2} \alpha(x)dx + \int_{1/2}^1 \alpha(x)dx \\ &\quad + \lim_{\varepsilon \to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx. \end{aligned}$$

Now since $c(\alpha) = \frac{1}{2}$, we have that $\int_0^{1/2} \alpha(x)dx = 1/2$ which yields

$$-\int_0^{1/2} \alpha(x)dx + \int_{1/2}^1 \alpha(x)dx = 0.$$

Hence, we get

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon \to 0} \int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= 0. \end{aligned}$$

Observe that in the case where $\alpha \equiv 1$ condition $\int_0^1 f_\varepsilon(x)\alpha(x)dx = 0$ is fulfilled for arbitrary ε. On the other hand,

$$\begin{aligned} \int_0^1 |f_\varepsilon(x)|\, dx &= 1 - \varepsilon \\ \int_0^1 \left|f_\varepsilon'(x)\right| dx &= \ \ 2. \end{aligned}$$

Then,we arrive at

$$\frac{\int_0^1 |f_\varepsilon(x)|\,dx}{\int_0^1 \left|f'_\varepsilon(x)\right|dx} = \frac{1-\varepsilon}{2} \to \frac{1}{2} \text{ when } \varepsilon \to 0,$$

which concludes the proof.

Now we state our first result whose proof follows immediately from Lemma 1.

Theorem 1 *If the weight function α from $\mathcal{N}$ satisfies* (11)*, then*

$$\int_0^1 |f(t)|\,dt \le \tfrac{1}{2}\int_0^1 \left|f'(t)\right|dt \tag{16}$$

for every function $f \in W^{1,1}[0,1]$ such that $\int_0^1 \alpha(x)f(x)dx = 0$. Moreover, if α does not satisfy (11), then the constant $\frac{1}{2}$ is optimal.

Proof The first part of the theorem is a direct consequence of Lemma 1. For the second part, without loss of generality, assume that $\int_0^{1/2}\alpha(x)dx > 1/2$. Take an arbitrary $\epsilon \in (0,1)$ and consider the sequence of functions

$$f_\varepsilon(x) = \begin{cases} -1 & \text{if } x \in \left[0, \frac{1}{2}-\varepsilon\right] \\ l(x) & \text{if } x \in \left[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon\right] \\ 1+2\delta & \text{if } x \in \left[\frac{1}{2}+\varepsilon, 1\right], \end{cases} \tag{17}$$

where l is the affine function

$$l(x) = -1 + \frac{1+\delta}{\varepsilon}(x - \frac{1}{2} + \varepsilon) \text{ and } \delta = \frac{2\int_0^{1/2}\alpha(x)dx - 1}{2\int_{1/2}^1 \alpha(x)dx}.$$

Observe that δ is positive and it satisfies

$$-\int_0^{1/2}\alpha(x)dx + (1+2\delta)\int_{1/2}^1 \alpha(x)dx = 0. \tag{18}$$

A simple calculation shows that

$$\begin{aligned} \lim_{\varepsilon\to 0}\int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon\to 0}\left(-\int_0^{1/2-\varepsilon}\alpha(x)dx + (1+2\delta)\int_{1/2+\varepsilon}^1 \alpha(x)dx\right) \\ &\quad + \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx \\ &= -\int_0^{1/2}\alpha(x)dx + (1+2\delta)\int_{1/2}^1 \alpha(x)dx \\ &\quad + \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx. \end{aligned}$$

Now using (18), we get

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^1 f_\varepsilon(x)\alpha(x)dx &= \lim_{\varepsilon\to 0}\int_{\frac{1}{2}-\varepsilon}^{\frac{1}{2}+\varepsilon} f_\varepsilon(x)\alpha(x)dx\\ &= \lim_{\varepsilon\to 0}\varepsilon\delta = 0.\end{aligned}$$

As we have

$$\begin{aligned}\int_0^1 |f_\varepsilon(x)|\,dx &= 1+\delta-\varepsilon-\tfrac{\delta\varepsilon}{1+\delta}\\ \int_0^1 \left|f_\varepsilon'(x)\right|dx &= \quad 2(1+\delta),\end{aligned}$$

it follows

$$\frac{\int_0^1 |f_\varepsilon(x)|\,dx}{\int_0^1 \left|f_\varepsilon'(x)\right|dx} = \frac{1+\delta-\varepsilon-\frac{\delta\varepsilon}{1+\delta}}{2(1+\delta)} \to \frac{1}{2}\text{ when }\varepsilon\to 0.$$

This proves the theorem.

Remark 1 A scaling argument shows that for a general interval $[a, b]$ the inequality (16) takes the form

$$\int_a^b |f(t)|\,dt \le \tfrac{(b-a)}{2}\int_a^b \left|f'(t)\right|dt.$$

We have the following result as a corollary to Theorem 1.

Corollary 1 *Suppose that α is any weight function from $\mathcal{N}$. Then, for every function $f \in W^{1,1}[0,1]$ satisfying $\int_0^1 \alpha(t)f(t)dt = 0$ the following inequality holds:*

$$\int_0^1 \alpha(t)\,|f(t)|\,dt \le \tfrac{1}{2}\int_0^1 \left|f'(t)\right|dt. \tag{19}$$

The constant $\frac{1}{2}$ cannot be improved.

Proof For a given function α in $\mathcal{N}$, we define

$$v(x) = \int_0^x \alpha(s)ds, \qquad x \in [0,1].$$

Clearly $v(0) = 0$, $v(1) = 1$ and v is non-decreasing on $[0,1]$. We associate with any function f, belonging to the class $W^{1,1}[0,1]$ on $[0,1]$ and satisfying the condition $\int_0^1 \alpha(x)f(x)dx = 0$, the following function

$$\tilde{f}(t) := f(v^{-1}(t)), \qquad t \in [0,1],$$

where v^{-1} is the inverse of v. Clearly $\tilde{f} \in W^{1,1}[0,1]$ and it satisfies the condition

$$\int_0^1 \tilde{f}(t)dt = \int_0^1 \tilde{f}(v(x))v'(x)dx = \int_0^1 \alpha(x)f(x)dx = 0.$$

Then, by Theorem 1, $\tilde{f}$ satisfies the inequality

$$\int_0^1 \left|\tilde{f}(t)\right| dt \le \tfrac{1}{2}\int_0^1 \left|\tilde{f}'(t)\right| dt.$$

Now by making the change of variables $t = v(x)$, we see that

$$\int_0^1 \alpha(x)\,|f(x)|\,dx \le \frac{1}{2}\int_0^1 \left|\tilde{f}'(v(x))\right| v'(x)dx = \frac{1}{2}\int_0^1 \left|f'(x)\right| dx,$$

which is the announced statement.

We observe that the integral on the right-hand side in the estimate (19) is not weighted. In view of this result the following question arises naturally:

- Does inequality (19) also hold with a weight function in the integral appearing on the right-hand side?

Acosta and Durán, see [1, Theorem 3.1], provide a positive answer to this question if the weight function is concave. We shall give a new and simpler proof of this result, and generalize it to a wide class of weight functions. From now on, we would like to consider the case when the weight function α is chosen in such a way that it belongs to the special class $\mathscr{M}$ defined by:

$$\mathscr{M} := \left\{\alpha \in \mathscr{N} : \int_0^{1/2} \alpha(t)dt = 1/2\right\}. \tag{20}$$

Recall that the best constant in the Poincaré inequality (2) is attained for this class of weight functions. This subset is nonempty, since it clearly contains the following two "broken" functions:

$$\alpha_2(x) = \begin{cases} 2-4x & \text{if } x \in [0, 1/2] \\ 4x-2 & \text{if } x \in [1/2, 1], \end{cases}$$

$$\alpha_3(x) = \begin{cases} 4x & \text{if } x \in [0, 1/2] \\ 4-4x & \text{if } x \in [1/2, 1]. \end{cases}$$

Moreover, since $\mathscr{M}$ is convex it is an infinite set. Note that the function α_2 is convex on [0, 1], while α_3 is concave on [0, 1].

Without concavity assumption on α the weighted Poincaré-type inequality (4) does not hold true in general. The question we now want to address is that of determining some appropriate assumptions on the weight function α like convexity, monotonicity, which ensure that (4) holds. To the best of our knowledge the problem involving finding the weight functions that produce the best constant $1/2$ in (4) was not considered previously. In doing so, we shall provide a new version and a different proof of the Acosta and Durán result for a general class of weight functions belonging to $\mathscr{M}$.

To this end we first make the following observation that gives a sufficient condition on the weight functions for inequality (4) to hold.

Remark 2 Let us first observe that if $T_\alpha[f] = 0$ then multiplying (7) by $\alpha \in \mathscr{N}$ and applying Fubini's theorem, we obtain

$$\int_0^1 \alpha(x)\,|f(x)|\,dx \leq 2\int_0^1 \left(\left(1-\int_0^x \alpha(t)dt\right)\int_0^x \alpha(t)dt\right)\left|f'(x)\right|dx. \tag{21}$$

As a consequence of the above inequality, the weighted Poincaré inequality (4) obviously holds for all $\alpha \in \mathscr{N}$ satisfying

$$\left(1-\int_0^x \alpha(t)dt\right)\int_0^x \alpha(t)dt \leq \frac{\alpha(x)}{4},\ \ x \in (0,1)\,. \tag{22}$$

Acosta and Durán have shown that the above inequality holds for every concave weight function $\alpha \in \mathscr{N}$, see [1, Lemma 3.1].

In what follows we will show that large subclasses of weight functions in $\mathscr{N}$ satisfy inequality (22). We now give a useful fact:

Remark 3 By using the identity $ab \leq \frac{(a+b)^2}{4}$, when $a, b \geq 0$, together with the fact that, for all x in the interval $(0, 1)$, $0 \leq \int_0^x \alpha(t)dt \leq 1$, we obtain that

$$\left(1-\int_0^x \alpha(t)dt\right)\int_0^x \alpha(t)dt \leq \frac{1}{4}. \tag{23}$$

Therefore, inequality (22) holds for every x on $(0, 1)$ such that $\alpha(x) \geq 1$.

To prove our next weighted Poincaré-type inequality, we make use of two general lemmas:

Lemma 2 *Let $\alpha \in \mathscr{N}$ be a non-decreasing weight function on I_0, where $I_0 := (0, t_\alpha)$ is any nonempty sub-interval of $(0, 1)$, then (22) holds for all values $x \in I_0 \cap (0, 1/2)$ satisfying*

$$\frac{4x-1}{4x^2} \leq \alpha(x). \tag{24}$$

If α does not satisfy (24), then there is a function from the described class for which (22) does not hold.

Proof We first observe that since α is a non-decreasing function on I_0 we have

$$\int_0^x \alpha(t)dt \le x\alpha(x), x \in I_0 \cap (0, 1/2). \tag{25}$$

According to Remark 3, it is enough to consider only those values of $x \in I_0$ for which $0 < \alpha(x) < 1$. Thus, under this condition, inequality (25) yields

$$\int_0^x \alpha(t)dt \le x\alpha(x) \le 1/2, x \in I_0 \cap (0, 1/2).$$

Therefore, since the function $h(x) = x(1-x)$ is non-decreasing on $[0, 1/2]$, we get

$$\left(1 - \int_0^x \alpha(t)dt\right) \int_0^x \alpha(t)dt \le x\alpha(x)(1 - x\alpha(x)), x \in I_0 \cap (0, 1/2),$$

and so to prove inequality (22) for a given fixed $x \in I_0 \cap (0, 1/2)$ it is enough to show that $x(1 - x\alpha(x)) \le 1/4$, which is in turn equivalent to $\frac{4x-1}{4x^2} \le \alpha(x)$ as was assumed. This proves the first part of the lemma.

For the second part, we may use a geometrically evident idea to construct, through any fixed point t in $(0, 1)$ such that

$$0 < \alpha(t) < \frac{4t-1}{4t^2},$$

the function

$$f_t(x) = \begin{cases} \alpha(t) & \text{if } x \in [0, t] \\ l(x) & \text{if } x \in [t, 1], \end{cases}$$

where l is the linear polynomial passing through the points

$$(t, \alpha(t)) \qquad and \qquad (1, \frac{2-(1+t)\alpha(t)}{1-t}).$$

It is easy to see that f_t is a non-decreasing weight function that belongs to $\mathscr{N}$. Also, we have

$$\left(1 - \int_0^t f_t(x)dx\right) \int_0^t f_t(x)dx = t\alpha(t)(1 - t\alpha(t)) > \frac{f_t(t)}{4} \ (:= \frac{\alpha(t)}{4}).$$

This shows that (22) is not satisfied at the point t for the function f_t and this completes the proof of our lemma.

If the weight function $\alpha \in \mathscr{N}$ and non-increasing, then we have the following:

Lemma 3 *Let $\alpha \in \mathscr{N}$ be a non-increasing weight function on I_1, where $I_1 := (t_\alpha, 1)$ is any nonempty sub-interval of $(0, 1)$, then (22) holds for all values $x \in I_1 \cap (1/2, 1)$ satisfying*

$$\frac{3-4x}{4(1-x)^2} \leq \alpha(x). \tag{26}$$

If α does not satisfy (26), then there is a function from the described class for which (22) does not hold.

Proof We prove only the first statement since the proof of the second is essentially the same as that of Lemma 2. We will show that this case can be reduced the one treated in the previous lemma. Indeed, let us fix a non-increasing weight function α from $\mathscr{N}$. Define $\tilde{\alpha}$ the function in the interval $(0, 1)$ by $\tilde{\alpha}(t) = \alpha(1-t)$. Then, a straightforward inspection shows that $\tilde{\alpha}$ is a non-decreasing function belonging to $\mathscr{N}$. Hence the desired result follows by applying Lemma 2 to $\tilde{\alpha}$. This completes the proof of Lemma 3.

Now we make some comments, containing consequences of Lemmas 2 and 3.

Remark 4 Let $\alpha \in \mathscr{N}$. Since α is nonnegative function on the interval $(0, 1)$ then (24) is automatically satisfied on $(0, 1/4]$, and hence, by Lemma 2, inequality (22) holds if α is a non-decreasing function on any interval $I_0 \subset (0, 1/4)$. The same is true if α is a non-increasing function on any interval $I_1 \subset (3/4, 1)$.

Hence, we have the following weighted Poincaré inequality:

Theorem 2 *Let α be any concave function belonging to $\mathscr{M}$. Then, for every function $f \in W^{1,1}[0, 1]$ satisfying $\int_0^1 \alpha(t) f(t) dt = 0$, the following weighted Poincaré-type inequality holds*

$$\int_0^1 \alpha(t) \left| f(t) \right| dt \leq \frac{1}{2} \int_0^1 \alpha(t) \left| f'(t) \right| dt. \tag{27}$$

The constant $\frac{1}{2}$ cannot be improved.

In order to prove the above theorem, we shall apply the results of the previous two lemmas, and some preliminary facts about concave functions on $[0, 1]$ that will often be used without explicit reference. A somewhat known result that we can use as a starting point is the following form of the right-hand side of the Hermite–Hadamard inequality. It says: If the function α is concave on $[0, 1]$, then

$$\int_0^x \alpha(t) dt \leq x\alpha(x/2), x \in [0, 1]. \tag{28}$$

In particular, for any concave weight function belonging to $\mathscr{N}$, the following inequality holds:

$$1 \leq \alpha(1/2).$$

Moreover, if α is concave and $\alpha \in \mathscr{M}$, then substituting $x = 1/2$ in (28) and taking account of the fact that $\int_0^{1/2} \alpha(t)dt = 1/2$, we have

$$1 \leq \alpha(1/4).$$

It is also easy to check that

$$1 \leq \alpha(3/4), \tag{29}$$

in fact since α is concave the following inequality holds

$$\int_x^1 \alpha(t)dt \leq (1-x)\alpha((1+x)/2), x \in [0, 1],$$

and so (29) follows immediately by substituting $x = 1/2$ in the above equation. Inequalities (28) have been extensively studied in the literature, see, e.g., [4–6].

We now turn to the proof of Theorem 2.

Proof Let us fix a function α from $\mathscr{M}$. We begin the proof of Theorem 2 by noting that since α is concave, then, there exists an t_α in $[0, 1]$ such that α is non-decreasing on $[0, t_\alpha]$ and non-increasing on $[t_\alpha, 1]$. We will distinguish the following two cases:

Case 1 $t_\alpha \in (0, 1/2)$. If $t_\alpha \in (0, 1/4]$, this case is easier to handle, indeed by Remark 4, there is nothing to prove, since (24) is automatically satisfied on $(0, t_\alpha]$, and hence inequality (22) holds on $(0, t_\alpha)$. We may therefore assume that $t_\alpha \in (1/4, 1/2)$. Let us denote I_0 the sub-interval $(0, t_\alpha)$. Then, since $1 \leq \alpha(1/4)$ and α is non-decreasing function on $(1/4, t_\alpha)$, it follows that $1 \leq \alpha(t)$ for all t in $(1/4, t_\alpha)$. Observe that for all $x \in (1/4, t_\alpha)$, $\frac{4x-1}{4x^2} \leq 1$, then inequality (24) is satisfied for all $x \in (0, t_\alpha]$. Hence, Lemma 2 applies, so inequality (22) holds on $(0, t_\alpha)$.

Now, we will use I_1 to denote the sub-interval $(1/2, 1)$. Then, since $1 \leq \alpha(1/2)$, $1 \leq \alpha(3/4)$, and α is non-increasing function on I_1, it follows that $1 \leq \alpha(t)$ for all t in $(1/2, 3/4)$. Then, arguing as before, we see that (26) is satisfied for all $x \in I_1$. Lemma 3 applies, consequently, inequality (22) also holds on I_1.

Finally, in the sub-interval $(t_\alpha, 1/2)$ we have nothing to prove, since $1 \leq \alpha(t)$ for all t in $(t_\alpha, 1/2)$.

Case 2 $t_\alpha \in [1/2, 1)$. The proof is similar to the proof of the above case and hence is omitted.

So altogether, inequality (22) holds on $(0, 1)$, then the weighted Poincaré constant in (27) follows from Remark 2.

We note that our proof given in case $\alpha \in \mathscr{M}$ is completely different from the one given in [1]. We observe also that if (27) is unweighted and we allow α to very freely over $\mathscr{N}$, then Lemma 2 and Theorem 2 inform us that the value 1/2 of the best weighted Poincaré constant in (27) and it is attained if and only if $\alpha \in \mathscr{M}$.

We may now ask the following question:

- Do we get similar result to Theorem 2 if concavity is replaced by convexity assumption?

We first make the following observation:

Remark 5 The inequality (27) cannot hold in Theorem 2 if concavity is replaced by convexity. Indeed, if α is a non-decreasing convex function on [0, 1], then applying the Hermite–Hadamard inequality on [0, 1], we have $\alpha(1/2) \leq 1$, and since α is non-decreasing we deduce that $\alpha(0) \leq 1$. These two conditions together imply that $\int_0^{1/2} \alpha(t)dt = 1/2$, which can only happen if α is the one constant weight. To see this is quite simple and becomes obvious on drawing a figure. So the weighted Poincaré inequality does not hold for non-constant weight convex functions, which are non-decreasing.

Thus, our aim, of course, is to find some subclasses of convex functions for which inequality (27) remains valid. The last observation motivates us to introduce the subsets of weight functions of $\mathcal{N}$ (satisfying the Dirichlet conditions)

$$\mathcal{N}_0 = \{\alpha \in \mathcal{N} : \alpha(0) = 0\}$$
$$\mathcal{N}_1 = \{\alpha \in \mathcal{N} : \alpha(1) = 0\}.$$

A weighted Poincaré inequality with a nonnegative weight function in $\mathcal{N}_0$ or $\mathcal{N}_1$ is sometimes referred to as Poincaré–Friedrichs inequality. In order to state our next theorem, we would like to point out that the Hermite–Hadamard inequality for convex functions gives

$$\left(1 - \int_0^x \alpha(t)dt\right) \int_0^x \alpha(t)dt \leq x(1-x)\frac{\alpha(1)+\alpha(x)}{2}\frac{\alpha(0)+\alpha(x)}{2}$$
$$\leq \frac{1}{4}\frac{\alpha(1)+\alpha(x)}{2}\frac{\alpha(0)+\alpha(x)}{2}.$$

By Remark 2 is enough to prove that (22) holds for $x \in (0, 1)$ such that $\alpha(x) \leq 1$. Therefore, we conclude that if $\alpha \in \mathcal{N}_0$ with $\alpha(1) \leq 3$ or $\alpha \in \mathcal{N}_1$ with $\alpha(0) \leq 3$ then we have

$$\left(1 - \int_0^x \alpha(t)dt\right) \int_0^x \alpha(t)dt \leq \frac{1}{4}\alpha(x).$$

Hence, by exactly the same argument as before we can show our modest extension of Theorem 2 when concavity is replaced by convexity.

Theorem 3 *Let α be any convex function belonging to $\mathcal{N}_0$ with $\alpha(1) \leq 3$, or $\mathcal{N}_1$ with $\alpha(0) \leq 3$. Then, for every function $f \in W^{1,1}[0, 1]$ such that $\int_0^1 \alpha(t) f(t)dt = 0$, the following weighted Poincaré-type inequality holds*

$$\int_0^1 \alpha(t) |f(t)| dt \leq \tfrac{1}{2} \int_0^1 \alpha(t) \left|f'(t)\right| dt. \tag{30}$$

The constant $\frac{1}{2}$ *cannot be improved.*

Conditions $\alpha(1) \leq 3$ or $\alpha(0) \leq 3$, required in Theorem 3, are not optimal. Indeed, let us consider the weight function

$$\alpha_m(x) = \begin{cases} mx & \text{if } x \in [0, 1/m], \\ \frac{mx+m(m-2)}{(m-1)^2} & \text{if } x \in [1/m, 1], \end{cases}$$

where $m \in (1, 2]$. Then it is easy to see that α_m is convex and that belongs to $\mathscr{N}_0$ for any $m \in (1, 2]$. Note also that $\alpha_m(x) < 1$ if $x \in [0, 1/m)$, and $\alpha_m(x) \geq 1$ if $x \in [1/m, 1]$. According to Remark 3, it is sufficient to consider interval $[0, 1/m)$. Therefore, after performing the integration inequality (22) simplifies to

$$\frac{x}{2}\left(1 - \frac{m}{2}x^2\right) \leq \frac{1}{4}, \;\; (x \in (0, 1/m)).$$

But now we can check that the latter holds if $m \geq \frac{32}{27}$. Hence, for any $m \in \left[\frac{32}{27}, 2\right]$ the weighted Poincaré inequality (30) holds for α_m. However, we have $\alpha_m(1) > 3$ for any $m \in [\frac{32}{27}, \frac{3}{2})$.

3 Sharp Inequalities for the Second Derivatives

In this section we discuss sharp weighted Poincaré inequality involving the second derivative. More precisely, we want to take advantage of a possible second order regularity of f and consider in this section estimates of the L^1-norm of a function f in terms of the L^1-norm of its second-order derivative. To state our main result we will use the following notation:

$$W^{2,1}[0, 1] = \left\{f : f', f'' \; abs.\, cont.,\; f'' \in L^1[0, 1)\right\}.$$

By applying twice inequality (16) for functions $f \in W^{2,1}[0, 1]$ which satisfy the conditions

$$\int_0^1 f(t)dt = \int_0^1 f'(t)dt = 0, \tag{31}$$

we conclude that

$$\int_0^1 |f(t)|\, dt \leq \frac{1}{2}\int_0^1 \left|f'(t)\right| dt \leq \frac{1}{4}\int_0^1 \left|f''(t)\right| dt. \tag{32}$$

As the next result shows the constant $1/4$, obtained by iteration from the Poincaré inequality of first order, is far from being the best one. The following alternative approach leads the optimal constant.

Theorem 4 *For all* $f \in W^{2,1}[0,1]$ *satisfying* (31) *the following inequality holds*

$$\int_0^1 |f(t)|\,dt \leq \frac{1}{16}\int_0^1 \left|f''(t)\right|dt. \tag{33}$$

The constant $1/16$ *is the smallest possible.*

Proof It can be easily verified that the operator

$$l_1\,[f]\,(x) := \int_0^1 f(t)dt + (x-\frac{1}{2})\int_0^1 f'(t)dt$$

reproduces the linear polynomials, that is,

$$l_1\,[f]\,(x) \equiv f(x)\ for\ f(t)=1\ and\ f(t)=t, t\in[0,1]\,.$$

Then, by Peano's kernel theorem, each function in $W^{2,1}[0,1]$ satisfying (31) can be represented in the form

$$f(x) = \int_0^1 \left[(x-t)_+ - l_1\left[(.-t)_+\right](x)\right] f''(t)dt,$$

where $x_+ := \frac{1}{2}(x+|x|)$. From this, in a standard way we derive the inequality

$$\int_0^1 |f(x)|\,dx \leq \max_{0\leq t\leq 1}\int_0^1 |K_1(x,t)|\,dx \int_0^1 \left|f''(t)\right|dt,$$

where the kernel $K_1(x,t)$ is defined by

$$\begin{aligned} K_1(x,t) &= (x-t)_+ - l_1\left[(.-t)_+\right](x) \\ &= (x-t)_+ - \textstyle\int_0^1 (s-t)_+ ds - (x-\frac{1}{2})\int_0^1 (s-t)_+^0 ds. \end{aligned}$$

In order to compute the integral $\int_0^1 |K_1(x,t)|\,dx$ we first note that

$$\begin{aligned} l_t(x) &:= \textstyle\int_0^1 (s-t)_+ ds + (x-\frac{1}{2})\int_0^1 (s-t)_+^0 ds \\ &= \textstyle\int_t^1 (s-t)ds + (x-\frac{1}{2})\int_t^1 ds \\ &= \frac{(1-t)^2}{2} + (x-\frac{1}{2})(1-t) \\ &= (1-t)(x-\frac{t}{2}). \end{aligned}$$

Therefore the line l_t crosses the x-axis at the point $x_1 = \frac{t}{2}$. We easily find also the point x_2 at which l_t intersects the truncated power function $(x - t)_+$. We should have

$$x - t = l_t(x) \; at \; x = x_2.$$

This immediately implies that

$$x_2 = \frac{1+t}{2} > t.$$

It is seen that $\int_0^1 |K_1(x,t)|\,dx$, which is the area between the functions l_t and $(. - t)_+$, equals 2 times the area $\mathscr{A}$, which satisfies the following equation

$$\begin{aligned} 2\mathscr{A} &= (x_2 - x_1)(x_2 - t) - (x_2 - t)^2 \\ &= \tfrac{1}{2}\tfrac{1-t}{2} - \left(\tfrac{1-t}{2}\right)^2 \\ &= \tfrac{(1-t)t}{4}. \end{aligned}$$

Since $(1 - t)t \leq 1/4$ for all $t \in [0, 1]$, we obtain

$$\max_{0 \leq t \leq 1} \int_0^1 |K_1(x,t)|\,dx = \frac{1}{16}$$

and thus the desired inequality (33) holds.

It remains to show that the constant $1/16$ cannot be improved. In order to see this we consider the function

$$f_0(x) := \begin{cases} \frac{1}{4} - x & on \;\; x \in \left[0, \frac{1}{2}\right] \\ x - \frac{3}{4} & on \;\; x \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We have $\int_0^1 |f_0(x)|\,dx = \frac{1}{8}$ while the variation $V(f_0')$ of f_0' on $[0, 1]$ is equal to 2. Thus, a smoothing of f_0 in a neighborhood $[-\varepsilon, \varepsilon]$ of the zero will produce a function f_ε from $W^{2,1}[0, 1]$ for which

$$\int_0^1 |f_\varepsilon(x)|\,dx \geq \frac{1}{8} - \varepsilon,\; V(f_\varepsilon') = \int_0^1 \left|f''(x)\right| dx = 2.$$

Thus inequality (33) with a constant smaller that $\frac{1}{16}$ does not hold true for $\varepsilon > 0$ sufficiently small. The proof is complete.

The common restrictions on f to estimate a certain norm of f in terms of a norm of higher derivative are of the form

$$f^{(i)}(\xi) = 0, \, j = 0, 1, \ldots, m,$$

with an appropriate m. Usually ξ is taken to be the middle of the interval considered. We can see as we did in Theorem 4 that the conditions

$$f(a) = f'(a) = 0 \; or \; f(b) = f'(b) = 0$$

imply the estimate

$$\int_a^b |f(t)| \, dt \leq \frac{(b-a)^2}{2} \int_a^b \left| f''(t) \right| dt.$$

Applying this inequality twice, on [0, 1/2] and [1/2, 1], we get under the conditions

$$f(\frac{1}{2}) = f'(\frac{1}{2}) = 0 \tag{34}$$

that

$$\int_0^1 |f(t)| \, dt \leq \frac{1}{8} \int_0^1 \left| f''(t) \right| dt.$$

Therefore the conditions we considered in Theorem 4 yield a better estimation of $\| f \|_1$ than the standard ones (34).

This arises the following question:

- Are there other functionals that produce the smallest constant?

It is difficult to characterize all of them as we did in the case $W^{1,1}[0, 1]$. Even if we restricted ourselves to the study of functionals of the form

$$\int_a^b \alpha(x) f(x) dx = 0, \int_a^b \alpha(x) f'(x) dx = 0,$$

with a certain weight function α on [0, 1], we would arrive at the problem of investigation of the corresponding kernel

$$K(x, t) = (x - t)_+ - l \, [(. - t)_+] \, (x)$$

where

$$l \, [f] \, (x) = \int_0^1 \alpha(s) f(s) ds + \left(x - \int_0^1 s\alpha(s) ds \right) \int_0^1 \alpha(s) f'(s) ds.$$

In this situation the line $l_t \, [(. - t)_+]$ intersects the truncated power function at the points

$$x_1(t) = j(t) / \textstyle\int_t^1 \alpha(s) ds$$
$$x_2(t) = (t - j(t)) / \textstyle\int_0^t \alpha(s) ds$$

where

$$j(t) = \int_0^t \alpha(s)ds \int_t^1 s\alpha(s)ds - \int_0^t s\alpha(s)ds \int_t^1 \alpha(s)ds.$$

Unfortunately, these expressions are too complicated to hope to get a complete characterization of the best weight functions, i. e., those that lead to

$$\int_0^1 |K(x,t)|\, dx = 1/16.$$

4 Conclusion

With direct and simple proofs, we establish, under the concavity, convexity, or monotonicity of the weight function, the best constants for a class of (weighted and non-weighted) new Poincaré-type inequalities. Finally, an (unweighted) inequality of a similar type involving the second derivative is studied. A sharp constant is determined.

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Operator Inequalities Involved Wiener–Hopf Problems in the Open Unit Disk

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Abstract In this effort, we employ some of the linear differential inequalities to achieve integral inequalities of the type Wiener–Hopf problems (WHP). We utilize the concept of subordination and its applications to gain linear integral operators in the open unit disk that preserve two classes of analytic functions with a positive real part. Linear second-order differential inequalities play a significant role in the field of complex differential equations. Our study is based on a neighborhood containing the origin. Therefore, the Wiener–Hopf problem is decomposed around the origin in the open unit disk using two different classes of analytic functions. Moreover, we suggest a generalization for WHP by utilizing some classes of entire functions. Special cases are given in the sequel as well. A necessary and sufficient condition for WHP to be averaging operator on a convex domain (in the open unit disk) is given by employing the subordination relation (inequality).

1 Introduction

The Wiener–Hopf problems (WHP) [1] is a mathematical method to solve systems of integral equations extensively used in the field of applied mathematics [2], specifically in optimization theory [3], control systems [4], electromagnetics [5], image processing [6], and cloud computing system [7]. The technique acts by developing the complex-holomorphic properties of transforming functions. The Wiener operator of absolutely convergent Taylor series of a complex variable is given by the formal

$$w(z) = \sum_{n\in\mathbb{N}} \omega_n z^n, \quad \texttt{with} \quad \|w\|_{\mathfrak{W}} = \sum_{n\in\mathbb{N}} |\omega_n| < \infty.$$

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It has been studied in many infinite spaces such as Hilbert spaces [8] and Banach spaces [9]. The main stage in many WHP is to decompose an arbitrary function into two functions. Overall, this can be done by putting

$$\Psi_{+}(\zeta) = \frac{1}{2\pi i} \int_{\Omega_1} \Psi(z) \frac{dz}{z-\zeta} \tag{1}$$

and

$$\Psi_{-}(\zeta) = -\frac{1}{2\pi i} \int_{\Omega_2} \Psi(z) \frac{dz}{z-\zeta}, \tag{2}$$

where the contours Ω_1, Ω_2 are parallel to the real line, but move above and below the point $z = \zeta$, respectively.

In this paper, we investigate some of the linear differential inequalities involving WHP. Our discussion is based on the concept of subordination: $\phi(z) \prec \psi(z)$, where $z \in U = \{z \in \mathbb{C} : |z| < 1\}$ (the open unit disk), if there occurs a Schwartz function $\sigma(z), \sigma(0) = 0, |\sigma(z)| < 1$ such that $\phi(z) = \psi(\sigma(z))$. We shall show that the integrals (1) and (2) preserve analytic functions with a positive real part. Special generalizations are provided involving entire functions. Moreover, we illustrate a necessary and sufficient condition for some convex inequalities containing (1) and (2).

Let $\mathfrak{H} = \mathfrak{H}(U)$ indicate the class of analytic functions in U. For a positive integer n and a complex number ϕ, let

$$\mathfrak{H}[\phi, n] = \{\varphi \in \mathfrak{H} : \varphi(z) = \phi + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots\}.$$

Define special classes of analytic functions

$$\mathfrak{P}_n = \{\varphi \in \mathfrak{H}[1, n] : \Re(\varphi(z)) > 0, \ \texttt{for} \ z \in U\}$$

$$\mathfrak{H}[0, n] = \{\varphi \in \mathfrak{H} : \varphi(z) = \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots\},$$

and

$$\mathfrak{A}_n = \{\varphi \in \mathfrak{H} : \varphi(z) = z + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots\},$$

where $\mathfrak{A}_1 = \mathfrak{A}$ is called the normalized class satisfying the normalized condition $\varphi(0) = \varphi'(0) - 1 = 0$ and taking the form

$$\mathfrak{A} = \{\varphi \in \mathfrak{H} : \varphi(z) = z + \phi_2 z^2 + \ldots\}.$$

Since our study is in the open unit disk, we need to define the following W-H operator (WHO)

$$W_\zeta(\varphi)(z) = \frac{1}{2\pi i}\int_0^z \varphi(\xi)\frac{d\xi}{\xi-\zeta}, \tag{3}$$

where $\varphi \in \mathfrak{H}[1,n]$ taking the expansion

$$\varphi(z) = 1 + \phi_n z^n + \phi_{n+1}z^{n+1} + \ldots, \quad z \in U$$

Denote $W_0(\varphi)(z) = W(\varphi)(z)$.

Definition 1 The integral operator WHO is called averaging operator, if $\varphi \in \mathfrak{K}$ (the class of convex function) satisfies

$$W(\varphi)(0 = \varphi(0)), \quad W(\varphi)(U) \subset co\varphi(U).$$

Remark 1 For the function $\varphi \in \mathfrak{A}$ which is starlike (S^*) on U, the operator WHO is also starlike. This result comes from equation (2.5–28) [10] when $\alpha = 1$.

2 Results

Our first result is that $W(\varphi)$ is closed in the space $\mathfrak{H}[1,n]$.

Proposition 1 *For analytic function* $\varphi \in \mathfrak{H}[1,n]$, *the operator* $W(\varphi) \in \mathfrak{H}[1,n]$.

Proof Let $\varphi(z) = 1 + \phi_n z^n + \phi_{n+1}z^{n+1} + \ldots$

$$\begin{aligned} W(\varphi)(z) &= \frac{1}{2\pi i}\int_0^z \varphi(\xi)\frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i}\int_0^z [1 + \phi_n\xi^n + \phi_{n+1}\xi^{n+1} + \ldots]\frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i}\int_0^z [\frac{1}{\xi} + \phi_n\xi^{n-1} + \phi_{n+1}\xi^n + \ldots]d\xi. \end{aligned}$$

Since dz/z is accurate in a cut plane, which means a plane eliminates some line moving from the origin to ∂U, we have

$$\int_0^z \frac{1}{\xi}d\xi = \int_{\partial U}\frac{1}{z}dz = 2\pi i.$$

Moreover, we have

$$\int_0^z \xi^{m-1}d\xi = \frac{\xi^m}{m}\Big|_0^z = \frac{z^m}{m}.$$

Hence, we attain

$$W(\varphi)(z) = \frac{1}{2\pi i}[2\pi i + \sum_{m\geq n} \frac{\phi_m z^m}{m}]$$
$$= 1 + \sum_{m\geq n} \frac{\phi_m}{2m\pi i} z^m,$$

which proves that $W(\varphi)$ is analytic in U. In other words $W(\varphi) \in \mathfrak{H}[1, n]$ taking the expansion

$$W(\varphi)(z) = 1 + \omega_n z^n + \omega_{n+1} z^{n+1} + \dots, \quad z \in U.$$

Proposition 2 *Let $\lambda \neq 0$ be a complex number with $\Re(\lambda) > 0$ and let n be a positive integer. If $\varphi \in \mathfrak{P}_n$ such that*

$$\left|\Im\Big(\frac{\lambda\, W(\varphi) + zW(\varphi)'}{\lambda\, W(\varphi)}\Big)\right| \leq n\Re(\frac{1}{\lambda}).$$

Then $W(\varphi) \in \mathfrak{P}_n$.

Proof Set the following functions

$$B(z) = \frac{1}{\lambda}, \quad C(z) = \frac{\lambda\, W(\varphi) + zW(\varphi)'}{\lambda\, W(\varphi)}.$$

Now,

$$\begin{aligned}
\Re\Big(B(z)\, zW(\varphi)' + C(z)W(\varphi)\Big) &= \Re\Big(\frac{1}{\lambda}\, zW(\varphi)' + \frac{\lambda\, W(\varphi) + zW(\varphi)'}{\lambda\, W(\varphi)} W(\varphi)\Big) \\
&= \Re\Big(\frac{\lambda\, W(\varphi) + 2zW(\varphi)'}{\lambda}\Big) \\
&= \Re\Big(W(\varphi)\Big) + 2\Re\Big(\frac{zW(\varphi)'}{\lambda}\Big) \\
&= \Re\Big(1 + \omega_n z^n + \omega_{n+1} z^{n+1} + \dots\Big) \\
&+ 2\Re\Big(\frac{n\omega_n}{\lambda} z^n + \frac{(n+1)\omega_{n+1}}{\lambda} z^{n+1} + \dots\Big) \\
&= 1 + \Re\Big((1 + \frac{2n}{\lambda})\omega_n z^n + (1 + \frac{2(n+1)}{\lambda})\omega_{n+1} z^{n+1} + \dots\Big).
\end{aligned}$$

By setting

$$\lambda = \frac{2m}{2\pi\, im - 1}, \quad m \geq n,$$

we have

$$\Re\Big(B(z)\, zW(\varphi)' + C(z)W(\varphi)\Big) = \Re(\varphi(z)) > 0.$$

Hence, in view of Corollary 4.1a.1 in [10], we obtain $W(\varphi) \in \mathfrak{P}_n$.

Proposition 3 *Let $\lambda \neq 0$ be a complex number with $\Re(\lambda) > -n$, where n is a positive integer. Let $\varphi \in \mathfrak{A}_n$ and*

$$\Re\Big(\lambda + n - \frac{zW(\lambda\varphi)'(z)}{W(\lambda\varphi)(z)}\Big) > 0.$$

If $|\varphi(z)| < M,\ M > 0$, then $W(\lambda\varphi) \in \mathfrak{A}_n$ and $|W(\lambda\varphi)(z)| < N,\ N > 0$.

Proof First, we show that $W(\varphi) \in \mathfrak{A}_n$. Let $\varphi(z) = z + \phi_n z^n + \phi_{n+1} z^{n+1} + \ldots$

$$\begin{aligned}
W(\lambda\varphi)(z) &= \frac{1}{2\pi i}\int_0^z \lambda\varphi(\xi)\frac{d\xi}{\xi} \\
&= \frac{\lambda}{2\pi i}\int_0^z [\xi + \phi_n\xi^n + \phi_{n+1}\xi^{n+1} + \ldots]\frac{d\xi}{\xi} \\
&= \frac{\lambda}{2\pi i}\int_0^z [1 + \phi_n\xi^{n-1} + \phi_{n+1}\xi^{n} + \ldots]d\xi.
\end{aligned}$$

By letting $\lambda = 2\pi i$, we have

$$W(\lambda\varphi)(z) = z + \omega_n z^n + \ldots \in \mathfrak{A}_n.$$

Assume the following functions:

$$B(z) = 1, \quad C(z) = \frac{\lambda W(\lambda\varphi)(z) - zW(\lambda\varphi)'(z)}{W(\lambda\varphi)(z)}, \quad D(z) = \varphi(z) - \lambda W(\lambda\varphi)(z)$$

$$\begin{aligned}
&|B(z)zW(\lambda\varphi)'(z) + C(z)W(\lambda\varphi)(z) + D(z)| \\
&= \Big|zW(\lambda\varphi)'(z) + \frac{\lambda W(\lambda\varphi)(z) - zW(\lambda\varphi)'(z)}{W(\lambda\varphi)(z)}W(\lambda\varphi)(z) + \varphi(z) - \lambda W(\lambda\varphi)(z)\Big| \\
&= |\varphi(z)| < M.
\end{aligned}$$

Hence, in view of Corollary 4.1b.1 in [10], we have

$$|W(\lambda\varphi)| < sup_{z\in U}\Big\{\frac{M + |D(z)|}{|nB(z) + C(z)|}\Big\} := N.$$

This completes the proof.

Proposition 4 *Let n be a positive integer and let* $\varphi \in \mathfrak{H}[0, n]$ *achieving*

$$\Re\Big(n - \frac{zW(\varphi)'(z)}{W(\varphi)(z)}\Big) \geq 0.$$

If $|\varphi(z)| < M, \ M > 0$ *and*

$$\Big|n - \frac{zW(\varphi)'(z)}{W(\varphi)(z)}\Big| \geq \frac{2M}{N}$$

then $W(\varphi) \in \mathfrak{H}[0, n]$ *and* $|W(\varphi)(z)| < N, \ N > 0.$

Proof First, we show that $W(\varphi) \in \mathfrak{H}[0, n]$. Let $\varphi(z) = \phi_n z^n + \phi_{n+1} z^{n+1} + \dots$

$$\begin{aligned} W(\varphi)(z) &= \frac{1}{2\pi i}\int_0^z \varphi(\xi)\frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i}\int_0^z [\phi_n \xi^n + \phi_{n+1}\xi^{n+1} + \dots]\frac{d\xi}{\xi} \\ &= \frac{1}{2\pi i}\int_0^z [\phi_n \xi^{n-1} + \phi_{n+1}\xi^{n} + \dots]d\xi. \end{aligned}$$

Thus, we obtain

$$W(\varphi)(z) = \omega_n z^n + \dots \in \mathfrak{H}[0, n].$$

Assume the following functions:

$$B(z) = 1, \quad C(z) = -\frac{zW(\varphi)'(z)}{W(\varphi)(z)}, \quad D(z) = \varphi(z)$$

$$\begin{aligned} &|B(z)zW(\lambda\varphi)'(z) + C(z)W(\varphi)(z) + D(z)| \\ &= \Big|zW(\lambda\varphi)'(z) - \frac{zW(\varphi)'(z)}{W(\varphi)(z)}W(\varphi)(z) + \varphi(z)\Big| \\ &= |\varphi(z)| < M. \end{aligned}$$

Hence, in view of Theorem 4.1b in [10], we have $|W(\varphi)(z)| < N$. This completes the proof.

Proposition 5 *Let* $M > 0, \ N > 0$ *and let* $\varphi \in \mathfrak{H}[0, 1]$ *achieving*

$$\Big|\Im(\frac{zW(\varphi)'(z)}{W(\varphi)(z)})\Big| \geq \frac{M}{N}.$$

Then $W(\varphi) \in \mathfrak{H}[0, 1]$ *and* $|W(\varphi)(z)| < N.$

Proof It is clear that $W(\varphi) \in \mathfrak{H}[0, 1]$. Consider the following functions:

$$B(z) = 1, \quad C(z) = -\frac{zW(\varphi)'(z)}{W(\varphi)(z)}$$

$$\begin{aligned} &|B(z)zW(\lambda\varphi)'(z) + C(z)W(\varphi)(z)| \\ &= \left|zW(\lambda\varphi)'(z) - \frac{zW(\varphi)'(z)}{W(\varphi)(z)}W(\varphi)(z)\right| \\ &= 0 < M. \end{aligned}$$

Hence, in view of Theorem 4.1c in [10], we have

$$|W(\varphi)(z)| < N := \sup_{z\in U}\{\frac{M}{|B(z).||\Im C(z)/B(z)|}\}.$$

This completes the proof.

Next, we discuss the upper bound of the WHO with respect to convex analytic function, by using the second-order differential subordination.

Theorem 1 *Let h be convex in U and let $\varphi \in \mathfrak{H}[h(0), 1]$ satisfying the subordination*

$$z^2W(\varphi)''(z) + zW(\varphi)'(z) + W(\varphi)(z) \prec h(z)$$

then $W(\varphi)(z) \prec h(z)$.

Proof Since h is convex then it has the normalized property $h(0) = 0$ then we have $W(\varphi)(z) \in \mathfrak{H}[0, 1]$ (Proposition 5). Consider the following functions:

$$A = 1, \quad B(z) = 1, \quad D(z) = 0.$$

Since $\Re(B(z)) = A = 1$ then in view of Theorem 4.1f [10], we have $W(\varphi)(z) \prec h(z)$.

Theorem 2 *Let $\varphi \in \mathfrak{H}[0, 1]$ satisfying the subordination*

$$z^2W(\varphi)''(z) + zW(\varphi)'(z) + W(\varphi)(z) \prec z$$

then $W(\varphi)(z) \prec \frac{z}{2}$ and $z/2$ is the best (0,1)-dominant.

Proof It is clear that $W(\varphi)(z) \in \mathfrak{H}[0, 1]$ (Proposition 5). Consider the following real numbers:

$$A = 1, \quad B = 1, \quad C = 1.$$

Then in view of Theorem 4.1g [10], we have $W(\varphi)(z) \prec \frac{z}{2}$ and $z/2$ is the best (0,1)-dominant.

Theorem 3 *Let n be a positive integer and $\varphi \in \mathfrak{H}[1, n]$ satisfying the linear first differential subordination*

$$zW(\varphi)'(z) + W(\varphi)(z) \prec [\frac{1+z}{1-z}]^{\alpha}$$

then

$$W(\varphi)(z) \prec [\frac{1+z}{1-z}]^{\beta}$$

where $\alpha := \beta + o(n) > 0$.

Proof It is clear that $W(\varphi)(z) \in \mathfrak{H}[1, n]$ (Proposition 1). According to Theorem 3.1c [10], we have

$$W(\varphi)(z) \prec [\frac{1+z}{1-z}]^{\beta}.$$

Theorem 4 *Let λ be a real number with $|\lambda| \leq 1$. If $\varphi \in \mathfrak{H}[1, n]$ satisfying $\Re(\varphi(z)) > 0$, then the generalized WHO achieves*

$$\Re(W_{\lambda}(\varphi)(z)) = \Re(\frac{1}{2\pi i e^{\lambda z^n}} \int_0^z \varphi(\xi) e^{\lambda \xi^n} \frac{d\xi}{\xi}) > 0$$

such that

$$|\Im(\frac{1}{2\pi i e^{\lambda z^n}})'| \leq n\Re(\frac{1}{2\pi i z\, e^{\lambda z^n}}).$$

Proof According to the relation 4.2–6 [10], we have the desire inequality.

Note that $W_0(\varphi)(z)) = W(\varphi)(z))$.

Theorem 5 *Let λ be a real number with $|\lambda| \leq 1$ and $\gamma > 0$. If $\varphi \in \mathfrak{H}[1, n]$ satisfying $\Re(\varphi(z)) > 0$ then the generalized WHO achieves*

$$\Re(W_{\lambda,\gamma}(\varphi)(z)) = \Re(\frac{1}{2\pi i z^{\gamma-1}\, e^{\lambda z^n}} \int_0^z \varphi(\xi)\xi^{\gamma-1}\, e^{\lambda \xi^n} \frac{d\xi}{\xi}) > 0.$$

Proof A direct application of the relation 4.2–4 [10], we have the desire inequality.

Note that $W_{0,1}(\varphi)(z)) = W(\varphi)(z))$. Theorems 4 and 5 show that the generalized WHO satisfies the relation

$$\varphi(z) \in \mathfrak{P}_n \Rightarrow W_{\lambda,\gamma}(\varphi)(z)) \in \mathfrak{P}_n.$$

Theorem 6 *Let φ be an analytic function in U with $\varphi(0) = 1$ ($\varphi \in \mathfrak{H}[1, n]$). If either of the following three conditions is achieved:*

- $$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec e^z, \quad \lambda > 1$$

- $$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec \frac{1 + Az}{1 + Bz}$$
$$\Big(-1 < B < A \leq 1, \ \ |\lambda| \geq \frac{A - B}{1 - |B|}\Big)$$

- $$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec \sqrt{1 + z}, \quad \lambda \geq 1$$

then

$$W(\varphi)(z) \prec e^z.$$

Proof According to Proposition 1, we have $W(\varphi)(z) \in \mathfrak{H}[1, n]$. Let $h(z)$ be the convex univalent function defined by $h(z) = e^z$. Then, obviously $\lambda\, z\, (h(z))'$ is starlike. The main aim of the proof reads on the information that if the subordination

$$1 + \lambda \frac{zW(\varphi)(z)'}{W(\varphi)(z)} \prec 1 + \lambda \frac{z\,(h(z))'}{h(z)} = 1 + \lambda z := \Theta(z)$$

is achieved, then $W(\varphi)(z) \prec h(z)$ (see Corollary 3.4h.1, p. 135 [10]). By Remark 2.1 in [11] and the first condition, we obtain

$$h(z) \prec \Theta(z) \Longrightarrow W(\varphi)(z) \prec h(z).$$

Now, let $\psi(z) := \dfrac{1 + Az}{1 + Bz}$ then $\psi^{-1}(\eta) = \dfrac{\eta - 1}{A - B\eta}$. But $\psi(z) \prec h(z)$ means $z \prec \psi^{-1}(\Theta(z))$ and

$$|\psi^{-1}(\Theta(e^{it})| = |\frac{\lambda\, e^{it}}{(A - B) - \lambda\, B\, e^{it}}| \geq \frac{\lambda}{A - B + \lambda |B|} \geq 1$$

for $\lambda \geq (A - B)(1 - |B|)$. Hence,

$$h(z) \prec \Theta(z) \Longrightarrow W(\varphi)(z) \prec h(z).$$

Finally, let $\Lambda(z) = \sqrt{1+z}$, where $\Lambda(U) \subset \Theta(U)$ then if $\lambda \geq 1$, we attain

$$h(z) \prec \Theta(z) \Longrightarrow W(\varphi)(z) \prec h(z).$$

A direct application of Lemma 4.4b in [10], we get the following outcome:

Theorem 7 *Let $\varphi \in \mathfrak{K}$ such that $\varphi(0) = 0$ and $h \in \mathfrak{K}$ such that $\varphi(z) \prec h(z)$. Then the WHO is averaging operator on $\mathfrak{K}$ satisfying $W(\varphi)(z) \prec h(z)$.*

Next, we discuss the case φ is not convex.

Theorem 8 *Let $\varphi \in \mathfrak{H}(U)$ and $h \in \mathfrak{K}$ such that $\varphi(z) \prec h(z)$ and*

$$\Re\Big(- \frac{W(\varphi)(z) - \varphi(z)}{zW(\varphi)(z)'}\Big) > 0.$$

Then the WHO is averaging operator on $\mathfrak{K}$ satisfying $W(\varphi)(z) \prec h(z)$.

Proof Since $\varphi \in \mathfrak{H}(U)$ then we obtain $W(\varphi) \in \mathfrak{H}(U)$. A computation leads to

$$W(\varphi)(z) - \frac{W(\varphi)(z) - \varphi(z)}{zW(\varphi)(z)'}.zW(\varphi)(z)' = \varphi(z) \prec h(z).$$

In view of Theorem 3.1a in [10], we get

$$\varphi(z) \prec h(z) \Longrightarrow W(\varphi)(z) \prec h(z),$$

which implies that WHO is averaging operator on $\mathfrak{K}$.

Theorem 9 *Let $\varphi \in \mathfrak{H}(U)$ and h is starlike on U. If $\varphi(z) \prec h(z)$, then*

$$W(\varphi)(z) \prec W(h)(z).$$

Proof By Remark 1, $W(h)(z)$ is starlike on U. Suppose that $W(\varphi)(z) \not\prec W(h)(z)$, then there occur some points $z_0 \in U$ and $\eta_0 \in \partial U$ such that $W(\varphi)(z_0) = W(h)(\eta_0)$ and $W(\varphi)(U_0) \subset W(h)(U)$. Thus, by Lemma 2.2c [10], we obtain

$$z_0\, W(\varphi)(z_0)' = k\eta_0 W(h)'(\eta_0), \quad k \geq 1.$$

This implies that

$$\varphi(z_0) = k\, h(\eta_0) \notin h(U),$$

which contradicts the assumption $\varphi(z) \prec h(z)$. Hence, $W(\varphi)(z) \prec W(h)(z)$.

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Some New Hermite–Hadamard Type Integral Inequalities via Caputo k-Fractional Derivatives and Their Applications

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Abstract The authors discover a general integral identity concerning $(n + 1)$-differentiable mappings defined on m-invex set via Caputo k-fractional derivatives. By using the notion of generalized $((h_1, h_2); (\eta_1, \eta_2))$-convex mappings and this integral equation as an auxiliary result, we derive some new estimates with respect to Hermite–Hadamard type inequalities via Caputo k-fractional derivatives. It is pointed out that some new special cases can be deduced from main results. At the end, some applications to special means for different positive real numbers are provided as well.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which

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generalize, improve, and extend the inequality (1) through various classes of convex functions, interested readers are referred to [1–30, 32–40, 42–46, 50, 52, 53].

Let us recall some special functions and evoke some basic definitions as follows.

Definition 1 The Euler beta function is defined for $a, b > 0$ as

$$\beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt. \tag{2}$$

Definition 2 The hypergeometric function ${}_2F_1(a, b; c; z)$ is defined by

$${}_2F_1(a, b; c; z) = \frac{1}{\beta(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt$$

for $c > b > 0$ and $|z| < 1$, where $\beta(x, y)$ is the Euler beta function for all $x, y > 0$.

Definition 3 For $k \in \mathbb{R}^+$ and $x \in \mathbb{C}$, the k-gamma function is defined by

$$\Gamma_k(x) = \lim_{n \longrightarrow +\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{(x)_{n,k}}. \tag{3}$$

Its integral representation is given by

$$\Gamma_k(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt. \tag{4}$$

One can note that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha).$$

For $k = 1$, (4) gives integral representation of gamma function.

Definition 4 For $k \in \mathbb{R}^+$ and $x, y \in \mathbb{C}$, the k-beta function with two parameters x and y is defined as

$$\beta_k(x, y) = \frac{1}{k} \int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1} dt. \tag{5}$$

For $k = 1$, (5) gives integral representation of beta function.

Theorem 2 *Let $x, y > 0$, then for k-gamma and k-beta function the following equality holds:*

$$\beta_k(x, y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}. \tag{6}$$

Definition 5 ([31]) Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$ such that $f^{(n)}$ exists and is continuous on $[a, b]$. The Caputo fractional derivatives of order α are defined as follows:

$$ {}^cD^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}}dt, \ \ x > a \tag{7}$$

and

$$ {}^cD^{\alpha}_{b-}f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}}dt, \ \ x < b. \tag{8}$$

If $\alpha = n \in \{1, 2, 3, \ldots\}$ and usual derivative of order n exists, then Caputo fractional derivative $\left({}^cD^{\alpha}_{a+}f\right)(x)$ coincides with $f^{(n)}(x)$. In particular we have

$$\left({}^cD^{0}_{a+}f\right)(x) = \left({}^cD^{0}_{b-}f\right)(x) = f(x) \tag{9}$$

where $n = 1$ and $\alpha = 0$.

Definition 6 ([12]) Let $\alpha > 0$, $k \geq 1$ and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$, $f \in C^n[a, b]$. The Caputo k-fractional derivatives of order α are defined as follows:

$$ {}^cD^{\alpha,k}_{a+}f(x) = \frac{1}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\frac{\alpha}{k}-n+1}}dt, \ \ x > a \tag{10}$$

and

$$ {}^cD^{\alpha,k}_{b-}f(x) = \frac{(-1)^n}{k\Gamma_k\left(n-\frac{\alpha}{k}\right)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\frac{\alpha}{k}-n+1}}dt, \ \ x < b. \tag{11}$$

Definition 7 ([51]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.

The invex set S is also termed an η-connected set.

Definition 8 ([35]) Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y, x)\big) \leq h(1-t)f(x) + h(t)f(y) \tag{12}$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 8, f becomes a preinvex function [41]. If the mapping $\eta(y, x) = y - x$ in Definition 8, then the non-negative function f reduces to h-convex mappings [48].

Definition 9 ([49]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s \in (0, 1]$, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)^s f(x) + t^s f(y). \tag{13}$$

Definition 10 ([38]) A function $f : K \longrightarrow \mathbb{R}$ is said to be s-Godunova–Levin–Dragomir-preinvex of second kind, if

$$f\big(x + t\eta(y, x)\big) \leq (1 - t)^{-s} f(x) + t^{-s} f(y), \tag{14}$$

for each $x, y \in K$, $t \in (0, 1)$ and $s \in (0, 1]$.

Definition 11 ([47]) A non-negative function $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be tgs-convex on K if the inequality

$$f\big((1 - t)x + ty\big) \leq t(1 - t)[f(x) + f(y)] \tag{15}$$

grips for all $x, y \in K$ and $t \in (0, 1)$.

Definition 12 ([32]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be MT-convex functions, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality

$$f(tx + (1 - t)y) \leq \frac{\sqrt{t}}{2\sqrt{1 - t}} f(x) + \frac{\sqrt{1 - t}}{2\sqrt{t}} f(y). \tag{16}$$

Definition 13 ([40]) Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y, mx)\big) \leq mh_1(t) f(x) + h_2(t) f(y) \tag{17}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

The concept of η-convex functions (at the beginning was named by φ-convex functions), considered in [15], has been introduced as the follows.

Definition 14 Consider a convex set $I \subseteq \mathbb{R}$ and a bifunction $\eta : f(I) \times f(I) \longrightarrow \mathbb{R}$. A function $f : I \longrightarrow \mathbb{R}$ is called convex with respect to η (briefly η-convex), if

$$f\big(\lambda x + (1 - \lambda)y\big) \leq f(y) + \lambda\eta(f(x), f(y)), \tag{18}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Geometrically it says that if a function is η-convex on I, then for any $x, y \in I$, its graph is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) +$

$\eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x, y) = x - y$ and the function reduces to a convex one. For more results about η-convex functions, see [7, 8, 14, 15].

Definition 15 ([1]) Let $I \subseteq \mathbb{R}$ be an invex set with respect to $\eta_1 : I \times I \longrightarrow \mathbb{R}$. Consider $f : I \longrightarrow \mathbb{R}$ and $\eta_2 : f(I) \times f(I) \longrightarrow \mathbb{R}$. The function f is said to be (η_1, η_2)-convex if

$$f\big(x + \lambda\eta_1(y, x)\big) \leq f(x) + \lambda\eta_2(f(y), f(x)), \tag{19}$$

is valid for all $x, y \in I$ and $\lambda \in [0, 1]$.

Motivated by the above works and the references therein, the main objective of this article is to apply the notion of generalized $((h_1, h_2); (\eta_1, \eta_2))$-convex mappings and an interesting lemma to establish some new estimates with respect to Hermite–Hadamard type inequalities via Caputo k-fractional derivatives. Also, some new special cases will be deduced. At the end, some applications to special means for different positive real numbers are given as well.

2 Main Results

Definition 16 ([10]) A set $K \subseteq \mathbb{R}^n$ is named as m-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0, 1]$, if $mx + t\eta(y, mx) \in K$ grips for each $x, y \in K$ and any $t \in [0, 1]$.

Remark 1 In Definition 16, under certain conditions, the mapping $\eta(y, mx)$ could reduce to $\eta(y, x)$. When $m = 1$, we get Definition 3.

For the simplicities of notations, let

$$\delta(\alpha, \xi) := \int_0^1 |t^\alpha - \xi| dt, \quad \varrho(\alpha, \xi, p) := \int_0^1 |t^\alpha - \xi|^p dt. \tag{20}$$

Lemma 1 *For* $0 \leq \xi \leq 1$, *we have*

(a)

$$\delta(\alpha, \xi) = \begin{cases} \dfrac{1}{\alpha + 1}, & \xi = 0; \\ \dfrac{2\alpha\xi^{1+\frac{1}{\alpha}} + 1}{\alpha + 1} - \xi, & 0 < \xi < 1; \\ \dfrac{\alpha}{\alpha + 1}, & \xi = 1. \end{cases}$$

(b)

$$\varrho(\alpha,\xi,p)=\begin{cases}\dfrac{1}{p\alpha+1}, & \xi=0;\\ \dfrac{\xi^{p+\frac{1}{\alpha}}}{\alpha}\beta\left(\dfrac{1}{\alpha},p+1\right)+\dfrac{(1-\xi)^{p+1}}{\alpha(p+1)} & \\ \times{}_2F_1\left(1-\dfrac{1}{\alpha},1;p+2;1-\xi\right), & 0<\xi<1;\\ \dfrac{1}{\alpha}\beta\left(p+1,\dfrac{1}{\alpha}\right), & \xi=1.\end{cases}$$

Proof These equalities follow from a straightforward computation of definite integrals. This completes the proof of the lemma.

We next introduce the concept of generalized $((h_1,h_2);(\eta_1,\eta_2))$-convex mappings.

Definition 17 Let $K\subseteq\mathbb{R}$ be an open m-invex set with respect to the mapping $\eta_1:K\times K\longrightarrow\mathbb{R}$. Suppose $h_1,h_2:[0,1]\longrightarrow[0,+\infty)$ and $\varphi:I\longrightarrow K$ are continuous. Consider $f:K\longrightarrow(0,+\infty)$ and $\eta_2:f(K)\times f(K)\longrightarrow\mathbb{R}$. The mapping f is said to be generalized $((h_1,h_2);(\eta_1,\eta_2))$-convex if

$$f\big(m\varphi(x)+t\eta_1(\varphi(y),m\varphi(x))\big)\leq\big[mh_1(t)f^r(x)+h_2(t)\eta_2(f^r(y),f^r(x))\big]^{\frac{1}{r}}, \tag{21}$$

holds for all $x,y\in I$, $r\neq0$, $t\in[0,1]$ and some fixed $m\in(0,1]$.

Remark 2 In Definition 17, if we choose $m=r=1$, $h_1(t)=1$, $h_2(t)=t$, $\eta_1(\varphi(y),m\varphi(x))=\varphi(y)-m\varphi(x)$, $\eta_2(f^r(y),f^r(x))=\eta(f^r(y),f^r(x))$ and $\varphi(x)=x$, $\forall x\in I$, then we get Definition 14. Also, in Definition 17, if we choose $m=r=1$, $h_1(t)=1$, $h_2(t)=t$ and $\varphi(x)=x$, $\forall x\in I$, then we get Definition 13. Under some suitable choices as we done above, we can also get Definitions 9 and 10.

Remark 3 Let us discuss some special cases in Definition 17 as follows.

1. Taking $h_1(t)=h(1-t)$, $h_2(t)=h(t)$, then we get generalized $((m,h);(\eta_1,\eta_2))$-convex mappings.
2. Taking $h_1(t)=(1-t)^s$, $h_2(t)=t^s$ for $s\in(0,1]$, then we get generalized $((m,s);(\eta_1,\eta_2))$-Breckner-convex mappings.
3. Taking $h_1(t)=(1-t)^{-s}$, $h_2(t)=t^{-s}$ for $s\in(0,1]$, then we get generalized $((m,s);(\eta_1,\eta_2))$-Godunova–Levin–Dragomir-convex mappings.
4. Taking $h_1(t)=h_2(t)=t(1-t)$, then we get generalized $((m,tgs);(\eta_1,\eta_2))$-convex mappings.
5. Taking $h_1(t)=\dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t)=\dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized $(m;(\eta_1,\eta_2))$-MT-convex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

Let us see the following example of a generalized $((h_1, h_2); (\eta_1, \eta_2))$-convex mapping which is not convex.

Example 1 Let us take $m = r = 1$, $h_1(t) = 1$, $h_2(t) = t$ and φ an identity function. Consider the function $f : [0, +\infty) \longrightarrow [0, +\infty)$ by

$$f(x) = \begin{cases} x, & 0 \leq x \leq 2; \\ 2, & x > 2. \end{cases}$$

Define two bifunctions $\eta_1 : [0, +\infty) \times [0, +\infty) \longrightarrow \mathbb{R}$ and $\eta_2 : [0, +\infty) \times [0, +\infty) \longrightarrow [0, +\infty)$ by

$$\eta_1(x, y) = \begin{cases} -y, & 0 \leq y \leq 2; \\ x + y, & y > 2, \end{cases}$$

and

$$\eta_2(x, y) = \begin{cases} x + y, & x \leq y; \\ 4(x + y), & x > y. \end{cases}$$

Then f is generalized $((1, t); (\eta_1, \eta_2))$-convex mapping. But f is not preinvex with respect to η_1 and also it is not convex (consider $x = 0$, $y = 3$, and $t \in (0, 1]$).

For establishing our main results regarding some new Hermite–Hadamard type integral inequalities associated with generalized $((h_1, h_2); (\eta_1, \eta_2))$-convexity via Caputo k-fractional derivatives, we need the following lemma.

Lemma 2 *Let $\alpha > 0$, $k \geq 1$, and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$. Suppose $K = [m\varphi(a), m\varphi(a) + \eta(\varphi(b), m\varphi(a))] \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, where $\eta(\varphi(b), m\varphi(a)) > 0$. Also, let $\varphi : I \longrightarrow K$ be a continuous function. Assume that $f : K \longrightarrow \mathbb{R}$ is a $(n + 1)$-differentiable mapping on K° such that $f \in C^{n+1}(K)$. Then for any $\lambda, \mu \in [0, 1]$ and $r \geq 0$, we have the following identity for Caputo k-fractional derivatives:*

$$\left(\frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)^{n-\frac{\alpha}{k}} \times \Bigg\{\lambda\left[f^{(n)}(m\varphi(a)) - f^{(n)}\left(m\varphi(a) + \frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)\right]$$

$$+\mu\left[f^{(n)}\left(m\varphi(a) + \frac{r}{r+1}\eta(\varphi(b), m\varphi(a))\right) - f^{(n)}(m\varphi(a) + \eta(\varphi(b), m\varphi(a)))\right]$$

$$+f^{(n)}\left(m\varphi(a) + \frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right) + f^{(n)}(m\varphi(a) + \eta(\varphi(b), m\varphi(a)))\Bigg\}$$

$$-(nk - \alpha)\Gamma_k\left(n - \frac{\alpha}{k}\right) \times \left[{}^{c}D^{\alpha,k}_{(m\varphi(a))^+} f\left(m\varphi(a) + \frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)\right.$$

$$+(-1)^n \times {}^c D^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(b),m\varphi(a)))^-} f\left(m\varphi(a) + \frac{r}{r+1}\eta(\varphi(b), m\varphi(a))\right)\Bigg]$$

$$= \left(\frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)^{n-\frac{\alpha}{k}+1}$$

$$\times\left\{\int_0^1 \left(1-\lambda-t^{n-\frac{\alpha}{k}}\right) f^{(n+1)}\left(m\varphi(a) + \left(\frac{1-t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)dt\right.$$

$$\left.+\int_0^1 \left(t^{n-\frac{\alpha}{k}}-\mu\right) f^{(n+1)}\left(m\varphi(a) + \left(\frac{r+t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)dt\right\}. \quad (22)$$

We denote

$$T^{\alpha,k}_f(\eta, \varphi; \lambda, \mu, n, r, m, a, b) := \left(\frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)^{n-\frac{\alpha}{k}+1} \quad (23)$$

$$\times\left\{\int_0^1 \left(1-\lambda-t^{n-\frac{\alpha}{k}}\right) f^{(n+1)}\left(m\varphi(a) + \left(\frac{1-t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)dt\right.$$

$$\left.+\int_0^1 \left(t^{n-\frac{\alpha}{k}}-\mu\right) f^{(n+1)}\left(m\varphi(a) + \left(\frac{r+t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)dt\right\}.$$

Proof Integrating by parts (23), we get

$$T^{\alpha,k}_f(\eta, \varphi; \lambda, \mu, n, r, m, a, b) = \left(\frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)^{n-\frac{\alpha}{k}+1}$$

$$\times\left\{\left[\frac{-(r+1)\left(1-\lambda-t^{n-\frac{\alpha}{k}}\right) f^{(n)}\left(m\varphi(a) + \left(\frac{1-t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)}{\eta(\varphi(b), m\varphi(a))}\Bigg|_0^1\right.\right.$$

$$\left.-\frac{(r+1)\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b), m\varphi(a))}\int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}\left(m\varphi(a) + \left(\frac{1-t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)dt\right]$$

$$+\left[\frac{(r+1)\left(t^{n-\frac{\alpha}{k}}-\mu\right) f^{(n)}\left(m\varphi(a) + \left(\frac{r+t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)}{\eta(\varphi(b), m\varphi(a))}\Bigg|_0^1\right.$$

$$\left.\left.-\frac{(r+1)\left(n-\frac{\alpha}{k}\right)}{\eta(\varphi(b), m\varphi(a))}\int_0^1 t^{n-\frac{\alpha}{k}-1} f^{(n)}\left(m\varphi(a) + \left(\frac{r+t}{r+1}\right)\eta(\varphi(b), m\varphi(a))\right)dt\right]\right\}$$

$$= \left(\frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)^{n-\frac{\alpha}{k}} \times \Bigg\{\lambda\Bigg[f^{(n)}(m\varphi(a)) - f^{(n)}\left(m\varphi(a) + \frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)\Bigg]$$

$$+\mu\Bigg[f^{(n)}\left(m\varphi(a) + \frac{r}{r+1}\eta(\varphi(b), m\varphi(a))\right) - f^{(n)}(m\varphi(a) + \eta(\varphi(b), m\varphi(a)))\Bigg]$$

$$+f^{(n)}\left(m\varphi(a) + \frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right) + f^{(n)}(m\varphi(a) + \eta(\varphi(b), m\varphi(a)))\Bigg\}$$

$$-(nk-\alpha)\Gamma_k\left(n - \frac{\alpha}{k}\right) \times \Bigg[{}^cD^{\alpha,k}_{(m\varphi(a))^+} f\left(m\varphi(a) + \frac{\eta(\varphi(b), m\varphi(a))}{r+1}\right)$$

$$+(-1)^n \times {}^c D^{\alpha,k}_{(m\varphi(a)+\eta(\varphi(b),m\varphi(a)))^-} f\left(m\varphi(a) + \frac{r}{r+1}\eta(\varphi(b), m\varphi(a))\right)\Bigg].$$

This completes the proof of our lemma.

Using Lemmas 1 and 2, we now state the following theorems for the corresponding version for power of $(n+1)$-derivative.

Theorem 3 *Let $\alpha > 0$, $k \geq 1$, $0 < r \leq 1$, and $\alpha \notin \{1, 2, 3, \ldots\}$, $n = [\alpha] + 1$. Suppose $K = [m\varphi(a), m\varphi(a) + \eta_1(\varphi(b), m\varphi(a))] \subseteq \mathbb{R}$ be an open m-invex subset with respect to $\eta_1 : K \times K \longrightarrow \mathbb{R}$ for some fixed $m \in (0, 1]$, where $\eta_1(\varphi(b), m\varphi(a)) > 0$. Also, let $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ and $\varphi : I \longrightarrow K$ are continuous. Assume that $f : K \longrightarrow (0, +\infty)$ is a $(n+1)$-differentiable mapping on K° such that $f \in C^{n+1}(K)$ and $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. If $\left(f^{(n+1)}\right)^q$ is generalized $((h_1, h_2); (\eta_1, \eta_2))$-convex mapping, $q > 1$, $p^{-1} + q^{-1} = 1$, then for any $\lambda, \mu \in [0, 1]$ and $r_1 \geq 0$, the following inequality for Caputo k-fractional derivatives holds:*

$$\left|T^{\alpha,k}_f(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right|$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\varrho^{\frac{1}{p}}\left(n - \tfrac{\alpha}{k}, 1-\lambda, p\right)\right. \tag{24}$$

$$\times\Bigg[m\left(f^{(n+1)}(a)\right)^{rq} I^r(h_1(t); r, r_1)$$

$$+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq}, \left(f^{(n+1)}(a)\right)^{rq}\right) I^r(h_2(t); r, r_1)\Bigg]^{\frac{1}{rq}}$$

$$+\varrho^{\frac{1}{p}}\left(n - \tfrac{\alpha}{k}, \mu, p\right) \times \Bigg[m\left(f^{(n+1)}(a)\right)^{rq} \overline{I}^r(h_1(t); r, r_1)$$

$$\left.+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq}, \left(f^{(n+1)}(a)\right)^{rq}\right) \overline{I}^r(h_2(t); r, r_1)\Bigg]^{\frac{1}{rq}}\right\},$$

where

$$I(h_i(t); r, r_1) := \int_0^1 h_i^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt,$$

$$\overline{I}(h_i(t); r, r_1) := \int_0^1 h_i^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt, \quad \forall\, i = 1, 2$$

and $\varrho\left(n-\frac{\alpha}{k}, 1-\lambda, p\right)$, $\varrho\left(n-\frac{\alpha}{k}, \mu, p\right)$ *are defined as in Lemma 1.*

Proof From Lemma 2, generalized $((h_1, h_2); (\eta_1, \eta_2))$-convexity of $\left(f^{(n+1)}\right)^q$, Hölder inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right| \leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}$$

$$\times\left\{\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right|dt\right.$$

$$\left.+\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right|\left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right|dt\right\}$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\left(\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|^p dt\right)^{\frac{1}{p}}\right.$$

$$\times\left(\int_0^1 \left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right|^p dt\right)^{\frac{1}{p}}$$

$$\left.\times\left(\int_0^1 \left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right)^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, 1-\lambda, p\right)\right.$$

$$\times\left[\int_0^1 \left[mh_1\left(\frac{1-t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}\right.\right.$$

$$+h_2\left(\frac{1-t}{r_1+1}\right)\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\Bigg]^{\frac{1}{r}}dt\Bigg]^{\frac{1}{q}}$$

$$\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\times\left[\int_0^1\left[mh_1\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}\right.\right.$$

$$\left.\left.+h_2\left(\frac{r_1+t}{r_1+1}\right)\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}\Bigg\}$$

$$\leq\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)$$

$$\times\Bigg[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q h_1^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1\eta_2^{\frac{1}{r}}\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)h_2^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\times\Bigg[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q h_1^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r$$

$$+\left(\int_0^1\eta_2^{\frac{1}{r}}\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)h_2^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}$$

$$=\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\Bigg\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)$$

$$\times\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1)$$

$$+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)I^r(h_2(t);r,r_1)\Bigg]^{\frac{1}{rq}}$$

$$+\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\times\Bigg[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1)$$

$$+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\overline{I}^r(h_2(t);r,r_1)\Bigg]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 3.

Corollary 1 *In Theorem 3 if we choose $\lambda = \mu = m = r = 1$ and $\eta_1(\varphi(b), m\varphi(a)) = \varphi(b) - m\varphi(a)$, $\varphi(x) = x$ for all $x \in I$, we get the following inequality for Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(1,1,n,r_1,1,a,b)\right| \leq \left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{ \left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}} \right. \tag{25}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q I(h_1(t);1,r_1) + \eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) I(h_2(t);1,r_1)\right]^{\frac{1}{q}}$$

$$+\left[\frac{1}{\left(n-\frac{\alpha}{k}\right)}\beta\left(p+1, \frac{1}{n-\frac{\alpha}{k}}\right)\right]^{\frac{1}{p}}$$

$$\left.\times\left[\left(f^{(n+1)}(a)\right)^q \overline{I}(h_1(t);1,r_1) + \eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) \overline{I}(h_2(t);1,r_1)\right]^{\frac{1}{q}} \right\}.$$

Corollary 2 *In Theorem 3 if we choose $\lambda = \mu = 0$, $m = r = 1$ and $\eta_1(\varphi(b), m\varphi(a)) = \varphi(b) - m\varphi(a)$, $\varphi(x) = x$ for all $x \in I$, we get the following inequality for Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(0,0,n,r_1,1,a,b)\right|$$

$$\leq \left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{ \left[\frac{1}{\left(n-\frac{\alpha}{k}\right)}\beta\left(p+1, \frac{1}{n-\frac{\alpha}{k}}\right)\right]^{\frac{1}{p}} \right. \tag{26}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q I(h_1(t);1,r_1) + \eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) I(h_2(t);1,r_1)\right]^{\frac{1}{q}}$$

$$+\left(\frac{1}{p\left(n-\frac{\alpha}{k}\right)+1}\right)^{\frac{1}{p}}$$

$$\left.\times\left[\left(f^{(n+1)}(a)\right)^q \overline{I}(h_1(t);1,r_1) + \eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) \overline{I}(h_2(t);1,r_1)\right]^{\frac{1}{q}} \right\}.$$

Corollary 3 *In Theorem 3 for $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ and $f^{(n+1)}(x) \leq L$, $\forall x \in I$, we get the following inequality for generalized $((m,h);(\eta_1,\eta_2))$-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1,\varphi;\lambda,\mu,n,r_1,m,a,b)\right|$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left[mL^{rq} + \eta_2\left(L^{rq}, L^{rq}\right)\right] \tag{27}$$

$$\times\left\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, 1-\lambda, p\right) I^{\frac{1}{q}}(h(t); r, r_1) + \varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, \mu, p\right) \overline{I}^{\frac{1}{q}}(h(t); r, r_1)\right\}.$$

Corollary 4 *In Corollary 3 for $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$, we get the following inequality for generalized $((m, s); (\eta_1, \eta_2))$-Breckner-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right| \leq \left(\frac{r}{r+s}\right)^{\frac{1}{q}} \left(\frac{1}{r_1+1}\right)^{\frac{s}{rq}} \tag{28}$$

$$\times \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left[mL^{rq} + \eta_2\left(L^{rq}, L^{rq}\right)\right]$$

$$\times\left\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, 1-\lambda, p\right) + \varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, \mu, p\right)\left[(r_1+1)^{\frac{s}{r}+1} - r_1^{\frac{s}{r}+1}\right]^{\frac{1}{q}}\right\}.$$

Corollary 5 *In Corollary 3 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $0 < s < r$, we get the following inequality for generalized $((m, s); (\eta_1, \eta_2))$-Godunova–Levin–Dragomir-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right| \leq \left(\frac{r}{r-s}\right)^{\frac{1}{q}} \left(\frac{1}{r_1+1}\right)^{\frac{s}{rq}} \tag{29}$$

$$\times \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left[mL^{rq} + \eta_2\left(L^{rq}, L^{rq}\right)\right]$$

$$\times\left\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, 1-\lambda, p\right) + \varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, \mu, p\right)\left[(r_1+1)^{\frac{s}{r}+1} - r_1^{\frac{s}{r}+1}\right]^{\frac{1}{q}}\right\}.$$

Corollary 6 *In Theorem 3 for $r = 1$, $h_1(t) = h_2(t) = t(1-t)$ and $f^{(n+1)}(x) \leq L$, $\forall x \in I$, we get the following inequality for generalized $((m, tgs); (\eta_1, \eta_2))$-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right| \leq \left(\frac{3r_1+1}{6(r_1+1)^2}\right)^{\frac{1}{q}} \tag{30}$$

$$\times \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left[mL^{q} + \eta_2\left(L^{q}, L^{q}\right)\right]$$

$$\times\left\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, 1-\lambda, p\right) + \varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k}, \mu, p\right)\right\}.$$

Corollary 7 *In Theorem 3 for* $h_1(t) = \frac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \frac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $f^{(n+1)}(x) \leq L, \forall x \in I$, *we get the following inequality for generalized* $(m;(\eta_1,\eta_2))$*-MT-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1,\varphi;\lambda,\mu,n,r_1,m,a,b)\right| \leq \tag{31}$$

$$\times\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left[mL^{rq}+\eta_2\left(L^{rq},L^{rq}\right)\right]$$

$$\times\left\{\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},1-\lambda,p\right)I^{\frac{1}{q}}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1\right)\right.$$

$$\left.+\varrho^{\frac{1}{p}}\left(n-\frac{\alpha}{k},\mu,p\right)\overline{I}^{\frac{1}{q}}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1\right)\right\}.$$

Theorem 4 *Let* $\alpha > 0$, $k \geq 1$, $0 < r \leq 1$, *and* $\alpha \notin \{1,2,3,\ldots\}$, $n = [\alpha]+1$. *Suppose* $K = [m\varphi(a), m\varphi(a)+\eta_1(\varphi(b),m\varphi(a))] \subseteq \mathbb{R}$ *be an open m-invex subset with respect to* $\eta_1 : K \times K \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$, *where* $\eta_1(\varphi(b),m\varphi(a)) > 0$. *Also, let* $h_1,h_2 : [0,1] \longrightarrow [0,+\infty)$ *and* $\varphi : I \longrightarrow K$ *are continuous. Assume that* $f : K \longrightarrow (0,+\infty)$ *is a* $(n+1)$*-differentiable mapping on* K° *such that* $f \in C^{n+1}(K)$ *and* $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$. *If* $\left(f^{(n+1)}\right)^q$ *is generalized* $((h_1,h_2);(\eta_1,\eta_2))$*-convex mapping and* $q \geq 1$, *then for any* $\lambda,\mu \in [0,1]$ *and* $r_1 \geq 0$, *the following inequality for Caputo k-fractional derivatives holds:*

$$\left|T_f^{\alpha,k}(\eta_1,\varphi;\lambda,\mu,n,r_1,m,a,b)\right|$$

$$\leq\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right. \tag{32}$$

$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq} I^r(h_1(t);r,r_1,\lambda,\alpha,k,n)\right.$$

$$\left.+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)I^r(h_2(t);r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\left[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1,\mu,\alpha,k,n)\right.$$

$$\left.\left.+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\overline{I}^r(h_2(t);r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\right\},$$

where

$$I\left(h_i(t); r, r_1, \lambda, \alpha, k, n\right) := \int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right| h_i^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right) dt,$$

$$\overline{I}\left(h_i(t); r, r_1, \mu, \alpha, k, n\right) := \int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right| h_i^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right) dt, \quad \forall i = 1, 2$$

and $\delta\left(n-\frac{\alpha}{k}, 1-\lambda\right)$, $\delta\left(n-\frac{\alpha}{k}, \mu\right)$ *are defined as in Lemma 1.*

Proof From Lemma 2, generalized $((h_1, h_2); (\eta_1, \eta_2))$-convexity of $\left(f^{(n+1)}\right)^q$, the well-known power mean inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right| \leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}$$

$$\times\left\{\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right| \left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right| dt\right.$$

$$\left.+\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right| \left|f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right| dt\right\}$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\left(\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right| dt\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right| \left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{1-t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right)^q dt\right)^{\frac{1}{q}}$$

$$+\left(\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right| dt\right)^{1-\frac{1}{q}}$$

$$\left.\times\left(\int_0^1 \left|t^{n-\frac{\alpha}{k}}-\mu\right| \left(f^{(n+1)}\left(m\varphi(a)+\left(\frac{r_1+t}{r_1+1}\right)\eta_1(\varphi(b), m\varphi(a))\right)\right)^q dt\right)^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k}, 1-\lambda\right)\right.$$

$$\times\left[\int_0^1 \left|1-\lambda-t^{n-\frac{\alpha}{k}}\right| \left[mh_1\left(\frac{1-t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}\right.\right.$$

$$\left.\left.+h_2\left(\frac{1-t}{r_1+1}\right)\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq}, \left(f^{(n+1)}(a)\right)^{rq}\right)\right]^{\frac{1}{r}} dt\right]^{\frac{1}{q}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\left[\int_0^1\left|t^{n-\frac{\alpha}{k}}-\mu\right|\left[mh_1\left(\frac{r_1+t}{r_1+1}\right)\left(f^{(n+1)}(a)\right)^{rq}\right.\right.$$

$$\left.\left.+h_2\left(\frac{r_1+t}{r_1+1}\right)\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\right]^{\frac{1}{r}}dt\right]^{\frac{1}{q}}\Bigg\}$$

$$\leq\left(\frac{\eta(\varphi(b),\varphi(a),m)}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right.$$

$$\times\left[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q\left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|h_1^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.+\left(\int_0^1\eta_2^{\frac{1}{r}}\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\left|1-\lambda-t^{n-\frac{\alpha}{k}}\right|h_2^{\frac{1}{r}}\left(\frac{1-t}{r_1+1}\right)dt\right)^r\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\left[\left(\int_0^1 m^{\frac{1}{r}}\left(f^{(n+1)}(a)\right)^q\left|t^{n-\frac{\alpha}{k}}-\mu\right|h_1^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right.$$

$$\left.\left.+\left(\int_0^1\eta_2^{\frac{1}{r}}\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\left|t^{n-\frac{\alpha}{k}}-\mu\right|h_2^{\frac{1}{r}}\left(\frac{r_1+t}{r_1+1}\right)dt\right)^r\right]^{\frac{1}{rq}}\right\}$$

$$=\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right.$$

$$\times\left[m\left(f^{(n+1)}(a)\right)^{rq}I^r(h_1(t);r,r_1,\lambda,\alpha,k,n)\right.$$

$$\left.+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)I^r(h_2(t);r,r_1,\lambda,\alpha,k,n)\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\left[m\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^r(h_1(t);r,r_1,\mu,\alpha,k,n)\right.$$

$$\left.\left.+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\overline{I}^r(h_2(t);r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\right\}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 4.

Corollary 8 *In Theorem 4 if we choose* $\lambda = \mu = m = r = 1$ *and* $\eta_1(\varphi(b), m\varphi(a)) = \varphi(b) - m\varphi(a)$, $\varphi(x) = x$ *for all* $x \in I$, *we get the following inequality for Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(1,1,n,r_1,1,a,b)\right| \leq \left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{ \left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \right. \tag{33}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q I(h_1(t); 1, r_1, 1, \alpha, k, n) \right.$$

$$\left. +\eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) I(h_2(t); 1, r_1, 1, \alpha, k, n)\right]^{\frac{1}{q}}$$

$$+\left(\frac{n-\frac{\alpha}{k}}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \times \left[\left(f^{(n+1)}(a)\right)^q \overline{I}(h_1(t); 1, r_1, 1, \alpha, k, n) \right.$$

$$\left.\left. +\eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) \overline{I}(h_2(t); 1, r_1, 1, \alpha, k, n)\right]^{\frac{1}{q}} \right\}.$$

Corollary 9 *In Theorem 4 if we choose* $\lambda = \mu = 0$, $m = r = 1$ *and* $\eta_1(\varphi(b), m\varphi(a)) = \varphi(b) - m\varphi(a)$, $\varphi(x) = x$ *for all* $x \in I$, *we get the following inequality for Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(0,0,n,r_1,1,a,b)\right| \leq \left(\frac{b-a}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{ \left(\frac{n-\frac{\alpha}{k}}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \right. \tag{34}$$

$$\times\left[\left(f^{(n+1)}(a)\right)^q I(h_1(t); 1, r_1, 0, \alpha, k, n) \right.$$

$$\left. +\eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) I(h_2(t); 1, r_1, 0, \alpha, k, n)\right]^{\frac{1}{q}}$$

$$+\left(\frac{1}{n-\frac{\alpha}{k}+1}\right)^{1-\frac{1}{q}} \times \left[\left(f^{(n+1)}(a)\right)^q \overline{I}(h_1(t); 1, r_1, 0, \alpha, k, n) \right.$$

$$\left.\left. +\eta_2\left(\left(f^{(n+1)}(b)\right)^q, \left(f^{(n+1)}(a)\right)^q\right) \overline{I}(h_2(t); 1, r_1, 0, \alpha, k, n)\right]^{\frac{1}{q}} \right\}.$$

Corollary 10 *In Theorem 4 for $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ and $f^{(n+1)}(x) \leq L$, $\forall x \in I$, we get the following inequality for generalized $((m, h); (\eta_1, \eta_2))$-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right|$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k}, 1-\lambda\right)\right. \tag{35}$$

$$\times\left[mL^{rq}I^r(h(1-t); r, r_1, \lambda, \alpha, k, n) + \eta_2\left(L^{rq}, L^{rq}\right)I^r(h(t); r, r_1, \lambda, \alpha, k, n)\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k}, \mu\right)$$

$$\left.\times\left[mL^{rq}\overline{I}^r(h(1-t); r, r_1, \mu, \alpha, k, n)+\eta_2\left(L^{rq}, L^{rq}\right)\overline{I}^r(h(t); r, r_1, \mu, \alpha, k, n)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 11 *In Corollary 10 for $h_1(t) = (1-t)^s$ and $h_2(t) = t^s$, we get the following inequality for generalized $((m, s); (\eta_1, \eta_2))$-Breckner-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right|$$

$$\leq \left(\tfrac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\delta^{1-\frac{1}{q}}\left(n-\tfrac{\alpha}{k}, 1-\lambda\right)\right. \tag{36}$$

$$\times\left[mL^{rq}I^r((1-t)^s; r, r_1, \lambda, \alpha, k, n) + \eta_2\left(L^{rq}, L^{rq}\right)I^r(t^s; r, r_1, \lambda, \alpha, k, n)\right]^{\frac{1}{rq}}$$

$$\delta^{1-\frac{1}{q}}\left(n-\tfrac{\alpha}{k}, \mu\right)$$

$$\left.\times\left[mL^{rq}\overline{I}^r((1-t)^s; r, r_1, \mu, \alpha, k, n) + \eta_2(L^{rq}, L^{rq})\overline{I}^r(t^s; r, r_1, \mu, \alpha, k, n)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 12 *In Corollary 10 for $h_1(t) = (1-t)^{-s}$, $h_2(t) = t^{-s}$ and $0 < s < r$, we get the following inequality for generalized $((m, s); (\eta_1, \eta_2))$-Godunova–Levin–Dragomir-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1, \varphi; \lambda, \mu, n, r_1, m, a, b)\right|$$

$$\leq \left(\frac{\eta_1(\varphi(b), m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1} \times \left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k}, 1-\lambda\right)\right. \tag{37}$$

$$\times\left[mL^{rq}I^{r}\left((1-t)^{-s};r,r_1,\lambda,\alpha,k,n\right)+\eta_2\left(L^{rq},L^{rq}\right)I^{r}\left(t^{-s};r,r_1,\lambda,\alpha,k,n\right)\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)$$

$$\times\left[mL^{rq}\overline{I}^{r}\left((1-t)^{-s};r,r_1,\mu,\alpha,k,n\right)+\eta_2\left(L^{rq},L^{rq}\right)\overline{I}^{r}\left(t^{-s};r,r_1,\mu,\alpha,k,n\right)\right]^{\frac{1}{rq}}\Bigg\}.$$

Corollary 13 *In Theorem 4 for* $h_1(t) = h_2(t) = t(1-t)$ *and* $f^{(n+1)}(x) \le L, \forall x \in I$, *we get the following inequality for generalized* $((m,tgs);(\eta_1,\eta_2))$*-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1,\varphi;\lambda,\mu,n,r_1,m,a,b)\right|$$

$$\le\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left[mL^{rq}+\eta_2\left(L^{rq},L^{rq}\right)\right] \tag{38}$$

$$\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)I^{\frac{1}{q}}\left(t(1-t);r,r_1,\lambda,\alpha,k,n\right)\right.$$

$$\left.+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\overline{I}^{\frac{1}{q}}\left(t(1-t);r,r_1,\mu,\alpha,k,n\right)\right\}.$$

Corollary 14 *In Theorem 4 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $f^{(n+1)}(x) \le L, \forall x \in I$, *we get the following inequality for generalized* $(m;(\eta_1,\eta_2))$*-MT-convex mappings via Caputo k-fractional derivatives:*

$$\left|T_f^{\alpha,k}(\eta_1,\varphi;\lambda,\mu,n,r_1,m,a,b)\right|$$

$$\le\left(\frac{\eta_1(\varphi(b),m\varphi(a))}{r_1+1}\right)^{n-\frac{\alpha}{k}+1}\times\left\{\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},1-\lambda\right)\right. \tag{39}$$

$$\times\left[mL^{rq}I^{r}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,r_1,\lambda,\alpha,k,n\right)+\eta_2\left(L^{rq},L^{rq}\right)I^{r}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1,\lambda,\alpha,k,n\right)\right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)$$

$$\left.\times\left[mL^{rq}\overline{I}^{r}\left(\frac{\sqrt{1-t}}{2\sqrt{t}};r,r_1,\mu,\alpha,k,n\right)+\eta_2\left(L^{rq},L^{rq}\right)\overline{I}^{r}\left(\frac{\sqrt{t}}{2\sqrt{1-t}};r,r_1,\mu,\alpha,k,n\right)\right]^{\frac{1}{rq}}\right\}.$$

Remark 4 For $k = 1$, by our Theorems 3 and 4, we can get some new special Hermite–Hadamard type inequalities associated with generalized $((h_1, h_2); (\eta_1, \eta_2))$-convex mappings via Caputo fractional derivatives of order α. The details are left to the interested reader.

3 Applications to Special Means

Definition 18 A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$ is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \le x'$ and $y \le y'$, then $M(x, y) \le M(x', y')$,
5. Internality: $\min\{x, y\} \le M(x, y) \le \max\{x, y\}$.

We consider some means for different positive real numbers α, β.

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \ge 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \ne \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \le G \le L \le I \le A$. Now, let a and b be positive real numbers such that $a < b$. Let us consider continuous functions $\varphi : I \longrightarrow K$, $\eta_1 : K \times K \longrightarrow \mathbb{R}$, $\eta_2 : f(K) \times f(K) \longrightarrow \mathbb{R}$ and $\overline{M} := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta_1(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta_1(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above-mentioned means. Therefore one can obtain various inequalities using the results of Sect. 2 for these means as follows. Replace $\eta_1(\varphi(b), m\varphi(a)) = M(\varphi(a), \varphi(b))$ for value $m = 1$, in (24) and (32), one can obtain the following interesting inequalities involving means:

$$\left|T_f^{\alpha,k}(\overline{M}, \varphi; \lambda, \mu, n, r_1, 1, a, b)\right| \le \left(\frac{\overline{M}}{r_1 + 1}\right)^{n - \frac{\alpha}{k} + 1} \times \left\{ \varrho^{\frac{1}{p}} \left(n - \frac{\alpha}{k}, 1 - \lambda, p\right) \right. \tag{40}$$
$$\times \left[\left(f^{(n+1)}(a)\right)^{rq} I^r(h_1(t); r, r_1) + \eta_2\left(\left(f^{(n+1)}(b)\right)^{rq}, \left(f^{(n+1)}(a)\right)^{rq}\right) I^r(h_2(t); r, r_1) \right]^{\frac{1}{rq}}$$
$$+ \varrho^{\frac{1}{p}}\left(n - \frac{\alpha}{k}, \mu, p\right) \times \left[\left(f^{(n+1)}(a)\right)^{rq} \overline{I}^r(h_1(t); r, r_1) \right.$$
$$\left. \left. + \eta_2\left(\left(f^{(n+1)}(b)\right)^{rq}, \left(f^{(n+1)}(a)\right)^{rq}\right) \overline{I}^r(h_2(t); r, r_1) \right]^{\frac{1}{rq}} \right\},$$

$$\left|T_f^{\alpha,k}(\overline{M}, \varphi; \lambda, \mu, n, r_1, 1, a, b)\right| \le \left(\frac{\overline{M}}{r_1 + 1}\right)^{n - \frac{\alpha}{k} + 1} \times \left\{ \delta^{1 - \frac{1}{q}} \left(n - \frac{\alpha}{k}, 1 - \lambda\right) \right. \tag{41}$$
$$\times \left[\left(f^{(n+1)}(a)\right)^{rq} I^r(h_1(t); r, r_1, \lambda, \alpha, k, n) \right.$$
$$\left. + \eta_2\left(\left(f^{(n+1)}(b)\right)^{rq}, \left(f^{(n+1)}(a)\right)^{rq}\right) I^r(h_2(t); r, r_1, \lambda, \alpha, k, n) \right]^{\frac{1}{rq}}$$

$$+\delta^{1-\frac{1}{q}}\left(n-\frac{\alpha}{k},\mu\right)\times\left[\left(f^{(n+1)}(a)\right)^{rq}\overline{I}^{r}(h_1(t);r,r_1,\mu,\alpha,k,n)\right.$$

$$\left.+\eta_2\left(\left(f^{(n+1)}(b)\right)^{rq},\left(f^{(n+1)}(a)\right)^{rq}\right)\overline{I}^{r}(h_2(t);r,r_1,\mu,\alpha,k,n)\right]^{\frac{1}{rq}}\Bigg\}.$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p$ in (40) and (41), we get the inequalities involving means for a particular choices of $\left(f^{(n+1)}\right)^q$ that are generalized $((h_1,h_2);(\eta_1,\eta_2))$-convex mappings.

4 Conclusion

In this article, we first presented a new identity concerning $(n+1)$-differentiable mappings defined on m-invex set via Caputo k-fractional derivatives. By using the notion of generalized $((h_1,h_2);(\eta_1,\eta_2))$-convexity and the obtained identity as an auxiliary result, some new estimates with respect to Hermite–Hadamard type inequalities via Caputo k-fractional derivatives are established. It is pointed out that some new special cases are deduced from main results. Motivated by this new interesting class of generalized $((h_1,h_2);(\eta_1,\eta_2))$-convex mappings we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite–Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of convex and preinvex functions involving local fractional integrals, fractional integral operators, q-calculus, (p,q)-calculus, time scale calculus, and conformable fractional integrals.

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Some New Hermite–Hadamard Type Integral Inequalities for Twice Differentiable Mappings and Their Applications

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Abstract The authors discover a general fractional integral identity regarding Hermite–Hadamard type inequality for twice differentiable functions. By using this integral equation, the authors derive some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings defined on **m**-invex set. It is pointed out that some new special cases can be deduced from main results. At the end, some applications to special means for different positive real numbers are provided as well.

1 Introduction

The following notations are used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$. For any subset $K \subseteq \mathbb{R}^n$, K° is the interior of K. The set of integrable functions on the interval $[a, b]$ is denoted by $L[a, b]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1 *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on I and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality (1) is also known as trapezium inequality.

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The trapezium type inequality has remained an area of great interest due to its wide applications in the field of mathematical analysis. For other recent results which generalize, improve and extend the inequality (1) through various classes of convex functions, interested readers are referred to [1–33, 35, 38, 39, 41–45, 49, 51, 52]. Let us recall some special functions and evoke some basic definitions as follows.

Definition 1 The incomplete beta function is defined for $a, b > 0$ as

$$\beta_x(a, b) = \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 < x \le 1. \tag{2}$$

Definition 2 ([50]) A set $S \subseteq \mathbb{R}^n$ is said to be invex set with respect to the mapping $\eta : S \times S \longrightarrow \mathbb{R}^n$, if $x + t\eta(y, x) \in S$ for every $x, y \in S$ and $t \in [0, 1]$.

The invex set S is also termed an η-connected set.

Definition 3 ([34]) Let $h : [0, 1] \longrightarrow \mathbb{R}$ be a non-negative function and $h \neq 0$. The function f on the invex set K is said to be h-preinvex with respect to η, if

$$f\big(x + t\eta(y, x)\big) \le h(1-t)f(x) + h(t)f(y) \tag{3}$$

for each $x, y \in K$ and $t \in [0, 1]$ where $f(\cdot) > 0$.

Clearly, when putting $h(t) = t$ in Definition 3, f becomes a preinvex function [40]. If the mapping $\eta(y, x) = y - x$ in Definition 3, then the non-negative function f reduces to h-convex mappings [47].

Definition 4 ([48]) Let $S \subseteq \mathbb{R}^n$ be an invex set with respect to $\eta : S \times S \longrightarrow \mathbb{R}^n$. A function $f : S \longrightarrow [0, +\infty)$ is said to be s-preinvex (or s-Breckner-preinvex) with respect to η and $s \in (0, 1]$, if for every $x, y \in S$ and $t \in [0, 1]$,

$$f\big(x + t\eta(y, x)\big) \le (1-t)^s f(x) + t^s f(y). \tag{4}$$

Definition 5 ([37]) A function $f : K \longrightarrow \mathbb{R}$ is said to be s-Godunova-Levin-Dragomir-preinvex of second kind, if

$$f\big(x + t\eta(y, x)\big) \le (1-t)^{-s} f(x) + t^{-s} f(y), \tag{5}$$

for each $x, y \in K$, $t \in (0, 1)$ and $s \in (0, 1]$.

Definition 6 ([46]) A non-negative function $f : K \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be tgs-convex on K if the inequality

$$f\big((1-t)x + ty\big) \le t(1-t)[f(x) + f(y)] \tag{6}$$

grips for all $x, y \in K$ and $t \in (0, 1)$.

Definition 7 ([31]) A function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ is said to be MT-convex functions, if it is non-negative and $\forall\, x, y \in I$ and $t \in (0, 1)$ satisfies the subsequent inequality

$$f(tx + (1-t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y). \tag{7}$$

Definition 8 ([39]) Let $K \subseteq \mathbb{R}$ be an open m-invex set respecting $\eta : K \times K \longrightarrow \mathbb{R}$ and $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$. A function $f : K \longrightarrow \mathbb{R}$ is said to be generalized (m, h_1, h_2)-preinvex, if

$$f\big(mx + t\eta(y, mx)\big) \leq mh_1(t) f(x) + h_2(t) f(y) \tag{8}$$

is valid for all $x, y \in K$ and $t \in [0, 1]$, for some fixed $m \in (0, 1]$.

Definition 9 ([32]) Let $f \in L[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+} f$ and $J^{\alpha}_{b-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a$$

and

$$J^{\alpha}_{b-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \int_0^{+\infty} e^{-u} u^{\alpha-1} du$. Here $J^{0}_{a+} f(x) = J^{0}_{b-} f(x) = f(x)$.

Note that $\alpha = 1$, the fractional integral reduces to the classical integral.

Motivated by the above works and the references therein, the main objective of this article is to apply the notion of generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings and an interesting lemma to establish some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings defined on **m**-invex set. Also, some new special cases will be deduced. At the end, some applications to special means for different positive real numbers will be given as well.

2 Main Results

The following definitions will be used in this section.

Definition 10 Let $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$ be a function. A set $K \subseteq \mathbb{R}^n$ is named as **m**-invex with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}^n$, if $\mathbf{m}(t)x + \xi\eta(y, \mathbf{m}(t)x) \in K$ holds for each $x, y \in K$ and any $t, \xi \in [0, 1]$.

Remark 1 In Definition 10, under certain conditions, the mapping $\eta(y, \mathbf{m}(t)x)$ for any $t, \xi \in [0, 1]$ could reduce to $\eta(y, mx)$. For example, when $\mathbf{m}(t) = m$ for all $t \in [0, 1]$, then the **m**-invex set degenerates an m-invex set on K.

Definition 11 ([24]) Let $K \subseteq \mathbb{R}$ be an open **m**-invex set with respect to the mapping $\eta : K \times K \longrightarrow \mathbb{R}$. Suppose $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$, $\varphi : I \longrightarrow K$ are continuous functions and $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. A mapping $f : K \longrightarrow (0, +\infty)$ is said to be generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex, if

$$f\big(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x))\big) \leq \Big[\mathbf{m}(\xi)h_1(\xi)f^r(x) + h_2(\xi)f^r(y)\Big]^{\frac{1}{r}} \tag{9}$$

holds for all $x, y \in I$ and $t, \xi \in [0, 1]$, where $r \neq 0$.

Remark 2 In Definition 11, if we choose $\mathbf{m} = m = r = 1$, this definition reduces to the definition considered by Noor in [36] and Fulga and Preda in [13].

Remark 3 In Definition 11, if we choose $\mathbf{m} = m = r = 1$ and $\varphi(x) = x$, then we get Definition 8.

Remark 4 Let us discuss some special cases in Definition 11 as follows.

1. Taking $h_1(t) = h(1 - t)$, $h_2(t) = h(t)$, then we get generalized relative semi-$(\mathbf{m}, h)$-preinvex mappings.
2. Taking $h_1(t) = (1 - t)^s$, $h_2(t) = t^s$ for $s \in (0, 1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Breckner-preinvex mappings.
3. Taking $h_1(t) = (1 - t)^{-s}$, $h_2(t) = t^{-s}$ for $s \in (0, 1]$, then we get generalized relative semi-$(\mathbf{m}, s)$-Godunova–Levin–Dragomir-preinvex mappings.
4. Taking $h_1(t) = h_2(t) = t(1-t)$, then we get generalized relative semi-$(\mathbf{m}, tgs)$-preinvex mappings.
5. Taking $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$, then we get generalized relative semi-**m**-MT-preinvex mappings.

It is worth to mention here that to the best of our knowledge all the special cases discussed above are new in the literature.

For establishing our main results regarding some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity via fractional integrals, we need the following lemma.

Lemma 1 *Let $\varphi : I \longrightarrow K$ be a continuous function and* $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. *Suppose $K \subseteq \mathbb{R}$ be an open* **m***-invex subset with respect to $\eta : K \times K \longrightarrow \mathbb{R}$ where $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) \neq 0$ and $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) \neq 0$ for all $t \in [0, 1]$. If $f : K \longrightarrow \mathbb{R}$ is a twice differentiable mapping on K° such that $f'' \in L(K)$, then for any $\alpha > 0$, the following identity holds:*

$$-\frac{(\alpha+1)}{2^{\alpha-1}}\frac{1}{\eta^2(\varphi(y),\mathbf{m}(t)\varphi(x))}f\left(\mathbf{m}(t)\varphi(x)+\frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)$$
$$-\frac{(\alpha+1)}{2^{\alpha-1}}\frac{1}{\eta^2(\varphi(x),\mathbf{m}(t)\varphi(y))}f\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)$$
$$+\frac{\Gamma(\alpha+2)}{\eta^{\alpha+2}(\varphi(y),\mathbf{m}(t)\varphi(x))}$$
$$\times\Big[J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x)+\frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^+}f(\mathbf{m}(t)\varphi(x)+\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))$$
$$+J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x)+\frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^-}f(\mathbf{m}(t)\varphi(x))\Big]$$
$$+\frac{\Gamma(\alpha+2)}{\eta^{\alpha+2}(\varphi(x),\mathbf{m}(t)\varphi(y))}$$
$$\times\Big[J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^+}f(\mathbf{m}(t)\varphi(y)+\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))$$
$$+J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^-}f(\mathbf{m}(t)\varphi(y))\Big]$$
$$=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))) \tag{10}$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi$$
$$+\int_{\frac{1}{2}}^1(1-\xi)^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi.$$

We denote

$$T_f^{\alpha}(\eta,\varphi,\mathbf{m};x,y):=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))) \tag{11}$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi$$
$$+\int_{\frac{1}{2}}^1(1-\xi)^{\alpha+1}\big[f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))$$
$$+f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\big]d\xi.$$

Proof

$$T_f^{\alpha}(\eta,\varphi,\mathbf{m};x,y)=T_{11}+T_{12}+T_{21}+T_{22},$$

where

$$T_{11}=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi;$$

$$T_{12}=\int_0^{\frac{1}{2}}\xi^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))\right)d\xi;$$

$$T_{21}=\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi;$$

$$T_{22}=\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\big[f''\left(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x))\right)d\xi.$$

Now, using twice integration by parts, we have

$$T_{11}=\frac{\xi^{\alpha+1}f'\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\Bigg|_0^{\frac{1}{2}}$$

$$-\frac{(\alpha+1)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\int_0^{\frac{1}{2}}\xi^{\alpha}f'\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi$$

$$=\frac{1}{2^{\alpha+1}}\frac{f'\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}-\frac{(\alpha+1)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}$$

$$\times\left\{\frac{\xi^{\alpha}f\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\Bigg|_0^{\frac{1}{2}}\right.$$

$$\left.-\frac{\alpha}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\int_0^{\frac{1}{2}}\xi^{\alpha-1}f\left(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y))\right)d\xi\right\}$$

$$=\frac{1}{2^{\alpha+1}}\frac{f'\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}-\frac{(\alpha+1)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\tag{12}$$

$$\times\left\{\frac{1}{2^{\alpha}}\frac{f\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}\right.$$

$$\left.-\frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(x),\mathbf{m}(t)\varphi(y))}\times J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y)+\frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^-}f(\mathbf{m}(t)\varphi(y))\right\}.$$

In a similar way, we find

$$T_{12} = \frac{1}{2^{\alpha+1}} \frac{f'\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} - \frac{(\alpha+1)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} \tag{13}$$

$$\times \left\{ \frac{1}{2^{\alpha}} \frac{f\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} \right.$$

$$\left. - \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(y), \mathbf{m}(t)\varphi(x))} \times J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^{-}} f(\mathbf{m}(t)\varphi(x)) \right\}.$$

$$T_{21} = -\frac{1}{2^{\alpha+1}} \frac{f'\left(\mathbf{m}(t)\varphi(y) + \frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x), \mathbf{m}(t)\varphi(y))} + \frac{(\alpha+1)}{\eta(\varphi(x), \mathbf{m}(t)\varphi(y))} \tag{14}$$

$$\times \left\{ -\frac{1}{2^{\alpha}} \frac{f\left(\mathbf{m}(t)\varphi(y) + \frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)}{\eta(\varphi(x), \mathbf{m}(t)\varphi(y))} + \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(x), \mathbf{m}(t)\varphi(y))} \times \right.$$

$$\left. J^{\alpha}_{\left(\mathbf{m}(t)\varphi(y) + \frac{\eta(\varphi(x),\mathbf{m}(t)\varphi(y))}{2}\right)^{+}} f(\mathbf{m}(t)\varphi(y) + \eta(\varphi(x), \mathbf{m}(t)\varphi(y))) \right\}.$$

$$T_{22} = -\frac{1}{2^{\alpha+1}} \frac{f'\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} + \frac{(\alpha+1)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} \tag{15}$$

$$\times \left\{ -\frac{1}{2^{\alpha}} \frac{f\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)}{\eta(\varphi(y), \mathbf{m}(t)\varphi(x))} + \frac{\Gamma(\alpha+1)}{\eta^{\alpha+1}(\varphi(y), \mathbf{m}(t)\varphi(x))} \times \right.$$

$$\left. J^{\alpha}_{\left(\mathbf{m}(t)\varphi(x) + \frac{\eta(\varphi(y),\mathbf{m}(t)\varphi(x))}{2}\right)^{+}} f(\mathbf{m}(t)\varphi(x) + \eta(\varphi(y), \mathbf{m}(t)\varphi(x))) \right\}.$$

Adding Eqs. (12)–(15), we get our lemma.

Remark 5 In Lemma 1, if we take $\alpha = 1$, $\mathbf{m}(t) \equiv 1$ for all $t \in [0, 1]$, $a < b$, $x = \mu a + (1-\mu)b$, $y = \mu b + (1-\mu)a$, where $\mu \in [0, 1]\backslash\left\{\frac{1}{2}\right\}$ and $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) = \varphi(x) - \mathbf{m}(t)\varphi(y)$, $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) = \varphi(y) - \mathbf{m}(t)\varphi(x)$, where $\varphi(x) = x$ for all $x \in I$, in identity (10), then it becomes identity of Lemma 2.1 in [41].

Theorem 2 *Let* $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ *and* $\varphi : I \longrightarrow K$ *are continuous functions and* $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. *Suppose* $K \subseteq \mathbb{R}$ *be an open* $\mathbf{m}$*-invex subset, where* $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) \neq 0$ *and* $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) \neq 0$ *for all* $t \in [0, 1]$. *Assume that* $f : K \longrightarrow (0, +\infty)$ *is a twice differentiable mapping on* K° *such that*

$f'' \in L(K)$. If f''^q is generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mapping, $0 < r \leq 1$ and $q > 1$, $p^{-1} + q^{-1} = 1$, then for any $\alpha > 0$, the following inequality for fractional integrals hold:

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right| \leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{16}$$

$$\times\Bigg\{\Big[\big(f''(x)\big)^{rq} I^r(h_1(\xi); \mathbf{m}(\xi), r) + \big(f''(y)\big)^{rq} I^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\big(f''(y)\big)^{rq} I^r(h_1(\xi); \mathbf{m}(\xi), r) + \big(f''(x)\big)^{rq} I^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\big(f''(x)\big)^{rq} J^r(h_1(\xi); \mathbf{m}(\xi), r) + \big(f''(y)\big)^{rq} J^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\big(f''(y)\big)^{rq} J^r(h_1(\xi); \mathbf{m}(\xi), r) + \big(f''(x)\big)^{rq} J^r(h_2(\xi); r)\Big]^{\frac{1}{rq}}\Bigg\},$$

where

$$I(h_1(\xi); \mathbf{m}(\xi), r) := \int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi) h_1^{\frac{1}{r}}(\xi) d\xi, \quad I(h_2(\xi); r) := \int_0^{\frac{1}{2}} h_2^{\frac{1}{r}}(\xi) d\xi;$$

and

$$J(h_1(\xi); \mathbf{m}(\xi), r) := \int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi) h_1^{\frac{1}{r}}(\xi) d\xi, \quad J(h_2(\xi); r) := \int_{\frac{1}{2}}^1 h_2^{\frac{1}{r}}(\xi) d\xi.$$

Proof From Lemma 1, generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvexity of f''^q, Hölder inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right|$$

$$\leq \int_0^{\frac{1}{2}} \xi^{\alpha+1}\big[|f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))|$$

$$+|f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))|\big]d\xi$$

$$+\int_{\frac{1}{2}}^1 (1-\xi)^{\alpha+1}\big[|f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))|$$

$$+|f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))|\big]d\xi$$

$$\leq \left(\int_0^{\frac{1}{2}} \xi^{p(\alpha+1)} d\xi\right)^{\frac{1}{p}} \times \left\{ \left(\int_0^{\frac{1}{2}} \left(f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_0^{\frac{1}{2}} \left(f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}} \right\}$$

$$+\left(\int_{\frac{1}{2}}^1 (1-\xi)^{p(\alpha+1)} d\xi\right)^{\frac{1}{p}} \times \left\{ \left(\int_{\frac{1}{2}}^1 \left(f''(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}} \right.$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \left(f''(\mathbf{m}(t)\varphi(y) + \xi\eta(\varphi(x), \mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}} \right\}$$

$$\leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}$$

$$\times \left\{ \left(\int_0^{\frac{1}{2}} \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq} + h_2(\xi)\left(f''(y)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}} \right.$$

$$+ \left(\int_0^{\frac{1}{2}} \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq} + h_2(\xi)\left(f''(x)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}}$$

$$+ \left(\int_{\frac{1}{2}}^1 \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq} + h_2(\xi)\left(f''(y)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}}$$

$$\left. + \left(\int_{\frac{1}{2}}^1 \left[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq} + h_2(\xi)\left(f''(x)\right)^{rq}\right]^{\frac{1}{r}} d\xi\right)^{\frac{1}{q}} \right\}$$

$$\leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}$$

$$\times \left\{ \left[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^q h_1^{\frac{1}{r}}(\xi) d\xi\right)^r + \left(\int_0^{\frac{1}{2}} \left(f''(y)\right)^q h_2^{\frac{1}{r}}(\xi) d\xi\right)^r \right]^{\frac{1}{rq}} \right.$$

$$+ \left[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^q h_1^{\frac{1}{r}}(\xi) d\xi\right)^r + \left(\int_0^{\frac{1}{2}} \left(f''(x)\right)^q h_2^{\frac{1}{r}}(\xi) d\xi\right)^r \right]^{\frac{1}{rq}}$$

$$+\left[\left(\int_{\frac{1}{2}}^{1}\mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^{q}h_1^{\frac{1}{r}}(\xi)d\xi\right)^{r}+\left(\int_{\frac{1}{2}}^{1}\left(f''(y)\right)^{q}h_2^{\frac{1}{r}}(\xi)d\xi\right)^{r}\right]^{\frac{1}{rq}}$$

$$+\left[\left(\int_{\frac{1}{2}}^{1}\mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^{q}h_1^{\frac{1}{r}}(\xi)d\xi\right)^{r}+\left(\int_{\frac{1}{2}}^{1}\left(f''(x)\right)^{q}h_2^{\frac{1}{r}}(\xi)d\xi\right)^{r}\right]^{\frac{1}{rq}}\Bigg\}$$

$$=\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}$$

$$\times\Bigg\{\left[\left(f''(x)\right)^{rq}I^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(y)\right)^{rq}I^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(y)\right)^{rq}I^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(x)\right)^{rq}I^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(x)\right)^{rq}J^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(y)\right)^{rq}J^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(y)\right)^{rq}J^{r}(h_1(\xi);\mathbf{m}(\xi),r)+\left(f''(x)\right)^{rq}J^{r}(h_2(\xi);r)\right]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 2.

Corollary 1 *In Theorem 2, if we take $\mathbf{m}(\xi)\equiv m\in(0,1]$ for all $\xi\in[0,1]$, $h_1(t)=h(1-t)$, $h_2(t)=h(t)$ and $f''(x)\leq L$, $\forall x\in I$, we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-$(\mathbf{m},h)$-preinvex mappings*

$$\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right|\leq 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{17}$$

$$\times\left\{\left[mI^{r}(h(t);r)+I^{r}(h(1-t);r)\right]^{\frac{1}{rq}}+\left[mI^{r}(h(1-t);r)+I^{r}(h(t);r)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 2 *In Corollary 1 for $h_1(t)=(1-t)^s$ and $h_2(t)=t^s$, we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-$(\mathbf{m},s)$-Breckner-preinvex mappings*

$$\begin{aligned}\left|T_f^{\alpha}(\eta,\varphi,m;x,y)\right|&\leq 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}\left(\frac{r}{(s+r)2^{\frac{s}{r}+1}}\right)^{\frac{1}{q}}\\&\times\left\{\left[m+\left(2^{\frac{s}{r}+1}-1\right)^{r}\right]^{\frac{1}{rq}}+\left[m\left(2^{\frac{s}{r}+1}-1\right)^{r}+1\right]^{\frac{1}{rq}}\right\}.\end{aligned} \tag{18}$$

Corollary 3 *In Corollary 1 for* $h_1(t) = (1-t)^{-s}$ *and* $h_2(t) = t^{-s}$ *and* $0 < s < r$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, s)$*-Godunova–Levin–Dragomir-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \le 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}\left(\frac{r}{(r-s)2^{1-\frac{s}{r}}}\right)^{\frac{1}{q}} \times\left\{\left[m+\left(2^{1-\frac{s}{r}}-1\right)^{r}\right]^{\frac{1}{rq}}+\left[m\left(2^{1-\frac{s}{r}}-1\right)^{r}+1\right]^{\frac{1}{rq}}\right\}. \tag{19}$$

Corollary 4 *In Theorem 2, if we take* $\mathbf{m}(\xi) \equiv m \in (0, 1]$ *for all* $\xi \in [0, 1]$, $h_1(t) = h_2(t) = t(1-t)$ *and* $f''(x) \le L$, $\forall x \in I$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, tgs)$*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \le 4L(m+1)^{\frac{1}{rq}}\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \times\beta_{1/2}^{\frac{1}{q}}\left(1+\frac{1}{r}, 1+\frac{1}{r}\right). \tag{20}$$

Corollary 5 *In Corollary 1 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r \in \left(\frac{1}{2}, 1\right]$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$\mathbf{m}$*-*MT*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \le 2L\left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}}\left(\frac{1}{2}\right)^{\frac{1}{rq}} \times\left\{\left[m\beta_{1/2}^{r}\left(1-\frac{1}{2r}, 1+\frac{1}{2r}\right)+\beta_{1/2}^{r}\left(1+\frac{1}{2r}, 1-\frac{1}{2r}\right)\right]^{\frac{1}{rq}} +\left[m\beta_{1/2}^{r}\left(1+\frac{1}{2r}, 1-\frac{1}{2r}\right)+\beta_{1/2}^{r}\left(1-\frac{1}{2r}, 1+\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right\}. \tag{21}$$

Theorem 3 *Let* $h_1, h_2 : [0, 1] \longrightarrow [0, +\infty)$ *and* $\varphi : I \longrightarrow K$ *are continuous functions and* $\mathbf{m} : [0, 1] \longrightarrow (0, 1]$. *Suppose* $K \subseteq \mathbb{R}$ *be an open* $\mathbf{m}$*-invex subset, where* $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) \neq 0$ *and* $\eta(\varphi(y), \mathbf{m}(t)\varphi(x)) \neq 0$ *for all* $t \in [0, 1]$. *Assume that* $f : K \longrightarrow (0, +\infty)$ *is a twice differentiable mapping on* K° *such that* $f'' \in L(K)$. *If* f''^{q} *is generalized relative semi-*$\mathbf{m}$*-*$(r; h_1, h_2)$*-preinvex mapping,*

$0 < r \leq 1$ *and* $q \geq 1$, *then for any* $\alpha > 0$, *the following inequality for fractional integrals hold:*

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right| \leq \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{22}$$

$$\times\left\{\left[\left(f''(x)\right)^{rq} F^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(y)\right)^{rq} F^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}\right.$$

$$+\left[\left(f''(y)\right)^{rq} F^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(x)\right)^{rq} F^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(x)\right)^{rq} G^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(y)\right)^{rq} G^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}$$

$$\left.+\left[\left(f''(y)\right)^{rq} G^r(h_1(\xi); \mathbf{m}(\xi), \alpha, r) + \left(f''(x)\right)^{rq} G^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}\right\},$$

where

$$F(h_1(\xi); \mathbf{m}(\xi), \alpha, r) := \int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\xi^{\alpha+1}h_1^{\frac{1}{r}}(\xi)d\xi;$$

$$F(h_2(\xi); \alpha, r) := \int_0^{\frac{1}{2}} \xi^{\alpha+1}h_2^{\frac{1}{r}}(\xi)d\xi,$$

and

$$G(h_1(\xi); \mathbf{m}(\xi), \alpha, r) := \int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi)(1-\xi)^{\alpha+1}h_1^{\frac{1}{r}}(\xi)d\xi;$$

$$G(h_2(\xi); \alpha, r) := \int_{\frac{1}{2}}^1 (1-\xi)^{\alpha+1}h_2^{\frac{1}{r}}(\xi)d\xi.$$

Proof From Lemma 1, generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity of f''^q, the well-known power mean inequality, Minkowski inequality, and properties of the modulus, we have

$$\left|T_f^{\alpha}(\eta, \varphi, \mathbf{m}; x, y)\right|$$

$$\leq \int_0^{\frac{1}{2}} \xi^{\alpha+1}\big[\left|f''\left(\mathbf{m}(t)\varphi(x) + \xi\eta(\varphi(y), \mathbf{m}(t)\varphi(x))\right)\right|$$

$$+|f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))|\Big]d\xi$$

$$+\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\Big[|f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))|$$

$$+|f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))|\Big]d\xi$$

$$\leq\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}\times\left\{\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}}\right\}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}d\xi\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(x)+\xi\eta(\varphi(y),\mathbf{m}(t)\varphi(x)))\right)^q d\xi\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\left(f''(\mathbf{m}(t)\varphi(y)+\xi\eta(\varphi(x),\mathbf{m}(t)\varphi(y)))\right)^q d\xi\right)^{\frac{1}{q}}\right\}$$

$$\leq\left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}$$

$$\times\left\{\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq}+h_2(\xi)\left(f''(y)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_0^{\frac{1}{2}}\xi^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq}+h_2(\xi)\left(f''(x)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(x)\right)^{rq}+h_2(\xi)\left(f''(y)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}$$

$$\left.+\left(\int_{\frac{1}{2}}^{1}(1-\xi)^{\alpha+1}\Big[\mathbf{m}(\xi)h_1(\xi)\left(f''(y)\right)^{rq}+h_2(\xi)\left(f''(x)\right)^{rq}\Big]^{\frac{1}{r}}d\xi\right)^{\frac{1}{q}}\right\}$$

$$\leq \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\Bigg[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^q \xi^{\alpha+1} h_1^{\frac{1}{r}}(\xi) d\xi\right)^r + \left(\int_0^{\frac{1}{2}} \left(f''(y)\right)^q \xi^{\alpha+1} h_2^{\frac{1}{r}}(\xi) d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[\left(\int_0^{\frac{1}{2}} \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^q \xi^{\alpha+1} h_1^{\frac{1}{r}}(\xi) d\xi\right)^r + \left(\int_0^{\frac{1}{2}} \left(f''(x)\right)^q \xi^{\alpha+1} h_2^{\frac{1}{r}}(\xi) d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[\left(\int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(x)\right)^q (1-\xi)^{\alpha+1} h_1^{\frac{1}{r}}(\xi) d\xi\right)^r$$

$$+\left(\int_{\frac{1}{2}}^1 \left(f''(y)\right)^q (1-\xi)^{\alpha+1} h_2^{\frac{1}{r}}(\xi) d\xi\right)^r\Bigg]^{\frac{1}{rq}}$$

$$+\Bigg[\left(\int_{\frac{1}{2}}^1 \mathbf{m}^{\frac{1}{r}}(\xi)\left(f''(y)\right)^q (1-\xi)^{\alpha+1} h_1^{\frac{1}{r}}(\xi) d\xi\right)^r$$

$$+\left(\int_{\frac{1}{2}}^1 \left(f''(x)\right)^q (1-\xi)^{\alpha+1} h_2^{\frac{1}{r}}(\xi) d\xi\right)^r\Bigg]^{\frac{1}{rq}}\Bigg\}$$

$$= \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}$$

$$\times\Bigg\{\Big[\left(f''(x)\right)^{rq} F^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r) + \left(f''(y)\right)^{rq} F^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} F^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r) + \left(f''(x)\right)^{rq} F^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(x)\right)^{rq} G^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r) + \left(f''(y)\right)^{rq} G^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}$$

$$+\Big[\left(f''(y)\right)^{rq} G^r(h_1(\xi);\mathbf{m}(\xi),\alpha,r) + \left(f''(x)\right)^{rq} G^r(h_2(\xi);\alpha,r)\Big]^{\frac{1}{rq}}\Bigg\}.$$

So, the proof of this theorem is completed.

We point out some special cases of Theorem 3.

Corollary 6 *In Theorem 3, if we take* $\mathbf{m}(\xi) \equiv m \in (0, 1]$ *for all* $\xi \in [0, 1]$, $h_1(t) = h(1-t)$, $h_2(t) = h(t)$ *and* $f''(x) \leq L$, $\forall x \in I$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, h)$*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{23}$$

$$\times\left\{\left[mF^r(h(1-t); \alpha, r) + F^r(h(t); \alpha, r)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[mG^r(h(1-t); \alpha, r) + G^r(h(t); \alpha, r)\right]^{\frac{1}{rq}}\right\}.$$

Corollary 7 *In Corollary 6 for* $h_1(t) = (1-t)^s$ *and* $h_2(t) = t^s$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, s)$*-Breckner-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{24}$$

$$\times\left\{\left[m\left(\frac{r}{(s+r(\alpha+2))2^{\frac{s}{r}+\alpha+2}}\right)^r + \beta_{1/2}^r\left(\alpha+2, 1+\frac{s}{r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m\beta_{1/2}^r\left(\alpha+2, 1+\frac{s}{r}\right) + \left(\frac{r}{(s+r(\alpha+2))2^{\frac{s}{r}+\alpha+2}}\right)^r\right]^{\frac{1}{rq}}\right\}.$$

Corollary 8 *In Corollary 6 for* $h_1(t) = (1-t)^{-s}$ *and* $h_2(t) = t^{-s}$ *and* $0 < s < r$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, s)$*-Godunova–Levin–Dragomir-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{25}$$

$$\times\left\{\left[m\left(\frac{r}{(r(\alpha+2)-s)2^{\alpha+2-\frac{s}{r}}}\right)^r + \beta_{1/2}^r\left(\alpha+2, 1-\frac{s}{r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m\beta_{1/2}^r\left(\alpha+2, 1-\frac{s}{r}\right) + \left(\frac{r}{(r(\alpha+2)-s)2^{\alpha+2-\frac{s}{r}}}\right)^r\right]^{\frac{1}{rq}}\right\}.$$

Corollary 9 *In Theorem 3, if we take* $\mathbf{m}(\xi) \equiv m \in (0, 1]$ *for all* $\xi \in [0, 1]$, $h_1(t) = h_2(t) = t(1-t)$ *and* $f''(x) \leq L$, $\forall x \in I$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$(\mathbf{m}, tgs)$*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 4L(m+1)^{\frac{1}{rq}} \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \times \beta_{1/2}^{\frac{1}{q}}\left(\alpha+2+\frac{1}{r}, 1+\frac{1}{r}\right). \tag{26}$$

Corollary 10 *In Corollary 6 for* $h_1(t) = \dfrac{\sqrt{1-t}}{2\sqrt{t}}$, $h_2(t) = \dfrac{\sqrt{t}}{2\sqrt{1-t}}$ *and* $r \in \left(\frac{1}{2}, 1\right]$, *we get the following Hermite–Hadamard type fractional inequality for generalized relative semi-*$\mathbf{m}$*-*MT*-preinvex mappings*

$$\left|T_f^{\alpha}(\eta, \varphi, m; x, y)\right| \leq 2L\left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}}\left(\frac{1}{2}\right)^{\frac{1}{rq}} \tag{27}$$

$$\times\left\{\left[m\beta_{1/2}^{r}\left(\alpha+2-\frac{1}{2r}, 1+\frac{1}{2r}\right)+\beta_{1/2}^{r}\left(\alpha+2+\frac{1}{2r}, 1-\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right.$$

$$\left.+\left[m\beta_{1/2}^{r}\left(\alpha+2+\frac{1}{2r}, 1-\frac{1}{2r}\right)+\beta_{1/2}^{r}\left(\alpha+2-\frac{1}{2r}, 1+\frac{1}{2r}\right)\right]^{\frac{1}{rq}}\right\}.$$

Remark 6 By applying our Theorems 2 and 3 for $\alpha = 1$, we can deduce some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-$\mathbf{m}$-$(r; h_1, h_2)$-preinvex mappings via classical integrals. The details are left to the interested reader.

3 Applications to Special Means

Definition 12 A function $M : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+$, is called a Mean function if it has the following properties:

1. Homogeneity: $M(ax, ay) = aM(x, y)$, for all $a > 0$,
2. Symmetry: $M(x, y) = M(y, x)$,
3. Reflexivity: $M(x, x) = x$,
4. Monotonicity: If $x \leq x'$ and $y \leq y'$, then $M(x, y) \leq M(x', y')$,
5. Internality: $\min\{x, y\} \leq M(x, y) \leq \max\{x, y\}$.

We consider some means for different positive real numbers α, β.

1. The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}.$$

2. The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}.$$

3. The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}.$$

4. The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \quad r \geq 1.$$

5. The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

6. The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln\beta - \ln\alpha}.$$

7. The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Let us consider continuous functions $\varphi : I \longrightarrow K$, $\eta : K \times K \longrightarrow \mathbb{R}$ and $\overline{M} := M(\varphi(a), \varphi(b)) : [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \times [\varphi(a), \varphi(a) + \eta(\varphi(b), \varphi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above-mentioned means. Therefore one can obtain various inequalities using the results of Sect. 2 for these means as follows. If we take $\mathbf{m}(t) \equiv 1$, $\forall t \in [0, 1]$ and replace $\eta(\varphi(x), \mathbf{m}(t)\varphi(y)) = \eta(\varphi(y), \mathbf{m}(t)\varphi(x)) = M(\varphi(x), \varphi(y))$ for all $x, y \in I$, in (16) and (22), one can obtain the following interesting inequalities involving means:

$$\left|T_f^{\alpha}(\overline{M}, \varphi, 1; a, b)\right| \leq \left(\frac{1}{(p(\alpha+1)+1)2^{p(\alpha+1)+1}}\right)^{\frac{1}{p}} \tag{28}$$

$$\times\left\{\left[\left(f''(a)\right)^{rq} I^r(h_1(\xi); r) + \left(f''(b)\right)^{rq} I^r(h_2(\xi); r)\right]^{\frac{1}{rq}}\right.$$

$$+\left[\left(f''(b)\right)^{rq} I^r(h_1(\xi); r) + \left(f''(a)\right)^{rq} I^r(h_2(\xi); r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(a)\right)^{rq} J^r(h_1(\xi); r) + \left(f''(b)\right)^{rq} J^r(h_2(\xi); r)\right]^{\frac{1}{rq}}$$

$$\left.+\left[\left(f''(b)\right)^{rq} J^r(h_1(\xi); r) + \left(f''(a)\right)^{rq} J^r(h_2(\xi); r)\right]^{\frac{1}{rq}}\right\},$$

$$\left|T_f^{\alpha}(\overline{M}, \varphi, 1; a, b)\right| \leq \left(\frac{1}{(\alpha+2)2^{\alpha+2}}\right)^{1-\frac{1}{q}} \tag{29}$$

$$\times\left\{\left[\left(f''(a)\right)^{rq} F^r(h_1(\xi); \alpha, r) + \left(f''(b)\right)^{rq} F^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}\right.$$

$$+\left[\left(f''(b)\right)^{rq} F^r(h_1(\xi); \alpha, r) + \left(f''(a)\right)^{rq} F^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}$$

$$+\left[\left(f''(a)\right)^{rq} G^r(h_1(\xi); \alpha, r) + \left(f''(b)\right)^{rq} G^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}$$

$$\left.+\left[\left(f''(b)\right)^{rq} G^r(h_1(\xi); \alpha, r) + \left(f''(a)\right)^{rq} G^r(h_2(\xi); \alpha, r)\right]^{\frac{1}{rq}}\right\}.$$

Letting $\overline{M} := A, G, H, P_r, I, L, L_p$ in (28) and (29), we get the inequalities involving means for a particular choices of f''^q that are generalized relative semi-1-$(r; h_1, h_2)$-preinvex mappings.

4 Conclusion

In this article, we first presented a new general fractional integral identity concerning twice differentiable mappings defined on **m**-invex set. By using the notion of generalized relative semi-**m**-$(r; h_1, h_2)$-preinvexity and lemma as an auxiliary result, some new estimates difference between the left and middle part in Hermite–Hadamard type integral inequality associated with twice differentiable generalized relative semi-**m**-$(r; h_1, h_2)$-preinvex mappings are established. It is pointed out that some new special cases are deduced from main results. At the end, some applications to special means for different positive real numbers are provided. Motivated by this interesting class we can indeed see to be vital for fellow researchers and scientists working in the same domain. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite–Hadamard, Ostrowski and Simpson type integral inequalities for various kinds of convex and preinvex functions involving local fractional integrals, fractional integral operators, Caputo k-fractional derivatives, q-calculus, (p, q)-calculus, time scale calculus, and conformable fractional integrals.

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Inequalities in Statistics and Information Measures

Christos P. Kitsos and Thomas L. Toulias

Abstract This paper presents and discusses a number of inequalities in the area of two distinct mathematical branches, with not that different line of thought: Statistics and Mathematical Information, which apply different "measures" to analyze the collected data. In principle, in these two fields, inequalities appear either as bounds in different measures or when different measures are compared. We discuss both and we prove new bounds for the Kullback–Leibler relative entropy measure, when the Generalized Normal distribution is involved.

1 Introduction

Inequalities play an important role in Mathematical Sciences. Provides bounds to the existing calculations, or even to the non-existing ones: we may not know the exact closed expression of a mathematical expression, but it is often possible to know the corresponding boundaries. Typical example is the bound of the n roots of an n-th degree polynomial, say

$$P_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n, \quad x \in \mathbb{C}.$$

Then, for $n > 4$, it is well known that there are no closed algebraic forms describing the root values, but we can have certain bounds for them. Indeed, if

$$A := \max\left\{|a_0|, |a_1|, \ldots, |a_{n-1}|\right\} \text{ and } B := \left\{|a_n|, |a_{n-1}|, \ldots, |a_1|\right\},$$

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then for the k-th root x_k, $k = 1, 2, \ldots, n$, it holds that

$$r := \frac{1}{1 + \frac{B}{|a_0|}} < |x_k| < 1 + \frac{A}{|a_n|} =: R.$$

Therefore, the roots x_k, $k = 1, 2, \ldots, n$, lie within the set-difference of the circles $C(O, R)$ and $C(O, r)$, i.e. $x_k \in C(O, R) \setminus C(O, r)$, $k = 1, 2, \ldots, n$.

Moreover, if the highest-order coefficient of the polynomial P_n as above is non-negative, i.e. $a_n > 0$, and $\delta := \max\left\{|a_k|, k \in \{0, 1, \ldots, n\} : \ a_k < 0\right\}$, then—according to the Lagrange theory for the positive roots of P_n—it holds that

$$0 < x_k \leq 1 + \sqrt[p]{\delta/a_n}, \quad k \in K \subseteq \{1, 2, \ldots, n\},$$

where p declares the position of the highest-order negative coefficient of P_n.

Inequalities appear in almost all the subject fields of Mathematics. The following Sect. 2 presents some classical inequalities in Mathematics, while Sect. 3 demonstrates the importance of inequalities in Statistics. Section 4 discusses certain inequalities that appear in Probability Theory. Section 5 shows some of the most important inequalities in Information Theory, while Sect. 6 briefly introduces the generalized Normal distribution and its relation to a generalized form of the logarithm Sobolev inequality, and to information measures in general. Finally, Sect. 7 proves and discusses some inequalities derived from the study of the information divergence between two generalized forms of the multivariate Normal distribution.

2 Fundamental Inequalities in Mathematics

Some of the main, in our opinion, inequalities widely used in Mathematics are presented in the following.

- *The Cauchy–Schwarz inequality.* Let f and g be two real functions defined on the interval $[a, b]$. Then, their inner product is defined to be

$$\langle f, g \rangle := \int_a^b f(x)\, g(x)\, w(x)\, \mathrm{d}x, \quad w(x) \geq 0.$$

The well-known Cauchy–Schwarz inequality is then formulated as

$$\langle f, g \rangle^2 \leq \langle f, f \rangle \langle g, g \rangle, \ \text{ or } \ \langle f, g \rangle \geq \|f\| \, \|g\|.$$

When f and g assumed to be n-dimensional vectors $\mathbf{a} := (a_i)$, $\mathbf{b} := (b_i) \in \mathbb{R}^n$ and $w \equiv 1$, their inner product is then given by the finite sum $\langle \mathbf{a}, \mathbf{b} \rangle = a_1 b_1 +$

$a_2\, b_2 + \cdots + a_n\, b_n$. As a result, the corresponding Cauchy–Schwarz inequality can then be written as $\big|\langle \mathbf{a}, \mathbf{b}\rangle\big| / (\|\mathbf{a}\|\, \|\mathbf{b}\|) \leq 1$.

- *The determinant inequality.* From Linear Algebra, it is known that the determinant of a square real matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is bounded. Indeed,

$$\frac{n}{\operatorname{tr}(\mathbf{A}^{-1})} \leq (\det \mathbf{A})^{1/n} \leq \tfrac{1}{n}\operatorname{tr}(\mathbf{A}).$$

- *The triangle inequality.* In Euclidian Plane Geometry, for every three non-collinear points A, B, and C, forming the triangle ABC, it holds that $|\overline{AC}| < |\overline{AB}| + |\overline{BC}|$, which is known as the *triangle inequality*. Considering now the Euclidian p-dimensional space, equipped with the usual Euclidian metric/norm, i.e. $\|\mathbf{a}\|^2 := a_1^2 + a_2^2 + \cdots + a_p^2$, $\mathbf{a} = (a_i) \in \mathbb{R}^p$, the triangle inequality holds, formulated as $\|\mathbf{a} + \mathbf{b}\| \leq \|\mathbf{a}\| + \|\mathbf{b}\|$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$. This is also one of the most widely known inequalities in Analytic/Convex Geometry as well as in the study of metric spaces.
- *The Minkowski inequality.* Triangle inequality can be considered as a special case of the Minkowski inequality $\|f + g\|_p \leq \|f\|_p + \|g\|_p$, $f, g \in \mathscr{L}^p(S)$, where S is a metric space with measure μ with $f + g \in \mathscr{L}^p(S)$, and where the p-norm $\|\cdot\|_p$ is defined as $\|f\|_p^p := \int |f|^p\, \mathrm{d}\mu$; see [30] among others. The equality holds for $f := \lambda\, g$, $\lambda \in \mathbb{R}^+$, or when $g \equiv 0$. Finally, if we are considering vectors, the Minkowski inequality is reduced to $\|\mathbf{a}+\mathbf{b}\|_p \leq \|\mathbf{a}\|_p + \|\mathbf{b}\|_p$, $\mathbf{a}, \mathbf{b} \in \mathbb{R}^p$, for the non-Euclidian p-norm $\|\mathbf{a}\|_p^p := |a_1|^p + |a_2|^p + \cdots + |a_p|^p$, $\mathbf{a} = (a_i) \in \mathbb{R}^p$.
- *Factorial bounds.* Two interesting inequalities are known as the lower and upper bounds for the factorial, i.e.

$$\left(\tfrac{n}{\mathrm{e}}\right)^n \sqrt{2\pi\, n} \leq n! \leq \mathrm{e}\left(\tfrac{n}{\mathrm{e}}\right)^n \sqrt{n}, \quad n \in \mathbb{N}, \tag{1}$$

or, generalizing via the Gamma function,

$$\left(\tfrac{x}{\mathrm{e}}\right)^x \sqrt{2\pi\, x} \leq \Gamma(x+1) \leq \mathrm{e}\left(\tfrac{x}{\mathrm{e}}\right)^x \sqrt{x}, \quad x \in \mathbb{R}^+. \tag{2}$$

Recall that the lower boundary of (1) is the well-known Stirling's approximation formula, $n! \overset{\text{asym.}}{\approx} (n/\mathrm{e})^n \sqrt{2\pi\, n}$, meaning that the quantities $n!$ and $(n/\mathrm{e})^n \sqrt{2\pi\, n}$ are asymptotically convergent. Historically speaking, the Stirling's formula was first introduced by Abraham de Moivre in the form of $n! \sim (\text{const.})\, (n/\mathrm{e})^n \sqrt{n}$, and later James Stirling evaluated the constant to be $\sqrt{2\pi}$. Note that the bounds in (1) shall be used later in Sect. 5. More precise bounds introduced by Robbins in [39] were formulated as

$$\mathrm{e}^{\frac{1}{12n+1}} \left(\tfrac{n}{\mathrm{e}}\right)^n \sqrt{2\pi\, n} < n! < \mathrm{e}^{\frac{1}{12n}} \left(\tfrac{n}{\mathrm{e}}\right)^n \sqrt{2\pi\, n}, \quad n \in \mathbb{N}^* := \mathbb{N} \setminus \{0\}. \tag{3}$$

Finally, Srinivasa Ramanujan, in his lost notebook, [36] provided some alternative bounds for the Gamma function, in the form of

$$\left(\tfrac{x}{\mathrm{e}}\right)^x \sqrt{\pi}\,\sqrt[6]{8x^3+4x^2+x+\tfrac{1}{100}} < \Gamma(x+1) < \left(\tfrac{x}{\mathrm{e}}\right)^x \sqrt{\pi}\,\sqrt[6]{8x^3+4x^2+x+\tfrac{1}{30}}, \quad x \in \mathbb{R}^+,$$

while Mortici proved in [33], some even stricter bounds for the Gamma function when $x \geq 8$, i.e.

$$\left(\tfrac{x}{\mathrm{e}}\right)^x \sqrt{\pi}\,\sqrt[6]{8x^3+4x^2+x+\tfrac{1}{30}-\tfrac{11}{240x}} < \Gamma(x+1) < \left(\tfrac{x}{\mathrm{e}}\right)^x \sqrt{\pi}\,\sqrt[6]{8x^3+4x^2+x+\tfrac{1}{30}-\tfrac{1}{24x}},$$

although the lower boundary actually holds for $x \geq 2$.

- *Rayleigh quotient.* Consider the Rayleigh quotient

$$R(\mathbf{A}) = R(\mathbf{A};\ \mathbf{x}) := \frac{\mathbf{x}^{\mathrm{H}}\,\mathbf{A}\,\mathbf{x}}{\mathbf{x}^{\mathrm{H}}\,\mathbf{x}}, \quad \mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\},$$

for the complex Hermitian (or self-adjoint) matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$, i.e. when $\mathbf{A} = \mathbf{A}^{\mathrm{H}}$, where $\mathbf{A}^{\mathrm{H}}$ denotes the *conjugate transpose* of matrix/vector $(a_{ij}) = \mathbf{A} \in \mathbb{C}^{m\times n}$, i.e. $\mathbf{A} = \mathbf{A}^{\mathrm{H}} := \overline{\mathbf{A}^{\mathrm{T}}} = (\overline{a_{ji}})$. For the case of a Hermitian (or real symmetric) matrix $\mathbf{A} \in \mathbb{R}^{n\times n}_{\mathrm{sym}}$, it holds $\mathbf{A} = \mathbf{A}^{\mathrm{T}}$ (symmetricity), while $\lambda_1 = \max_{\mathbf{x}}\{R(\mathbf{x})\}$ and $\lambda_n = \min_{\mathbf{x}}\{R(\mathbf{x})\}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the n real eigenvalues of matrix $\mathbf{A}$.
- *Error bounds.* In all the approximation problems, there are bounds for the existing errors. Practically speaking, for the exact solution $y(x_k)$ at point x_k, the total truncation error ϵ_k is then given by $\epsilon_k = y_k - y(x_k)$, where y_k is the exact value (corresponding to x_k) which would be resulting from an algorithm. We usually calculate some value, say y_k^*, which approximates the exact y_k value, and thus the corresponding rounding error ϵ_k^* is $\epsilon_k^* = y_k^* - y_k$. Therefore, the total error r_k is given by $|r_k| \leq |\epsilon_k| + |\epsilon_k^*|$. Both the forms of truncation error and the propagation error need particular inequalities; see [13, 17].
- *Error control.* When the simultaneous equations $\mathbf{A}\,\mathbf{x} = \mathbf{b}$ are asked to be solved, where $\mathbf{A} \in \mathbb{R}^{m\times n}$, $\det\mathbf{A} \neq \mathbf{0}$, $\mathbf{x}, \mathbf{b} \in \mathbb{R}^{n\times 1}$, errors may occur in both left- and right-hand side. These equations can then be written as $(\mathbf{A} + \delta\,\mathbf{A})\,(\mathbf{x} + \delta\,\mathbf{x}) = \mathbf{b} + \delta\,\mathbf{b}$. Froberg in [13] calculated the relative error $\epsilon_{\mathbf{x}}$ of the solution $\mathbf{x}$ and proved that it is bounded, i.e.

$$\epsilon_{\mathbf{x}} \leq \frac{c}{1 - c\epsilon_{mathbf A}}(\epsilon_{\mathbf{A}} + \epsilon_{\mathbf{b}}),$$

where $c := \mathrm{cond}(\mathbf{A}) = \|\mathbf{A}\|\,\|\mathbf{A}^{-1}\|$ is the conditional number of matrix $\mathbf{A}$, and the corresponding relative errors of $\mathbf{A}$, $\mathbf{x}$, and $\mathbf{b}$ are given, respectively, by $\epsilon_{\mathbf{A}} := \|\delta\,\mathbf{A}\|\big/\|\mathbf{A}\|$, $\epsilon_{\mathbf{x}} := \|\delta\,\mathbf{x}\|\big/\|\mathbf{x}\|$ and $\epsilon_{\mathbf{b}} := \|\delta\,\mathbf{b}\|\big/\|\mathbf{b}\|$. For a number of evaluated bounds in Numerical Analysis, see [17] among others.

- *Stochastic approximation*. For the numerical solution of an equation, there are numerous methods in literature, with the most popular being the Newton–Raphson (and its various forms) and, alternatively, the bisection method (in cases where the differentiation is fairly complicated or not available). The stochastic approximation method, [17], introduced by Robbins and Monro in [40] provides a statistical iterative approach for the solution of $\mathbf{M}(\mathbf{x}) = \boldsymbol{\theta}$, and evaluates maximum or minimum of a function, since the problem cannot adopt the line of thought in [37]. If we assume that an experiment is performed with response y at point x, i.e. $y = Y(x)$, and probability $H(y|x) := \Pr(Y(x) \le y)$ with expected value of random variable (r.v.) X (which measures x) of the form $\mathrm{E}(X) = \int_{\mathbb{R}} y \, \mathrm{d}H(y|x)$, it is then asked to solve the equation $\mathbf{M}(\mathbf{x}) = \boldsymbol{\theta}$. Under a certain number of restrictions (i.e. inequalities), the sequence $x_{n+1} = x_n + a_n(b - y_n)$ converges to x^*, where x^* is a solution of $\mathbf{M}(\mathbf{x}^*) = \boldsymbol{\theta}$, with a_n being an arbitrary sequence of real numbers. Kitsos in [22] applied the method for non-linear models. But why to adopt a Newton–Raphson framework in a statistical point estimation problem, under certain restrictions, and not the bisection method. The answer is that: The bisection approach leads to a (minimax) Decision Theory reasoning, and not to the classical statistical way of thinking; see Theorem 4 in Appendix 2. Stochastic Approximation is a particular method concerning statistical point estimation. Other methods were also developed; see, for example, [32, 55] for methods related to epidemiological problems.

3 Main Inequalities in Statistics

As far as the Statistics is concerned, the inequalities are strongly related to the development of the field. In the following, we present and discuss some widely used inequalities.

- *The Markov inequality*. Let X be a non-negative random variable (r.v.) with finite mean μ. Then, for every non-negative c, it holds

$$\Pr(X \ge c) \le \frac{\mu}{c}.$$

  The extra knowledge of variance results the following:
- *The Chebyshev's inequality*. Let X be an r.v. with given both finite mean μ and finite variance σ^2. Then, for every non-negative c, it holds that

$$\Pr(|X - \mu| \ge c) \le \left(\frac{\sigma}{c}\right)^2.$$

  The well-known Jensen's inequality relates the influence of a convex function when acting on the expected value operator. In particular:

- *The Jensen's inequality*. Let g be a convex function on a convex subset $\Omega \subseteq \mathbb{R}^k$, and suppose that $\Pr(X \in \Omega) = 1$. If the expected value $\mathrm{E}(X)$ of an r.v. X is finite, then $g\big(\mathrm{E}(X)\big) \leq \mathrm{E}\left(g(X)\right)$.

  The Cauchy–Schwartz inequality, mentioned in Sect. 2, is transferred in Statistics as:
- *The Statistical form of the Cauchy–Schwartz inequality*. Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two random vectors of the same dimension such that $\mathrm{E}\left(\|\mathbf{X}_i\|^2\right)$, $i = 1, 2$, are finite. Then,

$$\mathrm{E}\left(\mathbf{X}_1^{\mathrm{T}}\mathbf{X}_2\right) \leq \sqrt{\mathrm{E}\left(\|\mathbf{X}_1\|^2\right)\,\mathrm{E}\left(\|\mathbf{X}_2\|^2\right)}.$$

  The Cauchy–Schwarz inequality provides food for thought on how Mathematics and Statistics communicate. In the following paragraph we discuss the sense of distance from a probabilistic point of view.
- *Distance in Probability Theory*. Let $(\Omega, \mathscr{A}, P)$ be a probability space consisting of the sample space Ω, the σ-algebra of "events" of Ω, and the probability measure P that maps each event to the real interval $[0, 1]$, i.e. $\mathscr{A} \ni A \overset{P}{\mapsto} P(A) \in [0, 1]$. Recall that $\mathscr{A} = \bigcup_{i\in\mathbb{N}} A_i$ with $A_i \bigcap A_j = \varnothing$, $i \neq j$, and $\sum_{i\in\mathbb{N}} P(A_i) = 1$. The *(probability) distance* D between two probability measures P and Q (of the same probability space) is denoted with $D(P, Q)$ and is defined as $D(P, Q) := \sup\left\{\left|P(A) - Q(A)\right|\right\}_{A\in\mathscr{A}}$. Note that the mapping D that assigns a real non-negative number to every pair of probability measures of Ω is—indeed—a distance metric. Furthermore, it is easy to see that $D(P, Q) \in [0, 1]$ for every P and Q, and the following holds.

Proposition 1 *The "exponentiated" distance D^* of a given bounded distance $0 \leq D \leq 1$, i.e. $D^*(P, Q) := \mathrm{e}^{D(P,Q)} - 1$, is also a distance metric.*

See Appendix 1 for the proof, where the exponential inequality $\mathrm{e}^x \geq (1 + x/n)^n$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, was applied. We assume now that for every probability measure P of Ω, i.e. $P \in \mathscr{P}(\Omega)$, there is a σ-finite measure μ such that $P < \mu$ with $P \ll \mu$, i.e. P is absolutely continuous with respect to μ (assuming that $\mathscr{P}$ is countable, μ always exists since μ can be considered as $\mu := \sum_i 2^{-i} P_i$). Then, from the Radon–Nikodym theorem, there exists an integrable function $f : \mathscr{A} \to \mathbb{R}$ such that $P(A) = \int_A f\,\mathrm{d}\mu$, and thus $f := \mathrm{d}P/\mathrm{d}\mu$. Therefore, $D(P, Q) = \int_A |f - g|\,\mathrm{d}\mu$ with $f := \mathrm{d}P/\mathrm{d}\mu$ and $g := \mathrm{d}Q/\mathrm{d}\mu$. It holds, also, that $H^2(P, Q) < D(P, Q)$, $P, Q \in \mathscr{P}(\Omega)$, where H denotes the *Hellinger distance* defined by $H(P, Q)^2 := \int\left(\sqrt{f} - \sqrt{g}\right)^2 \mathrm{d}\mu = 2\big[1 - A(P, Q)\big]$, with $A(P, Q) := \int\sqrt{f\,g}\,\mathrm{d}\mu$ being the *affinity* between probability measures P and Q. This is true, since $H(P, Q)^2 < \int\left(\sqrt{f} - \sqrt{g}\right)\left(\sqrt{f} + \sqrt{g}\right)\mathrm{d}\mu \leq \int|f - g|\,\mathrm{d}\mu \leq D(P, Q)$ for every $P, Q \in \mathscr{P}(\Omega)$. For a study of the Hellinger distance between two generalized normal distributions, see [25].

- *Hypothesis testing for a mean*. In principle, if $\bar{\mathbf{x}} \in \mathbb{R}^{n\times 1}$ is the mean sample of n observations from the multivariate Normal distribution with mean vector

$\boldsymbol{\mu} \in \mathbb{R}^{n\times 1}$ and variance–covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{n\times n}$, the known region

$$n\,(\overline{\mathbf{x}} - \boldsymbol{\mu}_0)^{\mathrm{T}}\,\boldsymbol{\Sigma}^{-1}\,(\overline{\mathbf{x}} - \boldsymbol{\mu}_0) \geq \chi^2_{p,\alpha},$$

is a critical region at the confidence level α for testing the hypothesis $\mathrm{H}: \ \boldsymbol{\mu} = \boldsymbol{\mu}_0$. As far as the confidence intervals are concerned in a Biostatistics level, there are different approaches for the Odds Ratio; see [32, 55]. As we have already mentioned, Statistical Inference is based on point estimation (see [50] for example) as well as on interval estimation. Note that the interval estimation by itself introduces the use of inequalities. The Likelihood method is still valid when the Maximum Likelihood Estimation (MLE), say $\hat{\boldsymbol{\theta}}$, of the unknown parameter vector $\boldsymbol{\theta} = (\theta_i) \in \Theta \subseteq \mathbb{R}^p$, with Θ being the parameter space, is subject to certain restrictions, say $h(\hat{\boldsymbol{\theta}}) = 0$. The well-known Lagrangian method is then applied, i.e.

$$\frac{\partial}{\partial\theta_i}\big[\ell(\boldsymbol{\theta}) - \lambda\, h(\boldsymbol{\theta})\big] = 0,$$

with $\ell(\boldsymbol{\theta})$ being the log-Likelihood function with regard to $\boldsymbol{\theta}$, and $\lambda \in \mathbb{R}$ the Lagrange multiplier. In such a case, still the estimate $\hat{\boldsymbol{\theta}}$ follows the (multivariate) Normal distribution with mean $\boldsymbol{\mu} = \boldsymbol{\theta}$ and the asymptotic variance–covariance matrix $\boldsymbol{\Sigma} = n\,\mathbf{I}^{-1}(\boldsymbol{\theta})$, i.e. $\hat{\boldsymbol{\theta}} \sim \mathscr{N}\big(\boldsymbol{\theta}, n\,\mathbf{I}^{-1}(\boldsymbol{\theta})\big)$, where $\mathbf{I} \in \mathbb{R}^{p\times p}$ denotes the Fisher's information matrix; see the early work of Silvey in [44] among others. Moreover, Anderson in [1] discussed a number of confidence intervals concerning Multivariate Statistics, Ferguson in [9] considered a Decision Theory point of view, while Fortuin et al. in [11] focused on a particular inequality problem.

Example 1 Let us consider the vector of n observations $\mathbf{X} = (x_1, x_2, \ldots, x_k)$ which follows the k-th degree multinomial distribution, i.e.

$$p(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1!\,x_2!\,\ldots\,x_k!}\theta_1^{x_1}\,\theta_2^{x_2}\,\ldots\,\theta_k^{x_k}, \ \text{ with } \sum_{i=1}^{k} x_i = n \ \text{ and } \sum_{i=1}^{k}\theta_i = 1,$$

while θ_i, $i = 1, 2, \ldots, k$, denote the involved parameters. Following, therefore, the typical procedure for the evaluation of the log-Likelihood under the restriction $h(\boldsymbol{\theta}) := \big(\sum\theta_i\big) - 1 = 0$, we can evaluate the expected value, variance, and covariance as

$$\mathrm{E}(x_i) = n\,\theta_i, \ \ \mathrm{Var}(x_i) = n\,\theta_1\,(1-\theta_i), \ \text{ and } \ \mathrm{Cov}(x_i, x_j) = -n\,\theta_i\,\theta_j, \ \ i \neq j,$$

and hence, the inverse of the Fisher's information matrix $\mathbf{I}^{-1}(\boldsymbol{\theta})$ is the variance–covariance matrix with elements $\big(I^{-1}\big)_{ii}(\boldsymbol{\theta}) = n^{-1}\,\theta_i(1-\theta_i)$, $i = 1, 2, \ldots, k$, and $\big(I^{-1}\big)_{ij}(\boldsymbol{\theta}) = n^{-1}\,\theta_i\,\theta_j$, $i \neq j = 1, 2, \ldots, k$.

Let $\mathbf{c}$ be now an appropriate constant vector for an approximate $(1-\alpha)\cdot 100\%$. The confidence interval for $\mathbf{c}^{\mathrm{T}}\hat{\boldsymbol{\theta}}$ is defined to be the real interval $\mathrm{CI}\big(\mathbf{c}^{\mathrm{T}}\hat{\boldsymbol{\theta}}\big) := \Big(\mathbf{c}^{\mathrm{T}}\hat{\boldsymbol{\theta}} - K_{\alpha/2}\big[\mathbf{c}^{\mathrm{T}}\mathbf{I}^{-1}\big(\hat{\boldsymbol{\theta}}\big)\,\mathbf{c}\big],\ \mathbf{c}^{\mathrm{T}}\hat{\boldsymbol{\theta}} + K_{\alpha/2}\big[\mathbf{c}^{\mathrm{T}}\mathbf{I}^{-1}\big(\hat{\boldsymbol{\theta}}\big)\,\mathbf{c}\big]\Big)$ with $\hat{\boldsymbol{\theta}}$ being an estimate of $\boldsymbol{\theta}$, and $K_{\alpha/2}$ the appropriate value for either standard Normal or t-distribution. Recalling the previous Example 1, notice that, although we assumed a multinomial distribution, the common marginal distribution of two components, say x_p and x_q, is a trinomial one with $x_p + x_q \leq n$, $1 \leq p, q \leq k$, $p \neq q$, while the probability distribution of $x_p + x_q = \xi$, $\xi = 0, 1, \ldots, n$, is binomial, since it is the probability distribution of x_i, $i = 1, 2, \ldots, k$, with different parameters; see also [32] for a special case in epidemiology. Notice also that the components of the corresponding Fisher's information matrix, as in Example 1, are non-linear functions of the unknown parameter vector $\boldsymbol{\theta}$. This creates a real problem regarding the calculations.

- *Sequential Probability Ratio Test (SPRT).* The pioneering work of Wald in [52] was based on changing the probability ratio test; see also [53]. The fundamental difference is that now there are three regions testing two simple hypothesis $\mathrm{H}_0 : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ vs. $\mathrm{H}_1 : \boldsymbol{\theta} = \boldsymbol{\theta}_1$, $\boldsymbol{\theta}_0 \neq \boldsymbol{\theta}_1$, there is a "continuation region" and the sample size is not fixed anymore but a random variable, say n, such that $\Pr(n < \infty \mid \boldsymbol{\theta}) = 1$. Moreover, the expected value $\mathrm{E}(n;\ \boldsymbol{\theta})$ exists and certain bounds for this were derived; see [14] for details, while when the average sample size is less than the appropriate sample size in a random sample see [54]. Usually, we denote the Operating Character (OC) function as $Q(\boldsymbol{\theta})$ and the power function as $R(\boldsymbol{\theta})\ \big(:= 1 - Q(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in \Theta\big)$. For given confidence levels, say α and β, for the above defined test, it is required that $Q(\boldsymbol{\theta}_0) \geq 1-\alpha$ and $Q(\boldsymbol{\theta}_1) \leq \beta$. Then, the logarithm of the probability ratio test at stage n is defined as

$$Z_n := \ln \frac{f_n(\mathbf{x}_n;\ \boldsymbol{\theta}_1)}{f_n(\mathbf{x}_n;\ \boldsymbol{\theta}_0)}, \quad n \geq 1, \quad \mathbf{x}_n = (x_1, x_2, \ldots, x_n).$$

Based on the SPRT, when two given numbers act as stopping bounds (B, A) with $-\infty < B < A < +\infty$, these numbers are defined through the decision rule:

1. Accept H_0 if $Z_n \leq B$,
2. Reject H_0 if $Z_n \geq A$, and
3. Continue by examining $\mathbf{x}_{n+1}$, i.e. $B < Z_{n+1} < A$.

The inequality $B < Z_{n+1} < A$ is known as the *critical inequality* and the test is denoted by $S(B, A)$. Following Ghosh in [14, Th. 3.2], the following is true.

Theorem 1 *The risk errors $\alpha(\boldsymbol{\theta}_0)$ and $\beta(\boldsymbol{\theta}_1)$ associated with the SPRT $S(B, A)$ for* $\mathrm{H} : \boldsymbol{\theta} = \boldsymbol{\theta}_0$ *vs.* $\mathrm{H} : \boldsymbol{\theta} = \boldsymbol{\theta}_1$*, with $B < A$ being any choice of stopping bounds, then the following inequalities hold:*

$$\ln \frac{\beta(\boldsymbol{\theta}_1)}{1-\alpha(\boldsymbol{\theta}_0)} \leq \min\{0, B\}, \quad \ln \frac{1-\beta(\boldsymbol{\theta}_1)}{\alpha(\boldsymbol{\theta}_0)} \geq \max\{0, A\}.$$

However, the optimum bounds, say (B^*, A^*) have not evaluated and, therefore, the pair $(\alpha, \beta), \alpha+\beta < 1$, the optimum bounds can be approximated by $B^* \approx \ln \beta/(1-\alpha)$ and $A^* \approx \ln(1-\beta)/\alpha$.

Example 2 Let $x_1, x_2, \ldots$ be some Bernoulli variables regarding the SPRT with p being the proportion of successes, i.e. $\mathrm{H}_0 : \ p \le p_0$ vs. $\mathrm{H}_1 : \ p \ge p_1$, with $0 \le p_0 < p_1 \le 1$. For each observation x_i it is $Z_i = \ln\left\{(1-p_1)/(1-p_0)\right\} + x_i \ln\left\{p_1(1-p_0)\big/[p_0(1-p_1)]\right\}$. As $Z_n = \prod_{i=1}^n z_i$ and $X_n = \sum_{i=1}^n x_i$ then the critical inequality for $S(B^*, a^*)$ is reduced to $\mathrm{K} + \Lambda\, n < X_n < \mathrm{M} + \Lambda\, n$ where

$$\mathrm{K} := \frac{B^*}{\ln \frac{p_1(1-p_0)}{p_0(1-p_1)}}, \quad \Lambda := \frac{\ln \frac{1-p_0}{1-p_1}}{\ln \frac{p_1(1-p_0)}{p_0(1-p_1)}}, \quad \text{and } \mathrm{M} := \frac{A^*}{\ln \frac{p_1(1-p_0)}{p_0(1-p_1)}},$$

see [14]. Moreover, the value $\mathrm{E}(n;\ p)$ is also bounded. In particular, $S \le \mathrm{E}(n;\ p) \le T$, where

$$S := \frac{Q(p)\left(\ln\frac{1-p_1}{1-p_0} + B - A\right) + A}{p\,\ln\frac{p_1(1-p_0)}{p_0(1-p_1)} + \ln\frac{1-p_1}{1-p_0}} \text{ and } T := \frac{Q(p)\left(\ln\frac{p_0}{p_1} + B - A\right) + \ln\frac{p_1}{p_0} + A}{p\,\ln\frac{p_1(1-p_0)}{p_0(1-p_1)} + \ln\frac{1-p_1}{1-p_0}}.$$

- *Sequential design methods*. The sequential methods are the key for testing more than two hypotheses. Moreover, they are related to decision problems; see [41]. The inequalities involved to the Decision Theory, their links to the Bayesian Decision Theory and the evaluated risks are presented in a compact form by [41, Ch. 3]. The sequential way of thinking has been adopted by Kitsos in [22, 23] as well by Ford et al. in [10] with regard to optimal non-linear Design Theory. Moreover, Kitsos proved in [23] that when the initial design is D-optimal, [43], and a stochastic approximation scheme is used, then the limiting design is also D-optimal (and hence G-optimal due to the Kiefer's Equivalence Theorem). The main results of Wynn in [57, 58] rule the sequential design approach. The link between the optimal Design Theory and the moment inequalities was investigated by Torsney in [48], where Hölder's and Minkowski's inequalities were also discussed. If ξ denotes a design measure, [43], and $\mathbf{M}$ is the average-per-observation information matrix $\mathbf{M} = n^{-1}\mathbf{I}$, then it can be written as $\mathbf{M}(\xi) = n^{-1}\mathbf{I}(\xi)$ for the linear case, and $\mathbf{M}(\boldsymbol{\theta}, \xi) = n^{-1}\mathbf{I}(\boldsymbol{\theta}, \xi)$ for the non-linear case, where matrix $\mathbf{I}$ is the Fisher's information matrix; see [10]. In linear theory, it has been proved in [56] that $\mathbf{M}(\xi_n) \to \mathbf{M}(\xi^*)$ when $\xi_n \to \xi^*$, i.e. when a sequence of design measures converges to the optimum design, then the corresponding measures of information "follow" the scheme. That is, when we are not at the limit, inequalities are hold. This result is similar to the Dominated converge principle for a sequence of integrable functions, say u_n converging to u, provided that an integrable function w such that $|u_n| \le w$ exist, then u is also integrable and $\mathrm{E}(u_n) \to \mathrm{E}(u)$. However, this is not true for the non-linear case: there is no limiting result for $\mathbf{M}(\xi_n, \boldsymbol{\theta})$ or $\mathbf{M}(\xi_n, \boldsymbol{\theta}_n)$. Moreover, in Design Theory there is not a similar to the Fatou's Lemma that $\mathrm{E}(\lim_{n\to\infty} u_n) \leq \lim_{n\to\infty} \mathrm{E}(u_n)$, $u_n \geq 0$. In particular, $\mathrm{E}(u) \leq \lim_{n\to\infty} \mathrm{E}(u_n)$ when $u_n \to u$.

- *Linear programming.* As far as the linear programming is concerned, the Simplex method solves linear inequalities problems, such as: evaluate max $\{y = f(\mathbf{x})\}$, $\mathbf{x} \in \mathbb{R}^p$, under $\mathbf{A}\,\mathbf{x}^{\mathrm{T}} \leq \mathbf{b}^{\mathrm{T}}$, where $\mathbf{A} \in \mathbb{R}^{p\times p}$ and $\mathbf{b} \in \mathbb{R}^p$ are known. Adding the so-called slack variables the inequalities are eventually transformed into equalities.

4 Inequalities in Probability Theory

In this section we present some essential inequalities used in Probability Theory, in order to clarify the importance of these inequalities to all the fields of Statistics.

- *Renewal Theory*. From the Renewal Theory [20], consider the elapsed number of generation, say $T(0)$, known also as a generation of equal components. Then for a finite population of constant size N, it can be proved that $\mathrm{E}\left(T(0)\right) \leq N^N$.
- *Doob's martingale*. Recall that a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ is called a *martingale* with respect to $\{Y_n\}_{n\in\mathbb{N}}$ if $\mathrm{E}\left\{|Y_n|\right\} < \infty$ and $\mathrm{E}\left(X_{n+1}(Y_0, Y_1, \ldots, Y_n)\right) = X_n$, $n \in \mathbb{N}$. In such a case, the "existing history" determines x_n in terms that, eventually, $\mathrm{E}(X_n) = \mathrm{E}(X_{n+1}| \;\; Y_0, Y_1, \ldots, Y_n) = \mathrm{E}(X_0)$ for every $n \in \mathbb{N}$. As far as the Doob's Martingale Process is concerned, the inequality is requested in its definition, as well as for the Radon–Nikodym derivatives; see [8]. Indeed, for a given r.v. X with $\mathrm{E}(|X|) < \infty$, and for an ordinary sequence of r.v.-s, say $Y_0, Y_1, \ldots, Y_n$, then from $X_n := \mathrm{E}(X \mid Y_0, Y_1, \ldots, Y_n)$, $n \in \mathbb{N}$, a martingale structure $\{X_n\}$ with respect to $\{Y_n\}$, is obtained when $\mathrm{E}(|X_n|) \leq \mathrm{E}(|X|) < \infty$ and $\mathrm{E}(X_{n+1} \mid Y_0, Y_1, \ldots, Y_n) = X_n$, known as Doob's process. Suppose, now, that U is a uniformly distributed r.v. on $[0,\ 1]$. We define $Y_n = k/2^n$, $k = k(n, U)$, unique such that $k/2^n \leq U \leq (k+1)/2^n$. Then, process $\{X_n\}$ defined as $X_n := 2^n\left[g(Y_n + 2^{-n}) - g(Y_n)\right]$ for $g|_{[0,\,1]}$ bounded forms a martingale; see [20]. Moreover, the sequence X_n is known as the Radon–Nikodym derivative of g evaluated at U.
- *Crossing inequality*. One of the well-known inequalities in Stochastic Process Theory, strongly related to Sequential Analysis, is the so-called Crossing Inequality. It counts the number of times a sub-martingale $\{X_n\}$, with respect to a sequence $\{Y_n\}$, crosses a given interval $(a,\, b) \subseteq \mathbb{R}$. That is, the number of crosses, say $N_{a,b}$, from the level below a to a level above b. In fact, $N_{a,b}$ is the number of pairs (i, j) such that $X_i \leq a$ and $X_j \geq b$ with $a < X_k < b$, $0 \leq i < j \leq N_j$, $i < k < j$. For sub-martingales $\{X_n\}$ with given T and T' Markov times and $q \in Z$ with $0 \leq T \leq T' \leq q$, then $\mathrm{E}(X_T) \leq \mathrm{E}(X_{T'})$. The Crossing Inequality is then formulated by $\mathrm{E}(N_{a,b}) \leq \left(\mathrm{E}\left[(X_N - a)^+\right] - \mathrm{E}\left[(X_0 - a)^+\right]\right)/(b-a)$. For the *backward* martingale $\{X_n\}_{n=0,-1,-2,\ldots}$ with respect to a σ-field $\mathscr{F}_n$, $n = 0, -1, -2, \ldots$ (generated by some jointly distributed r.v.-s), the Crossing Inequality is reduced to $\mathrm{E}(N_{a,b}) \leq \mathrm{E}\left[(X_0 - a)^+\right]/(b-a)$ with the

only new restriction $N \le i < j \le 0$. For a given martingale $\{X_n\}$ satisfying $E\big(|X_n|^k\big) < \infty$ for every $k > 1$ and $n \in \mathbb{N}$, it can be proved that

$$\mathrm{E}\left(\max_{0\le r\le n}\big\{|X_r|\big\}\right) \le \tfrac{k}{k-1}\ \mathrm{E}\big(|X_k|^k\big)^{1/k} \ \text{ and }\ \mathrm{E}\left(\max_{0\le r\le n}\big\{|X_r|^k\big\}\right) \le \big(\tfrac{k}{k-1}\big)^k\ \mathrm{E}\left(|X_n|^k\right),$$

see [20] for details. When restrictions are imposed to expected value and variance, i.e. $\mathrm{E}(X_n) = 0$ and hence $\sigma^2 = \mathrm{E}(X_n^2) < \infty$ for every n, then

$$\Pr\left(\max_{0\le r\le n}\big\{|X_r|\big\} > k\right) \le \tfrac{\sigma^2}{\sigma^2+k}, \quad k > 0.$$

- *Chebyshev's and Kolmogorov's inequalities*. Chebyshev's Inequality provides food for thought when an extension, known as the Kolmogorov's Inequality, is considered. For given two independent and identically distributed (i.i.e.) r.v-s $X_1, X_2, \dots,$ with mean $\mu = \mathrm{E}(X_i) = 0$ and variance $\sigma^2 = \mathrm{E}(X_i^2) < \infty$, $i = 1, 2, \dots,$ we define $S_n = X_1 + X_2 + \cdots + X_n$, $n = 1, 2, \dots,$ and $S_0 = 0$. Then, Chebyshev's Inequality is formulated by

$$\epsilon^2\ \Pr\left(|S_n| > \epsilon\right) \le n\,\sigma^2 = \mathrm{Var}(S_n),$$

while Kolmogorov's Inequality is written as

$$\epsilon^2\ \Pr\left(\max_{k\le n}\big\{|S_k|\big\} > \epsilon\right) \le n\,\sigma^2 = \mathrm{Var}(S_n).$$

- *Maximal inequalities*. A number of inequalities are based on Kolmogorov's Inequality for the (sub-)martingales, and are known as the Maximal Inequalities; see [8, 20].

1. Let $\{X_n\}$ be a martingale and $k \ge 0$. Then, $k\ \Pr\left(\max_{0\le r\le n}\big\{|X_r|\big\} > k\right) \le \mathrm{E}\left(\big\{|X_n|\big\}\right)$.
2. Let $\{X_n\}$ be a sub-martingale with $X_n \ge 0$, $n \in \mathbb{N}$, and $k \ge 0$. Then, $k\ \Pr\left(\max_{0\le r\le n}\{X_r\} > k\right) \le \mathrm{E}\left(\{X_n\}\right)$.
3. When $\{X_n\}_{n=0,-1,-2,\dots}$ is a backward martingale with respect to a σ-field, say $\mathscr{F}_n$, $n = 0, -1, -1, \dots,$ generated by some jointly distributed r.v.-s $\{Y_n, Y_{n-1}, \dots\}$, then $k\ \Pr\left(\max_{0\le r\le n}\{X_r\} > k\right) \le \mathrm{E}\left(\{X_0\}\right)$.
4. Let $\{X_n\}$ be a martingale. Then, $k\ \Pr\left(\min_{0\le r\le n}\{X_r\} < -k\right) \le \mathrm{E}\left(\{X_n^+\} - \mathrm{E}(X_0)\right)$.
5. Let $\{X_n\}$ be a super-martingale with $X_n \ge 0$, $n \in \mathbb{N}$. In such a case, $k\ \Pr\left(\max_{0\le r\le n}\{X_r\} \ge k\right) \le \mathrm{E}(X_0)\big)$.

The above inequalities from the Probability Theory provide evidence of how really useful inequalities can be, in terms of offering bounds, for most of the involved "sequences," such as martingales. In the next section, we present the existence of certain bounds related to information measures.

- *Distance in navigation*. Franceschetti and Meester in [12], working in similar line of thought as in [42] and [35], consider the Euclidian distance between a source point and a target, in navigation in random networks, and presented a number of interesting inequalities for the ϵ-delivery time of a decentralized algorithm. This refers to the number of steps required for the message, originating at point s to reach an ϵ-neighborhood of point t. Moreover, working on network topology, they introduced a new distance measure, the chemical distance between two points x and y (and by considering the existence of a path connecting x with y), with a number of inequalities obtained through Probability Theory: for a random grid and given points x and y, probability assigned to be 1 if $|x-y|=1$, and $1-\exp\big(-\beta/|x-y|^a\big)$ if $|x-y|>1$, $a,b>0$. Their results are related to the percolation models; see [18, 31]. Although the evolution of ideas from Shannon's work in [42] to Navigation in Random Networks is important, it has attracted the interest of Engineers rather than Mathematicians, as the former pay more attention to the information flow in random networks; see [45]. We present here an important—in our opinion—inequality related to the Phase Transition: There is an interest to express positive correlations between increasing events, say A and B, so that $\Pr(A\cap B)\geq\Pr(A)\,\Pr(B)$; see [18, 31]. Then, for increasing events $A_1, A_2, \dots, A_n$, all having the same probability, it holds that

$$1-\left[1-\Pr\left(\bigcup_{i=1}^{n}A_i\right)\right]^{1/n}\leq\Pr(A_1).$$

Indeed, due to $\Pr(A\cap B)\geq\Pr(A)\,\Pr(B)$ and some set-theoretic algebra,

$$1-\Pr\left(\bigcup_{i=1}^{n}A_i\right)=\Pr\left(\bigcap_{i=1}^{n}A_i\right)\geq\prod_{i=1}^{n}\Pr\left(A_i^{\mathrm{c}}\right)=\left[\Pr\left(A_i^{\mathrm{c}}\right)\right]^n=\left[1-\Pr(A_1)\right]^n,$$

since we assumed that $\Pr(A_i)=\Pr(A_j)$, $i\neq j=1,2,\dots,n$.

5 Information Measures and Inequalities

In the following we shall try to investigate certain bounds concerning generalized entropy type information measures from the Information Theory.

New entropy type information measures were introduced in [24], generalizing the known Fisher's entropy type information measure; see also [5, 26–29, 49]. The introduced new entropy type measure of information $J_\alpha(X)$ is a function of the density f of the p-variate random variable r.v. X defined as, [24],

$$J_\alpha(X):=\mathrm{E}\left(\big\|\nabla\log f(X)\big\|^\alpha\right)=\int_{\mathbb{R}^p}f(x)\big\|\nabla\log f(x)\big\|^\alpha\,\mathrm{d}x,\quad\alpha>1,\tag{4}$$

where $\|\cdot\|$ is the usual two-norm of $\mathscr{L}^2(\mathbb{R}^p)$. Notice that $J_2 = J$, with J being the known Fisher's entropy type information measure.

In his pioneering work [42], Shannon introduced the notion of Entropy in an Information Theory context giving a new perspective to the study of Information Systems, Signal Processing and Cryptography among other fields of application. *Shannon entropy*, or *differential entropy*, denoted by $H(X)$, measures the average uncertainty of an r.v. X and is given by

$$H(X) := -\operatorname{E}\big(\log f(X)\big) = -\int_{\mathbb{R}^p} f(x)\log f(x)\,\mathrm{d}x, \tag{5}$$

with f being the probability density function (p.d.f.) of r.v. X; see [6, 42]. In Information Theory, it is the minimum number of bits required, on the average, to describe the value x of the r.v. X. In Cryptography, entropy gives the ultimately achievable error-free compression in terms of the average codeword length symbol per source; see [21] among others.

For the Shannon entropy $H(X)$ of any multivariate r.v. X with zero mean vector and covariance matrix $\boldsymbol{\Sigma}$, an upper bound exists,

$$H(X) \le \tfrac{1}{2}\log\big\{(2\pi\,\mathrm{e})^p|\det\boldsymbol{\Sigma}|\big\}, \tag{6}$$

where the equality holds if and only if X is a normally distributed r.v., i.e. $X \sim \mathscr{N}(\mathbf{0},\boldsymbol{\Sigma})$; see [6]. Note that the Normal distribution is usually adopted as the description variable for noise, and acts additively to the input variable when an input–output discrete time channel is formed. The known entropy power, dented by $N(X)$, and defined through the Shannon entropy $H(X)$, has been extended to

$$N_\alpha(X) := \nu_\alpha \exp\Big\{\tfrac{\alpha}{p}\mathrm{H}(X)\Big\}, \tag{7}$$

where

$$\nu_\alpha := \big(\tfrac{\alpha-1}{\mathrm{e}}\big)\pi^{-\alpha/2}\left[\frac{\Gamma\left(\frac{p}{2}+1\right)}{\Gamma\left(p\frac{\alpha-1}{\alpha}+1\right)}\right]^{\frac{\alpha}{p}}, \quad \alpha > 1, \tag{8}$$

see [24] for details. Notice that $\nu_2 = (2\pi e)^{-1}$ and hence $N_2 = N$. It can be proved that [24],

$$J_\alpha(X)\,N_\alpha(X) \ge p, \quad \alpha > 1, \tag{9}$$

which extends the well-known Information Inequality, i.e. $J(X)\,N(X) \ge p$, obtained from (9) by setting $\alpha := 2$.

The so-called Cramér–Rao Inequality, [6, Th. 11.10.1], is generalized due to the introduced information measures, [24], and is given by

$$\sqrt{\tfrac{2\pi\,\mathrm{e}}{p}\,\mathrm{Var}(X)}\Big[\tfrac{\nu_\alpha}{p}\,J_\alpha(X)\Big]^{1/\alpha}\geq 1,\quad \alpha>1. \tag{10}$$

When $\alpha := 2$ we have $\mathrm{Var}(X)\,J_2(X)\geq p^2$, which is the known Cramér–Rao inequality, $\mathrm{Var}(X)\,J(X)\geq 1$ for the univariate case. The lower boundary B_α for the introduced generalized information $J_\alpha(X)$ is then

$$\tfrac{p}{\nu_\alpha}\left[\tfrac{2\pi\,\mathrm{e}}{p}\,\mathrm{Var}(X)\right]^{-\alpha/2} =: B_\alpha\leq J_\alpha(X),\quad \alpha>1. \tag{11}$$

Finally, the classical Entropy Inequality,

$$\mathrm{Var}(X)\geq p\,N(X)=\tfrac{p}{2\pi\,\mathrm{e}}\exp\left\{\tfrac{2}{p}\,H(X)\right\}, \tag{12}$$

can be extended, adopting the extended entropy power as in (7), to the general form

$$\mathrm{Var}(X)\geq \tfrac{p}{2\pi\,\mathrm{e}}\nu_\alpha^{-2/\alpha}\,N_\alpha^{2/\alpha}(X),\quad \alpha>1. \tag{13}$$

Under the "normal" parameter value $\alpha := 2$, inequality (13) is reduced to (12).

The Blachman–Stam Inequality [2, 3, 47] is generalized through the generalized J_α measure. Indeed: For given two independent r.v.-s X and Y of the same dimension, it holds

$$J_\alpha\left(\lambda^{1/\alpha}X+(1-\lambda)^{1/\alpha}Y\right)\leq \lambda J_\alpha(X)+(1-\lambda)J_\alpha(Y),\quad \lambda\in(0,1),$$

where the equality holds for X and Y normally distributed r.v.-s with the same covariance matrix; see [26] for the proof. For parameter value $\alpha := 2$ we are reduced to the well-known Blachman–Stam Inequality, since $J_2 = J$.

Let now $X_1, X_2, \ldots, X_n$ be some n independent and identically distributed (i.i.d.) univariate random variables with mean 0 and variance σ^2, having density function $f(x)$ satisfying Poincaré conditions with finite restricted Poincaré constant c_P. If $\phi(x)$ denotes the corresponding probability density of $\mathscr{N}(0,\sigma^2)$, then the *Fisher's information distance* (or *standardized information*) of some univariate r.v. X (with mean 0 and variance σ^2) is defined to be

$$J_\phi(X) := \sigma^2\ \mathrm{E}\left[\tfrac{\mathrm{d}}{\mathrm{d}x}\log f(X)-\tfrac{\mathrm{d}}{\mathrm{d}x}\log\phi(X)\right]^2=\sigma^2\,J(X)-1,$$

with J being the known Fisher's (entropy type) information. Notice that $J_\phi(\lambda\,X) = J_\phi(X)$, so J_ϕ is scale invariant and, moreover, provides a measure of distance of "how far $f(x)$ is from normality," i.e. from $\phi(x)$. Then, for the sum $Y_n := \left(\sqrt{n}\sigma\right)^{-1}\sum_{i=1}^n X_i$, it can be proved that for every n

$$J(Y_n) = \frac{2c_P}{2c_P + (n-1)\sigma^2} J(X_1).$$

Moreover, if $\phi(x)$ represents the probability density of the standard Normal distribution, then it holds that

$$\sup_{x\in\mathbb{R}} \{|f(x)-\phi(x)|\} \leq (1+\sqrt{\sigma}/\pi)\sqrt{J(X)}, \quad \int_{\mathbb{R}} |f(x)-\phi(x)|\,\mathrm{d}x \leq 2H(f,\phi) \leq \sqrt{2J(X)},$$

with $H^2(f,\phi) := \int \left|\sqrt{f(x)} - \sqrt{\phi(x)}\right|^2 \mathrm{d}x$ being the *Hellinger distance* between densities f and ϕ; see [19] for details.

6 The Generalized Normal (GN) Distribution

The Logarithmic Sobolev Inequalities (LSI) attempt to estimate the lower-order derivatives of a given function in terms of higher-order derivatives. The well-known LSI was introduced in 1938 and translated in English 1963 as appeared in [46]; see also [16, 26] for details. The introductory and well-known Sobolev Inequality (SI) is of the form

$$\left(\int_{\mathbb{R}^p} |f(x)|^{\frac{2p}{p-2}}\,\mathrm{d}x \right)^{\frac{p-2}{2p}} \leq c_S \left(\int_{\mathbb{R}^p} |\nabla f(x)|^2\,\mathrm{d}x \right)^{\frac{1}{2}}, \tag{14}$$

or, using the two-norm notation, $\|f\|_q \leq c_S \|\nabla f\|_2$, with the constant $c_S > 0$ is known as the *Sobolev constant*.

Kitsos and Tavoularis [24] introduced and studied an exponential-power generalized form of the multivariate Normal distribution, denoted as $\mathscr{N}_\gamma(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$, called the γ-order Generalized Normal (γ-GN) distribution; see also [27, 28] for further reading. The derivation of this three-parameter extended Normal distribution came up an extremal of a generalized Euclidian LSI introduced by Del Pino et al. in [7], which can be written as

$$\int_{\mathbb{R}^p} |u|^\gamma \log|u|\,\mathrm{d}\mathbf{x} \leq \tfrac{p}{\gamma^2} \log\left\{ K_\gamma \int_{\mathbb{R}^p} |\nabla u|^\gamma\,\mathrm{d}\mathbf{x} \right\}, \tag{15}$$

where $u = u(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^p$, belongs to the Sobolev space $H^{1/2}(\mathbb{R}^p)$ with $\|u\|_\gamma = \int_{\mathbb{R}^p} |g(\mathbf{x})|^\gamma\,\mathrm{d}\mathbf{x} = 1$. The optimal constant K_γ is being equal to

$$K_\gamma := \tfrac{\gamma}{p}\left(\tfrac{\gamma-1}{\mathrm{e}}\right)^{\gamma-1} \pi^{-\gamma/2} \left[\frac{\Gamma(\frac{p}{2}+1)}{\Gamma\left(p\frac{\gamma-1}{\gamma}+1\right)} \right]^{\gamma/p}. \tag{16}$$

The equality in (15) holds, [24], when u is considered to be the p.d.f. of an r.v. X following γ-GN distribution as defined below.

Definition 1 The p-variate random variable X follows the γ*-order generalized Normal* (γ-GN) distribution, i.e. $X \sim \mathscr{N}_\gamma(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, with location parameter vector $\boldsymbol{\mu} \in \mathbb{R}^p$, shape parameter $\gamma \in \mathbb{R} \setminus [0, 1]$, and positive definite scale parameter matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$, when the density function f_X of X is of the form

$$f_X(\mathbf{x}) = f_X(\mathbf{x};\ \boldsymbol{\mu}, \boldsymbol{\Sigma}, \gamma, p) := C(\boldsymbol{\Sigma}) \exp\left\{-\tfrac{\gamma-1}{\gamma} Q(\mathbf{x})^{\frac{\gamma}{2(\gamma-1)}}\right\}, \quad \mathbf{x} \in \mathbb{R}^p, \tag{17}$$

where Q is the p-quadratic form $Q(\mathbf{x}) = Q(\mathbf{x};\ \boldsymbol{\mu}, \boldsymbol{\Sigma}) := (\mathbf{x} - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})^{\mathrm{T}}$, $\mathbf{x} \in \mathbb{R}^p$, while the normalizing factor C is defined as

$$C(\boldsymbol{\Sigma}) = C(\boldsymbol{\Sigma};\ \gamma, p) := \frac{\Gamma\left(\frac{p}{2} + 1\right)}{\pi^{p/2}\,\Gamma\left(p\frac{\gamma-1}{\gamma} + 1\right)\sqrt{|\boldsymbol{\Sigma}|}} \left(\tfrac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}}, \tag{18}$$

where $|\boldsymbol{\Sigma}|$ denotes the determinant $\det \boldsymbol{\Sigma}$ of the scale matric $\boldsymbol{\Sigma} \in \mathbb{R}^{p\times p}$.

From the p.d.f. f_X as above, notice that the location vector of X is essentially the mean vector of X, i.e. $\boldsymbol{\mu} = \boldsymbol{\mu}_X := \mathrm{E}(X)$. Moreover, for the shape parameter value $\gamma = 2$, $\mathscr{N}_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is reduced to the well-known multivariate normal distribution, where $\boldsymbol{\Sigma}$ is now the covariance of X, i.e. $\operatorname{Cov} X = \boldsymbol{\Sigma}$. Recall that

$$\operatorname{Cov}(X) = \frac{\Gamma\left((p+2)\frac{\gamma-1}{\gamma}\right)}{p\,\Gamma^3\left(p\frac{\gamma-1}{\gamma}\right)} \left(\tfrac{\gamma}{\gamma-1}\right)^{2\frac{\gamma-1}{\gamma}} \boldsymbol{\Sigma}, \tag{19}$$

for the positive definite scale matrix $\boldsymbol{\Sigma}$; see [28].

Note that there are several other exponential-power generalizations of the usual Normal distribution, see [4, 15, 34], and [59] among others. Those generalizations are technically obtained and, thus, they have no specific physical interpretation. On the contrary, the γ-GN distribution has a strong information-theoretic background. Indeed, the most significant fact about the γ-GN family is that—at least for the spherically contoured case—acts to the generalized Information Inequality, the same way as the usual Normal distribution acts (i.e. providing equality) to the usual Information Inequality. In fact, the generalized form of the Information Inequality in (9) is reduced to equality for every spherically contoured γ-order normally distributed r.v., as it holds that $J_\alpha(X)\, N_\alpha(X) = p$ for $X \sim \mathscr{N}_\alpha\left(\boldsymbol{\mu}, \sigma^2\mathbb{I}_p\right)$; see [24, Cor. 3.2] for details. Moreover, the equality in the generalized Cramér–Rao Inequality as in (10) is achieved for r.v. X following the γ-GN distribution as above, i.e. it behaves the same way the usual Normal distribution does on the usual Cramér–Rao inequality. Indeed, using the fact that $J_\alpha(X)\, N_\alpha(X) = p$ holds for $X \sim \mathscr{N}_\alpha\left(\boldsymbol{\mu}, \sigma^2\mathbb{I}_p\right)$, as well as the extended Entropy Inequality as in (13), the equality of (10) can then be deduced, for the spherically contoured case; see also [26].

The family of multivariate γ-GN distributions, i.e. the family of the elliptically contoured γ-order generalized Normals, provides a smooth bridging between some important multivariate (and elliptically countered) distributions. Indeed:

1. *Case* $\gamma := 0$. For the limiting case when the shape parameter $\gamma \to 0^-$, the degenerate Dirac distribution $\mathscr{D}(\boldsymbol{\mu})$ with pole at point $\boldsymbol{\mu} \in \mathbb{R}^p$ is derived for dimensions $p := 1, 2$, while for $p \geq 3$ the corresponding p.d.f. "vanishes," i.e. $f_X \equiv 0$ for $X \sim \mathscr{N}_0(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.
2. *Case* $\gamma := 1$. For the limiting case when $\gamma \to 1^+$, the elliptically contoured Uniform distribution $\mathscr{U}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is obtained, which is defined over the p-ellipsoid $Q(\mathbf{x}) \leq 1$, $\mathbf{x} \in \mathbb{R}^p$.
3. *Case* $\gamma := 2$. For the "normality" case of $\gamma := 2$ the usual p-variate Normal distribution $\mathscr{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is obtained.
4. *Case* $\gamma := \pm\infty$. For the limiting case when $\gamma \to \pm\infty$ the elliptically contoured Laplace distribution $\mathscr{L}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ is derived.

See [28] for details. Therefore, one of the merits of the γ-GN family is that it can provide "heavy-" or "light-tailed" distributions as the change of shape parameter γ influences the "amount" of probability at the tails.

7 Information Divergencies

The informational divergence between two r.v.-s is usually calculated through the Kullback–Leibler (KL) divergence, which is acting as an "discrimination" measure of information. Recall that the *KL divergence* (also known as *relative entropy*), usually denoted by $D_{\mathrm{KL}}(X \| Y)$, of an r.v. X over an r.v. Y (of the same dimension), measures the amount of information "gained" when r.v. Y is replaced by X (say in an I/O system), and is defined by, [6],

$$D_{\mathrm{KL}}(X \| Y) := \int f_X \log \frac{f_X}{f_Y}, \tag{20}$$

where f_X and f_Y denote the corresponding density functions of r.v.-s X and Y.

In this section, we shall investigate the KL divergence measure of the multivariate γ-order normally distributed $X \sim \mathscr{N}_\gamma(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1)$ over the multivariate t_ν-distributed $Y \sim t_\nu(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2)$; see [51] for the univariate case. Recall the p.d.f. f_Y of the multivariate (and scaled) t_ν-distributed r.v. Y with $\nu \geq 1$ degrees of freedom, mean vector $\boldsymbol{\mu}_2 \in \mathbb{R}^p$, and scale matrix $\boldsymbol{\Sigma}_2 \in \mathbb{R}^{p \times p}$, which is given by

$$f_Y(\mathbf{y}) = f_Y(\mathbf{y};\ \boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2, \nu) := C_2 \left[1 + \tfrac{1}{\nu} Q_2(\mathbf{y})\right]^{-\frac{\nu+p}{2}}, \quad \mathbf{y} \in \mathbb{R}^p, \tag{21}$$

with normalizing factor

$$C_2 = C_2(\mathbf{\Sigma}_2;\ \nu, p) := \frac{(\pi\,\nu)^{-p/2}\ \Gamma\big(\frac{\nu+p}{2}\big)}{\Gamma\big(\frac{\nu}{2}\big)\sqrt{|\mathbf{\Sigma}_2|}}, \tag{22}$$

and p-quadratic form $Q_2(\mathbf{y}) := (\mathbf{y}-\boldsymbol{\mu}_2)\,\mathbf{\Sigma}_2^{-1}\,(\mathbf{y}-\boldsymbol{\mu}_2)^{\mathrm{T}}$, $\mathbf{y}\in\mathbb{R}^p$. Note that parameter ν can be also a positive real $\mathbb{R}^+ \ni \nu \geq 1$.

The following theorem provides an upper bound for the "gained" information when the t_ν-distribution is replaced by a γ-GN distribution. Note that we often rely on inequalities when it comes to the calculation of information divergencies (including KL) between certain r.v.-s, since the integrals involved cannot usually be solved in a closed form.

Theorem 2 *The KL divergence $D_{\mathrm{KL}} := D_{\mathrm{KL}}(X\|Y)$, of a multivariate spherically contoured γ-order normally distributed r.v. $X \sim \mathcal{N}_\gamma\big(\boldsymbol{\mu}, \sigma_1^2\mathbb{I}_p\big)$ over a t_ν-distributed r.v. $Y \sim t_\nu\big(\boldsymbol{\mu}, \sigma_2^2\mathbb{I}_p\big)$, of the same mean $\boldsymbol{\mu}\in\mathbb{R}^p$, has the following upper bound,*

$$D_{\mathrm{KL}} \leq \log K + p\left(\log\tfrac{\sigma_2}{\sigma_1} - \tfrac{\gamma-1}{\gamma}\right) + \tfrac{\nu+p}{2\nu}\left(\tfrac{\sigma_1}{\sigma_2}\right)^2\left(\tfrac{\gamma}{\gamma-1}\right)^{2\frac{\gamma-1}{\gamma}}\frac{\Gamma\big((p+2)\,\frac{\gamma-1}{\gamma}\big)}{\Gamma\big(p\frac{\gamma-1}{\gamma}\big)}, \tag{23}$$

where

$$K = K(\gamma,\nu,p) := \frac{\nu^{p/2}\ \Gamma\big(\frac{p}{2}\big)\ \Gamma\big(\frac{\nu}{2}\big)}{2\,\Gamma\big(p\frac{\gamma-1}{\gamma}\big)\ \Gamma\big(\frac{\nu+p}{2}\big)}\left(\tfrac{\gamma-1}{\gamma}\right)^{p\frac{\gamma-1}{\gamma}-1}. \tag{24}$$

Proof From the definition of the KL divergence (20) and the probability densities f_X and f_Y, as in (17) and (21), with K, C_1, and C_2 are defined as in (24), (18), and (22), respectively, while $Q_i(\mathbf{x}) := (\mathbf{x}-\boldsymbol{\mu})\,\mathbf{\Sigma}_i^{-1}\,(\mathbf{x}-\boldsymbol{\mu})^{\mathrm{T}}$, $\mathbf{x}\in\mathbb{R}^p$, $i=1,2$, with $\mathbf{\Sigma}_1 := \sigma_1^2\,\mathbb{I}_p$, $\mathbf{\Sigma}_2 := \sigma_2^2\,\mathbb{I}_p$, it holds

$$D_{\mathrm{KL}} = C_1\left[\left(\log K + p\,\log\tfrac{\sigma_2}{\sigma_1}\right)I_1 - g\,I_2 + \tfrac{p+\nu}{2}\,I_3\right], \tag{25}$$

where

$$\begin{aligned} I_1 &:= \int_{\mathbb{R}^p}\exp\left\{-g\left\|\frac{\mathbf{x}-\boldsymbol{\mu}}{\sigma_1}\right\|^{1/g}\right\}\,\mathrm{d}\mathbf{x} \\ I_2 &:= \int_{\mathbb{R}^p}\exp\left\{-g\left\|\frac{\mathbf{x}-\boldsymbol{\mu}}{\sigma_1}\right\|^{1/g}\right\}\left\|\frac{\mathbf{x}-\boldsymbol{\mu}}{\sigma_1}\right\|^{1/g}\mathrm{d}\mathbf{x},\quad \text{and} \\ I_3 &:= \int_{\mathbb{R}^p}\exp\left\{-g\left\|\frac{\mathbf{x}-\boldsymbol{\mu}}{\sigma_1}\right\|^{1/g}\right\}\,\log\left(1+\frac{1}{\nu}\left\|\frac{\mathbf{x}-\boldsymbol{\mu}}{\sigma_2}\right\|^2\right)\mathrm{d}\mathbf{x}, \end{aligned}$$

and $g = g(\gamma) := (\gamma - 1)/\gamma$. Applying the linear transformation $\mathbf{z} = \mathbf{z}(\mathbf{x}) := g^g\,(\mathbf{x} - \boldsymbol{\mu})/\sigma_1$, $\mathbf{x} \in \mathbb{R}^p$, with $\mathrm{d}\mathbf{x} = g^{-p\,g}\,\sigma_1^p\,\mathrm{d}\mathbf{z}$, the above three multiple integrals are then written as

$$I_1 = g^{-p\,g}\,\sigma_1^p \int_{\mathbb{R}^p} \mathrm{e}^{-\|\mathbf{z}\|^{1/g}}\,\mathrm{d}\mathbf{z}, \tag{26a}$$

$$I_2 = g^{-p\,g}\sigma_1^p \int_{\mathbb{R}^p} \|\mathbf{z}\|^{1/g}\,\mathrm{e}^{-\|\mathbf{z}\|^{1/g}}\,\mathrm{d}\mathbf{z}, \text{ and} \tag{26b}$$

$$I_3 = g^{-p\,g}\,\sigma_1^p \int_{\mathbb{R}^p} \mathrm{e}^{-\|\mathbf{z}\|^{1/g}}\,\log\Big(1 + \tfrac{g^{-2g}}{\nu}\big(\tfrac{\sigma_1}{\sigma_2}\big)^2\|\mathbf{z}\|^2\Big)\,\mathrm{d}\mathbf{z}. \tag{26c}$$

Applying then the known integrals

$$\int_{\mathbb{R}^p} \mathrm{e}^{-\|\mathbf{z}\|^\beta}\,\mathrm{d}\mathbf{z} = \frac{2\pi^{p/2}\,\Gamma\big(\frac{p}{\beta}\big)}{\beta\,\Gamma\big(\frac{p}{2}\big)} \text{ and } \int_{\mathbb{R}^p} \|\mathbf{z}\|^\beta\,\mathrm{e}^{-\|\mathbf{z}\|^\beta}\,\mathrm{d}\mathbf{z} = \tfrac{p}{\beta}\int_{\mathbb{R}^p} \mathrm{e}^{-\|\mathbf{z}\|^\beta}\,\mathrm{d}\mathbf{z}, \tag{27}$$

with $\beta \in \mathbb{R}^{*+} := \mathbb{R}^+ \setminus \{0\}$, integrals (26a) and (26b) are then calculated as

$$I_1 = g^{-p\,g}\,\sigma_1^p\,\frac{2\pi^{p/2}}{\Gamma(p/2)}\,g\,\Gamma(p\,g) \text{ and } I_2 = p\,g\,I_1, \tag{28}$$

, respectively. Thus, (25) is reduced to

$$D_{\mathrm{KL}} = C_1\,\Big(\log K + p\,\log\tfrac{\sigma_2}{\sigma_1} - p\,g\Big)\,I_1 + \tfrac{p+\nu}{2}\,C_1\,I_3.$$

Substituting I_1 from (28) and using C_1 from (18), and applying the Gamma function additive identity, the above is reduced to

$$D_{\mathrm{KL}} = \log K + p\,\Big(\log\tfrac{\sigma_2}{\sigma_1} - g\Big) + \frac{p+\nu}{4\big(\sqrt{\pi}\,\sigma_1\big)^p}\,\frac{\Gamma(p/2)}{\Gamma(p\,g)}\,g^{p\,g-1}\,I_3. \tag{29}$$

Notice that the function in the integral of (26c) is positive, and so, using the known logarithmic inequality $\log(x+1) \le x$, $x > -1$, relation (26c) implies

$$I_3 \le g^{-(p+2)\,g}\,\frac{\sigma_1^{p+2}}{\nu\,\sigma_2^2}\int_{\mathbb{R}^p} \|\mathbf{z}\|^2\,\mathrm{e}^{-\|\mathbf{z}\|^{1/g}}\,\mathrm{d}\mathbf{z}. \tag{30}$$

We calculate now the first and the third integral of the above inequality by switching to hyperspherical coordinates, while the second integral is calculated using the relation first of (27). Recall the known hyperspherical transformation

$H_p : S_p \to \mathbb{R}^p$, where $S_p := \mathbb{R}^+ \times [0, \pi)^{p-2} \times [0, 2\pi)$, in which $S_p \ni (\rho, \varphi_1, \varphi_2, \ldots, \varphi_{p-1}) \overset{H_p}{\longmapsto} (z_1, z_2, \ldots, z_p) \in \mathbb{R}^p$, is given by

$$z_1 = \rho \cos\varphi_1, \tag{31a}$$

$$z_i = \rho \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{i-1} \cos\varphi_i, \quad i = 2, 3, \ldots, p-1, \tag{31b}$$

$$z_p = \rho \sin\varphi_1 \sin\varphi_2 \cdots \sin\varphi_{p-2} \sin\varphi_{p-1}, \tag{31c}$$

where $\rho \in \mathbb{R}^+$, $\varphi_1, \varphi_2, \ldots, \varphi_{p-2} \in [0, \pi)$, and $\varphi_{p-1} \in [0, 2\pi)$. It holds that $\|\mathbf{z}\|^2 = z_1^2 + z_2^2 + \cdots + z_p^2 = \rho^2$, $\mathbf{z} \in \mathbb{R}^p$, while the volume element $\mathrm{d}\mathbf{z} = \mathrm{d}z_1\,\mathrm{d}z_2 \cdots \mathrm{d}z_p$ of the p-dimensional Euclidean space is given in hyperspherical coordinates as

$$\mathrm{d}\mathbf{z} = \mathrm{J}(H_p)\,\mathrm{d}\rho\,\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{p-1} = \rho^{p-1} \left(\prod_{k=1}^{p-2} \sin^{p-k-1}\varphi_k \right) \mathrm{d}\rho\,\mathrm{d}\varphi_1 \cdots \mathrm{d}\varphi_{p-1}, \tag{32}$$

where $\mathrm{J}(H_p)$ is the Jacobian determinant of the transformation H_p, i.e.

$$\mathrm{J}(H_p) := \left| \det \frac{\partial(z_1, z_2, \ldots, z_p)}{\partial(\rho, \varphi_1, \ldots, \varphi_{p-1})} \right| = \rho^{p-1} \sin^{p-2}\varphi_1 \sin^{p-3}\varphi_2 \cdots \sin\varphi_{p-2}, \tag{33}$$

Moreover, the volume element of the $(p-1)$-sphere is given by

$$\mathrm{d}^{p-1}V = \sin^{p-2}\varphi_1 \sin^{p-3}\varphi_2 \cdots \sin\varphi_{p-2}\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2 \cdots \mathrm{d}\varphi_{p-1}.$$

Thus the corresponding volume is then $V_{p-1} = 2\pi^{p/2}/\Gamma(p/2)$. Therefore, the multiple integral in (30) is transformed to

$$I := \int_{\mathbb{R}^p} \|z\|^2 \,\mathrm{e}^{-\|z\|^{1/g}}\,\mathrm{d}z = V_{p-1} \int_{\mathbb{R}^+} \rho^2 \rho^{p-1} \,\mathrm{e}^{-\rho^{1/g}}\,\mathrm{d}\rho. \tag{34}$$

Applying the transformation $u = u(\rho) := \rho^{1/g}$, $\rho \in \mathbb{R}^+$, with $\mathrm{d}\rho = g u^{g-1}\,\mathrm{d}u$, the integral (34) is then calculated, via the definition of the Gamma function, as

$$I = g\,V_{p-1} \int_{\mathbb{R}^+} u^{(p+2)\,g-1}\,\mathrm{e}^{-u}\,\mathrm{d}u = g\,V_{p-1}\,\Gamma\big((p+2)\,g\big), \tag{35}$$

hence, the inequality (30) is then reduced to

$$I_3 \le 2g^{1-(p+2)\,g}\,\frac{\pi^{p/2}\sigma_1^{p+2}\,\Gamma\big((p+2)\,g\big)}{\nu\,\sigma_2^2\,\Gamma(p/2)}. \tag{36}$$

Applying (36) to (29) we finally derive the upper bound of D_{KL} as in (23).

Consider now the (multivariate) Normal distribution instead of the t_ν distribution. Then, following Theorem 2, we can derive an exact form of the KL divergence of the γ-GN over the usual Normal distribution, extending the corresponding univariate result in [51]. Note that, in order to achieve this result, the inequality proved in Theorem 2 is studied in limit, showing that the upper bounds in (23) increase along with the degrees of freedom ν of the t_ν-distribution, until they reach a supremum. Hence, when ν tends to infinity we are approaching normality as well as equality for (23).

Theorem 3 *The KL divergence of a p-variate r.v.* $X \sim \mathcal{N}_\gamma\big(\boldsymbol{\mu}, \sigma_1^2 \mathbb{I}_p\big)$, $\boldsymbol{\mu} \in \mathbb{R}^p$, $\sigma > 0$, *over a p-variate normally distributed r.v.* $N \sim \mathcal{N}\big(\boldsymbol{\mu}, \sigma_2^2 \mathbb{I}_p\big)$, *is given by*

$$D_{\mathrm{KL}}(X\|N) = \log\left\{\frac{2^{p/2-1}\,\Gamma(p/2)}{\Gamma\big(p\frac{\gamma-1}{\gamma}\big)}\big(\tfrac{\gamma-1}{\gamma}\big)^{p\frac{\gamma-1}{\gamma}-1}\right\} + p\left(\log\tfrac{\sigma_2}{\sigma_1} - \tfrac{\gamma-1}{\gamma}\right) + \big(\tfrac{\gamma}{\gamma-1}\big)^{2\frac{\gamma-1}{\gamma}}\big(\tfrac{\sigma_1}{\sigma_2}\big)^2\frac{\Gamma\big((p+2)\frac{\gamma-1}{\gamma}\big)}{2\,\Gamma\big(p\frac{\gamma-1}{\gamma}\big)}. \tag{37}$$

Proof Firstly, by substituting of (26c) to (29), we obtain

$$D_{\mathrm{KL}}(X\|Y_\nu) = \log K + p\left(\log\tfrac{\sigma_1}{\sigma_2} - g\right) + \frac{\Gamma(p/2)}{4\pi^{p/2} g\,\Gamma(pg)}\, I, \tag{38}$$

where $g := (\gamma-1)/\gamma$, $Y_\nu \sim t_\nu\big(\boldsymbol{\mu}, \sigma_2^2\mathbb{I}_p\big)$, $\nu \in \mathbb{N}^*$, and

$$I := \int_{\mathbb{R}^p} \mathrm{e}^{-\|\mathbf{z}\|^{1/g}} \log\left\{1 + \tfrac{1}{\nu}\big(\tfrac{\sigma_2}{\sigma_1}\big)^2 g^{-2g}\|\mathbf{z}\|^2\right\}^{p+\nu} \mathrm{d}\mathbf{z}. \tag{39}$$

For the KL divergence of $X \sim \mathcal{N}_\gamma\big(\boldsymbol{\mu}, \sigma_1^2\mathbb{I}_p\big)$ over the p-variate normally distributed r.v. $N \sim \mathcal{N}\big(\boldsymbol{\mu}, \sigma_2^2\mathbb{I}_p\big)$, it holds that $D_{\mathrm{KL}}(X\|N) = \lim_{\nu\to\infty} D_{\mathrm{KL}}(X\|Y_\nu)$, as the scaled spherically contoured $t_\nu\big(\mu, \sigma_2^2\mathbb{I}_p\big)$ distribution is, in limit, the normal distribution $\mathcal{N}\big(\boldsymbol{\mu}, \sigma_2^2\mathbb{I}_p\big)$ when $\nu \to \infty$. As a result, the sequence

$$b_\nu := \frac{\nu^{p/2}\,\Gamma(\nu/2)}{\Gamma\big(\frac{\nu+p}{2}\big)}, \quad \nu, p \in \mathbb{N}^*, \tag{40}$$

converges to $2^{p/2}$ as $\nu \to \infty$, since $\lim_{\nu\to\infty} f_{Y_\nu} = f_N$, where f_{Y_ν} and f_N are the probability densities of the t_ν-distributed r.v. Y_ν and the normally distributed r.v. N, respectively. Indeed, $b_\nu \to 2^{p/2}$, as $\nu \to \infty$, due to the fact that the normalizing factor $C_2(\sigma_2^2\mathbb{I}_p)$ of f_{Y_ν} converges to the normalizing factor $C_1(\sigma_2^2\mathbb{I}_p)$ of f_N, i.e. (18) and (22) yield $\pi^{-p/2}\lim_{\nu\to\infty} b_\nu^{-1} = (2\pi)^{-p/2}$, or equivalently $\lim_{\nu\to\infty} b_\nu = 2^{p/2}$.

Therefore, substituting $C_2(\sigma_2^2\mathbb{N}_p)$ from (18) into (38), and then computing the limit for $\nu \to \infty$, we derive, using the limit in (40) as well as the well-known exponential limit $\lim_{\nu\to\infty}(1+\nu^{-1})^{\nu} = \mathrm{e}$, that

$$D_{\mathrm{KL}}(X\|N) = \log\left\{2^{(p/2)-1}\frac{\Gamma(p/2)}{\Gamma(p\,g)}g^{p\,g-1}\right\} + p\Big(\log\frac{\sigma_2}{\sigma_1} - g\Big) + \frac{\Gamma(p/2)}{4\pi^{p/2}\,g\;\Gamma(p\,g)}\,I, \tag{41}$$

where

$$I = \big(\tfrac{\sigma_1}{\sigma_2}\big)^2 g^{-2g}\int_{\mathbb{R}^p}\|\mathbf{z}\|^2\,\mathrm{e}^{-\|\mathbf{z}\|^{1/g}}\,\mathrm{d}\mathbf{z}. \tag{42}$$

Calculating the above integral (42) with the help of (27), we derive

$$I = \frac{2\pi^{p/2}}{\Gamma(p/2)}\big(\tfrac{\sigma_1}{\sigma_2}\big)^2 g^{1-2g}\,\Gamma\big((p+2)g\big).$$

By substitution in (41), we finally obtain (37) using the known Gamma function additive identity, i.e. $\Gamma(x+1) = x\;\Gamma(x)$, $x \in \mathbb{R}^{*+}$.

The following investigates the order behavior of the upper bounds in (23).

Proposition 2 *When the degrees of freedom $\nu \in \mathbb{N}^*$ rise, the upper bound value, say $B_{\gamma,\nu}$ of (23) approximate better the KL divergence D_{KL} for all parameters $\gamma \in \mathbb{R}\setminus[0,1]$. Furthermore, for the univariate and the bivariate case, the corresponding bounds $B_{\gamma,\nu}$ have a strict descending order converging to the D_{KL} measure of r.v. $X \sim \mathscr{N}_\gamma\big(\boldsymbol{\mu}, \sigma_1^2\mathbb{I}_p\big)$ over the normally distributed r.v. $N \sim \mathscr{N}\big(\boldsymbol{\mu}, \sigma_2^2\,\mathbb{I}_p\big)$ as υ rises, i.e. $B_{\gamma,1} < B_{\gamma,2} < \cdots < B_{\gamma,\infty} = D_{\mathrm{KL}}(X\|N)$ for $p = 1, 2$.*

Proof Consider the sequence $a_\nu := (\nu+1)/\nu$, $nu \in \mathbb{N}^*$. Then a_ν and b_ν, as in (40), converge both to 1 as $\nu \to \infty$. Considering the bounds $B_{\gamma,\nu}$ as in (23) when $\nu \to \infty$, it holds that $B_{\gamma,\infty}$ approaches the KL divergence as in (37). Thus, the equality in (23) is obtained in limit as $\nu \to \infty$, i.e. $D_{\mathrm{KL}}(X\|N) = B_{\gamma,\infty}$, and therefore the bounds $B_{\gamma,\nu}$ approximate better the KL divergence $D_{\mathrm{KL}}(X\|Y)$ as υ rises, until $B_{\gamma,\nu}$ coincides eventually with D_{KL} of Theorem 3 for all parameter γ values.

Especially for the bivariate case of $p := 2$, the sequence b_ν is constant, i.e. $b_\nu = 2$, $\nu \in \mathbb{N}^*$, while for univariate case of $p := 1$, sequence b_ν is descending with $b_\nu \ge \lim_{\nu\to\infty} b_\nu = \sqrt{2}$. Indeed,

$$\frac{b_{2\nu+1}}{b_{2\nu}} = \frac{1}{\nu}\sqrt{\frac{2\nu+1}{2\nu}}\,\frac{\Gamma^2\big(\nu+\frac{1}{2}\big)}{\Gamma^2(\nu)}, \quad \nu \in \mathbb{N}^*.$$

By applying the known result of Gamma function,

$$\Gamma\big(k+\tfrac{1}{2}\big) = \tfrac{(2k-1)!!}{2^k}\sqrt{\pi} = \tfrac{(2k)!}{2^{2k}\,k!}\sqrt{\pi}, \quad k \in \mathbb{N}, \tag{43}$$

we obtain

$$\frac{b_{2\nu+1}}{b_{2\nu}} = \pi\,\nu\,\sqrt{\tfrac{2\nu+1}{2\nu}}\left[\frac{(2\nu)!}{2^{2\nu}\,(\nu!)^2}\right]^2. \tag{44}$$

Finally, utilizing the known bounds for the factorial in (1), the ratio in (44) is less than 1, as

$$\frac{b_{2\nu+1}}{b_{2\nu}} \leq \tfrac{e^2}{4\pi}\sqrt{\tfrac{2\nu+1}{2\nu}} \leq \tfrac{e^2}{4\pi}\sqrt{\tfrac{3}{2}} \approx 0.72015 < 1.$$

Therefore, for dimensions $p = 1$ and $p = 2$, and from the form of bounds in Theorem 23, we derive that $B_{\gamma,1} < B_{\gamma,2} < \cdots < B_{\gamma,\infty}$. That is, as t_ν-distribution approaches the Normal distribution (as $\nu \to \infty$), the bounds $B_{\gamma,\nu}$ have a strictly descending order converging to $B_{\gamma,\infty}$, i.e. to $D_{\mathrm{KL}}(X\|N)$.

8 Discussion

Inequalities cover all the Mathematical disciplines, either as bounds to different quantities or measures—with typical example being the error control, as described in Sect. 2, or confidence intervals in Sect. 3—or as an attempt to compare different measures, like the notion of distance in Probability Theory, the SPRT method in Sect. 3, the various forms of triangle inequality given in Sect. 2, or in Information Theory as discussed in Sects. 5, 6 and 7. There are cases were the inequalities are involved either in definition, as in SPRT, or imposed as restriction to the developed theory, as in Stochastic Approximation. In Statistics, inequalities are often related with the interval estimation for the estimated parameters, usually through the Maximum Likelihood methodology. Sequences under imposed assumptions create different approaches in Statistics, with the main ones being the Sequential approach and the Stochastic processes.

A number of inequalities were presented in this paper. For example, consider the maximal inequalities in Sect. 4, or the Crossing Inequality that measures the times we can exceed the imposed bounds in a stochastic process; in the SPRT case, if this happens once, the method stops. Similar inequalities can also be considered under different lines of thought, with typical example being the Cauchy–Schwarz inequality in Sect. 2, which can be also be transferred and used in Statistics as shown in Sect. 3.

The inequalities in Information theory are more "mathematically oriented" and well-known bounds have been extended, with typical examples being the Information Inequality, the Cramér–Rao Inequality, or the Blachman–Stam Inequality. The upper bound of the Kullback–Leibler divergence, as proved in Sect. 7, is essential, we believe in the sense that offers a way of approximating "how far" can be the family of the generalized Normal distributions from the multivariate Student's t-distribution, since the involved integrals cannot be computed in a closed form.

Moreover, Proposition 2 gives us an idea of how those bounds behave in relation to the degrees of freedom of the considered t-distribution.

This paper can also be considered as an attempt to increase the existed inequality problems, collected by Rassias in [38].

Appendix 1

Proof of Proposition 1 It is easy to see that D^* satisfies the positive-definiteness and symmetricity conditions, and therefore—in order to prove that D^* is indeed a proper distance metric—the triangle inequality (or subadditivity) must be fulfilled. For this purpose, three arbitrary probability measures $P, Q, R \in \mathscr{P}(\Omega)$ are considered. Applying the exponential inequality $e^x \geq (1 + x/n)^n$, $x \in \mathbb{R}$, with $n := 3$, to the definition of D^*, we get

$$D^*(P, Q)+D^*(Q, R) = e^{D(P,Q)} + e^{D(Q,R)} - 2 \geq \left[1+\tfrac{1}{3}D(P, Q)\right]^3 + \left[1+\tfrac{1}{3}D(Q, R)\right]^3 - 2,$$

and using the simplified notations $a := D(P, Q)$, $b := d(Q, R)$ and $c := d(P, R)$,

$$\begin{aligned} D^*(P, Q) + D^*(Q, R) &\geq \tfrac{1}{27}\big(a^3 + b^3\big) + \tfrac{1}{3}\big(a^2 + b^2\big) + a + b \\ &= \tfrac{1}{27}(a + b)^3 - \tfrac{1}{9}a\, b\, (a + b) + \tfrac{1}{3}(a + b)^2 - \tfrac{2}{3}a\, b + a + b \\ &\geq \tfrac{1}{27}(a + b)^3 - \tfrac{1}{36}(a + b)^3 + \tfrac{1}{3}(a + b)^2 - \tfrac{1}{6}(a + b)^2 + a + b \\ &\geq \tfrac{1}{3}c^3 + \tfrac{1}{6}c^2 + c, \end{aligned} \tag{45}$$

where the triangle inequality of metric D was used as well as the inequality $\sqrt{ab} \leq \frac{1}{2}(a+b)$, $a, b \in \mathbb{R}^+$. By expressing D in terms of D^*, through the definition of D^*, relation (45) yields

$$D^*(P, Q)+D^*(Q, R) \geq \tfrac{1}{3}\log^3\big(1+D^*(P, R)\big)+\tfrac{1}{6}\log^2\big(1+D^*(P, R)\big)+\log\big(1+D^*(P, R)\big). \tag{46}$$

Consider now the function $f(x) := \frac{1}{3}\log^3(1+x) + \frac{1}{6}\log^2(1+x) + \log(1+x) - x$, $x \in \mathbb{R}^+$. Assuming that $f' \leq 0$, i.e. $\log^2(1 + x) + \frac{1}{3}\log(1 + x) - x \leq 0$, the logarithm identity $\log x \geq (x-1)/x$, $x \in \mathbb{R}^{*+} := \mathbb{R}^+ \setminus \{0\}$ gives $4x^2 - 2x - 3 \leq 0$, which holds for $x \geq x_0 := \frac{1}{4}\big(2+\sqrt{28}\big) \approx 1.822$. Therefore, f has a global maxima at x_0, and as $x_1 = 0 = f(0)$ is one of the two roots $x_1, x_2 \in \mathbb{R}^+$ of f, the fact that $0 = x_1 \leq x_0$ means that $f(x) \geq 0$ for $x \in [0, x_2]$, where $x_2 \approx 3.5197$ (numerically computed). Therefore, the fact that metric $D \leq 1$ implies $0 \leq D^* \leq e - 1 \approx 1.718 < x_2$, resulting (from the above discussion) that $f\big(D^*(P, Q)\big) \geq 0$ which is equivalent, through (46), to the requested triangle inequality $D^*(P, Q) + D^*(Q, R) \geq D^*(P, Q)$.

Appendix 2

Some introductory definitions from the Statistical Decision Theory are needed.

Definition 2 (Decision Problem and Rules) A general *decision problem* is defined to be a triplet (Θ, D, ℓ) and a random variable X, known as *data*, following the probability distribution $F(x \mid \theta), \theta \in \Theta$, with Θ being a parameter space. Moreover, θ is called as the *state of nature* with $\ell = \ell(\theta, d)$ denoting the *loss function*, while d is a *decision* from the *decision space* D. A *non-randomized decision rule* is a function $q(\cdot)$ such that $X \ni x \overset{q}{\mapsto} q(x) = d \in D$, while a *randomized decision rule* $q(x)$ specifies a probability distribution according to which a member, say d, of D is to be chosen.

Definition 3 (Risk Function) The *risk function* $r_\theta(q)$ of a decision rule q, for a decision problem (Θ, D, ℓ), is defined by $r_\theta(q) := \mathrm{E}\big(\ell\big(\theta, q(X)\big)\big)$, when θ is referring to the true state of nature, as appeared in the expected (or average) loss in the definition. To assign an *order* to decision rules, we assume that $r_\theta(q_1) > r_\theta(q_2)$, for every $\theta \in \Theta$, and say that q_2 is more preferable than q_1.

Now, let $\mathfrak{F}$ be the class of monotone non-decreasing functions f with $f(x) = 0$ on $I := [0, 1]$. Let D^* be a collection of sub-intervals of I, and Q_n be the set of decision rules q. We try to estimate $q(f)$ with $f(x) = 0$ and n observations, with the final decision to be $q(f) \in d$ for a particular $d \in D^*$. Obviously, $q(f) = d \in D^*$ defines a decision rule with n observations. We also consider the set, say Q_n^*, of all procedures $q \in Q_n$ for which $q(f) \in D$. An *optimum procedure* q_n^* is imposed as a minimax decision procedure, depending on the length $L(f, q)$ of d, with $f \in \mathscr{F}$, $q \in Q_n^*$ of the form

$$\sup_{f \in \mathfrak{F}} L\big(f, q_n^*\big) = \inf_{q \in Q_n^*} \sup_{f \in \mathfrak{F}} L(f, q).$$

Theorem 4 *The bisection method is a q_n^* minimax procedure.*

Proof Consider the iterative procedure $x_k := (\alpha_{k-1} + \beta_{k-1})/2,\ k = 2, 3, \ldots,$ with initial value $x_1 := 1/2$. We assume that the value α_{k-1} corresponds to the largest previously observed value of x, with $f(x) = 0$ if there is no largest value for x, and $f(x) < 0$ if x is the largest value. We assume also that the value β_{k-1} corresponds to the smallest value of x, for which $f(x) = 1$ if there is no such x, and $f(x) > 0$ if x is the largest value. Let $d = [\alpha_n, \beta_n]$ be each time interval. In such a procedure with $n \geq 1$, any procedure $q \in Q_n^*$ with $x_1 \neq 1/2$ shall provide a larger $\sup_{f \in \mathfrak{F}} L(f, q)$. By induction, if we accept that theorem holds for $\nu = 1, 2, \ldots, n-1$, we shall try to prove it for $\nu = n$. Any procedure q with $n-1$ evaluations at $(x_1, x_2, \ldots, x_{n-1})$ does best to adopt the x_ν value midway between $\alpha_{\nu-1}$ and $\beta_{\nu-1}$, both evaluated via $(x_1, x_2, \ldots, x_{n-1})$. Hence, x_ν reaches a minimax length of $(\alpha_{\nu-1} - \beta_{\nu-1})/2$. However, by taking into account the values $x_1, x_2, \ldots, x_{\nu-1}$, then we are in accordance with $q_{\nu-1}^*$ which minimizes $\beta_{\nu-1} - \alpha_{\nu-1}$.

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Multiple Hardy–Littlewood Integral Operator Norm Inequalities

J. C. Kuang

Abstract How to obtain the sharp constant of the Hardy–Littlewood inequality remains unsolved. In this paper, the new analytical technique is to convert the exact constant factor to the norm of the corresponding operator, the multiple Hardy–Littlewood integral operator norm inequalities are proved. As its generalizations, some new integral operator norm inequalities with the radial kernel on n-dimensional vector spaces are established. The discrete versions of the main results are also given.

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1 Introduction

Throughout this paper, we write

$$E_n(\alpha) = \{x = (x_1, x_2, \cdots, x_n) : x_k \geq 0, 1 \leq k \leq n, \|x\|_\alpha = (\sum_{k=1}^{n} |x_k|^\alpha)^{1/\alpha}, \alpha > 0\},$$

$E_n(\alpha)$ is an n-dimensional vector space, when $1 \leq \alpha < \infty$, $E_n(\alpha)$ is a normed vector space. In particular, $E_n(2)$ is an n-dimensional Euclidean space $\mathbb{R}_+^n$.

$$\|f\|_{p,\omega} = (\int_{E_n(\alpha)} |f(x)|^p \omega(x) dx)^{1/p},$$

$$L^p(\omega) = \{f : f \ is \ measurable, \ and \|f\|_{p,\omega} < \infty\},$$

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where, ω is a non-negative measurable function on $E_n(\alpha)$. If $\omega(x) \equiv 1$, we will denote $L^p(\omega)$ by $L^p(E_n(\alpha))$, and $\|f\|_{p,1}$ by $\|f\|_p$. $\Gamma(\alpha)$ is the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \ (\alpha > 0).$$

$B(\alpha, \beta)$ is the Beta function:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \ (\alpha, \beta > 0).$$

The celebrated Hardy–Littlewood inequality (see [1], Theorem 401 and [2–4]) asserts that if f and g are non-negative, and $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} \geq 1$, $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $\delta < 1 - \frac{1}{p}$, $\beta < 1 - \frac{1}{q}$, $\delta + \beta \geq 0$, and $\delta + \beta > 0$, if $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\delta y^\beta |x-y|^{\lambda-\delta-\beta}} dxdy \leq c\left(\int_0^\infty f^p dx\right)^{1/p}\left(\int_0^\infty g^q dx\right)^{1/q}. \tag{1}$$

Here, c denotes a positive number depending only on the parameters of the theorem (here p, q, δ, β). But Hardy was unable to fix the constant c in (1). We note that (1) is equivalent to

$$\|T_0 f\|_p \leq c\|f\|_p, \tag{2}$$

where,

$$T_0(f, x) = \int_0^\infty \frac{1}{x^\delta y^\beta |x-y|^{\lambda-\delta-\beta}} f(y) dy. \tag{3}$$

Hence, $c = \|T_0\|$ in (2) is the sharp constant for (1) and (2). Under the above conditions, Hardy–Littlewood [2] proved that there exists a positive constant c_1 such that

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x^\delta y^\beta |x-y|^{\lambda-\delta-\beta}} dxdy \leq c_1 \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda} dxdy. \tag{4}$$

The following Hardy–Littlewood–Pólya inequality was proved in [5] and [6]:

Theorem 1 *Let* $f \in L^p(0, \infty)$, $g \in L^q(0, \infty)$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} > 1$, $0 < \lambda < 1$, $\lambda = 2 - \frac{1}{p} - \frac{1}{q}$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda} dxdy \leq c_2 \|f\|_p \|g\|_q, \tag{5}$$

where,

$$c_2 = c_2(p, q, \lambda) = \frac{1}{1-\lambda}\{(\frac{p}{p-1})^{p(1-\frac{1}{q})} + (\frac{q}{q-1})^{q(1-\frac{1}{p})}\}. \tag{6}$$

Let

$$T_1(f, x) = \int_0^\infty \frac{1}{|x-y|^\lambda} f(y)dy. \tag{7}$$

Then (5) is equivalent to

$$\|T_1 f\|_{p_1} \le c_2 \|f\|_p, \tag{8}$$

where, $1 < p < \infty, 1 - \frac{1}{p} < \lambda < 1$, $\frac{1}{p_1} = \frac{1}{p} + \lambda - 1$, c_2 is given by (6). For a function $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$, define its potential of order λ as

$$T_2(f, x) = \int_{\mathbb{R}^n} \frac{1}{\|x-y\|_2^\lambda} f(y)dy, \ \ 0 < \lambda < n. \tag{9}$$

Theorem 2 ([6, pp. 412–413]) *There exists a constant c_3 depending only upon n, p, and λ, such that*

$$\|T_2 f\|_{p_2} \le c_3 \|f\|_p, \tag{10}$$

where, $\frac{1}{p_2} = \frac{1}{p} + \frac{\lambda}{n} - 1$.

Theorem 3 ([7–10]) *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty, 0 < \lambda < n$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, then there exists a constant $c_4 = c_4(p, \lambda, n)$ (depending only upon n, p, and λ), such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x-y\|_2^\lambda} dxdy \le c_4 \|f\|_p \|g\|_q, \tag{11}$$

where,

$$c_4 \le \frac{n}{pq(n-\lambda)}(\frac{S_n}{n})^{\lambda/n}\{(\frac{\lambda/n}{1-(1/p)})^{\lambda/n} + (\frac{\lambda/n}{1-(1/q)})^{\lambda/n}\},$$

and S_n is the surface areas of the unit sphere in $\mathbb{R}^n$. In particular, for $p = q = \frac{2n}{2n-\lambda}$,

$$c_4 = \pi^{\lambda/2}\frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n-\frac{\lambda}{2})}\{\frac{\Gamma(\frac{n}{2})}{\Gamma(n)}\}^{\frac{\lambda}{n}-1}$$

is the best possible constant.

But when $p \ne q$, the best possible value of c_4 is also unknown.

In 2017, the author Kuang [14] established the norm inequality of operator T_2.

Theorem 4 ([11]) *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty$, $0 < \lambda < n$, $\delta + \beta \geq 0$, $1 - \frac{1}{p} - \frac{\lambda}{n} < \frac{\delta}{n} < 1 - \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda+\delta+\beta}{n} = 2$, then there exists a constant $c_5 = c_5(p, \delta, \beta, \lambda, n)$, such that*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda}} dxdy \leq c_5\|f\|_p\|g\|_q. \tag{12}$$

Remark 1 Inequality (12) can be given an equivalent form

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta}} dxdy \leq c_5\|f\|_p\|g\|_q, \tag{13}$$

then the conditions $1 - \frac{1}{p} - \frac{\lambda}{n} < \frac{\delta}{n} < 1 - \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda+\delta+\beta}{n} = 2$ are replaced by

$$\frac{\delta}{n} < 1 - \frac{1}{p} < \frac{\lambda}{n} - \frac{\beta}{n}, \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2.$$

The multiple Hardy–Littlewood integral operator T_3 defined by

$$T_3(f, x) = \int_{\mathbb{R}^n} \frac{f(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta}} dy. \tag{14}$$

Then (13) is equivalent to

$$\|T_3 f\|_p \leq c_5\|f\|_p. \tag{15}$$

But, the problem of determining the best possible constants in (13) and (15) remains unsolved. In this paper, the new analytical technique is to convert the exact constant factor to the norm $c_5 = \|T_3\|$ of the corresponding operator T_3. Hence, we consider operator norm inequality (15). Without loss of generality, we may consider that the multiple Hardy–Littlewood integral operator T_4 defined by

$$T_4(f, x) = \int_{\mathbb{R}^n_+} \frac{f(y)}{\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta}} dy \tag{16}$$

and f be a nonnegative measurable function on $\mathbb{R}^n_+$, thus, by the triangle inequality, we have

$$|\|x\|_2 - \|y\|_2| \leq \|x-y\|_2 \leq \|x\|_2 + \|y\|_2.$$

Let

$$K_4(x, y) = (\|x\|_2^{\delta}\|y\|_2^{\beta}\|x-y\|_2^{\lambda-\delta-\beta})^{-1},$$

$$K_5(x, y) = (\|x\|_2^\delta \|y\|_2^\beta (\|x\|_2 + \|y\|_2)^{\lambda-\delta-\beta})^{-1},$$
$$K_6(x, y) = (\|x\|_2^\delta \|y\|_2^\beta (|\|x\|_2 - \|y\|_2|)^{\lambda-\delta-\beta})^{-1},$$

$$T_j(f, x) = \int_{\mathbb{R}_+^n} K_j(x, y) f(y) dy, \tag{17}$$

$$\|T_j\| = \sup_{f \neq 0} \frac{\|T_j f\|_{p,\omega}}{\|f\|_p}, \quad j = 4, 5, 6,$$

where, ω is a nonnegative measurable weight function on $\mathbb{R}_+^n$. If $\delta > 0$, $\beta > 0$, $\lambda - \delta - \beta > 0$, then

$$T_5(f, x) \leq T_4(f, x) \leq T_6(f, x),$$

and therefore,

$$\|T_5\| \leq \|T_4\| \leq \|T_6\|. \tag{18}$$

Thus, we may use the norms $\|T_5\|$, $\|T_6\|$ of the operator T_5, T_6 with the radial kernels to find the norm inequality of the multiple Hardy–Littlewood integral operator T_4. As their generalizations, we introduce the new integral operator T defined by

$$T(f, x) = \int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha) f(y) dy, \quad x \in E_n(\alpha), \tag{19}$$

where, the radial kernel $K(\|x\|_\alpha, \|y\|_\alpha)$ is a nonnegative measurable function defined on $E_n(\alpha) \times E_n(\alpha)$, which satisfies the following condition:

$$K(\|x\|_\alpha, \|y\|_\alpha) = \|x\|_\alpha^{-\lambda} K(1, \|y\|_\alpha \|x\|_\alpha^{-1}), \quad x, y \in E_n(\alpha), \lambda > 0. \tag{20}$$

Equation (19) includes many famous operators as special cases. In particular, for $n = 1$, we have

$$T(f, x) = \int_0^\infty K(x, y) f(y) dy, \quad x > 0, \tag{21}$$

and

$$K(x, y) = x^{-\lambda} K(1, yx^{-1}), \quad x, y > 0, \lambda > 0. \tag{22}$$

The kernel in (3)

$$K(x,y)=\frac{1}{x^{\delta}y^{\beta}|x-y|^{\lambda-\delta-\beta}}$$

satisfies (22). In 2016, the author Kuang [12] proved that if $f \in L^p(\omega_0)$, $g \in L^q(\omega_0)$, $1 < p < \infty$, $\frac{1}{p}+\frac{1}{q}=1$, $\omega_0(x)=x^{1-\lambda}$, and

$$\max\{\frac{1}{p},\delta+\beta+\frac{1}{q}\}<\lambda<1+\delta+\beta<1+\frac{1}{p},$$

then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x^{\delta}y^{\beta}|x-y|^{\lambda-\delta-\beta}}dxdy \le c_0\|f\|_{p,\omega_0}\|g\|_{q,\omega_0},$$

which is equivalent to

$$\|T_0 f\|_p \le c_0\|f\|_{p,\omega_0},$$

where, T_0 is defined by (3) and

$$c_0 = B(\lambda-\frac{1}{p},1-\lambda+\delta+\beta)+B(\frac{1}{p}-\delta-\beta,1-\lambda+\delta+\beta)$$
$$+B(\frac{1}{q},1-\lambda+\delta+\beta)+B(\lambda-\delta-\beta-\frac{1}{q},1-\lambda+\delta+\beta). \quad (23)$$

We define $\omega_1 = x^{\lambda-1}$, then the above norm inequality is also equivalent to

$$\|T_0 f\|_{p,\omega_1} \le c_0\|f\|_p. \quad (24)$$

The celebrated Hardy–Littlewood inequality (1) and (2) are important in analysis mathematics and its applications. In this paper, we give some new improvements and extensions of (24). As some further generalizations of the above results, the norm inequalities of the multiple integral operators with the radial kernels on n-dimensional vector spaces $E_n(\alpha)$ are established. In particular, using new analytical techniques, we convert the exact constant factor we are looking for into the norm of the corresponding operator, under a somewhat different hypothesis, we get lower and upper bounds of the sharp constant of the multiple Hardy–Littlewood inequality. Finally, the discrete versions of the main results are also given in Sect. 6.

2 Main Results

Our main results read as follows.

Theorem 5 *Let* $1 < p, q < \infty, \lambda \geq n > 1$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, $0 \leq \delta < 1 - \frac{1}{q}$, $0 \leq \beta < 1 - \frac{1}{p}$, *and*

$$\max\{\beta + 1 - \frac{1}{q}, \delta + n(1 - \frac{1}{p})\} < \lambda < \min\{\frac{\delta+\beta}{1-(1/n)}, \frac{\delta+\beta}{1-\frac{1}{pn(1-(1/q))}}\}.$$

If $f \in L^p(\mathbb{R}^n_+)$, $f(x) \geq 0$, $x \in \mathbb{R}^n_+$, $\omega(x) = \|x\|_2^{p(\lambda-n)}$, *then the multiple Hardy–Littlewood integral operator* T_4 *is defined by (16):* $T_4 : L^p(\mathbb{R}^n_+) \to L^p(\omega)$ *exists as a bounded operator and*

$$c_3 \leq \|T_4\| \leq c_1^{1-(1/p)} c_2^{1/p},$$

where,

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))$$
$$+ B(\frac{n}{\lambda}(\lambda - \delta - \frac{1}{q} + 1) - n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))\},$$
$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{pn}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))$$
$$+ B(\frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - 1 + \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))\},$$
$$c_3 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}B(n(1 - \frac{1}{p}) - \beta, \lambda - \delta - n(1 - \frac{1}{p})).$$

For $n = 1$, we have

Theorem 6 *Let* $1 < p, q < \infty, \lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $0 \leq \beta < 1 - \frac{1}{p}$, $0 \leq \delta < 1 - \frac{1}{q}$, *and*

$$\max\{\beta + 1 - \frac{1}{q}, \delta + 1 - \frac{1}{p}\} < \lambda < \frac{\delta+\beta}{1-\frac{1}{p(1-(1/q))}}.$$

If $f \in L^p(0,\infty)$, $f(x) \geq 0$, $x \in (0,\infty)$, $\omega(x) = x^{p(\lambda-1)}$, *then the Hardy–Littlewood integral operator* T_0 *is defined by (3):*$T_0 : L^p(0,\infty) \to L^p(\omega)$ *exists as a bounded operator and*

$$c_3 \leq \|T_0\| \leq c_1^{1-(1/p)} c_2^{1/p}$$

where,

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q} - 1 - \beta) + 1, \frac{\delta+\beta}{\lambda}) + B(\frac{1}{\lambda}(1 - \delta - \frac{1}{q}), \frac{\delta+\beta}{\lambda})$$

$$c_2 = B(\frac{p}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - (1 - \frac{\delta+\beta}{\lambda})p(1 - \frac{1}{q}))$$
$$+ B(p(1 - \frac{1}{q})(1 - \frac{1}{\lambda}(\delta + 1 - \frac{1}{p})), 1 - (1 - \frac{\delta+\beta}{\lambda})p(1 - \frac{1}{q})),$$

$$c_3 = B(1 - \beta - \frac{1}{p}, \lambda - \delta - 1 + \frac{1}{p})$$

Corollary 1 *Let* $1 < p < \infty, \frac{1}{p}+\frac{1}{q} = 1, 0 \le \delta < \frac{1}{p}, 0 \le \beta < \frac{1}{q}, \delta+\beta > 0, \lambda = 1$. *If* $f \in L^p(0,\infty)$, $f(x) \ge 0, x \in (0,\infty)$, *then the integral operator* T_0 *is defined by (3):* $T_0 : L^p(0,\infty) \to L^p(0,\infty)$ *exists as a bounded operator and*

$$B(\frac{1}{p} - \delta, \frac{1}{q} - \beta) \le \|T_0\| \le B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta).$$

As some further generalizations of the above results, we have

Theorem 7 *Let* $1 < p < \infty, 1 < q < \infty$, $\delta, \beta \ge 0, \lambda \ge n$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, $\omega(x) = \|x\|_\alpha^{p(\lambda-n)}$, *the radial kernel* $K(\|x\|_\alpha, \|y\|_\alpha)$ *satisfies (20).*

(i) If

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty (K(1,u))^{n/\lambda} u^{\frac{n}{\lambda}(\frac{1}{q}-1)+n-1} du < \infty, \tag{25}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty (K(1,u))^{\frac{pn}{\lambda}(1-\frac{1}{q})} u^{\frac{n(p-1)(q-1)}{\lambda q}-1} du < \infty, \tag{26}$$

then the integral operator T *is defined by (19):* $T : L^p(E_n(\alpha)) \to L^p(\omega)$ *exists as a bounded operator and*

$$\|Tf\|_{p,\omega} \le c\|f\|_p. \tag{27}$$

This implies that

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \le c, \tag{28}$$

where,

$$c = c_1^{(1-(1/p))} c_2^{1/p}. \tag{29}$$

(ii) If

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1,u)u^{(1-\frac{1}{p})n-1} du < \infty, \tag{30}$$

then

$$\|T\| \geq c_3. \tag{31}$$

In particular, for $n = 1$, by Theorem 7, we get

Theorem 8 *Let* $1 < p < \infty, 1 < q < \infty$, $\delta, \beta \geq 0$, $1 \leq \lambda = 2 - \frac{1}{p} - \frac{1}{q}$, $\omega(x) = x^{p(\lambda-1)}$, *the radial kernel* $K(x, y)$ *satisfies (22).*

(i) If

$$c_1 = \int_0^\infty (K(1,u))^{1/\lambda} u^{\frac{1}{\lambda}(\frac{1}{q}-1)} du < \infty, \tag{32}$$

$$c_2 = \int_0^\infty (K(1,u))^{\frac{p}{\lambda}(1-\frac{1}{q})} u^{\frac{(p-1)(q-1)}{\lambda q}-1} du < \infty, \tag{33}$$

then the integral operator T *is defined by (21):*$T : L^p(0,\infty) \to L^p(\omega)$ *exists as a bounded operator and*

$$\|Tf\|_{p,\omega} \leq c\|f\|_p. \tag{34}$$

This implies that

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \leq c, \tag{35}$$

where,

$$c = c_1^{(1-(1/p))} c_2^{1/p}. \tag{36}$$

(ii) If

$$c_3 = \int_0^\infty K(1,u)u^{-\frac{1}{p}} du < \infty, \tag{37}$$

then

$$\|T\| \geq c_3. \tag{38}$$

For $\lambda = n$, we have $\frac{1}{p} + \frac{1}{q} = 1$, and by Theorem 7, we get

Theorem 9 *Let* $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\delta, \beta \geq 0$, *the radial kernel* $K(\|x\|_\alpha, \|y\|_\alpha)$ *satisfies:*

$$K(\|x\|_\alpha, \|y\|_\alpha) = \|x\|_\alpha^{-n} K(1, \|y\|_\alpha \|x\|_\alpha^{-1}), \ \ x, y \in E_n(\alpha).$$

(i) If

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, u) u^{-(1/p)+n-1} du < \infty, \tag{39}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, u) u^{-(1/p)} du < \infty, \tag{40}$$

then the integral operator T *is defined by (19):* $T : L^p(E_n(\alpha)) \to L^p(E_n(\alpha))$ *exists as a bounded operator and*

$$\|Tf\|_p \leq c\|f\|_p. \tag{41}$$

This implies that

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|_p}{\|f\|_p} \leq c, \tag{42}$$

where,

$$c = c_1^{(1/q)} c_2^{(1/p)}. \tag{43}$$

(ii) If

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, u) u^{(n/q)-1} du < \infty, \tag{44}$$

then,

$$\|T\| \geq c_3. \tag{45}$$

In particular, for $n = 1$, by Theorem 9, we get

$$c = c_1 = c_2 = c_3 = \int_0^\infty K(1, u) u^{-(1/p)} du, \tag{46}$$

then by (42), (45), and (46), we get

$$\|T\| = c = \int_0^\infty K(1, u) u^{-(1/p)} du. \tag{47}$$

Thus, we get the following

Corollary 2 *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, *the kernel* $K(x, y)$ *satisfies (22). Then the integral operator* T *is defined by (21):* $T : L^p(0, \infty) \to L^p(0, \infty)$ *exists as a bounded operator and*

$$\|Tf\|_p \leq c\|f\|_p, \tag{48}$$

where $\|T\| = c = \int_0^\infty K(1, u)u^{-(1/p)}du$ *is the sharp constant.*

3 Proofs of Theorems

We require the following lemmas to prove our results:

Lemma 1 ([4, 13]) *If* $a_k, b_k, p_k > 0, 1 \leq k \leq n, f$ *is a measurable function on* $(0, \infty)$*, then*

$$\int_{\mathbb{R}_+^n} f\Big(\sum_{k=1}^{n}(\frac{x_k}{a_k})^{b_k}\Big)x_1^{p_1-1}\cdots x_n^{p_n-1}dx_1\cdots dx_n$$
$$= \frac{\prod_{k=1}^n a_k^{p_k}}{\prod_{k=1}^n b_k} \times \frac{\prod_{k=1}^n \Gamma(\frac{p_k}{b_k})}{\Gamma(\sum_{k=1}^n \frac{p_k}{b_k})}\int_0^\infty f(t)t^{(\sum_{k=1}^n \frac{p_k}{b_k}-1)}dt.$$

We get the following Lemma 2 by taking $a_k = 1, b_k = \alpha > 0, p_k = 1, 1 \leq k \leq n$, in Lemma 1.

Lemma 2 *Let* f *be a measurable function on* $(0, \infty)$*, then*

$$\int_{E_n(\alpha)} f(\|x\|_\alpha^\alpha)dx = \frac{(\Gamma(1/\alpha))^n}{\alpha^n\Gamma(n/\alpha)}\int_0^\infty f(t)t^{(n/\alpha)-1}dt. \tag{49}$$

Proof of Theorem 7

(i) Let

$$p_1 = \frac{p}{p-1},\ q_1 = \frac{q}{q-1},$$

thus, we have

$$\frac{1}{p_1} + \frac{1}{q_1} + (1 - \frac{\lambda}{n}) = 1,\ \frac{p}{q_1} + p(1 - \frac{\lambda}{n}) = 1.$$

By Hölder's inequality, we get

$$\begin{aligned}T(f,x)&=\int_{E_n(\alpha)}K(\|x\|_\alpha,\|y\|_\alpha)f(y)dy\\&=\int_{E_n(\alpha)}\{\|y\|_\alpha^{(\frac{n}{p_1\lambda})}K^{n/\lambda}(\|x\|_\alpha,\|y\|_\alpha)f^p(y)\}^{1/q_1}\\&\times\{K^{n/\lambda}(\|x\|_\alpha,\|y\|_\alpha)\|y\|_\alpha^{-(\frac{n}{q_1\lambda})}\}^{1/p_1}\{f(y)\}^{p(1-\frac{\lambda}{n})}dy\\&\le\{\int_{E_n(\alpha)}\|y\|_\alpha^{\frac{n}{p_1\lambda}}K^{n/\lambda}(\|x\|_\alpha,\|y\|_\alpha)|f(y)|^pdy\}^{1/q_1}\\&\times\{\int_{E_n(\alpha)}\|y\|_\alpha^{-(\frac{n}{q_1\lambda})}K^{n/\lambda}(\|x\|_\alpha,\|y\|_\alpha)dy\}^{1/p_1}\|f\|_p^{p(1-\frac{\lambda}{n})}\\&=I_1^{1/q_1}\times I_2^{1/p_1}\times\|f\|_p^{p(1-\frac{\lambda}{n})},\end{aligned}\tag{50}$$

where,

$$I_1=\int_{E_n(\alpha)}\|y\|_\alpha^{(\frac{n}{p_1\lambda})}K^{n/\lambda}(\|x\|_\alpha,\|y\|_\alpha)|f(y)|^pdy,$$

$$I_2=\int_{E_n(\alpha)}\|y\|_\alpha^{-(\frac{n}{q_1\lambda})}K^{n/\lambda}(\|x\|_\alpha,\|y\|_\alpha)dy.$$

In I_2, by using Lemma 2, and letting $u=\|x\|_\alpha^{-1}t^{1/\alpha}$, and use (20), (49), and (25), we get

$$\begin{aligned}I_2&=\|x\|_\alpha^{-n}\int_{E_n(\alpha)}\|y\|_\alpha^{-(\frac{n}{q_1\lambda})}K^{n/\lambda}(1,\|y\|_\alpha\cdot\|x\|_\alpha^{-1})dy\\&=\|x\|_\alpha^{-n}\frac{(\Gamma(1/\alpha))^n}{\alpha^n\Gamma(n/\alpha)}\int_0^\infty t^{-(\frac{n}{q_1\lambda\alpha})}K^{n/\lambda}(1,t^{1/\alpha}\|x\|_\alpha^{-1})\times t^{\frac{n}{\alpha}-1}dt\\&=\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\|x\|_\alpha^{-(\frac{n}{q_1\lambda})}\int_0^\infty K^{\frac{n}{\lambda}}(1,u)u^{-(\frac{n}{q_1\lambda})+n-1}du\\&=c_1\|x\|_\alpha^{-(\frac{n}{q_1\lambda})}.\end{aligned}\tag{51}$$

Hence, by (50) and (51), we conclude that

$$\begin{aligned}\|Tf\|_{p,\omega}&=(\int_{E_n(\alpha)}|T(f,x)|^p\omega(x)dx)^{1/p}\le(\int_{E_n(\alpha)}I_1^{\frac{p}{q_1}}I_2^{\frac{p}{p_1}}\|f\|_p^{p^2(1-\frac{\lambda}{n})}\omega(x)dx)^{1/p}\\&=c_1^{\frac{1}{p_1}}\|f\|_p^{p(1-\frac{\lambda}{n})}\{\int_{E_n(\alpha)}(\int_{E_n(\alpha)}\|y\|_\alpha^{\frac{n}{p_1\lambda}}K^{\frac{n}{\lambda}}(\|x\|_\alpha,\|y\|_\alpha)|f(y)|^pdy)^{\frac{p}{q_1}}\\&\times\|x\|_\alpha^{-\frac{pn}{p_1q_1\lambda}}\omega(x)dx\}^{1/p}.\end{aligned}\tag{52}$$

Using the Minkowski's inequality for integrals (see [3]):

$$\{\int_X (\int_Y |f(x,y)|dy)^p \omega(x)dx\}^{1/p} \le \int_Y \{\int_X |f(x,y)|^p \omega(x)dx\}^{1/p} dy,\ 1 \le p < \infty,$$

and letting $v = \|y\|_\alpha t^{-(1/\alpha)}$, we obtain

$$\begin{aligned}
\|Tf\|_{p,\omega} &\le c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times (\int_{E_n(\alpha)} K^{\frac{pn}{q_1\lambda}}(\|x\|_\alpha, \|y\|_\alpha) \|x\|_\alpha^{-\frac{pn}{p_1 q_1 \lambda}} \omega(x)dx)^{\frac{q_1}{p}} dy\}^{1/q_1} \\
&= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times (\int_{E_n(\alpha)} K^{\frac{pn}{q_1\lambda}}(1, \|y\|_\alpha \cdot \|x\|_\alpha^{-1}) \|x\|_\alpha^{-\frac{pn}{q_1} - \frac{pn}{p_1 q_1 \lambda} + p(\lambda-n)} dx)^{\frac{q_1}{p}} dy\}^{\frac{1}{q_1}} \\
&= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times \Big(\frac{(\Gamma(1/\alpha))^n}{\alpha^n \Gamma(n/\alpha)} \int_0^\infty K^{\frac{pn}{q_1\lambda}}(1, \|y\|_\alpha \cdot t^{-\frac{1}{\alpha}}) t^{-\frac{pn}{q_1\alpha} - \frac{pn}{\lambda\alpha p_1 q_1} + \frac{p(\lambda-n)}{\alpha}} t^{\frac{n}{\alpha}-1} dt\Big)^{\frac{q_1}{p}} dy\}^{\frac{1}{q_1}} \\
&= c_1^{\frac{1}{p_1}} \|f\|_p^{p(1-\frac{\lambda}{n})} \{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{p_1\lambda}} |f(y)|^p \\
&\times \Big(\|y\|_\alpha^{-\frac{pn}{q_1 p_1 \lambda}} \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K^{\frac{pn}{q_1\lambda}}(1, v) v^{\frac{pn}{p_1 q_1 \lambda} - 1} dv\Big)^{\frac{q_1}{p}} dy\}^{\frac{1}{q_1}} \\
&= c_1^{1/p_1} c_2^{1/p} \|f\|_p^{p(1-\frac{\lambda}{n})} \|f\|_p^{\frac{p}{q_1}} = c_1^{(1-(1/p))} c_2^{1/p} \|f\|_p.
\end{aligned}$$

Thus,

$$\|Tf\|_{p,\omega} \le c\|f\|_p. \tag{53}$$

(ii) For proving (31), we take

$$f_\varepsilon(x) = \|x\|_\alpha^{-(n/p)-\varepsilon} \varphi_{B^c}(x),$$

$$g_\varepsilon(x) = (p\varepsilon)^{1/p_1} \{\frac{\alpha^{n-1}\Gamma(n/\alpha)}{(\Gamma(1/\alpha))^n}\}^{1/p_1} \|x\|_\alpha^{-\frac{n}{p_1} - (p-1)\varepsilon} \varphi_{B^c}(x),$$

where, $\varepsilon > 0, B = B(0,1) = \{x \in E_n(\alpha) : \|x\|_\alpha < 1\}$, φ_{B^c} is the characteristic function of the set $B^c = \{x \in E_n(\alpha) : \|x\|_\alpha \ge 1\}$, that is

$$\varphi_{B^c}(x) = \begin{cases} 1, x \in B^c \\ 0, x \in B. \end{cases}$$

Thus, we get

$$\|f_\varepsilon\|_p = \big(\frac{(\Gamma(1/\alpha))^n}{p\varepsilon\alpha^{n-1}\Gamma(n/\alpha)}\big)^{1/p},$$

$$\|g_\varepsilon\|_{p_1}^{p_1} = (p\varepsilon)(\frac{\Gamma^n(1/\alpha)}{\alpha^{n-1}\Gamma(n/\alpha)})^{-1}\int_{B^c}\|x\|_\alpha^{-n-(p-1)p_1\varepsilon}dx$$

$$= (p\varepsilon)\frac{1}{\alpha}\int_1^\infty t^{-\frac{p\varepsilon}{\alpha}-1}dt = 1.$$

Using the sharpness in Hölder's inequality (see [13]):

$$\|Tf\|_{p,\omega} = \sup\{|\int_{E_n(\alpha)} T(f,x)g(x)(\omega(x))^{1/p}dx| : \|g\|_{p_1} \le 1\},$$

where, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p_1} = 1$, thus, if $\|g\|_{p_1} \le 1$, then

$$|\int_{E_n(\alpha)} T(f,x)g(x)\{\omega(x)\}^{1/p}dx| \le \|Tf\|_{p,\omega}. \tag{54}$$

By (54) and (19), we get

$$\|Tf_\varepsilon\|_{p,\omega} \ge \int_{E_n(\alpha)} T(f_\varepsilon,x)g_\varepsilon(x)\{\omega(x)\}^{1/p}dx$$

$$= \int_{E_n(\alpha)}\int_{E_n(\alpha)} K(\|x\|_\alpha,\|y\|_\alpha)f_\varepsilon(y)g_\varepsilon(x)\|x\|_\alpha^{\lambda-n}dydx$$

$$= (p\varepsilon)^{1/p_1}\{\frac{\alpha^{n-1}\Gamma(n/\alpha)}{(\Gamma(1/\alpha))^n}\}^{1/p_1}$$

$$\times \int_{B^c}\{\int_{B^c} K(\|x\|_\alpha,\|y\|_\alpha)\|y\|_\alpha^{-(n/p)-\varepsilon}dy\}\|x\|_\alpha^{-\frac{n}{p_1}-(p-1)\varepsilon+\lambda-n}dx. \tag{55}$$

Letting $u = t^{1/\alpha}\|x\|_\alpha^{-1}$, and using (20), we have

$$\int_{B^c} K(\|x\|_\alpha,\|y\|_\alpha)\|y\|_\alpha^{-(n/p)-\varepsilon}dy$$

$$= \frac{(\Gamma(1/\alpha))^n}{\alpha^n\Gamma(n/\alpha)}\|x\|_\alpha^{-\lambda}\int_1^\infty K(1,t^{1/\alpha}\|x\|_\alpha^{-1})t^{-(\frac{n}{p\alpha})-\frac{\varepsilon}{\alpha}+\frac{n}{\alpha}-1}dt$$

$$= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\|x\|_\alpha^{-\lambda+\frac{n}{p_1}-\varepsilon}\int_{\|x\|_\alpha^{-1}}^\infty K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}du. \tag{56}$$

We insert (56) into (55) and use Fubini's theorem to obtain

$$\|Tf_\varepsilon\|_{p,\omega} \geq (p\varepsilon)^{1/p_1}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{1/p}$$

$$\times \int_{B^c} \|x\|_\alpha^{-p\varepsilon-n}(\int_{\|x\|_\alpha^{-1}}^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}du)dx$$

$$= (p\varepsilon)^{1/p_1}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{1/p}$$

$$\times \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\int_{\beta(u)}^{\infty} \|x\|_\alpha^{-p\varepsilon-n}dx)du$$

$$= (p\varepsilon)^{1/p_1}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{(1/p)+1} \times \frac{1}{\alpha}$$

$$\times \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\int_{\beta(u)}^{\infty} t^{-(p\varepsilon)/\alpha-1}dt)du$$

$$= (p\varepsilon)^{-(1/p)}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{(1/p)+1}$$

$$\times \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\beta(u))^{-(p\varepsilon)/\alpha}du,$$

where, $\beta(u) = \max\{1, u^{-1}\}$. Thus, we get

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \geq \frac{\|Tf_\varepsilon\|_{p,\omega}}{\|f_\varepsilon\|_p}$$

$$\geq \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-\varepsilon-1}(\beta(u))^{-(p\varepsilon)/\alpha}du. \tag{57}$$

By letting $\varepsilon \to 0^+$ in (57) and using Fatou lemma, we get

$$\|T\| \geq \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^{\infty} K(1,u)u^{\frac{n}{p_1}-1}du = c_3.$$

The proof is complete.

4 Some Applications

As applications, a large number of known and new results have been obtained by proper choice of kernel K. In this section we present some model applications which display the importance of our results.

Example 1 Let $h : E_n(\alpha) \times E_n(\alpha) \to \mathbb{R}_+$ be a measurable function. K_7 is defined by

$$K_7(\|x\|_\alpha, \|y\|_\alpha) = \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta |\|x\|_\alpha - \|y\|_\alpha|^{\lambda-\delta-\beta}}, \tag{58}$$

and let

$$T_7(f, x) = \int_{E_n(\alpha)} \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta |\|x\|_\alpha - \|y\|_\alpha|^{\lambda-\delta-\beta}} f(y)dy.$$

If p, q, λ, and ω satisfy the conditions of Theorem 7, and

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta |1-u|^{\lambda-\delta-\beta}}\}^{\frac{n}{\lambda}} u^{\frac{n}{\lambda}(\frac{1}{q}-1)+n-1} du < \infty, \tag{59}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta |1-u|^{\lambda-\delta-\beta}}\}^{\frac{pn}{\lambda}(1-\frac{1}{q})} u^{\frac{n}{\lambda}(p-1)(1-\frac{1}{q})-1} du < \infty, \tag{60}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{h(u)}{u^\beta |1-u|^{\lambda-\delta-\beta}} u^{n(1-\frac{1}{p})-1} du < \infty, \tag{61}$$

then by Theorem 7, we get

$$c_3 \le \|T_7\| \le c_1^{(1-(1/p))} c_2^{1/p}. \tag{62}$$

Setting $h(u) = 1$, we distinguish four cases:

(i) The case $n > 1$. Let $0 \le \delta < 1 - \frac{1}{q}$, $0 \le \beta < 1 - \frac{1}{p}$, and

$$\max\{\beta+1-\frac{1}{q}, \delta+n(1-\frac{1}{p})\} < \lambda < \min\{1+\delta+\beta, \frac{\delta+\beta}{1-(1/n)}, \frac{\delta+\beta}{1-\frac{1}{pn(1-(1/q))}}\},$$

then by (59), (60), and (61), we get

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \{B(\frac{n}{\lambda}(\frac{1}{q}-1-\beta)+n, 1-\frac{n}{\lambda}(\lambda-\delta-\beta)) + B(\frac{n}{\lambda}(\lambda-\delta-\frac{1}{q}+1)-n, 1-\frac{n}{\lambda}(\lambda-\delta-\beta))\}, \tag{63}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \{B(\frac{pn}{\lambda}(1-\frac{1}{q})(1-\beta-\frac{1}{p}), 1-\frac{pn}{\lambda}(1-\frac{1}{q})(\lambda-\delta-\beta)) + B(\frac{pn}{\lambda}(1-\frac{1}{q})(\lambda-\delta-1+\frac{1}{p}), 1-\frac{pn}{\lambda}(1-\frac{1}{q})(\lambda-\delta-\beta))\}, \tag{64}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(n(1-\frac{1}{p})-\beta, 1-\lambda+\delta+\beta)$$

$$+ B(\lambda-\delta-n(1-\frac{1}{p}), 1-\lambda+\delta+\beta)\}. \quad (65)$$

(ii) The case $n = 1$. Let $0 \le \beta < 1 - \frac{1}{p}$,$0 \le \delta < 1 - \frac{1}{q}$,$\delta + \beta > 0$, and

$$\max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\} < \lambda < \min\{1 + \delta + \beta, \frac{\delta + \beta}{1 - \frac{1}{p(1-(1/q))}}\},$$

then by (59), (60), and (61), we get

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q} - 1 - \beta) + 1, \frac{\delta+\beta}{\lambda}) + B(\frac{1}{\lambda}(1 - \delta - \frac{1}{q}), \frac{\delta+\beta}{\lambda}), \quad (66)$$

$$c_2 = B(\frac{p}{\lambda}(1-\frac{1}{q})(1-\beta-\frac{1}{p}), 1-(1-\frac{\delta+\beta}{\lambda})p(1-\frac{1}{q}))$$

$$+ B(p(1-\frac{1}{q})(1-\frac{1}{\lambda}(\delta+1-\frac{1}{p})), 1-(1-\frac{\delta+\beta}{\lambda})p(1-\frac{1}{q})), \quad (67)$$

$$c_3 = B(1-\beta-\frac{1}{p}, 1-\lambda+\delta+\beta) + B(\lambda-\delta-1+\frac{1}{p}, 1-\lambda+\delta+\beta). \quad (68)$$

(iii) The case $\lambda = n$, this implies that $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le \delta < \min\{\frac{1}{p}, n - \frac{1}{q}\}$, $0 \le \beta < \min\{\frac{1}{q}, n - \frac{1}{p}\}$, $n - 1 < \delta + \beta$, then by (59), (60), and (61), we get

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(n-\frac{1}{p}-\beta, 1-n+\delta+\beta)$$

$$+ B(\frac{1}{p}-\delta, 1-n+\delta+\beta)\}, \quad (69)$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(\frac{1}{q}-\beta, 1-n+\delta+\beta)$$

$$+ B(n-\delta-\frac{1}{q}, 1-n+\delta+\beta)\}, \quad (70)$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\{B(\frac{n}{q}-\beta, 1-n+\delta+\beta)$$

$$+ B(\frac{n}{p}-\delta, 1-n+\delta+\beta)\}. \quad (71)$$

(iv) The case $\lambda = n = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$, $\delta + \beta > 0$, then by (69), (70), and (71), we get

$$\|T_7\| = B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta). \tag{72}$$

Example 2 Let $h : E_n(\alpha) \times E_n(\alpha) \to \mathbb{R}_+$ be a measurable function. K_8 is defined by

$$K_8(\|x\|_\alpha, \|y\|_\alpha) = \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta (\|x\|_\alpha + \|y\|_\alpha)^{\lambda-\delta-\beta}}, \tag{73}$$

and let

$$T_8(f, x) = \int_{E_n(\alpha)} \frac{h(\|y\|_\alpha \cdot \|x\|_\alpha^{-1})}{\|x\|_\alpha^\delta \|y\|_\alpha^\beta (\|x\|_\alpha + \|y\|_\alpha)^{\lambda-\delta-\beta}} f(y)dy.$$

If p, q, λ, and ω satisfy the conditions of Theorem 7, and

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta (1+u)^{\lambda-\delta-\beta}}\}^{\frac{n}{\lambda}} u^{\frac{n}{\lambda}(\frac{1}{q}-1)+n-1} du < \infty, \tag{74}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta (1+u)^{\lambda-\delta-\beta}}\}^{\frac{pn}{\lambda}(1-\frac{1}{q})} u^{\frac{n}{\lambda}(p-1)(1-\frac{1}{q})-1} du < \infty, \tag{75}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \{\frac{h(u)}{u^\beta (1+u)^{\lambda-\delta-\beta}}\} u^{n(1-\frac{1}{p})-1} du < \infty, \tag{76}$$

then by Theorem 7, we get

$$c_3 \le \|T_8\| \le c_1^{(1-(1/p))} c_2^{1/p}.$$

Setting $h(u) = 1$, we distinguish four cases:

(i) The case $n > 1$. Let $0 \le \delta < 1 - \frac{1}{q}$, $0 \le \beta < 1 - \frac{1}{p}$, and

$$\lambda > \max\{\beta + 1 - \frac{1}{q}, \delta + n(1 - \frac{1}{p})\},$$

then by (74), (75), and (76), we get

$$c_1 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, \frac{n}{\lambda}(\lambda - \delta + 1 - \frac{1}{q}) - n), \tag{77}$$

$$c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{pn}{\lambda}(1-\frac{1}{q})(1-\frac{1}{p}-\beta), \frac{pn}{\lambda}(1-\frac{1}{q})(\lambda-\delta-1+\frac{1}{p})), \tag{78}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(n(1-\frac{1}{p})-\beta, \lambda-\delta-n(1-\frac{1}{p})). \tag{79}$$

(ii) The case $n = 1$. Let $0 \le \beta < 1 - \frac{1}{p}$, $0 \le \delta < 1 - \frac{1}{q}$, and $\lambda > \max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\}$, then by (74), (75), and (76), we get

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q}-1-\beta)+1, \frac{1}{\lambda}(1-\delta-\frac{1}{q})), \tag{80}$$

$$c_2 = B(\frac{p}{\lambda}(1-\frac{1}{q})(1-\frac{1}{p}-\beta), \frac{p}{\lambda}(1-\frac{1}{q})(\lambda-\delta-1+\frac{1}{p})), \tag{81}$$

$$c_3 = B(1-\beta-\frac{1}{p}, \lambda-\delta-1+\frac{1}{p}). \tag{82}$$

(iii) The case $\lambda = n$, this implies that $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le \beta < \frac{1}{q}$, $0 \le \delta < \min\{\frac{n}{p}, n - \frac{1}{q}\}$, then by (74), (75), and (76), we get

$$c_1 = c_2 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{1}{q}-\beta, n-\delta-\frac{1}{q}), \tag{83}$$

$$c_3 = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{n}{q}-\beta, \frac{n}{p}-\delta), \tag{84}$$

and

$$c_3 \le \|T_8\| \le c_1. \tag{85}$$

(iv) The case $\lambda = n = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$, $\alpha + \beta > 0$, and $\max\{\beta + \frac{1}{p}, \delta + \frac{1}{q}\} < 1$, then by (83), (84), and (85), we get

$$\|T_8\| = B(\frac{1}{p}-\delta, \frac{1}{q}-\beta). \tag{86}$$

5 Multiple Hardy–Littlewood Integral Operator Norm Inequalities

In Examples 1 and 2, setting $h(u) = 1, \alpha = 2$, thus, $E_n(\alpha)$ reduces to $E_n(2) = \mathbb{R}^n_+$, T_7, T_8 reduces to T_6, T_5, respectively. Assume $f \in L^p(\mathbb{R}^n_+)$, $f(x) \geq 0, x \in \mathbb{R}^n_+$, $1 < p, q < \infty, \lambda \geq n, \delta, \beta \geq 0, \frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$. The multiple Hardy–Littlewood integral operator T_4 is defined by (16):$T_4 : L^p(\mathbb{R}^n_+) \to L^p(\omega)$, where $\omega(x) = \|x\|_2^{p(\lambda-n)}$ and

$$\|T_4\| = \sup_{f \neq 0} \frac{\|T_4\|_{p,\omega}}{\|f\|_p}.$$

We distinguish four cases:

(i) The case $n > 1$. Let $0 \leq \delta < 1 - \frac{1}{q}, 0 \leq \beta < 1 - \frac{1}{p}$, and

$$\max\{\beta + 1 - \frac{1}{q}, \delta + n(1 - \frac{1}{p})\} < \lambda < \min\{\frac{\delta + \beta}{1 - (1/n)}, \frac{\delta + \beta}{1 - \frac{1}{pn(1-(1/q))}}\},$$

then by (18), (63), (64) and (79), we get

$$c_3 \leq \|T_4\| \leq c_1^{1-(1/p)} c_2^{1/p}, \tag{87}$$

where

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{n}{\lambda}(\frac{1}{q} - 1 - \beta) + n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta)) + B(\frac{n}{\lambda}(\lambda - \delta - \frac{1}{q} + 1) - n, 1 - \frac{n}{\lambda}(\lambda - \delta - \beta))\}, \tag{88}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{pn}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta)) + B(\frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - 1 + \frac{1}{p}), 1 - \frac{pn}{\lambda}(1 - \frac{1}{q})(\lambda - \delta - \beta))\}, \tag{89}$$

$$c_3 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(n(1 - \frac{1}{p}) - \beta, \lambda - \delta - n(1 - \frac{1}{p})). \tag{90}$$

(ii) The case $n = 1$. Let $0 \leq \beta < 1 - \frac{1}{p}, 0 \leq \delta < 1 - \frac{1}{q}$, and

$$\max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\} < \lambda < \frac{\delta + \beta}{1 - \frac{1}{p(1-(1/q))}},$$

then by (18), (66), (67) and (82), we get

$$c_3 \le \|T_4\| \le c_1^{1-(1/p)} c_2^{1/p},$$

where,

$$c_1 = B(\frac{1}{\lambda}(\frac{1}{q} - 1 - \beta) + 1, \frac{\delta+\beta}{\lambda}) + B(\frac{1}{\lambda}(1 - \delta - \frac{1}{q}), \frac{\delta+\beta}{\lambda}), \tag{91}$$

$$c_2 = B(\frac{p}{\lambda}(1 - \frac{1}{q})(1 - \beta - \frac{1}{p}), 1 - (1 - \frac{\delta+\beta}{\lambda})p(1 - \frac{1}{q})) + B(p(1 - \frac{1}{q})(1 - \frac{1}{\lambda}(\delta + 1 - \frac{1}{p})), 1 - (1 - \frac{\delta+\beta}{\lambda})p(1 - \frac{1}{q})), \tag{92}$$

$$c_3 = B(1 - \beta - \frac{1}{p}, \lambda - \delta - 1 + \frac{1}{p}). \tag{93}$$

(iii) The case $\lambda = n$, this implies that $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$, and

$$\max\{\beta + \frac{1}{p}, \delta + \frac{1}{q}\} < n < 1 + \delta + \beta,$$

then by (18), (69), (70), and (84), we get

$$c_3 \le \|T_4\| \le c_1^{1-(1/p)} c_2^{1/p},$$

where

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(n - \frac{1}{p} - \beta, 1 - n + \delta + \beta) + B(\frac{1}{p} - \delta, 1 - n + \delta + \beta)\}, \tag{94}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)}\{B(\frac{1}{q} - \beta, 1 - n + \delta + \beta) + B(n - \delta - \frac{1}{q}, 1 - n + \delta + \beta)\}, \tag{95}$$

$$c_3 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(\frac{n}{q} - \beta, \frac{n}{p} - \delta). \tag{96}$$

(iv) The case $\lambda = n = 1$. Let $0 \le \delta < \frac{1}{p}$, $0 \le \beta < \frac{1}{q}$,$\delta+\beta > 0$, then by (18), (72), and (86), we get

$$B(\frac{1}{p} - \delta, \frac{1}{q} - \beta) \le \|T_4\| \le B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta). \tag{97}$$

We have thus also proved that Theorems 5 and 6 are correct.

6 The Discrete Versions of the Main Results

Let $a = \{a_m\}$ be a sequence of real numbers, we define

$$\|a\|_{p,\omega} = \{\sum_{m=1}^{\infty} |a_m|^p \omega(m)\}^{1/p},\ \ l^p(\omega) = \{a = \{a_m\} : \|a\|_{p,\omega} < \infty\}.$$

If $\omega(m) \equiv 1$, we will denote $l^p(\omega)$ by l^p, and $\|a\|_{p,1}$ by $\|a\|_p$. Defining f, K by $f(x) = a_m,\ K(x, y) = K(m, n)(m - 1 \le x < m, n - 1 \le y < n)$, respectively, we obtain the corresponding series form of (21):

$$T(a, m) = \sum_{n=1}^{\infty} K(m, n) a_n. \tag{98}$$

Then by Theorem 8, we get

Theorem 10 *Let* $1 < p < \infty$, $1 < q < \infty$, $\delta, \beta \ge 0$, $\delta+\beta > 0$, $1 \le \lambda = 2-\frac{1}{p}-\frac{1}{q}$, $\omega(m) = m^{p(\lambda-1)}$, *the kernel* $K(m, n)$ *satisfies*

$$K(m, n) = m^{-\lambda} K(1, nm^{-1}). \tag{99}$$

(i) If

$$c_1 = \int_0^{\infty} (K(1, u))^{\frac{1}{\lambda}} u^{\frac{1-q}{\lambda q}} du < \infty, \tag{100}$$

$$c_2 = \int_0^{\infty} (K(1, u))^{\frac{p}{\lambda}(1-\frac{1}{q})} u^{\frac{(p-1)(q-1)}{\lambda q}-1} du < \infty, \tag{101}$$

then the integral operator T *is defined by (98):* $T : l^p \to l^p(\omega)$ *exists as a bounded operator and*

$$\|Ta\|_{p,\omega} \le c\|a\|_p. \tag{102}$$

This implies that

$$\|T\| = \sup_{a \neq 0} \frac{\|Ta\|_{p,\omega}}{\|a\|_p} \leq c, \tag{103}$$

where

$$c = c_1^{(1-(1/p))} c_2^{1/p}. \tag{104}$$

(ii) If

$$c_3 = \int_0^\infty K(1,u) u^{-\frac{1}{p}} du < \infty, \tag{105}$$

then

$$\|T\| \geq c_3. \tag{106}$$

For $\lambda = 1$, we have $\frac{1}{p} + \frac{1}{q} = 1$ and by Theorem 10, we get

$$\|Ta\|_p \leq c\|a\|_p, \tag{107}$$

where $c = \|T\| = \int_0^\infty K(1,u)u^{-(1/p)}du$ is the sharp constant. In particular, let

$$K(m,n) = \frac{1}{m^\delta n^\beta |m-n|^{\lambda-\delta-\beta}},$$

if $0 \leq \beta < 1 - \frac{1}{p}$, $0 \leq \delta < 1 - \frac{1}{q}$, $\delta + \beta > 0$, and

$$\max\{\delta + 1 - \frac{1}{p}, \beta + 1 - \frac{1}{q}\} < \lambda < \min\{1 + \delta + \beta, \frac{\delta+\beta}{1 - \frac{1}{p(1-(1/q))}}\},$$

then by Example 1, we get

$$c_3 \leq \|T\| \leq c_1^{1-(1/p)} c_2^{1/p}, \tag{108}$$

where c_1, c_2, and c_3 are defined by (66), (67), and (68), respectively.

If $\lambda = 1$, that is, $0 \leq \delta < \frac{1}{p}$, $0 \leq \beta < \frac{1}{q}$, $\delta + \beta > 0$, then by (72), we have

$$\|T\| = B(\frac{1}{p} - \delta, \delta + \beta) + B(\frac{1}{q} - \beta, \delta + \beta). \tag{109}$$

Remark 2 In 2016, the author Kuang [12] proved that if $a = \{a_m\} \in l^p(\omega_0)$, $b = \{b_n\} \in l^q(\omega_0)$, $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\omega_0(m) = m^{1-\lambda}$, and

$$\max\{\frac{1}{p}, \delta + \beta + \frac{1}{q}\} < \lambda < 1 + \beta + \delta < 1 + \frac{1}{p}$$

then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m^\delta n^\beta |m-n|^{\lambda-\delta-\beta}} \leq c_0 \|a\|_{p,\omega_0} \|b\|_{q,\omega_0}, \tag{110}$$

where c_0 is defined by (23). Inequality (110) is equivalent to

$$\|T_9(a)\|_p \leq c_0 \|a\|_{p,\omega_0}, \tag{111}$$

where,

$$T_9(a, m) = \sum_{n=1}^{\infty} \frac{a_n}{m^\delta n^\beta |m-n|^{\lambda-\delta-\beta}}$$

is the Hardy–Littlewood operator. We define $\omega_1(m) = m^{\lambda-1}$, then the above norm inequality is also equivalent to

$$\|T_9(a)\|_{p,\omega_1} \leq c_0 \|a\|_p. \tag{112}$$

Hence, (108) and (109) are new improvements and extensions of (112).

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Norm Inequalities for Generalized Fractional Integral Operators

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Abstract Some new norms of the integral operator with the radial kernel on n-dimensional vector spaces are deduced. These norms used then to establish some new norm inequalities for generalized fractional integral operators and the Riesz potential operators.

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1 Introduction

Throughout this paper, we write

$$E_n(\alpha) = \{x = (x_1, x_2, \cdots, x_n) : x_k \geq 0, 1 \leq k \leq n, \|x\|_\alpha = (\sum_{k=1}^{n} |x_k|^\alpha)^{1/\alpha}, \alpha > 0\},$$

$E_n(\alpha)$ is an n-dimensional vector space, when $1 \leq \alpha < \infty$, $E_n(\alpha)$ is a normed vector space. In particular, $E_n(2)$ is an n-dimensional Euclidean space $\mathbb{R}^n_+$.

$$\|f\|_{p,\omega} = (\int_{E_n(\alpha)} |f(x)|^p \omega(x) dx)^{1/p},$$

$$L^p(\omega) = \{f : f \ is \ measurable, \ and \|f\|_{p,\omega} < \infty\},$$

where ω is a nonnegative measurable function on $E_n(\alpha)$. If $\omega(x) \equiv 1$, we will denote $L^p(\omega)$ by $L^p(E_n(\alpha))$, and $\|f\|_{p,1}$ by $\|f\|_p$. $\Gamma(\alpha)$ is the Gamma function:

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$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \ (\alpha > 0).$$

$B(\alpha, \beta)$ is the Beta function:

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx \ (\alpha, \beta > 0).$$

For a given function $\Omega : (0, \infty) \to (0, \infty)$, the generalized fractional integral operator T_0 defined by (see [4])

$$T_0(f, x) = \int_{\mathbb{R}^n} \frac{\Omega(\|x-y\|_2)}{\|x-y\|_2^n} f(y) dy. \tag{1}$$

We may consider that the integral operator T_1 defined by

$$T_1(f, x) = \int_{\mathbb{R}^n} \frac{\Omega(\|x-y\|_2)}{\|x-y\|_2^\lambda} f(y) dy, \ \ \lambda > 0. \tag{2}$$

In particular, when $\Omega \equiv 1, 0 < \lambda < n$, then (2) reduces to the Riesz potential operator of order λ:

$$T_2(f, x) = \int_{\mathbb{R}^n} \frac{1}{\|x-y\|_2^\lambda} f(y) dy. \tag{3}$$

The following Hardy–Littlewood–Pólya inequality was proved in ([3, 5, 7]):

Theorem 1 *Let* $f \in L^p(0, \infty), g \in L^q(0, \infty),$ $1 < p, q < \infty, \frac{1}{p} + \frac{1}{q} > 1,$ $0 < \lambda < 1, \lambda = 2 - \frac{1}{p} - \frac{1}{q}$, *then*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{|x-y|^\lambda} dx dy \le c_3 \|f\|_p \|g\|_q, \tag{4}$$

where

$$c_3 = c_3(p, q, \lambda) = \frac{1}{1-\lambda}\{(\frac{p}{p-1})^{p(1-\frac{1}{q})} + (\frac{q}{q-1})^{q(1-\frac{1}{p})}\}. \tag{5}$$

Let

$$T_3(f, x) = \int_0^\infty \frac{1}{|x-y|^\lambda} f(y) dy. \tag{6}$$

Then (4) is equivalent to

$$\|T_3 f\|_{p_1} \le c_3 \|f\|_p, \tag{7}$$

where $1 < p < \infty$, $1 - \frac{1}{p} < \lambda < 1$, $\frac{1}{p_1} = \frac{1}{p} + \lambda - 1$, c_3 is given by (5), but it is not asserted that the constant c_3 is the best possible.

Theorem 2 ([3], pp. 412–413) *If $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty, 0 < \lambda < n$, then the Riesz potential operator T_2 is defined by (3): $T_2 : L^p(\mathbb{R}^n) \to L^{p_2}(\mathbb{R}^n)$ exists as a bounded operator and*

$$\|T_2 f\|_{p_2} \leq c_2 \|f\|_p, \tag{8}$$

where $\frac{1}{p_2} = \frac{1}{p} + \frac{\lambda}{n} - 1$, and the constant c_2 depending only upon n, p, and λ.

Theorem 3 ([6, 7, 12, 13, 16]) *Let $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$, $1 < p, q < \infty, 0 < \lambda < n$, $\frac{1}{p} + \frac{1}{q} + \frac{\lambda}{n} = 2$, then there exists a constant $c_2 = c_2(p, \lambda, n)$ (depending only upon n, p, and λ), such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)g(y)}{\|x-y\|_2^{\lambda}} dxdy \leq c_2 \|f\|_p \|g\|_q, \tag{9}$$

where

$$c_2 \leq \frac{n}{pq(n-\lambda)} (\frac{S_n}{n})^{\lambda/n} \{ (\frac{\lambda/n}{1-(1/p)})^{\lambda/n} + (\frac{\lambda/n}{1-(1/q)})^{\lambda/n} \},$$

and S_n is the surface areas of the unit sphere in $\mathbb{R}^n$.

In particular, when $p = q = \frac{2n}{2n-\lambda}$,

$$c_2 = \pi^{\lambda/2} \frac{\Gamma(\frac{n-\lambda}{2})}{\Gamma(n - \frac{\lambda}{2})} \{ \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \}^{\frac{\lambda}{n} - 1}$$

is the best possible constant. But when $p \neq q$, the best possible value of c_2 is also unknown. Note that (9) is equivalent to

$$\|T_2 f\|_p \leq c_2 \|f\|_p. \tag{10}$$

There are many works about the boundedness of the operator T_j on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$, and their weighted versions, that is,

$$\|T_j f\|_{p,\omega_1} \leq c_j \|f\|_{p,\omega_2}, \quad j = 0, 1, 2, 3, \tag{11}$$

where $c_j > 0$, with some appropriate conditions on $\Omega, \omega_1, \omega_2$. The boundedness of these operators on more general spaces than $L^p(\mathbb{R}^n)$, for example, Orlicz spaces, Lorentz spaces, generalized Morrey spaces, generalized Campanato spaces and their weighted versions, as well as on general Banach spaces, has been investigated by various authors (see, e.g.,[1–5, 10–15, 17–19] and the references cited therein). But, the problem of determining the best possible constants in (11) remains unsolved. In fact, in the research of the boundedness of various operators, c only means a positive constant independent of the main parameters and may change from one occurrence to another.

In this paper, by means of the new analysis technique of the sharp constant factor is changed into the corresponding operator norm

$$c_j = \|T_j\| = \sup_{f \neq 0} \frac{\|T_j f\|_{p,\omega_1}}{\|f\|_{p,\omega_2}}.$$

Without loss of generality, we may consider that the generalized fractional integral operator T_4 defined by

$$T_4(f, x) = \int_{\mathbb{R}_+^n} \frac{\Omega(\|x - y\|_2)}{\|x - y\|_2^\lambda} f(y) dy, \tag{12}$$

where $\Omega : (0, \infty) \to (0, \infty)$ is increasing and homogeneous of degree 0, thus, by the triangle inequality, we have

$$|\|x\|_2 - \|y\|_2| \leq \|x - y\|_2 \leq \|x\|_2 + \|y\|_2.$$

This implies that

$$\Omega(|\|x\|_2 - \|y\|_2|) \leq \Omega(\|x - y\|_2) \leq \Omega(\|x\|_2 + \|y\|_2).$$

Let

$$K_4(x, y) = \frac{\Omega(\|x - y\|_2)}{\|x - y\|_2^\lambda},$$

$$K_5(x, y) = \frac{\Omega(|\|x\|_2 - \|y\|_2|)}{(\|x\|_2 + \|y\|_2)^\lambda},$$

$$K_6(x, y) = \frac{\Omega(\|x\|_2 + \|y\|_2)}{(|\|x\|_2 - \|y\|_2|)^\lambda}.$$

$$T_j(f, x) = \int_{\mathbb{R}_+^n} K_j(x, y) f(y) dy, \tag{13}$$

$$\|T_j\| = \sup_{f \neq 0} \frac{\|T_j f\|_{p,\omega}}{\|f\|_p}, \quad j = 4, 5, 6,$$

where ω be a nonnegative measurable weight function on $\mathbb{R}_+^n$. If f be a nonnegative measurable function on $\mathbb{R}_+^n$, and $\lambda > 0$, then

$$T_5(f, x) \leq T_4(f, x) \leq T_6(f, x),$$

and therefore,

$$\|T_5\| \leq \|T_4\| \leq \|T_6\|. \tag{14}$$

Thus, we may use the norms $\|T_5\|$, $\|T_6\|$ of the operator T_5, T_6 with the radial kernels to find the norm inequality of the generalized fractional integral operator T_4. In particular, we may consider that the Riesz potential operator of order λ:

$$T_7(f, x) = \int_{\mathbb{R}^n_+} \frac{1}{\|x - y\|_2^{\lambda}} f(y)dy. \tag{15}$$

As their further generalizations, we introduce the new integral operator T defined by

$$T(f, x) = \int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha) f(y)dy, \ \ x \in E_n(\alpha), \tag{16}$$

where the radial kernel $K(\|x\|_\alpha, \|y\|_\alpha)$ is a nonnegative measurable function defined on $E_n(\alpha) \times E_n(\alpha)$, which satisfies the following condition:

$$K(\|x\|_\alpha, \|y\|_\alpha) = \|x\|_\alpha^{-\lambda} K(1, \|y\|_\alpha \|x\|_\alpha^{-1}), \ \ x, y \in E_n(\alpha), \lambda > 0. \tag{17}$$

Equation (16) includes many famous operators as special cases. In particular, for $n = 1$, we have

$$T(f, x) = \int_0^\infty K(x, y) f(y)dy, \ \ x > 0, \tag{18}$$

and

$$K(x, y) = x^{-\lambda} K(1, yx^{-1}), \ \ x, y > 0, \lambda > 0. \tag{19}$$

In this paper, some new norms

$$\|T\| = \sup_{f \neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p}$$

of the integral operator T defined by (16) with the radial kernel on n-dimensional vector spaces are deduced. These norms used then to establish some new generalized fractional integral operator norm inequalities and the Riesz potential operator norm inequalities.

2 Statement of the Main Results

Our main results read as follows.

Theorem 4 *Let* $1 < p < \infty, \frac{1}{p} + \frac{1}{q} = 1, \lambda > 0, \omega(x) = \|x\|_2^{p(\lambda-n)}$, *and* $\Omega : (0, \infty) \to (0, \infty)$ *be an increasing and homogeneous of degree* 0*. If* f *be a nonnegative measurable function on* $\mathbb{R}_+^n$, *and*

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty \frac{\Omega(|1-u|)}{(1+u)^\lambda} u^{\frac{n}{q}-1} du < \infty, \tag{20}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty \frac{\Omega(1+u)}{|1-u|^\lambda} u^{\frac{n}{q}-1} du < \infty, \tag{21}$$

then the generalized fractional integral operator T_4 *is defined by (12):* $T_4 : L^p(\mathbb{R}_+^n) \to L^p(\omega)$ *exists as a bounded operator and*

$$c_1\|f\|_p \le \|T_4 f\|_{p,\omega} \le c_2\|f\|_p. \tag{22}$$

This implies that

$$c_1 \le \|T_4\| = \sup_{f\neq 0} \frac{\|T_4 f\|_{p,\omega}}{\|f\|_p} \le c_2, \tag{23}$$

where c_1 and c_2 are defined by (20) and (21), respectively.

In particular, for $n = 1$, we get

$$\int_0^\infty \frac{\Omega(|1-u|)}{(1+u)^\lambda} u^{-(\frac{1}{p})} du \le \|T_4\| \le \int_0^\infty \frac{\Omega(1+u)}{|1-u|^\lambda} u^{-(\frac{1}{p})} du.$$

In Theorem 4, setting $\Omega \equiv 1$, we get

Corollary 1 *Suppose that* p, q, λ, ω, *and* f *are as in Theorem 4, and* $0 < \frac{n}{q} < \lambda < 1$, *then the Riesz potential operator* T_7 *is defined by (15):* $T_7 : L^p(\mathbb{R}_+^n) \to L^p(\omega)$ *exists as a bounded operator and*

$$c_1\|f\|_p \le \|T_7 f\|_{p,\omega} \le c_2\|f\|_p. \tag{24}$$

This implies that

$$c_1 \le \|T_7\| = \sup_{f\neq 0} \frac{\|T_7 f\|_{p,\omega}}{\|f\|_p} \le c_2, \tag{25}$$

where

$$c_1 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(\frac{n}{q}, \lambda - \frac{n}{q}), \tag{26}$$

$$c_2 = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \{B(\frac{n}{q}, 1-\lambda) + B(\lambda - \frac{n}{q}, 1-\lambda)\}. \tag{27}$$

In particular, for $n = 1$, we get

$$B(\frac{1}{q}, \lambda - \frac{1}{q}) \leq \|T_7\| \leq B(\frac{1}{q}, 1-\lambda) + B(\lambda - \frac{1}{q}, 1-\lambda).$$

As some further generalizations of the above results, we have

Theorem 5 *Let* $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\omega(x) = \|x\|_\alpha^{p(\lambda-n)}$, *and the radial kernel* $K(\|x\|_\alpha, \|y\|_\alpha)$ *satisfies (17). If*

$$c = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1,u) u^{\frac{n}{q}-1} du < \infty, \tag{28}$$

then the integral operator T *is defined by (16):* $T : L^p(E_n(\alpha)) \to L^p(\omega)$ *exists as a bounded operator and*

$$\|Tf\|_{p,\omega} \leq c\|f\|_p, \tag{29}$$

where

$$c = \|T\| = \sup_{f \neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \tag{30}$$

be the sharp constant defined by (28).

3 Proof of Theorems

Theorem 4 is proved in Sect. 5. In order to prove Theorem 5, we require the following Lemmas:

Lemma 1 ([8, 9, 19]) *If* $a_k, b_k, p_k > 0$, $1 \leq k \leq n$, f *is a measurable function on* $(0, \infty)$*, then*

$$\int_{\mathbb{R}_+^n} f\Big(\sum_{k=1}^n (\frac{x_k}{a_k})^{b_k}\Big) x_1^{p_1-1} \cdots x_n^{p_n-1} dx_1 \cdots dx_n$$

$$= \frac{\prod_{k=1}^{n} a_k^{p_k}}{\prod_{k=1}^{n} b_k} \times \frac{\prod_{k=1}^{n} \Gamma(\frac{p_k}{b_k})}{\Gamma(\sum_{k=1}^{n} \frac{p_k}{b_k})} \int_0^\infty f(t) t^{(\sum_{k=1}^{n} \frac{p_k}{b_k} - 1)} dt.$$

We get the following Lemma 2 by taking $a_k = 1, b_k = \alpha > 0, p_k = 1, 1 \le k \le n$, in Lemma 1.

Lemma 2 *Let f be a measurable function on $(0, \infty)$, then*

$$\int_{E_n(\alpha)} f(\|x\|_\alpha^\alpha) dx = \frac{(\Gamma(1/\alpha))^n}{\alpha^n \Gamma(n/\alpha)} \int_0^\infty f(t) t^{(n/\alpha)-1} dt. \tag{31}$$

Proof of Theorem 5. By Hölder's inequality, we get

$$\begin{aligned} T(f, x) &= \int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha) f(y) dy \\ &= \int_{E_n(\alpha)} \|y\|_\alpha^{-(\frac{n}{pq})} \{K(\|x\|_\alpha, \|y\|_\alpha)\}^{1/q} \|y\|_\alpha^{(\frac{n}{pq})} \{K(\|x\|_\alpha, \|y\|_\alpha)\}^{1/p} f(y) dy \\ &\quad \le \{\int_{E_n(\alpha)} \|y\|_\alpha^{-(\frac{n}{p})} K(\|x\|_\alpha, \|y\|_\alpha) dy\}^{1/q} \\ &\quad \times \{\int_{E_n(\alpha)} \|y\|_\alpha^{n/q} K(\|x\|_\alpha, \|y\|_\alpha) |f(y)|^p dy\}^{1/p} \\ &= I_1^{1/q} \times I_2^{1/p}. \end{aligned} \tag{32}$$

In I_1, by using lemma 2 and letting $u = \|x\|_\alpha^{-1} t^{1/\alpha}$, we get

$$\begin{aligned} I_1 &= \int_{E_n(\alpha)} \|y\|_\alpha^{-(n/p)} K(\|x\|_\alpha, \|y\|_\alpha) dy \\ &= \|x\|_\alpha^{-\lambda} \frac{(\Gamma(1/\alpha))^n}{\alpha^n \Gamma(n/\alpha)} \int_0^\infty t^{-(\frac{n}{p\alpha})} K(1, \|x\|_\alpha^{-1} t^{1/\alpha}) \times t^{(\frac{n}{\alpha})-1} dt \\ &= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1} \Gamma(n/\alpha)} \|x\|_\alpha^{(n/q)-\lambda} \int_0^\infty K(1, u) u^{(n/q)-1} du \\ &= c\|x\|_\alpha^{(n/q)-\lambda}. \end{aligned} \tag{33}$$

Hence, by (32), (33) and the Fubini theorem and letting $v = \|y\|_\alpha t^{-(1/\alpha)}$, we conclude that

$$\begin{aligned} \|Tf\|_{p,\omega} &= (\int_{E_n(\alpha)} |T(f, x)|^p \omega(x) dx)^{1/p} \le (\int_{E_n(\alpha)} I_1^{p/q} I_2 \omega(x) dx)^{1/p} \\ &= c^{1/q} \{\int_{E_n(\alpha)} \|x\|_\alpha^{(\frac{n}{q}-\lambda)\frac{p}{q}} \big(\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{q}} K(\|x\|_\alpha, \|y\|_\alpha) |f(y)|^p dy\big) \|x\|_\alpha^{p(\lambda-n)} dx\}^{1/p} \end{aligned}$$

$$= c^{1/q}\{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{q}} \big(\int_{E_n(\alpha)} \|x\|_\alpha^{(\frac{n}{q}-\lambda)(p-1)+p(\lambda-n)} K(\|x\|_\alpha, \|y\|_\alpha)dx\big)|f(y)|^p dy\}^{1/p}$$

$$= c^{1/q}\{\int_{E_n(\alpha)} \|y\|_\alpha^{\frac{n}{q}} \big(\frac{(\Gamma(1/\alpha))^n}{\alpha^n\Gamma(n/\alpha)} \int_0^\infty t^{\frac{n}{\alpha q}-\frac{n}{\alpha}} K(1, t^{-(1/\alpha)}\|y\|_\alpha)t^{\frac{n}{\alpha}-1}dt\big)|f(y)|^p dy\}^{1/p}$$

$$= c^{1/q}\{\int_{E_n(\alpha)} |f(y)|^p dy\}^{1/p} \times \{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1, v)v^{(\frac{n}{q}-1)}dv\}^{1/p}$$

$$= c\|f\|_p. \tag{34}$$

This implies that

$$\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \leq c. \tag{35}$$

To prove the opposite inequality $\|T\| \geq c$, we take

$$f_\varepsilon(x) = \|x\|_\alpha^{-(n/p)-\varepsilon}\varphi_{B^c}(x),$$

$$g_\varepsilon(x) = (p\varepsilon)^{1/q}\{\frac{\alpha^{n-1}\Gamma(n/\alpha)}{(\Gamma(1/\alpha))^n}\}^{1/q}\|x\|_\alpha^{-(\frac{n}{q})-(p-1)\varepsilon}\varphi_{B^c}(x),$$

where $\varepsilon > 0$, $B = B(0, 1) = \{x \in E_n(\alpha) : \|x\|_\alpha < 1\}$, φ_{B^c} is the characteristic function of the set $B^c = \{x \in E_n(\alpha) : \|x\|_\alpha \geq 1\}$, that is

$$\varphi_{B^c}(x) = \begin{cases} 1, x \in B^c \\ 0, x \in B. \end{cases}$$

Thus, we get

$$\|f_\varepsilon\|_p = \big(\frac{(\Gamma(1/\alpha))^n}{p\varepsilon\alpha^{n-1}\Gamma(n/\alpha)}\big)^{1/p}, \|g_\varepsilon\|_q = 1.$$

Using the sharpness in Hölder's inequality (see [8]):

$$\|Tf\|_{p,\omega} = \sup\{|\int_{E_n(\alpha)} T(f, x)g(x)\{\omega(x)\}^{1/p}dx| : \|g\|_q \leq 1\},$$

where $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, thus, if $\|g\|_q \leq 1$, then

$$|\int_{E_n(\alpha)} T(f, x)g(x)\{\omega(x)\}^{1/p}dx| \leq \|Tf\|_{p,\omega}. \tag{36}$$

Hence, By (36) and (16), we get

$$\|Tf_\varepsilon\|_{p,\omega} \geq \int_{E_n(\alpha)} T(f_\varepsilon, x)g_\varepsilon(x)\{\omega(x)\}^{1/p}dx$$
$$= \int_{E_n(\alpha)}\int_{E_n(\alpha)} K(\|x\|_\alpha, \|y\|_\alpha)f_\varepsilon(y)g_\varepsilon(x)\|x\|_\alpha^{\lambda-n}dydx$$
$$= (p\varepsilon)^{1/q}\{\frac{\alpha^{n-1}\Gamma(n/\alpha)}{(\Gamma(1/\alpha))^n}\}^{1/q}$$
$$\times \int_{B^c}\{\int_{B^c} K(\|x\|_\alpha, \|y\|_\alpha)\|y\|_\alpha^{-(n/p)-\varepsilon}dy\}\|x\|_\alpha^{-(n/q)-(p-1)\varepsilon+\lambda-n}dx. \quad (37)$$

Letting $u = t^{1/\alpha}\|x\|_\alpha^{-1}$, and using (31), we have

$$\int_{B^c} K(\|x\|_\alpha, \|y\|_\alpha)\|y\|_\alpha^{-(n/p)-\varepsilon}dy$$
$$= \|x\|_\alpha^{-\lambda}\frac{(\Gamma(1/\alpha))^n}{\alpha^n\Gamma(n/\alpha)}\int_1^\infty K(1, t^{1/\alpha}\|x\|_\alpha^{-1})t^{(\frac{n}{q\alpha})-\frac{\varepsilon}{\alpha}+\frac{n}{\alpha}-1}dt$$
$$= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\|x\|_\alpha^{\frac{n}{q}-\lambda-\varepsilon}\int_{\|x\|_\alpha^{-1}}^\infty K(1,u)u^{(n/q)-\varepsilon-1}du. \quad (38)$$

We insert (38) into (37) and use Fubini's theorem to obtain

$$\|Tf_\varepsilon\|_{p,\omega} \geq (p\varepsilon)^{1/q}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{1/p}$$
$$\times \int_{B^c}\|x\|_\alpha^{-p\varepsilon-n}(\int_{\|x\|_\alpha^{-1}}^\infty K(1,u)u^{(n/q)-\varepsilon-1}du)dx$$
$$\geq (p\varepsilon)^{1/q}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{1/p}$$
$$\times \int_0^\infty K(1,u)u^{(n/q)-\varepsilon-1}(\int_{\beta(u)}^\infty \|x\|_\alpha^{-p\varepsilon-n}dx)du$$
$$= (p\varepsilon)^{1/q}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{(1/p)+1}$$
$$\times\frac{1}{\alpha}\int_0^\infty K(1,u)u^{(n/q)-\varepsilon-1}(\int_{\beta(u)}^\infty t^{-(p\varepsilon)/\alpha-1}dt)du$$
$$= (p\varepsilon)^{-(1/p)}\{\frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)}\}^{(1/p)+1}$$
$$\times\int_0^\infty K(1,u)u^{(n/q)-\varepsilon-1}(\beta(u))^{-(p\varepsilon)/\alpha}du,$$

where $\beta(u) = \max\{1, u^{-1}\}$. Thus, we get

$$\begin{aligned}\|T\| &= \sup_{f\neq 0} \frac{\|Tf\|_{p,\omega}}{\|f\|_p} \geq \frac{\|Tf_\varepsilon\|_{p,\omega}}{\|f_\varepsilon\|_p} \\ &\geq \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \\ &\times \int_0^\infty K(1,u)u^{(n/q)-\varepsilon-1}(\beta(u))^{-(p\varepsilon)/\alpha}du. \end{aligned} \tag{39}$$

By letting $\varepsilon \to 0^+$ in (39) and using the Fatou lemma, we get

$$\|T\| \geq \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty K(1,u)u^{\frac{n}{q}-1}du = c.$$

The proof is complete.

4 Some Applications

As applications, a large number of known and new results have been obtained by proper choice of kernel K. In this section we present some model and interesting applications which display the importance of our results. Also these examples are of fundamental importance in analysis.

Example 1 Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\omega(x) = \|x\|_\alpha^{p(\lambda-n)}$, and $\Omega : (0,\infty) \to (0,\infty)$ be an increasing and homogeneous of degree 0. Take K to be defined by

$$K(\|x\|_\alpha, \|y\|_\alpha) = \frac{\Omega(|\|x\|_\alpha - \|y\|_\alpha|)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda}, \tag{40}$$

and let

$$T_8(f,x) = \int_{E_n(\alpha)} \frac{\Omega(|\|x\|_\alpha - \|y\|_\alpha|)}{(\|x\|_\alpha + \|y\|_\alpha)^\lambda} f(y)dy.$$

By Theorem 5, we get

$$c = \|T_8\| = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{\Omega(|1-u|)}{(1+u)^\lambda} u^{(n/q)-1}du. \tag{41}$$

In particular, for $n = 1$, we get

$$c = \|T_8\| = \int_0^\infty \frac{\Omega(|1-u|)}{(1+u)^\lambda} u^{-(1/p)} du.$$

For $\Omega \equiv 1$, if $\lambda > n/q$, then by (41), we get

$$\begin{aligned} c = \|T_8\| &= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{u^{(n/q)-1}}{(1+u)^\lambda} du \\ &= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} B(\frac{n}{q}, \lambda - \frac{n}{q}). \end{aligned} \tag{42}$$

In particular, for $n = 1$, we get

$$c = \|T_8\| = B(\frac{1}{q}, \lambda - \frac{1}{q}).$$

Example 2 Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $\omega(x) = \|x\|_\alpha^{p(\lambda-n)}$, and $\Omega : (0, \infty) \to (0, \infty)$ be an increasing and homogeneous of degree 0. Take K to be defined by

$$K(\|x\|_\alpha, \|y\|_\alpha) = \frac{\Omega(\|x\|_\alpha + \|y\|_\alpha)}{(|\|x\|_\alpha - \|y\|_\alpha|)^\lambda}, \tag{43}$$

and let

$$T_9(f, x) = \int_{E_n(\alpha)} \frac{\Omega(\|x\|_\alpha + \|y\|_\alpha)}{|\|x\|_\alpha - \|y\|_\alpha|^\lambda} f(y) dy.$$

By Theorem 5, we get

$$c = \|T_9\| = \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{\Omega(1+u)}{|1-u|^\lambda} u^{(n/q)-1} du. \tag{44}$$

In particular, for $n = 1$, we get

$$c = \|T_9\| = \int_0^\infty \frac{\Omega(1+u)}{|1-u|^\lambda} u^{-(1/p)} du.$$

For $\Omega \equiv 1$, if $0 < n/q < \lambda < 1$, then by (44), we get

$$\begin{aligned} c = \|T_9\| &= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \int_0^\infty \frac{u^{(n/q)-1}}{|1-u|^\lambda} du \\ &= \frac{(\Gamma(1/\alpha))^n}{\alpha^{n-1}\Gamma(n/\alpha)} \{B(\frac{n}{q}, 1-\lambda) + B(\lambda - \frac{n}{q}, 1-\lambda)\}. \end{aligned} \tag{45}$$

In particular, for $n = 1$, we get

$$c = \|T_9\| = B(\frac{1}{q}, 1-\lambda) + B(\lambda - \frac{1}{q}, 1-\lambda).$$

Remark 1 Defining other forms of the kernel K, we can obtain new results of interest.

5 Some Integral Operator Norm Inequalities

In Examples 1–2, setting $\alpha = 2$, thus, $E_n(\alpha)$ reduces to $E_n(2) = \mathbb{R}^n_+$, T_8, T_9 reduces to T_5, T_6, respectively. Assume $f \in L^p(\mathbb{R}^n_+)$, $f(x) \geq 0, x \in \mathbb{R}^n_+$, $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$. $\Omega : (0, \infty) \to (0, \infty)$ be an increasing and homogeneous of degree 0, $\omega(x) = \|x\|_2^{p(\lambda - n)}$, then by (41) and (44), we get

$$\|T_5\| = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty \frac{\Omega(|1-u|)}{(1+u)^\lambda} u^{(n/q)-1} du; \tag{46}$$

$$\|T_6\| = \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty \frac{\Omega(1+u)}{|1-u|^\lambda} u^{(n/q)-1} du. \tag{47}$$

In particular, for $n = 1$, we get

$$\|T_5\| = \int_0^\infty \frac{\Omega(|1-u|)}{(1+u)^\lambda} u^{-(1/p)} du;$$

$$\|T_6\| = \int_0^\infty \frac{\Omega(1+u)}{|1-u|^\lambda} u^{-(1/p)} du.$$

For $\Omega \equiv 1$, if $0 < n/q < \lambda < 1$, then by (46) and (47), we get

$$\begin{aligned} \|T_5\| &= \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty \frac{u^{(n/q)-1}}{(1+u)^\lambda} du \\ &= \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} B(\frac{n}{q}, \lambda - \frac{n}{q}); \end{aligned} \tag{48}$$

$$\begin{aligned} \|T_6\| &= \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \int_0^\infty \frac{u^{(n/q)-1}}{|1-u|^\lambda} du \\ &= \frac{\pi^{n/2}}{2^{n-1}\Gamma(n/2)} \{B(\frac{n}{q}, 1-\lambda) + B(\lambda - \frac{n}{q}, 1-\lambda)\}. \end{aligned} \tag{49}$$

In particular, for $n = 1$, we get

$$\|T_5\| = B(\frac{1}{q}, \lambda - \frac{1}{q}); \tag{50}$$

$$\|T_6\| = B(\frac{1}{q}, 1 - \lambda) + B(\lambda - \frac{1}{q}, 1 - \lambda). \tag{51}$$

By using (14), (46), (47), (48), (49) and (50), (51), we have thus also proved that Theorem 4 and Corollary 1 are correct.

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Application of Davies–Petersen Lemma

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Abstract In this paper we have shown how use of a simple lemma first proved by Davies and Petersen and later extended by Mohapatra and Russell can be used effectively to prove three main results which can yield integral inequalities of Hardy and Copson. We have also shown how those results can be used to obtain many known results obtained by Levinson, Pachpatte, Chan, etc. by carefully manipulating these three theorems. A look at this paper will also reveal that there can be simple proofs of sophisticated results after they have been proved by exploiting the important points that make things work. It does not take away the value of the original contributions. We have also mentioned that it has not been possible to deduce other known results by using our results.

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1 Introduction

With a view to providing an alternative proof of the discrete version of Hilbert's inequality G. H. Hardy proved the following inequality:

Theorem 1.1 ([13] p. 239, Theorem 326) *If* $p > 1$, $a_n > 0$, $n = 0, 1, 2, \ldots,$ *then*

$$\sum_{n=0}^{\infty}\left((n+1)^{-1}\sum_{k=0}^{n} a_k\right)^p \leq q^p \sum_{n=0}^{\infty} a_n^p, \tag{1}$$

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with $\frac{1}{p} + \frac{1}{q} = 1$. The constant q^p on the right-hand side of (1) is best possible. In [23, Theorem 1] Davies and Petersen proved the following result.

Theorem 1.2 ([7], Theorem 1) *Suppose $A = (a_{mn})$ be an infinite matrix with*

$$a_{mn} > 0 \quad (n \leq m), a_{mn} = 0 \quad (n > m), m, n = 1, 2, \ldots, \tag{2}$$

Further assume that

$$0 \leq \frac{a_{mn}}{a_{kn}} \leq K, \quad (0 \leq n \leq k \leq m) \tag{3}$$

and $\frac{a_{mn}}{a_{kn}}$ is a decreasing sequence as n increases in (3) with $0 \leq n \leq k \leq m$.

Let us also assume that there exists an $f(m)$ ($f(m) \to \infty$ as $m \to \infty$) such that the matrix (c_{mn}) defined by $c_{mn} = f(m)a_{mn}$ $(n = 1, 2, \ldots)$ has properties (2) and (3) mentioned before with perhaps a different constant k in (2).

If $x_n \geq 0$ $(n = 1, 2, \ldots,)$ and if

$$\sum_{k=1}^{\infty} a_{k1} \{f(k)\}^{1-p} \tag{4}$$

converges and

$$\sum_{k=n}^{\infty} a_{k1} \{f(k)\}^{1-p} \leq M a_{n1} \{f(n)\}^{1-p}, \tag{5}$$

then

$$\sum_{m=1}^{\infty} \left\{ \sum_{n=1}^{m} a_{mn} x_n \right\}^p \leq C \sum_{m=1}^{\infty} \{x_m f(m) a_{mn}\}^p, \tag{6}$$

where p is an integer and C is an arbitrary constant.

In [9, Theorem 2], Davies and Petersen extended Theorem 1.2 to all real $p > 1$. This extension was, in fact, a consequence of the following lemma which we name as Davies–Petersen lemma for sequences.

Lemma 1.1 ([9], Lemma 1) *If $p > 1$ and $z_n \geq 0$ $(n = 1, 2, \ldots,)$, then*

$$\left(\sum_{k=1}^{n} z_k \right)^p \leq p \sum_{k=1}^{n} z_k \left(\sum_{j=1}^{k} z_j \right)^{p-1}. \tag{7}$$

Using this lemma, Davies and Petersen proved an analogue of the Theorem 1.2 for all real $p > 1$. Johnson and Mohapatra [15] proved discrete inequalities for a class of matrices and called such inequalities as Hardy–Davies–Petersen inequality. For details, we refer the reader to [15].

An analogue of the Lemma 1.1 for integrals was proved in [9, Lemma 2] by Davies and Petersen.

Lemma 1.2 ([9], Lemma 2) *Let $p > 1$ and $z(x)$ be any positive integral function of x. Then*

$$\left(\int_0^x z(x)dx\right)^p = p\int_0^x z(x)\left(\int_0^x z(t)dt\right)^{p-1} dx. \tag{8}$$

Davies and Petersen used Lemma 1.2 to prove an integral inequality (see [9, Theorem 4]) involving μ-kernel which is defined below.

Let an μ-kernel $a(x, y)$ satisfy the following conditions:

$$\begin{cases} a(x, y) > 0 \ (y \leq x) \\ \\ a(x, y) = 0 \ (y > x) \end{cases} \qquad (x, y) \geq 0. \tag{9}$$

Also

$$0 \leq \frac{a(x_0, y)}{a(x_1, y)} \leq K \qquad (0 \leq y \leq x_1 \leq x_0), \tag{10}$$

where K is an absolute constant. Further let there exist a function $f(x)$ ($f(x) \to \infty$ as $x \to \infty$) such that $c(x, y) = f(x)a(x, y)$ is an μ-kernel. Davies and Petersen proved

Theorem 1.3 ([9], Theorem 4) *Let $a(x, y)$ be an μ-kernel and $u(y) \geq 0$ ($y \geq 0$). Then if*

$$\int_0^\infty \{f(x)\}^{-p}\, dx \tag{11}$$

exists and

$$\int_{x_0}^\infty \{f(x)\}^{-p}\, dx \leq M\, [f(x_0)]^{1-p}, \tag{12}$$

we have

$$\int_0^\infty \left\{\int_0^x a(x, y)u(y)dy\right\}^p dx \leq C \int_0^\infty \{u(x)f(x)a(x, x)\}^p\, dx, \tag{13}$$

where $p \geq 1$ and C is a constant which depends on p.

The intent of Davies and Petersen in proving Theorem 1.3 was to provide a generalization of Hardy's integral inequality:

Theorem 1.4 ([13], Theorem 327) *If* $p > 1$, $f(x) \geq 0$ *for* $0 < x < \infty$

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx < q^p \int_0^\infty f(x)^p dx \tag{14}$$

unless $f(x)$ *is identically zero, with* $q = p/(p-1)$.

Over the years Theorem 1.4 has been generalized in several directions and many research papers have been written on this inequality (see [1–12, 14–18, 20–32] and all the references in those papers).

Mohapatra and Russell [21] mentioned Lemma 1.2 as Davies–Petersen lemma and remarked what happens when $0 < p < 1$. They wrote the following:

Lemma 1.3 ([21], Lemma 1)

(i) *Let* $1 \leq p < \infty$ *and* $Z(t)$ *be non-negative and integrable over* $0 < t < x$. *Then*

$$\left(\int_0^x Z(t)dt\right)^p = p\int_0^x Z(t)\left\{\int_0^t Z(u)du\right\}^{p-1} dt. \tag{15}$$

The result holds for $0 < p < 1$ *provided* $\int_0^t Z(u)du > 0$ *for* $0 < t < x$.

(ii) *Let* $1 \leq p < \infty$ *and* $Z(t)$ *be an integrable function for* $x < t < \infty$. *Then*

$$\left(\int_x^\infty Z(t)dt\right)^p = p\int_x^\infty Z(t)\left\{\int_t^\infty Z(u)du\right\}^{p-1} dt. \tag{16}$$

The result holds for $0 < p < 1$ *provided that* $\int_t^\infty Z(u)du > 0$ *for* $x < t < \infty$.

Proof of (15) was given by Davies and Petersen [9] and proof of (16) was given by Mohapatra and Russell (see [21, Lemma 1, p. 201]).

The main objective of this chapter is to use the above lemma (hereafter called as Davies–Petersen lemma) to obtain three theorems which will yield many known results as corollaries.

2 Known Integral Inequalities

In this section we give a number of generalizations of Hardy's, Copson's and Levinson's integral inequalities. The results of Copson and Levinson were proved to provide generalization of Hardy's integral inequality. We state these below:

Theorem 2.1 (Hardy [13]) *Let* $p > 1$, $c \neq 1$, *and* $h(x)$ *be non-negative and Lebesgue integrable on* $[0, a]$ *or* $[a, \infty]$ *for every* $a > 0$ *according as* $c > 1$ *or*

$c < 1$. *If, we define,*

$$F(x) = \begin{cases} \int_0^x h(t)dt, & c > 1; \\ \int_x^\infty h(t)dt, & 0 < c < 1; \end{cases}$$

then

$$\int_0^\infty x^{-c}\{F(x)\}^p\,dx \le \left(\frac{p}{|c-1|}\right)^p \int_0^\infty x^{-c}\{xh(x)\}^p\,dx. \tag{17}$$

In [8] Copson has proved integral inequality with a view to generalizing Hardy's inequality. One such result is

Theorem 2.2 ([8], Theorem 1) *Let* $\phi(x)$, $f(x)$ *be non-negative for* $x \ge 0$ *and be continuous in* $[0, \infty)$. *Let* $p \ge 1, c > 1$. *If* $0 < b \le \infty$ *and*

$$\int_0^b F(x)^p \Phi(x)^{-c}\phi(x)dx \tag{18}$$

converges at the lower limit of integration, then

$$\int_0^b F(x)^p \Phi(x)^{-c}\phi(x)dx \le \left(\frac{p}{c-1}\right)^p \int_0^b f(x)^p \Phi(x)^{p-c}\phi(x)dx \tag{19}$$

where $\Phi(x) = \int_o^x \phi(t)dt,\ F(x) = \int_o^x f(t)\phi(t)dt.$

Remark 2.1 The case $c = p > 1$ and $\phi(x) = 1$ is Hardy's classical integral inequality [13, Theorem 327] which inspired numerous researchers including Copson. In fact, Theorem 2.2 mentioned above is one of the six inequalities established in [8]. Beesack [2] has proved six similar inequalities two of which provide alternative proofs of [8, Theorem 5 and Theorem 6]. Independent generalization of Copson's inequalities has been done by Love [18], Mohapatra and Russel [21], and Mohapatra and Vajravelu [22].

With a view to generalizing Hardy's inequality, Levinson established the following results

Theorem 2.3 ([17], Theorem 4, p. 329) *Let* $p > 1$, $f(x) \ge 0$ *and let* $r(x)$ *be positive and locally absolutely continuous in* $(0, \infty)$. *In addition, let*

$$\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \ge \frac{1}{\lambda} \tag{20}$$

for some $\lambda > 0$ *and for almost all* x. *If*

$$H(x) = \frac{\int_0^x r(t)f(t)dt}{xr(x)}, \tag{21}$$

then

$$\int_0^\infty H(x)^p dx \leq \lambda^p \int_0^\infty f(x)^p dx. \tag{22}$$

Theorem 2.4 ([1], Theorem 5, p. 393) *Let $p > 1$, $f(x) \geq 0$, $r(x)$ be locally absolutely continuous for $x > 0$. Let*

$$\frac{xr'(x)}{r(x)} - \frac{p-1}{p} \geq \frac{1}{\lambda} \tag{23}$$

for some $\lambda > 0$. If

$$J(x) = \frac{r(x)}{x} \int_x^\infty \frac{f(t)}{r(t)} dt, \tag{24}$$

then

$$\int_0^\infty J(x)^p dx \leq \lambda^p \int_0^\infty f(x)^p dx, \tag{25}$$

Remark 2.2 If in Theorem 2.4, we take $r(x) = 1$ and $\lambda = \frac{p}{p-1}$, then we get Hardy's integral inequality [13, Theorem 327]. If $r(x) = x$ and $\lambda = p$, then (25) reduces to the dual inequality related to that of Hardy.

3 Main Results

In this section, we shall prove three integral inequalities from which we shall be able to deduce a number of known results as corollaries. These theorems will be proved by using Davies–Petersen lemma and careful use of H*ö*lder inequalities.

Since some of the research papers consider the interval of integration as (a, b) in place of $(0, \infty)$, our next theorem will be established for (a, b). Although these were proved in [6], we give complete proofs and apply them to get many known results as corollaries.

Thus, this chapter shows how simple techniques can yield nice results.

Theorem 3.1 (See [6], Theorem A) *Let $0 \leq a < b \leq \infty$ and h be a non-negative function which is Lebesgue integrable in (x, b), and u is a positive function with*

$$U(x) = \int_a^x u(t)dt \tag{26}$$

is finite for each x in $a < x < b$. Then the following inequality holds:

(i) When $1 < p < \infty$,

$$\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx \le p^p \int_a^b u(x)\left(\frac{h(x)U(x)}{u(x)}\right)^p dx. \tag{27}$$

(ii) If $0 < p < 1$, then the inequality $\le$ is replaced by $\ge$. If $p = 1$, then the inequality (27) reduces to an equality.

Proof

Case 1 If $p = 1$,

$$\int_a^b u(x)\left(\int_x^b h(t)dt\right) dx = \int_a^b h(t)\left(\int_a^t u(x)dx\right) dt = \int_a^b h(t)U(t)dt. \tag{28}$$

This completes the proof for the case $p = 1$.

Case 2 If $1 < p < \infty$, if the left-hand side of (27) is infinite, then apply the following with b replaced by c with $a < c < b$, and the let c approach b from below.

Using Davies–Petersen lemma (Lemma 1.7),

$$\begin{aligned}\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx &= p\int_a^b u(x)\int_x^b h(t)dt\left(\int_t^b h(s)ds\right)^{p-1} dx\\ &= p\int_a^b h(t)\left(\int_t^b h(s)ds\right)^{p-1} dt\int_a^t u(x)dx\\ &= p\int_a^b h(t)U(t)\left(\int_t^b h(s)ds\right)^{p-1} dt.\end{aligned} \tag{29}$$

Now, let us write the expressing on the right-hand side of (29) as

$$p\int_a^b \frac{h(t)U(t)}{u(t)}u(t)\left(\int_t^b h(s)ds\right)^{p-1} dt \tag{30}$$

and apply Hölder's inequality to (30).

Hence, (30) is not greater than

$$p\left[\int_a^b u(t)\left(\int_t^b h(s)ds\right)^p dt\right]^{\frac{1}{p'}}\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}. \tag{31}$$

Now, collecting results from (29)–(31), we have, after dividing both sides by $\left[\int_a^b \left(\int_t^b h(s)ds\right)^p u(t)dt\right]^{\frac{1}{p'}}$

$$\left[\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx\right]^{\frac{1}{p'}} \leq p\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}, \qquad (32)$$

which yields the required result for Theorem 1.1, when $1 < p < \infty$.

Case 3 When $0 < p < 1$. In this case the Hölder inequality applied to the expression (30) in case 2 yields

$$\left[\int_a^b u(x)\left(\int_x^b h(t)dt\right)^p dx\right]^{\frac{1}{p}} \geq p\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}$$

and the result follows.

Note that in cases 2 and 3, if the expression by which we are dividing both sides is zero, then the inequality is automatically satisfied because both sides of the inequality to be proved are zero. □

Theorem 3.2 (See [6], Theorem B) *Let* $0 \leq a < b \leq \infty$, *h be a non-negative function which is Lebesgue integrable in* $a < x < b$ *and u be a positive function such that*

$$U(x) = \int_x^b u(t)dt$$

is finite for $a < x < b$. *Then for* $1 \leq p < \infty$,

$$\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx \leq p^p \int_a^b u(x)\left(\frac{h(x)U(x)}{u(x)}\right)^p dx. \qquad (33)$$

If $0 < p < 1$, *then the inequality in* (33) *becomes* $\geq$.

Proof

Case 1 $p = 1$. In this case change of order of integration for the double integral on the left-hand side of (33) yields the equality.

Case 2 $1 < p < \infty$, if the left-hand side of (33) is infinite, we apply the proof given below with a replaced by c, $a < c < b$ and then let $c \to a$ from above. Hence we assume, for the rest of the proof, the left-hand side of (33) as finite.

By Davies–Petersen Lemma (Lemma 1.2), we have

$$\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx = p\int_a^b u(x)\left[\int_a^x h(t)\left(\int_a^t h(s)ds\right)^{p-1} dt\right]dx$$

$$= p\int_a^b h(t)\left(\int_a^t h(s)ds\right)^{p-1}\int_t^b u(x)dx\, dt \quad (34)$$

By applying Hölder's inequality in the same manner, as in the proof of Theorem 3.1, we obtain with $p' = p/(p-1)$,

$$\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx \le p\left[\int_a^b u(t)\left(\int_a^t h(s)ds\right)^p dt\right]^{\frac{1}{p'}}$$

$$\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}. \quad (35)$$

Since the integral $\int_a^b u(t)\left(\int_a^t h(s)ds\right)^p dt$ is finite, we divide both sides of (35) by that integral to get

$$\left[\int_a^b u(x)\left(\int_a^x h(t)dt\right)^p dx\right]^{\frac{1}{p}} \le p\left[\int_a^b u(t)\left(\frac{h(t)U(t)}{u(t)}\right)^p dt\right]^{\frac{1}{p}}. \quad (36)$$

Raising both sides of (36) to power p the result follows.

Case 3 $0 < p < 1$. In this case the Hölder's inequality yields the required results because $\le$ is replaced by $\ge$. □

Our next theorem is Levinson type generalization of Hardy's inequality. We first state Levinson's result for the reader to appreciate our next theorem.

Theorem 3.3 (See Levinson [17], Theorem 2) *Let* $0 \le a < b \le \infty$, $\phi(u) \ge 0$ *and* $\phi''(u) \ge 0$ *when* $0 \le u \le b$, $p > 1$. *Let*

$$\phi(x)\phi''(x) \ge \left(1 - \frac{1}{p}\right)\left(\phi'(x)\right)^2 \quad for \quad a < x < b. \quad (37)$$

At end points of the interval $a < x < b$, *let* $\phi(x)$ *take its limiting values, finite or infinite. For* $x > 0$, *let* $r(x)$ *be continuous and non-decreasing, and let*

$$R(x) = \int_0^x r(t)dt. \quad (38)$$

Then

$$\int_0^\infty \phi\left(\int_0^x \frac{r(t)f(t)}{R(x)}\right)dx \le \left(\frac{p}{p-1}\right)^p \int_0^\infty \phi(f(x))dx. \quad (39)$$

Remark 3.1 When $\phi(u) = u^p$, $p > 1$, (39) is automatically satisfied and the inequality (39) reduces to Hardy's inequality when $r(t) \equiv 1$.

We prove the following generalization of Theorem 3.3.

Theorem 3.4 (See [6] Theorem C) *Let $p \geq 1$ and let ϕ, R, r, and f be defined as in Theorem 3.3 so that the hypotheses of Theorem 3.3 are satisfied. Further assume that g is a positive function which is Lebesgue integrable over the interval $(0, b)$, and*

$$U(x) = \int_x^b \frac{g(t)}{(R(t))^p} dt. \tag{40}$$

Then

$$\int_0^b g(x)\, \phi\left(\int_0^x \frac{r(x) f(t)}{R(x)}\right) dx$$
$$\leq p^p \int_a^b (g(x))^{p-1} \left\{R(x)^{p-1} r(x) U(x)\right\}^p \phi(f(x)) dx \tag{41}$$

Proof Let us write $\eta(t)^p = \phi(t)$. Then

$$\phi(x)\phi''(t) \geq \left(1 - \frac{1}{p}\right) \left(\phi'(t)\right)^2$$

means

$$p(p-1)\eta^{2(p-1)} \left(\eta'(t)\right)^2 + p\eta(t)^{2p-1}\eta''(t) \geq (p-1)p\eta^{2(p-1)} \left(\eta'(t)\right)^2.$$

This implies that $\eta(t)\eta''(t) \geq 0$. Since $\phi(t) \geq 0$, $\eta(t) \geq 0$ and consequently, the condition (37) amounts to $\eta''(t) \geq 0$. Hence, the function $\eta(t)$ is convex. Now, by Jensen's in equality applied to $\int_0^x \frac{r(t)f(t)}{R(x)} dt$ yields

$$\eta\left(\int_0^x \frac{r(t) f(t)}{R(x)} dt\right) \leq \int_0^x \frac{r(t)\eta(f(t))}{R(x)} dt \tag{42}$$

Substituting $\phi(t)^{1/p}$ for $\eta(t)$ in (42) and raising both sides to power p, we get

$$\phi\left(\int_0^x \frac{r(t) f(t)}{R(x)} dt\right) \leq \left(\int_0^x \frac{r(t)\phi(f(t))^{1/p}}{R(x)} dt\right)^p \tag{43}$$

since g is a positive function (43) yields

$$\int_0^b g(x)\phi\left(\int_0^x \frac{r(t)f(t)}{R(x)}dt\right)dx \le \int_0^b g(x)\left(\int_0^x \frac{r(t)\phi(f(t))^{1/p}}{R(x)}dt\right)^p \quad (44)$$

Now, we can apply Theorem 3.2 to the left-hand side of (44) with $h(t) = r(t)\phi(f(t))^{1/p}$ and $u(x) = \frac{g(x)}{(R(x))^p}$, and $a = 0$. We will get

$$\int_0^b g(x)\phi\left(\int_0^x \frac{r(t)f(t)}{R(x)}dt\right)dx \le \int_0^b \frac{g(x)}{R(x)^p}\left(\frac{r(x)\phi(f(x))^{1/p}U(x)R(x)^p}{g(x)}\right)^p dx$$
$$\le p^p \int_0^b g(x)^{1-p}\left\{R(x)^{p-1}r(x)U(x)\right\}^p \phi(f(x))dx, \quad (45)$$

after simplification. This completes the proof of Theorem 3.4. □

Remark 3.2 Theorem 3.3 can be obtained from Theorem 3.4 by setting $g(x) \equiv 1$. Since $r(x)$ is non-decreasing, we estimate $u(x)$ from

$$U(x) \le \frac{1}{r(x)}\int_x^b \frac{r(t)}{R(t)^p}dt \le \frac{1}{p-1}\left(\frac{1}{r(x)R(x)^{p-1}}\right) \quad (46)$$

Then (39) follows.

4 Corollaries from Theorems 3.1 and 3.2

Corollary 4.1 (See Chan [7], Theorem 1, p. 165) *If $h(t)$ is Lebesgue integrable in (x, ∞) for every $x \in (1, \infty)$, and $h(t) > 0$ for all $t \in (1, \infty)$, then for $1 < p < \infty$,*

$$\int_1^\infty \frac{1}{x}\left(\int_0^\infty h(t)dt\right)^p dx \le p^p \int_1^\infty \frac{1}{x}(x\ \ln x\ h(x))^p dx. \quad (47)$$

The inequality is reversed if $0 < p < 1$ and yields equality when $p = 1$.

Proof Equation (47) follows from Theorem 3.1 by setting $u(x) = \frac{1}{x}$ and $U(x) = \ln(x)$, $a = 1$ and $b = \infty$. □

Corollary 4.2 (See Chan [7], Theorem 2, p. 166) *Suppose $h(t)$ is Lebesgue integrable over $(0, x)$ for each $x \in (0, 1)$, and $h(t) \ge 0$ for all $t \in (0, 1)$. Then for $1 < p < \infty$,*

$$\int_0^1 \frac{1}{x}\left(\int_0^x h(t)dt\right)^p dx \le p^p \int_0^1 \frac{1}{x}(x\ |\ln x|\ h(x))^p dx. \quad (48)$$

The inequality is reversed if $0 < p < 1$ and yields equality when $p = 1$.

Proof In Theorem 3.2, take $a = 0$, $b = 1$, $u(x) = \frac{1}{x}$, $0 < x < 1$. Then

$$U(x) = \int_x^1 \frac{1}{x} dx = -\ln x = |\ln x|, \tag{49}$$

with these, we see that the Corollary 4.2 holds. □

Corollary 4.3 (See Chen [7], Theorem 3, p. 166) *Suppose $h(t)$ is integrable in the sense of Lebesgue over $(1, x)$ for each $x \in (1, \infty)$, and $h(t) \geq 0$ for all $t \in (1, \infty)$. Then for $1 < p < \infty$,*

$$\int_1^\infty \frac{1}{x} (\ln x)^{-p} \left(\int_1^x h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_1^\infty \frac{1}{x} (x h(x))^p \, dx. \tag{50}$$

Proof In Theorem 3.2, set $u(x) = x^{-1}(\ln x)^{-p}$, $1 < p < \infty$, $a = 1$, and $b = \infty$. Then $U(x) = (p-1)^{-1}(\ln x)^{-p+1}$ and Corollary 4.3 follows. □

Corollary 4.4 (See Chen [7], Theorem 4, p. 167) *Let $h(t)$ be Lebesgue integrable over $(x, 1)$ for each $x \in (0, 1)$. Then for $1 < p < \infty$,*

$$\int_0^1 x^{-1}(\ln x)^{-p} \left(\int_x^1 h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^1 x^{-1} (x h(x))^p \, dx. \tag{51}$$

Proof In Theorem 3.1, set $u(x) = x^{-1}(\ln x)^{-p}$ and $1 < p < \infty$. Let a approach zero from above and $b = 1$. Clearly,

$$U(x) = \left| \int_0^x \frac{dt}{t |\ln t|^p} \right| = \frac{1}{(p-1)|\ln x|^{p-1}}, \tag{52}$$

Equation (51) follows from Theorem 3.1. □

Corollary 4.5 *Let $h(t)$ be as in Corollary 4.3. Then for $1 < p, q < \infty$,*

$$\int_0^b \frac{1}{x|\ln x|^q} \left(\int_a^b h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_0^b x^{p-1} |\ln x|^{p-q} h(x)^p dx \tag{53}$$

Corollary 4.6 *Let $h(t)$ be as in corollary 4.4. Then for $1 < p, q < \infty$,*

$$\int_a^\infty \frac{1}{x|\ln x|^q} \left(\int_a^x h(t) dt \right)^p dx \leq \left(\frac{p}{p-1} \right)^p \int_a^\infty x^{p-1} |\ln x|^{p-q} h(x)^p dx \tag{54}$$

Remark 4.1 Corollaries 4.5 and 4.6 are obtained from Theorems 3.1 and 3.2 by choosing $u(x) = \frac{1}{x|\ln x|^q}$. Also both inequalities are reversed when $0 < p < 1$.

Corollary 4.7 (Hardy [12]) *Let $p > 1$, $r \neq 1$, and $h(t)$ be a non-negative function which is integrable on the interval $(0, a]$ or $[a, \infty)$ for every $a > 0$, according as $r > 1$ or $r < 1$. If $F(x)$ is defined by*

$$F(x) := \begin{cases} \int_0^x h(t)dt & r > 1; \\ \int_x^\infty h(t)dt & r < 1; \end{cases}$$

then

$$\int_0^\infty x^{-r} F(x)^p dx \leq \left(\frac{p}{|r-1|}\right)^p \int_0^\infty x^{-r}(xh(x))^p dx \tag{55}$$

Proof Inequality (55) is deducible from Theorems 3.1 and 3.2 by taking $u(x) = x^{-r}$ according to the value of r. If $r < 1$, it is deduced from Theorem 3.1 but when $r > 1$, it can be obtained using Theorem 3.2. $U(x)$ is easily calculated to complete the proofs for both cases. □

Corollary 4.8 (Levinson [17], Theorem 4) *Suppose $1 < p < \infty$, $f(x) \geq 0$ and $r(x)$ is positive and locally absolutely continuous in $(0, \infty)$. Further assume that $r(x)$ satisfies the following:*

$$\frac{p-1}{p} + \frac{xr'(x)}{r(x)} \geq \frac{1}{\lambda}, \tag{56}$$

for some $\lambda > 0$ and for almost all x.

If we define

$$H(x) = \frac{1}{xr(x)} \int_0^x r(t)f(t)dt, \tag{57}$$

then

$$\int_0^\infty H(x)^p dx \leq \lambda^p \int_0^\infty f(x)^p dx. \tag{58}$$

Remark 4.2 If $r(x) \equiv 1$, $\lambda = \frac{p}{p-1}$, then (58) reduces to well-known Hardy's inequality [12, Theorem 327].

Proof (of Corollary 4.8) We shall use Theorem 3.2 with $a = 0$, $h(x) = r(x)f(x)$ and $u(x) = (xr(x))^{-p}$.

Applying integration by parts,

$$U(x) = \int_x^b \frac{dt}{t^p r(t)^p} = \left[\left(\frac{t^{-p+1}}{-p+1}\frac{1}{r(t)^p}\right)\right]_x^b - \frac{p}{p-1}\int_x^b \frac{t^{-p+1}}{r(t)^{p+1}} r'(t)dt. \tag{59}$$

Since $r(b) > 0$, $b > 0$ and $1 < p < \infty$.

$$\frac{b^{-p+1}}{(-p+1)r(b)^p} < 0. \tag{60}$$

Hence, (59) yields

$$U(x) \le \frac{x}{(p-1)x^p r(x)^p} - \frac{p}{(p-1)} \int_x^b \left[\frac{tr'(t)}{r(t)}\right] \frac{dt}{(tr(t))^p}. \tag{61}$$

Using

$$-\frac{tr'(t)}{r(t)} \le \frac{p-1}{p} - \frac{1}{\lambda} \tag{62}$$

in the integral on the right-hand side of (61), we get by (59) and (61)

$$U(x) \le \frac{x}{(p-1)(xr(x))^p} + U(x) + \frac{p}{(p-1)\lambda} U(x)$$

or

$$U(x) \le \frac{\lambda x}{p(xr(x))^p} = \frac{\lambda x U(x)}{p}, \tag{63}$$

since $u(x) = (xr(x))^{-p}$. Now, for $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$u(x)^{-1/p'} U(x) r(x) f(x) \le u(x)^{-1/p'} \frac{\lambda x u(x)}{p} r(x) f(x) = \frac{\lambda}{p} f(t), \tag{64}$$

since $r(x)^{-1} = u(x)^{1/p}$. Now applying Theorem 3.2 and letting $b \to \infty$, the result follows. □

Corollary 4.9 (See Levinson [17], Theorem 5) *Suppose* $f(x) \ge 0$, $r(x) > 0$, $1 < p < \infty$, *and* $r(x)$ *be locally absolutely continuous for* $x > 0$.

Let

$$\frac{xr'(x)}{r(x)} - \frac{p-1}{p} \ge \frac{1}{\lambda} \tag{65}$$

for some $\lambda > 0$. *If, we write*

$$J(x) = \frac{r(x)}{x} \int_x^\infty \frac{f(t)}{r(t)} dt \tag{66}$$

then

$$\int_0^\infty J(x)^p dx \le \lambda^p \int_0^\infty f(x)^p dx \tag{67}$$

Proof In Theorem 3.1, take $b = \infty$ and $u(x) = (r(x)/x)^p$, $1 < p < \infty$ and $h(t) = f(t)/r(t)$. Now follow the method of proof of Corollary 4.8 and use Theorem 3.1 with $a \to 0$. □

Remark 4.3 If, in Corollary 4.9, $r(x) = x$ and $\lambda = p$, then Corollary 4.9 reduces to the dual inequality related to Hardy's inequality.

Pachpatte [27] followed the method of Levinson [17] to obtain generalization of two theorems of Chan [7]. We can deduce the results of Pachpatte from our Theorems 3.1 and 3.2 by appropriate choice of $u(x)$.

Corollary 4.10 (See Pachpatte [27], Theorem 1) *Let f be a non-negative and Lebesgue integrable function over the interval $[1, b)$, $1 < b \leq \infty$. Let $1 < p < \infty$ and $r(x)$ be a positive and locally absolutely continuous function on the interval $[1, b)$. Let*

$$1 - px(\ln x)\frac{r'(x)}{r(x)} \geq \frac{1}{\alpha} \tag{68}$$

for almost all x in $[1, b)$ and for some constant $\alpha > 0$. If $F(x)$ is given by

$$F(x) = \frac{1}{r(x)} \int_x^b \frac{r(t)f(t)}{t} dt, \quad x \in [1, b), \tag{69}$$

then

$$\int_1^b x^{-1} F(x)^p dx \leq (\alpha p)^p \int_1^b x^{-1} [\ln x f(x)]^p \, dx. \tag{70}$$

Proof In Theorem 3.1 take $u(x) = x^{-1}(r(x))^{-p}$, $h(x) = \frac{r(x)f(x)}{x}$ and $a = 1$. Then, by integrating by parts,

$$U(x) = \int_1^x \frac{dt}{t(r(t))^p} = \frac{\ln x}{(r(x))^p} - (-p)\int_1^x \frac{(lnx)r'(t)}{r(t)^{p+1}} dt. \tag{71}$$

Using (68) in the integral on the right-hand side of (71), we get, after some calculation,

$$U(x) \leq \int_1^x \frac{dt}{t(r(t))^p} = \frac{\ln x}{r(x)^p} + p\int_1^x \frac{(\ln t)r'(t)}{r(t)^{p+1}} dt \tag{72}$$

using (68) to the integral on the right-hand side of (72), and observing that

$$U(x) = \int_1^x \frac{dt}{t(r(t))^p},$$

we will get

$$U(x) \le \frac{\alpha \ln x}{(r(x))^p}. \tag{73}$$

Now the corollary can be deduced from Theorem 3.1. □

Remark 4.4 If, in Corollary 4.10, we take $r(t) \equiv 1$ in $[1, b)$ and $f(t) = tg(t)$, then $\alpha = 1$ yield Theorem 1 (1a) of Chan [7].

Corollary 4.11 (See Pachpatte [27], Theorem 2) *Let $p \ge 1$ and f be a non-negative and integrable function on* $(0, 1)$. *Let r be a positive and locally absolutely continuous function on* $(0, 1)$. *Suppose further that*

$$1 - px(\ln x)\frac{r'(x)}{r(x)} \ge \frac{1}{\alpha} \tag{74}$$

for almost all x in $(0, 1)$ *and for some constant* $\alpha > 0$.

If, we define,

$$F(x) = \frac{1}{r(x)} \int_0^x \frac{r(t)f(t)}{t} dt, \quad x \in (0, 1), \tag{75}$$

then

$$\int_0^1 \frac{(F(x))^p}{x} dx \le (\alpha p)^p \int_0^1 x^{-1} \left[|\ln x| f(x)\right]^p dx. \tag{76}$$

Proof Use Theorem 3.2 with $a = 0$, $b = 1$, $h(t) = r(t)f(t)/t$ and $u(x) = x^{-1}r(t)^{-p}$. Then follow the method used in the proof of Corollary 4.10 to get the required results of Corollary 4.11. We leave the details to the interested reader. □

Remark 4.5 The case $r(x) = 1$ for all x in $(0, 1)$, $\alpha = 1$ and $f(x) = xg(x)$, reduces Corollary 4.11 to Theorem 2 of the Chan [7].

Our next set of corollaries will be concerned with inequalities of the type proved by Copsen [8] and Beesack [2]. When those results were proved one felt that they are unique in their findings and could not be unified. We shall show that they follow from our Theorems 3.1 and 3.2 by choosing $u(x)$ appropriately.

Let f and ϕ be positive and measurable functions on $(0, \infty)$ and let us suppose that $\Phi(x) = \int_0^x \phi(t)dt$ exists for all x in $0 < x < \infty$. Whenever the integrals written below have finite values, we can write

$$G_1(x) = \int_0^x f(t)\phi(t)dt, \tag{77}$$

and

$$G_2(x) = \int_x^{\infty} f(t)\phi(t)dt. \tag{78}$$

Corollary 4.12 (Copson [8], Theorem 1) *If* $0 < b < \infty$, $1 < p < \infty$, *then*

$$\int_0^b G_1(x)\Phi(x)^{-c}\phi(x)dx < \left(\frac{p}{c-1}\right)^p \int_0^b f(x)^p \Phi(x)^{p-c}\phi(x)dx \tag{79}$$

if $0 \le a < \infty$, $0 < p \le 1$, $c > 1$, *and* $\lim_{x\to\infty} \Phi(x) = \infty$, *then*

$$\int_a^{\infty} G_1(x)^p\Phi(x)^{-c}\phi(x)dx \ge \left(\frac{p}{p-1}\right)^p \int_a^{\infty} f(x)^p \Phi(x)^{p-c}\phi(x)dx. \tag{80}$$

Proof In Theorem 3.2, take $u(x) = \frac{\phi(x)}{\Phi(x)^c} \quad (x > 0)$. Then

$$U(x) = \frac{1}{(c-1)}\left[\Phi(x)^{1-c} - \Phi(b)^{1-c}\right].$$

□

Since $c > 1$ and $\Phi(x)$ is a monotonically increasing function of x, we can conclude that

$$U(x) \le \frac{1}{(c-1)}\left[\Phi(x)\right]^{1-c}.$$

Now (79) follows from Theorem 3.2 when we substitute for $u(t)$ and $h(t)$.

To obtain (80), we need to note that $\Phi(\infty) = \infty$ and that gives

$$u(t) \le \frac{1}{(c-1)}\left[\Phi(t)\right]^{1-c}, \quad c > 1.$$

Then we use Theorem 3.1 to get the required result.

Corollary 4.13 (Copson [8], Theorems 3 and 4)

(i) If $1 \le p < \infty$, $c > 1$, $0 \le a < \infty$, *then*

$$\int_a^{\infty} G_2(x)^p\Phi(x)^{-c}\phi(x)dx < \left(\frac{p}{1-c}\right)^p \int_a^{\infty} f(x)^p \Phi(x)^{p-c}\phi(x)dx \tag{81}$$

(ii) if $0 < p \le 1$, $c < 1$, $0 < b \le \infty$, *then*

$$\int_0^b G_2(x)^p \Phi(x)^{-c}\phi(x)dx \geq \left(\frac{p}{1-c}\right)^p \int_0^b f(x)^p \Phi(x)^{p-c}\phi(x)dx. \quad (82)$$

Proof Use the method outlined in the proof of Corollary 4.12 and use Theorem 3.1 instead of Theorem 3.2. □

Corollary 4.14 (Beesack [2], Results (33))

(i) If $0 < a < \infty$, $1 < p < \infty$, *then*

$$\int_a^\infty G_2(x)^p \Phi(x)^{-1}\phi(x)dx \leq p^p \int_a^\infty f(x)^p \Phi(x)^{p-1} \left\{\ln \frac{\Phi(x)}{\phi(x)}\right\}^p \phi(x)dx \quad (83)$$

(ii) [Copson [8], Theorem 6 and Beesack [2], result (33)]. If $0 < p < 1$, *and* $0 < a < \infty$, *then the inequality in* (83) *is reserved. We also get equality when* $p = 1$.

Proof In Theorem 3.1, let $b \to \infty$, and $u(x) = \phi(x)/\Phi(x)$ and $h(x) = f(x)\phi(x)$. Then

$$U(x) = \int_a^x \frac{\phi(x)}{\Phi(x)} dt = \ln \left\{\frac{\Phi(x)}{\Phi(a)}\right\},$$

since $\phi(t) = \Phi'(t)$ almost everywhere. The results can now be obtained. □

Corollary 4.15 (See Copson [8], Theorem 5 and Beesack [2], Result (28)) *If* $0 < b < \infty$, $1 < p < \infty$, *then*

$$\int_0^b G_1(x)^p \Phi(x)^{-1}\phi(x)dx \leq p^p \int_0^b f(x)^p \Phi(x)^{p-1} \left\{\ln \frac{\Phi(b)}{\Phi(x)}\right\}^p \Phi(x)dx, \quad (84)$$

if $0 < p < 1$ *and* $0 < b < \infty$, *the inequality* (84) *is reserved. If* $p = 1$, (84) *reduces to an equality.*

Proof In Theorem 3.2, let us take $a = 0$, $u(x) = \frac{\phi(x)}{\Phi(x)}$ and $h(t) = f(x)\phi(x)$. Then

$$U(x) = \int_x^b \phi(t)\Phi(t)^{-1}dt = \ln \left\{\frac{\Phi(b)}{\Phi(x)}\right\} \quad (85)$$

since $\Phi'(x) = \phi(x)$ almost everywhere. Now the corollary follows. □

5 Conclusion

As we have obtained many known results from Theorem 3.1–3.3 in Sect. 4, we can also obtain results proved in Mohapatra and Russell [21] and Mohapatra and

Vajravelu [22]. We can also obtain results proved by same authors given in the references. However at this time we are unable to deduce results proved by Love [18] and [19] and Bicheng et al. [3].

A look at Theorem 2.1 of [3] shows that it is an improvement of Hardy's inequality. It will be instructive to see how are can generalize the results proved in [3] and many other papers in the reference so that a number of inequalities can be unified.

The objective of this chapter is to demonstrate that simple use of integration by parts and Hőlder's inequality which led to Davies–Peterson lemma can deliver fairly general results which unify variants of inequalities of Hardy, Copson, and Levinson types.

One can think of generalizing the results of Sect. 3 to Orlicz spaces and try to obtain interesting results. A look at the result of Love [20] where he has proved Hardy inequalities in Orlicz and Luxemburg norms shows that one can think of generalizations of theorems in Sect. 3 to more general norms as a field for future research. We should also look at the paper of Andersen and Heinig [1] where nice results involving integral operators have been proved. Equally instructive are also the papers of Boas [4], Boas and Imoru [5], and Nemeth [23] where nice results are established. It will be interesting to see if some unification of these results is possible.

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Double-Sided Taylor's Approximations and Their Applications in Theory of Analytic Inequalities

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Abstract In this paper the double-sided Taylor's approximations are studied. A short proof of a well-known theorem on the double-sided Taylor's approximations is introduced. Also, two new theorems are proved regarding the monotonicity of such approximations. Then we present some new applications of the double-sided Taylor's approximations in the theory of analytic inequalities.

1 Introduction

Consider a real function $f\ :\ (a,b)\ \longrightarrow\ \mathbb{R}$ such that there exist finite limits $f^{(k)}(a+) = \lim\limits_{x\to a+} f^{(k)}(x)$, for $k = 0, 1, \ldots, n$. Let us denote by $T_n^{f,\,a+}(x)$ Taylor's polynomial of degree n, $n \in \mathbb{N}_0$, for the function $f(x)$ in the right neighborhood of a:

$$T_n^{f,\,a+}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k.$$

We will call $T_n^{f,\,a+}(x)$ the *first Taylor's approximation in the right neighborhood of* a.

Similarly, the *first Taylor's approximation in the left neighborhood of* b is defined by:

$$T_n^{f,\,b-}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(b-)}{k!}(x-b)^k,$$

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where $f^{(k)}(b-) = \lim\limits_{x \to b-} f^{(k)}(x)$, for $k = 0, 1, \dots, n$.

Also, for $n \in \mathbb{N}$, the following functions:

$$R_n^{f,\,a+}(x) = f(x) - T_{n-1}^{f,\,a+}(x)$$

and

$$R_n^{f,\,b-}(x) = f(x) - T_{n-1}^{f,\,b-}(x)$$

are called the *remainder of the first Taylor's approximation in the right neighborhood of* a, and the *remainder of the first Taylor's approximation in the left neighborhood of* b, respectively.

Polynomials:

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) = \begin{cases} T_{n-1}^{f,\,a+}(x) + \dfrac{1}{(b-a)^n} R_n^{f,\,a+}(b-)(x-a)^n & : n \geq 1 \\ f(b-) & : n = 0, \end{cases}$$

and

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) = \begin{cases} T_{n-1}^{f,\,b-}(x) + \dfrac{1}{(a-b)^n} R_n^{f,\,b-}(a+)(x-b)^n & : n \geq 1 \\ f(a+) & : n = 0, \end{cases}$$

are called the *second Taylor's approximation in the right neighborhood of* a, and the *second Taylor's approximation in the left neighborhood of* b, respectively, $n \in \mathbb{N}_0$.

Theorem 2 in [26] provides an important result regarding Taylor's approximations. We cite it below:

Theorem 1 *Suppose that $f(x)$ is a real function on (a, b), and that n is a positive integer such that $f^{(k)}(a+)$, $f^{(k)}(b-)$, for $k \in \{0, 1, 2, \dots, n\}$, exist.*

(i) *Supposing that $(-1)^{(n)} f^{(n)}(x)$ is increasing on (a, b), then for all $x \in (a, b)$ the following inequality holds:*

$$\mathbb{T}_n^{f;\,b-,\,a+}(x) < f(x) < T_n^{f,\,b-}(x). \tag{1}$$

Furthermore, if $(-1)^n f^{(n)}(x)$ is decreasing on (a, b), then the reversed inequality of (1) *holds.*

(ii) *Supposing that $f^{(n)}(x)$ is increasing on (a, b), then for all $x \in (a, b)$ the following inequality also holds:*

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) > f(x) > T_n^{f,\,a+}(x). \tag{2}$$

Furthermore, if $f^{(n)}(x)$ is decreasing on (a, b), then the reversed inequality of (2) holds.

Let us name this theorem the *Theorem on double-sided Taylor's approximations*. In papers [16, 20, 23, 24] and [8] Theorem 1 was denoted by Theorem WD. Let us note that the proof of Theorem 1 (Theorem 2 in [26]) was based on the L'Hospital's rule for monotonicity. The same method was used in proofs of some theorems in [25, 27] and [28], that had been published earlier.

Here, we cite a theorem (Theorem 1.1. from [23]) that represents a natural extension of Theorem 1 over the set of real analytic functions.

Theorem 2 *For the function $f : (a, b) \longrightarrow \mathbb{R}$ let there exist the power series expansion*:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, \tag{3}$$

for every $x \in (a, b)$, where $\{c_k\}_{k \in \mathbb{N}_0}$ is the sequence of coefficients such that there is only a finite number of negative coefficients, and their indices are all in the set $J = \{j_0, \ldots, j_\ell\}$.

Then, for the function

$$F(x) = f(x) - \sum_{i=0}^{\ell} c_{j_i} (x-a)^{j_i} = \sum_{k \in N_0 \setminus J} c_k (x-a)^k, \tag{4}$$

and the sequence $\{C_k\}_{k \in \mathbb{N}_0}$ of the non-negative coefficients defined by

$$C_k = \begin{cases} c_k & : c_k > 0, \\ 0 & : c_k \le 0; \end{cases} \tag{5}$$

holds that:

$$F(x) = \sum_{k=0}^{\infty} C_k (x-a)^k, \tag{6}$$

for every $x \in (a, b)$.

Also, $F^{(k)}(a+) = k!\, C_k$, for $k \in \{0, 1, 2, \ldots, n\}$, and the following inequalities hold:

$$\begin{aligned} &\sum_{k=0}^{n} C_k (x-a)^k < F(x) < \\ &< \sum_{k=0}^{n-1} C_k (x-a)^k + \frac{1}{(b-a)^n} \left(F(b-) - \sum_{k=0}^{n-1} (b-a)^k C_k \right) (x-a)^n, \end{aligned} \tag{7}$$

i.e.:

$$\sum_{k=0}^{n} C_k (x-a)^k + \sum_{i=0}^{\ell} c_{j_i} (x-a)^{j_i} < f(x) <$$
$$< \sum_{k=0}^{n-1} C_k (x-a)^k + \sum_{i=0}^{\ell} c_{j_i} (x-a)^{j_i} + \frac{(x-a)^n}{(b-a)^n} \left(f(b-) - \sum_{k=0}^{n-1} C_k (b-a)^k - \sum_{i=0}^{\ell} c_{j_i} (b-a)^{j_i} \right), \tag{8}$$

for every $x \in (a, b)$.

Corollary 1 *Let there hold the conditions from the previous theorem. If*

$$n > \max\{j_0, \ldots, j_\ell\}, \tag{9}$$

then the following holds:

$$\sum_{k=0}^{n} c_k (x-a)^k < f(x) <$$
$$< \sum_{k=0}^{n-1} c_k (x-a)^k + \frac{1}{(b-a)^n} \left(f(b-) - \sum_{k=0}^{n-1} c_k (b-a)^k \right) (x-a)^n, \tag{10}$$

for every $x \in (a, b)$.

2 Some New Results on Double-Sided Taylor's Approximations

Consider a real function $f : (a, b) \longrightarrow \mathbb{R}$ such that there exist its first and second Taylor's approximations on both sides, for some $n \in \mathbb{N}$. Let us recall the remainders in Lagrange and the integral form, respectively, [22]:

$$R_n^{f,\,a+}(x) = \frac{f^{(n)}(\xi_{a,x})}{n!}(x-a)^n,$$

for some $\xi_{a,x} \in (a, x)$, and

$$R_n^{f,\,a+}(x) = \frac{(x-a)^n}{(n-1)!} \int_0^1 f^{(n)}(a + (x-a)t)(1-t)^{n-1}\, dt.$$

2.1 A New Proof of Theorem 1

We consider the case when $f^{(n)}(x)$ is a monotonically increasing function on (a, b) for some $n \in \mathbb{N}$. Other cases from Theorem 1 are proved similarly.

From the Lagrange form of the remainder and monotonicity of $f^{(n)}(x)$ on (a, b) we get:

$$\frac{f^{(n)}(a+)}{n!} < \frac{f^{(n)}(\xi_{a,x})}{n!} = \frac{f(x) - T_{n-1}^{f,\, a+}(x)}{(x-a)^n} \quad \Longrightarrow \quad T_n^{f,\, a+}(x) < f(x).$$

since $\xi_{a,x} \in (a, x)$ for all $x \in (a, b)$.

Using the integral form of the remainder we obtain the following inequality for all $x \in (a, b)$:

$$\begin{aligned} R_n^{f,\, a+}(x) &= \frac{(x-a)^n}{(n-1)!} \int_0^1 f^{(n)}(a + (x-a)t)(1-t)^{n-1}\, dt \\ &< \frac{(x-a)^n}{(n-1)!} \int_0^1 f^{(n)}(a + (b-a)t)(1-t)^{n-1}\, dt \\ &= \frac{(x-a)^n}{(b-a)^n} R_n^{f,\, a+}(b-) \qquad \Longrightarrow \qquad f(x) < \mathbb{T}_n^{f;\, a+,\, b-}(x). \end{aligned}$$

This completes the proof.

2.2 Monotonicity of Double-Sided Taylor's Approximations

Proposition 1 *Consider a real function $f : (a, b) \longrightarrow \mathbb{R}$ such that there exist its first and second Taylor's approximations on both sides, for some $n \in N_0$. Then,*

$$\operatorname{sgn}\Big(\mathbb{T}_n^{f,\, a+,\, b-}(x) - \mathbb{T}_{n+1}^{f,\, a+,\, b-}(x)\Big) = \operatorname{sgn}\Big(f(b-) - T_n^{f,\, a+}(b)\Big), \tag{11}$$

for all $x \in (a, b)$.

Proof From the definitions of the first and second Taylor's approximations we have:

$$\begin{aligned}
\mathbb{T}_{n+1}^{f,a+,b-}(x) &= T_n^{f,\,a+}(x) + \left(\frac{x-a}{b-a}\right)^{n+1} \cdot \Big(f(b-) - T_n^{f,\,a}(b)\Big) \\
&= T_{n-1}^{f,\,a+}(x) + \frac{f^{(n)}(a+)}{n!}(x-a)^n \\
&\quad + \left(\frac{x-a}{b-a}\right)^n \left(\frac{x-a}{b-a} - 1 + 1\right)\Big(f(b-) - T_n^{f,\,a+}(b)\Big) \\
&= T_{n-1}^{f,\,a+}(x) + \left(\frac{x-a}{b-a}\right)^n \Big(f(b-) - T_{n-1}^{f,\,a+}(b) - \frac{f^{(n)}(a+)}{n!}(b-a)^n\Big) + \\
&\quad + \frac{f^{(n)}(a+)}{n!}(x-a)^n + \left(\frac{x-a}{b-a}\right)^n \left(\frac{x-a}{b-a} - 1\right)\Big(f(b-) - T_n^{f,\,a+}(b)\Big) \\
&= \mathbb{T}_n^{f,\,a+,\,b-}(x) - \frac{b-x}{b-a}\left(\frac{x-a}{b-a}\right)^n \Big(f(b-) - T_n^{f,\,a+}(b)\Big).
\end{aligned}$$

Thus we have:

$$\mathbb{T}_n^{f,a+,b-}(x) - \mathbb{T}_{n+1}^{f,\,a+,\,b-}(x) = \frac{b-x}{b-a}\left(\frac{x-a}{b-a}\right)^n \Big(f(b-) - T_n^{f,\,a+}(b)\Big), \tag{12}$$

and the equality (11) immediately follows. □

Now, let us notice that if the real analytic function $f : (a, b) \longrightarrow \mathbb{R}$ satisfies the condition $(\forall n \in \mathbb{N}_0)\; f^{(n)}(a+) \geq 0$, then, from Proposition 1 directly follows:

$$(\forall n \in N_0)(\forall x \in (a, b))\; \mathbb{T}_n^{f,\,a+,\,b-}(x) > \mathbb{T}_{n+1}^{f,\,a+,\,b-}(x).$$

Theorem 3 *Consider a real function $f : (a, b) \longrightarrow \mathbb{R}$ such that the derivatives $f^{(k)}(a+)$, $k \in \{0, 1, 2, \ldots, n+1\}$ exist, for some $n \in \mathbb{N}$.*

Suppose that $f^{(n)}(x)$ and $f^{(n+1)}(x)$ are increasing on (a, b), then for all $x \in (a, b)$ the following inequalities hold:

$$T_n^{f,\,a+}(x) < T_{n+1}^{f,\,a+}(x) < f(x) < \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) < \mathbb{T}_n^{f;\,a+,\,b-}(x), \tag{13}$$

for all $x \in (a, b)$. If $f^{(n)}(x)$ and $f^{(n+1)}(x)$ are decreasing on (a, b), then for all $x \in (a, b)$ the reversed inequalities hold.

Case of the Real Analytic Functions

In applications, of special interest are the real analytic functions.

Theorem 4 *Consider the real analytic functions $f : (a, b) \longrightarrow \mathbb{R}$:*

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, \tag{14}$$

where $c_k \in \mathbb{R}$ and $c_k \geq 0$ for all $k \in \mathbb{N}_0$. Then,

$$
\begin{aligned}
T_0^{f,\,a+}(x) \le \ldots \le T_n^{f,\,a+}(x) \le T_{n+1}^{f,\,a+}(x) \le \ldots \\
\ldots \le f(x) \le \ldots \\
\ldots \le \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) \le \mathbb{T}_n^{f;\,a+,\,b-}(x) \le \ldots \le \mathbb{T}_0^{f;\,a+,\,b-}(x),
\end{aligned}
\tag{15}
$$

for all $x \in (a, b)$. If $c_k \in \mathbb{R}$ and $c_k \le 0$ for all $k \in \mathbb{N}_0$, then the reversed inequalities hold.

3 An Application of the Theorem on Double-Sided Taylor's Approximations

In this section we discuss an implementation of the Theorem on double-sided Taylor's approximations applied to the sequence of functions:

$$h_n(x) = \frac{\tan x - T_{2n-1}^{\tan,\,0}(x)}{x^{2n}\tan x} : \left(0, \frac{\pi}{2}\right) \longrightarrow R, \tag{16}$$

for $n \in \mathbb{N}$. This sequence of functions was considered in papers [4, 29]. In order to obtain estimates of functions $h_n(x)$, we use the well-known series expansions:

$$\tan x = \sum_{i=1}^{\infty} \frac{2^{2i}(2^{2i}-1)|\boldsymbol{B}_{2i}|}{(2i)!} x^{2i-1}, \tag{17}$$

where $|x| < \frac{\pi}{2}$ and $\boldsymbol{B}_k$ is the k-th Bernoulli number. Then:

$$T_{2n-1}^{\tan,\,0}(x) = \sum_{i=1}^{n} \frac{2^{2i}(2^{2i}-1)|\boldsymbol{B}_{2i}|}{(2i)!} x^{2i-1}, \tag{18}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. The main results on the functions $h_n(x)$, presented in the paper [29] (see also [4]), are cited below in the following two theorems.

Theorem 5 *For $x \in \left(0, \frac{\pi}{2}\right)$ and $n \in \mathbb{N}$, we have:*

$$h_n(x) = \sum_{j=1}^{n} \frac{2^{2(n-j+1)}(2^{2(n-j+1)}-1)|\boldsymbol{B}_{2(n-j+1)}|}{(2(n-j+1))!} \sum_{k=j}^{\infty} \frac{2^{2k}|\boldsymbol{B}_{2k}|}{(2k)!} x^{2(k-j)}. \tag{19}$$

Theorem 6 *For $x \in \left(0, \dfrac{\pi}{2}\right)$ and $n \in \mathbb{N}$, we have:*

$$\frac{2^{2(n+1)}(2^{2(n+1)}-1)|\boldsymbol{B}_{2(n+1)}|}{(2n+2)!} < h_n(x) < \left(\frac{2}{\pi}\right)^{2n}, \tag{20}$$

where the scalars $\dfrac{2^{2(n+1)}(2^{2(n+1)}-1)|\boldsymbol{B}_{2(n+1)}|}{(2n+2)!}$ and $\left(\dfrac{2}{\pi}\right)^{2n}$ in (20) are the best possible.

From Theorem 5, using the change of variables and some algebraic transformations, immediately follows the next theorem.

Theorem 7 *For $x \in \left(0, \dfrac{\pi}{2}\right)$ and $n \in \mathbb{N}$, functions $h_n(x)$ are real analytic functions and have the following Taylor series expansions:*

$$h_n(x) = \sum_{i=0}^{\infty}\sum_{j=1}^{n} \frac{2^{2(n+i+1)}(2^{2(n-j+1)}-1)\,|\boldsymbol{B}_{2(n-j+1)}|\,|\boldsymbol{B}_{2(i+j)}|}{(2(n-j+1))!\,(2(i+j))!}\,x^{2i}. \tag{21}$$

Let us notice that the Taylor series expansions of the functions $h_n(x)$ satisfy the conditions of Theorem 4.

Thus, we get the improvement of the results of Theorem 6:

Theorem 8 *For $x \in \left(0, \dfrac{\pi}{2}\right)$ and $n \in \mathbb{N}$, we have*

$$\begin{aligned} T_0^{h_n(x),\,0+}(x) = \frac{2^{2(n+1)}(2^{2(n+1)}-1)|\boldsymbol{B}_{2(n+1)}|}{(2n+2)!} < \\ < T_2^{h_n(x),\,0+}(x) < \ldots < T_{2m}^{h_n(x),\,0+}(x) < T_{2m+2}^{h_n(x),\,0+}(x) < \ldots \\ \ldots < h_n(x) < \ldots \\ \ldots < \mathbb{T}_{2m+2}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) < \mathbb{T}_{2m}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) < \ldots < \mathbb{T}_{2}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) < \\ < \mathbb{T}_{0}^{h_n(x);\,0+,\,\frac{\pi}{2}-}(x) = \left(\frac{2}{\pi}\right)^{2n}. \end{aligned} \tag{22}$$

4 More Examples of Double-Sided Taylor's Approximations

In this section we give two examples of some analytic inequalities recently proved using the results of Theorem 1. Also, we illustrate the application of double-sided

Taylor approximations and Theorem 4 in the generalizations and improvements of some analytic inequalities.

Example 1 In [6] the following improvement of Stečkin's inequality, in the left neighborhood of $b = \frac{\pi}{2}$, was proposed and proved:

$$Q_1(x) = \frac{2}{\pi} - \frac{1}{2}\left(\frac{\pi}{2} - x\right) < \tan x - \frac{4x}{\pi(2\pi - x)} < \frac{2}{\pi} - \frac{1}{3}\left(\frac{\pi}{2} - x\right) = R_1(x), \tag{23}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. In [20] the inequality (23) was further generalized. The starting point was the following real function:

$$g(t) = \cot t - \frac{1}{t} + \frac{2}{\pi} : \left(0, \frac{\pi}{2}\right) \longrightarrow R, \tag{24}$$

for which it is fulfilled

$$g\left(\frac{\pi}{2} - x\right) = \tan x - \frac{4x}{\pi(2\pi - x)}, \tag{25}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. It has been shown that the function $g(t)$ satisfies the conditions of Theorem 1. Namely, it has the following power series expansion

$$g(t) = \frac{2}{\pi} - \sum_{k=1}^{\infty} \frac{2^{2k}|\boldsymbol{B}_{2k}|}{(2k)!} t^{2k-1} \tag{26}$$

which converges for $t \in \left(0, \frac{\pi}{2}\right)$, and it is true

$$g(0+) = \lim_{t \to 0+} g(t) = \frac{2}{\pi} \quad \text{and} \quad g\left(\frac{\pi}{2}-\right) = \lim_{t \to \pi/2-} g(t) = 0.$$

The function $g(t)$ also satisfies the conditions of Theorem 4. Based on this, the following result was proposed in [20] (Theorem 3) for the function $f(x) = g\left(\frac{\pi}{2} - x\right)$:

Theorem 9 *For every $x \in \left(0, \frac{\pi}{2}\right)$ and $m \in N$, $m \geq 2$, the following inequalities hold:*

$$\mathbb{T}_{2m-1}^{g;\,0+,\,\pi/2-}\left(\frac{\pi}{2} - x\right) < f(x) < T_{2m-1}^{g,\,0}\left(\frac{\pi}{2} - x\right), \tag{27}$$

where

$$\mathbb{T}_{2m-1}^{g;\,0+,\,\pi/2-}\left(\frac{\pi}{2}-x\right)=$$
$$=\frac{2}{\pi}-\sum_{k=1}^{m-1}\frac{2^{2k}|\boldsymbol{B}_{2k}|}{(2k)!}\left(\frac{\pi}{2}-x\right)^{2k-1}+\sum_{k=1}^{m-1}\frac{2^{2k}|\boldsymbol{B}_{2k}|}{(2k)!}\left(\frac{2}{\pi}\right)^{2(m+k-1)}\left(\frac{\pi}{2}-x\right)^{2m-1} \tag{28}$$

and

$$T_{2m-1}^{g,\,0}\left(\frac{\pi}{2}-x\right)=\frac{2}{\pi}-\sum_{k=1}^{m}\frac{2^{2k}|\boldsymbol{B}_{2k}|}{(2k)!}\left(\frac{2}{\pi}-x\right)^{2k-1}. \tag{29}$$

It is easy to check that the function $g(t)$ also satisfies the conditions of Theorem 4. Therefore, for the function $f(x)=g\left(\frac{\pi}{2}-x\right)$ the following assertion directly follows:

Theorem 10 *For every* $x\in\left(0,\frac{\pi}{2}\right)$ *and* $m\in N$, $m\geq 2$, *the following inequalities hold*:

$$T_1^{g,\,0+}\left(\frac{\pi}{2}-x\right)\leq\ldots\leq T_{2m-1}^{g,\,0+}\left(\frac{\pi}{2}-x\right)\leq T_{2m+1}^{g,\,0+}\left(\frac{\pi}{2}-x\right)\leq\ldots$$
$$\ldots\leq f(x)\leq\ldots$$
$$\ldots\leq\mathbb{T}_{2m+1}^{g;\,0+,\,\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right)\leq\mathbb{T}_{2m-1}^{g;\,0+,\,\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right)\leq\ldots\leq\mathbb{T}_{1}^{g;\,0+,\,\frac{\pi}{2}-}\left(\frac{\pi}{2}-x\right). \tag{30}$$

Finally, from the previous two theorems an improvement of the inequality (23) directly follows. For example, if $m=1$, we have:

$$\begin{aligned}\mathbb{T}_{1}^{g;\,0+,\,\pi/2-}\left(\frac{\pi}{2}-x\right)=\frac{2}{\pi}-\frac{4}{\pi^2}\left(\frac{\pi}{2}-x\right)\ &\leq\\ \leq\tan x-\frac{4x}{\pi(2\pi-x)}\ &\leq\\ \leq\frac{2}{\pi}-\frac{1}{3}\left(\frac{\pi}{2}-x\right)=T_1^{g,\,0}\left(\frac{\pi}{2}-x\right),&\end{aligned}$$

which further implies the following:

$$Q_1(x)<\mathbb{T}_{1}^{g;\,0+,\,\pi/2-}\left(\frac{\pi}{2}-x\right)\leq\tan x-\frac{4x}{\pi(2\pi-x)}\leq T_1^{g,\,0+}\left(\frac{\pi}{2}-x\right)=R_1(x),$$

for $x\in\left(0,\frac{\pi}{2}\right)$.

Note that the same approach (based on Theorems 1 and 4) enables generalizations of the inequalities from [20] connected with the function

$$f(x) = \left(\pi^2 - 4x^2\right) \frac{\tan x}{x} : \left(0, \frac{\pi}{2}\right) \longrightarrow R.$$

Example 2 In [19] (Theorem 5) the following inequality was proved:

$$2 + \frac{2}{45} x^4 < \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \quad \text{for} \quad 0 < x < \pi/2. \tag{31}$$

In order to refine the previous inequality, the following real function was considered in [16]:

$$f(x) = \left(\frac{x}{\sin x}\right)^2 + \frac{x}{\tan x} \quad \text{for} \quad 0 < x < \pi/2.$$

It has been shown that the above function satisfies the conditions of Theorem 1.

Namely, it has the following power series expansion

$$f(x) = 2 + \sum_{k=2}^{\infty} \frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!} x^{2k}, \tag{32}$$

which converges for $x \in \left(0, \frac{\pi}{2}\right)$, and it is true

$$f(0+) = \lim_{x \to 0+} f(x) = 2 \quad \text{and} \quad f\left(\frac{\pi}{2}-\right) = \lim_{x \to \pi/2-} f(x) = \frac{\pi^2}{4}.$$

Based on this, the following result was proposed and proved in [16] (Theorem 5):

Theorem 11 *For every $x \in \left(0, \frac{\pi}{2}\right)$ and $m \in N$, $m \geq 2$, the following inequalities hold:*

$$T_{2m}^{f,\,0}(x) < f(x) < \mathbb{T}_{2m}^{f;\,0+,\,\pi/2-}(x), \tag{33}$$

where

$$T_{2m}^{f,\,0}(x) = 2 + \sum_{k=2}^{m} \frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!} x^{2k} \tag{34}$$

and

$$\mathbb{T}_{2m}^{f;\,0+,\,\pi/2-}(x) =$$
$$= 2 + \sum_{k=2}^{m-1} \frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!} x^{2k} + \left(\frac{2}{\pi}\right)^{2m} \left(\frac{\pi^2}{4} - 2 - \sum_{k=2}^{m-1} \frac{|\boldsymbol{B}_{2k}|\,(2k-2)4^k}{(2k)!} \left(\frac{\pi}{2}\right)^{2k}\right) x^{2m}. \tag{35}$$

In [16] the polynomials $T_m^{f,\,0+}(x)$ and $\mathbb{T}_m^{f;\,0+,\,\pi/2-}(x)$ are calculated and the concrete inequalities

$$T_m^{f,\,0+}(x) < f(x) < \mathbb{T}_m^{f;\,0+,\,\pi/2-}(x)$$

are given for $x \in \left(0, \dfrac{\pi}{2}\right)$ and for $m = 2, 3, 4, 5$.

It is easy to check that the function $f(x)$ also satisfies the conditions of Theorem 4, and hence the following generalizations of the inequality (33) i.e. of the inequality (31) are true:

Theorem 12 *For every* $x \in \left(0, \dfrac{\pi}{2}\right)$ *and* $m \in N$, $m \geq 2$, *the following inequalities hold:*

$$\begin{array}{c} T_0^{f,\,0+}(x) \leq \ldots \leq T_{2m}^{f,\,0+}(x) \leq T_{2m+2}^{f,\,0+}(x) \leq \ldots \\ \ldots \leq f(x) \leq \ldots \\ \ldots \leq \mathbb{T}_{2m+2}^{f;\,0+,\,\pi/2-}(x) \leq \mathbb{T}_{2m}^{f;\,0+,\,\pi/2-}(x) \leq \ldots \leq \mathbb{T}_0^{f;\,0+,\,\pi/2-}(x) \end{array} \tag{36}$$

The same approach, based on Theorem 1 and Theorem 4, provides generalizations of the inequalities from [16] related to the function

$$f(x) = 3\frac{x}{\sin x} + \cos x : \left(0, \frac{\pi}{2}\right) \longrightarrow R.$$

5 Conclusion

Even though Taylor's approximations represent a few centuries old topic, they are still present in research in many areas of science and engineering. Let us note that many results regarding Taylor's approximations are presented in well-known monographs [18] and [17]. Historically speaking, the second Taylor's approximation was mentioned in 1851 in the proof of the Taylor's formula with the Lagrange remainder in the paper [5] by Cox, see also [22].

Let us mention that in papers [1, 9, 23, 24, 30] and [8] double-sided Taylor's approximations are used to obtain corresponding inequalities. Results of these papers can be further organized and made more precise using Theorem 4 so we get the order among the functions occurring within these inequalities. Similar to double-sided Taylor's approximations, in papers [2, 3, 7, 10–14, 21] i [15] the finite expansions are used in the proofs of some mixed-trigonometric polynomial inequalities, as well as in some inequalities which can be reduced to mixed-trigonometric polynomial inequalities.

Currently, we are working on developing a computer system for automatic proving of some classes of analytic inequalities based on the results in the mentioned papers.

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The Levin–Stečkin Inequality and Simple Quadrature Rules

Peter R. Mercer

Abstract We obtain an error term for an extension of the Levin–Stečkin Inequality, which yields the error terms for the Midpoint, Trapezoid, and Simpson's rules.

1 Preliminaries

For an interval $[a, b]$, we denote its midpoint by $c = (a + b)/2$. If a function p defined on $[a, b]$ satisfies $p(a + b - x) = p(x)$ for $x \in [a, c]$, we shall say that p is even$_c$. If p satisfies $p(a + b - x) = -p(x)$ for $x \in [a, c]$, we shall say that p is odd$_c$. So an even function is even$_0$ and an odd function is odd$_0$.

Let p be an integrable even$_c$ function on $[a, b]$ which is increasing on $[a, c]$. We recall the **Levin–Stečkin Inequality** [1–3]: *For f convex on $[a, b]$, we have*

$$\int_a^b f(x)p(x)\,dx \;\leq\; \tfrac{1}{b-a}\int_a^b f(x)\,dx \int_a^b p(x)\,dx\,.$$

Here is a quick proof, more or less contained in [4]: Set $h(t) = f(t) + f(a+b-t)$. Since f is convex the slope of its chords is increasing, so we have for $c < x < y < b$,

$$h(y) - h(x) = f(y) + f(a+b-y) - (f(x) + f(a+b-x)) \geq 0,$$

so h is increasing on $[c, b]$. Then since h and p are even$_c$, it is easily verified that

$$\int_a^b fp - \tfrac{1}{b-a}\int_a^b f \int_a^b p = \tfrac{1}{b-a}\int_c^b \int_c^b \left[\big(h(x) - h(y)\big)\big(p(x) - p(y)\big)\right]dx\,dy,$$

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and this is ≤ 0, since p is decreasing on $[c, b]$.

Contained in the Levin–Stečkin Inequality is the term

$$A \;=\; \tfrac{1}{b-a}\int_a^b p(x)\,dx\;,$$

which is, of course, the average value of p. If p is even$_c$ and increasing on $[a, c]$, then p crosses over this value just twice; once in $[a, c]$ and once in $[c, b]$. The idea here is to allow p to cross its average value somewhat more often. We provide below an *error term* for the Levin–Stečkin Inequality—while requiring less from p, but (necessarily) a little more from f. As a consequence, we obtain the familiar error terms for the Midpoint, Trapezoid, and Simpson's Rules.

2 Main Result

Theorem *Let p be an integrable even$_c$ function on* $[a, b]$. *Let* $A_p = \frac{1}{b-a}\int\limits_a^b p(x)dx$ *and suppose that*

$$P(x) \;=\; \int_a^x \big[p(u) - A_p\big]du$$

does not change sign on [a,c]. Let f'' *be continuous on* $[a, b]$. *Then there is* $\xi \in (a, b)$ *such that*

$$\int_a^b f(x)p(x)\,dx \;-\; \tfrac{1}{b-a}\int_a^b f(x)\,dx\int_a^b p(x)\,dx \;=\; Kf''(\xi)\;,$$

where

$$K = \int_a^b\int_a^x P(t)\,dt\,dx\,.$$

Proof By replacing p with $p - A_p$, we may assume that $A_p = 0$. With P as defined above, i.e.,

$$P(x) \;=\; \int_a^x p(u)\,du\;,$$

let

$$Q(x) \;=\; \int_a^x P(t)\,dt\,.$$

Clearly we have $P(a) = P(b) = 0$. (And incidentally, $P(c) = 0$.) But since p is even$_c$, P is odd$_c$, so that Q is even$_c$, with $Q(a) = Q(b) = 0$. Integration by parts gives

$$\int_a^b f(x)p(x)\,dx = f(x)P(x)\Big|_a^b - \int_a^b f'(x)P(x)\,dx = -\int_a^b f'(x)P(x)\,dx$$

$$= -f'(x)Q(x)\Big|_a^b + \int_a^b f''(x)Q(x)\,dx = \int_a^b f''(x)Q(x)\,dx\,.$$

Now if P does not change sign on $[a, c]$ then Q does not change sign on $[a, b]$. Therefore, since f'' is continuous, by the Mean Value Theorem for Integrals there is $\xi \in [a, b]$ such that

$$\int_a^b f(x)p(x)\,dx = f''(\xi)\int_a^b Q(x)\,dx\,,$$

as desired. □

We make a few observations regarding the theorem.

(a) If p defined on $[a, b]$ is even$_c$ and monotonic on $[a, c]$, then the hypotheses of the theorem are satisfied. If, in particular, p is increasing on $[a, c]$, then

$$K = \int_a^b Q(x)\,dx \le 0\,.$$

This yields the Levin–Stečkin Inequality, at least for $f'' \ge 0$.

(b) Setting $f(x) = x^2$ in the conclusion of the theorem we get (as may be expected),

$$K = \frac{1}{2}\left(\int_a^b x^2 p(x)\,dx - \frac{1}{b-a}\int_a^b x^2\,dx\int_a^b p(x)\,dx\right).$$

This was obtained in an entirely different way in [5], but only for p as in the hypothesis of the Levin–Stečkin Inequality. Notice that K is independent of f. Under most circumstances this is a more manageable definition for K than the one provided in the statement of the theorem.

(c) The theorem may be viewed as a sort of a Mean Value-type theorem. Here is a very simple example: If f'' is continuous on [0, 1], then since $\int\limits_0^1 \cos(2\pi u)du = 0$ and $\int\limits_0^x \cos(2\pi u)du = \frac{1}{2\pi}\sin(2\pi x)$ does not change sign on [0, 1/2], there is $\xi \in [0, 1]$ such that

$$\int_0^1 f(x)\cos(2\pi x)dx = \tfrac{f''(\xi)}{2}\int_0^1 x^2\cos(2\pi x)dx = \tfrac{f''(\xi)}{4\pi^2}.$$

If f is convex, then $\int\limits_0^1 f(x)\cos(2\pi x)dx \geq 0$.

3 Quadrature Rules

Let us denote by χ_S the characteristic function for the set S.

Midpoint Rule Let $M_n = (b-a)\frac{n}{2}\chi_{[c-1/n,\,c+1/n]}$. Then M_n is even$_c$ and $A_{M_n} = \frac{1}{b-a}\int\limits_a^b M_n(x)dx = 1$. Here, $\int\limits_a^x [M_n(u)-1]du$ does not change sign on $[a,c]$, and the theorem gives

$$\int_a^b f(x)M_n(x)\,dx - \int_a^b f(x)\,dx = K_n f''(\xi_n).$$

Therefore if f'' is continuous then, as $n \to \infty$, we get

$$f(c)[b-a] - \int_a^b f(x)\,dx = Kf''(\xi_1).$$

The K_n's can be computed explicitly along the way; consequently so can $K = \lim K_n$. However, it is easier to obtain K by setting $f(x) = x^2$. This gives the familiar

$$f(c)[b-a] - \int_a^b f(x)\,dx = -\tfrac{1}{3}\left(\tfrac{b-a}{2}\right)^3 f''(\xi_1).$$

Trapezoid Rule Let $T_n = (b-a)\frac{n}{2}\chi_{[a,\,a+1/n]\cup[b-1/n,\,b]}$. Then T_n is even$_c$ and, again, $A_{T_n} = 1$. Here, $\int_a^x [T_n(u)-1]du$ does not change sign on $[a,c]$ and the theorem gives

$$\int_a^b f(x)T_n(x)\,dx - \int_a^b f(x)\,dx = K_n f''(\xi_n).$$

If f'' is continuous, then as $n \to \infty$, we get

$$\tfrac{f(a)+f(b)}{2}[b-a] - \frac{1}{b-a}\int_a^b f(x)\,dx = Kf''(\xi_2).$$

As before, K can be obtained via $K = \lim K_n$, though it is easier to set $f(x) = x^2$. This gives the familiar

$$\tfrac{f(a)+f(b)}{2}[b-a] \;-\; \int_a^b f(x)\,dx \;=\; \tfrac{2}{3}\left(\tfrac{b-a}{2}\right)^3 f''(\xi_2)\,.$$

Simpson's Rule Even though $\xi_1 \neq \xi_2$ in the error terms above, their coefficients suggest consideration of a quadrature rule which arises from

$$S_n \;=\; \tfrac{2}{3}M_n \;+\; \tfrac{1}{3}T_n\,.$$

Then again, S_n is even$_c$ and $\frac{1}{b-a}\int\limits_a^b S_n(x)\,dx = 1$. The *beginning* of the proof of the theorem gives

$$\int_a^b f(x)S_n(x)\,dx \;-\; \int_a^b f(x)\,dx \;=\; \int_a^b f''(x)Q_n(x)\,dx\,,$$

where

$$Q_n(x) = \int_a^x \int_a^t [S_n(u)-1]\,du\,dt\,.$$

Then Q_n is even$_c$ and $A_{Q_n} \;=\; \frac{1}{b-a}\int\limits_a^b Q_n(x)dx \;=\; \frac{1}{b-a}\frac{-n+2}{12n^2}$. To verify that $\int\limits_a^x [Q_n(u) - A_{Q_n}]du$ does not change sign on $[a,c]$ is routine (but tedious) and the theorem applied to Q_n gives

$$\int_a^b f''(x)Q_n(x)\,dx \;=\; K_n f^{(iv)}(\xi_n) \;+\; \tfrac{1}{b-a}\int_a^b f''(x)\,dx \int_a^b Q_n(x)\,dx\,.$$

That is,

$$\int_a^b f(x)S_n(x)\,dx - \int_a^b f(x)\,dx = K_n f^{(iv)}(\xi_n) + \tfrac{1}{b-a}\int_a^b f''(x)\,dx \int_a^b Q_n(x)\,dx\,.$$

If $f^{(iv)}$ is continuous, then since $\int\limits_a^b Q_n(x)\,dx \to 0$ as $n \to \infty$, we get

$$\left(\tfrac{2}{3}f(c) \;+\; \tfrac{1}{3}\tfrac{f(a)+f(b)}{2}\right)[b-a] \;-\; \int_a^b f(x)\,dx \;=\; Kf^{(iv)}(\xi_3).$$

Again, the K_n's can be computed explicitly (this is *very* tedious), so K can be obtained via $K = \lim K_n$. It is easier to set $f(x) = x^4$. This yields the familiar

$$\tfrac{1}{6}\big(f(a) + 4f(c) + f(b)\big)[b - a] \; - \int_a^b f(x)\,dx \;=\; \tfrac{1}{90}\left(\tfrac{b-a}{2}\right)^5 f^{(iv)}(\xi_3)\,.$$

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(p, q)-Laplacian Equations with Convection Term and an Intrinsic Operator

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Abstract The paper introduces a new type of nonlinear elliptic Dirichlet problem driven by the (p, q)-Laplacian where the reaction term is in the convection form (meaning that it exhibits dependence on the solution and its gradient) composed with a (possibly nonlinear) general map called intrinsic operator on the Sobolev space. Under verifiable hypotheses, we establish the existence of at least one (weak) solution. A second main result deals with the uniqueness of solution. Finally, a third result provides the existence and uniqueness of solution to a problem of this type involving a translation viewed as an intrinsic operator. Examples show the applicability of these results.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a nonempty bounded open set with Lipschitz boundary $\partial\Omega$, let the real numbers p, q, μ with $1 < q < p < N$ and $\mu \geq 0$, and let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function, i.e., $f(\cdot, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $f(x, \cdot, \cdot)$ is continuous for a.e. $x \in \Omega$. It is supposed to have $p < N$ only for avoiding to repeat certain arguments. The case $p \geq N$ can be handled similarly. By $W_0^{1,p}(\Omega)$ we denote the usual Sobolev space equipped with the norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx \right)^{\frac{1}{p}}.$$

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In the sequel, corresponding to any real number $r \in (1, +\infty)$ we denote by r' its Hölder conjugate, i.e., $r' = \frac{r}{r-1}$.

Given a continuous map $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ called intrinsic operator, we consider the following Dirichlet problem

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, T(u), \nabla T(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

In (1) we have the negative p-Laplacian $-\Delta_p : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ given by

$$\left\langle -\Delta_p u, v \right\rangle = \int_\Omega |\nabla u(x)|^{p-2} \nabla u(x) \cdot \nabla v(x)\, dx, \qquad \forall v \in W_0^{1,p}(\Omega),$$

and the negative q-Laplacian $-\Delta_q : W_0^{1,q}(\Omega) \to W^{-1,q'}(\Omega)$ given by

$$\left\langle -\Delta_q u, v \right\rangle = \int_\Omega |\nabla u(x)|^{q-2} \nabla u(x) \cdot \nabla v(x)\, dx, \qquad \forall v \in W_0^{1,q}(\Omega).$$

Since $\Omega \subset \mathbb{R}^N$ is a bounded domain and $p > q$, it holds the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, so it is well defined the sum of operators $-\Delta_p - \mu \Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ for all μ. Important special cases are when $\mu = 0$, i.e., the negative p-Laplacian $-\Delta_p$, and when $\mu = 1$, i.e., the negative (p, q)-Laplacian $-\Delta_p - \Delta_q$. In this way it is achieved the unifying presentation of two very different operators p-Laplacian and (p, q)-Laplacian.

We say that $u \in W_0^{1,p}(\Omega)$ is a weak solution of problem (1) if it holds

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v\, dx + \mu \int_\Omega |\nabla u|^{q-2} \nabla u \cdot \nabla v\, dx - \int_\Omega f(x, T(u), \nabla T(u)) v\, dx = 0$$

for all $v \in W_0^{1,p}(\Omega)$ provided that the third integral makes sense. This occurs under suitable growth conditions for f and T as will be supposed later on. The aim of this note is to seek for weak solutions of problem (1).

A relevant feature of problem (1) is the fact that the right-hand side of the equation is expressed as the composition of the Nemytskii operator associated to the function $f(x, s, \xi)$ giving rise to the convection term $f(x, u, \nabla u)$ (i.e., an expression depending on the solution u and its gradient ∇u) with a prescribed continuous map $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ that we call intrinsic operator for problem (1). Generally, a convection term prevents to have a variational structure, so the variational methods are not applicable, which complicates considerably the study of such problems. Inserting the map T in the convection term makes the problem substantially more general and more difficult. The case where T is the identity map is just the (p, q)-Laplacian problem with convection term, a topic extensively studied (see, e.g., [1, 2, 5–9, 11, 13]). We emphasize that the operator T

cannot be incorporated in the function $f(x, s, \xi)$ because T acts on the whole space $W_0^{1,p}(\Omega)$ and not pointwise.

A novelty of this work with respect to other papers in the field is the presence of the (possibly nonlinear) intrinsic operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ in the statement of problem (1), which makes the problem nonstandard and more complex. Under adequate assumptions we present three main results. Theorem 2 provides the existence of a (weak) solution. Theorem 3 constitutes a uniqueness result for problem (1). An existence and uniqueness result is formulated in Theorem 4 for a particular case of problem (1). These results are accompanied by examples. An essential role in our developments is played by the theory of pseudomonotone operators whose basic needed elements are recalled in the next section for the sake of clarity.

2 Preliminary Tools

In view of Rellich–Kondrachov Theorem, one has that the Sobolev space $W_0^{1,p}(\Omega)$ is compactly embedded into $L^\theta(\Omega)$ for every $1 \le \theta < p^*$ and continuously embedded for $\theta = p^*$, where p^* stands for the Sobolev critical exponent, that is $p^* = \frac{Np}{N-p}$ (if $N > p$, as supposed). Thus for every $1 \le \theta \le p^*$ there exists a positive constant S_θ such that

$$\|u\|_\theta \le S_\theta \|u\|, \quad \forall u \in W_0^{1,p}(\Omega), \tag{2}$$

where $\|u\|_\theta$ denotes the norm in $L^\theta(\Omega)$.

We also recall that the first eigenvalue $\lambda_{1,r}$ of the negative r-Laplacian $-\Delta_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ has the following variational characterization

$$\lambda_{1,r} = \inf_{u \in W_0^{1,r}(\Omega),\, u \neq 0} \frac{\|\nabla u\|_r^r}{\|u\|_r^r}. \tag{3}$$

Notice that, for $\theta = p$ in (2), the best constant S_p is $S_p = (1/\lambda_{1,p})^{1/p}$ (see (3)).

For a later use, we mention a few basic facts about pseudomonotone operators that are necessary in the rest of the paper. Let X be a real reflexive Banach space with the norm $\|\cdot\|$ and its dual space X^*. Denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X and X^*. The norm convergence in X and X^* is denoted by $\to$ and the weak convergence by $\rightharpoonup$.

A map $A : X \to X^*$ is called:

- continuous, if $u_n \to u$ implies $Au_n \to Au$;
- bounded, if A maps bounded sets into bounded sets;
- coercive if

$$\lim_{\|u\|\to+\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty;$$

- pseudomonotone if $u_n \rightharpoonup u$ in X and

$$\limsup_{n\to+\infty} \langle Au_n, u_n - u\rangle \le 0$$

imply

$$\langle Au_n, u - w\rangle \le \liminf_{n\to+\infty} \langle Au_n, u_n - w\rangle\,, \ \forall w \in X.$$

The main theorem on pseudomonotone operators is now stated. More details can be found in [3, 12].

Theorem 1 *Let X be a real reflexive Banach space, let $A : X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and let $b \in X^*$. Then a solution $u \in X$ of the equation $Au = b$ exists.*

3 Existence of Solutions

We formulate the following assumptions on the Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and the continuous operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$:

(i) There exist constants $a_1 \ge 0$, $a_2 \ge 0$, $\alpha \in (0, p^* - 1)$, $\beta \in (0, \frac{p}{(p^*)'})$, and a function $\sigma_1 \in L^{r'}(\Omega)$ with $r \in [1, p^*)$ such that

$$|f(x, s, \xi)| \le \sigma_1(x) + a_1|s|^\alpha + a_2|\xi|^\beta$$

for a.a. $x \in \Omega$, all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^N$.

(ii) There exist constants $K_1, K_2, K_3 \ge 0$ such that

$$\|T(u)\|_{p^*}^\alpha \le K_1\|u\|^{p-1} + K_3 \quad \text{and} \quad \|T(u)\|^\beta \le K_2\|u\|^{p-1} + K_3$$

for all $u \in W_0^{1,p}(\Omega)$ and

$$a_1 K_1 S_{\frac{p^*}{p^*-\alpha}} + a_2 K_2 S_{\frac{p}{p-\beta}} < 1 \tag{4}$$

(see (2)). In particular, this assumption is fulfilled if there are constants $K_1', K_2' \ge 0$ and $\gamma \in [0, p-1)$ such that

$$\max\{\|T(u)\|^\alpha, \|T(u)\|^\beta\} \le K_1'\|u\|^\gamma + K_2' \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{5}$$

Our result on the existence of weak solutions to problem (1) is as follows.

Theorem 2 *Assume that conditions* (i) *and* (ii) *are verified. Then problem* (1) *admits at least one (weak) solution.*

Proof On the basis of assumption (i), the Nemytskii operator $N_f : W_0^{1,p}(\Omega) \to L^{(p^*)'}(\Omega)$ related with the function f, that is

$$N_f(u) = f(x, u, \nabla u),$$

is well defined and continuous. Then using the intrinsic operator, i.e. the map $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, we construct the nonlinear operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by

$$Au = -\Delta_p u - \mu \Delta_q u - N_f \circ T(u). \tag{6}$$

Here the continuity of the embedding $L^{(p^*)'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$ was used, too. The fact that $u \in W_0^{1,p}(\Omega)$ is a weak solution for problem (1) is equivalent to have that u is a zero of the operator A, which means that

$$\langle Au, v\rangle = 0, \quad \forall\, v \in W_0^{1,p}(\Omega). \tag{7}$$

Our aim is to apply Theorem 1, from which we will get the surjectivity of the operator $A : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ defined by (6), thus (7) ensues. We divide the proof into six steps.

Claim 1 For every $u,\ w \in W_0^{1,p}(\Omega)$ we have the estimate

$$\begin{aligned} &\left| \int_\Omega f(x, T(u), \nabla T(u)) w \, dx \right| \\ &\le \|\sigma_1\|_{r'} \|w\|_r + a_1 \|T(u)\|_{p^*}^{\alpha} \|w\|_{\frac{p^*}{p^*-\alpha}} + a_2 \|T(u)\|^{\beta} \|w\|_{\frac{p}{p-\beta}}. \end{aligned} \tag{8}$$

Estimate (8) is obtained from assumption (i) in conjunction with Hölder's inequality.

Claim 2 The operator A is continuous and bounded.

Since $q < p$, one has the continuous embedding $W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,q}(\Omega)$, so the operator $-\Delta_p - \mu\Delta_q : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous (see, e.g., [4, Lemma 2.111]). Moreover, as noted above, the Nemytskii operator N_f is continuous from $W_0^{1,p}(\Omega)$ to $L^{(p^*)'}(\Omega)$. Hence, due to the continuity of the embedding $L^{(p^*)'}(\Omega) \hookrightarrow W^{-1,p'}(\Omega)$, the operator $N_f \circ T : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ is continuous as composition of continuous maps. Consequently, the operator A is continuous as being a sum of continuous operators. Taking into account that the operators $-\Delta_p - \mu\Delta_q, N_f : W_0^{1,p}(\Omega) \to W^{-1,p'}(\Omega)$ are bounded, as well as the operator T (see (ii)), we infer that A is bounded.

Claim 3 If $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$, then it holds

$$\lim_{n\to+\infty} \int_\Omega f(x, T(u_n), \nabla T(u_n))(u_n - u)\, dx = 0. \tag{9}$$

The sequence (u_n) is bounded in $W_0^{1,p}(\Omega)$, so due to the boundedness of the operator T, which follows from assumption (ii), there is a constant $c_1 > 0$ such that

$$\|T(u_n)\| \le c_1, \quad \forall n.$$

From this, owing to (8) with u and w replaced by u_n and $u_n - u$, respectively, we get the estimate

$$\left| \int_\Omega f(x, T(u_n), \nabla T(u_n))(u_n - u)\, dx \right|$$
$$\le \|\sigma_1\|_{r'} \|u_n - u\|_r + a_1 S_{p^*}^\alpha c_1^\alpha \|u_n - u\|_{\frac{p^*}{p^*-\alpha}} + a_2 c_1^\beta \|u_n - u\|_{\frac{p}{p-\beta}}$$

for every n (with S_{p^*} as in (2)). In view of the conditions imposed in assumption (i) on r, α, and β, the space $W_0^{1,p}(\Omega)$ is compactly embedded in $L^r(\Omega)$, $L^{p^*/(p^*-\alpha)}(\Omega)$, and $L^{p/(p-\beta)}(\Omega)$. Therefore, passing to the limit as $n \to +\infty$, we arrive at (9).

Claim 4 If $u_n \rightharpoonup u$ in $W_0^{1,p}(\Omega)$ and

$$\limsup_{n\to+\infty} \langle Au_n, u_n - w\rangle \le 0, \tag{10}$$

then there hold $u_n \to u$ in $W_0^{1,p}(\Omega)$, $Au_n \to Au$ in $W^{-1,p'}(\Omega)$ and

$$\lim_{n\to+\infty} \langle Au_n, u_n\rangle = \langle Au, u\rangle . \tag{11}$$

In particular, the operator A is pseudomonotone.

On the basis of Claim 3, inequality (10) reads as

$$\limsup_{n\to+\infty} \left\langle -\Delta_p u_n - \mu \Delta_q u_n, u_n - u\right\rangle \le 0.$$

Invoking the (S_+)-property of the operator $-\Delta_p - \mu\Delta_q$ on $W_0^{1,p}(\Omega)$ (see, e.g., [4]), it follows that $u_n \to u$ in $W_0^{1,p}(\Omega)$. Moreover, we know by Claim 2 that A is continuous. Hence $Au_n \to Au$ in $W^{-1,p'}(\Omega)$ and (11) holds, too.

Claim 5 There exist constants $\delta \in [0, 1)$ and $\gamma \geq 0$ such that

$$\int_\Omega f(x, T(u), \nabla T(u))u\, dx \leq \delta \|u\|^p + \gamma \quad \text{for all } u \in W_0^{1,p}(\Omega). \tag{12}$$

Using Claim 1, hypothesis (ii), and Young's inequality, we find that

$$\begin{aligned}
&\int_\Omega f(x, T(u), \nabla T(u))u\, dx \\
&\leq \|\sigma_1\|_{r'} \|u\|_r + a_1 (K_1 \|u\|^{p-1} + K_3)\|u\|_{\frac{p^*}{p^*-\alpha}} + a_2(K_2\|u\|^{p-1} + K_3)\|u\|_{\frac{p}{p-\beta}} \\
&\leq (a_1 K_1 S_{\frac{p^*}{p^*-\alpha}} + a_2 K_2 S_{\frac{p}{p-\beta}} + \epsilon)\|u\|^p + \gamma
\end{aligned}$$

with a constant $\gamma > 0$ and an $\epsilon > 0$ small so that $\delta := a_1 K_1 S_{\frac{p^*}{p^*-\alpha}} + a_2 K_2 S_{\frac{p}{p-\beta}} + \epsilon < 1$ (see (4)). This establishes Claim 5.

Claim 6 The operator A is coercive.

Using Claim 5, we find that

$$\langle Au, u\rangle = \left\langle -\Delta_p u - \mu \Delta_q u, u\right\rangle - \int_\Omega f(x, T(u), \nabla T(u))u\, dx \geq (1 - \delta)\|u\|^p - \gamma$$

for all $u \in W_0^{1,p}(\Omega)$. Since $p > 1$ and $\delta < 1$, we derive that

$$\lim_{\|u\| \to +\infty} \frac{\langle Au, u\rangle}{\|u\|} = +\infty,$$

thus Claim 6 holds true.

At this moment we are able to conclude. Claims 2, 4, and 6 entail that the operator A satisfies all the assumptions of Theorem 1. The conclusion follows by applying this theorem since then the operator A is surjective. □

Remark 1 A careful reading of the proof shows that hypothesis (ii) can be replaced by requiring the (continuous) operator T to be bounded and to fulfill the non-local condition (12) for constants $\delta \in [0, 1)$ and $\gamma \geq 0$.

Here are two examples where Theorem 2 can be applied. In the first example, T is a typical truncation operator (thus defined locally) while in the second example $T(u)$ cannot be defined locally (as being a solution of an auxiliary problem).

Example 1 Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying condition (i) with $\alpha, \beta < p - 1$. Then the following problem

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = f(x, u^+, \nabla(u^+)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{13}$$

possesses at least a weak solution. Here u^+ stands for the positive part of u, that is $u^+ = \max\{u, 0\}$. In order to obtain the stated conclusion, we consider the continuous intrinsic operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ given by $T(u) = u^+$ for all $u \in W_0^{1,p}(\Omega)$. We note that since we supposed $\alpha, \beta < p-1$ in (i), condition (ii) is fulfilled since T satisfies (5) with $\gamma = \max\{\alpha, \beta\}$. We can thus apply Theorem 2, which yields the existence of a weak solution $u \in W_0^{1,p}(\Omega)$ to problem (13).

Example 2 In this example we argue with a continuous function $g : \mathbb{R} \to \mathbb{R}$ which satisfies the following growth condition

$$|g(s)| \leq a|s|^t + b \quad \text{for all } s \in \mathbb{R} \tag{14}$$

for constants $a, b \geq 0$ and $t \in (0, p^* - 1)$. Under this condition, for every $u \in W_0^{1,p}(\Omega)$, the problem

$$\begin{cases} -\Delta_p v = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{15}$$

has a unique weak solution $v =: T(u)$. We consider the mapping $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ so obtained.

Let a Carathéodory function $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ which satisfies (i) and assume in addition that

$$t \max\{\alpha, \beta\} < (p-1)^2 \tag{16}$$

(this holds, for instance, if $\max\{t, \alpha, \beta\} < p - 1$, or if $t = 1$ and $\max\{\alpha, \beta\} < (p-1)^2$). Under these conditions we claim that problem (1) (for the operator T defined above) has a weak solution. To see this, let us note that T is continuous and satisfies condition (5) (and thus condition (ii)) so that Theorem 2 can be applied. Since $0 < t < p^* - 1$ we can find $r \in (1, p^*)$ such that $tr' < p^*$. Let $u \in W_0^{1,p}(\Omega)$. On the one hand by (14) we have

$$\|g(u)\|_{r'}^{r'} \leq \tilde{a}\|u\|_{tr'}^{tr'} + \tilde{b} \leq \tilde{a} S_{tr'}^{tr'} \|u\|^{tr'} + \tilde{b}$$

for constants $\tilde{a}, \tilde{b} > 0$ independent of u (where we also use the constant S_θ of (2)). On the other hand using $T(u)$ as test function in (15) leads to

$$\|T(u)\|^p = \int_\Omega g(u)T(u)\,dx \leq \|g(u)\|_{r'}\,\|T(u)\|_r \leq S_r\|g(u)\|_{r'}\,\|T(u)\|$$

whence

$$\|T(u)\|^{p-1} \leq S_r\|g(u)\|_{r'} \leq a'\|u\|^t + b'$$

for constants $a', b' > 0$ independent of u. By (16) we can find γ such that $\frac{t \max\{\alpha,\beta\}}{p-1} < \gamma < p - 1$, so that the previous relation yields

$$\max\{\|T(u)\|^{\alpha}, \|T(u)\|^{\beta}\} \le \tilde{a}'\|u\|^{\gamma} + \tilde{b}' \tag{17}$$

for some constants $\tilde{a}', \tilde{b}' > 0$ independent of u. Thus (5) is satisfied.

Moreover, the mapping $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ is continuous. This can be seen as follows. Let $u_n \to u$ in $W_0^{1,p}(\Omega)$, thus $u_n \to u$ in $L^{p^*}(\Omega)$ by the Sobolev embedding theorem, which guarantees through the Krasnoselskii theorem and (14) that $g(u_n) \to g(u)$ in $L^{\frac{p^*}{t}}(\Omega)$. Due to (17), the sequence $(w_n) := (T(u_n))$ is bounded in $W_0^{1,p}(\Omega)$. Passing to a relabeled subsequence we have $w_n \rightharpoonup w$ in $W_0^{1,p}(\Omega)$ and $w_n \to w$ in $L^{(\frac{p^*}{t})'}(\Omega)$, with some $w \in W_0^{1,p}(\Omega)$. Therefore we infer that

$$\langle -\Delta_p w_n, w_n - w\rangle = \int_{\Omega} g(u_n)(w_n - w)\,dx \to 0 \ \text{ as } n \to \infty.$$

Then the (S_+)-property of the operator $-\Delta_p$ on $W_0^{1,p}(\Omega)$ (see, e.g., [4]) ensures that $w_n \to w$ in $W_0^{1,p}(\Omega)$. From here it is straightforward to deduce that $w = T(u)$, whence the continuity of the mapping T follows. The application of Theorem 2 ensues.

4 Uniqueness

In this section we present a uniqueness result for problem (1). The main features of this result are that $p > q \ge 2$ and the right-hand side of the equation is written as a sum $f(x, s, \xi) = g(x, s, \xi) + h(x, s, \xi)$ with the terms $g(x, s, \xi)$ and $h(x, s, \xi)$ having different behavior.

We recall that for every $r \ge 2$ there is a constant $c_r > 0$ such that

$$(|\xi|^{r-2}\xi - |\eta|^{r-2}\eta) \cdot (\xi - \eta) \ge c_r|\xi - \eta|^r \ \text{ for all } \xi, \eta \in \mathbb{R}^N$$

so that

$$\langle -\Delta_r u_1 + \Delta_r u_2, u_1 - u_2\rangle \ge c_r\|\nabla u_1 - \nabla u_2\|_r^r, \ \ \forall\, u_1, u_2 \in W_0^{1,r}(\Omega). \tag{18}$$

Note that for $r = 2$ the constant is $c_2 = 1$. Our uniqueness statement is as follows.

Theorem 3 *Let $p > q \ge 2$. Assume that the Carathéodory functions $g, h : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and the continuous map $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ satisfy the conditions:*

(j) *there is a constant* $m_1 \in [0, c_p]$ *such that for all* $v_1, v_2 \in W_0^{1,p}(\Omega)$*, one has*

$$\int_\Omega (g(x, T(v_1), \nabla T(v_1)) - g(x, T(v_2), \nabla T(v_2)))(v_1 - v_2)\, dx \leq m_1 \|v_1 - v_2\|^p;$$

(jj) *there is a constant* $m_2 \in [0, \mu c_q]$ *such that for all* $v_1, v_2 \in W_0^{1,p}(\Omega)$*, one has*

$$\int_\Omega (h(x, T(v_1), \nabla T(v_1)) - h(x, T(v_2), \nabla T(v_2)))(v_1 - v_2)\, dx$$
$$\leq m_2 \int_\Omega |\nabla(v_1 - v_2)|^q\, dx;$$

(jjj) $m_1 < c_p$ *or* $m_2 < \mu c_q$.

Then the Dirichlet problem (of type (1)*)*

$$\begin{cases} -\Delta_p u - \mu \Delta_q u = g(x, T(u), \nabla T(u)) + h(x, T(u), \nabla T(u)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{19}$$

has at most one weak solution.

Proof Suppose $u_1, u_2 \in W_0^{1,p}(\Omega)$ are both weak solutions to problem (19). Then by (18) and hypotheses (j) and (jj) we have

$$c_p \|u_1 - u_2\|^p + \mu c_q \int_\Omega |\nabla(u_1 - u_2)|^q\, dx$$
$$\leq \left\langle -\Delta_p u_1 + \Delta_p u_2, u_1 - u_2 \right\rangle + \mu \left\langle -\Delta_q u_1 + \Delta_q u_2, u_1 - u_2 \right\rangle$$
$$= \int_\Omega (g(x, T(u_1), \nabla T(u_1)) - g(x, T(u_2), \nabla T(u_2)))(u_1 - u_2)\, dx$$
$$+ \int_\Omega (h(x, T(u_1), \nabla T(u_1)) - h(x, T(u_2), \nabla T(u_2)))(u_1 - u_2)\, dx$$
$$\leq m_1 \|u_1 - u_2\|^p + m_2 \int_\Omega |\nabla(u_1 - u_2)|^q\, dx.$$

This results in

$$(c_p - m_1)\|u_1 - u_2\|^p + (\mu c_q - m_2) \int_\Omega |\nabla(u_1 - u_2)|^q\, dx \leq 0.$$

Invoking (jjj), we infer that $u_1 = u_2$, which completes the proof. □

Remark 2 Assumptions (j) and (jj) express properties of interaction between the nonlinearity $f(x, s, \xi) = g(x, s, \xi) + h(x, s, \xi)$ and the intrinsic operator T :

$W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$. Uniqueness results for elliptic problems with convection terms (i.e., the corresponding vector field fully depends on the solution and its gradient) can be found in [2, Theorem 2] (see also [10] and the references therein where the nonlinearity does not depend on the gradient of the solution). The novelty here is that the result involves the intrinsic operator T.

We provide a simple example of application of Theorem 3.

Example 3 Fix an element $u_0 \in W_0^{1,p}(\Omega)$ and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that for a.e. $x \in \Omega$ the function $g(x, \cdot)$ is nonincreasing, so

$$(g(x, s_1) - g(x, s_2))(s_1 - s_2) \leq 0, \ \forall s_1, s_2 \in \mathbb{R}. \tag{20}$$

Then Theorem 3 with $p > q = 2$ applies to the problem

$$\begin{cases} -\Delta_p u - \mu \Delta u = g(x, u + u_0) + \frac{\partial}{\partial x_1}(u + u_0) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{21}$$

ensuring that, for μ sufficiently large, problem (21) possesses at most a weak solution. Indeed, in order to apply Theorem 3 we take the intrinsic operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$ to be the translation $T(v) = v + u_0$. In view of assumption (20), condition (j) (with the function $g(x, s, \xi) := g(x, s)$) is verified with the constant $m_1 = 0 < c_p$. Next we point out that condition (jj) is verified by the function $h(x, s, \xi) = h(\xi) := \xi_1$ for all $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$, provided that the parameter $\mu \geq \frac{1}{2}(1 + \frac{1}{\lambda_{1,2}})$ with $\lambda_{1,2}$ as in (3). This follows from the estimate

$$\begin{aligned} &\int_\Omega (h(\nabla T(v_1)) - h(\nabla T(v_2)))(v_1 - v_2)\, dx \\ &= \int_\Omega \frac{\partial(v_1 - v_2)}{\partial x_1}(v_1 - v_2)\, dx \\ &\leq \frac{1}{2} \int_\Omega \left(|\nabla(v_1 - v_2)|^2 + (v_1 - v_2)^2\right) dx \leq \frac{1}{2}\left(1 + \frac{1}{\lambda_{1,2}}\right) \int_\Omega |\nabla(v_1 - v_2)|^2\, dx \end{aligned}$$

for all $v_1, v_2 \in W_0^{1,p}(\Omega)$, where (3) with $r = 2$ has been used. Theorem 3 implies that problem (21) admits at most one solution whenever $\mu \geq \frac{1}{2}(1 + \frac{1}{\lambda_{1,2}})$.

Finally, we present a result involving convection term with an intrinsic operator where Theorems 2 and 3 jointly apply, thus obtaining an existence and uniqueness result. We do this in the case of problem (21) in Example 3 by strengthening the hypotheses therein.

Theorem 4 *Assume that $p > q = 2$, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying* (20) *together with the growth condition*

(i') *there exist constants* $a_0, a_1 \geq 0$ *and* $\alpha \in (0, p^* - 1)$ *such that*

$$|g(x, s)| \leq a_0 + a_1|s|^\alpha \quad \textit{for a.e. } x \in \Omega \textit{ and all } s \in \mathbb{R},$$

and let $u_0 \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. *If* $\mu \geq 0$, *then problem* (21) *has at least one weak solution. If* $\mu \geq \frac{1}{2}(1 + \frac{1}{\lambda_{1,2}})$, *then the weak solution is unique.*

Proof The uniqueness part follows from Example 3. In order to prove the existence of a weak solution to problem (21) we check the conditions required to address Theorem 2 (in the refined version pointed out in Remark 1). To this end we set

$$f(x, s, \xi) = g(x, s) + \xi_1$$

for a.e. $x \in \Omega$, all $s \in \mathbb{R}$, and $\xi = (\xi_1, \ldots, \xi_N) \in \mathbb{R}^N$. From assumption (i') we obtain the estimate

$$|f(x, s, \xi)| \leq |g(x, s)| + |\xi_1| \leq a_0 + a_1|s|^\alpha + |\xi|,$$

which shows that assumption (i) of Theorem 2 is fulfilled. The operator $T : W_0^{1,p}(\Omega) \to W_0^{1,p}(\Omega)$, $u \mapsto u + u_0$ corresponding to problem (21) is clearly bounded. On the basis of (20) we have the estimate

$$\begin{aligned}\int_\Omega f(x, T(u), \nabla T(u))u\,dx &= \int_\Omega g(x, u + u_0)u\,dx + \int_\Omega \frac{\partial(u + u_0)}{\partial x_1} u\,dx \\ &\leq \int_\Omega g(x, u_0)u\,dx + \int_\Omega \frac{\partial(u + u_0)}{\partial x_1} u\,dx \\ &\leq \delta\|u\|^2 + c \leq \delta'\|u\|^p + c' \ \text{ for all } u \in W_0^{1,p}(\Omega)\end{aligned}$$

with constants $\delta, c, c' > 0$ and $\delta' \in (0, 1)$. In view of Remark 1 the conclusion of Theorem 2 is valid. This completes the proof. □

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Iterative Methods for Variational Inequalities

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Abstract Variational inequalities can be viewed as novel and significant extension of variational principles. A wide class of unrelated problems, which arise in various branches of pure and applied sciences are being investigated in the unified framework of variational inequalities. It is well known that variational inequalities are equivalent to the fixed point problems. This equivalent fixed point formulation has played not only a crucial part in studying the qualitative behavior of complicated problems, but also provide us numerical techniques for finding the approximate solution of these problems. Our main focus is to suggest some new iterative methods for solving variational inequalities and related optimization problems using projection methods, Wiener–Hopf equations and dynamical systems. Convergence analysis of these methods is investigated under suitable conditions. Some open problems are also discussed and highlighted for future research.

1 Introduction

Variational inequality theory contains a wealth of new ideas and techniques. Variational inequality theory, which was introduced and considered in the early 1960s by Stampacchia [43], can be viewed as a natural extension and generalization of the variational principles. It is well known that the minimum $u \in K$ of a differentiable convex functions on the convex set K can be characterized by an inequality of the type:

$$\langle f'(u), v - u\rangle \geq 0, \quad \forall v \in K,$$

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which is called the variational inequality. It is amazing that a wide class of unrelated problems, which arise in various different branches of pure and applied sciences, can be studied in the general and unified framework of variational inequalities. For the applications, motivation, numerical results, and other aspects of variational inequalities, see [1–48] and the references therein.

It is worth mentioning that a convex function f is a convex function, if and only if, it satisfies the inequality of the type:

$$f\Big(\frac{a+b}{2}\Big) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \forall a, b \in [a, b],$$

which is called the Hermite–Hadamard inequality. Such type of inequalities have important and fundamental applications in various fields of pure and applied sciences.

It is well known that the variational inequalities are equivalent to the fixed point problems. This alternative formulation is used not only to study the existence theory of the solution of the variational inequalities, but also to develop several iterative methods such as projection method, implicit methods, and their variant modifications. The convergence analysis of the projection method requires that the underlying operator must be strongly monotone and Lipschitz continuous, which are strict conditions. To overcome these drawbacks, Korpelevich [16] suggested the extragradient method, convergence of which requires only the monotonicity and Lipschitz continuity. Noor [29] has proved that the convergence analysis of the extragradient method only requires the monotonicity. This result can be viewed as the significant refinement of a result of Korpelevich [16], see also Noor et al. [37, 38].

The Wiener–Hopf equations were introduced and studied by Shi [42] and Robinson [41]. The technique of Wiener–Hopf equations is quite general and unifying one. This technique has been used to study the existence of a solution as well as to develop various iterative methods for solving the variational inequalities. Noor [23, 25, 26, 32] has used the Wiener–Hopf equations technique to suggest iterative method and study the sensitivity, stability analysis, and dynamical systems of the variational inequalities.

The alternative fixed point technique is used to establish the equivalence between the variational inequalities and dynamical systems. This equivalence has been used to study the existence and stability of the solution of variational inequalities. Bin-Mohsin et al. [1] have been shown that the dynamical system can be used to suggest some implicit iterative method for solving variational inequalities. For the applications and numerical methods of the dynamical systems, see [1, 6–9, 17–19, 32, 47].

In this paper, we use the fixed point formulation, Wiener–Hopf equations, and dynamical systems to suggest some implicit and explicit methods for solving the variational inequalities. The convergence criteria of the proposed implicit method is discussed under some mild conditions. An example is given to illustrate the implementation and efficiency of the proposed method for solving variational inequalities. Using the techniques and ideas of this paper, one can suggest a wide class of iterative schemes for solving different classes of variational inequalities, equilibrium and optimization problems.

2 Formulations and Basic Facts

Let H be a real Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

Let $T : H \to H$ be a nonlinear operator and f be a linear continuous functional. Let K be a closed and convex set in H. We consider the problem of finding $u \in K$, such that

$$\langle Tu, v-u\rangle \geq \langle f, v-u\rangle, \quad \forall v \in K, \tag{1}$$

which is called the variational inequalities, introduced and studied by Karamardian [14]. A wide class of problems arising in pure and applied sciences can be studied via variational inequalities (1), see [1–7, 26, 32, 33, 40–42, 48].

If $K^* = \{u \in H : \langle u, v\rangle \geq 0, \forall v \in K, \}$ is a polar(dual) cone, then problem (1) is equivalent to finding

$$u \in K, \quad Tu - f \in K^*, \quad \langle Tu - f, u\rangle = 0, \tag{2}$$

which is known as the nonlinear complementarity problem, introduced by Karamardian [14]. For the applications and other aspects of the complementarity problems in engineering and applied sciences, see [1, 6–12, 32, 35, 47] and the references therein.

If $K = H$, then problem (1) is equivalent to finding $u \in H$, such that

$$\langle Tu, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{3}$$

which is known as the general Lax–Milgram Lemma, see [20, 40] and the references therein, We would like to point out that, if $T = I$, the identity operator, then problem (3) is known as the Riesz –Frechet representation theorem, see, for example, [17, 20, 40].

If the operator T is linear, symmetric, and positive definite, then problem (1) is equivalent to finding the minimum of the functional $I[v]$ on the convex set K, where

$$I[v] = \langle Tv, v\rangle - 2\langle f, v\rangle, \quad \forall v \in H,$$

which is the energy functional. Consequently, it is clear that all these problems are closely related to the quadratic programming and optimization theory.

Definition 1 An operator $T : H \to H$ is said to be:

1. Strongly monotone, if there exists a constant $\alpha > 0$, such that

$$\langle Tu - Tv, u - v \rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

2. Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\|Tu - Tv\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

3. Monotone, if

$$\langle Tu - Tv, u - v \rangle \geq 0, \quad \forall u, v \in H.$$

Remark 1 Every strongly monotone operator is monotone but the converse is not true.

We also need the following result, known as the Projection Lemma, which plays a crucial part in establishing the equivalence between the variational inequalities and the fixed point problem. This result can be used in analyzing the convergence analysis of the projective implicit and explicit methods for solving the variational inequalities and related optimization problems.

Lemma 1 ([10, 15]) *Let K be a closed and convex set in H. Then, for a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geq 0, \quad \forall v \in K, \tag{4}$$

if and only if

$$u = P_K(z),$$

where P_K is the projection operator.

It is well known that the projection operator P_K is nonexpansive, that is,

$$\|P_K(u) - P_K(v)\| \leq \|u - v\|, \forall u, v \in H.$$

Lemma 1 has played a crucial part in the development of numerical methods, sensitivity analysis, dynamical systems, convergence analysis , and other aspects of variational inequalities and related problems.

3 Projection Methods

In this section, we use the fixed point formulation to suggest and analyze some new implicit methods for solving the variational inequalities.

Using Lemma 1, one can show that the variational inequalities are equivalent to the fixed point problems.

Lemma 2 ([32]) *The function $u \in K$ is a solution of the variational inequalities (1), if and only if, $u \in K$ satisfies the relation*

$$u = P_K[u - \rho T u], \tag{5}$$

where P_K is the projection operator and $\rho > 0$ is a constant.

Lemma 2 implies that the variational inequality (1) is equivalent to the fixed point problem (5). This equivalent fixed point formulation was used to suggest some implicit iterative methods for solving the variational inequalities. One uses (5) to suggest the following iterative methods for solving variational inequalities.

Algorithm 1 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T u_n], \quad n = 0, 1, 2, \ldots \tag{6}$$

which is known as the projection method and has been studied extensively, see [26, 32, 35].

Algorithm 2 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots \tag{7}$$

which is known as the extragradient method, which was suggested and analyzed by Korpelevich [16] and has been studied extensively. Noor [29] has proved the convergence of the extragradient for pseudomonotone operators.

Algorithm 3 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_{n+1} - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots \tag{8}$$

which is known as the modified projection method and has been studied extensively, see [26, 32].

We can rewrite Eq. (5) as:

$$u = P_K[\frac{u+u}{2} - \rho T u]. \tag{9}$$

This fixed point formulation was used to suggest the following implicit method, which is due to Noor et al. [38].

Algorithm 4 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[\frac{u_n + u_{n+1}}{2} - \rho T u_{n+1}], \quad n = 0, 1, 2, \ldots \tag{10}$$

For the implementation and numerical performance of Algorithm 4, Noor et al. [38] used the predictor-corrector technique to suggest the following two-step iterative method for solving variational inequalities.

Algorithm 5 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$y_n = P_K[u_n - \rho T u_n]$$
$$u_{n+1} = P_K[\frac{y_n + u_n}{2} - \rho T y_n], \quad \lambda \in [0, 1], \quad n = 0, 1, 2, \ldots$$

which is a two-step iterative method:

From Eq. (5), we have

$$u = P_K[u - \rho T(\frac{u + u}{2})]. \tag{11}$$

This fixed point formulation is used to suggest the implicit method for solving the variational inequalities as

Algorithm 6 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[u_n - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, \ldots. \tag{12}$$

which is another implicit method, see Noor et al. [37, 38]. To implement this implicit method, one can use the predictor-corrector technique to rewrite Algorithm 3 as equivalent two-step iterative method:

Algorithm 7 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$y_n = P_K[u_n - \rho T u_n],$$
$$u_{n+1} = P_K[u_n - \rho T(\frac{u_n + y_n}{2})], \quad n = 0, 1, 2, \ldots.$$

which was suggested and studied by Noor et al. [10] and is known as the mid-point implicit method for solving variational inequalities. For the convergence analysis and other aspects of Algorithm 3, see Noor et al. [10].

It is obvious that Algorithms 4 and 5 have been suggested using different variant of the fixed point formulations of Eq. (5). It is natural to combine these fixed point formulation to suggest a hybrid implicit method for solving the variational inequalities and related optimization problems, which is the main motivation of this paper.

One can rewrite Eq. (5) as

$$u = P_K[\frac{u+u}{2} - \rho T(\frac{u+u}{2})]. \tag{13}$$

This equivalent fixed point formulation enables to suggest the following method for solving the variational inequalities.

Algorithm 8 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K[\frac{u_n + u_{n+1}}{2} - \rho T(\frac{u_n + u_{n+1}}{2})], \quad n = 0, 1, 2, \ldots . \tag{14}$$

which is an implicit method.

We would like to emphasize that Algorithm 8 is an implicit method. To implement the implicit method, one uses the predictor-corrector technique. We use Algorithm 1 as the predictor and Algorithm 8 as corrector. Thus, we obtain a new two-step method for solving variational inequalities.

Algorithm 9 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= P_K[\left(\frac{y_n + u_n}{2}\right) - \rho T\left(\frac{y_n + u_n}{2}\right)], \quad n = 0, 1, 2, \ldots \end{aligned}$$

which is two-step method introduced.

From the above discussion, it is clear that Algorithms 8 and 9 are equivalent. It is enough to prove the convergence of Algorithm 8, which is the main motivation of our next result.

Theorem 1 *Let the operator T be strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, respectively. Let $u \in K$ be solution of (1) and u_{n+1} be an approximate solution obtained from Algorithm 8. If there exists a constant $\rho > 0$, such that*

$$0 < \rho < \frac{2\alpha}{\beta^2}, \tag{15}$$

then the approximate solution u_{n+1} converges to the exact solution $u \in K$.

Proof Let $u \in K$ be a solution of (1) and u_{n+1} be the approximate solution obtained from Algorithm 8. Then, from Eqs. (13) and (14), we have

$$\begin{aligned} \|u_{n+1} - u\|^2 &= \|P_K[(\frac{u_n + u_{n+1}}{2}) - \rho T(\frac{u_n + u_{n+1}}{2})] - P_K[\frac{u+u}{2}) - \rho T(\frac{u+u}{2})]\|^2 \\ &\leq \|\frac{u_{n+1} + u_n}{2} - \frac{u+u}{2}) - \rho(T(\frac{u_{n+1} + u_n}{2}) - T(\frac{u+u}{2}))\|^2 \end{aligned}$$

$$\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|\frac{u_n - u}{2} + \frac{u_{n+1} - u}{2}\|^2, \tag{16}$$

where we have used the fact that the operator T is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous constant $\beta > 0$, respectively.

Thus, from (16), we have

$$\begin{aligned} \|u_{n+1} - u\| &\leq \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\{\|\frac{u_n - u}{2}\| + \|\frac{u_{n+1} - u}{2}\|\} \\ &= \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_n - u\| \\ &\quad + \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\|u_{n+1} - u\|, \end{aligned} \tag{17}$$

which implies that

$$\begin{aligned} \|u_{n+1} - u\| &\leq \frac{\frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1 - \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}\|u_n - u\| \\ &= \theta\|u_n - u\|, \end{aligned} \tag{18}$$

where

$$\theta = \frac{\frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}{1 - \frac{1}{2}\sqrt{1 - 2\rho\alpha + \rho^2\beta^2}}.$$

From (15), it follows that $\theta < 1$. This shows that the approximate solution u_{n+1} obtained from Algorithm 8 converges to the exact solution $u \in K$ satisfying the variational inequality (1).

4 Wiener–Hopf Equations Technique

We now consider the problem of solving the Wiener–Hopf equations related to the variational inequalities. Let T be an operator and $Q_K = I - P_K$, where I is the identity operator and P_K is the projection of H onto the closed convex set K. We consider the problem of finding $z \in H$ such that

$$TP_K z + \rho^{-1} 1 Q_K z = 0. \tag{19}$$

The equations of the type (19) are called the Wiener–Hopf equations, which were introduced and studied by Shi [42] and Robinson [41] independently. It has been shown that the Wiener–Hopf equations play an important part in the developments of iterative methods, sensitivity analysis, and other aspects of the variational inequalities, see [23–27, 30, 32, 39–42] and the references therein.

Lemma 3 *The element $u \in K$ is a solution of variational inequality (1), if and only if $z \in H$ satisfies the Wiener–Hopf equation (19), where*

$$u = P_K z, \tag{20}$$

$$z = u - \rho T u, \tag{21}$$

where $\rho > 0$ is a constant.

From Lemma 3, it follows that the variational inequalities (1) and the Wiener–Hopf equations (19) are equivalent. This alternative equivalent formulation has been used to suggest and analyze a wide class of efficient and robust iterative methods for solving variational inequalities and related optimization problems, see [6–11] and the references therein.

We use the Wiener–Hopf equations (19) to suggest some new iterative methods for solving the variational inequalities. From (20) and (21),

$$\begin{aligned} z &= P_K z - \rho T P_K z \\ &= P_K[u - \rho T u] - \rho T P_K[u - \rho T u]. \end{aligned}$$

Thus, we have

$$u = \rho T u + \big[P_K[u - \rho T u] - \rho T P_K[u - \rho T u + P_K[u - \rho T u] - u].$$

Consequently, for a constant $\alpha_n > 0$, we have

$$\begin{aligned} u &= (1 - \alpha_n)u + \alpha_n\{P_K[P_K[u - \rho T u] + \rho T u - \rho T P_K[u - \rho T u] + P_K[u - \rho T u] - u]\} \\ &= (1 - \alpha_n)u + \alpha_n\{P_K[y - \rho T y + \rho T u + y - u]\}, \end{aligned} \tag{22}$$

where

$$y = P_K[u - \rho T u]. \tag{23}$$

Using (22) and (23), we can suggest the following new predictor-corrector method for solving variational inequalities.

Algorithm 10 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\begin{aligned} y_n &= P_K[u_n - \rho T u_n] \\ u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n\Big\{P_K[y_n - \rho T y_n + y_n - (u_n - \rho T u_n)]\Big\}. \end{aligned} \tag{24}$$

Algorithm 10 can be rewritten in the following equivalent form:

Algorithm 11 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n\{P_K[P_K[u_n - \rho T u_n] - \rho T P_K[u_n - \rho T u_n] \\ + P_K[u_n - \rho T u_n) - (u_n - \rho T u_n])\},$$

which is an explicit iterative method and appears to be a new one.

If $\alpha_n = 1$, then Algorithm 11 reduces to

Algorithm 12 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$y_n = P_K[u_n - \rho T u_n]$$
$$u_{n+1} = P_K[y_n - \rho T y_n + y_n - (u_n - \rho T u_n])\}, \quad n = 0, 1, 2, \ldots,$$

which appears to be a new one.

5 Dynamical Systems Technique

In this section, we consider the projected dynamical systems associated with variational inequalities. We investigate the convergence analysis of these new methods involving only the monotonicity of the operator.

We now define the residue vector $R(u)$ by the relation

$$R(u) = u - P_K[u - \rho T u]. \tag{25}$$

Invoking Lemma 2, one can easily conclude that $u \in K$ is a solution of (1), if and only if, $u \in K$ is a zero of the equation

$$R(u) = 0. \tag{26}$$

We now consider a projected dynamical system associated with the variational inequalities. Using the equivalent formulation (2), we suggest a class of projected dynamical systems as

$$\frac{du}{dt} = \lambda P_K[u - \rho T u] - u\}, \quad u(t_0) = u_0 \in K, \tag{27}$$

where λ is a parameter. The system of type (27) is called the projected dynamical system associated with variational inequalities (1). Here the right hand is related to the resolvent and is discontinuous on the boundary. From the definition, it is clear that the solution of the dynamical system always stays in H. This implies that the qualitative results such as the existence, uniqueness, and continuous dependence of the solution of (27) can be studied. These projected dynamical systems are associated with the variational inequalities (1), which have been studied extensively.

We use the projected dynamical system (27) to suggest some iterative for solving variational inequalities (1). These methods can be viewed in the sense of Korpelevich [16] and Noor [26, 32] involving the double resolvent operator.

For simplicity, we consider the dynamical system (27)

$$\frac{du}{dt} + u = P_K[u - \rho Tu], \quad u(t_0) = \alpha. \tag{28}$$

We construct the implicit iterative method using the forward difference scheme. Discretizing the equation (28), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_n - \rho Tu_{n+1}], \tag{29}$$

where h is the step size. Now, we can suggest the following implicit iterative method for solving the variational inequality (1).

Algorithm 13 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\left[u_n - \rho Tu_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

This is an implicit method and is quite different from the implicit method [16, 32]. Using Lemma 1, Algorithm 13 can be rewritten in the equivalent form as:

Algorithm 14 For a given $u_0 \in H$, compute u_{n+1} by the iterative scheme

$$\langle \rho Tu_{n+1} + \frac{1+h}{h}(u_{n+1} - u_n), v - u_{n+1}\rangle \geq 0, \quad \forall v \in K, \quad n = 0, 1, 2, \ldots. \tag{30}$$

We now study the convergence analysis of algorithm 14 under some mild conditions.

Theorem 2 *Let $u \in K$ be a solution of variational inequality (1). Let u_{n+1} be the approximate solution obtained from (30). If T is monotone, then*

$$\|u - u_{n+1}\|^2 \leq \|u - u_n\|^2 - \|u_n - u_{n+1}\|^2. \tag{31}$$

Proof Let $u \in K$ be a solution of (1). Then

$$\langle Tv, v - u\rangle \geq 0, \quad \forall v \in K, \tag{32}$$

since T is a monotone operator.

Set $v = u_{n+1}$ in (32), to have

$$\langle Tu_{n+1}, u_{n+1} - u\rangle \geq 0. \tag{33}$$

Taking $v = u$ in (30), we have

$$\langle \rho Tu_{n+1} + \{\frac{(1+h)u_{n+1} - (1+h)u_n}{h}\}, u - u_{n+1}\rangle \geq 0. \tag{34}$$

From (33) and (34), we have

$$\langle (1+h)(u_{n+1} - u_n), u - u_{n+1}\rangle \geq 0. \tag{35}$$

From (35) and using $2\langle a, b\rangle = \|a+b\|^2 - \|a\|^2 - \|b\|^2, \quad \forall a, b \in H$, we obtain

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2 - \|u_{n+1} - u_n\|^2. \tag{36}$$

the required result.

Theorem 3 *Let $u \in K$ be the solution of variational inequality (1). Let u_{n+1} be the approximate solution obtained from (30). If T is a monotone operator, then u_{n+1} converges to $u \in H$ satisfying (1).*

Proof Let T be a monotone operator. Then, from (31), it follows the sequence $\{u_i\}_{i=1}^{\infty}$ is a bounded sequence and

$$\sum_{i=1}^{\infty} \|u_n - u_{n+1}\|^2 \leq \|u - u_0\|^2,$$

which implies that

$$\lim_{n\to\infty} \|u_{n+1} - u_n\|^2 = 0. \tag{37}$$

Since sequence $\{u_i\}_{i=1}^{\infty}$ is bounded, so there exists a cluster point $\hat{u}$ to which the subsequence $\{u_{ik}\}_{k=1}^{\infty}$ converges. Taking limit in (30) and using (37), it follows that $\hat{u} \in K$ satisfies

$$\langle T\hat{u}, v - \hat{u}\rangle \geq 0, \quad \forall v \in K,$$

and

$$\|u_{n+1} - u\|^2 \leq \|u - u_n\|^2.$$

Using this inequality, one can show that the cluster point $\hat{u}$ is unique and

$$\lim_{n\to\infty} u_{n+1} = \hat{u}.$$

We now suggest another implicit iterative method for solving (1). Discretizing (30), we have

$$\frac{u_{n+1} - u_n}{h} + u_{n+1} = P_K[u_{n+1} - \rho T u_{n+1}], \tag{38}$$

where h is the step size.

This formulation enables us to suggest the following iterative method.

Algorithm 15 For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\left[u_{n+1} - \rho T u_{n+1} - \frac{u_{n+1} - u_n}{h}\right], \quad n = 0, 1, 2, \ldots.$$

Using Lemma 1, Algorithm 15 can be rewritten in the equivalent form as:

Algorithm 16 For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$\langle \rho T u_{n+1} + \{\frac{u_{n+1} - u_n}{h}\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{39}$$

Again using the dynamical systems, we can suggest some iterative methods for solving the variational inequalities and related optimization problems.

Algorithm 17 For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$u_{n+1} = P_K\left[\frac{(h+1)(u_n - u_{n+1})}{h} - \rho T u_n\right], \quad n = 0, 1, 2, \ldots,$$

which can be written in the equivalent form as

Algorithm 18 For a given $u_0 \in K$, compute u_{n+1} by the iterative scheme

$$\langle \rho T u_n + \{\frac{h+1}{h}(u_{n+1} - u_n)\}, v - u_{n+1}\rangle \geq 0, \quad \forall v \in K. \tag{40}$$

In a similar way, one can suggest a wide class of implicit iterative methods for solving variational inequalities and related optimization problems. The comparison of these methods with other methods is an interesting problem for future research.

6 Computational Results

We now explain Algorithm 8 as follows, which is used to illustrate the efficiency of Algorithm 8.

Step 0 Let $\rho_0 > 0, \delta := 0.95 < 1, \epsilon > 0, k = 0$ and $u^0 \in K$.
Step 1 If $\||r(u^k, \rho_k)\|_\infty \le \epsilon$, then stop. Otherwise, go to Step 2.
Step 2
$y^k = P_K[u^k - \rho_k T(u^k)], \qquad \varepsilon^k = \rho_k(T(\tilde{u}^k) - T(u^k)),$
$r = \frac{\|\varepsilon^k\|}{\|u^k - \tilde{u}^k\|}.$
While $(r > \delta)$
$\rho_k = \frac{0.8}{r} * \rho_k, \qquad y^k = P_K[u^k - \rho_k T(u^k)],$
$\varepsilon^k = \rho_k(T(\tilde{u}^k) - T(u^k)), \qquad r = \frac{\|\varepsilon^k\|}{\|u^k - \tilde{u}^k\|}.$
end While
Step 3
$u^{k+1} = P_K\Big[\frac{u^k + y^k}{2} - \rho T(\frac{u^k + y^k}{2})\Big].$
Step 4 $\rho_{k+1} = \begin{cases} \frac{\rho_k * 0.7}{r} & \text{if} \quad r \le 0.5; \\ \rho_k & \text{otherwise.} \end{cases}$
Step 5 k:=k+1; go to Step 1.

We now consider an example to illustrate the implementation and efficiency of the proposed Algorithm 8. In order to verify the theoretical assertions, we consider the variational inequality (1), where

$$T(u) = D(u) + Mu + q, \tag{41}$$

$D(u)$ and $Mu + q$ are the nonlinear part and the linear part of $T(u)$, respectively.

We form the linear part in the test problems similarly as in Harker and Pang [11]. The matrix $M = A^T A + B$, where A is an $n \times n$ matrix whose entries are randomly generated in the interval $(-5, +5)$ and a skew-symmetric matrix B is generated in the same way. The vector q is generated from a uniform distribution in the interval $(-500, 500)$. In $D(u)$, the nonlinear part of $T(u)$, the components are chosen to be $D_j(u) = d_j * \arctan(u_j)$, where d_j is a random variable in $(0, 1)$. A similar type of problems was tested in [18] and [44].

In all tests we take $\delta = 0.95$ and $\gamma = 1.98$. All iterations start with $u^0 = (1, \ldots, 1)^T$ and $\rho_0 = 1$, and stopped whenever $\|r(u^k, 1)\|_\infty \le 10^{-7}$. All codes are written in Matlab. The iteration numbers and the computational time for Algorithm 3.2, the methods in [5] and in [1] with different dimensions are given in Table 1.

Table 1 Numerical results for problem (1)

	Algorithm 3.2		The method in [6]		The method in [3]	
Dimension of the problem	No. It.	CPU (s)	No. It.	CPU (s)	No. It.	CPU (s)
$n = 100$	122	0.009	261	0.06	164	0.04
$n = 200$	194	0.022	402	0.59	250	0.53
$n = 300$	178	0.047	442	1.94	282	1.30
$n = 500$	221	90.107	496	5.91	312	3.29
$n = 700$	181	0.169	479	16.99	310	10.68

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Recent Developments of Lyapunov-Type Inequalities for Fractional Differential Equations

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Abstract A survey of results on Lyapunov-type inequalities for fractional differential equations associated with a variety of boundary conditions is presented. This includes Dirichlet, mixed, Robin, fractional, Sturm–Liouville, integral, nonlocal, multi-point, anti-periodic, conjugate, right-focal, and impulsive conditions. Furthermore, our study includes Riemann–Liouville, Caputo, Hadamard, Prabhakar, Hilfer, and conformable type fractional derivatives. Results for boundary value problems involving fractional p-Laplacian, fractional operators with nonsingular Mittag–Leffler kernels, q-difference, discrete, and impulsive equations are also taken into account.

1 Introduction and Preliminaries

Integral inequalities are fundamental in the study of quantitative properties of solutions of differential and integral equations. The Lyapunov-type inequality is one of such inequalities when investigating the zeros of solutions of differential equations. A method for deriving a Lyapunov-type inequality for boundary value problems dates back to Nehari [1] and is based on the idea of converting the given problem into an integral equation. To illustrate this method, let us consider the following boundary value problem:

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{1}$$

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where $a, b \in \mathbb{R}$, $a < b$ are consecutive zeros of $y(t)$ and $y(t) \not\equiv 0$, for all $t \in (a, b)$. It can easily be shown that problem (1) is equivalent to the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{2}$$

where $G(t, s)$ is the Green's function given by

$$G(t,s) = -\begin{cases} \dfrac{(t-a)(b-s)}{b-a}, & a \leq s \leq t \leq b, \\ \dfrac{(s-a)(b-t)}{b-a}, & a \leq t \leq s \leq b. \end{cases} \tag{3}$$

Taking the absolute value of both sides of Eq. (2), and taking into account that y does not have any zeros in (a, b),we get

$$1 \leq \max_{a \leq t \leq b} \int_a^b |G(t,s)||q(s)|ds, \tag{4}$$

which yields the desired Lyapunov inequality

$$\int_a^b |q(s)|ds \geq \frac{1}{\max_{(t,s)\in[a,b]\times[a,b]} |G(t,s)|}. \tag{5}$$

In the special case where can find $H(s)$ explicitly such that $\max_{t\in[a,b]} |G(t,s)| \leq H(s)$, then we obtain the following inequality:

$$1 \leq \int_a^b H(s)|q(s)|ds.$$

Clearly the function $H(t)$ for problem (1) is $\dfrac{(t-a)(b-t)}{b-a}$. Moreover, if we take the absolute maximum of the function $H(t)$ for all $t \in [a, b]$, then it is obtained the following well-known Lyapunov inequality [2].

Theorem 1 *If the boundary-value problem (1) has a nontrivial solution, where q is a real and continuous function, then*

$$\int_a^b |q(s)|ds > \frac{4}{b-a}. \tag{6}$$

The factor 4 in the above inequality is sharp and cannot be replaced by a larger number.

Later Wintner in [3] and more authors thereafter generalized this result by replacing the function $|q(t)|$ in (6) by the function $q^+(t)$, $q^+(t) = \max\{q(t), 0\}$, where now the resulting inequality reads:

$$\int_a^b q^+(s)ds > \frac{4}{b-a}, \tag{7}$$

with $q^+(t) = \max\{q(t), 0\}$.

In [4], Hartman expanded further this result with the following inequality:

$$\int_a^b (b-t)(t-a)q^+(t)dt > (b-a), \tag{8}$$

which is sharper than both (6) and (7).

Clearly, (8) implies (7) as $(b-t)(t-a) \leq \dfrac{(b-a)^2}{4}$ for all $t \in [a, b]$ and the equality holds when $t = \dfrac{a+b}{2}$.

It is worth mentioning that inequality (6) has found many practical applications in differential equations (oscillation theory, disconjugacy, eigenvalue problems, etc.), for instance, see [5–11] and the references therein. A thorough literature review dealing with continuous and discrete Lyapunov inequalities and their applications can be found in [12] and [13] (which also includes an excellent account on the history of such inequalities).

In many engineering and scientific disciplines such as physics, chemistry, aerodynamics, electrodynamics of complex media, polymer rheology, economics, control theory, signal and image processing, biophysics, blood flow and related phenomena, fractional differential and integral equations represent processes in a more effective manner than their integer-order counterparts. This aspect has led to the increasing popularity in the study of fractional order differential and integral equations among mathematicians and researchers. In view of their extensive applications in various fields, the topic of inequalities for fractional differential equations has also attracted a significant attention in recent years.

This survey article is organized as follows. In Sect. 2 we introduce the reader to some basic concepts of fractional calculus. In Sect. 3 we summarize Lyapunov-type inequalities for fractional boundary value problems with different kinds of boundary conditions. In Sect. 4 we consider the inequalities for nonlocal and multi-point boundary value problems. Results on p-Laplacians are discussed in Sect. 5, while results on mixed fractional derivatives are given in Sect. 6. Section 7 deals with Lyapunov-type inequalities for Hadamard fractional differential equations. In Sect. 8, inequalities involving Prabhakar fractional differential equations are discussed. Section 9 contains the results on fractional q-difference equations, while Sect. 10 consists of the results involving fractional derivatives with respect to a certain function. Inequalities involving left and right derivatives, operators with nonsingular Mittag–Leffler kernels, discrete fractional differential equations, and impulsive fractional boundary value problems are, respectively, given in Sects. 11, 12, 13 and 14, respectively. We include the results for Hilfer and Katugampola fractional differential equations in Sects. 15 and 16, respectively, and conclude with Sect. 17 with results on conformable fractional differential equations. Note that our goal here is a more complete and comprehensive review and as such the choice is made to include as many results as possible to illustrate the progress on the matter. Any proofs (which are rather long) are omitted, for this matter, and the reader is referred to the relative article accordingly.

2 Fractional Calculus

Here we introduce some basic definitions of fractional calculus [14, 15] and recall some results that we need in the sequel.

Definition 1 (Riemann–Liouville Fractional Integral) Let $\alpha \geq 0$ and f be a real function defined on $[a, b]$. The Riemann–Liouville fractional integral of order α is defined by $(I^0 f)(x) = f(x)$ and

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds, \quad \alpha > 0, \quad t \in [a, b]$$

provided the right-hand side is point-wise defined on $[0, \infty)$, where $\Gamma(\alpha)$ is the Euler Gamma function: $\Gamma(\alpha) = \displaystyle\int_0^\infty t^{\alpha-1} e^{-t} dt$.

Definition 2 (Riemann–Liouville Fractional Derivative) The Riemann–Liouville fractional derivative of order $\alpha \geq 0$ is defined by $(D^0 f)(t) = f(t)$ and

$$(D^\alpha f)(t) = (D^m I^{m-\alpha} f)(t) \text{ for } \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Definition 3 (Caputo Fractional Derivative) The Caputo fractional derivative of order $\alpha \geq 0$ is defined by $({}^C D^0 f)(t) = f(t)$ and

$$({}^C D^\alpha f)(t) = (I^{m-\alpha} D^m f)(t) \text{ for } \alpha > 0,$$

where m is the smallest integer greater than or equal to α.

Notice that the differential operators of arbitrary order are nonlocal in nature and appear in the mathematical modeling of several real-world phenomena due to this characteristic (see, e.g., [14]).

If $f \in C([a, b], \mathbb{R})$, then the Riemann–Liouville fractional integral of order $\gamma > 0$ exists on $[a, b]$. On the other hand, following [14, Lemma 2.2, p. 73], we know that the Riemann–Liouville fractional derivative of order $\gamma \in [n-1, n)$ exists almost everywhere (a.e.) on $[a, b]$ if $f \in AC^n([a, b], \mathbb{R})$, where $C^k([a, b], \mathbb{R})$ $(k = 0, 1, \ldots)$ denotes the set of k times continuously differentiable mappings on $[a, b]$, $AC([a, b], \mathbb{R})$ is the space of functions which are absolutely continuous on $[a, b]$ and $AC^{(k)}([a, b], \mathbb{R})$ $(k = 1, \ldots)$ is the space of functions f such that $f \in C^{k-1}([a, b], \mathbb{R})$ and $f^{(k-1)} \in AC([a, b], \mathbb{R})$. In particular, $AC([a, b], \mathbb{R}) = AC^1([a, b], \mathbb{R})$. (We recall here that $AC([a, b], \mathbb{R})$ is the space of functions f which are absolutely continuous on $[a, b]$, and $AC^n([a, b], \mathbb{R})$ the space of functions f which have continuous derivatives up to order $n-1$ on $[a, b]$ such that $f^{(n-1)}(t) \in AC([a, b], \mathbb{R})$).

Now we enlist some important results involving fractional order operators [14].

Proposition 1 *Let f be a continuous function on some interval J and $p, q > 0$. Then*

$$(I^p\, I^q f)(t) = (I^{p+q} f)(t) = (I^q\, I^p f)(t) \text{ on } J.$$

Proposition 2 *Let f be a continuous function on some interval I and $\alpha \geq 0$. Then*

$$(\mathscr{D}^\alpha I^\alpha f)(t) = f(t) \text{ on } I,$$

with $\mathscr{D}$ being the Riemann–Liouville or Caputo fractional derivative operator.

Proposition 3 *The general solution of the following fractional differential equation*

$$(D^q y)(t) = f(t),\ t > a,\ 0 < q \leq 1,$$

is $y(t) = c(t-a)^{q-1} + (I^q f)(t),\ c \in \mathbb{R}$.

Proposition 4 *The general solution of the following fractional differential equation*

$$({}^C D^q y)(t) = f(t),\ t > a,\ 0 < q \leq 1,$$

is $y(t) = c + (I^q f)(t),\ c \in \mathbb{R}$.

3 Lyapunov-Type Inequalities for Fractional Differential Equations with Different Boundary Conditions

Lyapunov-type inequalities involving fractional differential operators have been investigated by many researchers in the recent years. In 2013, Ferreira [16] derived a Lyapunov-type inequality for Riemann–Liouville fractional differential equation with *Dirichlet boundary conditions*:

$$\begin{cases} D^{\alpha} y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{9}$$

where D^{α} is the Riemann–Liouville fractional derivative of order $1 < \alpha \le 2$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

An appropriate approach for obtaining the Lyapunov inequality within the framework of fractional differential equations relies on the idea of converting the boundary value problem into an equivalent integral equation and then finding the maximum value of its kernel function (Green's function).

It is straightforward to show that the boundary value problem (9) is equivalent to the integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds, \tag{10}$$

where $G(t, s)$ is the Green's function defined by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a \le t \le s \le b, \\ \dfrac{(b-s)^{\alpha-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} - (t-s)^{\alpha-1}, & a \le s \le t \le b. \end{cases} \tag{11}$$

Observe that the Green's function (11) satisfies the following properties:

1. $G(t, s) \ge 0, \forall t, s \in [a, b]$;
2. $\max\limits_{s \in [a,b]} G(t, s) = G(s, s), s \in [a, b]$;
3. $G(s, s)$ has a unique maximum, given by

$$\max_{s \in [a,b]} G(s, s) = G\Big(\frac{a+b}{2}, \frac{a+b}{2}\Big) = \frac{1}{\Gamma(\alpha)}\Big(\frac{b-a}{4}\Big)^{\alpha-1}.$$

The Lyapunov inequality for problem (9) can be expressed as follows.

Theorem 2 *If y is a nontrivial solution of the boundary value problem (9), then*

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{12}$$

Here we remark that Lyapunov's standard inequality (6) follows by taking $\alpha = 2$ in the above inequality. Also, inequality (12) can be used to determine intervals for the real zeros of the Mittag–Leffler function:

$$E_{\alpha}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(k\alpha + \alpha)}, \; z \in \mathbb{C}, \; \Re(\alpha) > 0.$$

For the sake of convenience, let us consider the following: fractional Sturm–Liouville eigenvalue problem (with $a = 0$ and $b = 1$):

$$\begin{cases} {}^C D^\alpha y(t) + \lambda y(t) = 0, & 0 < t < 1, \\ y(0) = y(1) = 0. \end{cases} \tag{13}$$

By Theorem 2, if $\lambda \in \mathbb{R}$ is an eigenvalue of (13), that is, if λ is a zero of equation $E_\alpha(-\lambda) = 0$, then $|\lambda| > \Gamma(\alpha)4^{\alpha-1}$. Therefore the Mittag–Leffler function $E_\alpha(z)$ has no real zeros for $|z| \le \Gamma(\alpha)4^{\alpha-1}$.

In 2014, Ferreira [17] replaced the Riemann–Liouville fractional derivative in problem (9) with Caputo fractional derivative ${}^C D^\alpha$ and derived the following Lyapunov-type inequality for the resulting problem:

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha-1)(b-a)]^{\alpha-1}}. \tag{14}$$

In 2015, Jleli and Samet [18] considered the fractional differential equation

$${}^C D^\alpha y(t) + q(t)y(t) = 0, \quad 1 < \alpha \le 2, \quad a < t < b, \tag{15}$$

associated with the *mixed boundary conditions*

$$y(a) = 0 = y'(b) \tag{16}$$

or

$$y'(a) = 0 = y(b). \tag{17}$$

An integral equation equivalent to problem (15) and (16) is

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{18}$$

where $G(t,s)$ is again the Green's function defined by

$$G(t,s) = \frac{H(t,s)}{\Gamma(\alpha)(b-s)^{2-\alpha}},$$

and

$$H(t,s) = \begin{cases} (\alpha-1)(t-\alpha), & a \le t \le s \le b, \\ (\alpha-1)(t-\alpha) - (t-s)^{\alpha-1}(b-s)^{2-\alpha}, & a \le s \le t \le b. \end{cases} \tag{19}$$

The function H satisfies the following inequality:

$$|H(t,s)| \le \max\{(2-\alpha)(b-s), (\alpha-1)(s-a)\} \text{ for all } (t,s) \in [a,b] \times [a,b].$$

In relation to problem (15) and (16), we have the following Lyapunov-type inequality.

Theorem 3 *If y is a nontrivial solution of the boundary value problem (15) and (16), then*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds > \frac{\Gamma(\alpha)}{\max\{\alpha-1, 2-\alpha\}(b-a)}. \tag{20}$$

In a similar manner, the Lyapunov-type inequality obtained for the boundary value problem (15)–(17) is

$$\int_a^b (b-s)^{\alpha-1}|q(s)|ds > \Gamma(\alpha). \tag{21}$$

As an application of Lyapunov-type inequalities (20) and (21), we can obtain the intervals, where certain Mittag–Leffler functions have no real zeros.

Corollary 1 *Let $1 < \nu \le 2$. Then the Mittag–Leffler function $E_\alpha(z)$ has no real zeros for*

$$z \in \left(-\Gamma(\alpha)\frac{(\alpha-1)}{\max\{\alpha-1, 2-\alpha\}}, 0\right].$$

The proof of the above corollary follows by applying inequality (20) to the following eigenvalue problem

$$\begin{cases} ({}^C D^\alpha y)(t) + \lambda y(t) = 0, & 0 < t < 1, \\ y(0) = y'(1) = 0. \end{cases} \tag{22}$$

Moreover, by applying inequality (21) to the following eigenvalue problem

$$\begin{cases} ({}^C D^\alpha y)(t) + \lambda y(t) = 0, & 0 < t < 1, \\ y'(0) = y(1) = 0, \end{cases} \tag{23}$$

we can obtain the following result:

Corollary 2 *Let $1 < \nu \le 2$. Then the Mittag–Leffler function $E_\alpha(z)$ has no real zeros for*

$$z \in (\alpha\Gamma(\alpha), 0].$$

In 2015, Jleli et al. [19] obtained a Lyapunov-type inequality:

$$\int_a^b (b-s)^{\alpha-2}(b-s+\alpha-1)|q(s)|ds \geq \frac{(b-a+2)\Gamma(\alpha)}{\max\{b-a+1, ((2-\alpha)/(\alpha-1))(b-a)-1\}}, \tag{24}$$

for the following problem with *Robin boundary conditions*

$$\begin{cases} {}^C D^\alpha y(t) + q(t)y(t) = 0, \ 1 < \alpha \leq 2, \quad a < t < b, \\ y(a) - y'(a) = y(b) + y'(b) = 0, \end{cases} \tag{25}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function.

Using the Lyapunov-type inequality (24), we can find an interval, where a linear combination of Mittag–Leffler functions $E_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+\beta)}$, $\alpha > 0, \beta > 0$, $z \in \mathbb{C}$ has no real zeros. In precise terms, we have the following result:

Theorem 4 *Let $a < \alpha \leq 2$. Then $E_{\alpha,2}(z) + E_{\alpha,1}(z) + zE_{\alpha,\alpha}(z)$ has no real zeros for*

$$z \in \left(\frac{-3\alpha\Gamma(\alpha)}{(1+\alpha)\max\{2, ((2-\alpha)/(\alpha-1)-1\}}, 0 \right].$$

In 2015, Rong and Bai [20] considered a boundary value problem with *fractional boundary conditions*:

$$\begin{cases} {}^C D^\alpha y(t) + q(t)y(t) = 0, \ 1 < \alpha \leq 2, \quad a < t < b, \\ y(a) = 0, \quad {}^C D^\beta y(b) = 0, \ 0 < \beta \leq 1, \end{cases} \tag{26}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function. The Lyapunov-type inequality derived for problem (26) is

$$\int_a^b (b-s)^{\alpha-\beta-1}|q(s)|ds > \frac{(b-a)^{-\beta}}{\max\left\{ \frac{1}{\Gamma(\alpha)} - \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)}, \frac{2-\alpha}{\alpha-1} \cdot \frac{\Gamma(2-\beta)}{\Gamma(\alpha-\beta)} \right\}}. \tag{27}$$

Later, Jleli and Samet [21] obtained a Lyapunov-type inequality for a boundary value problems with *Sturm–Liouville boundary conditions*

$$\begin{cases} {}^C D^\alpha y(t) + q(t)y(t) = 0, \ 1 < \alpha \leq 2, \quad a < t < b, \\ py(a) - ry'(a) = y(b) = 0, \ p > 0, r \geq 0, \end{cases} \tag{28}$$

where $q : [a, b] \to \mathbb{R}$ is a continuous function. The integral equation equivalent to the problem (28) is

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{29}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{(r/p+t-a)}{r/p+b-a}(b-s)^{\alpha-1}, & a \le t \le s \le b, \\ \frac{(r/p+t-a)}{r/p+b-a}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s \le t \le b. \end{cases} \tag{30}$$

In order to estimate this Green's function, we distinguish two cases:

1. If $\frac{r}{p} > \frac{b-a}{\alpha-1}$, then $0 \le G(t,s) \le G(s,s)$, $\ (t,s) \in [a,b] \times [a,b]$ with

$$\max_{s\in[a,b]} G(t,s) = \frac{1}{\Gamma(\alpha)} \frac{(r/p)(b-a)^{\alpha-1}}{\left(r/p+b-a\right)}.$$

2. If $0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}$, then $\Gamma(\alpha)G(t,s) \le \max\{\mathscr{A}(\alpha, r/p), \mathscr{B}(\alpha, r/p)\}$, where

$$\mathscr{A}(\alpha, r/p) = \frac{(b-a)^{\alpha-1}}{(r/p+b-a)}\left(\left(\frac{(b-a)^{\alpha-1}}{(r/p+b-a)(\alpha-1)^{\alpha-1}}\right)^{\frac{1}{\alpha-2}}(2-\alpha) - \frac{r}{p}\right),$$

$$\mathscr{B}(\alpha, r/p) = \left(\frac{r}{p}+b-a\right)^{\alpha-1}\frac{(\alpha-1)\alpha-1}{\alpha^\alpha}.$$

The Lyapunov inequalities corresponding to the above cases are given in the following result.

Theorem 5 *If there exists a nontrivial continuous solution of the fractional boundary value problem (28), then*

$$(i)\ \int_a^b |q(s)|ds > \left(1+\frac{p}{r}(b-a)\right)\frac{\Gamma(\alpha)}{(b-a)^{\alpha-1}} \quad \textit{when} \quad p>0,\ \frac{r}{p} > \frac{b-a}{\alpha-1}; \tag{31}$$

$$(ii)\ \int_a^b |q(s)|ds > \frac{\Gamma(\alpha)}{\max\{\mathscr{A}(\alpha, r/p), \mathscr{B}(\alpha, r/p)\}} \quad \textit{when} \quad p>0,\ 0 \le \frac{r}{p} \le \frac{b-a}{\alpha-1}. \tag{32}$$

Using the above Lyapunov-type inequalities, we can find intervals, where linear combinations of some Mittag–Leffler functions have no real zeros.

Corollary 3 *Let* $1 < \alpha < 2,\ p > 0,\ \dfrac{r}{p} > \dfrac{1}{\alpha - 1}$. *Then the linear combination of Mittag–Leffler functions given by*

$$pE_{\alpha,2}(z) + qrE_{\alpha,1}(z)$$

has no real zeros for

$$z \in \Big(-\Big(1 + \frac{p}{r}\Big)\Gamma(\alpha), 0\Big].$$

This corollary can be established by considering the following fractional Sturm–Liouville eigenvalue problem:

$$\begin{cases} {}^{C}D^{\alpha}y(t) + \lambda y(t) = 0, \quad 0 < t < 1, \\ py(0) - ry'(0) = y(1) = 0. \end{cases}$$

We can apply the foregoing Lyapunov-type inequalities to study the nonexistence of solutions for certain fractional boundary value problems. For example, the problem (28) with $p = 1, r = 2, a = 0, b = 1, 3/2 < \alpha < 2$, has no nontrivial solution if $\int_0^1 |q(s)|ds < \frac{3}{2}\Gamma(\alpha)$. As a second example, there is no nontrivial solution for the problem (28) with $p = 2, r = 1, a = 0, b = 1, 1 < \alpha < 2$, provided that

$$\int_0^1 |q(s)|ds < \frac{\Gamma(\alpha)}{\max\{\mathscr{A}(\alpha, 1/2), \mathscr{B}(\alpha, 1/2)\}}.$$

In 2015, O'Regan and Samet [22] obtained a Lyapunov-type inequality for the fractional boundary value problem:

$$\begin{cases} D^{\alpha}y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y'(a) = y''(a) = y''(b) = 0, \end{cases} \tag{33}$$

where D^{α} is the standard Riemann–Liouville fractional derivative of fractional order $3 < \alpha \le 4$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

The integral equation associated with problem (33) is

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds, \tag{34}$$

where $G(t, s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-3}}{(b-a)^{\alpha-3}}, & a \le t \le s \le b, \\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-3}}{(b-a)^{\alpha-3}} - (t-s)^{\alpha-1}, & a \le s \le t \le b. \end{cases} \tag{35}$$

The Green's function defined in (35) satisfies the following inequality:

$$0 \le G(t,s) \le G(b,s) = \frac{(b-s)^{\alpha-3}(s-a)(2b-a-s)}{\Gamma(\alpha)}, \quad (t,s) \in [a,b] \times [a,b].$$

The Lyapunov inequality for the problem (33) is as follows.

Theorem 6 *If there exists a nontrivial continuous solution of the fractional boundary value problem (33), then*

$$\int_a^b (b-s)^{\alpha-3}(s-a)(2b-a-s)|q(s)|ds \ge \Gamma(\alpha). \tag{36}$$

To demonstrate an application of the above inequality, we consider the eigenvalue problem:

$$\begin{cases} D^\alpha y(t) + \lambda y(t) = 0, \quad 0 < t < 1, \ 3 < \alpha \le 4, \\ y(0) = y'(0) = y''(0) = y''(1) = 0. \end{cases} \tag{37}$$

Corollary 4 *If λ is an eigenvalue of the problem (37), then*

$$|\lambda| \ge \frac{\Gamma(\alpha)}{2B(2, \alpha-2)},$$

where B is the beta function defined by $B(x,y) = \int_0^1 s^{x-1}(1-s)^{y-1}ds, \ x, y > 0.$

Sitho et al. [23] established some Lyapunov-type inequalities for the following *hybrid fractional boundary value problem*

$$\begin{cases} D_a^\alpha \left[\dfrac{y(t)}{f(t, y(t))} - \displaystyle\sum_{i=1}^n I_a^\beta h_i(t, y(t)) \right] + g(t)y(t) = 0, \quad t \in (a,b), \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{38}$$

where D_a^α denotes the Riemann–Liouville fractional derivative of order $\alpha \in (2,3]$ starting from a point a, the functions $g \in L^1((a,b], \mathbb{R})$, $f \in C^1([a,b] \times \mathbb{R}, \mathbb{R} \setminus \{0\})$, $h_i \in C([a,b] \times \mathbb{R}, \mathbb{R})$, $\forall i = 1, 2, \ldots, n$ and I_a^β is the Riemann–Liouville fractional integral of order $\beta \ge \alpha$ with the lower limit at the point a. We consider two cases: (I) $h_i = 0, i = 1, 2, \ldots, n$ and (II) $h_i \ne 0, i = 1, 2, \ldots, n$.

Case I $h_i = 0, i = 1, 2, \ldots, n$. We consider the problem (38) with $h_i(t, \cdot) = 0$ for all $t \in [a, b]$. For $\alpha \in (2, 3]$, we first construct the Green's function for the following boundary value problem

$$\begin{cases} D_a^\alpha \left[\dfrac{y(t)}{f(t, y(t))} \right] + g(t)y(t) = 0, \quad t \in (a, b), \\ y(a) = y'(a) = y(b) = 0, \end{cases} \tag{39}$$

with the assumption that f is continuously differentiable and $f(t, y(t)) \neq 0$ for all $t \in [a, b]$.

Let $y \in AC([a, b], \mathbb{R})$ be a solution of the problem (39). Then the function y satisfies the following integral equation:

$$y = f(t, y) \int_a^b G(t, s)g(s)y(s)ds, \tag{40}$$

where $G(t, s)$ is the Green's function defined by (11) and satisfies the following properties:

1. $G(t, s) \geq 0, \forall t, s \in [a, b]$.
2. $G(t, s) \leq H(s) := \dfrac{(b-s)^{\alpha-1}}{\Gamma(\alpha-1)}$.
3. $\max\limits_{s \in [a,b]} H(s) = \dfrac{(b-a)^{\alpha-1}}{\Gamma(\alpha-1)}$.

Theorem 7 *The necessary condition for existence of a nontrivial solution for the boundary value problem* (39) *is*

$$\frac{\Gamma(\alpha-1)}{\|f\|} \leq \int_a^b (b-s)^{\alpha-1}|g(s)|ds, \tag{41}$$

where $\|f\| = \sup\limits_{t \in [a,b], y \in \mathbb{R}} |f(t, y)|$.

Case II $h_i \neq 0, i = 1, 2, \ldots, n$.

Let $y \in AC[a, b]$ be a solution of the problem (38). Then the function y can be written as

$$y(t) = f(t, y(t)) \left[\int_a^b G(t, s)g(s)y(s)ds - \sum_{i=1}^n \int_a^b G^*(t, s)h_i(s, y(s))ds \right], \tag{42}$$

where $G(t, s)$ is defined as in (11) and $G^*(t, s)$ is defined by

$$G^*(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}\dfrac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}-(t-s)^{\beta-1}, & a\le s\le t\le b,\\ \dfrac{(b-s)^{\beta-1}(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}}, & a\le t\le s\le b.\end{cases} \tag{43}$$

The Green's function $G^*(t,s)$, which is given by (43), satisfies the following inequalities:

1. $G^*(t,s)\ge 0,\quad \forall t,s\in[a,b]$;
2. $G^*(t,s)\le J(s):=\dfrac{(\alpha-1)(b-s)^{\beta-1}}{\Gamma(\beta)}$.

Also we have

3. $\max\limits_{s\in[a,b]} J(s)=\dfrac{(\alpha-1)(b-a)^{\beta-1}}{\Gamma(\beta)}$.

Theorem 8 *Assume that* $|h_i(t,y(t))|\le |x_i(t)||y(t)|$, *where* $x_i\in C([a,b],\mathbb{R})$, $i=1,2,\ldots,n$ *and* $[a,b]=[0,1]$. *The necessary condition for existence of a nontrivial solution for the problem* (38) *on* $[0,1]$ *is*

$$\Gamma(\alpha-1)\left(\frac{1}{\|f\|}-\frac{(\alpha-1)}{\Gamma(\beta+1)}\sum_{i=1}^{n}\|x_i\|\right)\le\int_0^1(1-s)^{\alpha-1}|g(s)|ds. \tag{44}$$

In 2016, Al-Qurashi and Ragoub [24] obtained a Lyapunov-type inequality for a boundary value problem with *integral boundary condition*

$$\begin{cases}{}^CD^\alpha y(t)+q(t)y(t)=0, & a<t<b,\\ y(a)+\mu\displaystyle\int_a^b y(s)q(s)ds=y(b),\end{cases} \tag{45}$$

where ${}^CD^\alpha$ is the Caputo fractional derivative of order $0<\alpha\le 1$, $q:[a,b]\to\mathbb{R}$ is a continuous function, a,b are consecutive zeros of the solution y and μ is positive.

The function $y\in C([a,b],\mathbb{R})$ is a solution of the boundary value problem (45) if and only if y satisfies the integral equation

$$y(t)=\int_a^b G(t,s)q(s)y(s)ds, \tag{46}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(b-s)^{\alpha-1}}{\mu(b-a)} - \dfrac{(b-s)^{\alpha}}{(b-a)\alpha}, & a \le t \le s \le b, \\ \dfrac{(b-s)^{\alpha-1}}{\mu(b-a)} - \dfrac{(b-s)^{\alpha}}{(b-a)\alpha} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \end{cases} \tag{47}$$

and satisfies the following properties:

1. $G(t,s) \ge 0$, for all $a \le t, s \le b$;
2. $\max_{t\in[a,b]} G(t,s) = G(b,s),\ s \in [a,b]$;
3. $G(b,s)$ has a unique maximum given by $\max\limits_{s\in[a,b]} G(b,s) = \dfrac{\alpha(b-a+\mu)(\alpha\mu+1)^{\alpha-1}}{\Gamma(\alpha)\mu^{\alpha}(b-a)}$, provident $0 < \mu(b-a) < \alpha$.

We describe the Lyapunov's inequality for the problem (45) as follows.

Theorem 9 *The boundary value problem (45) has a nontrivial solution provided that the real and continuous function q satisfies the following inequality*

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\mu^{\alpha}(b-a)}{\alpha(b-a+\mu)(\alpha\mu+1)^{\alpha-1}}. \tag{48}$$

In 2016, Fereira [25] obtained a Lyapunov-type inequality for a *sequential fractional boundary value problem*

$$\begin{cases} (D^{\alpha}D^{\beta}y)(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{49}$$

where $D^{\delta},\ \delta = \alpha, \beta$ stands for the Riemann– Liouville fractional derivative and $q : [a,b] \to \mathbb{R}$ is a continuous function. Assuming that (49) has a nontrivial solution $y \in C[a,b]$ of the form

$$y(t) = c\frac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1} - (I^{\alpha+\beta}qy)(t),$$

it follows by Proposition 3 and the fact $I^{\beta}(t-a)^{\alpha-1} = \dfrac{\Gamma(\alpha)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}$ that y' is integrable in $[a,b]$. Then, as argued in [14, Section 2.3.6–2.3.7], we have

$$(D^{\alpha}D^{\beta}y)(t) = (D^{\alpha+\beta}y)(t).$$

The following result is therefore an immediate consequence of Theorem 2.

Theorem 10 (Riemann–Liouville Case) *Let* $0 < \alpha, \beta \leq 1$ *with* $1 < \alpha + \beta \leq 2$. *If there exists a nontrivial continuous solution of the fractional boundary value problem (49), then*

$$\int_a^b |q(s)|ds > \Gamma(\alpha+\beta)\Big(\frac{4}{b-a}\Big)^{\alpha+\beta-1}. \tag{50}$$

As an application we consider the following sequential fractional differential equation

$$(D^\alpha D^\alpha y)(t) + \lambda^2 y(t) = 0, \quad \lambda \in \mathbb{R}, \quad t \in (0,1), \quad \frac{1}{2} < \alpha \leq 1. \tag{51}$$

The fundamental set of solutions to (51) is

$$\{\cos_\alpha(\lambda t), \sin_\alpha(\lambda t)\},$$

where

$$\cos_\alpha(\lambda t) = \sum_{j=0}^{\infty}(-1)^j \lambda^{2j}\frac{t^{(2j+1)\alpha-1}}{\Gamma((2j+1)\alpha)} \text{ and } \sin_\alpha(\lambda t) = \sum_{j=0}^{\infty}(-1)^j \lambda^{2j+1}\frac{t^{(j+1)2\alpha-1}}{\Gamma((j+1)2\alpha)}.$$

Therefore the general solution of (51) can be written as

$$y(t) = c_1\cos_\alpha(\lambda t) + c_2\sin_\alpha(\lambda t), \quad c_1, c_2 \in \mathbb{R}.$$

Now, the nontrivial solutions of (51) for which the boundary conditions $y(0) = 0 = y(1)$ hold satisfy $\sin_\alpha(\lambda) = 0$, where λ is a real number different from zero (the eigenvalue of the problem). By Theorem 10, the following inequality then holds:

$$\lambda^2 > \Gamma(2\alpha)4^{2\alpha-1},$$

which can alternatively be expressed in form of the following result.

Corollary 5 *Let* $\frac{1}{2} < \alpha \leq 1$. *If*

$$|t| \leq \sqrt{\Gamma(2\alpha)4^{2\alpha-1}}, \quad t \neq 0,$$

then $\sin_\alpha(t)$ *has no real zeros.*

In [25], Fereira replaced the Riemann –Liouville fractional derivative in the problem (49) with the Caputo fractional derivative and obtained the following Lyapunov-type inequality:

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha+\beta)}{(b-a)^{\alpha+\beta-1}} \frac{(\alpha+2\beta-1)^{\alpha+2\beta-1}}{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta}. \tag{52}$$

In 2016, Al-Qurashi and Ragoub [26] obtained a Lyapunov-type inequality for a *fractional boundary value problem:*

$$\begin{cases} D^\alpha y(t) + q(t)y(t) = 0, \quad 1 < t < e, \\ y(a) = y(b) = y''(a) = y''(b) = 0, \end{cases} \tag{53}$$

where D^α is the standard Riemann–Liouville fractional derivative of order $3 < \alpha \leq 4$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

The function $y \in C([a, b], \mathbb{R})$ is a solution of the boundary value problem (53) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds, \tag{54}$$

where $G(t, s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -(t-a)(b-s)^{\alpha-1} \\ +\dfrac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)\Big[1-\dfrac{(t-a)^2}{(b-a)^2}\Big], & a \leq t \leq s \leq b, \\ (t-s)^{\alpha-1} - (t-a)(b-s)^{\alpha-1} \\ +\dfrac{\alpha(\alpha-1)(b-a)(b-s)^{\alpha-3}}{6}(t-a)\Big[1-\dfrac{(t-a)^2}{(b-a)^2}\Big], & a \leq s \leq t \leq b, \end{cases} \tag{55}$$

and satisfies the relation:

$$0 \leq G(t,s) \leq G(b,s) = \frac{(1-(b-a))(b-a)^{\alpha-1}}{\Gamma(\alpha)}, \quad (t,s) \in [a,b] \times [a,b].$$

The Lyapunov-type inequality for the problem (53) is given in the following result.

Theorem 11 *If there exists a nontrivial continuous solution to the fractional boundary value problem (53), then*

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)}{(1-(b-a))(b-a)^{\alpha-1}}. \tag{56}$$

In order to illustrate Theorem 11, we apply the Lyapunov-type inequality (56) to find a bound for λ so that the following eigenvalue problem has a nontrivial solution:

$$\begin{cases} D^{\alpha}y(t)+\lambda y(t)=0, \quad 0<t<\frac{1}{2},\ 3<\alpha\leq 4, \\ y(1)=y\Big(\dfrac{1}{2}\Big)=y''(1)=y''\Big(\dfrac{1}{2}\Big)=0. \end{cases} \tag{57}$$

Corollary 6 *If λ is an eigenvalue of the fractional boundary value problem (57), then the following inequality holds:*

$$|\lambda|\geq \frac{\Gamma(\alpha)}{2^{-\alpha}}.$$

In 2016, Dhar et al. [27] derived Lyapunov-type inequalities for the following boundary value problem:

$$\begin{cases} D^{\alpha}y(t)+q(t)y(t)=0, \quad a<t<b,\ 1<\alpha\leq 2, \\ D^{\alpha-2}y(a)=D^{\alpha-2}y(b)=0, \end{cases} \tag{58}$$

where D^{α} is the Riemann–Liouville fractional derivative of order α $(1<\alpha\leq 2)$, $q\in L([a,b],\mathbb{R})$. Their main result on fractional Lyapunov-type inequalities is the following.

Theorem 12

(a) If the problem (58) has a nontrivial solution, then

$$\max_{t\in[a,b]}\left\{\int_a^b |D^{2-\alpha}[G(t,s)q(s)]|ds\right\}>1,$$

where $D^{2-\alpha}[G(t,s)q(s)]$ is the right-sided fractional derivative of $G(t,s)q(s)$ with respect to s with

$$G(t,s)=\frac{1}{b-a}\begin{cases}(t-a)(b-s), & a\leq t\leq s\leq b,\\ (s-a)(b-t), & a\leq s\leq t\leq b.\end{cases}$$

(b) If problem (58) has a nontrivial solution and $D^{\alpha-2}y(t)\neq 0$ on (a,b), then

$$\max_{t\in[a,b]}\left\{\int_a^b \Big[D^{2-\alpha}[G(t,s)q(s)]\Big]_+ ds\right\}>1,$$

where $D^{2-\alpha}[G(t,s)q(s)]_+$ is the positive part of $D^{2-\alpha}[G(t,s)q(s)]$.

As a special case we have the following corollary.

Corollary 7 *Assume that $D_{b-}^{2-\alpha}[G(t,s)q(s)]\geq 0$ for $t,s\in[a,b]$ so that the problem (58) has a nontrivial solution. Then*

$$\int_a^b q_+(t)dt > \frac{\alpha^\alpha \Gamma(\alpha-1)}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}.$$

Next we consider the sequential fractional boundary value problem

$$\begin{cases} \left[\left(D^{\beta}_{a+}(D^{\alpha}y)\right)\right](t) + q(t)y(t) = 0, & a < t < b,\ 0 < \alpha, \beta \le 1, \\ (D^{\alpha-1}y)(a^+) = (D^{\alpha-1}y)(b) = 0, \end{cases} \tag{59}$$

which is equivalent to the integral equation

$$y(t) = \int_a^b G(t,s)q(s)D^{1-\alpha}_{a+}y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\beta+1)} \begin{cases} \dfrac{(t-a)^\beta(b-s)^\beta}{(b-a)^\beta}, & a \le t \le s \le b, \\ \dfrac{(t-a)^\beta(b-s)^\beta}{(b-a)^\beta} - (t-s)^\beta, & a \le s \le t \le b. \end{cases}$$

In the following result, we express the fractional Lyapunov-type inequalities for problem (59).

Theorem 13

(a) If problem (59) has a nontrivial solution, then

$$\max_{t\in[a,b]} \left\{ \int_a^b |D^{1-\alpha}[G(t,s)q(s)]|ds \right\} > 1.$$

(b) If problem (59) has a nontrivial solution and $(D^{\alpha-1}_{a+}y)(t) \neq 0$ *on* (a,b)*, then*

$$\max_{t\in[a,b]} \left\{ \int_a^b \left[D^{1-\alpha}[G(t,s)q(s)]\right]_+ ds \right\} > 1.$$

As a special case we have the following corollary.

Corollary 8 *Assume that* $D^{1-\alpha}[G(t,s)q(s)] \ge 0$ *for* $t,s \in [a,b]$, $1 < \alpha+\beta \le 2$ *and the problem (59) has a nontrivial solution. Then*

$$\int_a^b q_+(t)dt > \frac{(\alpha+2\beta-1)^{\alpha+2\beta-1}\Gamma(\alpha)\Gamma(\beta+1)}{(\alpha+\beta-1)^{\alpha+\beta-1}\beta^\beta(b-a)^{\alpha+\beta-1}}.$$

For some similar results on fractional boundary value problems of order $\alpha \in (2, 3]$, see [28].

In 2017, Jleli et al. [29] obtained Lyapunov-type inequality for *higher order fractional boundary value problem*

$$\begin{cases} D^{\alpha} y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y'(a) = \ldots = y^{(n-2)}(a) = 0, \; y(b) = I^{\alpha}(yh)(b), \end{cases} \tag{60}$$

where $n \in \mathbb{N}, n - 1 < \alpha < n$, D^{α} is the standard Riemann–Liouville fractional derivative of order α, I^{α} denotes the Riemann–Liouville fractional integral of order α, and $q, h : [a, b] \to \mathbb{R}$ are continuous functions.

The function y is a solution of the boundary value problem (60) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t, s)(q(s) + h(s))y(s)ds + \frac{1}{\Gamma(\alpha)} \int_a^t (t - s)^{\alpha-1} h(s)y(s)ds, \tag{61}$$

where $G(t, s)$ is the Green's function given by (11) such that

$$0 \le G(t, s) \le G(s^*, s) = \frac{(s - a)^{\alpha-1}(b - s)^{\alpha-1}}{\Gamma(\alpha)(b - a)^{\alpha-1}\left[1 - \left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}\right]^{\alpha-2}}, \; a < s < b,$$

with

$$s^* = \frac{s - a\left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}}{1 - \left(\frac{b-s}{b-a}\right)^{\frac{\alpha-1}{\alpha-2}}}.$$

The following result presents the Lyapunov-type inequality for problem (60).

Theorem 14 *Let $n \in \mathbb{N}$ with $n \ge 3$. If y is a nontrivial solution of the fractional boundary value problem (60), then*

$$\int_a^b \left(|q(s) + h(s)| + \frac{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}{z_\alpha^{\alpha-1}(1 - z_\alpha)^{\alpha-1}} |h(s)| \right) ds \ge \frac{\Gamma(\alpha)}{(b - a)^{\alpha-1}} \frac{\left(1 - z_\alpha^{\frac{\alpha-1}{\alpha-2}}\right)^{\alpha-2}}{z_\alpha^{\alpha-1}(1 - z_\alpha)^{\alpha-1}}, \tag{62}$$

where z_α is the unique zero of the nonlinear algebraic equation

$$z^{\frac{2\alpha-3}{\alpha-2}} - 2z + 1 = 0$$

in the interval $\left(0, \left(\frac{2\alpha-4}{2\alpha-3}\right)^{\frac{\alpha-2}{\alpha-1}}\right)$.

Theorem 15 *Let $n = 2$. If y is a nontrivial solution of the fractional boundary value problem*

$$\begin{cases} D^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \\ y(a) = 0, \ y(b) = I^\alpha (hy)(b), \end{cases} \tag{63}$$

then

$$\int_a^b \Big(|q(s) + h(s)| + 4^{\alpha-1} h(s)|\Big) ds \geq \Gamma(\alpha)\Big(\frac{4}{b-a}\Big)^{\alpha-1}. \tag{64}$$

In 2017, Cabrera et al. [30] obtained a Lyapunov-type inequality for a sequential fractional boundary value problem

$$\begin{cases} {}^C D^\alpha y(t) + q(t)y(t) = 0, & a < t < b, \ \ \alpha \in (n-1, n], \ n \geq 4, \\ y^i(a) = y''(b) = 0, \ \ 0 \leq i \leq n-1, \ i \neq 2, \end{cases} \tag{65}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative of fractional order $\alpha \geq 0$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

The function $y \in C([a, b], \mathbb{R})$ is a solution of the boundary value problem (65) if and only if it satisfies the integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds, \tag{66}$$

where $G(t, s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \frac{1}{2}(\alpha-1)(\alpha-2)(t-a)^2(b-s)^{\alpha-3} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \\ \frac{1}{2}(\alpha-1)(\alpha-2)(t-a)^2(b-s)^{\alpha-3}, & a \leq t \leq s \leq b, \end{cases} \tag{67}$$

such that $G(t, s) \geq 0$ for $t, s \in [a, b]$, $|G(t, s)| \leq G(b, s)$ for $t, s \in [a, b]$ and

$$|G(t,s)| \leq G(b,s) = \frac{1}{2}(\alpha-1)(\alpha-2)(b-a)^2(b-s)^{\alpha-3} - (b-s)^{\alpha-1}, \ \ (t,s) \in [a,b]\times[a,b].$$

Their result is as follows.

Theorem 16 *If there exists a nontrivial continuous solution of the fractional boundary value problem (65), then*

$$\int_a^b \Big[\frac{1}{2}(\alpha-1)(\alpha-2)(b-a)^2(b-s)^{\alpha-3} - (b-s)^{\alpha-1}\Big]|q(s)|ds \geq \Gamma(\alpha). \tag{68}$$

In 2017, Wang et al. [31] obtained a Lyapunov-type inequality for the *higher order fractional boundary value problem*

$$\begin{cases} D^{\alpha} y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) = y'(a) = \ldots = y^{(n-2)}(a) = 0, \ y^{(n-2)}(b) = 0, \end{cases} \tag{69}$$

where $n \in \mathbb{N}, 2 < n - 1 < \alpha \leq n$, D^{α} is the standard Riemann–Liouville fractional derivative of order α, and $q : [a, b] \to \mathbb{R}$ is a continuous function.

The function y is a solution of the boundary value problem (69) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds,$$

where $G(t, s)$ is the Green's function given by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-n+1}}{(b-a)^{\alpha-n+1}}, & a \leq t \leq s \leq b, \\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-n+1}}{(b-a)^{\alpha-n+1}} - (t-s)^{\alpha-1}, & a \leq s \leq t \leq b, \end{cases}$$

such that

$$0 \leq G(t, s) \leq G(b, s) = \frac{(b-s)^{\alpha-n+1}(s-a)}{\Gamma(\alpha)} \sum_{i=1}^{n-2} (-1)^{i-1} C_{n-2}^{i} (b-a)^{n-2-i} (s-a)^{i-1},$$

$(t, s) \in [a, b] \times [a, b]$ and C_{n-2}^{i} is the binomial coefficient.

Their Lyapunov-type inequality for the problem (69) is given in the following theorem.

Theorem 17 *If there exists a nontrivial continuous solutions y of the fractional boundary value problem (69), and q is a real continuous function, then*

$$\int_a^b (b-s)^{\alpha-n+1}(s-a) \sum_{i=1}^{n-2} (-1)^{i-1} C_{n-2}^{i} (b-a)^{n-2-i} (s-a)^{i-1} |q(s)| ds \geq \Gamma(\alpha).$$

Corollary 9 *If the fractional boundary value problem (69) has a nontrivial continuous solution, then*

$$\int_a^b |q(s)| ds \geq \frac{\Gamma(\alpha)(\alpha - n + 2)^{\alpha-n+2}}{(n-2)(\alpha-n+1)^{\alpha-n+1}(b-a)^{\alpha-1}}.$$

The following result shows the application of the above Lyapunov-type inequality to eigenvalue problems.

Corollary 10 *If* λ *is an eigenvalue to the fractional boundary value problem*

$$\begin{cases} D^{\alpha}y(t) + \lambda y(t) = 0, \quad a < t < b, \\ y(a) = y'(a) = \ldots = y^{(n-2)}(a) = 0, \ y^{(n-2)}(b) = 0, \end{cases}$$

then

$$|\lambda| \geq \frac{\Gamma(\alpha)(\alpha - n + 3)(\alpha - n + 2)}{n - 2}.$$

In 2017, Fereira [32] obtained a Lyapunov-type inequality for the so-called *anti-periodic* boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha}y(t) + q(t)y(t) = 0, \quad a < t < b, \\ y(a) + y(b) = 0 = y'(a) + y'(b), \end{cases} \tag{70}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $1 < \alpha \leq 2$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

Then $y \in C([a, b], \mathbb{R})$ is a solution of the boundary value problem (70) if and only if it satisfies the integral equation

$$y(t) = \int_a^b (b - s)^{\alpha-2} H(t, s) q(s) y(s) ds, \tag{71}$$

where $H(t, s)$ is defined by

$$H(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(-\frac{b-a}{4} + \frac{t-a}{2}\right)(\alpha - 1) + \frac{b-s}{2}, & a \leq t \leq s \leq b, \\ \left(-\frac{b-a}{4} + \frac{t-a}{2}\right)(\alpha - 1) + \frac{b-s}{2} - \frac{(t-s)^{\alpha-1}}{(b-a)^{\alpha-2}}, & a \leq s \leq t \leq b. \end{cases} \tag{72}$$

Here the function H satisfies the following property:

$$|H(t, s)| \leq \frac{(b-a)(3-\alpha)}{4}, \quad (t, s) \in [a, b] \times [a, b].$$

The Lyapunov-type inequality for the problem (70) is given in the following result.

Theorem 18 *If (70) admits a nontrivial continuous solution, then*

$$\int_a^b (b - s)^{\alpha-2} |q(s)| ds \geq \frac{4}{(b-a)(3-\alpha)}. \tag{73}$$

Inequality (73) is useful in finding a bound for the possible eigenvalues of the fractional boundary value problem:

$$\begin{cases} D^{\alpha} y(t) + \lambda y(t) = 0, \ a < t < b, \\ y(a) + y(b) = 0 = y'(a) + y'(b), \end{cases}$$

that is, an eigenvalue $\lambda \in \mathbb{R}$ satisfies the inequality

$$|\lambda| > \frac{4(\alpha - 1)}{(b-a)^{\alpha}(3-\alpha)}.$$

In 2017, Agarwal and Zbekler [33] obtained a Lyapunov-type inequality for the following fractional boundary value problem with *Dirichlet boundary conditions*

$$\begin{cases} (D^{\alpha} y)(t) + p(t)|y(t)|^{\mu-1}y(t) + q(t)|y(t)|^{\gamma-1}y(t) = f(t), \quad a < t < b, \\ y(a) = 0, \ \ y(b) = 0, \end{cases} \tag{74}$$

where D^{α} is the Riemann–Liouville fractional derivative, $p, q, f \in C[t_0, \infty)$ and $0 < \gamma < 1 < \mu < 2$. No sign restrictions are imposed on the potentials p and q, and the forcing term f.

The problem (74) is equivalent to the following integral equation

$$y(t) = \int_a^b G(t,s)[p(s)y^{\mu}(s) + q(s)y^{\gamma}(s) - f(s)]ds,$$

where $G(t,s)$ is the Green's function defined by (11).

Their Lyapunov-type inequality for the problem (74) is as follows.

Theorem 19 *Let y be a nontrivial solution of the problem (74). If $y(t) \neq 0$ in (a,b), then the inequality*

$$\Big(\int_a^b [p^+(t) + q^+(t)]dt\Big)\Big(\int_a^b [\mu_0 p^+(t) + \gamma_0 q^+(t) + |f(t)|]dt\Big) > \frac{4^{2\alpha-3}\Gamma^2(\alpha)}{(b-a)^{2\alpha-2}} \tag{75}$$

holds, where $u^+ = \max\{u, 0\}$, $u = p, q$ and

$$\mu_0 = (2-\mu)\mu^{\mu/(2-\mu)}2^{2/(\mu-2)} > 0, \ \ \gamma_0 = (2-\gamma)\gamma^{\gamma/(2-\gamma)}2^{2/(\gamma-2)} > 0.$$

In 2017, Zhang and Zheng [34] considered the Riemann–Liouville fractional differential equations with mixed nonlinearities of order $\alpha \in (n-1, n]$ for $n \geq 3$

$$(D^{\alpha} y)(t) + p(t)|y(t)|^{\mu-1}y(t) + q(t)|y(t)|^{\gamma-1}y(t) = f(t), \tag{76}$$

where $p, q, f \in C([t_0, \infty), \mathbb{R})$ and the constants satisfy $0 < \gamma < 1 < \mu < n$ $(n \geq 3)$. Equation (76) subjects to the following two kinds boundary conditions, respectively:

$$y(a) = y'(a) = y''(a) = \ldots = y^{(n-2)}(a) = y(b) = 0, \tag{77}$$

and

$$y(a) = y'(a) = y''(a) = \ldots = y^{(n-2)}(a) = y'(b) = 0, \tag{78}$$

where a and b are two consecutive zeros of the function y.

Obviously, it is easy to see that Eq. (76) has two special forms: one is the forced *sub-linear* ($p(t) = 0$) fractional equation

$$(D^{\alpha} y)(t) + q(t)|y(t)|^{\gamma-1} y(t) = f(t), \ 0 < \gamma < 1, \tag{79}$$

and the other is the forced *super-linear* ($q(t) = 0$) fractional equation

$$(D^{\alpha} y)(t) + q(t)|y(t)|^{\gamma-1} y(t) = f(t), \ 1 < \mu < n. \tag{80}$$

Their Lyapunov-type inequalities for the problems (76)–(77) and (76)–(78) are, respectively, the following:

Theorem 20 *Let y be a positive solution of the boundary value problem (76)–(77) in (a, b). Then*

$$\begin{aligned}
&\left(\int_a^b [(b-s)(s-a)]^{\alpha-1} \left[1 - \left(\frac{b-s}{b-a} \right)^{\frac{\alpha-1}{\alpha-2}} \right]^{2-\alpha} [\mu_0 p^+(s) + \gamma_0 q^+(s) + f^-(s)] ds \right) \\
&\times \left(\int_a^b [(b-s)(s-a)]^{\alpha-1} \left[1 - \left(\frac{b-s}{b-a} \right)^{\frac{\alpha-1}{\alpha-2}} \right]^{2-\alpha} [p^+(s) + q^+(s)] ds \right) \\
&> [\Gamma(\alpha)(b-a)^{\alpha-1}]^{\frac{n}{n-1}} (n-1) n^{\frac{n}{1-n}},
\end{aligned}$$

where $\mu_0 = (n-\mu)\mu^{\frac{\mu}{n-\mu}} n^{\frac{n}{n-\mu}}$ and $\gamma_0 = (n-\gamma)\gamma^{\frac{\gamma}{n-\gamma}} n^{\frac{n}{\gamma-n}}$.

Theorem 21 *Let y be a positive solution of the boundary value problem (76)–(78) in (a, b). Then*

$$\begin{aligned}
&\left(\int_a^b [(b-s)^{\alpha-2}(s-a)][\mu_0 p^+(s) + \gamma_0 q^+(s) + f^-(s)] ds \right) \\
&\times \left([b-s)^{\alpha-2}(s-a)][p^+(s) + q^+(s)] ds \right)^{\frac{1}{\mu-1}} \\
&> \Gamma(\alpha)^{\frac{n}{n-1}} (n-1) n^{\frac{n}{1-n}},
\end{aligned}$$

where μ_0 and γ_0 are the same as in Theorem 21.

In 2017, Chidouh and Torres [35] extended the linear term $q(t)y(t)$ to a nonlinear term of the form $q(t)f(y(t))$ and obtained a generalized Lyapunov's inequality for the fractional boundary value problem

$$\begin{cases} (D^\alpha y)(t) + q(t)f(y(t)) = 0, & a < t < b, \quad 1 < \alpha \le 2, \\ y(a) = y(b) = 0, \end{cases} \tag{81}$$

where D^α is the Riemann–Liouville fractional derivative of order α, $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous, and $q : [a, b] \to \mathbb{R}^+$ is a Lebesgue integrable function.

An integral equation equivalent to the Problem (81) is

$$y(t) = \int_a^b G(t, s)q(s)f(y(s))ds,$$

where $G(t, s)$ is the Green's function defined by (11) and satisfies the following properties:

1. $G(t, s) \ge 0$, for all $(t, s) \in [a, b] \times [a, b]$;
2. $\max_{t\in[a,b]} G(t, s) = G(s, s), \; s \in [a, b]$;
3. $G(s, s)$ has a unique maximum given by $\max\limits_{s\in[a,b]} G(s, s) = \dfrac{(b-a)^{\alpha-1}}{4^{\alpha-1}\Gamma(\alpha)}$.

The Lyapunov-type inequality for the problem (81) is as follows.

Theorem 22 *Let q be a real nontrivial Lebesgue integrable function. Assume that $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a concave and nondecreasing function. If the fractional boundary value problem (81) has a nontrivial solution y, then*

$$\int_a^b q(s)ds > \frac{4^{\alpha-1}\Gamma(\alpha)\eta}{(b-a)^{\alpha-1}f(\eta)}. \tag{82}$$

where $\eta = \max_{t\in[a,b]} y(t)$.

In 2016, Ma [36] obtained a generalized form of Lyapunov's inequality for the fractional boundary value problem

$$\begin{cases} (D^\alpha y)(t) + q(t)f(y(t)) = 0, & a < t < b, \quad 1 < \alpha \le 2, \\ y(a) = y(b) = y''(a) = 0, \end{cases} \tag{83}$$

where D^α is the Riemann–Liouville fractional derivative of order α, $q : [a, b] \to \mathbb{R}^+$ is a Lebesgue integrable function, and $f : \mathbb{R}^+ \to \mathbb{R}^+$ is continuous.

The function $y \in C([a, b], \mathbb{R})$ is a solution to the problem (83) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t,s)q(s)f(y(s))ds,$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(b-s)^{\alpha-1}(t-a)}{(b-a)}, & a \le t \le s \le b, \\ \dfrac{(b-s)^{\alpha-1}(t-a)}{(b-a)} - (t-s)^{\alpha-1}, & a \le s \le t \le b. \end{cases} \tag{84}$$

The above Green's function satisfies the following properties:

1. $G(t,s) \ge 0$, for all $(t,s) \in [a,b] \times [a,b]$;
2. For any $s \in [a,b]$,

$$\max_{t\in[a,b]} G(t,s) = G(t_0,s) = \frac{(s-a)(b-s)^{\alpha-1}}{\Gamma(\alpha)(b-a)} + \frac{(\alpha-2)(b-s)^{\frac{(\alpha-1)^2}{\alpha-2}}}{(\alpha-1)^{\frac{\alpha-1}{\alpha-2}}(b-a)^{\frac{\alpha-1}{\alpha-2}}\Gamma(\alpha)},$$

   where $t_0 = s + \left(\dfrac{(b-s)^{\alpha-1}}{(b-a)(\alpha-1)}\right)^{\frac{1}{\alpha-2}} \in [s,b]$;
3. $\max\limits_{s\in[a,b]} G(t_0,s) \le \dfrac{(b-a)^{\alpha-1}}{\Gamma(\alpha)}$;
4. $G(t,s) \ge \dfrac{(t-a)(b-t)}{(b-a)^2} G(t_0,s)$ for all $a \le t, s \le b$.

The following Lyapunov-type inequalities are given in [36].

Theorem 23 *Assume that f is bounded by two lines, that is, there exist two positive constants M and N such that $Ny \le f(y) \le My$ for any $y \in \mathbb{R}^+$. If (83) has a solution in $E^+ = \{y \in C[a,b], y(t) \ge 0,$ for any $t \in [a,b]$ and $\|y\| \neq 0\}$, then the following Lyapunov-type inequalities hold:*

(i) $\displaystyle\int_a^b q(s)ds > \frac{\Gamma(\alpha)}{M(b-a)^{\alpha-1}}$;

(ii) $\displaystyle\int_a^b (s-a)^2(b-s)^{\alpha} q(s)ds \le \frac{4\Gamma(\alpha)(b-a)^3}{N}$;

(iii) $\displaystyle\int_a^b (s-a)(b-s)^{\frac{\alpha^2-\alpha-1}{\alpha-2}} q(s)ds \le \frac{4\Gamma(\alpha)(b-a)^{\frac{3\alpha-2}{\alpha-2}}(\alpha-1)^{\frac{\alpha-1}{\alpha-2}}}{(\alpha-2)N}$.

The applications of the above inequalities are given in the following corollaries.

Corollary 11 *For any $\lambda \in [0, \Gamma(\nu)] \cup \left(\dfrac{4\Gamma(\nu)}{B(3,\nu+1)}, +\infty\right)$, where $B(x,y) = \displaystyle\int_0^1 s^{x-1}(1-s)^{y-1}ds$, $x > 0, y > 0$, the eigenvalue for the problem*

$$\begin{cases} (D^\nu y)(t) + \lambda y(t) = 0, \quad 0 < t < 1,\ 2 < \nu \le 3, \\ y(0) = y(1) = y''(0) = 0, \end{cases} \tag{85}$$

has no corresponding eigenfunction $y \in E^+$.

In the next corollary we obtain an interval in which the Mittag–Leffler function $E_{\nu,2}(z)$ with $\beta = 2$, $2 < \nu \le 3$ has no real zeros.

Corollary 12 *Let* $2 < \nu \le 3$. *Then the Mittag–Leffler function* $E_{\nu,2}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\nu+2)}$ *has no real zeros for* $z \in \Big(-\infty, -\frac{4\Gamma(\nu)}{B(3,\nu+1}, \Big) \cup [-\Gamma(\nu), +\infty)$.

In 2017, Ru et al. [37] obtained the Lyapunov-type inequality for the following *fractional Sturm–Liouville boundary value problem*

$$\begin{cases} D^\alpha_{0^+}(p(t)y'(t)) + q(t)y(t) = 0, \quad 0 < t < 1,\ 1 < \alpha \le 2, \\ ay(0) - bp(0)y'(0) = 0, \ \ cy(1) + dp(1)y'(1) = 0, \end{cases} \tag{86}$$

where $a, b, c, d > 0$ D^α is the standard Riemann–Liouville fractional derivative of order α, $p : [0, 1] \to (0, \infty)$ and $q : [0, 1] \to \mathbb{R}$ is a nontrivial Lebesgue integrable function.

The solution of the boundary value problem (86) in terms of the integral equation is

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds,$$

where $G(t, s)$ is the Green's function given by

$$G(t,s) = \frac{1}{\rho\Gamma(\alpha)} \begin{cases} \Big[b + a\int_0^t \frac{d\tau}{p(\tau)}\Big]\Big[d(1-s)^{\alpha-1} + c\int_s^1 \frac{(\tau-s)^{\alpha-1}d\tau}{p(\tau)}\Big], & 0 \le t \le s \le 1, \\ \Big[b + a\int_0^t \frac{d\tau}{p(\tau)}\Big]\Big[d(1-s)^{\alpha-1} + c\int_s^1 \frac{(\tau-s)^{\alpha-1}d\tau}{p(\tau)}\Big] - H(t,s), & 0 \le s \le t \le 1, \end{cases}$$

$$\rho = bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau + ad, \quad H(t,s) = a\Big[d + c\int_t^1 \frac{d\tau}{p(\tau)}\Big]\int_0^t \frac{(\tau-s)^{\alpha-1}}{p(\tau)}d\tau.$$

Further, the above Green's function $G(t, s)$ satisfies the following properties:

1. $G(t, s) \ge 0$ for $0 \le t, s \le 1$;
2. The maximum value of $G(t, s)$ is

$$\overline{G} = \max_{0\le t,s\le 1} G(t,s) = \max\{\max_{s\in[0,1]} G(s,s), \max_{s\in[0,1]} G(t_0(s), s)\}, \tag{87}$$

where

$$t_0(s) = s + \left[\frac{ad(1-s)^{\alpha-1} + ac\int_s^1 \frac{(\tau-s)^{\alpha-1}}{p(\tau)}d\tau}{\rho}\right]^{\frac{1}{\alpha-1}}.$$

They obtained the following Lyapunov-type inequality for the problem (86).

Theorem 24 *For any nontrivial solutions y of the fractional boundary value problem (86), the following Lyapunov-type inequality holds:*

$$\int_0^1 |q(s)|ds > \frac{1}{\overline{G}},$$

where $\overline{G}$ is defined by (87).

In [37], the authors also considered the *generalized fractional Sturm–Liouville boundary value problem*:

$$\begin{cases} D_{0+}^{\alpha}(p(t)y'(t)) + q(t)f(y(t)) = 0, & 0 < t < 1,\ 1 < \alpha \le 2, \\ ay(0) - bp(0)y'(0) = 0, \ \ cy(1) + dp(1)y'(1) = 0, \end{cases} \tag{88}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, and obtained the Lyapunov-type inequality for this problem as follows.

Theorem 25 *Let f be a positive function on $\mathbb{R}$. For any nontrivial solutions y of the fractional boundary value problem (88), the following Lyapunov-type inequality will be satisfied*

$$\int_0^1 |q(s)|ds > \frac{y^*}{\overline{G}\max_{y\in[y_*.y^*]} f(y)},$$

where $\overline{G}$ is defined by (87) and $y_* = \min_{t\in[0,1]} y(t)$, $y^* = \max_{t\in[0,1]} y(t)$.

4 Lyapunov Inequalities for Nonlocal Boundary Value Problems

In 2017, Cabrera et al. [38] obtained Lyapunov-type inequalities for a *nonlocal* fractional boundary value problem

$$\begin{cases} {}^C D^{\alpha} y(t) + q(t)y(t) = 0, & a < t < b, \\ y'(a) = 0, \ \ \beta^C D^{\alpha-1}y(b) + y(\eta) = 0, \end{cases} \tag{89}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative of fractional order $1 < \alpha \leq 2$, $\beta > 0, a < \eta < b$, $\beta > \dfrac{(\beta - \eta)^{\alpha-1}}{\Gamma(\alpha)}$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

Note that problem (89) is the fractional analogue of the classical nonlocal boundary value problem

$$\begin{cases} y''(t) + q(t)y(t) = 0, \quad 0 < t < 1, \\ y'(0) = 0, \ \ \beta y'(1) + y(\eta) = 0, \ 0 < \eta < 1, \end{cases} \tag{90}$$

which represents a thermostat model insulated at $t = 0$ with a controller dissipating heat at $t = 1$ depending on the temperature detected by a sensor at $t = \eta$ [39].

The function $y \in C([a, b], \mathbb{R})$ is a solution of the boundary value problem (89) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds, \tag{91}$$

where $G(t, s)$ is the Green's function defined by

$$G(t, s) = \beta + H_\eta(s) - H_t(s), \tag{92}$$

for $r \in [a, b]$, $H_r : [a, b] \to \mathbb{R}$ is

$$H_r(s) = \begin{cases} \dfrac{(r - s)^{\alpha-1}}{\Gamma(\alpha)}, & a \leq s \leq r \leq b, \\ 0, & a \leq r \leq s \leq b. \end{cases}$$

The Green's function defined in (92) satisfies the relation

$$|G(t, s)| \leq \beta + \frac{(\eta - a)^{\alpha-1}}{\Gamma(\alpha)}, \ \ (t, s) \in [a, b] \times [a, b].$$

The Lyapunov-type inequality derived for the problem (89) is given in the following result.

Theorem 26 *If there exists a nontrivial continuous solution of the fractional boundary value problem (89), then*

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)}{\beta\Gamma(\alpha) + (\eta - \alpha)^{\alpha-1}}. \tag{93}$$

In 2017, Cabrera et al. [40] obtained a Lyapunov-type inequality for the following *nonlocal fractional boundary value problem*

$$\begin{cases} (D^\alpha y)(t) + q(t)y(t) = 0, & a < t < b, \ 2 < \alpha \le 3, \\ y(a) = y'(a) = 0, \ \ y'(b) = \beta y(\xi), \end{cases} \tag{94}$$

where D^α is the Riemann–Liouville fractional derivative of order α, $a < \xi < b$, $0 \le \beta(\xi - a)^{\alpha-1} < (\alpha-1)(b-a)^{\alpha-2}$, and $q : [a, b] \to \mathbb{R}$ is a continuous function.

The unique solution of the nonlocal boundary value problem (94) is given by

$$y(t) = \int_a^b G(t, s)q(s)y(s)ds + \frac{\beta(t-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}} \int_a^b G(\xi, s)q(s)y(s)ds,$$

where $G(t, s)$ is the Green's function defined by

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(b-s)^{\alpha-2}(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}}, & a \le t \le s \le b, \\ \dfrac{(b-s)^{\alpha-2}(t-a)^{\alpha-1}}{(b-a)^{\alpha-2}} - (t-s)^{\alpha-1}, & a \le s \le t \le b. \end{cases} \tag{95}$$

The Green's function defined in (95) satisfies the following properties:

1. $G(t, s) \ge 0$, for all $(t, s) \in [a, b] \times [a, b]$;
2. $G(t, s)$ is non-decreasing with respect to the first variable;
3. $0 \le G(a, s) \le G(t, s) \le G(b, s), \ (t, s) \in [a, b] \times [a, b]$.

Their Lyapunov-type inequality for the problem (94) is expressed as follows.

Theorem 27 *If the problem (94) has a nontrivial solution, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}} \left(1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}}\right)^{-1}. \tag{96}$$

As an application of Theorem 27, we consider the following eigenvalue problem:

$$\begin{cases} D^\alpha y(t) + \lambda y(t) = 0, & a < t < b, \ 2 < \alpha \le 3, \\ y(a) = y'(a) = 0, y'(b) = \beta y(\xi), \end{cases} \tag{97}$$

where $a < \xi < b$ and $0 \le \beta(\xi - a)^{\alpha-1} < (\alpha - 1)(b - a)^{\alpha-2}$. If λ is an eigenvalue of problem (97), then

$$|\lambda| \ge \frac{\alpha(\alpha-1)\Gamma(\alpha)}{(b-a)^\alpha} \left(1 + \frac{\beta(b-a)^{\alpha-1}}{(\alpha-1)(b-a)^{\alpha-2} - \beta(\xi-a)^{\alpha-1}}\right)^{-1}.$$

This is an immediate consequence of Theorem 27.

Very recently, Wang and Wang [41] obtained Lyapunov-type inequalities for the fractional differential equations with *multi-point boundary conditions*

$$\begin{cases} D^{\alpha}y(t)+q(t)y(t)=0, \quad a<t<b,\ 2<\alpha\le 3,\\ y(a)=y'(a)=0, \quad (D^{\beta+1}y)(b)=\displaystyle\sum_{i=1}^{m-2}b_i(D^{\beta}y)(\xi_i), \end{cases} \tag{98}$$

where D^{α} denotes the standard Riemann–Liouville fractional derivative of order α, $\alpha>\beta+2$, $0<\beta<1$, $a<\xi_1<\xi_2<\ldots<\xi_{m-2}<b$, $b_i\ge 0 (i=1,2,\ldots,m-2)$, $0\le\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}<(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}$ and $q:[a,b]\to\mathbb{R}$ is a continuous function.

The solution of the boundary value problem (98) in terms of the integral equation is

$$y(t)=\int_a^b G(t,s)q(s)y(s)ds+T(t)\int_a^b\Big(\sum_{i=1}^{m-2}b_iH(\xi,s)q(s)y(s)\Big)ds,,$$

where $G(t,s)$, $H(t,s)$ and $T(t)$ defined by

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}-(t-s)^{\alpha-1}, & a\le s\le t\le b,\\ \dfrac{(t-a)^{\alpha-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, & a\le t\le s\le b.\end{cases}$$

$$H(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\dfrac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}-(t-s)^{\alpha-\beta-1}, & a\le s\le t\le b,\\ \dfrac{(t-a)^{\alpha-\beta-1}(b-s)^{\alpha-\beta-2}}{(b-a)^{\alpha-\beta-2}}, & a\le t\le s\le b.\end{cases}$$

$$T(t)=\frac{(t-a)^{\alpha-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2}b_i(\xi_i-a)^{\alpha-\beta-1}},\quad t\ge a.$$

Further, the above functions $G(t,s)$ and $H(t,s)$ satisfy the following properties:

1. $G(t,s)\ge 0$ for $a\le t,s\le b$;
2. $G(t,s)$ is non-decreasing with respect to the first variable;
3. $0\le G(a,s)\le G(t,s)\le G(b,s)=\frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-\beta-2}[(b-a)^{\beta+1}-(b-s)^{\beta+1}]$, $(t,s)\in[a,b]\times[a,b]$;
4. for any $s\in[a,b]$,

$$\max_{s\in[a,b]} G(b,s) = \frac{\beta+1}{\alpha-1}\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{(\alpha-\beta-2)/(\beta+1)} \frac{(b-a^{\alpha-1}}{\Gamma(\alpha)};$$

5. $H(t,s) \geq 0$ for $a \leq t, s \leq b$;
6. $H(t,s)$ is non-decreasing with respect to the first variable;
7. $0 \leq H(a,s) \leq H(t,s) \leq H(b,s) = \frac{1}{\Gamma(\alpha)}(b-s)^{\alpha-\beta-2}(s-a)$, $(t,s) \in [a,b]\times[a,b]$;
8.

$$\max_{s\in[a,b]} H(b,s) = H(b,s^*) = \frac{(\alpha-\beta-2)^{\alpha-\beta-2}}{\Gamma(\alpha)}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1},$$

where $s^* = \dfrac{\alpha-\beta-2}{\alpha-\beta-1}a + \dfrac{1}{\alpha-\beta-1}b.$

They obtained the following Lyapunov-type inequalities.

Theorem 28 *If the fractional boundary value problem*

$$\begin{cases} D^\alpha y(t) + q(t)y(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \\ y(a) = y'(a) = 0, \quad (D^{\beta+1}y)(b) = \sum_{i=1}^{m-2} b_i (D^\beta y)(\xi_i), \end{cases}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b (b-s)^{\alpha-\beta-2}\Big[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \sum_{i=1}^{m-2} b_i T(b)(s-a)\Big]|q(s)|ds \geq \Gamma(\alpha).$$

Note that

$$\begin{aligned} &\Gamma(\alpha)\Big[G(b,s) + \sum_{i=1}^{m-2} b_i T(b) H(b,s)\Big] \\ &\leq \Gamma(\alpha)\Big[\max_{s\in[a,b]} G(b,s) + \sum_{i=1}^{m-2} b_i T(b) \max_{s\in[a,b]} H(b,s)\Big] \\ &= \frac{\beta+1}{\alpha-1}\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{(\alpha-\beta-2)/(\beta+1)} (b-a)^{\alpha-1} \\ &\quad + \sum_{i=1}^{m-2} b_i T(b)(\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}. \end{aligned}$$

Thus we have

Corollary 13 *If the fractional boundary value problem*

$$\begin{cases} D^{\alpha}y(t)+q(t)y(t)=0, & a<t<b,\ 2<\alpha\le 3, \\ y(a)=y'(a)=0, & (D^{\beta+1}y)(b)=\displaystyle\sum_{i=1}^{m-2} b_i(D^{\beta}y)(\xi_i), \end{cases}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds$$
$$\ge \frac{\Gamma(\alpha)}{\frac{\beta+1}{\alpha-1}\left(\frac{\alpha-\beta-2}{\alpha-1}\right)^{\frac{\alpha-\beta-2}{\beta+1}}(b-a)^{\alpha-1}+\sum_{i=1}^{m-2} b_i T(b)(\alpha-\beta-2)^{\alpha-\beta-2}\left(\frac{b-a}{\alpha-\beta-1}\right)^{\alpha-\beta-1}}.$$

If $\beta=0$ in Theorem 28 we obtain

Corollary 14 *If the fractional boundary value problem*

$$\begin{cases} D^{\alpha}y(t)+q(t)y(t)=0, & a<t<b,\ 2<\alpha\le 3, \\ y(a)=y'(a)=0, & y'(b)=\displaystyle\sum_{i=1}^{m-2} b_i y(\xi_i), \end{cases}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\int_a^b (b-s)^{\alpha-2}(s-a)|q(s)|ds$$
$$\ge \frac{\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_i T(b)}$$
$$= \frac{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}+\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2}-\sum_{i=1}^{m-2} b_i(\xi_i-a)^{\alpha-\beta-1}+\sum_{i=1}^{m-2} b_i(b-a)^{\alpha-1}}\Gamma(\alpha).$$

If $\beta=0$ in Corollary 13 we have

Corollary 15 *If the fractional boundary value problem*

$$\begin{cases} D^{\alpha}y(t)+q(t)y(t)=0, & a<t<b,\ 2<\alpha\le 3, \\ y(a)=y'(a)=0, & y'(b)=\displaystyle\sum_{i=1}^{m-2} b_i y(\xi_i), \end{cases}$$

has a nontrivial solution, where q is a real and continuous function, then

$$\begin{aligned}
&\int_a^b |q(s)|ds \\
&\geq \frac{\Gamma(\alpha)}{1+\sum_{i=1}^{m-2} b_i T(b)} \cdot \frac{(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}} \\
&= \frac{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1}}{(\alpha-\beta-1)(b-a)^{\alpha-\beta-2} - \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1} + \sum_{i=1}^{m-2} b_i (b-a)^{\alpha-1}} \\
&\quad \times \frac{\Gamma(\alpha)(\alpha-1)^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.
\end{aligned}$$

5 Lyapunov Inequalities for Fractional p-Laplacian Boundary Value Problems

In this section we present Lyapunov-type inequalities for fractional p-Laplacian boundary value problems.

In 2016, Al Arifi et al. [42] considered the nonlinear fractional boundary value problem

$$\begin{cases} D^\beta(\Phi_p((D^\alpha y(t))) + q(t)\Phi_p(y(t)) = 0, \quad a < t < b, \\ y(a) = y'(a) = y'(b) = 0, \;\; D^\alpha y(a) = D^\alpha y(b) = 0, \end{cases} \tag{99}$$

where $2 < \alpha \leq 3$, $1 < \beta \leq 2$, D^α, D^β are the Riemann–Liouville fractional derivatives of orders α and β, respectively, $\Phi_p(s) = |s|^{p-2}s,\ p > 1$ is p-Laplacian operator, and $q : [a, b] \to \mathbb{R}$ is a continuous function.

For $h \in C([a, b], \mathbb{R})$, the linear variant of the problem (99):

$$\begin{cases} D^\beta(\Phi_p((D^\alpha y(t))) + h(t) = 0, \quad a < t < b, \\ y(a) = y'(a) = y'(b) = 0, \;\; D^\alpha y(a) = D^\alpha y(b) = 0, \end{cases} \tag{100}$$

has the unique solution

$$y(t) = \int_a^b G(t,s)\Phi_q\Big(\int_a^b H(s,\tau)h(\tau)d\tau\Big)ds,$$

where

$$H(t,s)=\frac{1}{\Gamma(\beta)}\begin{cases}\left(\dfrac{b-s}{b-a}\right)^{\beta-1}(t-a)^{\beta-1}, & a\le t\le s\le b,\\ \left(\dfrac{b-s}{b-a}\right)^{\beta-1}(t-a)^{\beta-1}-(t-s)^{\beta-1}, & a\le s\le t\le b,\end{cases} \tag{101}$$

and $G(t,s)$ is the Green's function for the boundary value problem

$$\begin{cases}D^{\beta}y(t)+h(t)=0, \quad 2<\alpha\le 3, \quad a<t<b,\\ y(a)=y'(a)=y'(b)=0,\end{cases} \tag{102}$$

which is given by

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases}\left(\dfrac{b-s}{b-a}\right)^{\alpha-2}(t-a)^{\alpha-1}, & a\le t\le s\le b,\\ \left(\dfrac{b-s}{b-a}\right)^{\alpha-2}(t-a)^{\alpha-1}-(t-s)^{\alpha-1}, & a\le s\le t\le b.\end{cases} \tag{103}$$

Observe that the following estimates hold:

(i) $0\le G(t,s)\le G(b,s), \quad (t,s)\in[a,b]\times[a,b]$,
(ii) $0\le H(t,s)\le H(s,s), \quad (t,s)\in[a,b]\times[a,b]$.

For the problem (99), the Lyapunov-type inequality is the following:

Theorem 29 *Let $2<\alpha\le 3$, $1<\beta\le 2$, $p>1$, and $q\in C[a,b]$. If (99) has a nontrivial solution, then*

$$\int_a^b (b-s)^{\beta-1}(s-a)^{\beta-1}|q(s)|ds\ge[\Gamma(\alpha)]^{p-1}\Gamma(\beta)(b-a)^{\beta-1}\left(\int_a^b(b-s)^{\alpha-2}(s-a)ds\right)^{1-p}. \tag{104}$$

Now we present an application of this result to eigenvalue problems.

Corollary 16 *Let λ be an eigenvalue of the problem*

$$\begin{cases}D_0^{\beta}(\Phi_p(D_{0+}^{\alpha}y(t)))+\lambda\Phi_p(y(t))=0, \quad 0<t<1,\\ y(0)=y'(0)=y'(1)=0, \quad D_{0+}^{\alpha}y(0)=D_{0+}^{\alpha}y(1)=0,\end{cases} \tag{105}$$

where $2<\alpha\le 3$, $1<\beta\le 2$, $p>1$, then

$$|\lambda|\ge\frac{\Gamma(2\beta)}{\Gamma(\beta)}\left(\frac{\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(\alpha-1)}\right)^{p-1}.$$

In particular, for $p=2$, that is, for $\Phi_p(y(t))=y(t)$, the bound on λ takes the form:

$$|\lambda|\ge\frac{\Gamma(\alpha)\Gamma(\alpha+1)\Gamma(2\beta)}{\Gamma(\alpha-1)\Gamma(\beta)}.$$

In 2017, Liu et al. [43] considered the nonlinear fractional p-Laplacian boundary value problem of the form:

$$\begin{cases} D^{\beta}(\Phi_p({}^C D^{\alpha} y(t))) - q(t) f(y(t)) = 0, \quad a < t < b, \\ y'(a) = {}^C D^{\alpha} y(a) = 0, \quad y(b) = {}^C D^{\alpha} y(b) = 0, \end{cases} \tag{106}$$

where $1 < \alpha, \beta \leq 2$, ${}^C D^{\alpha}$, D^{β} are the Riemann–Liouville fractional derivatives of orders α and β, respectively, $\Phi_p(s) = |s|^{p-2}s$, $p > 1$, and $q : [a, b] \to \mathbb{R}$ is a continuous function.

An integral equation equivalent to the problem (106) is

$$y(t) = \int_a^b G(t, s)\Phi_q\Big(\int_a^b H(s, \tau) f(y(\tau))d\tau\Big)ds,$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (b - s)^{\alpha-1}, & a \leq t \leq s \leq b, \\ (b - s)^{\alpha-1} - (t - s)^{\alpha-1}, & a \leq s \leq t \leq b, \end{cases}$$

and

$$H(t, s) = \frac{1}{\Gamma(\beta)} \begin{cases} \Big(\dfrac{s - a}{b - a}\Big)^{\beta-1} (b - s)^{\beta-1}, & a \leq t \leq s \leq b, \\ \Big(\dfrac{s - a}{b - a}\Big)^{\beta-1} (b - s)^{\beta-1} - (t - s)^{\beta-1}, & a \leq s \leq t \leq b. \end{cases} \tag{107}$$

Moreover, the following estimates hold:

1. $H(t, s) \geq 0$ for all $a \leq t, s \leq b$;
2. $\max_{t\in[a,b]} H(t, s) = H(s, s), \ s \in [a, b]$;
3. $H(t, s)$ has a unique maximum given by

$$\max_{s\in[a,b]} H(s, s) = \frac{(b - a)^{\beta-1}}{4^{\beta-1}\Gamma(\beta)};$$

4. $0 \leq G(t, s) \leq G(s, s) = \dfrac{1}{\Gamma(\alpha)}(b - s)^{\alpha-1}$ for all $a \leq t, s \leq b$;
5. $G(t, s)$ has a unique maximum given by

$$\max_{s\in[a,b]} G(s, s) = \frac{1}{\Gamma(\alpha)}(b - a)^{\alpha-1}.$$

The Lyapunov-type inequalities for the problem (106) are as follows.

Theorem 30 *Let $p : [a, b] \to \mathbb{R}^+$ be a real Lebesgue function. Suppose that there exists a positive constant M satisfying $0 \leq f(x) \leq M\Phi_p(x)$ for any $x \in \mathbb{R}^+$. If (106) has a nontrivial solution in $E^+ = \{y \in C[a, b], y(t) \geq 0,$ for any $t \in [a, b]$ and $\|y\| \neq 0\}$, then the following Lyapunov inequality holds:*

$$\int_a^b q(s)ds > \frac{4^{\beta-1}\Gamma(\beta)}{M(b-a)^{\beta-1}}\Phi_p\Big(\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\Big). \tag{108}$$

Theorem 31 *Let $p : [a, b] \to \mathbb{R}^+$ be a real Lebesgue function. Assume that $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ is a concave and nondecreasing function. If (106) has a nontrivial solution in $E^+ = \{y \in C[a, b], y(t) \geq 0,$ for any $t \in [a, b]$ and $\|y\| \neq 0\}$, then the following Lyapunov inequality holds:*

$$\int_a^b q(s)ds > \frac{4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))\Phi_p(\eta)}{M(b-a)^{\alpha p+\beta-\alpha-1}f(\eta)}, \tag{109}$$

where $\eta = \max_{t\in[a,b]} y(t)$.

As an application of the foregoing results, we give the following corollary.

Corollary 17 *If $\lambda \in [0, 4^{\beta-1}\Gamma(\beta)\Phi_p(\Gamma(\alpha+1))]$, then the following eigenvalue problem*

$$\begin{cases} D^{\beta}(\Phi_p({}^C D^{\alpha}_{0+}y(t))) - \lambda\Phi_p(y(t)) = 0, \quad 0 < t < 1, \\ y'(0) = ^C D^{\alpha} = 0, \quad y(1) = ^C D^{\alpha}D^{\alpha}y(0) = 0, \end{cases} \tag{110}$$

has no corresponding eigenfunction $y \in E^+$, where $1 < \alpha, \beta \leq 2$, and $p > 1$.

6 Lyapunov Inequalities for Boundary Value Problems with Mixed Fractional Derivatives

In 2017, Guezane-Lakoud et al. [44] obtained a Lyapunov-type inequality for the following problem involving both right Caputo and left Riemann–Liouville fractional derivatives:

$$\begin{cases} -{}^C D^{\alpha}_{b-}D^{\beta}_{a+}y(t) + q(t)y(t), \ t \in [a, b], \\ \\ y(a) = D^{\beta}_{a+}y(b) = 0, \end{cases} \tag{111}$$

where $0 < \alpha, \beta < 1, 1 < \alpha + \beta \leq 2$, ${}^C D^{\alpha}_{b-}$ denotes the right Caputo fractional derivative, D^{β}_{a+} denotes left Riemann–Liouville fractional derivative, and $q : [a, b] \to \mathbb{R}^+$ is a continuous function.

The left and right Riemann–Liouville fractional integrals of order $p > 0$ for a function $g : (0, \infty) \to \mathbb{R}$ are, respectively, defined by

$$I_{a+}^{p} g(t) = \int_a^t \frac{(t-s)^{p-1}}{\Gamma(p)} g(s)ds,$$

$$I_{b-}^{p} g(t) = \int_t^b \frac{(s-t)^{p-1}}{\Gamma(p)} g(s)ds,$$

provided the right-hand sides are point-wise defined on $(0, \infty)$, where Γ is the Gamma function.

The left Riemann–Liouville fractional derivative and the right Caputo fractional derivative of order $p > 0$ for a continuous function $g : (0, \infty) \to \mathbb{R}$ are, respectively, given by

$$D_{a+}^{p} g(t) = \frac{d^n}{dt^n} (I_{a+}^{n-p})(t),$$

$${}^{c}D_{b-}^{p} g(t) = (-1)^n I_{b-}^{n-p} g^{(n)}(t),$$

where $n - 1 < p < n$.

The function $y \in C[a, b]$ is a solution to the problem (111) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t, s) q(s) f(y(s)) ds,$$

where $G(t, s)$ is the Green's function defined by

$$G(t, s) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \begin{cases} \int_a^r (t-s)^{\beta-1}(r-s)^{\alpha-12}, & a \le r \le t \le b, \\ \int_a^t (t-s)^{\beta-1}(r-s)^{\alpha-1}, & a \le t \le s \le b. \end{cases} \quad (112)$$

The above Green's function satisfies the following properties:

1. $G(t, s) \ge 0$, for all $a \le r \le t \le b$;
2. $\max_{t \in [a,b]} G(t, r) = G(r, r)$ for all $r \in [a, b]$;
3. $\max\limits_{r \in [a,b]} G(r, r) = \dfrac{(b-a)^{\alpha+\beta-1}}{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}$.

The following result describes the Lyapunov inequality for problem (111).

Theorem 32 *Assume that* $0 < \alpha, \beta < 1$ *and* $1 < \alpha + \beta \leq 2$. *If the fractional boundary value problem (111) has a nontrivial continuous solution, then*

$$\int_a^b |q(r)|dr \geq \frac{(\alpha+\beta-1)\Gamma(\alpha)\Gamma(\beta)}{(b-a)^{\alpha+\beta-1}}.$$

7 Lyapunov Inequality for Hadamard Type Fractional Boundary Value Problems

Let us begin this section with some fundamental definitions.

Definition 4 ([14]) The Hadamard derivative of fractional order q for a function $g : [1, \infty) \to \mathbb{R}$ is defined as

$${}^H D^q g(t) = \frac{1}{\Gamma(n-q)} \left(t\frac{d}{dt}\right)^n \int_1^t \left(\log\frac{t}{s}\right)^{n-q-1} \frac{g(s)}{s} ds, \quad n-1 < q < n, \ n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$ is the usual Napier logarithm.

Definition 5 ([14]) The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)} \int_1^t \left(\log\frac{t}{s}\right)^{q-1} \frac{g(s)}{s} ds, \quad q > 0,$$

provided the integral exists.

In 2017, Ma et al. [45] obtained a Lyapunov-type inequality for a *Hadamard fractional boundary value problem*

$$\begin{cases} {}^H D^\alpha y(t) - q(t)y(t) = 0, & 1 < t < e, \\ y(1) = y(e) = 0, \end{cases} \tag{113}$$

where ${}^H D^\alpha$ is the fractional derivative in the sense of the Hadamard of order $1 < \alpha \leq 2$ and $q : [a, b] \to \mathbb{R}$ is a continuous function.

The function $y \in C([1, e], \mathbb{R})$ is a solution of the boundary value problem (113) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{114}$$

where $G(t, s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -\Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{(\log t)^{\alpha-1}}{s}, & 1 \le t \le s \le e, \\ -\Big(\log\frac{e}{s}\Big)^{\alpha-1}\frac{(\log t)^{\alpha-1}}{s} + \Big(\log\frac{t}{s}\Big)^{\alpha-1}\frac{1}{s}, & 1 \le s \le t \le e. \end{cases} \tag{115}$$

such that

$$|G(t,s)| \le \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(1-\lambda)^{\alpha-1}\exp(-\lambda),$$

with

$$\lambda = \frac{1}{2}\Big(2\alpha - 1 - \sqrt{(2\alpha-2)^2+1}\Big). \tag{116}$$

The result concerning the Lyapunov-type inequality for the problem (113) is as follows.

Theorem 33 *If there exists a nontrivial continuous solution of the fractional boundary value problem (113), then*

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\lambda^{1-\alpha}.(1-\lambda)^{1-\alpha}\exp\lambda, \tag{117}$$

where λ is defined by (116).

For recent results on Hadamard type fractional boundary value problems, we refer the interested reader to the book [46].

8 Lyapunov Inequality for Boundary Value Problems with Prabhakar Fractional Derivative

In [47], the authors discussed Lyapunov-type inequality for the following fractional boundary value problem involving the k-Prabhakar derivative:

$$\begin{cases} ({}_kD^{\gamma}_{\rho,\beta,\omega,a+}y)(t) + q(t)f(y(t)) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{118}$$

where ${}_kD^{\gamma}_{\rho,\beta,\omega,a+}$ is the k-Prabhakar differential operator of order $\beta \in (1,2]$, $k \in \mathbb{R}^+$ and $\rho, \gamma, \omega \in \mathbb{C}$. The k-Prabhakar integral operator is defined as

$$({}_kP_{\alpha,\beta,\omega}\phi)(t)=\int_0^x\frac{(x-t)^{\frac{\beta}{k}-1}}{k}E_{k,\alpha,\beta}[\omega(x-t)^{\frac{\alpha}{k}}]\phi(t)dt,\quad x>0,$$

where $E_{k,\alpha,\beta}$ is the k-Mittag–Leffler function given by

$$E^{\gamma}_{k,\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{n,k}z^n}{\Gamma_k(\alpha n+\beta)n!},$$

$\Gamma_k(x)$ is the k-Gamma function $\Gamma_k(x)=\lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}}$ and $(\gamma)_{n,k}=\frac{\Gamma_k(\gamma+nk)}{\Gamma_k(\gamma)}$ is the Pochhammer k-symbol.

The k-Prabhakar derivative is defined as

$${}_kD^{\gamma}_{\rho,\beta,\omega}f(x)=\left(\frac{d}{dx}\right)^m k^m_k P^{-\gamma}_{\rho,mk-\beta,\omega}f(x),$$

where $m=[\beta/k]+1$.

An integral equation related to the problem (118) is

$$y(t)=\int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s)=\begin{cases}\dfrac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})(b-s)^{\frac{\beta}{k}-1}}{(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})k}E^{\gamma}_{k,\rho,\beta}(\omega(b-s)^{\frac{\rho}{k}}), & a\le t\le s\le b,\\ \dfrac{(t-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(t-a)^{\frac{\rho}{k}})(b-s)^{\frac{\beta}{k}-1}}{(b-a)^{\frac{\beta}{k}-1}E^{\gamma}_{k,\rho,\beta}(\omega(b-a)^{\frac{\rho}{k}})k}E^{\gamma}_{k,\rho,\beta}(\omega(b-s)^{\frac{\rho}{k}} & \\ -\frac{(t-s)^{\frac{\beta}{k}-1}}{k}E^{\gamma}_{k,\rho,\beta}(\omega(t-s)^{\frac{\rho}{k}}), & a\le s\le t\le b,\end{cases}\tag{119}$$

which satisfies the following properties:

1. $G(t,s)\ge 0$ for all $a\le t,s\le b$;
2. $\max_{t\in[a,b]}G(t,s)=G(s,s)$ for all $s\in[a,b]$;
3. $G(t,s)$ has a unique maximum given by

$$\max_{s\in[a,b]}G(s,s)=\left(\frac{b-a}{4}\right)^{\frac{\beta}{k}-1}\frac{E^{\gamma}_{k,\rho,\beta}\Big(\omega\Big(\frac{b-a}{2}\Big)^{\frac{\rho}{k}}\Big)E^{\gamma}_{k,\rho,\beta}\Big(\omega\Big(\frac{b-a}{2}\Big)^{\frac{\rho}{k}}\Big)}{kE^{\gamma}_{k,\rho,\beta}\Big(\omega(b-a)^{\frac{\rho}{k}}\Big)}.$$

The Lyapunov-type inequality for the problem (118) is given in the following result.

Theorem 34 *If the problem (118) has a nontrivial solution, then*

$$\int_a^b |q(s)|ds \geq \left(\frac{4}{b-a}\right)^{\frac{\beta}{k}-1} \frac{kE^{\gamma}_{k,\rho,\beta}\left(\omega(b-a)^{\frac{\rho}{k}}\right)}{E^{\gamma}_{k,\rho,\beta}\left(\omega\left(\frac{b-a}{2}\right)^{\frac{\rho}{k}}\right)E^{\gamma}_{k,\rho,\beta}\left(\omega\left(\frac{b-a}{2}\right)^{\frac{\rho}{k}}\right)}.$$

The special case $k = 1$ for the problem (118) has recently been studied in [48].

In 2017, Pachpatte et al. [49] established some Lyapunov-type inequalities for the following *hybrid fractional boundary value problem*

$$\begin{cases} D^{\gamma}_{\rho,\beta,\omega}\left[\dfrac{y(t)}{f(t,y(t))} - \displaystyle\sum_{i=1}^{n} E^{\gamma}_{\rho,\beta,\omega} h_i(t,y(t))\right] + g(t)y(t) = 0, \quad t \in (a,b), \\ y(a) = y(b) = 0, \end{cases} \tag{120}$$

where $D^{\gamma}_{\rho,\beta,\omega}$ denotes the Prabhakar fractional derivative of order $\beta \in (1,2]$ starting from a point a, $y \in C([a,b],\mathbb{R})$, $g \in L^1((a,b],\mathbb{R})$, $f \in C^1([a,b]\times\mathbb{R},\mathbb{R}\setminus\{0\})$, $h_i \in C([a,b]\times\mathbb{R},\mathbb{R})$, $\forall i = 1,2,\ldots,n$ and $E^{\alpha}_{\rho,\mu,\omega}$ is the Prabhakar fractional integral of order μ with the lower limit at the point a.

We consider two cases: (I) $h_i = 0, i = 1,2,\ldots,n$ and (II) $h_i \neq 0, i = 1,2,\ldots,n$.

Case I $h_i = 0, i = 1,2,\ldots,n$. We consider the problem (120) with $h_i(t,\cdot) = 0$ for all $t \in [a,b]$. For $\alpha \in (2,3]$, we first construct a Green's function for the following boundary value problem

$$\begin{cases} D^{\alpha}_{\rho,\mu,\omega}\left[\dfrac{y(t)}{f(t,y(t))}\right] + g(t)y(t) = 0, \quad t \in (a,b), \\ y(a) = y(b) = 0, \end{cases} \tag{121}$$

with the assumption that f is continuously differentiable and $f(t,y(t)) \neq 0$ for all $t \in [a,b]$. Let $y \in AC([a,b],\mathbb{R})$ be a solution of the problem (121). Then the function y satisfies the following integral equation:

$$y = f(t,y)\int_a^b G(t,s)g(s)y(s)ds, \tag{122}$$

where $G(t,s)$ is the Green's function defined by (119). The Lyapunov-type inequality for this case is as follows.

Theorem 35 *If the problem (121) has a nontrivial solution, then*

$$\int_a^b |q(s)|ds \geq \frac{1}{\|f\|}\Big(\frac{4}{b-a}\Big)^{\beta-1}\frac{E^{\gamma}_{\rho,\beta}\Big(\omega(b-a)^{\rho}\Big)}{E^{\gamma}_{\rho,\beta}\Big(\omega\Big(\frac{b-a}{2}\Big)^{\rho}\Big)E^{\gamma}_{\rho,\beta}\Big(\omega\Big(\frac{b-a}{2}\Big)^{\rho}\Big)},$$

where $\|f\| = \sup_{t\in[a,b],y\in\mathbb{R}} |f(t,y)|$.

Case II $h_i \neq 0, i = 1, 2, \ldots, n$. Let $y \in AC[a,b]$ be a solution of the problem (120) given by

$$y(t) = f(t, y(t))\int_a^b G(t,s)g\Big[(s)y(s) - \sum_{i=1}^{n}\int_a^b h_i(s, y(s))\Big]ds, \tag{123}$$

where $G(t,s)$ is defined as in (119).

Theorem 36 (Lyapunov-Type Inequality) *Assume that* $|q(t)y(t) - \sum_{i=1}^{n} h_i(t, y(t))| \leq K|qt)|\|y\|$, $K \in \mathbb{R}$. *If a nontrivial solution for the problem* (120) *exists, then*

$$\int_a^b |q(s)|ds \geq \frac{1}{K\|f\|}\Big(\frac{4}{b-a}\Big)^{\beta-1}\frac{E^{\gamma}_{\rho,\beta}\Big(\omega(b-a)^{\rho}\Big)}{E^{\gamma}_{\rho,\beta}\Big(\omega\Big(\frac{b-a}{2}\Big)^{\rho}\Big)E^{\gamma}_{\rho,\beta}\Big(\omega\Big(\frac{b-a}{2}\Big)^{\rho}\Big)}.$$

9 Lyapunov Inequality for q-Difference Boundary Value Problems

Let a q-real number denoted by $[a]_q$ be defined by

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}, \ q \in \mathbb{R}^+ \setminus \{1\}.$$

The q-analogue of the Pochhammer symbol (q-shifted factorial) is defined as

$$(a;q)_0 = 1, \quad (a;q)_k = \prod_{i=0}^{k-1}(1-aq^i), \ k \in \mathbb{N} \cup \{\infty\}.$$

The q-analogue of the exponent $(x-y)^k$ is

$$(x-y)_a^{(0)} = 1, \quad (x-y)_a^{(k)} = \prod_{j=0}^{k-1}((x-a)-(y-a)q^j), \ k \in \mathbb{N}, \ x, y \in \mathbb{R}.$$

More generally, if $\gamma \in \mathbb{R}$, then

$$(x-y)_a^{(\gamma)} = (x-a)^\gamma \prod_{i=0}^{\infty} \frac{(x-a)-q^i(y-a)}{(x-a)-q^{\gamma+1}(y-a)}.$$

The q-Gamma function $\Gamma_q(y)$ is defined as

$$\Gamma_q(y) = \frac{(1-q)_0^{(y-1)}}{(1-q)^{y-1}},$$

where $y \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}$. Observe that $\Gamma_q(y+1) = [y]_q \Gamma_q(y)$.

The q-derivative of a function $f : [a; b] \to \mathbb{R}(a < b)$ is defined by

$$({}_aD_q f)(t) = \frac{f(t) - f(qt + (1-q)a}{(1-q)(t-a)}, \quad t \neq a$$

and

$$({}_aD_q f)(a) = \lim_{t \to a} ({}_aD_q f)(t).$$

In 2016, Jleli and Samet [50] established a Lyapunov-type inequality for a Dirichlet boundary value problem of fractional q-difference equations given by

$$\begin{cases} ({}_aD^\alpha y)(t) + \phi(t)y(t) = 0, & a < t < b, \; q \in [0, 1), \; 1 < \alpha \leq 2, \\ y(a) = y(b) = 0, \end{cases} \tag{124}$$

where ${}_aD^\alpha$ denotes the fractional q-derivative of Riemann–Liouville type and $\phi : [a, b] \to \mathbb{R}$ is a continuous function.

The solution $y \in C([a, b], \mathbb{R})$ of the problem (124) satisfies the integral equation

$$y(t) = \int_a^t G_1(t, qs + (1-q)a)\phi(s)y(s)\, {}_ad_q s + \int_t^b G_2(t, s)\phi(s)y(s)\, {}_ad_q s, \quad a \leq t \leq b,$$

where

$$G_1(t, s) = \frac{1}{\Gamma_q(\alpha)} \Big(\frac{(t-a)^{\alpha-1}}{(b-a)^{\alpha-1}} (b-s)_a^{(\alpha-1)} - (t-s)_a^{(\alpha-1)} \Big), \; a \leq s \leq t \leq b,$$

$$G_2(t, s) = \frac{(t-a)^{\alpha-1}}{\Gamma_q(\beta)(b-a)^{\alpha-1}} (b - (qs + (1-q)a)_a^{(\alpha-1)}, \; a \leq t \leq s \leq b,$$

satisfying the relations

1. $0 \leq G_1(t, qs + (1-q)a) \leq G_2(s, s), \quad a < s \leq t \leq b;$
2. $G_2(a, s) = 0 \leq G_2(t, s) \leq G_2(s, s), \quad a \leq t \leq s \leq b.$

Theorem 37 (Lyapunov-Type Inequality) *If the problem (124) has a nontrivial solution, then*

$$\int_a^b (s-a)^{\alpha-1}(b-(qs+(1-q)a))_a^{(\alpha-1)}|\phi(s)|\,{}_ad_qs \geq \Gamma_q(\alpha)(b-a)^{\alpha-1}.$$

Taking $\alpha = 2$ in the above theorem we have the following corollary.

Corollary 18 *If a nontrivial continuous solution to the q-difference boundary value problem,*

$$\begin{cases} ({}_aD_q^2 y)(t) + \phi(t)y(t) = 0, & a < t < b,\ q \in (0,1), \\ y(a) = y(b) = 0, \end{cases} \tag{125}$$

exists, where $\phi : [a,b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b (s-a)(b-(qs+(1-q)a))|\phi(s)|\,{}_ad_qs \geq (b-a).$$

Some recent work on q-difference boundary value problems can be found in [51].

10 Lyapunov Inequality for Boundary Value Problems Involving a Fractional Derivative with Respect to a Certain Function

In 2017, Jleli et al. [52] considered the following fractional boundary value problem involving a fractional derivative with respect to a certain function g

$$\begin{cases} (D_{a,g}^{\alpha} y)(t) + q(t)y(t) = 0, & a < t < b,\ 1 < \alpha \leq 2, \\ y(a) = y(b) = 0, \end{cases} \tag{126}$$

where $D_{a,g}^{\alpha}$ is the fractional derivative operator of order α with respect to a nondecreasing function $g \in C^1([a,b],\mathbb{R})$ with $g'(x) > 0$, for all $x \in [a,b]$, and $q : [a,b] \to \mathbb{R}$ is a continuous function.

Let $f \in L^1((a,b),\mathbb{R})$. The fractional integral of order $\alpha > 0$ of f with respect to the function g is defined by

$$(I_{a,g}^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s)f(s)}{(g(t)-g(s))^{1-\alpha}} ds, \text{ a.e. } t \in [a,b].$$

Let $\alpha > 0$ and n be the smallest integer greater than or equal to α. Let $f : [a,b] \to \mathbb{R}$ be a function such that $((1/g'(t))(d/dt)^n I_{a,g}^{n-\alpha} f$ exists almost everywhere on

$[a, b]$. Then the fractional derivative of order α of f with respect to the function g is defined as

$$D^{\alpha}_{a,g} f(t) = \Big(\frac{1}{g'(t)}\frac{t}{dt}\Big)^n I^{n-\alpha}_{a,g} f(t)$$
$$= \frac{1}{\Gamma(n-\alpha)}\Big(\frac{1}{g'(t)}\frac{t}{dt}\Big)^n \int_a^t \frac{g'(s)f(s)}{(g(t)-g(s))^{\alpha-n+1}} ds, \text{ for a.e } t \in [a, b].$$

Theorem 38 *Assume that $q \in C([a, b], \mathbb{R})$ and $g \in C^1([a, b], \mathbb{R})$ be a nondecreasing function with $g'(x) > 0$, for all $x \in [a, b]$. If the problem (126) has a nontrivial solution, then*

$$\int_a^b [(g(s) - g(a)((g(b) - g(s))]^{\alpha-1} g'(s)|q(s)|ds \geq \Gamma(\alpha)(g(b) - g(a))^{\alpha-1}. \tag{127}$$

From inequality (127), we can obtain Lyapunov-type inequalities for different choices of the function g. For instance, for $g(x) = x^{\beta}, x \in [a, b]$ and $g(x) = \log x, x \in [a, b]$ we have, respectively, the following results.

Corollary 19 *If the problem (126) has a nontrivial solution and $g(x) = x^{\beta}, x \in [a, b], 0 < a < b$, then*

$$\int_a^b |q(s)|ds \geq \frac{\Gamma(\alpha)(b^{\beta} - a^{\beta})^{\alpha-1}}{\beta\phi_{\alpha,\beta}(s^*(\alpha, \beta))},$$

where $\phi_{\alpha,\beta}(s^*(\alpha, \beta)) = \max\{\phi_{\alpha,\beta}(s) : s \in [a, b]\} > 0$.

Taking $g(x) = \log x, x \in [a, b], 0 < a < b$, in Theorem 38, we deduce the following Hartman–Wintner-type inequality

$$\int_a^b \Big[\Big(\log\frac{s}{a}\Big)\Big(\log\frac{b}{s}\Big)\Big]^{\alpha-1}\frac{|q(s)|}{s} ds \geq \Gamma(\alpha)\Big(\log\frac{b}{a}\Big)^{\alpha-1},$$

for the Hadamard fractional boundary value problem of the form:

$$\begin{cases} ({}^H D^{\alpha}_a y)(t) + q(t)y(t) = 0, \quad a < t < b, \ 1 < \alpha \leq 2, \\ y(a) = y(b) = 0. \end{cases} \tag{128}$$

In order to demonstrate the application of Theorem 38, we consider the eigenvalue problem

$$\begin{cases} D^{\alpha}_{a,g} y(t) + \lambda y(t) = 0, \quad a < t < b, \\ y(a) = y(b) = 0, \end{cases} \tag{129}$$

and use the Lyapunov-type inequality (127) to obtain the following result.

Theorem 39 *If λ is an eigenvalue of fractional boundary value problem (129), then the following inequality holds:*

$$|\lambda| \geq \frac{\Gamma(\alpha)(g(b)-g(a))^{\alpha-1}}{\int_{g(a)}^{g(b)}(x-g(a))^{\alpha-1}(g(b)-x)^{\alpha-1}dx}.$$

11 Lyapunov Inequality for Boundary Value Problems Involving Left and Right Derivatives

The left and right Caputo fractional derivatives are defined via the Riemann–Liouville fractional derivatives (see [14, p. 91]). In particular, they are defined for a class of absolutely continuous functions.

Definition 6 (Left and Right Riemann–Liouville Fractional Integrals [14]) Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional integrals of order γ for function f denoted by I_{a+}^{γ} and I_{b-}^{γ}, respectively, are defined by

$$I_{a+}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_a^t (t-s)^{\gamma-1} f(s)ds,\ t \in [a,b],\ \gamma > 0,$$

and

$$I_{b-}^{\gamma} f(t) = \frac{1}{\Gamma(\gamma)} \int_t^b (t-s)^{\gamma-1} f(s)ds,\ t \in [a,b],\ \gamma > 0,$$

provided the right-hand sides are point-wise defined on $[a, b]$, where $\Gamma > 0$ is the gamma function.

Definition 7 (Left and Right Riemann–Liouville Fractional Derivatives [14]) Let f be a function defined on $[a, b]$. The left and right Riemann–Liouville fractional derivatives of order γ for function f denoted by ${}_aD_t^{\gamma} f(t)$ and ${}_tD_b^{\gamma} f(t)$, respectively, are defined by

$${}_aD_t^{\gamma} f(t) = \frac{d^n}{dt^n}\,{}_aD_t^{\gamma-n} f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n}\Big(\int_a^t (t-s)^{n-\gamma-1} f(s)ds\Big)$$

and

$${}_tD_b^{\gamma} f(t)=(-1)^n \frac{d^n}{dt^n}\,{}_tD_b^{\gamma-n} f(t) = (-1)^n \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n}\Big(\int_t^b (t-s)^{n-\gamma-1} f(s)ds\Big)$$

where $t \in [a, b]$, $n - 1 \leq \gamma < n$ and $n \in \mathbb{N}$.

In 2017, Chen et al. [53] obtained a Lyapunov-type inequality for the following boundary value problem

$$\begin{cases} \frac{d}{dt}\Big(\frac{1}{2}I_{a+}^{\beta}y'(t)+\frac{1}{2}I_{b-}^{\beta}y'(t)\Big)+q(t)y(t)=0, \quad a<t<b, \\ y(a)=y(b)=0, \end{cases} \tag{130}$$

where I_{a+}^{β} and I_{b-}^{β} denote the left and right Riemann–Liouville fractional integrals of order $0 \le \beta < 1$,, respectively, and $q \in L^1([a,b],\mathbb{R})$.

Theorem 40 *Let $q \in L^1([a,b],\mathbb{R})$ be a nonnegative function and there exists a nontrivial solution for the boundary value problem (130). Then*

$$\int_a^b |q(s)|dt \ge \left(\frac{2(b-a)^{\alpha-1/2}}{\Gamma(\alpha)(2\alpha-1)^{1/2}}\frac{1}{|\cos(\pi\alpha)|^{1/2}}\right)^{-2}, \quad \alpha = 1-\frac{\beta}{2}.$$

Jleli et al. [54] considered the following quasilinear problem involving both left and right Riemann–Liouville fractional derivative operators:

$$\begin{cases} \frac{1}{2}\Big({}_tD_b^{\alpha}(|{}_aD_t^{\alpha}|_a^{p-2}D_t^{\alpha}y)+{}_a\,D_t^{\alpha}(|{}_tD_b^{\alpha}y|_t^{p-2}D_b^{\alpha}y)\Big)=q(t)|y|^{p-2}y, \quad a<t<b, \\ y(a)=y(b)=0, \end{cases} \tag{131}$$

where ${}_aD_t^{\alpha}$ and ${}_tD_b^{\alpha}$ denote the left Riemann–Liouville fractional derivative of order $\alpha \in (0,1)$ and the right Riemann–Liouville fractional derivative of order α, respectively. Note that for $\alpha = 1$, problem (131) reduces to the p-Laplacian problem

$$\begin{cases} (|y'|^{p-2}y)'+q(t)|y|^{p-2}y=0, \quad a<t<b, \quad p>1, \\ y(a)=y(b)=0, \end{cases} \tag{132}$$

The Lyapunov-type inequality for the problem (131) is given in the following theorem.

Theorem 41 *Assume that $0\frac{1}{p} < \alpha < 1$ and $q \in L^1(a,b)$. If problem (131) admits a nontrivial solution $y \in AC_a^{\alpha,p}[a,b] \cap AC_b^{\alpha,p}[a,b] \cap C[a,b]$ such that $|y(c)| = \|y\|_{\infty}$, $c \in (a,b)$, then*

$$\int_a^b q^+(s)ds \ge \left(\frac{2(\alpha p-1)}{p-1}\right)^{p-1}\frac{[\Gamma(\alpha)]^p}{\Big((c-a)^{\frac{\alpha p-1}{p-1}}+(b-c)^{\frac{\alpha p-1}{p-1}}\Big)^{p-1}},$$

where $q^+(t)=\max\{q(t),0\}$ for $t \in [a,b]$, $AC_a^{\alpha,p}[a,b]=\{y \in L^1(a,b) :_a D_t^{\alpha}y \in L^p(a,b)\}$ and $AC_b^{\alpha,p}[a,b]=\{y \in L^1(a,b) :_t D_b^{\alpha}y \in L^p(a,b)\}$.

12 Lyapunov Inequality for Boundary Value Problems with Nonsingular Mittag–Leffler Kernel

In 2017, Abdeljawad [55] proved a Lyapunov-type inequality for the following Riemann–Liouville type fractional boundary value problem of order $2 < \alpha \leq 3$ in terms of Mittag–Leffler kernels:

$$\begin{cases} ({}_a^{ABR}D^{\alpha} y)(t) + q(t)y(t) = 0, \quad a < t < b,\ 2 < \alpha \leq 3, \\ y(a) = y(b) = 0, \end{cases} \tag{133}$$

where ${}_a^{ABR}D^{\alpha}$ denotes the left Riemann–Liouville fractional derivative defined by

$$({}_a^{ABR}D^{\alpha} f)(t) = \frac{B(\alpha)}{1-\alpha}\frac{d}{dt}\int_a^t f(x)E_{\alpha}\Big(-\alpha\frac{(t-x)^{\alpha}}{1-\alpha}\Big)ds,$$

where $B(\alpha)$ is a normalization function such that $B(0) = B(1) = 1$, and E_{α} is the generalized Mittag–Leffler function given by $E_{\alpha}(-t^{\alpha}) = \sum_{k=0}^{\infty} \frac{(-t)^{\alpha k}}{\Gamma(\alpha k + 1)}$.

The integral equation equivalent to the boundary value problem (133) is

$$y(t) = \int_a^b G(t,s)R(t, y(s))ds,$$

where

$$G(t,s) = \begin{cases} \dfrac{(t-a)(b-s)}{b-a}, & a \leq t \leq s \leq b, \\ \dfrac{(t-a)(b-s)}{b-a} - (t-s), & a \leq s \leq t \leq b, \end{cases}$$

and

$$R(t, y(t)) = \frac{1-\beta}{B(\beta)}q(t)y(t) + \frac{\beta}{B(\beta)}\big({}_aI^{\beta}q(\cdot)y(\cdot)\big)(t), \quad \beta = \alpha - 2.$$

The Green's function $G(t,s)$ defined above has the following properties:

1. $G(t,s) \geq 0$ for all $a \leq t, s \leq b$;
2. $\max_{t\in[a,b]} G(t,s) = G(s,s)$ for $s \in [a,b]$;
3. $G(s,s)$ has a unique maximum, given by

$$\max_{s\in[a,b]} G(s,s) = G\Big(\frac{a+b}{2}, \frac{a+b}{2}\Big) = \frac{b-a}{4}.$$

The Lyapunov inequality for the problem (133) is given in the following result.

Theorem 42 *If the boundary value problem (133) has a nontrivial solution, where q is a real-valued continuous function on $[a, b]$, then*

$$\int_a^b \Big[\frac{3-\alpha}{B(\alpha-2)}|q(t)| + \frac{\alpha-2}{B(\alpha-2)}({}_aI^{\alpha-2}|q(\cdot)|)(t)\Big]ds > \frac{4}{b-a}. \tag{134}$$

Remark 1 For $\alpha \to 2+$, notice that $\frac{3-\alpha}{B(\alpha-2)}|q(t)|+\frac{\alpha-2}{B(\alpha-2)}({}_aI^{\alpha-2}|q(\cdot)|)(t) \to |q(t)|$ and hence the inequality (134) reduces to the classical Lyapunov inequality (6).

13 Lyapunov Inequality for Discrete Fractional Boundary Value Problems

Let us begin this section with the definitions of integral and derivative of arbitrary order for a function defined on a discrete set. For details, see [56].

The power function is defined by

$$x^{(y)} = \frac{\Gamma(x+1)}{\Gamma(x+1-y)}, \text{ for } x, x-y \in \mathbb{R} \setminus \{\ldots, -2, -1\}.$$

For $a \in \mathbb{R}$, we define the set $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$. Also, we use the notation $\sigma(s) = s+1$ for the shift operator and $(\Delta f)(t) = f(t+1) - f(t)$ for the forward difference operator. Notice that $(\Delta^2 f)(t) = (\Delta\Delta f)(t)$.

For a function $f : \mathbb{N}_a \to \mathbb{R}$, the discrete fractional sum of order $\alpha \geq 0$ is defined as

$$({}_a\Delta^0 f)(t) = f(t), \quad t \in \mathbb{N}_a,$$
$$({}_a\Delta^{-\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)}, f(s), \; t \in \mathbb{N}a+\alpha, \; \alpha > 0.$$

The discrete fractional derivative of order $\alpha \in (1, 2]$ is defined by

$$({}_a\Delta^\alpha f)(t) = (\Delta^2 {}_a\Delta^{-(2-\alpha)} f)(t), \quad t \in \mathbb{N}_{a+2-\alpha}.$$

In 2015, Ferreira [57] studied the following conjugate boundary value problem

$$\begin{cases} (\Delta^\alpha y)(t) + q(t+\alpha-1)y(t+\alpha-1) = 0, & t \in [0, b+1]_{\mathbb{N}_0}, \\ y(\alpha-2) = 0 = y(\alpha+b+1). \end{cases} \tag{135}$$

The function y is a solution of the boundary value problem (135) if and only if y satisfies the integral equation

$$y(t) = \sum_{0}^{b+1} G(t,s)q(s+\alpha-1)f(y(s+\alpha-1)), \tag{136}$$

where $G(t, s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{t^{(\alpha-1)}(\alpha+b-\sigma(s))^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}} \\ -(t-\sigma(s))^{(\alpha-1)}, & s < t-\alpha+1 < b+1, \\ \dfrac{t^{(\alpha-1)}(\alpha+b-\sigma(s))^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}}, & t-\alpha+1 \le s \le b+1, \end{cases} \tag{137}$$

and that

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s,s) = G\Big(\frac{b}{2}+\alpha-1, \frac{b}{2}\Big), \text{ if } b \text{ is even,}$$

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s,s) = G\Big(\frac{b+1}{2}+\alpha-1, \frac{b+1}{2}\Big), \text{ if } b \text{ is odd,}$$

The Lyapunov inequality for the problem (135) is as follows.

Theorem 43 *If the discrete fractional boundary value problem (135) has a nontrivial solution, then*

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha) \frac{\Gamma(b+\alpha+2)\Gamma^2\big(\frac{b}{2}+2\big)}{(b+2\alpha)(b+2)\Gamma^2\big(\frac{b}{2}+\alpha\big)\Gamma(b+3)}, \text{ if } b \text{ is even};$$

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha) \frac{\Gamma(b+\alpha+2)\Gamma^2\big(\frac{b+3}{2}\big)}{\Gamma(b+3)(\Gamma^2\big(\frac{b+1}{2}+\alpha\big)}, \text{ if } b \text{ is odd.}$$

As a simple application, consider the right-focal boundary value problem in Theorem 43 with $q = \lambda \in \mathbb{R}$. Then an eigenvalue of the boundary value problem

$$\begin{cases} (\Delta^{\alpha} y)(t) + \lambda y(t+\alpha-1) = 0, & t \in [0, b+1]_{\mathbb{N}_0}, \\ y(\alpha-2) = 0 = \Delta y(\alpha+b), \end{cases} \tag{138}$$

must necessarily satisfy the following inequality

$$|\lambda| \ge \frac{1}{\Gamma(\alpha-1)(b+2)^2}.$$

Ferreira [57] also studied the following right-focal boundary value problem

$$\begin{cases} (\Delta^{\alpha} y)(t) + q(t+\alpha-1)y(t+\alpha-1) = 0, & t \in [0, b+1]_{\mathbb{N}_0}, \\ y(\alpha-2) = 0 = \Delta y(\alpha+b). \end{cases} \tag{139}$$

The function y is a solution of the boundary value problem (139) if and only if y satisfies the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds, \tag{140}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{\Gamma(b+3)t^{(\alpha-1)}(\alpha+b-\sigma(s))^{(\alpha-2)}}{\Gamma(\alpha+b+1)} \\ -(t-\sigma(s))^{(\alpha-1)}, & s < t-\alpha+1 < b+1, \\ \dfrac{\Gamma(b+3)t^{(\alpha-1)}(\alpha+b-\sigma(s))^{(\alpha-2)}}{\Gamma(\alpha+b+1)}, & t-\alpha+1 \le s \le b+1, \end{cases} \tag{141}$$

with

$$\max_{s\in[0,b+1]_{\mathbb{N}_0}} G(s+\alpha-1,s) = G(b+\alpha, b+1) = \Gamma(\alpha-1)(b+2).$$

The Lyapunov inequality for the problem (139) is presented as follows.

Theorem 44 *If the discrete fractional boundary value problem (139) has a nontrivial solution, then*

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > \frac{1}{\Gamma(\alpha-1)(b+2)}.$$

In 2017, Chidouh and Torres [58] studied the following conjugate boundary value problem

$$\begin{cases} (\Delta^{\alpha} y)(t) + q(t+\alpha-1)f(y(t+\alpha-1)) = 0, & t \in [0, b+1]_{\mathbb{N}_0}, \\ y(\alpha-2) = 0 = y(\alpha+b+1), \end{cases} \tag{142}$$

where $f \in C(\mathbb{R}^+, \mathbb{R}^+)$ is nondecreasing and $q : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}^+$ is a nontrivial function.

The function y is a solution of the boundary value problem (142) if and only if y satisfies the integral equation

$$y(t) = \sum_{0}^{b+1} G(t,s)q(s+\alpha-1)f(y(s+\alpha-1)), \tag{143}$$

where $G(t,s)$ is the Green's function defined by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{t^{(\alpha-1)}(\alpha+b-s)^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}} - (t-s-1)^{(\alpha-1)}, & s < t-\alpha+1 < b+1, \\ \dfrac{t^{(\alpha-1)}(\alpha+b-s)^{(\alpha-1)}}{(\alpha+b+1)^{(\alpha-1)}}, & t-\alpha+1 \le s \le b+1. \end{cases} \tag{144}$$

Moreover, the function G satisfies the following properties:

1. $G(t,s) > 0$ for all $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_0}$ and $s \in [1, b+1]_{\mathbb{N}_1}$;
2. $\max_{[\alpha-1,\alpha+b]_{\mathbb{N}_0}} G(t,s) = G(s+\alpha-1, s),\ \ s \in [1, b+1]_{\mathbb{N}_1}$;
3. $G(s+\alpha-1)$ has a unique maximum given by

$$\max_{s\in[1,b+1]_{\mathbb{N}_1}} G(s+\alpha-1) = \begin{cases} \dfrac{1}{4}\dfrac{(b+2\alpha)(b+2)\Gamma^2\left(\frac{b}{2}+\alpha\right)\Gamma(b+3)}{\Gamma(\alpha)\Gamma(b+\alpha+2)\Gamma^2\left(\frac{b}{2}+\alpha\right)} & if\ b \text{ is even}, \\ \dfrac{1}{\Gamma(\alpha)}\dfrac{(b+3)\Gamma^2\left(\frac{b+1}{2}+\alpha\right)}{\Gamma(b+\alpha+2)\Gamma^2\left(\frac{b+3}{2}\right)} & if\ b \text{ is odd}. \end{cases}$$

The Lyapunov inequality for the problem (142) is expressed as follows.

Theorem 45 *If the discrete fractional boundary value problem (142) has a nontrivial solution, then*

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2\left(\frac{b}{2}+2\right)\eta}{(b+2\alpha)(b+2)\Gamma^2\left(\frac{b}{2}+\alpha\right)\Gamma(b+3)f(\eta)}, \ \ if\, b\ is\ even,$$

and

$$\sum_{s=0}^{b+1} |q(s+\alpha-1)| > 4\Gamma(\alpha)\frac{\Gamma(b+\alpha+2)\Gamma^2\left(\frac{b+3}{2}\right)\eta}{\Gamma(b+3)(\Gamma^2\left(\frac{b+1}{2}+\alpha\right)f(\eta)}, \ \ if\, b\ is\ odd,$$

where $\eta = \max_{[\alpha-1,\alpha+b]_{\mathbb{N}_{\alpha-1}}} y(s+\alpha-1)$.

In 2017, Ghanbari and Gholami [59] presented the Lyapunov-type inequalities for two special classes of Sturm–Liouville problems equipped with fractional Δ-difference operators, which are given in the next two results.

Theorem 46 *Assume that* $p : [a, b]_{\mathbb{N}_0} \to \mathbb{R}^+$ *and* $q : [\alpha+a-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}$ *are real-valued functions. If* y *defined on* $[\alpha+a-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}}$ *is a nontrivial solution to the fractional Sturm–Liouville problem*

$$\begin{cases} \Delta_{b-}^{\alpha}(p(t)\Delta_{a+}^{\alpha}y(t)) + [q(t+\alpha-1)-\lambda]y(t+\alpha-1) = 0, \quad t \in (a, b), \\ y(\alpha+a-1) = 0, \ y(\alpha+b) = 0, \end{cases} \tag{145}$$

where $\alpha \in (1/2, 1)$ *and* $t = a, a+1, \ldots, b$, $a, b \in \mathbb{Z}$, $\lambda \in \mathbb{R}$ *such that* $a \geq 1$, $b \geq 3$, *then the following Lyapunov-type inequality holds:*

$$\sum_{s=a}^{b}\sum_{w=a}^{b}\left(\frac{|q(w+\alpha-1)-\lambda|}{p(s)}\right) \geq \frac{1}{2}.$$

Theorem 47 *Suppose that* $q : [\alpha+a-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}$ *is a real-valued function for* $1 < \alpha \leq 2$. *Assume that* y *defined on* $[\alpha+a-2, \alpha+b+1]_{\mathbb{N}_{\alpha-2}}$ *is a nontrivial solution to the fractional* Δ*-difference boundary value problem:*

$$\begin{cases} \Delta_{a+}^{\alpha}y(t) + [q(t+\alpha-1)-\lambda]y(t+\alpha-1) = 0, \quad t \in (a, b), \\ y(\alpha+a-2) = 0, \ y(\alpha+b+1) = 0, \end{cases} \tag{146}$$

where $t = a, a+1, \ldots, b, b+1$ $a, b \in \mathbb{Z}$, $\lambda \in \mathbb{R}$ *such that* $a \geq 1$, $b \geq 2$, *then the following Lyapunov-type inequalities hold:*

$$\sum_{a}^{b+1}|q(s+\alpha-1)-\lambda| \geq \Gamma(\alpha)\frac{b-a+2}{b-a+2\alpha}\,\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a}{2}+\alpha\right)},$$

if $a+b$ *is even and*

$$\sum_{a}^{b+1}|q(s+\alpha-1)-\lambda| \geq \Gamma(\alpha)\frac{b-a+3}{b-a+2\alpha+1}\,\frac{\Gamma(\alpha+b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+1\right)}{\Gamma(b-a+2)\Gamma^2\left(\frac{b-a+1}{2}+\alpha\right)},$$

if $a+b$ *is odd.*

14 Lyapunov-Type Inequality for Fractional Impulsive Boundary Value Problems

In 2017, Kayar [60] considered the following impulsive fractional boundary value problem

$$\begin{cases} ({}^C D^\alpha y)(t) + q(t)y(t) = 0, \quad t \neq \tau_i, \ \ a < t < b, 1 < \alpha < 2, \\ \Delta y|_{t=\tau_i} := y(\tau_i^+) - y(\tau_i^-), \ \ i = 1, 2, \ldots, p, \\ \Delta y'_{t=\tau_i} = -\dfrac{\gamma_i}{\beta_i} y(\tau_i^-), \ \ i = 1, 2, \ldots, p, \\ y(a) = y(b) = 0, \end{cases} \tag{147}$$

where ${}^C D^\alpha$ is the Caputo fractional derivative of order α $(1 < \alpha \leq 2)$, $q : PLC[a, b] \to \mathbb{R}$ is a continuous function, $a = \tau_0 < \tau_1 < \ldots < \tau_p < \tau_{p+1} = b$, $PLC[a, b] = \{y : [a, b] \to \mathbb{R}$ is continuous on each interval (τ_i, τ_{i+1}), the limits $y(\tau_i^\pm)$ exist and $y(\tau_i^-) = y(\tau_i)$ for $i = 1, 2, \ldots, p\}$, and $PLC^1[a, b] = \{y : [a, b] \to \mathbb{R}, y' \in PLC[a, b]\}$.

$y \in PLC^1([a, b], \mathbb{R})$ is a solution of the boundary value problem (147) if and only if y satisfies the following integral equation

$$y(t) = -\int_a^b G(t, s)q(s)y(s)ds - \sum_{a \leq \tau_i < b} H(t, \tau_i)\frac{\gamma_i}{\beta_i} y(\tau_i),$$

where

$$G(t, s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{a - t}{b - a}(b - s)^{\alpha-1} & a \leq t \leq s \leq b, \\ \dfrac{a - t}{b - a}(b - s)^{\alpha-1} - (t - s)^{\alpha-1}, & a \leq s \leq t \leq b, \end{cases}$$

and

$$H(t, \tau_i) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{a - t}{b - a}(b - \tau_i) & a \leq t \leq \tau_i \leq b, \\ \dfrac{a - \tau_i}{b - a}(b - t), & a \leq \tau_i \leq t \leq b. \end{cases}$$

Furthermore, the functions G and H satisfy the following properties:

1. $G(t, s)| \leq \frac{1}{\Gamma(\alpha)} \frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}(b - a)^{\alpha-1}$, for all $a \leq t, s \leq b$;
2. $H(t, \tau_i) \leq 0$ and $|H(t, \tau_i)| \leq \frac{b-a}{4}$, for all $a \leq t, \tau_i \leq b$.

The Lyapunov inequality for the problem (147) is the following.

Theorem 48 (Lyapunov Inequality) *If the problem (147) has a nontrivial solution $y(t) \neq 0$ on (a, b), then*

$$\int_a^b |q(s)|ds + \sum_{a \leq \tau_i < b} \left(\frac{\gamma_i}{\beta_i}\right)^+ > \min\left\{\frac{4}{b - a}, \frac{\Gamma(\alpha)\alpha^\alpha}{[(\alpha - 1)(b - a)]^{\alpha-1}}\right\},$$

where $\left(\frac{\gamma_i}{\beta_i}\right)^+ = \max\left\{\frac{\gamma_i}{\beta_i}, 0\right\}$.

15 Lyapunov Inequality for Boundary Value Problems Involving Hilfer Fractional Derivative

A generalization of both Riemann–Liouville and Caputo derivatives, known as *generalized Riemann–Liouville derivative of order* $\alpha \in (0, 1)$ *and type* $\beta \in [0, 1]$, was proposed by Hilfer in [61]. Such a derivative interpolates between the Riemann–Liouville and Caputo derivative in some sense. For properties and applications of the Hilfer derivative, see [62, 63] and the references cited therein.

Definition 8 The generalized Riemann–Liouville fractional derivative or Hilfer fractional derivative of order α and parameter β for a function u is defined as

$$^{H}D^{\alpha,\beta}u(t) = I^{\beta(n-\alpha)}D^{n}I^{(1-\beta)(n-\alpha)}u(t),$$

where $n-1 < \alpha < n, \quad 0 \le \beta \le 1, \quad t > a, \quad D = \dfrac{d}{dt}.$

Remark 2 The Hilfer fractional derivative corresponds to the Riemann–Liouville fractional derivative for $\beta = 0$, that is, $^{H}D^{\alpha,0}u(t) = D^{n}I^{n-\alpha}u(t)$, while it corresponds to the Caputo fractional derivative for $\beta = 1$ given by $^{H}D^{\alpha,1}u(t) = I^{n-\alpha}D^{n}u(t)$.

In 2016, Pathak [64] studied Lyapunov-type inequalities for fractional boundary value problems involving Hilfer fractional derivative and Dirichlet, mixed Dirichlet, and Neumann boundary conditions.

Let us first consider the Dirichlet boundary value problem given by

$$\begin{cases} (D^{\alpha,\beta}y)(t) + g(t)y(t) = 0, \quad t \in (a,b),\ 1 < \alpha \le 2,\ 0 \le \beta \le 1, \\ y(a) = y(b) = 0, \end{cases} \tag{148}$$

which is equivalent to the integral equation

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \left(\dfrac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}(b-s)^{\alpha-1}, & a \le t \le s \le b, \\ \left(\dfrac{t-a}{b-a}\right)^{1-(2-\alpha)(1-\beta)}(b-s)^{\alpha-1} - (t-s)^{\alpha-1}, & a \le s \le t \le b, \end{cases}$$

is the Green's function satisfying the property:

$$|G(t,s)| \le \frac{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}[\alpha-1]^{\alpha-1}}{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}},$$
$$(t,s) \in [a,b]\times[a,b].$$

Theorem 49 (Lyapunov-Type Inequality) *If a nontrivial continuous solution of the problem (148) exists, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)[\alpha-(2-\alpha)(1-\beta)]^{\alpha-(2-\alpha)(1-\beta)}}{(b-a)^{\alpha-1}[\alpha-1+\beta(2-\alpha)]^{\alpha-1+\beta(2-\alpha)}(\alpha-1)^{\alpha-1}}.$$

Next we consider a fractional boundary value problems involving Hilfer fractional derivative and mixed Dirichlet and Neumann boundary conditions:

$$\begin{cases} (D^{\alpha,\beta}y)(t)+q(t)y(t)=0, & t\in(a,b),\ 1<\alpha\le 2,\ 0\le\beta\le 1,\\ y(a)=y'(b)=0, \end{cases} \tag{149}$$

which is equivalent to the integral equation

$$y(t)=\int_a^b G(t,s)q(s)y(s)ds,$$

where $G(t,s)=\dfrac{H(t,s)}{\Gamma(\alpha)(b-s)^{2-\alpha}}$ and

$$H(t,s)=\begin{cases} \dfrac{(\alpha-1)(t-a)^{1-(2-\alpha)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{1-(2-\alpha)(1-\beta)}, & a\le t\le s\le b,\\ \dfrac{(\alpha-1)(t-a)^{1-(2-\alpha)(1-\beta)}(b-a)^{(2-\alpha)(1-\beta)}}{1-(2-\alpha)(1-\beta)} & \\ -(t-s)^{\alpha-1}(b-s)^{2-\alpha}, & a\le s\le t\le b. \end{cases}$$

The function H satisfies the following property:

$$|H(t,s)| \le \frac{b-a}{\alpha-1+\beta(2-\alpha)}\max\{\alpha-1,\beta(2-\alpha)\},\ (t,s)\in[a,b]\times[a,b].$$

Theorem 50 (Lyapunov-Type Inequality) *If a nontrivial continuous solution of the problem (149) exists, then*

$$\int_a^b (b-s)^{\alpha-2}|q(s)|ds \ge \frac{\Gamma(\alpha)(\alpha-1+\beta(2-\alpha))}{(b-a)\max\{\alpha-1,\beta(2-\alpha)\}}.$$

Now we establish a Lyapunov-type inequality for another fractional boundary value problems with Hilfer fractional derivative and mixed Dirichlet and Neumann boundary conditions:

$$\begin{cases} (D^{\alpha,\beta}y)(t) + q(t)y(t) = 0, \quad t \in (a,b),\ 1 < \alpha \le 2,\ 0 \le \beta \le 1, \\ y(a) = y'(a) = y'(b) = 0, \end{cases} \tag{150}$$

which can be transformed to the integral equation:

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{1-(3-\alpha)(1-\beta)}[2-(3-\alpha)(1-\beta)}, & a \le t \le s \le b, \\ \dfrac{(\alpha-1)(t-a)^{2-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{1-(3-\alpha)(1-\beta)}[2-(3-\alpha)(1-\beta)} \\ -(t-s)^{\alpha-1}, & a \le s \le t \le b, \end{cases}$$

is the Green's function satisfying the property:

$$|G(t,s)| \le \frac{2(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}{\Gamma(\alpha)[2-(3-\alpha)(1-\beta)]^{\alpha-1}}, \ (t,s) \in [a,b] \times [a,b].$$

Theorem 51 (Lyapunov-Type Inequality) *If a nontrivial continuous solution of the problem (150) exists, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)[2-(3-\alpha)(1-\beta)]^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.$$

Finally we consider the following fractional boundary value problem with Hilfer fractional derivative and a mixed set of fractional Dirichlet, Neumann, and fractional Neumann boundary conditions

$$\begin{cases} (D^{\alpha,\beta}y)(t) + q(t)y(t) = 0, \quad a < t < b,\ 2 < \alpha \le 3,\ 0 \le \beta \le 1, \\ \left(I^{(3-\alpha)(1-\beta)}y\right)(a) = 0,\ y'(b) = 0,\ \dfrac{d^2}{dt^2}\left(I^{(3-\alpha)(1-\beta)}y\right)(a) = 0. \end{cases} \tag{151}$$

The integral equation equivalent to Problem (151) is

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} \dfrac{(\alpha-1)(t-a)^{1-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{-(3-\alpha)(1-\beta)}[1-(3-\alpha)(1-\beta)]}, & a \le t \le s \le b, \\ \dfrac{(\alpha-1)(t-a)^{1-(3-\alpha)(1-\beta)}(b-s)^{\alpha-2}}{(b-a)^{-(3-\alpha)(1-\beta)}[1-(3-\alpha)(1-\beta)]} \\ -(t-s)^{\alpha-1}, & a \le s \le t \le b, \end{cases}$$

such that

$$|G(t,s)| \le \frac{(\alpha-2)^{\alpha-2}(b-a)^{\alpha-1}}{\Gamma(\alpha)[1-(3-\alpha)(1-\beta)]^{\alpha-1}}, \quad (t,s) \in [a,b] \times [a,b].$$

Theorem 52 (Lyapunov Inequality) *If a nontrivial continuous solution of the problem (151) exists, then*

$$\int_a^b |q(s)|ds \ge \frac{\Gamma(\alpha)[1-(3-\alpha)(1-\beta)]^{\alpha-1}}{(b-a)^{\alpha-1}(\alpha-2)^{\alpha-2}}.$$

In 2017, Kirane and Torebek [65] obtained Lyapunov-type inequalities for the following *fractional boundary value problem*

$$\begin{cases} D_a^{\alpha,\gamma} y(t) + q(t)f(y(t)) = 0, \quad a < t < b, \ 1 < \alpha \le \gamma < 2, \\ y(a) = y(b) = 0, \end{cases} \tag{152}$$

where $D_a^{\alpha,\gamma}$ is a generalized Hilfer fractional derivative of order $\alpha \in \mathbb{R}$ ($m-1 < \alpha < m, m \in \mathbb{N}$) and type γ, defined as

$$D_a^{\alpha,\gamma} f(t) = I_a^{\gamma-\alpha} \frac{d^m}{dt^m} I_a^{m-\gamma} f(t),$$

and $q : [a,b] \to \mathbb{R}$ is a nontrivial Lebesgue integrable function.

The integral representation for the solution of the boundary value problem (152) is

$$y(t) = \int_a^b G(t,s)q(s)f(y(s))ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s)=\begin{cases}\left(\frac{t-a}{b-a}\right)^{\gamma-1}\frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}-\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & a\le s\le t\le b,\\ \left(\frac{t-a}{b-a}\right)^{\gamma-1}\frac{(b-s)^{\alpha-1}}{\Gamma(\alpha)}, & a\le t\le s\le b.\end{cases}$$

Further, the above Green's function $G(t,s)$ satisfies the following properties:

1. $G(t,s)\ge 0$ for $a\le t,s\le b$;
2. $\max_{a\le t\le b}G(t,s)=G(s,s),\ s\in[a,b]$;
3. $G(s,s)$ has a unique maximum, given gy

$$\max_{a\le s\le b}G(s,s)=\frac{(\alpha-1)^{\alpha-1}}{(\gamma+\alpha-2)^{\gamma+\alpha-2}}\frac{((\gamma-1)b-(\alpha-1)a)^{\gamma-1}}{\Gamma(\alpha)(b-a)^{\gamma-\alpha}}.$$

They obtained the following Lyapunov-type inequalities.

Theorem 53 *If the fractional boundary value problem* (152) *has a nontrivial solution for a real-valued continuous function q, then*

$$\int_a^b|q(s)|ds>\frac{(\gamma+\alpha-2)^{\gamma+\alpha-2}}{(\alpha-1)^{\alpha-1}}\frac{\Gamma(\alpha)(b-a)^{\gamma-\alpha}}{((\gamma-1)b-(\alpha-1)a)^{\gamma-1}}.$$

Theorem 54 *Let $q:[a,b]\to\mathbb{R}$ be a real nontrivial Lebesgue integrable function and $f\in C(\mathbb{R}_+,\mathbb{R}_+)$ be a concave and nondecreasing function. If there exists a nontrivial solution y for the problem (152), then*

$$\int_a^b|q(s)|ds>\frac{(\gamma+\alpha-2)^{\gamma+\alpha-2}}{(\alpha-1)^{\alpha-1}}\frac{\Gamma(\alpha)(b-a)^{\gamma-\alpha}}{((\gamma-1)b-(\alpha-1)a)^{\gamma-1}}\frac{\omega}{f(\omega)},$$

where $\omega=\max_{t\in[a,b]}y(t)$.

Theorem 55 (Hartman–Wintner Type Inequality) *Let the functions q and f satisfy the conditions of Theorem 54. Suppose that the fractional boundary value problem (152) has a nontrivial solution. Then*

$$\int_a^b(s-a)^{\gamma-1}(b-s)^{\alpha-1}q^+(s)ds>\frac{\|y\|}{f(\|y\|)}\Gamma(\alpha)(b-a)^{\gamma-1}.$$

Corollary 20 *If $f(y)=y$ (linear case) and $q\in L^1([a,b],\mathbb{R}_+)$, then*

$$\int_a^b(s-a)^{\gamma-1}(b-s)^{\alpha-1}q^+(s)ds>\Gamma(\alpha)(b-a)^{\gamma-1}.$$

16 Lyapunov-Type Inequality with the Katugampola Fractional Derivative

In 2018, Lupinska and Odzijewicz [66] obtained a Lyapunov-type inequality for the following *fractional boundary value problem*

$$\begin{cases} D_{a+}^{\alpha,\rho} y(t) + q(t)y(t) = 0, & a < t < b, \ \alpha > 0, \rho > 0, \\ y(a) = y(b) = 0, \end{cases} \tag{153}$$

where $D_{a+}^{\alpha,\rho}$ is the Katugampola fractional derivative of order α, defined as

$$D_{a+}^{\alpha,\gamma} f(t) = \Big(t^{1-\alpha}\frac{d}{dt}\Big)^n I_{a+}^{n-\alpha} f(t),$$

for $t \in (a,b)$, $n = [\alpha]+1$, $0 < a < t < b \le \infty$ and $q : [a,b] \to \mathbb{R}$ is a continuous function. Here $I_{a+}^{\alpha,\rho}$ is the Katugampola fractional integral defined by

$$I_{a+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}} f(s)ds.$$

The integral representation for the solution of the boundary value problem (153) is

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \begin{cases} \dfrac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \Big(\dfrac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1}, & a \le t \le s \le b, \\ \dfrac{s^{\rho-1}}{(b^\rho - s^\rho)^{1-\alpha}} \Big(\dfrac{t^\rho - a^\rho}{b^\rho - a^\rho}\Big)^{\alpha-1} - \dfrac{s^{\rho-1}}{(t^\rho - s^\rho)^{1-\alpha}}, & a \le s \le t \le b, \end{cases}$$

which satisfies the following properties:

1. $G(t,s) \ge 0$ for $a \le t, s \le b$;
2. $\max\limits_{a \le t \le b} G(t,s) = G(s,s) \le \dfrac{\max\{a^{\rho-1}, b^{\rho-1}\}}{\Gamma(\alpha)} \Big(\dfrac{b^\rho - a^\rho}{4\rho}\Big)^{\alpha-1}, \ s \in [a,b]$.

They obtained the following Lyapunov-type inequality.

Theorem 56 *If the fractional boundary value problem* (153) *has a nontrivial solution for a real-valued continuous function q, then*

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)}{\max\{a^{\rho-1}, b^{\rho-1}\}}\left(\frac{4\rho}{b^\rho - a^\rho}\right)^{\alpha-1}.$$

Remark 3 In the special case when $\rho = 1$ in Theorem 56, we get the following result

$$\int_a^b |q(s)|ds \geq \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1},$$

which is Theorem 2, while taking $\rho \to 0^+$ in Theorem 56, we have the Lyapunov's type inequality for the Hadamard fractional derivative:

$$\int_a^b |q(s)|ds \geq \alpha\Gamma(\alpha)\left(\frac{\log(b/a)}{4}\right)^{1-\alpha}.$$

17 Lyapunov Inequality for a Boundary Value Problem Involving the Conformable Derivative

Recently, Khalil et al. [67] introduced a new derivative, which appears in the form of a limit like the classical derivative and is known as the conformable derivative. Later, this new local derivative was improved by Abdeljawad [68]. The importance of the conformable derivative is that it has properties similar to the ones of the classical derivative. However, the conformable derivative does not satisfy the index law [69, 70] and the zero order derivative property, that is, the zero order derivative of a differentiable function does not return to the function itself.

In 2017, Khaldi et al. [71] obtained a Lyapunov-type inequality for the following boundary value problem involving the conformable derivative of order $1 < \alpha < 2$ and Dirichlet boundary conditions:

$$\begin{cases} T_\alpha y(t) + q(t)y(t) = 0, & t \in (a, b), \\ y(a) = y'(b) = 0, \end{cases} \tag{154}$$

where T_α denotes the conformable derivative of order α and $q : [a, b] \to \mathbb{R}$ is a real continuous function.

The conformable derivative of order $0 < \alpha < 1$ for a function $g : [a, \infty) \to \mathbb{R}$ is defined by

$$T_\alpha g(t) = \lim_{\varepsilon\to 0} \frac{g\left(t + \varepsilon(t-a)^{1-\alpha}\right) - g(t)}{\varepsilon}, \quad t > a.$$

If $T_\alpha g(t)$ exists on (a, b), $b > a$ and $\lim_{t\to a+} T_\alpha g(t)$ exists, then we define $T_\alpha g(a) = \lim_{t\to a+} T_\alpha g(t)$.

The conformable derivative of order $n < \alpha < n+1$ of a function $g : [a, \infty) \to \mathbb{R}$, when $g^{(n)}$ exists, is defined as

$$T_\alpha g(t) = T_\beta g^{(n)}(t),$$

where $\beta = \alpha - n \in (0, 1)$.

The solution y of the problem (154) can be written as

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where

$$G(t,s) = \frac{1}{b-a}\begin{cases}(b-s)(t-a), & a \le t \le s \le b,\\ -(b-a)(t-s) + (b-s)(t-a), & a \le s \le t \le b,\end{cases}$$

is the Green's function, which is nonnegative, continuous and satisfies the property:

$$0 \le G(t,s) \le b-a, \text{ for all } t, s \in [a,b].$$

Theorem 57 (Lyapunov Inequality) *Let $q \in C([a,b], \mathbb{R})$. If the boundary value problem (154) has a solution $y \in AC^2([a,b], \mathbb{R})$ such that $y(t) \neq 0$ a.e. on (a,b), then*

$$\int_a^b |q(s)|(s-a)^{\alpha-2}ds \ge \frac{4}{b-a}.$$

In 2017, Abdeljawad et al. [72] obtained Lyapunov-type inequality for a Dirichlet boundary value problem involving conformable derivative of order $1 < \alpha < 2$:

$$\begin{cases}T_\alpha y(t) + q(t)y(t) = 0, & t \in (a,b),\\ y(a) = y(b) = 0,\end{cases} \tag{155}$$

where T_α denotes the conformable derivative of order α and $q : [a,b] \to \mathbb{R}$ is a real continuous function.

The solution for the boundary value problem (155) is

$$y(t) = \int_a^b G(t,s)q(s)y(s)ds,$$

where $G(t,s)$ is the Green's function given by

$$G(t,s)=\begin{cases}\dfrac{(t-a)(b-s)}{b-a}\cdot(s-a)^{\alpha-2}, & a\le t\le s\le b,\\ \left(\dfrac{(t-a)(b-s)}{b-a}-(t-s)\right)\cdot(s-a)^{\alpha-2}, & a\le s\le t\le b,\end{cases}$$

which satisfies the properties:

1. $G(t,s)\ge 0$ for all $a\le t,s\le b$;
2. $\max_{t\in[a,b]} G(t,s)=G(s,s)$ for $s\in[a,b]$;
3. $G(t,s)$ has a unique maximum, given by

$$\max_{s\in[a,b]} G(s,s)=G\left(\frac{a+(\alpha-1)b}{\alpha},\frac{a+(\alpha-1)b}{\alpha}\right)=\frac{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}}{\alpha^{\alpha}}.$$

The Lyapunov inequality for the problem (155) is given in the following result.

Theorem 58 *If the problem (155) has a nontrivial solution, where q is a real-valued continuous function on $[a,b]$, then*

$$\int_0^1 |q(s)|ds>\frac{\alpha^{\alpha}}{(b-a)^{\alpha-1}(\alpha-1)^{\alpha-1}}.$$

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Hypersingular Integrals in Integral Equations and Inequalities: Fundamental Review Study

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Abstract The present review deals with the fundamental approaches that cover the numerical solutions of singular and hypersingular integrals. The contribution of this work is to highlight and gather the most important background with the current modification of such work and provide the reader with an accurate image of today's knowledge regarding the approximate solutions of singular integrals. The review provides a clear understanding of various numerical approaches from the 1960s up to the present day. Some interesting applications in physics and engineering are also given.

1 Introduction

Singular integrals are usually defined for unbounded integrands or over unbounded ranges of integration. These integrals do not exist as proper or improper Riemann integrals, but are defined as limits of certain proper integrals [1]. However, in numerical analysis, numerical integration constitutes a broad family of algorithms for calculating the numerical value of a definite integral [2, 3]. It is known that two- and higher-dimensional integration is called curvature formula, whereas the quadrature reflects more realizable meaning for higher dimensional integration as well.

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The basic problem considered by numerical integration is to compute an approximate solution for a definite integral of the form

$$\int_a^b f(x)dx.$$

If $f(x)$ is a smooth well-behaved function and integrated over a small number of dimensions with bounded limits of integration, then there are many methods of approximating the integral with arbitrary precision. It is vital to mention that there are several reasons for carrying out numerical integration. The integrated $f(x)$ may be known only at certain points, such as obtained by sampling. Some embedded systems and other computer applications may need numerical integration for this reason. Moreover, a formula for the integrated may be known, but it may be difficult or impossible to find an antiderivative which is an elementary function. Sometimes, computing the numerical approximation of the integral is easier than computing the antiderivative which is given as an infinite series or product, or if its evaluation requires a special function which is not available. Furthermore, numerical integration methods can generally be described as combining evaluations of the integrated to get an approximation to the integral. Integral equation and inequalities, in the sense of Cauchy principal value, with integrals having a singularity in the domain of integration is called Cauchy singular integral equations and inequalities [4].

2 Singular Integrals

Integral inequalities are very beneficial in the qualitative theory and analysis of both differential and integral equations. Integral equation is called singular if either the range of the integration is infinite or the kernel has singularities within the range of integration. Such equations occur rather frequently in mathematical physics and possess very unusual properties [4, 5]. Indeed, a differential equation can be replaced by an integral equation that incorporates its boundary conditions [6]. As such, each solution of the integral equation automatically satisfies these boundary conditions. One can also consider integral equations in which the unknown function is dependent not only on one variable but also on several variables, for example, the equation

$$g(x) = f(x) + \lambda \int_L K(x,t)g(t)dt, \qquad x \in L,$$

where x and t are n-dimensional vectors and L is the region of an n-dimensional space. If the limits of the integral are fixed, then it is called a Fredholm integral equation. It is called a Volterra integral equation, if one of the limits is a variable. Moreover, if the unknown function is only under the integral sign, the equation is said to be of the first kind. If the function is both inside and outside, then the equation is called of the second kind.

In the same vein, we can consider system of integral equations with several unknown functions. Lipovan [7] introduced and studied different types of integral inequalities when $\lambda = 1$ taking the form

$$g(x) \le f(x) + \int_L K(x,t)g(t)dt, \qquad x \in L$$

It has been shown that

$$g(x) \le Fe^{\int_L K(x,t)dt}, \quad F := max_x f(x).$$

Also, the author extended the inequality for λ as a function of t

$$g(x) \le f(x) + \lambda(t) \int_L K(x,t)g(t)dt, \qquad x \in L.$$

Assuming $\lambda(t) = t$ we obtain Morro's inequality [8]. Applications of integral inequalities can be found in [9–11]. Recently, El-Deeb and Ahmad [12] introduced a generalization of the form

$$g(x)^p \le f(x) + \int_L K(x,t)g(t)dt + \int_L \omega(t)g^p(t)dt, \qquad x \in L = C^1(J, R_+).$$

As a good application of singular integral is the jump variational inequalities. These inequalities have been studied in various recent investigations of probability, ergodic theory, and harmonic analysis. The first variational inequality was given by Lepingle [13], followed by works due to Pisier and Xu [14] as well as by Bourgain [15] by reducing different problems to linked jump inequalities. Newly, Liu [16] proved the jump inequalities for singular integrals and averages of Radon type with rough kernels : Let $P(x) = (P_1(x), \ldots, P_n(x))$ be a polynomial mapping with components $Pj(y)$ that are real valued polynomials of $x \in \mathbb{R}^k$ such that $P_j(0) = 0$. For $t > 0$, we define the truncated singular Radon transforms T_t^P as

$$(T_t^P f)(y) = \int_{|x|>t} f(y - P(x))\,.\frac{\Omega(x/|x|)}{|x|^n}, \quad \Omega \in L.$$

In addition, there is a connection between singular integral and Jensen inequalities [17] for a function $f : [a,b] \to \mathbb{R}$

$$\Theta\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right) \le \frac{1}{b-a}\int_a^b \Theta(f(x))\,dx,$$

where f is a non-negative Lebesgue-integrable function. It has been shown that the operator norms of singular integral is approximated to the maximal value of some classes of functions. This estimate leads to reverse Jensen inequalities. Dragomir et al. [18] refined the Jensen inequalities by using a measure space (Ω, Λ, μ):

$$\Theta\left(\frac{1}{\mu(\Omega)}\int_{\Omega} f\,d\mu\right) \leq \frac{1}{\mu(\Omega)}\int_{\Omega}\Theta(f)\,d\mu.$$

In this study we focus on two kinds of singular integrals, namely the Cauchy principal value integrals (CPVI) and the hypersingular integrals (HSI) and their related inequalities, respectively.

2.1 Cauchy Principal Value Integrals (CPVI)

Consider equations that involve integration of the type

$$I(u, x) = \int_L (t-x)^{-1} u(t)dt, \qquad x \in L, \tag{1}$$

where the kernel is not integrable over any interval that includes the point $t = x$ and x is a point on the contour L outside its nodes. Consider a circle with center x and small radius $\epsilon > 0$ that intersects L at two points t' and t''. Denote by ℓ the arc $t't'' \subset L$. If the integral in (1) has a finite limit $U(x)$ as $\epsilon \to 0$, this limit is called the Cauchy principal value of the singular integral [19]

$$U(x) = \lim_{\epsilon \to 0} \int_{L/\ell} \frac{u(t)}{t-x} dt = \int_L \frac{u(t)}{t-x} dt. \tag{2}$$

A special condition for the function $u(t)$ that is needed for integrals with this kernel is called Hölder condition.

Definition 2.1 *A function $u(t)$ defined on a set D (on the complex plane, in general) is said of class $H(\alpha)$ on D, or said to satisfy the Hölder condition with exponent α if for any $t, x \in D$, the inequality*

$$|u(t) - u(x)| \leq A|t-x|^{\alpha},$$

holds with $0 < \alpha \leq 1$, and A is a positive constant. These constants are called the coefficient and the exponent in the Hölder condition (see [4]).

The function $u(t)$ is also said to be Hölder continuous and we usually write $u(t) \in H(\alpha)$ or $u(t) \in H^{(\alpha)}(A, D)$. If L is a single arc ab, then the formula in (2) reads [20, 21]

$$\fint_a^b \frac{u(t)}{t-x}dt = \lim_{\varepsilon\to 0}\left[\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right)\frac{u(t)}{t-x}dt\right], \quad x \in (a,b). \tag{3}$$

If D is a measurable subset of $\mathbb{R}^n$ with the Lebesgue measure, and ϕ and ψ are measurable real- or complex-valued functions on D, then Hölder integral inequality is

$$\int_S \left|\phi(x)\psi(x)\right| \mathrm{d}x \le \left(\int_S |\phi(x)|^p \,\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_S |\psi(x)|^q \,\mathrm{d}x\right)^{\frac{1}{q}},$$

where $p, q \in [1, \infty)$ satisfying $1/p + 1/q \le 1$. Arkhipova [22] used the Hölder integral inequality to find the upper bound of solutions of boundary value problems with integral boundary. Ding [23] obtained a local weighted Caccioppoli-type estimate and showed the weighted version of the weak reverse Holder inequality for A-harmonic tensors. Recently, Wang [24] employed the Hölder integral inequality in a class of fractional integral.

2.2 Hypersingular Integrals

Hypersingular Integrals (HSI) are integrals with strong singularities whose convergence is understood in the sense of Hadamard finite part. Integral equations with such integrals are also called hypersingular [25]. This concept was introduced in the 1930s by Hadamard in connection with the Cauchy problem for equations of hyperbolic type, and originally it was narrowly specific. The singular integral in Hadamard type is defined as below.

Definition 2.2 *Consider the improper integral on the interval* [a,b] *such that*

(i) the integrand has a singularity of the type $\frac{1}{(t-x)^2}$ *at an interior point* $a < x < b$, *and*

(ii) the regular part of the integrand is a function $\varphi(t),\ a \le t \le b$, *which satisfies a Hölder-continuous first derivative*

$$|\varphi(t) - \varphi(x) - \varphi'(x)(t-x)| \le A|t-x|^{\alpha+1}, \tag{4}$$

where $|A| < \infty$ *and* $0 < \alpha < 1$. *Then the Hadamard finite part integral or hypersingular integral is defined as*

$$F(x) = \int_a^b \frac{\varphi(t)}{(t-x)^2}dt = \lim_{\varepsilon\to 0}\left[\int_a^{x-\varepsilon}\frac{\varphi(t)}{(t-x)^2}dt + \int_{x+\varepsilon}^b \frac{\varphi(t)}{(t-x)^2}dt - \frac{2\varphi(x)}{\varepsilon}\right], \tag{5}$$

where the neighborhood ε is symmetric about the singular points x, and is defined [1, 20]

$$F(x) = F.P. \int_a^b \frac{\varphi(t)}{(t-x)^2} dt = \fint_a^b \frac{\varphi(t)}{(t-x)^2} dt. \tag{6}$$

One should clarify in which sense hypersingular integrals as given by Eq. (6) may be treated, since they do not exist either as improper integrals of the first kind or as Cauchy-type singular integral. At least three various definitions of HSIs are known [26–28]:

1. The differentiation of Cauchy principal value integral (CPVI) with respect to x yields

$$\int_a^b \frac{\varphi(t)}{(t-x)^2} dt = \frac{d}{dx} \int_a^b \frac{\varphi(t)}{t-x} dt. \tag{7}$$

2. The integral is treated as a Hadamard principal value

$$\int_a^b \frac{\varphi(t)}{(t-x)^2} dt = \lim_{\varepsilon \to 0} \left[\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{\varphi(t) dt}{(t-x)^2} - \frac{2\varphi(x)}{\varepsilon} \right]. \tag{8}$$

3. The integral is a residue value, in the sense of generalized functions,

$$\int_a^b \mid t-x \mid^\alpha \varphi(t) dt = \int_a^x (t-x)^\alpha dt + \int_x^b (x-t)^\alpha dt, \tag{9}$$

where it exists in the classical sense when the value of $\alpha = -2$. The following shows that the three different approaches are equivalent to each other, when $\varphi(x) \equiv 1$.

1.

$$\begin{aligned} \fint_a^b \frac{dt}{(t-x)^2} &= \frac{d}{dx} \fint_a^b \frac{dt}{t-x} \\ &= \frac{d}{dx} \ln \left(\frac{b-x}{x-a} \right) = \frac{b-a}{(a-x)(b-x)}. \end{aligned}$$

2.

$$\begin{aligned} \fint_a^b \frac{dt}{(t-x)^2} &= \lim_{\varepsilon \to 0} \left[\left(\frac{1}{a-x} + \frac{1}{\varepsilon} + \frac{1}{\varepsilon} - \frac{1}{b-x} \right) - \frac{2}{\varepsilon} \right] \\ &= \frac{b-a}{(a-x)(b-x)}. \end{aligned}$$

3.

$$\int_a^b | t - x |^\alpha \, dt = \int_a^x (t-x)^\alpha dt + \int_x^b (x-t)^\alpha dt =$$
$$= -\frac{(a-x)^{\alpha+1}}{\alpha+1} - \frac{(x-b)^{\alpha+1}}{\alpha+1},$$

when the value of $\alpha = -2$ it gives

$$\int_a^b \frac{dt}{(x-t)^2} = \lim_{\alpha \to -2} \left[-\frac{(a-x)^{\alpha+1}}{\alpha+1} - \frac{(x-b)^{\alpha+1}}{\alpha+1} \right]$$
$$= (a-x)^{-1} + (x-b)^{-1} = \frac{b-a}{(a-x)(b-x)}.$$

Then the three definitions for the case of $\varphi(x) = 1$ are equivalent to each other. If the density function $\varphi(x)$ is differentiable (i.e. analytical) on the open interval (a,b), then

$$\frac{d}{dx} \int_a^b \frac{\varphi(t)}{t-x} dt = \lim_{\varepsilon \to 0} \frac{d}{dx} \left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{\varphi(t)}{t-x} dt$$
$$= \lim_{\varepsilon \to 0} \left[\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) \frac{\varphi(t)dt}{(t-x)^2} - \frac{2\varphi(x)}{\varepsilon} \right],$$

and

$$\int_a^b | t - x |^\alpha \varphi(t) dt = \lim_{\varepsilon \to 0} \left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b \right) | t - x |^\alpha \varphi(t) dt$$
$$= \lim_{\varepsilon \to 0} \left[\int_a^{x-\varepsilon} (t-x)^\alpha \varphi(t) dt + \int_{x+\varepsilon}^b (x-t)^\alpha \varphi(t) dt \right]$$
$$= \lim_{\varepsilon \to 0} \frac{d}{dx} \left[-\int_a^{x-\varepsilon} \frac{(t-x)^{\alpha+1}}{\alpha+1} \varphi(t) dt \right.$$
$$\left. - \int_{x+\varepsilon}^b \frac{(x-t)^{\alpha+1}}{\alpha+1} \varphi(t) dt \right].$$

Then, by applying analytical continuation to the last relation, one can see that the right-hand side yields (7). Therefore, equivalence of the first, second, and third definitions is evident.

If $\varphi(x) \in C^2(a,b)$, then the finite limit in expression (8) exists, and the integral $\int_a^b \frac{\varphi(x)}{(t-x)^2} dt$ is finite for $x \in (a,b)$, then Eq. (8) becomes

$$\ \text{⨎}_a^b \frac{\varphi(t)dt}{(t-x)^2} = \lim_{\varepsilon\to 0}\left(\int_a^{x-\varepsilon} + \int_{x+\varepsilon}^b\right)\frac{1}{(t-x)^2}[\varphi(t)-\varphi(x)-\varphi'(x)(t-x)]dt$$
$$+\varphi(x)\frac{b-a}{(a-x)(b-x)} + \varphi'(x)\ln\frac{b-x}{a-x},$$

which has a finite limit at $\varepsilon \to +0$.

Samko et al. [29] studied some properties of fractional integrals and hypersingular integrals in variable order Holder spaces on homogeneous spaces.

SIEs occur widely in diverse areas of applied mathematics and physics. They offer a powerful (sometimes the only) technique for solving a variety of practical problems. One reason for this utility is that all of the conditions specifying an initial value or boundary value problem for a differential equation can often be condensed into a single integral equation. In the case of partial differential equations the dimension of the problem is reduced in this process so that, for example, a boundary value problem for a partial differential equation in two independent variables transforms into an integral equation involving an unknown function of only one variable. Whether one is looking for an exact solution to a given problem or having to settle for an approximation to it, an integral equation formulation can often prove to be a useful way forward. For this reason integral equations have attracted attention for most of this century and their theory is well-developed.

A rich literature of applications involve with the numerical evaluation of SI, HSI, SIE, and HSIE can be found in the following section.

3 Numerical Solutions

Clenshaw [30] proposed a quadrature scheme based on the practical abscissas $x = \cos(i\pi/n);\ i = 0,\ldots,n$, a much simpler technique based on sampling the integrand at Chebyshev points. In this method the function f which is continuous and bounded on the interval [a, b] can be expanded in the form

$$f(x) \equiv F(t) = \frac{1}{2}a_0 + a_1T_1(t) + a_2T_2(t) + \cdots, \qquad a \le x \le b, \tag{10}$$

where $T_i(t)$ is a Chebyshev polynomial of the first kind

$$T_i(t) = \cos(i \arccos t), \qquad t = \frac{2x-(b+a)}{b-a}. \tag{11}$$

Integrating $f(x)$ in (10) (see [31]) yields

$$\frac{2}{b-a}\int_a^x f(x)dx = \int_{-1}^t F(t)dt = \frac{1}{2}b_0 + b_1T_1(t) + b_2T_2(t) + \cdots, \tag{12}$$

where [31]

$$b_r = \frac{a_{r-1} - a_{r+1}}{2r}, \qquad r = 1, 2, \ldots . \tag{13}$$

The value of b_0 is determined by the lower limit of integration, thus

$$b_0 = 2b_1 - 2b_2 - 2b_3 - \cdots . \tag{14}$$

The definite integral is given by

$$\begin{aligned} \frac{2}{b-a}\int_a^b f(x)dx = \int_{-1}^{1} F(t)dt &= \frac{1}{2}b_0 + b_1 + b_2 + \cdots \\ &= 2(b_1 + b_3 + b_5 + \cdots). \end{aligned} \tag{15}$$

The coefficients in Eq. (10) may be calculated after first observing that any polynomial of degree N in x may be written in the form

$$\begin{aligned} f(x) \equiv F(t) &= \frac{1}{2}a_0 + a_1 T_1(t) + \cdots + a_{N-1}T_{N-1}(t) + \frac{1}{2}a_N T_N(t) \\ &= \sum_{r=0}^{N}{}'' a_r T_r(t), \qquad -1 \le t \le 1. \end{aligned} \tag{16}$$

where $\sum''$ denotes the finite sum whose first and last terms are to be halved. The function f interpolated at the abscissae $t_j^N = \cos(\pi j/N)$, $(0 \le j \le N)$, which are the zeros of the polynomial $\omega_{N+1}(t)$ defined by

$$\omega_{N+1}(t) = T_{N+1}(t) - T_{N-1}(t) = 2(t^2 - 1)U_{N-1}(t), \quad N \ge 1, \tag{17}$$

where $U_i(t) = \sin(i+1)\theta/\sin\theta$ is a Chebyshev polynomial of the second kind. The coefficients a_r in (16) are given by

$$a_r = \frac{2}{N}\sum_{j=0}^{N}{}'' F_j \cos\frac{\pi r j}{N}, \tag{18}$$

where

$$F_j = F(\cos\frac{\pi j}{N}). \tag{19}$$

They considered a definite integral of the form

$$\frac{2}{b-a}\int_a^b f(x)dx = \int_{-1}^{1} F(t)dt, \tag{20}$$

This method is based on the term-by-term integration of the expansion of $f(x)$ in Chebyshev polynomials, where in the evaluation of a definite integral in (20), alternate terms vanish, and they have suggested as a criterion that three successive non-zero coefficients should be small. This number of three coefficients may be increased if they desire to reduce even further the possibility of an erroneous result. This method is written briefly as CC method.

Paget and Elliott [32] have described an algorithm for the evaluation of the Cauchy principal value integral

$$⨍_{-1}^{1} \frac{w(x)f(x)}{x-a}dx, \qquad -1 < a < 1, \tag{21}$$

with a non-negative weight function $w(x)$. Their method consists of approximating $f(x)$ by a suitable finite series of polynomials orthonormal on $(-1, 1)$ with respect to the weight function $w(x)$, and then evaluating the coefficients in terms of "functions of the second kind" with a suitable interpolation method.

Lifanov and Polonskii [33] presented a numerical method for the real singular integral equation of the first kind of the form

$$\int_a^b \gamma(x)\frac{K(x_0, x)}{x-x_0}dx = f(x_0), \qquad a < x_0 < b, \tag{22}$$

where $f(x_0)$ satisfies the Hölder continuous condition, with exponent α, and $K(x_0, x)$ satisfies the condition $H(\alpha)$ with respect to x_0 and x in the region $a \leq x_0, x \leq b$, $\gamma(x)$ is the unknown function to be determined in the class of functions which are bounded for $x = b$ and unbounded for $x = a$. They solved Eq. (22) by using the method of discrete vortices to compute the integral in the left-hand side and choose the collocation points as $x_{0j} = \dfrac{x_j + x_{j+1}}{2}$ to obtain the following system of $n \times n$ linear equations

$$\sum_{i=1}^{n} \gamma(x_i)\frac{K(x_{0j}, x_i)}{x_i - x_{0j}}h = f(x_{0j}), \qquad j = 1, 2, \ldots, n, \tag{23}$$

where the solution gives the values of the unknown function γ at the computational points x_i, $i = 1, 2, \ldots, n$ and $x_i = a + ih : h = \frac{2}{n+1}$. They also solved the singular integral equations

$$\int_a^b \frac{\gamma(x)}{x-x_0}dx + \int_a^b K(x_0, x)\gamma(x)dx = \varphi(x_0), \tag{24}$$

with the condition

$$K_0(x_0, x) = \frac{K(x_0, x) - K(x_0, x_0)}{x - x_0}, \qquad j = 1, 2, \ldots, n, \tag{25}$$

and

$$\varphi(x_0) = \frac{f(x_0)}{K(x_0, x_0)}. \tag{26}$$

where $K(x_0, x_0) \neq 0$ for $a \leq x_0 \leq b$.

Their system of linear integral equations is of the form

$$\sum_{i=1}^{n} \frac{\gamma(x_i)}{x_i - x_{0j}} h = \varphi(x_{0j}) - \sum_{p=1}^{n} K(x_{0j}, x_p)\gamma(x_p)h, \qquad j = 1, 2, \ldots, n, \tag{27}$$

where $K(x_{0j}, x_i)$ and $\varphi(x_{0j})$ are the values of the related functions in Eq. (25) and Eq. (26), respectively. They showed that by increasing the values of n the solutions of the systems (23) and (27) approximate those of the singular integral Eqs (22) and (24), respectively.

Ioakimidis and Theocaris [34] considered the direct numerical solution of Cauchy type singular integral equations of the first kind

$$K(y) + \int_{-1}^{1} \frac{k(x, t)y(t)}{\sqrt{1 - t^2}} dt = g(x), \qquad -1 < x < 1, \tag{28}$$

where

$$K(f) = \frac{1}{\pi} \fint_{-1}^{1} \frac{f(t)}{\sqrt{1 - t^2}(x - t)} dt, \tag{29}$$

and is subject to

$$\int_{-1}^{1} \frac{y(t)}{\sqrt{1 - t^2}} dt = N. \tag{30}$$

They used the following approximation for the function $f(x)$:

$$f_N^T(x) = {\sum_{k=0}^{N}}' c_k T_k(x), \qquad f_{N-1}^U(x) = \sum_{k=1}^{N} P_k U_{k-1}(x), \tag{31}$$

where $T_k(x)$ and $U_k(x)$ denote the Chebyshev polynomials of degree k of the first and second kind, respectively. In the approximation of Eqs. (28) and (30), they replaced the functions $g(x)$ by $g_n(x)$, $k(x,t)$ by $k_n(x,t)$ and $y(t)$ by $y_n(t)$ using the approximation in (31). By defining

$$||f||_\infty = \sup_{-1\le x\le 1} |f(x)|,$$

they proved that for the direct Galerkin method of numerical solution of (28) and (30) based on Gauss–Chebyshev, as

$$n \to \infty,\ g_n(x) \to g(x),\ k_n(x,t) \to k(x,t) \text{ and } y_n(t) \to y(t),$$

and if $g \in C^{p_1+1}(-1,1),\ p_1 \ge 1$, and $k(x,t) \in C^{p_2+1}(-1,1),\ p_2 \ge 1$, then

$$||y - y_n||_\infty \le Cn^{-p},\ p = \min(p_1, p_2),$$

for a sufficiently large n.

In [34] paper another method of Lobatto–Chebyshev for the numerical solution of Eqs. (28) and (30) was also proposed, where they have used the following approximation for the function $f(x)$

$$f(x) \simeq \sum_{k=0}^{\infty}{}' c_k T_k(x) = \sum_{k=1}^{\infty} P_k U_{k-1}(x), \tag{32}$$

and based on the corresponding quadrature rule

$$\int_{-1}^{1} (1-t^2)^{-\frac{1}{2}} f(t)dt \cong \frac{\pi}{n} \sum_{\alpha=0}^{n}{}'' f(t_\alpha), \tag{33}$$

where t_α are the roots of the polynomial $(1-x^2)U_{n-1}(x)$ and the corresponding collocation points x_β are the roots of $T_n(x)$. The reduction of such an integral equation to a system of linear equations was proved for the convergence problem under appropriate conditions [35].

Ioakimidis [36] used the classical collocation and Galerkin methods for the numerical solution of Fredholm integral equations of the first kind with a double pole singularity of the form

$$\frac{1}{\pi} \mathop{\not\!\!=\!\!\!\int}_{-1}^{1} \frac{\sqrt{1-t^2}g(t)}{(x-t)^2}dt + \int_{-1}^{1} \sqrt{1-t^2}K(t,x)g(t)dt = -f(x), \quad -1 < x < 1, \tag{34}$$

where the density function $g(t)$ is proportional to the crack opening displacement function along the crack. Ioakimidis investigated the following integral equation

$$\mathop{\not\!\!=\!\!\!\int}_{-1}^{1} \frac{g(t)}{(t-x)^2}dt = f(x), \quad x \in (-1,1). \tag{35}$$

He applied the above integral with the values of the stress intensity factors $K(t)$ at the crack tips

$$K(\pm 1) = g(\pm 1), \tag{36}$$

taking into account that

$$U_n(\pm 1) = (\pm 1)^n(n+1), \tag{37}$$

implies

$$K(\pm 1) = \sum_{i=0}^{n} (\pm 1)^i (i+1) a_i, \tag{38}$$

where a_i are the coefficients to be determined. Equation (38) determines both the numerical values for stress intensity factors at the crack tips $K(\pm 1)$ [37] and also the unknown coefficients a_i, where

$$K(1) = 1.83122498, \qquad K(-1) = 0.70090677. \tag{39}$$

Hasegawa et al. [38] presented an algorithm to generate the sequence of interpolation polynomials by increasing the number of sample points in arithmetic progression, where they obtained an AQS which overcomes the drawback in the CC method that the number of sample points is increased in geometric progression. They considered the following integral

$$Q^l(f) = \int_{-1}^{1} P_l(x)dx, \tag{40}$$

where $P_l(x)$ is an interpolatory polynomial of degree $(l+1)N-2$ of the form

$$P_l(x) = \sum_{k=1}^{N-1} A_{0,k} U_{k-1}(x) + \sum_{i=1}^{l} \omega_{i-1}(T_N(x)) + {\sum_{k=0}^{N-1}}' A_{i,k} T_k(x), \tag{41}$$

which satisfy

$$P_l(x_i) = f(x_i), \qquad i = 1, 2, \ldots, (l+1)N-2. \tag{42}$$

Their lth quadrature rule for the integration interval $[-1, 1]$ is defined as

$$Q^l(f) = \int_{-1}^{1} P_l(x)dx = \sum_{k=1}^{N-1} A_{0,k} W_{0,k} + \sum_{i=1}^{l} {\sum_{k=0}^{N-1}}' A_{i,k} W_{i,k}, \tag{43}$$

where $A_{i,k}$ and $W_{i,k}$ are independent of l. The adequate error estimation $E_l(f)$ of the approximation $Q^l(f)$ is given by

$$E_l(f) = \int_{-1}^{1} U_{N-1}(x)\omega_l(T_N(x))2^{1-(l+1)N} f[x, x_l, \ldots, x_{(l+1)N-1}]dx, \tag{44}$$

which is very important for the automatic quadrature scheme, where $f[x, x_l, \ldots, x_{(l+1)N-1}]$ are the divided difference of order $(l+1)N - 1$.

Golberg [39] established the convergence rate for solving a class of Hadamard singular integral equation

$$\frac{1}{\pi} \oint_{-1}^{1} \frac{\sqrt{1-\xi^2}}{(x-\xi)^2} u(\xi)d\xi + \int_{-1}^{1} \sqrt{1-\xi^2} K(x,\xi)u(\xi)d\xi = f(x), \qquad -1 < x < 1, \tag{45}$$

for the real functions $f \in C^r([-1, 1])$ and $K \in C^r([-1, 1] \times [-1, 1]);\ r \geq 3$. By the relation (7), the first integral in Eq. (45) is treated as

$$Hu = \frac{1}{\pi} \oint_{-1}^{1} \frac{\sqrt{1-\xi^2}}{(x-\xi)^2} u(\xi)d\xi = \frac{1}{\pi}\frac{d}{dx} \oint_{-1}^{1} \frac{\sqrt{1-\xi^2}}{x-\xi} u(\xi)d\xi, \tag{46}$$

where the integral is a Cauchy principal value. He used the Chebyshev polynomial of the second kind to approximate the unknown function u, which implies the following system of equations

$$(Hu_n + Ku_n - f)(x_k) = 0 \tag{47}$$

where $\{x_k\}_k^n = 0$ are the zeros of $U_{n+1}(x)$, and the operator $K : L(P) \to L(P)$ defined by

$$(Ku)(x) = \int_{-1}^{1} \sqrt{1-\xi^2} K(x,\xi)u(\xi)d\xi. \tag{48}$$

The above system of equation is solved by both Galerkin and collocation methods and he proved that $\{u_n\}$ converges uniformly to u with a rate of convergence in the $L_{2,w}$ norm which is

$$||u - u_n||_\infty = O(n^{-r+2}), \tag{49}$$

for $n \geq \max(n_0, r+1)$, where $r > 3$ and n_0 comes from Jackson's theorem.

Hasegawa and Torii [40] extended the CC method in [30] for the integral $\int_{-1}^{1} f(t)dt$ to a problem of the form

$$Q(f;c) = \oint_{-1}^{1} \frac{f(t)}{t-c} dt, \qquad -1 < c < 1, \tag{50}$$

where $⨍$ is a Cauchy principal value integral. They offered a set of approximations $Q_N(f;c)$ to the integral in (50) by using automatic quadrature method based on Chebyshev polynomials. To subtract out the singularity, $Q(f,x)$ in (50) can be written as

$$Q(f,x) = \int_{-1}^{1} \frac{f(t)-f(x)}{t-x} dt + f(x) \log\left(\frac{1-x}{1+x}\right). \tag{51}$$

By using the approximate polynomial $P_N(t)$

$$P_N(t) = \sum_{k=0}^{N}{}'' a_k^N T_k(t), \qquad -1 \le t \le 1, \tag{52}$$

to interpolate f, Eq. (51) becomes

$$Q(f,x) \approx Q_N(f,x) = \int_{-1}^{1} \frac{P_N(t)-P_N(x)}{t-x} dt + f(x) \log\left(\frac{1-x}{1+x}\right). \tag{53}$$

The integrated in (53) can be written as

$$\frac{P_N(t)-P_N(x)}{t-x} = \sum_{k=0}^{N-1}{}' d_k T_k(t), \tag{54}$$

Integrating term-by-term and substituting the result into Eq. (53) gives an automatic quadrature scheme (AQS) for Cauchy principal value integral (50)

$$Q_N(f,x) \approx ⨍_{-1}^{1} \frac{f(t)}{t-x} dt \simeq 2 \sum_{k=0}^{[\frac{N}{2}-1]}{}' \frac{d_{2k}}{1-4k^2} + f(x) \log\left(\frac{1-x}{1+x}\right), \tag{55}$$

where the prime means that the first term is halved and assuming that N is even. The polynomial coefficients d_k in (54) can be stably calculated by using recurrence relation

$$d_{k+1} - 2x d_k + d_{k-1} = 2a_k^N, \qquad k = N, N-1, \ldots, 1, \tag{56}$$

in the backward direction with the starting condition $d_N = d_{N+1} = 0$, and by using the interpolation condition

$$P_N(\cos \pi j/N) = f(\cos \pi j/N), \qquad 0 \le j \le N, \tag{57}$$

the coefficients a_k^N in (56) are determined as follows:

$$a_k^N = \frac{2}{N} \sum_{j=0}^{N}{}'' f(\cos \pi j/N) \cos(\pi k j/N), \; 0 \le k \le N. \tag{58}$$

The error of the approximate integral $Q_N(f, x)$ given by Eq. (55) is bounded and independently of x, i.e.,

$$|Q(f, x) - Q_N(f, x)| \le 8 \sum_{k=0}^{\infty}{}' |V_k^N(f)|, \tag{59}$$

where

$$V_k^N(f) = \frac{1}{\pi^2 i} \oint_{\varepsilon p} \frac{\breve{U}_k(z) f(z) dz}{\omega_{N+1}(z)}, \qquad k \ge 0,$$

and $\breve{U}_k(z)$ is the Chebyshev function of the second kind defined by

$$\breve{U}_k(z) = \int_{-1}^{1} \frac{T_k(t)dt}{(z-t)\sqrt{1-t^2}} = \frac{\pi}{\sqrt{z^2-1}w^k} = \frac{2\pi}{(w - w^{-1})w^k},$$

where $w = z + \sqrt{z^2 - 1}$ and $|w| > 1$ for $z \notin [-1, 1]$.

Thus, from Eq. (59), they estimated the error of $Q_N(f, x)$ as follows:

$$|Q(f, x) - Q_N(f, x)| \lesssim 4|V_0^N(f)| \left(\frac{r+1}{r-1}\right),$$

where $|V_k^N(f)| = O(r^{-k-N})$ and

$$r = \min_{1 \le m \le M} |z_m + \sqrt{z_m^2 - 1}| > 1.$$

Rassias [41] selected applications of polynomials (Chebyshev polynomial) in approximation theory and computer aided geometric design (CAGD).

Ashour [42] used the general theory of approximations for unbounded operators to discuss and obtain convergence of the mechanical quadratures method for solving Hadamard singular integral equation

$$KX = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-\tau}X(\tau)}{(\tau - t)^2} d\tau + T(X, t) = y \tag{60}$$

where T is the given continuous operator. The solution X is approximated by Chebyshev polynomial of the second kind, and according to his method, the system of n linear algebraic equations in n unknowns take the form

$$X_n(t) = \sum_{k=1}^{n} \alpha_k U_{k-1}(t), \qquad -1 \le t \le 1.$$

Several algorithms that solve a class of Hadamard singular integral equations

$$\sum_{i=1}^{n} \alpha_i \left(KU_{i-1}, KU_{j-1}\right)_p = \left(y, KU_{j-1}\right)_p, \qquad 1 \le j \le n,$$

which is equivalent to

$$\sum_{k=1}^{n} \alpha_k \Big((TU_{k-1}, TU_{j-1})_p - j(TU_{k-1}, U_{j-1})_p - k(TU_{j-1}, U_{k-1})_p \Big) + j^2 \alpha_j$$
$$= \left(y, KU_{j-1}\right)_p, \qquad 1 \le j \le n. \tag{61}$$

If the following conditions hold where $x_n^* \in L_{2,p}$ is the solution of Eq. (60) for a given function $y \in L_{2,p}$, then for all $n \in \mathbb{N}$, Eq. (61) has a unique solution $\{\alpha_k^*\}_1^n$ and if KU_{i-1} is closed in $L_{2,p}$. The rate of convergence in $C_p[-1, 1]$ space is as follow

$$||r_n||_p = ||y - Kx_n^*|| \to 0 \quad \text{as} \quad n \to \infty,$$
$$x_n^* = \sum_{k=1}^{n} \alpha_k^* U_{k-1}(t).$$

Hui and Shia [43] constructed a set of Gaussian quadrature formulae for the approximation of hypersingular integrals of the form

$$I(t) = F.P. \oint_a^b \frac{f(x)}{(x-t)^2} dx, \qquad a < t < b, \tag{62}$$

where F.P. defined the finite part integral, the regular function $f(x)$ is defined over the interval (a,b) and $t \in (a, b)$. For the evaluations of hypersingular integrals in (62), they developed the Gaussian quadrature formula for the Cauchy principal value integrals with a weight function $\omega(x)$ of the form

$$\int_a^b \omega(x) \frac{f(x)}{x-t} dx \cong -\frac{2f(t)}{\varphi_n(t)} q_n(t) + \sum_{k=1}^{n} \lambda_k \frac{f(x_k)}{(x_k - t)}, t \ne x_k, \quad t \in (a, b), \tag{63}$$

where x_k are the roots of polynomial φ_n which satisfies the orthonormal relation [44]

$$\int_a^b \omega(x)\varphi_n(x)\varphi_m(x)dx = \delta_{mn}, \tag{64}$$

$q_n(t)$ is defined by

$$q_n(t) = \frac{1}{2}\int_a^b \frac{\omega(x)\varphi_n(x)}{t-x}dx \tag{65}$$

and λ_k is the kth weight given by

$$\lambda_k = \frac{-k_{n+1}}{k_n\varphi_n'(x_k)\varphi_{n+1}(x_k)}, \tag{66}$$

and k_n is defined by $\varphi_n(x) = k_n x^n + k_{n-1}x^{n-1} + \cdots$.

With the help of the orthogonality property in Eq. (63), the Gaussian quadrature formula for hypersingular integrals with second-order singularities (62) is

$$\int_a^b \omega(x)\frac{f(x)}{(x-t)^2}dx \cong -\frac{2f'(t)}{\varphi_n(t)}q_n(t) - \frac{2W(\varphi_n, q_n)}{\varphi_n^2(t)} + \sum_{k=1}^{n}\lambda_k\frac{f(x_k)}{(x_k-t)^2}, \tag{67}$$

where $W(\varphi_n, q_n) \equiv \varphi_n q_n' - q_n\varphi_n'$ is the Wronskian. Specializing (67) to Legendre and Chebyshev polynomials, respectively, we have

$$\int_{-1}^1 \frac{f(x)}{(x-t)^2}dx \cong -\frac{2f'(t)}{P_n(t)}Q_n(t) - \frac{2f(t)(1-t^2)^{-1}}{P_n^2(t)} + \sum_{k=1}^{n}\lambda_k\frac{f(x_k)}{(x_k-t)^2}, \tag{68}$$

$$\int_{-1}^1 \frac{\sqrt{(1-x^2)}f(x)}{(x-t)^2}dx \cong -\frac{\pi f'(t)}{U_{n-1}(t)}T_n(t) - \frac{\pi f(t)W(U_{n-1}, T_n)}{U_{n-1}^2(t)} + \sum_{k=1}^{n}\lambda_k\frac{f(x_k)}{(x_k-t)^2}. \tag{69}$$

Moreover, they showed that by choosing the appropriate weight function, their numerical experiments in some cases performed better, compared with Kutt's quadrature formula which is given by

$$I_0(f) \cong \sum_{k=1}^{n}\omega_k\left[f(-x_k) + f(x_k)\right], \tag{70}$$

where ω_k is the weight function and x_k represents the station. The numerical results of both Eq. (68) and Kutt's method (70) seem to be diverging slowly, which means the values are getting greater as the number of the quadrature points n is increased.

Iovane et al. [28] approximated the hypersingular integral equation with the characteristic kernel of the form

$$\neq\!\!\!\!\int_a^b \frac{g(t)}{(x-t)^2}dt = f'(x), \quad x \in (a,b), \quad f(x) \in C^2(a,b). \tag{71}$$

The bounded solution of Eq. (71) is unique and given by

$$g(x) = \frac{\sqrt{(x-a)(b-x)}}{\pi^2} \int_a^b \frac{f(t)}{\sqrt{(t-a)(b-t)}(x-t)}. \tag{72}$$

According to the definition of HSIs in Eqs. (7) and (71) is equivalent to

$$\neq\!\!\!\!\int_a^b \frac{g(t)}{x-t}dt = -f(x) + C, \qquad x \in (a,b),$$

where the constant C is defined as

$$C = \frac{1}{\pi^2} \int_a^b \frac{f(t)}{\sqrt{(t-a)(b-t)}}.$$

It is clear that any bounded solution of Eq. (71) vanishes as $x \to a, b$.

The direct collocation technique of solving Eq. (71) for an arbitrary right-hand side, they divided the interval (a,b) into n small equal subintervals of the length $h = \dfrac{b-a}{n}$, by the nodes $a = t_0, t_1, \ldots, t_{n-1}, t_n = b,\ t_j = a + jh,\ j = 0, 1, \ldots, n$. The central points of each subinterval (t_{i-1}, t_i) are denoted by x_i, where $x_i = a + \left(i - \frac{1}{2}\right) h,\ i = 1, 2, \ldots, n$. They approximated the integral in the left-hand side of (71) using the finite sum for $x = x_i$ as

$$\begin{aligned}
\neq\!\!\!\!\int_a^b \frac{g(t)}{(x_i-t)^2}dt &\simeq \sum_{j=0}^{n} g(t_j) \neq\!\!\!\!\int_{t_{j-1}}^{t_j} \frac{1}{(x_i-t)^2}dt \\
&= g(t_0) \int_{-h/2}^{h/2} \frac{dt}{t^2} + \sum_{j\neq i, i=1}^{n} g(t_j)\Big(\frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}}\Big) \\
&= \sum_{j=1}^{n} g(t_j)\Big(\frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}}\Big).
\end{aligned}$$

where

$$\left.\begin{aligned}
&t_j = a + jh, \quad j = 0, 1, \cdots, n, \\
&x_i = a + \left(i - \tfrac{1}{2}\right) h, \quad i = 1, 2, \cdots, n, \\
&h = \tfrac{b-a}{n}.
\end{aligned}\right\}$$

They obtained the following linear algebraic system

$$\sum_{j=i}^{n} g(t_j)\Big(\frac{1}{x_i - t_j} - \frac{1}{x_i - t_{j-1}}\Big) = f'(x_i), \qquad i = 1, 2, \cdots, n. \tag{73}$$

When $x = x_l \in (a, b)$ is fixed, the difference between the solution $g(x_l)$ of the system (73) and the analytical solution (72) tends to zero, provided $n \to \infty$. The system in (73) solved by Cramer's method where they have shown that their proposed method is an efficient alternative to a standard reduction to infinite systems of linear algebraic equations. Its principal merit is that there is no need for numerical computations when calculating elements of the respective matrix.

Mandal and Bera [45] presented a simple method based on the polynomial approximation of a function that is obtained by approximate solutions of hypersingular integral equation of the second kind over a finite interval

$$\phi(x) - \frac{\alpha\sqrt{1-x^2}}{\pi} \,=\!\!\!\!\!\!\int_{-1}^{1} \frac{\phi(t)}{(x-t)^2} dt = f(x), \quad x \in (-1, 1), \tag{74}$$

and the Cauchy type singular integro-differential equation

$$2\frac{d\phi}{dx} - \lambda \,-\!\!\!\!\!\!\int_{-1}^{1} \frac{\phi(t)}{x-t} dt = f(x), \quad x \in (-1, 1), \qquad \lambda > 0, \tag{75}$$

with the condition

$$\phi(\pm 1) = 0.$$

They approximated the unknown function $\phi(x)$ in Eq. (74) by

$$\phi(x) = \sqrt{1-x^2}\psi(x), \quad -1 < x < 1,$$

where

$$\psi(x) \approx \sum_{j=0}^{n} a_j x^j.$$

Then Eq. (74) reduces to

$$\sum_{j=0}^{n} a_j C_j(x) = F(x), \qquad -1 < x < 1, \tag{76}$$

where

$$\left.\begin{aligned} &C_j(x) = x^j - \tfrac{\alpha}{\pi} A'_j(x),\\ &A_0(x) = -\pi x,\\ &A_j(x) = -\pi x^{j+1} + \sum_{j=0}^{j-1} \tfrac{1+(-1)^i}{4} \tfrac{\Gamma(\frac{1}{2})\Gamma\left(\frac{i+1}{2}\right)}{\Gamma\left(\frac{i+4}{2}\right)},\\ &F(x) = \tfrac{f(x)}{\sqrt{1-x^2}}. \end{aligned}\right\}$$

They determined the unknown coefficients $a_j, \;\; j = 0, 1, \ldots, n$ by solving the following system of linear equations obtained by using $x = x_i (i = 0, 1, \ldots, n)$ as the collocation points in Eq. (76), where x_i's are suitable distinct collocation points in the interval $(-1, 1)$ which is

$$\sum_{j=0}^{n} a_j C_j(x_i) = F(x_i), \qquad i = 0, 1, \ldots, n. \tag{77}$$

In the solution of Eq. (75), they approximated the function $\phi(x)$ as in Eq. (74) and obtained

$$\sum_{j=0}^{n} a_j B_j(x) = G(x), \qquad -1 < x < 1,$$

where

$$\left.\begin{aligned} &B_0(x) = -\left(\tfrac{x}{\sqrt{1-x^2}} + \tfrac{\lambda\pi x}{2}\right),\\ &B_j(x)) = \left(\tfrac{jx^{j-1}-(j+1)x^{j+1}}{\sqrt{1-x^2}}\right) + \tfrac{\lambda}{2} A_j(x), \quad j = 1, 2, \cdots, n,\\ &G(x) = \tfrac{f(x)}{2}. \end{aligned}\right\}$$

They obtained the unknown coefficients a_j, $j = 0, 1, \ldots, n$ by solving the system of linear equations.

Mandal and Bhattacharya [46] obtained the approximate numerical solutions of two classes of integral equations. The first class involves Fredholm integral equation of second kind of the form

$$\phi(x) + \int_a^b K(x,t)\phi(t)dt = f(x), \qquad a < x < b, \tag{78}$$

where $\phi(t)$ is the unknown function to be determined, $K(x, t)$ is the regular kernel, and $f(x)$ is a known function. To find appropriate solution to the integral in (78), the function ϕ is approximated in the Bernstein polynomial basis in [a,b] of degree n of the form

$$\phi(x) = \sum_{i=0}^{n} a_i B_{i,n}(t), \tag{79}$$

where a_i are the unknown coefficients to be determined, and the Bernstein polynomial $B_{i,n}$

$$B_{i,n}(x) = \binom{n}{i} \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n}, \qquad i = 0, 1, \ldots, n. \tag{80}$$

Furthermore, the same basis in (79) are used for the numerical solution of the other class of Fredholm integral equation of second kind with hypersingular kernel as well as a simple hypersingular integral equation of the form

$$\,⨎_{-1}^{1} \frac{\phi(t)}{(t-x)^2} dt = f(x), \quad -1 \le x \le 1, \tag{81}$$

by imposing two additional conditions

$$\phi(-1) = \phi(1) = 0. \tag{82}$$

The integral in Eq. (81) must be interpreted in the sense of Hadamard finite-part integral, defined by

$$⨎_{-1}^{1} \frac{\phi(t)}{(t-x)^2} dt = \lim_{\varepsilon \to 0+} \left[\left(\int_{-1}^{x-\varepsilon} + \int_{x+\varepsilon}^{1} \right) \frac{\phi(t)}{(t-x)^2} dt - \frac{\phi(x-\varepsilon) + \phi(x+\varepsilon)}{\varepsilon} \right], \tag{83}$$

where $x \in (-1, 1)$ and the function ϕ is required to verify Hölder condition, $\phi \in C^{1,\alpha}(-1, 1)$.

Many applications generalized Eq. (81) in the following operator form

$$(H + K)\phi = f, \tag{84}$$

where K is another linear operator in Eq. (78). The regularized form of Eq. (84), [47] and [48].

$$(I + H^{-1}K)\phi = H^{-1} f. \tag{85}$$

The airfoil equation of the form

$$\frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{t-x} dt = g(x), \quad -1 < x < 1, \tag{86}$$

represents the simplest singular integral equation over a finite interval. By restricting g to have Hölder continuous condition, $g \in C^{0,\alpha}[-1, 1]$, we can write the general solution of Eq. (86) as (see [49, pp. 173–180])

$$f(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2} g(t)}{\sqrt{1-x^2}(t-x)} dt + \frac{A}{\sqrt{1-x^2}}, \tag{87}$$

where A is an arbitrary constant. Generally, $\phi(x)$ has inverse square-root singularity at $x = -1$ and $x = 1$ and for the unique solution, ϕ should be bounded at one end-point. Applying Eqs. (7)–(81) yields

$$\frac{1}{\pi} \int_{-1}^{1} \frac{f(t)}{t-x} dt = g(x) + B, \tag{88}$$

where B is any integration constant. To solve the above Eq. (87) is applied and gives

$$\phi(x) = -\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}(B - f(t))}{\sqrt{1-x^2}(t-x)} dt + \frac{A}{\sqrt{1-x^2}}. \tag{89}$$

It is not difficult to show

$$\begin{aligned} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} dt &= -\sqrt{1-t^2} + x \arcsin t \\ &= -\sqrt{1-t^2} \log \left(\frac{|x-t|}{1 - xt - \sqrt{(1-x^2)(1-t^2)}} \right). \end{aligned} \tag{90}$$

Using the well-known Cauchy singular integral formula [47]

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} U_n(t) dt = -T_{n+1}(x), \qquad n \geq 0, \tag{91}$$

for $n = 0$, results

$$\frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{t-x} dt = -x. \tag{92}$$

Substituting Eq. (92) into Eq. (89) gives the general solution of Eq. (81)

$$\phi(x) = \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-t^2} f(t)}{\sqrt{1-x^2}(t-x)} dt + \frac{A - Bx}{\sqrt{1-x^2}}. \tag{93}$$

The exact solution of Eq. (81) is obtained by exploiting Eq.(90) into Eq. (93), yields

$$\phi(x) = \frac{1}{\pi} \int_{-1}^{1} f(t) \ln \left| \frac{x-t}{1 - xt - \sqrt{(1-x^2)(1-t^2)}} \right| dt + \frac{A - Bx}{\sqrt{1-x^2}}, \tag{94}$$

where A and B are arbitrary constants. For the unique solution, we need an extra condition in (82), which gives the exact solution of Eq. (81) (see [48–50])

$$\phi(x) = \frac{1}{\pi^2}\int_{-1}^{1} f(t)\ln\left|\frac{x-t}{1-xt-\sqrt{(1-x^2)(1-t^2)}}\right|dt, \qquad -1 \le x \le 1. \tag{95}$$

The exact solution in Eq. (95) is specialized for the case of $f(x) = 1$, which give

$$\phi(x) = -\frac{1}{\pi}\sqrt{(1-x^2)}. \tag{96}$$

The proof of the convergence problem in terms of truncated Bernstein polynomials is also presented for a general Fredholm integral equation of the second kind and hypersingular integral equation.

Capobianco et al. [51] applied Newton's method and its modified version to solve the equations obtained by applying a collocation method to a nonlinear hypersingular integral equations (NHIEs) problems, of the form

$$\gamma(x, g(x)) - \frac{\epsilon}{\pi} \,\,=\!\!\!\!\!\!\!\int_{-1}^{1} \frac{g(t)}{(t-x)^2}dt = f(x) \tag{97}$$

where $0 < \epsilon < 1$, and the unknown function g satisfies the boundary conditions $g(\pm 1) = 0$. They approximated the density function g by the normalized Chebyshev polynomial of the second kind of the form

$$P_n^{\varphi}(\cos s) = \sqrt{\frac{2}{\pi}}\frac{\sin(n+1)s}{\sin s}, \; n = 0, 1, \cdots,$$

Furthermore, they proved the related convergence results in L^2 norm for both of Newton's method and its modified method.

Mahiub et al. [52] discussed the numerical solution of Cauchy type singular integral equations of the first kind of the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt + \int_{-1}^{1} K(t,x)\varphi(t)dt = f(x), \qquad -1 < x < 1, \tag{98}$$

with the condition

$$\varphi(\pm 1) = 0,$$

where $K(t, x)$ and $f(x)$ are given real function satisfying Hölder continuous condition, and by applying Fredholm integral equation theory along with the exact solutions of the characteristic integral equation

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\varphi(t)}{t-x}dt = f(x), \qquad -1 < x < 1, \tag{99}$$

where the unknown function φ is approximated by Chebyshev polynomial of the first kind

$$\varphi(x) = \frac{1}{\sqrt{1-x^2}}\sum_{j=0}^{n}\beta_j T_j(x). \tag{100}$$

By substituting (100) into Eq. (98), with the use of several Chebyshev properties, the system of linear algebraic equations becomes

$$\pi\sum_{j=1}^{n}\beta_j U_{j-1}(x_i) + \sum_{j=1}^{n}\beta_j\psi_j(x_i) = f(x_i), \tag{101}$$

where x_i are the collocation points that had chosen as the zeros of Chebyshev polynomial of the second kind U_n, of the form

$$x_i = \cos\left(\frac{i\pi}{n+1}\right), \quad i = 1, 2, \ldots, n, \tag{102}$$

and

$$\psi_j(x) = \int_{-1}^{1}\frac{1}{\sqrt{1-x^2}}K(t,x)T_j(t)dt. \tag{103}$$

The solution of the system in Eq. (101) gives the unknown coefficients β_j, $j = 1, 2, \ldots, n$, while the integral in (103) can be evaluated analytically or numerically using a quadrature formula.

Jung and Rassias [53] solved the inhomogeneous Chebyshev's differential equation and employ this outcome for approximating analytic functions by the Chebyshev functions. The Chebyshev's differential equation has regular singular points at -1, 1, and ∞, and it plays a great role in physics and engineering. In particular, this equation is most important for handling the boundary value problems exhibiting certain symmetries.

Obaiys et al. [54] used the classical Galerkin method for the numerical solution of the characteristic hypersingular integral equation of the form

$$\frac{1}{\pi}\,\neq\!\!\!\!\!\!\int_{-1}^{1}\frac{g(t)}{(t-x)^2}dt = f(x), \quad x \in (-1, 1), \tag{104}$$

where $g(t) \in C^{1,\alpha}[-1, 1]$. The construction of the numerical solutions is obtained by expanding both of the hypersingular kernel and the density function by using the sum of Chebyshev polynomials. They reformulated the main problem in (104) as a set of linear algebraic system that solved by applying the Galerkin method. For the characteristic HSIE in (104), the exactness of the approximate method is also given.

Chen and Jiang [55] proposed Taylor expansion method for solving a mixed linear Volterra–Fredholm integral equation of the second kind of the form

$$\lambda u(x) = f(x) + \int_a^x k_1(x, y)u(y)dy + \int_a^b k_2(x, y)u(y)dy, \quad x \in [a, b], \tag{105}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$ is a known constant, $\mathbb{R}$ is the real number set, f, k_1, k_2 are known continuous functions. They defined

$$(\mathbb{L}u)(x) = \lambda u(x) - \int_a^x k_1(x, y)u(y)dy - \int_a^b k_2(x, y)u(y)dy. \tag{106}$$

The unknown continuous function u is approximated by the square approximation

$$u(x) = \sum_{i=0}^{n} c_{i,n}^*(x-a)^i, \tag{107}$$

where the set of unknown coefficients $\{c_{i,n}^*\}_{i=1}^n$ satisfies

$$||f - \sum_{i=0}^{n} c_{i,n}^* f_i||_c = \min_{c_{i,n}} ||f - \sum_{i=0}^{n} c_{i,n} f_i||_c, \tag{108}$$

and $f_i(x) = [\mathbb{L}y(y-a)^i](x), i = 0, 1, 2, \ldots$. They declared that for any given $\epsilon > 0$, there exists a positive integer N such that for every fixed $n > N$, there exists a polynomial

$$u_n(x) = \sum_{i=0}^{n} c_{i,n}(x-a)^i \tag{109}$$

that satisfies

$$||u - u_n(x)||_c \le \frac{\epsilon}{|\lambda| + 2M(b-a)}. \tag{110}$$

They have showed that the approximation in (109) is an ϵ-approximate solution of Eq. (105) verifying

$$\left|\left|\mathbb{L}\sum_{i=0}^{n} c_{i,n}^*(x-a)^i - f\right|\right|_c = \left|\left|\sum_{i=0}^{n} c_{i,n}^* f_i - f\right|\right|_c \le \epsilon. \tag{111}$$

Obaiys et al. [56] developed automatic quadrature scheme (AQS) for the numerical solutions of the first kind bounded hypersingular integral equation of the form

$$\frac{1}{\pi} \displaystyle{\not}\int_{-1}^{1} \frac{Q(t)}{(t-x)^2} dt + \int_{-1}^{1} K(t,x) Q(t) dt = f(x), \quad x \in (-1,1), \tag{112}$$

where $f(x)$ is a given function, the unknown function Q satisfies the boundary conditions $Q(\pm 1) = 0$, and the kernel function $K(t,x)$ satisfies a Hölder continuous first-derivative condition.

The unknown function Q in (112) is approximated by using the finite sum of Chebyshev polynomial of the second kind of the form

$$Q_n(t) = \sqrt{(1-t^2)} \sum_{i=0}^{n} C_i U_i(t), \quad |t| < 1, \tag{113}$$

where C_i, $i = 0, 1, 2, \cdots, n$, are the unknown coefficients and $U_n(t)$ is the Chebyshev polynomial of the second kind defined by Mason and Handscomb [47]

$$U_n(t) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad t = \cos\theta, \quad 0 \le \theta \le \pi.$$

The initial integral problem in (112) is successfully reduced to a linear finite algebraic system of $n+1$ equations with $n+1$ unknown coefficients C_i of the form

$$\sum_{i=0}^{n} C_i [-(i+1)U_i(x_j) + \frac{\pi}{2} \rho_i(x_j)] = f(x_j), \tag{114}$$

where x_j are the collocation points of the roots of the Chebyshev polynomial of the first kind $T_{n+1}(x)$ along the interval $[-1, 1]$, which are

$$x_j = \cos\left(\frac{2j-1}{2(n+1)}\pi\right), \ j = 0, 1, \cdots, n. \tag{115}$$

and $\rho_j(x)$ is the linear system defined by

$$\rho_j(x) = \frac{2}{\pi} \int_{-1}^{1} \sqrt{1-t^2} K(t,x) U_j(t) dt. \tag{116}$$

This method showed that the calculation of C_i endorses the evaluation of $Q_n(t)$ in (113).

In [57], the author paid particular attention to the error estimate of the developed AQS, where the HSIE problem of the form in Eq. (112) recapped into the following operator form

$$(H + K)Q(x) = f(x), \tag{117}$$

where both $HQ(x)$ and $KQ(x)$ are defined as follows:

$$HQ(x) = \frac{1}{\pi} \fint_{-1}^{1} \frac{\sqrt{1-t^2}Q(t)}{(t-x)^2} dt, \tag{118}$$

and

$$KQ(x) = \int_{-1}^{1} \sqrt{1-t^2} K(t, x) Q(t) dt. \tag{119}$$

Moreover, the convergence problem of $Q_n(t)$ is solved by the proof of the following theorem.

Theorem 3.1 *If* $f(x) \in C^{\ell}[-1, 1]$ *and* $k(t, x) \in C^{\ell}\Big([-1, 1] \times [-1, 1]\Big)$, $\ell \geq 1$, *then*

$$||Q - Q_n||_{L_{2,\sqrt{1-t^2}}} = O(n^{-\ell}).$$

where $L_{2,w}$ denotes the space of real valued functions square integrable with respect to the weight function $w = \sqrt{1-x^2}$. The space of functions $Q(x)$ defined on $[-1, 1]$ and square integrable with respect to the corresponding weight function w, denoted by $L_{2,w}$:

$$L_{2,w} = \Big\{h(x)\Big| \int_{-1}^{1} w(x)|h(x)|^2 dx < \infty\Big\} \tag{120}$$

and the Chebyshev norm takes the form

$$\| e_N \|_c = \max_{-1 \leq a \leq t \leq b \leq 1} |f(t) - P_N(t)|. \tag{121}$$

By using Jackson's theorem [39], the convergence of the function $f \in C^{\ell}([-1, 1])$ in the class of function $H^{\alpha}([-1, 1])$ and $L_2([-1, 1])$ takes the form

$$||Q - Q_n||_2 = C(n^{-\ell}). \tag{122}$$

where the error in (122) decreases very quickly and the convergence is very fast to the exact solution even when x is close to the end points. In particular, if ℓ in the relation (122) can be chosen to be any large positive number, then the error decreases rapidly as n increases. Then the sequence $\{Q_n\}$ converges uniformly in $L_{2,w}$ norm to $\{Q\}$.

Dragomir [58] surveyed different classes of integral inequalities of Chebyshev functional including in measurable spaces.

Ioakimidis [59] suggested singular integral equations of crack problems under parametric inequality constraints with the Fredholm kernels

$$K(t,x) = \frac{a}{b} cot \frac{\pi a(t-x)}{b} - \frac{1}{\pi(t-x)}$$

and

$$K(t,x) = \frac{a}{b}[2coth \frac{\pi a(t-x)}{b} - \frac{\pi a(t-x)}{b} csch^2 \frac{\pi a(t-x)}{b}] - \frac{1}{\pi(t-x)}.$$

Boukov et al. [60] proposed a new technique for finding linear and nonlinear hypersingular integral equations using classical integral inequalities. For nonlinear equations the importance of the technique is in rather weak requirements for the nonlinear operator conduct in the vicinity of the outcome.

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Exact Bounds on the Zeros of Solutions of Second-Order Differential Inequalities

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Abstract Exact upper bounds on the zeros of solutions of a certain class of second-order differential inequalities are obtained.

Let a function $x\colon [0,\infty) \to \mathbb{R}$ be a solution of the differential inequality

$$x''(t) - 2a(t)x'(t) + [a(t)^2 + b(t, x(t))^2]\, x(t) \le 0 \tag{1}$$

for $t \in [0,\infty)$ satisfying initial conditions

$$x(0) = x_0 \quad \text{and} \quad x'(0) = y_0, \tag{2}$$

where functions $a \in C^1(\mathbb{R})$ and $b \in C(\mathbb{R}^2)$ are such that

$$b_*^2 := \inf_{(t,u)\in\mathbb{R}^2} [a'(t) + b(t,u)^2] > 0; \tag{3}$$

let then $b_* := \sqrt{b_*^2}$. Consider the smallest positive root

$$t_1 := \inf\{t \in (0,\infty)\colon x(t) = 0\} \tag{4}$$

of the function x; here, one may recall the general convention $\inf\emptyset := \infty$.

Theorem 1 *Suppose that $x(t) > 0$ for all t in a right neighborhood of 0; that is, $x_0 \ge 0$, and $x_0 = 0$ implies $y_0 > 0$. Then*

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$$t_1 \leq T_1 := \begin{cases} \dfrac{\pi}{b_*} - t_* & \textit{if } \ y_{a,0} > 0, \\ t_* & \textit{if } \ y_{a,0} \leq 0, \end{cases} \tag{5}$$

where

$$y_{a,0} := y_0 - a(0)x_0 \tag{6}$$

and

$$t_* := \frac{1}{b_*} \arcsin \frac{b_* x_0}{\sqrt{b_*^2 x_0^2 + y_{a,0}^2}}.$$

Moreover, for each given quadruple of values $(a(0), b_*, x_0, y_0)$*, the upper bound* T_1 *on* t_1 *is exact: if inequality* (1) *is replaced by the corresponding equality and* (3) *is replaced by the identity* $a'(t) + b(t, u)^2 = b_*^2$ *for all* $(t, u) \in \mathbb{R}^2$ *(which obtains, e.g., when a and b are constant), then* $t_1 = T_1$*.*

One may note that, if $y_{a,0} = 0$, then $\frac{\pi}{b_*} - t_* = t_* = \frac{\pi}{2b_*}$.

Immediately from Theorem 1 we obtain the following.

Corollary 1 *If in* (1) *we replace "*≤ 0*" by "*$= H(t, x(t))$*," where a function* $H \in C(\mathbb{R}^2)$ *is such that* $uH(t, u) \leq 0$ *for all* $(t, u) \in [0, \infty) \times \mathbb{R}$*, then the resulting solution x has infinitely many zeros on* $[0, \infty)$ *and the distance between any consecutive nonnegative zeros of x is bounded from above by* π/b_**.*

Corollary 1 is illustrated in Fig. 1.

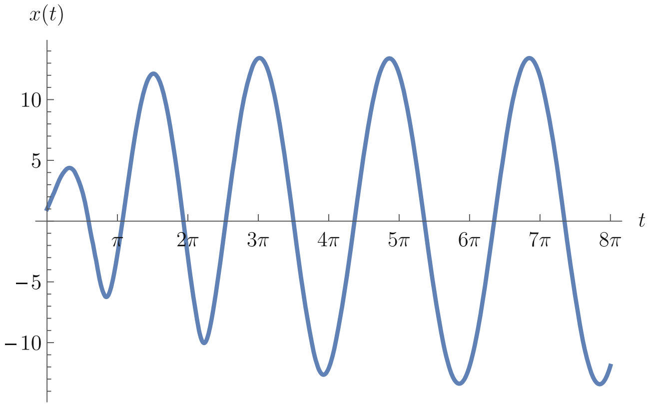

Fig. 1 Graph $\{(t, x(t)) : t \in [0, 8\pi]\}$ for the solution x of (1)–(2) with "≤ 0" in (1) replaced by "$= H(t, x(t))$", $H(t, u) = -\frac{u}{2} e^{-k(t+u)}$, $a(t) = e^{-kt}$, $b(t, u) = 1 + \frac{7}{10} e^{-kt/2}$, $k = \frac{6}{10}$ (so that $b_* = 1$), $x_0 = 1$, $y_0 = 4$

Proof of Theorem 1 Introduce

$$r(t) := x(t) \exp\left\{ -\int_0^t a(u)\,du \right\}. \tag{7}$$

Then inequality (1) and definitions (4) and (6) can be rewritten as

$$r''(t) + [a'(t) + b(t, x(t))^2]r(t) \le 0, \tag{8}$$

$$t_1 = \inf\{t \in (0, \infty)\colon r(t) = 0\},$$

and

$$y_{a,0} = r'(0), \tag{9}$$

respectively. In particular, $r > 0$ on the interval $(0, t_1)$. Therefore, on $(0, t_1)$ we have $r'' \le -b_*^2 r < 0$ and hence r is strictly concave.

So, if $t_1 = \infty$, the condition that $x(t) > 0$ for all t in a right neighborhood of 0 implies that r is increasing on $[0, \infty)$, and hence for all real $t \ge 1$ we have $r''(t) \le -b_*^2 r(1) =: -c < 0$ and $r(t) \le r(1) + r'(1)(t-1) - c(t-1)^2/2 \to -\infty$ as $t \to \infty$, which contradicts the condition that r is increasing on $[0, \infty)$. We conclude here that $t_1 \in (0, \infty)$ and hence $r(t_1) = 0$.

Now consider separately the two cases mentioned in (5), depending on whether $y_{a,0} > 0$.

Case 1: $y_{a,0} > 0$ That is, by (9), here $r'(0) > 0$. Since $r'' < 0$ on $(0, t_1)$, we have $r'(t_1) < 0$—because otherwise we would have $r' > 0$ on $(0, t_1)$ and hence $0 = r(t_1) > r(0) \ge 0$, a contradiction. So, there exists a unique $s \in (0, t_1)$ such that $r'(t)$ (strictly) decreases from $r'(0) > 0$ to $r'(s) = 0$ to $r'(t_1) < 0$ as t increases from 0 to s to t_1. So, $r' > 0$ on $(0, s)$ and $r' < 0$ on (s, t_1), which in turn implies that r continuously increases on $[0, s]$ from $r(0) = r_0 \in [0, \infty)$ to

$$r_{\max} := r(s) > r_0 \ge 0,$$

and r continuously decreases on $[s, t_1]$ from $r_{\max} > 0$ to $r(t_1) = 0$.

Consider now the functions $\rho_1\colon I_1 \to K_1$ and $\rho_2\colon I_2 \to K_2$ defined by the formula $\rho_j(t) := r(t)$ for $t \in I_j$, where $I_1 := [0, s]$, $I_2 := [s, t_1]$, $K_1 := [r_0, r_{\max}]$, and $K_2 := [0, r_{\max}]$; here and in the sequel, $j = 1, 2$. It follows that the corresponding inverse functions $\rho_j^{-1}\colon K_j \to I_j$ are well defined, and then one can also introduce the functions p_j by the formula $p_j(\rho) := r'(\rho_j^{-1}(\rho))$ for $\rho \in K_j$, so that for $t \in I_j$ we have $r'(t) = p_j(\rho_j(t)) = p_j(r(t))$ and hence, by the chain rule,

$$r''(t) = p_j'(r(t))r'(t) = p_j'(r(t))p_j(r(t)) = q_j'(r(t))/2$$

for $t \in I_j$, where

$$q_j(\rho) := p_j(\rho)^2. \tag{10}$$

$\big($The functions p_j and $q_j/2$ may be interpreted as the momentum and kinetic energy, respectively, of a particle of unit mass, with position $r(t)$ at time t.$\big)$ Thus, formulas (8) and (3) imply

$$q_j'(\rho) \le -2b_*^2\rho \tag{11}$$

for $\rho \in K_j$. Integrating (11), we have

$$q_1(\rho) \le q_1(r_0) - b_*^2(\rho^2 - r_0^2) = y_{a,0}^2 - b_*^2(\rho^2 - r_0^2) \tag{12}$$

for $\rho \in K_1 = [r_0, r_{\max}]$; here, we used that $q_1(r_0) = q_1(r(0)) = p_1(r(0))^2 = r'(0)^2 = y_{a,0}^2$. Since $p_j(r_{\max}) = p_j(\rho_j(s)) = r'(s) = 0$ and hence $q_j(r_{\max}) = 0$, it follows from (12) (with $\rho = r_{\max}$) that

$$r_{\max} \le r_* := \sqrt{r_0^2 + y_{a,0}^2/b_*^2}. \tag{13}$$

Another integration of (11) yields

$$-q_j(\rho) = q_j(r_{\max}) - q_j(\rho) \le -b_*^2(r_{\max}^2 - \rho^2) \tag{14}$$

for $\rho \in K_j$. We have $q_j(r(t)) = p_j(r(t))^2 = r'(t)^2$ for $t \in I_j$. Also, $r' \ge 0$ on $I_1 = [0, s]$ and $r' \le 0$ on $I_2 = [s, t_1]$. So, (14) can be rewritten as

$$r' \ge b_*\sqrt{r_{\max}^2 - r^2} \text{ on } I_1 = [0, s] \quad \text{and} \quad r' \le -b_*\sqrt{r_{\max}^2 - r^2} \text{ on } I_2 = [s, t_1] \tag{15}$$

– or, equivalently, as

$$\frac{d}{dt}\arcsin\frac{r(t)}{r_{\max}} \ge b_* \text{ for } t \in (0, s) \quad \text{and} \quad \frac{d}{dt}\arcsin\frac{r(t)}{r_{\max}} \le -b_* \text{ for } t \in (s, t_1).$$

Integrating these inequalities over the corresponding intervals and recalling that $r(s) = r_{\max}$ and $r(t_1) = 0$, we get

$$\frac{\pi}{2} - \arcsin\frac{r_0}{r_{\max}} \ge b_*s \quad \text{and} \quad -\frac{\pi}{2} \le -b_*(t_1 - s).$$

Eliminating s from these two inequalities and recalling (13) (and the equality $r_0 = x_0$), we finally get

$$t_1 \leq \frac{\pi}{b_*} - \frac{1}{b_*} \arcsin \frac{r_0}{r_{\max}} \leq \frac{\pi}{b_*} - \frac{1}{b_*} \arcsin \frac{b_* x_0}{\sqrt{b_*^2 x_0^2 + y_{a,0}^2}} = t_*$$

—in Case 1. Moreover, following the lines of the above reasoning, we see that if inequality (1) is replaced by the corresponding equality and (3) is replaced by the identity $a'(t) + b(t,u)^2 = b_*^2$ for all $(t,u) \in \mathbb{R}^2$, then $t_1 = \frac{\pi}{b_*} - t_*$, which shows that the upper bound t_* on t_1 is exact in Case 1.

Case 2: $y_{a,0} \leq 0$ This case is only simpler than Case 1. Here $r' < 0$ on the entire interval $(0, t_1)$. Accordingly, here we need to consider only one branch of the inverse function, $p := r^{-1}$, of the function r on $[0, t_1]$. So, with $q := p^2$, instead of (14) here we have

$$y_{a,0}^2 - r'(t)^2 = q(r_0) - q(r(t)) \leq -b_*^2(r_0^2 - r(t)^2)$$

for $t \in [0, t_1]$. So, in place of (15), here we have $r' \leq -b_*\sqrt{r_*^2 - r^2}$ on $[0, t_1]$, where r_* is as in (13). Thus, we similarly get

$$t_1 \leq \frac{1}{b_*} \arcsin \frac{b_* x_0}{\sqrt{b_*^2 x_0^2 + y_{a,0}^2}} = t_*$$

—in Case 2. Moreover, following the lines of the above reasoning, we see that if inequality (1) is replaced by the corresponding equality and (3) is replaced by the identity $a'(t) + b(t,u)^2 = b_*^2$ for all $(t,u) \in \mathbb{R}^2$, then $t_1 = t_*$, which shows that the upper bound t_* on t_1 is exact in Case 2.

Theorem 1 is now completely proved. □

Theorem 1 is a generalization and refinement of a result in the recent paper by Riely [3], devoted to sharp inequalities in harmonic analysis. More specifically, in [3, Lemma 3.4] the differential inequality (1) is considered with $a = 0$ and constant $b > 0$, with the resulting upper bound $\frac{\pi}{b}$ on t_1. The method of proof in [3] (is different from the one presented in this note and) apparently does not work in our more general setting. Thus, Theorem 1 may lead to more general applications in harmonic analysis.

Another potential wide field of applications of Theorem 1 is to various special functions that are solutions of second-order differential equations; see, e.g., survey [1] by Muldoon.

Studies of differential inequalities go back to at least as far as the seminal paper [2] by Petrovitch. However, our method, based on the use of the mentioned "kinetic energy" $q_j = p_j^2/2$, appears to differ from the one in [2].

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Variational Methods for Emerging Real-Life and Environmental Conservation Problems

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Abstract Variational methods and duality theory are tools of paramount importance in many areas of mathematics, and are fruitfully used in many different applications. In this survey paper, we aim at discussing recent developments in duality theory with the idea of unifying certain basic duality results in nonlinear optimization, and showing main properties and powerful results.

1 Introduction

In this paper, we present some problems of paramount importance for the human survival, such as the pollution emission problem, the Kyoto Protocol commitments, and the Walrasian equilibrium problem, and provide several effective answers and strategies to decision-makers, governments, managers, etc. We deal with these problems using notable variational methods, which imply the transformation of variational and quasi-variational inequalities, as well as infinite dimensional duality theory. In addition, these tools allow us to find and interpret the dual variables, and formulate the problems as general nonlinear systems.

It is worth mentioning that the above problems are considered in an infinite dimensional setting in order to capture their evolutionary nature. If a static approach is performed, then situations that naturally evolve over time are studied only in a fixed moment of their evolution. On the contrary, the evolutionary framework studies the evolution of systems over time, and provides curves of equilibria that unveil important features of the models. For this reason, we explicitly take into account the dependence on time of variables (see [1, 10, 25, 37, 38]) and choose as our functional setting the space L^2. This choice allows us to handle both smooth and non-smooth functions, and prove the existence of solutions under minimal assumptions.

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We note that problems discussed in our survey are represented not only by a minimization problem of a convenient utility function, but also by generalized complementarity conditions, which, roughly speaking, control when an equilibrium threshold is overcome. Starting from the equilibrium conditions, by means of duality theory, we are able to transform the problems into variational inequalities on convex sets with side constraints. In this context, surprisingly, we find a new complementarity law which connects the dual variables with the corresponding constraints, so as to control when a constraint is active or not (see also [9, 14]).

We remark that in infinite dimensional spaces, the classical theorems that satisfy strong duality and ensure existence of multipliers require that the interior of the ordering cone be nonempty. From these theorems, in the finite dimensional case, the well-known Slater conditions and KKT conditions can be derived. Unfortunately, in most infinite dimensional cases where the functional space is L^2 or a Sobolev space, the ordering cone has empty interior. Therefore, the classical results cannot be applied. We then adopt the approach as in [11] in which the so-called *Assumption S* is used as a suitable constraint qualification, and results to be necessary and sufficient to have strong duality. Once that *Assumption S* is verified, the existence of Lagrange multipliers is guaranteed. It is worth noting that other results in the literature provide the existence of the Lagrange multipliers and the nonnegativity of the Fréchet derivative of the Lagrange functional (see, for instance, [28]). However, these conditions, which are generally only sufficient, require a lot of regularity assumptions on data involving directional derivatives. The use of the separation condition *Assumption S* as a constraint qualification does not require regularity assumptions.

The paper is organized as follows. In Sect. 2, we present some preliminary recent results in infinite dimensional duality. In Sect. 3, we discuss the pollution emission problem formulated as a bilevel problem, in which the government chooses the optimal price of the pollution emission with consideration to firms' response to the price; on the other hand, firms choose the optimal quantities of production to maximize their profits, given the price of pollution emission. Therefore, the government's problem is the upper-level problem and the firm's problem is the lower-level problem, see [34]. In Sect. 4, we examine the problem in which different countries, aiming at reducing pollution emissions according to Kyoto Protocol commitments, accept to coordinate both emissions and investment strategies in order to optimize jointly their welfare, see [35]. In Sect. 5, we present the evolutionary Walrasian price equilibrium problem with memory term, in which the excess demand function depends on the current price and on previous events of the market, see [23]. For each model, we provide an example to show the feasibility of the approach. Even if they are low dimensional examples, they allow us to validate the theoretical achievements.

2 Preliminary Duality Results

In this section, we present some infinite dimensional Lagrange duality results that have been recently achieved (see [4, 11–13, 20–22, 24, 26, 33]). For reader's convenience, we first recall some typical concepts in duality theory, see [28]. Let X denote a real normed space and X^* the topological dual of all continuous linear functionals on X.

Definition 1 Let C be a nonempty subset of X and x a given element. The set

$$T_C(x) := \Big\{ h \in X : h = \lim_{n \to \infty} \lambda_n (x_n - x), \lambda_n \in \mathbb{R}, \lambda_n > 0 \; \forall n \in \mathbb{N}, \\ x_n \in C \; \forall n \in \mathbb{N}, \; \lim_{n \to \infty} x_n = x \Big\}$$

is called the contingent cone to C at x.

Of course, if $T_C(x) \neq \emptyset$, then x belongs to the closure of C, denoted by cl C. If $x \in \text{cl}\, C$ and C is convex, then (see [3])

$$T_C(x) = \text{cl cone}(C - \{x\}),$$

where

$$\text{cone}(C) := \{\lambda x : x \in C, \lambda \in \mathbb{R}, \lambda \geq 0\}.$$

Finally, we recall the definition of convex-like function.

Definition 2 Let S be a nonempty subset of a linear real space X and let Y be a linear real space partially ordered by the cone C. A function $g : S \to Y$ is called convex-like if and only if the set $g(S) + C$ is convex.

We now present the statement of Theorem 3.2 in [33]. Let X be a real linear topological space and S a nonempty subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C. Let $f : S \to \mathbb{R}$ and $g : S \to Y$ be two functions such that the function (f, g) is convex-like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$. Let us consider the primal problem

$$\min_{x \in K} f(x), \tag{1}$$

where

$$K := \{x \in S : g(x) \in -C\},$$

and the dual problem

$$\max_{u \in C^*} \inf_{x \in S} \{f(x) + \langle u, g(x) \rangle\}, \tag{2}$$

where

$$C^* := \{u \in Y^* : \langle u, y \rangle \geq 0, \forall y \in C\}$$

is the dual cone of C.

It is said that *Assumption S* is fulfilled at a point $x_0 \in K$ if and only if it results:

$$T_{\widetilde{M}}(f(x_0), 0_Y) \cap] - \infty, 0[\times \{0_Y\} = \emptyset,$$

where

$$\widetilde{M} := \{(f(x) - f(x_0) + \alpha, g(x) + y) : x \in S \setminus K, \alpha \geq 0, y \in C\}.$$

Then, in [33] the following theorem is proved.

Theorem 1 *Let us assume that the function* $(f, g) : S \to \mathbb{R} \times Y$ *be convex-like. Then, if problem* (1) *is solvable and Assumption S is fulfilled at the extremal solution* $x_0 \in S$*, also problem* (2) *is solvable, the extreme values of both problems are equal and it results:*

$$\langle \overline{u}, g(x_0) \rangle = 0,$$

where $\overline{u}$ *is the extremal point of problem* (2).

The following result entitles us to characterize a solution of problem (1) as a saddle point of the Lagrange function, see [11].

Theorem 2 *Let us assume that assumptions of Theorem 1 be satisfied. Then,* $x_0 \in S$ *is a minimal solution to problem* (1) *if and only if there exist* $\overline{u} \in C^*$ *such that* $(x_0, \overline{u})$ *is a saddle point of the Lagrange function, namely*

$$\mathscr{L}(x_0, u) \leq \mathscr{L}(x_0, \overline{u}) \leq \mathscr{L}(x, \overline{u}), \forall x \in S, u \in C^*,$$

and $\langle \overline{u}, g(x_0) \rangle = 0$.

3 A New Approach to the Bilevel Pollution Emission Problem

In this section, we present the pollution emission price problem formulated as a bilevel problem, see [34]. In particular, we suppose that the government chooses the optimal price of the pollution emission with consideration to firms' response to the price; on the other hand, firms choose the optimal quantities of production to maximize their profits, given the price of pollution emission. Therefore, the government's problem represents the leader's problem (or upper-level problem) with the goal of maximizing the social welfare via taxation, and the firm's problem describes the follower's problem (or lower-level problem).

Bilevel programming problems are hierarchical optimization problems, where a subset of the variables of a decision-maker is constrained to be a solution of an optimization problem of another decision-maker. The bilevel programming problems have been extensively studied in [15–17] and the references cited there. In [29], the authors give a comprehensive review for solving bilevel problems in finite-dimensional spaces. Other valuable contributions are given in [18, 19, 42, 44].

Bilevel programming problems in finite-dimensional spaces are often reformulated using Karush–Kuhn–Tucker conditions for the lower-level problem, under a Slater constraint qualification, obtaining a mathematical program with equilibrium constraints; see [36]. However, in infinite dimensional problems, it is necessary to apply a different approach. In [46], the author considers, like in our case, a bilevel problem in an infinite dimensional setting, and reformulates the bilevel problem as a single level optimal control problem, by using first order optimality conditions for the lower-level problem as additional constraints for the upper-level problem. In virtue of the functional constraints, the bilevel dynamic problem is written as an infinite dimensional optimization problem and the first order optimality conditions, involving Lagrange multipliers, are obtained using a constraint qualification of calmness type [45, 47, 48]. The functional setting is the space of continuous mapping and the Lagrange multipliers result to be Radon measures.

We now present an evolutionary model in the time interval $[0, \bar{t}]$, with $\bar{t} > 0$. We consider a model in which m firms are involved in the production of a homogeneous product. We denote a typical firm by i.

In order to describe our model, we introduce the following notations:

- $q_i(t)$ is the production output of firm i at time $t \in [0, \bar{t}]$;
- $\overline{q}_i(t)$ is the feasible maximum production of firm i at time $t \in [0, \bar{t}]$;
- $p_i(t)$ is the price per unit product manufactured by firm i at time $t \in [0, \bar{t}]$;
- $c_i(t)$ is the production cost of firm i at time $t \in [0, \bar{t}]$;
- $h_i(t)$ is the quantity of pollution emitted by firm i at time $t \in [0, \bar{t}]$;
- $\pi(t)$ is the price per unit product of the pollution emission imposed by the government to firm i at time $t \in [0, \bar{t}]$;
- $\overline{\pi}(t)$ is the feasible maximum price of the pollution emission imposed by the government to firm i at time $t \in [0, \bar{t}]$.

We consider the situation in which the production cost of firm i may depend upon the production of firm i, that is $c_i(t) = c_i(q_i(t))$. However, it would also be possible to study a more general formulation in which $c_i(t) = c_i(q_1(t), \ldots, q_m(t))$.

Moreover, we denote by $\kappa(t) = \sum_{i=1}^{m} h_i(q_i(t)) \in L^2([0, \bar{t}])$, $\kappa(t) \geq 0$ a.e. in $[0, \bar{t}]$, the total quantity of pollution emitted by firms, where $q_i(t)$ is the response to the price $\pi(t)$ at time t. We denote by $C(t) = C\Big(\sum_{i=1}^{m} h_i(q_i(t))\Big)$ the cost of abating pollution at time $t \in [0, \bar{t}]$. We also assume that C is an increasing function and that abating pollution exhibits an economy of scale. Finally, we assume that all the price and cost functions are Carathéodory and continuously differentiable operators, and belong to the functional space.

We now describe the behavior of the firms and the government. We then reformulate the lower-level problem into the equilibrium conditions and the bilevel problem into a one level problem.

3.1 The Firm's Problem

Firm i, $i = 1, \ldots, m$, chooses the quantity of production $q_i(t)$, given the price of pollution emission $\pi(t)$ imposed by the government, to maximize the profit.

The total costs incurred by a firm i are given by the firm's production cost and the price of the pollution emission charged by the government times the quantity of pollution emitted. The revenue is generated by the selling of the products. The i-th firm's profit function is:

$$f_i(\pi, q_i) = p_i(t)q_i(t) - \pi(t)h_i(q_i(t)) - c_i(q_i(t)).$$

The maximization problem of firm i, for $i = 1, \ldots, m$, can be expressed as follows:

$$\max_{q_i \in K_i} \int_0^{\bar{t}} f_i(\pi, q_i)dt = - \min_{q_i \in K_i} \int_0^{\bar{t}} \Big(- f_i(\pi, q_i)\Big)dt, \tag{3}$$

where

$$K_i = \Big\{ q_i \in L^2([0, \bar{t}]) : 0 \leq q_i(t) \leq \overline{q}_i(t), \text{ a.e. in } [0, \bar{t}] \Big\}.$$

We note that in the following we will adopt the minimization formulation of (3): this will be useful for applying the variational inequality approach.

Definition 3 A vector $q_i^* \in K_i$, for $i = 1, \ldots, m$, is an equilibrium production of the pollution emission price problem of firm i, if and only if it satisfies the following conditions:

$$-\frac{\partial f_i(\pi, q_i^*)}{\partial q_i} + \mu_i(t) \geq 0, \text{ a.e in } [0, \bar{t}], \ i = 1, \ldots, m,$$

$$\Big(- \frac{\partial f_i(\pi, q_i^*)}{\partial q_i} + \mu_i(t)\Big) q_i^*(t) = 0, \text{ a.e in } [0, \bar{t}], \ i = 1, \ldots, m,$$

$$\mu_i(t) \geq 0, \ \mu_i(t)(q_i^*(t) - \overline{q}_i(t)) = 0, \text{ a.e in } [0, \bar{t}], \ i = 1, \ldots, m,$$

where $\mu_i \in L^2([0, \bar{t}])$ is the Lagrange function.

The meaning of this definition is the following: to each firm i, we associate the function $\mu_i(t)$, related to the production $q_i(t)$, that represents the disutility or shadow price associated with a unitary extra production. The equilibrium conditions state that if there is positive production, then the marginal profit of firm i must be equal to disutility $\mu_i(t)$; if the disutility exceeds the marginal profit of the firm, then it will be unfeasible for the firm to produce.

The function $\mu_i(t)$ is the Lagrange multiplier associated with the upper bound on the product shipment that is unknown a priori. However, this does not affect the results, since the above conditions are equivalent to a variational inequality where $\mu_i(t)$ does not appear.

3.2 *The Government's Problem*

The government chooses the price of pollution imposed to firms to maximize the profit. The total cost incurred by the government is equal to the cost of abating pollution. The revenue is equal to the social profits, given by

$$\sum_{i=1}^{m} \Big(p_i(t) q_i(t) - c_i(q_i(t)) \Big),$$

plus the total price of the pollution emission times the quantity of pollution emitted by all firms. We observe that the social profit has the meaning that the government encourages manufacturing production and progress for our better living.

Thus, the government's profit function is:

$$F(\pi, \kappa(q)) = \sum_{i=1}^{m} \Big(p_i(t) q_i(t) - c_i(q_i(t)) \Big) + \pi(t) \sum_{i=1}^{m} h_i(q_i(t))$$
$$- C\Big(\sum_{i=1}^{m} h_i(q_i(t)) \Big).$$

The government is also a profit-maximizer; then the optimization problem is:

$$\max_{\pi \in P} \int_0^{\bar{t}} F(\pi, \kappa(q)) dt, \tag{4}$$

where

$$P = \Big\{ \pi \in L^2([0, \bar{t}]) : 0 \le \pi(t) \le \overline{\pi}(t), \text{ a.e. in } [0, \bar{t}] \Big\}.$$

Now we can give the bilevel programming model as follows:

$$\max_{\pi \in P} \int_0^{\bar{t}} F(\pi, \kappa(q))dt$$
$$= \max_{\pi \in P} \int_0^{\bar{t}} \Big(\sum_{i=1}^{m} \Big(p_i(t) q_i(t) - c_i(q_i(t)) \Big) + \pi(t) \sum_{i=1}^{m} h_i(q_i(t)) - C\Big(\sum_{i=1}^{m} h_i(q_i(t)) \Big) \Big) dt,$$

where q_i solves the following problem

$$\max_{q_i \in K_i} \int_0^{\bar{t}} f_i(\pi, q_i)dt$$
$$= \max_{q_i \in K_i} \int_0^{\bar{t}} \Big(\sum_{j=1}^{n} p_i(t) q_i(t) - \pi(t) h_i(q_i(t)) - c_i(q_i(t)) \Big) dt,$$
$$i = 1, \ldots, m.$$

The government's problem represents the upper-level problem, whereas the firm's problem defines the lower-level problem. We assume that for each parameter value π, the lower-level problem is a concave problem in the variables $q_i \in K_i$. If the functions $f_i(\pi, q_i)$ and $F(\pi, \kappa(q))$ are strongly concave or strictly concave with respect to q_i and $\kappa(q)$, respectively (in the convex set K_i and $L^2([0, \bar{t}])$, respectively), then for each π the lower-level problem has a unique solution and the bilevel problem is well-defined. Without these conditions in general the solution map is a set-valued map, which makes the objective function of the upper-level problem also a set-valued one.

3.3 The Equilibrium Condition Reformulation

By applying the duality framework to the lower-level problem expressed as a minimization problem, it is possible to reformulate the lower-level problem into the equilibrium conditions and then the bilevel problem into a one level problem; see [34].

Theorem 3 *Let us assume that $f_i(\pi, q_i)$ is strictly concave and differentiable with respect to q_i, for $i = 1, \ldots, m$. Then a vector $q_i^* \in K_i$ is an optimal solution of the lower-level problem (3) if and only if it is a solution to the following evolutionary parametric variational inequality*

$$\int_0^{\bar{t}} \Big(-\frac{\partial f_i(\pi, q_i^*)}{\partial q_i} \Big) (q_i(t) - q_i^*(t)) dt$$
$$= \int_0^{\bar{t}} \Big(-p_i(t) + \frac{\partial c_i(q_i^*)}{\partial q_i} + \pi(t) \frac{\partial h_i(q_i^*)}{\partial q_i} \Big) (q_i(t) - q_i^*(t)) dt \geq 0,$$
$$\forall q_i \in K_i. \qquad (5)$$

Now, we set, for $i = 1, \ldots, m$:

$$\Psi_i(\pi, q_i) = \int_0^{\bar{t}} \Big(- p_i(t) + \frac{\partial c_i(q_i^*)}{\partial q_i} + \pi(t) \frac{\partial h_i(q_i^*)}{\partial q_i} \Big)(q_i(t) - q_i^*(t))dt,$$

and observe that variational inequality (5) is equivalent to the minimization problem

$$\min_{q_i \in K_i} \Psi_i(\pi, q_i) = \Psi_i(\pi, q_i^*) = 0. \tag{6}$$

For $i = 1, \ldots, m$, we consider the Lagrange function associated with optimization problem (6)

$$\mathscr{L}_i(\pi, q_i, \lambda_i, \mu_i) = \Psi_i(\pi, q_i) - \int_0^{\bar{t}} \lambda_i(t) q_i(t) dt + \int_0^{\bar{t}} \mu_i(t)(q_i(t) - \overline{q}_i(t))dt,$$

$\forall q_i \in L^2([0, \bar{t}]), \lambda_i, \mu_i \in L^2([0, \bar{t}])_+$, where

$$L^2([0, \bar{t}])_+ := \{f \in L^2([0, \bar{t})] : f(t) \geq 0 \text{ a.e in } [0, \bar{t}]\}.$$

We now give two preliminary outcomes.

Lemma 1 *Let $q_i^* \in K_i$, for $i = 1, \ldots, m$, be a solution to variational inequality* (5) *and let us introduce the following sets:*

$$I_- := \Big\{ t \in [0, \bar{t}] : q_i^*(t) = 0, \Big\},$$
$$I_0 := \Big\{ t \in [0, \bar{t}] : 0 < q_i^*(t) < \overline{q}_i(t) \Big\},$$
$$I_+ := \Big\{ t \in [0, \bar{t}] : q_i^*(t) = \overline{q}_i(t) \Big\}.$$

Then, it results that

$$\frac{\partial f_i(\pi, q_i^*)}{\partial q_i} \leq 0, \ \text{a.e. in } I_-, \tag{7}$$

$$\frac{\partial f_i(\pi, q_i^*)}{\partial q_i} = 0, \ \text{a.e. in } I_0, \tag{8}$$

$$\frac{\partial f_i(\pi, q_i^*)}{\partial q_i} \geq 0, \ \text{a.e. in } I_+. \tag{9}$$

In order to apply Theorem 3.2 in [33], we set:

$$X = Y = S = L^2([0, \bar{t}], \mathbb{R}^n);$$
$$C = C^* = \{w \in L^2([0, \bar{t}], \mathbb{R}^n) : w(t) \geq 0 \text{ a.e. in } [0, \bar{t}]\};$$
$$g(q_i) = (g_1(q_i), g_2(q_i)) = (-q_i(t), q_i(t) - \overline{q}_i(t)).$$

Theorem 4 *Problem (6) verifies Assumption S at the minimal point q_i^*, for $i = 1, \ldots, m$.*

The following theorem ensures the equivalence of the variational inequality and the equilibrium conditions.

Theorem 5 *q_i^* is a solution to variational inequality (5) if and only if for $i = 1, \ldots, m$ $\exists \lambda_i^*, \mu_i^* \in L^2([0, \bar{t}]) = \mathbb{L}$ such that a.e. in $[0, \bar{t}]$*

(i) $\lambda_i^*(t), \mu_i^*(t) \geq 0$, *a.e. in* $[0, \bar{t}]$,
(ii) $\lambda_i^*(t) q_i^*(t) = 0$, *a.e. in* $[0, \bar{t}]$, $\mu_i^*(t)(q_i^*(t) - \overline{q}_i(t)) = 0$, *a.e. in* $[0, \bar{t}]$,
(iii) $-\dfrac{\partial f_i(\pi, q_i^*)}{\partial q_i} - \lambda_i^*(t) + \mu_i^*(t) = 0$, *a.e. in* $[0, \bar{t}]$.

Now, the bilevel problem can be reformulated into a one-level problem, replacing the lower-level problem with the equilibrium conditions.

Definition 4 The equilibrium condition formulation of the bilevel problem is as follows:

$$\max_{(\pi, q, \lambda, \mu) \in P \times K \times \mathbb{L} \times \mathbb{L}} \int_0^{\bar{t}} F(\pi, \kappa(q)) dt$$
s.t.
$$-\frac{\partial f_i(\pi, q_i)}{\partial q_i} - \lambda_i(t) + \mu_i(t) = 0, \text{ a.e in } [0, \bar{t}], \; i = 1, \ldots, m,$$
$$\lambda_i(t) q_i(t) = 0, \; \mu_i(t)(q_i(t) - \overline{q}_i(t)) = 0, \text{ a.e in } [0, \bar{t}], \; i = 1, \ldots, m,$$
$$\lambda_i(t) \geq 0, \; \mu_i(t) \geq 0, \text{ a.e in } [0, \bar{t}], \; i = 1, \ldots, m. \quad (10)$$

Taking into account (i) and (ii) in Theorem 5, the bilevel problem can be also formulated as follows:

$$\max_{(\pi, q, \mu) \in P \times K \times \mathbb{L}} \int_0^{\bar{t}} F(\pi, \kappa(q)) dt$$
s.t.
$$-\frac{\partial f_i(\pi, q_i)}{\partial q_i} + \mu_i(t) \geq 0, \text{ a.e in } [0, \bar{t}], \; i = 1, \ldots, m,$$

$$\Big(- \frac{\partial f_i(\pi, q_i)}{\partial q_i} + \mu_i(t) \Big) q_i(t) = 0, \text{ a.e in } [0, \bar{t}], \ i = 1, \dots, m,$$

$$\mu_i(t) \geq 0, \ \mu_i(t)(q_i(t) - \overline{q}_i(t)) = 0, \text{ a.e in } [0, \bar{t}], \ i = 1, \dots, m. \quad (11)$$

Theorem 6 *Let q_i^* be a solution to (3) for $i = 1, \dots, m$, then there exist $\pi^* \in L^2([0, \bar{t}])$ and $\mu^* = (\mu_i^*)_i \in \mathbb{L}_+$ such that a.e. in $[0, \bar{t}]$ (π^*, q^*, μ^*) is an optimal solution to problem (11). Conversely, assume that (π^*, q^*, μ^*) be an optimal solution to problem (11), then q_i^* is an optimal solution to (3) for $i = 1, \dots, m$.*

We note that the constraint set of problem 11 is not convex, due to the presence of the nonlinear constraint $\Big(- \frac{\partial f_i(\pi, q_i)}{\partial q_i} + \mu_i(t) \Big) q_i(t) = 0$. A possible way to overcome this problem could be to consider a sequence of convex approximating constraints and apply a regularization procedure. Thus, one should be able to construct an approximate solution and deduce the final solution. We also note that, by Theorem 3, the solution of the lower-level problem can be computed by solving an evolutionary variational inequality; see [8, 10].

3.4 Numerical Results

We now present a small example to show the feasibility of the model, see [34]. We note that the setting is not restrictive since the real situation can be viewed as an iteration of this case. The time horizon is the interval [0, 1]. There are assumed to be two firms producing the quantities $q_i(t)$, $i = 1, 2$, respectively, of a homogeneous product and emitting the same pollution. The prices of the per unit production are $p_1(t)$ and $p_2(t)$, respectively. The charge of the per unit pollution emission is denoted by $\pi(t)$. Firms' productions conditions are as follows:

$$\begin{array}{ll} p_1(t) = 10 + \frac{1}{4}t; & p_2(t) = 8 - \frac{1}{2}t; \\ h_1(q_1) = 2q_1^2(t); & h_2(q_2) = 4q_2(t); \\ c_1(q_1) = 3q_1(t); & c_2(q_2) = q_2^2(t); \\ \overline{q}_1(t) = 10 + t; & \overline{q}_2(t) = 10 + \frac{1}{2}t; \\ \overline{\pi}(t) = 8. & \end{array}$$

The total quantity of pollution is $\kappa(q) = 2q_1^2(t) + 4q_2(t)$ and the abatement cost is $C(\kappa) = 1000 - 4q_1^2(t) - 8q_2(t)$.

The sets of feasible product shipments are given by:

$$K_1 = \Big\{ q_1 \in L^2([0, 1]) : 0 \leq q_1(t) \leq \overline{q}_1(t), \text{ a.e. in } [0, 1] \Big\},$$

$$K_2 = \Big\{ q_2 \in L^2([0, 1]) : 0 \leq q_2(t) \leq \overline{q}_2(t), \text{ a.e. in } [0, 1] \Big\}.$$

Concerning the government's problem, the set of feasible prices is given by:

$$P = \Big\{\pi \in L^2([0, 1]) : 0 \le \pi(t) \le \overline{\pi}(t), \text{ a.e. in } [0, 1]\Big\}.$$

Firms' profit functions are as follows:

$$f_1(\pi, q_1) = p_1(t)q_1(t) - \pi(t)g_1(q_1) - c_1(q_1) = \Big(10 + \frac{1}{4}t\Big)q_1(t) - 2\pi(t)q_1^2(t) - 3q_1(t);$$

$$f_2(\pi, q_2) = p_2(t)q_2(t) - \pi(t)g_2(q_2) - c_2(q_2) = \Big(8 - \frac{1}{2}t\Big)q_2(t) - 4\pi(t)q_2(t) - q_2^2(t).$$

Government's profit function is given by:

$$F(\pi, \kappa(q)) = p_1(t)q_1(t) - c_1(q_1) + p_2(t)q_2(t) - c_2(q_2) + \pi(t)\Big(g_1(q_1) + g_2(q_2)\Big) - C(q)$$

$$= (2p + 4)q_1^2(t) + \Big(7 - \frac{1}{4}t\Big)q_1(t) - q_2^2(t) + \Big(16 + \frac{1}{2}t + 4p\Big)q_2(t) - 1000.$$

Hence, the bilevel model is

$$\max_{\pi \in P} \int_0^1 F(\pi, \kappa(q))dt,$$

where $q_i(t)$, $i = 1, 2$ solves the following problem

$$\max_{q_i \in K_i} \int_0^1 f_i(p_i, q_i)dt.$$

According to (10), the above problem can be formulated as follows:

$$\max_{(\pi,q,\lambda,\mu) \in P \times K \times \mathbb{L} \times \mathbb{L}} \int_0^1 \Big((2p + 4)q_1^2(t) + \Big(7 - \frac{1}{4}t\Big)q_1(t) - q_2^2(t) + \Big(16 + \frac{1}{2}t + 4p\Big)q_2(t) - 1000\Big)dt$$

$$-\Big(7 + \frac{1}{4}t - 8\pi(t)q_1(t)\Big) - \lambda_1(t) + \mu_1(t) = 0, \text{ a.e in } [0, 1], \tag{12}$$

$$-\Big(8 - \frac{1}{2}t - 4\pi(t) - 2q_2(t)\Big) - \lambda_2(t) + \mu_2(t) = 0, \text{ a.e in } [0, 1], \tag{13}$$

$$\lambda_1(t)q_1(t) = 0, \ \lambda_2(t)q_2(t) = 0, \text{ a.e in } [0, 1], \tag{14}$$

$$\mu_1(t)(q_1(t) - \overline{q}_1(t)) = 0, \ \mu_2(t)(q_2(t) - \overline{q}_2(t)) = 0, \text{ a.e in } [0, 1], \tag{15}$$

$$\lambda(t), \ \mu(t) \ge 0, \text{ a.e in } [0, 1], . \tag{16}$$

We note that complementarity conditions (14) and (15) make the feasible set of the above maximization problem nonconvex. Thus, we study all possible combinations of active and non-active multipliers $\lambda_i(t)$, $\mu_i(t)$, $i = 1, 2$. There are fifteen possible cases, but only three feasible cases result in positive profits. They are described in the following.

Case 1. If $\lambda_1(t) = \lambda_2(t) = 0$, $\mu_1(t) \neq 0$, $\mu_2(t) \neq 0$, from (15) it follows that $q_1^*(t) = \overline{q}_1(t)$ and $q_2^*(t) = \overline{q}_2(t)$. We find:

$$F(\pi, \kappa(q^*)) = F(\pi) = \left(240+42\,t+2\,t^2\right) p-470+\frac{165}{2}\,t+\frac{15}{4}\,t^2, \text{ a.e in } [0, 1];$$

hence,

$$\max_{\pi \in P} F(\pi) = F(\overline{\pi}) = 1450 + \frac{837}{2}\,t + \frac{79}{4}\,t^2, \text{ a.e in } [0, 1].$$

Finally,

$$\int_0^1 F(\overline{\pi})dt = \frac{9995}{6}.$$

Case 2. If $\lambda_1(t) = 0$, $\lambda_2(t) \neq 0$, $\mu_1(t) \neq 0$, $\mu_2(t) = 0$, from (14), (15) it follows that $q_1^*(t) = \overline{q}_1(t)$ and $q_2^*(t) = 0$. We find:

$$F(\pi, \kappa(q^*)) = F(\pi)$$
$$= \left(200 + 40\,t + 2\,t^2\right) p - 530 + \frac{179}{2}\,t + \frac{17}{4}\,t^2, \text{ a.e in } [0, 1];$$

hence,

$$\max_{\pi \in P} F(\pi) = F(\overline{\pi}) = 1070 + \frac{819}{2}\,t + \frac{81}{4}\,t^2, \text{ a.e in } [0, 1].$$

Finally,

$$\int_0^1 F(\overline{\pi})dt = \frac{2563}{2}.$$

Case 3. If $\lambda_1(t) = \lambda_2(t) = 0$, $\mu_1(t) \neq 0$, $\mu_2(t) = 0$, from (15) it follows that $q_1^*(t) = \overline{q}_1(t)$ and $q_2^*(t) = \frac{16-t-8p}{4}$. We find:

$$F(\pi, \kappa(q)) = F(\pi)$$
$$= -12\,p^2 + \left(39\,t + 200 + 2\,t^2\right) p - 482 + \frac{171}{2}\,t + \frac{69}{16}\,t^2, \text{ a.e in } [0, 1];$$

hence

$$\max_{\pi \in P} F(\pi) = F(\overline{\pi}) = 350 + \frac{795}{2}t + \frac{325}{16}t^2, \text{ a.e in } [0, 1].$$

Finally

$$\int_0^1 F(\overline{\pi})dt = \frac{26665}{48}.$$

Then the optimal profit is given by

$$\max\left(\frac{9995}{6}, \frac{2563}{2}, \frac{26665}{48}\right) = \frac{9995}{6},$$

that corresponds to the optimal values along the time interval

$$q_1^* = \frac{21}{2},\ q_2^* = \frac{41}{4}t,\ \pi^* = 8.$$

We note that the used procedure allows us to overcome the difficulty due to the nonconvexity of the feasible set; hence, it could suggest an alternative method with respect to the method of convex approximations to solve the initial problem.

4 A Cooperative Approach to Kyoto Protocol Commitments

In this section, we present an evolutionary variational inequality approach to the study of cooperative games in pollution control problems as in [35]. In particular, we study the investment strategies for reducing greenhouse gas emissions, as requested by Kyoto Protocol; see [41]. The Kyoto Protocol prescribes that "Annex I Parties," a list of industrialized countries, must reduce their greenhouse gas emissions below a fixed level. Its first commitment period started in 2008 and ended in 2012. Now, new commitments for Annex I Parties are requested for the period from 1 January 2013 to 31 December 2020. Under the Treaty, countries have disposal of three market-based mechanisms: the Emissions Trading, the Clean Development mechanism, and the Joint Implementation (JI), which is the focus of this model. The JI mechanism allows countries with high abatement costs to reach their targets by investing in countries where the abatement costs are lower. The investor country gets a credit, referred to as emission reduction units (ERUs), for carrying out environmental projects in foreign countries.

We examine the situation in which different countries, aiming at reducing pollution emissions, agree to play a cooperative game, namely, they accept to coordinate both emissions and investment strategies in order to optimize jointly their welfare. The optimal welfare is obtained considering the global maximization of the profit. In order to keep simple the economics of the pollution model, we assume that investments in environmental projects are operated by companies that are then aggregated together and represented by the country to which they belong. For this reason, the country welfare can be assimilated to the total companies' profit. The most general case, in which each country may reduce emissions in a noncooperative manner by investing in both local and foreign projects, is discussed in [40]. Environmental models have been extensively studied in [5–7] and the references cited there. In [2, 27] the authors give a comprehensive review of optimization models in pollution control and other contributions are given in [30].

We now present the cooperative evolutionary pollution control problem and give the equilibrium conditions governing the model (see also [38–40]). We study the system in the finite time horizon $[0, \bar{t}]$, with $\bar{t} > 0$. Let N be the number of countries involved in the Treaty that accept to coordinate their emissions and investment strategies to jointly optimize their welfare, and agree on a rule for sharing the total cooperative reward.

Let $e^i(t)$ denote the gross emissions resulting from the industrial production of country $i \in I$ at time $t \in [0, \bar{t}]$. We assume that the emissions of each country are proportional to the industrial output. Thus, we define the revenue R_i of country i as the function

$$R_i : [0, \bar{t}] \times \mathbb{R}_+ \to \mathbb{R}_+.$$

Let $I^i_j(t)$ be the investment in an environmental project held by country i in country $j = 1, \ldots, N$ at time $t \in [0, \bar{t}]$. We note that whenever $j = i$ variable $I^i_i(t)$ denotes the investment of country i in its own country. We further group the total gross emissions in the column vector $e(t) = (e^1(t), \ldots, e^N(t))^T$. In the JI mechanism, the benefits of the investment lie in the acquisition of emission reduction units (ERUs), which are assumed to be proportional to the investment, namely, the emission reduction units for country i, denoted by ERUs^i, are given by $\text{ERUs}^i = \gamma_{ij}(t) I^i_j(t)$. Here, $\gamma_{ij}(t)$ is a positive technological efficiency parameter depending on both the investor i and the host country j, because in general there is a dependence on both the investor's technologies and laws, and the situation in the host country. The net emissions in country i, namely, the difference between the gross emissions and the reduction resulting from local and foreign investments in the same country, is given by

$$e^i(t) - \sum_{j=1}^{N} \gamma_{ji}(t) I^j_i(t) \geq 0. \tag{17}$$

Let $E_i(t) > 0$ denote a prescribed level of net emissions for each country i, namely, for $i = 1, \ldots, N$ it results

$$e^i(t) - \sum_{j=1}^{N} \gamma_{ji}(t) I^j_i(t) \leq E_i(t). \tag{18}$$

Now, we assume that countries jointly optimize their welfare under the following collective environmental constraint

$$\sum_{i=1}^{N}\Big(e^i(t)-\sum_{j=1}^{N}\gamma_{ji}(t)I_i^j(t)\Big)\le\sum_{i=1}^{N}E_i(t). \tag{19}$$

Constraint (19) imposes a total emission cap on the whole market that is given by the sum of country cap E_i. The above constraint describes an instantaneous relationship, in the sense that a tolerable level of emissions is requested at time t and the constraint must be satisfied at the same time. Clearly, a delayed reaction, namely, requested at time t and verified at time $t+\delta(t)$, where $\delta(t)$ is a nonnegative delay factor, would be more realistic and will be investigated in the future.

We observe that constraints (18) and (19) have redundant variables, thus we simplify the model denoting by $\displaystyle\sum_{j=1}^{N} I_i^j(t) = I_i(t)$ the total investment held by all the countries in country i, and setting $\sum_{j=1}^{N}\gamma_{ji}(t) = \gamma_i(t)$. We further group the total investments in all the countries in the vector $I(t)=(I_1(t),\dots,I_N(t))^T$.

We introduce the investment cost C_i of country i as the function

$$C_i : [0,\bar{t}]\times\mathbb{R}_+\to\mathbb{R}_+,$$

and the damage from pollution as the function D_i, where

$$D_i : [0,\bar{t}]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}_+.$$

We choose as our functional setting the Hilbert space $L^2([0,\bar{t}],\mathbb{R}^{2N})$ of square-integrable functions defined in the closed interval $[0,\bar{t}]$, endowed with the scalar product $\langle\cdot,\cdot\rangle_{L^2}=\int_0^{\bar{t}}\langle\cdot,\cdot\rangle dt$ and the usual associated norm $\|\cdot\|_{L^2}$.

Thus, the set of pairs of feasible emissions and investments is given by:

$$\begin{aligned}K=\Big\{&(e,I)\in L^2([0,\bar{t}],\mathbb{R}^{2N}) : e^i(t)\ge 0, I_i(t)\ge 0, i=1,\dots,N, \text{ a.e. in } [0,\bar{t}];\\&e^i(t)-\gamma_i(t)I_i(t)\ge 0, i=1,\dots,N, \text{ a.e. in } [0,\bar{t}];\\&\sum_{i=1}^{N}\Big(e^i(t)-\gamma_i(t)I_i(t)-E_i(t)\Big)\le 0 \text{ a.e. in } [0,\bar{t}]\Big\}.\end{aligned} \tag{20}$$

We observe that, in the cooperative game, the investment decisions are given by the total investment $\sum_{j=1}^{N} I_i^j(t) = I_i(t)$. Therefore, even if players are jointly implementing environmental projects, joint implementation only takes into account the total investment in country i, that is $I_i(t)$. Thus, cost C_i depends on I_i. If, for instance, we consider a quadratic cost function for country i, it has the form

$$\frac{1}{2}a\Big(I_i^1(t)+\cdots+I_i^N(t)\Big)^2=\frac{1}{2}a\Big(\sum_{j=1}^{N}I_i^j(t)\Big)^2=\frac{1}{2}a(I_i(t))^2, a>0.$$

We also assume that pollution in one country can affect also other countries; hence, damages from pollution of each country, represented by D_i, depend on the net emissions as well as the total investments in all the countries.

Our assumptions are that the functions $R_i(\cdot,\cdot)$, $C_i(\cdot,\cdot)$ and $D_i(\cdot,\cdot,\cdot)$ are measurable in the first variable and continuous in the others.

Moreover, we assume that $\frac{\partial R_i(t,e^i)}{\partial e^i}$, $\frac{\partial C_i(t,I_i)}{\partial I_i}$, $\frac{\partial D_i(t,e,I)}{\partial e^i}$ and $\frac{\partial D_i(t,e,I)}{\partial I_i}$ do exist and be measurable in t and continuous in the other variables. In addition, we require the following conditions:

$$\exists\delta_1^i\in L^2([0,\bar{t}]):\left|\frac{\partial R_i(t,e^i)}{\partial e^i}\right|\le\delta_1^i(t)+|e^i|, \tag{21}$$

$$\exists\delta_2^i\in L^2([0,\bar{t}]):\left|\frac{\partial C_i(t,I_i)}{\partial I_i}\right|\le\delta_2^i(t)+|I_i|, \tag{22}$$

$$\exists\delta_3^i\in L^2([0,\bar{t}]):\left|\frac{\partial D_i(t,e,I)}{\partial e^i}\right|\le\delta_3^i(t)+|e|, \tag{23}$$

$$\exists\delta_4^i\in L^2([0,\bar{t}]):\left|\frac{\partial D_i(t,e,I)}{\partial I_i}\right|\le\delta_4^i(t)+|I|. \tag{24}$$

The goal of countries consists in maximizing the sum of their welfare functions, subject to the collective environmental constraint. Therefore, the global optimization problem in which countries form a coalition and act as a single country is given by

$$\max_{(e,I)\in K}\int_0^{\bar{t}}W(t,e,I)dt, \tag{25}$$

where

$$W(t,e,I)=\sum_{i=1}^{N}W_i(t,e,I),$$

$$W_i(t,e,I)=R_i(t,e^i)-C_i(t,I_i)-D_i(t,e,I).$$

We note that conditions (21)–(24) ensure that $\int_0^{\bar{t}}W(t,e,I)dt$ is well-defined. Finally, we require that function W is concave.

We also remark that it is not possible to study problem (25) considering, instead of the maximum of the integral, the integral of the maximum for each t, because the maximum of the function W in the interval $[0,\bar{t}]$ could not be an integrable function.

4.1 Equilibrium Conditions

We now state the equilibrium conditions governing the model, using complementarity conditions.

Definition 5 A vector of emissions and investments $(e^*, I^*) \in K$ is an equilibrium of the cooperative evolutionary pollution control problem if and only if, for each $i = 1, \ldots, N$, and a.e. in $[0, \bar{t}]$, it satisfies the system of inequalities

$$-\frac{\partial W(t, e^*, I^*)}{\partial e^i} + \overline{\nu}(t) - \overline{\tau_i}(t) \geq 0, \quad (26)$$

$$-\frac{\partial W(t, e^*, I^*)}{\partial I_i} - \gamma_i \overline{\nu}(t) + \gamma_i \overline{\tau_i}(t) \geq 0, \quad (27)$$

and equalities

$$\left(-\frac{\partial W(t, e^*, I^*)}{\partial e^i} + \overline{\nu}(t) - \overline{\tau_i}(t) \right) e^{*i}(t) = 0, \quad (28)$$

$$\left(-\frac{\partial W(t, e^*, I^*)}{\partial I_i} - \gamma_i \overline{\nu}(t) + \gamma_i \overline{\tau_i}(t) \right) I_i^*(t) = 0, \quad (29)$$

simultaneously, where $\overline{\nu} \in L^2([0, \bar{t}])_+$ is the dual variable related to the collective environmental constraint and $\overline{\tau_i} \in L^2([0, \bar{t}])_+$, $i = 1, \ldots, N$, is the dual variable related to nonnegativity of net emissions. As usual, $L^2([0, \bar{t}])_+$ denotes the positive cone of $L^2([0, \bar{t}])$.

The dual variable $\overline{\nu}$ can be viewed as the marginal abatement cost of all the countries. We can also refer to the Emission Trading System that puts a limit on overall emissions from covered installations which is reduced each year. Within this limit, companies or countries can buy and sell emission allowances as needed. This *cap-and-trade* approach gives companies the flexibility they need to cut their emissions in the most cost-effective way. Under this perspective, the dual variable $\overline{\nu}$ can be interpreted as the price of emission allowances that are exchanged in order to cover the emissions. In addition, countries can buy credits from certain types of approved emission-saving projects around the world, namely, using the JI mechanism. On the other hand, $\overline{\tau_i}$ can be viewed as the marginal abatement revenue of country i. It is equal to zero if the net emissions are positive; whereas if it is positive, then the net emissions are minimal and equal to zero.

As a consequence, conditions (26) and (28) have the following interpretation.

If the total marginal welfare in case country i of the coalition emits pollution equals the difference between the marginal abatement cost of all the countries and the marginal abatement revenue of country i, namely,

$$\frac{\partial W(t, e^*, I^*)}{\partial e^i} = \overline{\nu}(t) - \overline{\tau_i}(t),$$

then by (28) we find that $e^{*i}(t) \geq 0$. This means that it is possible for the coalition of countries to emit in country i. The total marginal welfare of emitting $\frac{\partial W(t,e^*,I^*)}{\partial e^i}$ represents the welfare or the advantage for the coalition with respect to the pollution emitted by country i. The difference $\overline{\nu}(t)-\overline{\tau_i}(t)$ can be regarded as the total pollution threshold under JI. Therefore, the coalition of countries can emit in country i if the pollution threshold is reached.

If the marginal welfare in case country i of the coalition emits pollution is less than the difference between the marginal abatement cost of all the countries and the marginal abatement revenue of country i, namely,

$$\frac{\partial W(t, e^*, I^*)}{\partial e^i} < \overline{\nu}(t) - \overline{\tau_i}(t),$$

then by (28) we find that $e^{*i}(t) = 0$. This means that it will not be convenient for the coalition to emit.

An analogous interpretation can be given for conditions (27) and (29).

If the total marginal welfare of investing in country i of the coalition equals the difference between the marginal abatement revenue of country i and the marginal abatement cost, also taking into account the technology of the country, namely,

$$\frac{\partial W(t, e^*, I^*)}{\partial I_i} = \gamma_i \overline{\tau_i}(t) - \gamma_i \overline{\nu}(t),$$

then by (29) we find $I_i^*(t) \geq 0$, and the coalition of countries is willing to invest in environmental projects. The total marginal welfare of investing $\frac{\partial W(t,e^*,I^*)}{\partial I_i}$ represents the welfare or the advantage for the coalition with respect to the investments in environmental projects in country i. The difference $\overline{\tau_i}(t) - \overline{\nu}(t)$ represents the effectiveness of investments under JI and measures the benefits coming from investments. Therefore, the coalition of countries will invest in country i if the effectiveness level is reached.

If the total marginal welfare of investing in country i of the coalition is less than the difference between the marginal abatement revenue of country i and the marginal abatement cost, also taking into account the technology of the country, namely,

$$\frac{\partial W(t, e^*, I^*)}{\partial I_i} < \gamma_i \overline{\tau_i}(t) - \gamma_i \overline{\nu}(t),$$

then by (29) we find $I_i^*(t) = 0$, and it will not be convenient for the coalition of countries to invest.

The equilibrium of the cooperative evolutionary pollution control problem can be characterized as a solution to an evolutionary variational inequality.

Theorem 7 *$(e^*, I^*) \in K$ is an equilibrium of the cooperative evolutionary pollution control problem if and only if it satisfies the following variational inequality:*

$$\sum_{i=1}^{N} \int_0^{\bar{t}} \left(-\frac{\partial W(t, e^*, I^*)}{\partial e^i}(e^i(t) - e^{*i}(t)) - \frac{\partial W(t, e^*, I^*)}{\partial I_i}(I_i(t) - I_i^*(t)) \right) dt \geq 0,$$

$$\forall (e, I) \in K. \tag{30}$$

Proof The proof uses arguments as in Section 4.1.2 in [8]. □

Now, we set:

$$\Psi(e, I) = \sum_{i=1}^{N} \int_0^{\bar{t}} -\frac{\partial W(t, e^*, I^*)}{\partial e^i}(e^i(t) - e^{*i}(t))dt$$
$$+ \sum_{i=1}^{N} \int_0^{\bar{t}} -\frac{\partial W(t, e^*, I^*)}{\partial I_i}(I_i(t) - I_i^*(t)dt,$$

and observe that variational inequality (30) is equivalent to the minimization problem

$$\min_{(e,I) \in K} \Psi(e, I) = \Psi(e^*, I^*) = 0. \tag{31}$$

We now apply the duality framework to our problem and prove that *Assumption S* is verified. We first recall two preliminary results.

Lemma 2 *Let $(e^*, I^*) \in K$ be a solution to variational inequality (30) and introduce the following sets for $i = 1, \ldots, N$*

$$V_1^i = \left\{ t \in [0, \bar{t}] : e^{*i}(t) = 0 \right\},$$
$$V_2^i = \left\{ t \in [0, \bar{t}] : 0 < \gamma_i(t) I_i^*(t) < e^{*i}(t) \right\},$$
$$V_3^i = \left\{ t \in [0, \bar{t}] : 0 < e^{*i}(t) = \gamma_i(t) I_i^*(t) \right\}.$$

Then, for $i = 1, \ldots, N$, it results that

$$-\frac{\partial W(t, e^*, I^*)}{\partial e^i} \geq 0 \text{ and } -\frac{\partial W(t, e^*, I^*)}{\partial I_i} = 0 \text{ a.e. in } V_1^i,$$
$$-\frac{\partial W(t, e^*, I^*)}{\partial e^i} = 0 \text{ and } -\frac{\partial W(t, e^*, I^*)}{\partial I_i} = 0 \text{ a.e. in } V_2^i,$$
$$-\frac{\partial W(t, e^*, I^*)}{\partial e^i}\gamma_i(t) - \frac{\partial W(t, e^*, I^*)}{\partial I_i} = 0 \text{ a.e. in } V_3^i,$$

$$-\frac{\partial W(t, e^*, I^*)}{\partial e^i} \leq 0 \; a.e. \; in \; V_3^i.$$

Theorem 8 *Problem (31) verifies Assumption S at the minimal point (e^*, I^*).*

4.2 Main Result

In this section we state the main theorem.

Theorem 9 *If $(e^*, I^*) \in K$ is a solution to (30), then there exist $\overline{\lambda}$, $\overline{\mu}$, $\overline{\tau} \in L^2([0, \overline{t}], \mathbb{R}^N)$, $\overline{\nu} \in L^2([0, \overline{t}])$ such that, a.e. in $[0, \overline{t}]$ and for $i = 1, \ldots, N$:*

(i) $\overline{\lambda_i}(t), \overline{\mu_i}(t), \overline{\nu}(t), \overline{\tau_i}(t) \geq 0$;

(ii) $$\begin{cases} \overline{\lambda_i}(t) e^{*i}(t) = 0\,, \overline{\mu_i}(t) I^{*i}(t) = 0; \\ \overline{\nu}(t) \sum_{i=1}^N \Big(e^{*i}(t) - \gamma_i(t) I_i^*(t) - E_i(t) \Big) = 0; \\ \overline{\tau_i}(t)(e^{*i}(t) - \gamma_i(t) I_i^*(t)) = 0; \end{cases}$$

(iii) $$\begin{cases} -\dfrac{\partial W(t, e^*, I^*)}{\partial e^i} + \overline{\nu}(t) - \overline{\tau_i}(t) = \overline{\lambda_i}(t); \\ -\dfrac{\partial W(t, e^*, I^*)}{\partial I_i} - \gamma_i(t)\overline{\nu}(t) + \gamma_i(t)\overline{\tau_i}(t) = \overline{\mu_i}(t); \\ \Big(-\dfrac{\partial W(t, e^*, I^*)}{\partial e^i} + \overline{\nu}(t) - \overline{\tau_i}(t) \Big) e^{*i}(t) = 0; \\ \Big(-\dfrac{\partial W(t, e^*, I^*)}{\partial I_i} - \gamma_i(t)\overline{\nu}(t) + \gamma_i(t)\overline{\tau_i}(t) \Big) I^{*i}(t) = 0. \end{cases}$$

We now describe some relevant consequences of the previous result and provide some suggestions to improve the environmental policies. Dual variables λ_i and μ_i regulate the whole pollution control system. In particular, λ_i represents a control variable on emissions; whereas μ_i is a control variable on investments. We can focus on some cases, considering active or non-active constraints.

(a) We assume that there exist a country $\hat{i}$ and a set $S_{\hat{i}}^1 \subset [0, \overline{t}]$ with positive measure $m(S_{\hat{i}}^1) > 0$, such that $\overline{\lambda}_{\hat{i}}(t) > 0$ in $S_{\hat{i}}^1$. From the first of (ii) it follows that $e^{*\hat{i}}(t) = 0$ a.e. in $S_{\hat{i}}^1$. From constraint (17), we find $I_{\hat{i}}^*(t) = 0$. Lemma 2 implies that $\frac{\partial W(t, e^*, I^*)}{\partial e^{\hat{i}}} \leq 0$, namely, emitting in country $\hat{i}$ is not convenient. In addition, from the first of (iii), we find

$$\frac{\partial W(t, e^*, I^*)}{\partial e^{\hat{i}}} < \overline{\nu}(t) - \overline{\tau}_{\hat{i}}(t),$$

namely, the advantage of emitting is below the pollution threshold within JI; hence, it is better not to change strategy and not to emit.

Thus, if there exists a country $\hat{i}$ such that

$$S_{\hat{i}}^1 = \{t \in [0, \bar{t}] : \overline{\lambda}_{\hat{i}}(t) > 0\} \neq \emptyset, \quad m(S_{\hat{i}}^1) > 0,$$

then set $S_{\hat{i}}^1$ represents the time in which emissions are null and there are no investments in environmental projects. This is an ideal situation of a virtuous country.

If we assume that there exist a country $\hat{i}$ and a set $S_{\hat{i}}^2 \subset [0, \bar{t}]$ with positive measure $m(S_{\hat{i}}^2) > 0$, such that $\overline{\lambda}_{\hat{i}}(t) = 0$ in $S_{\hat{i}}^2$, then $e^{*\hat{i}}(t) \geq 0$ a.e. in $S_{\hat{i}}^2$.

Now, if $e^{*\hat{i}}(t) - \gamma_{\hat{i}}(t) I_{\hat{i}}^*(t) > 0$, from Lemma 2, we find

$$\frac{\partial W(t, e^*, I^*)}{\partial e^{\hat{i}}} = \frac{\partial W(t, e^*, I^*)}{\partial I\hat{i}} = 0,$$

namely, there is no advantage of emitting and no advantage of investing in country $\hat{i}$. In addition, $\overline{\tau}_{\hat{i}}(t) = 0$ and from the first of (iii), we deduce that also

$$0 = \frac{\partial W(t, e^*, I^*)}{\partial e^{\hat{i}}} = \overline{\nu}(t) - \overline{\tau}_{\hat{i}}(t) = \overline{\nu}(t).$$

In this case, the advantage coming from emissions of country $\hat{i}$ equals the pollution threshold, and the coalition is indifferent to follow JI.

If $e^{*\hat{i}}(t) - \gamma_{\hat{i}}(t) I_{\hat{i}}^*(t) = 0$, then the gross emissions are balanced by the ERUs. From Lemma 2, we find

$$\frac{\partial W(t, e^*, I^*)}{\partial e^{\hat{i}}} \geq 0,$$

namely, the coalition can take advantage from emitting in country $\hat{i}$. In addition, from the first of (iii), we find

$$\frac{\partial W(t, e^*, I^*)}{\partial e^{\hat{i}}} = \overline{\nu}(t) - \overline{\tau}_{\hat{i}}(t).$$

In this case, the welfare coming from emissions of country $\hat{i}$ equals the pollution level suggested by JI and there is an incentive to follow JI.

Thus, if there exists a country $\hat{i}$ such that

$$S_{\hat{i}}^2 = \{t \in [0, \bar{t}] : \overline{\lambda}_{\hat{i}}(t) = 0\} \neq \emptyset, \quad m(S_{\hat{i}}^2) > 0,$$

then set $S^2_{\hat{i}}$ represents the time in which emissions are nonnegative and JI implementation is desirable.

(b) We assume that there exist a country $\hat{i}$ and a set $S^3_{\hat{i}} \subset [0, \bar{t}]$ with positive measure $m(S^3_{\hat{i}}) > 0$, such that $\overline{\mu}_{\hat{i}}(t) > 0$ in $S^3_{\hat{i}}$. From the first of (ii) it follows that $I^*_{\hat{i}}(t) = 0$. From the last of (ii) we find that $\overline{\tau}_{\hat{i}}(t)e^{*\hat{i}}(t) = 0$. If $e^{*\hat{i}}(t) > 0$, then $\overline{\tau}_{\hat{i}}(t) = 0$ and from the second of (iii), we find

$$\frac{\partial W(t, e^*, I^*)}{\partial I_{\hat{i}}} < -\gamma_{\hat{i}}(t)\overline{\nu}(t),$$

namely, the advantage of investing in country $\hat{i}$ decreases with the technology of the country and is negatively proportional to the marginal abatement cost. If $e^{*\hat{i}}(t) = 0$, we find a previous examined case. If there exists a country $\hat{i}$ such that

$$S^3_{\hat{i}} = \{t \in [0, \bar{t}] : \overline{\mu}_{\hat{i}}(t) > 0\} \neq \emptyset \quad m(S^3_{\hat{i}}) > 0,$$

then set $S^3_{\hat{i}}$ represents the time in which the coalition does not invest in country $\hat{i}$ with pollution control projects and emissions are at a tolerable level.

If there exist a country $\hat{i}$ and a set $S^4_{\hat{i}} \subset [0, \bar{t}]$ with positive measure $m(S^4_{\hat{i}}) > 0$, such that $\overline{\mu}_{\hat{i}}(t) = 0$ in $S^4_{\hat{i}}$. From the first of (ii) it follows that $I^*_{\hat{i}}(t) \geq 0$. From the second of (ii), we find

$$\frac{\partial W(t, e^*, I^*)}{\partial I_{\hat{i}}} = \gamma_{\hat{i}}(t)(\overline{\tau}_{\hat{i}}(t) - \overline{\nu}(t)),$$

namely, the advantage of investing increases with the technology of the country and is proportional to the effectiveness of JI. If there exists a country $\hat{i}$ such that

$$S^4_{\hat{i}} = \{t \in [0, \bar{t}] : \overline{\mu}_{\hat{i}}(t) = 0\} \neq \emptyset \quad m(S^4_{\hat{i}}) > 0,$$

then set $S^4_{\hat{i}}$ represents the time in which country $\hat{i}$ invests and investments are convenient if

$$\frac{\partial W(t, e^*, I^*)}{\partial I_{\hat{i}}} = \gamma_{\hat{i}}(t)(\overline{\tau}_{\hat{i}}(t) - \overline{\nu}(t)).$$

Theorem 10 *If there exist* $\overline{\lambda}$, $\overline{\mu}$, $\overline{\tau} \in L^2([0, \bar{t}], \mathbb{R}^N_+)$, $\overline{\nu} \in L^2([0, \bar{t}])_+$ *satisfying conditions* (i)–(iii) *of Theorem* 9, (e^*, I^*) *is a cooperative evolutionary pollution control equilibrium.*

Theorem 11 *The following conditions are equivalent:*

- (e^*, I^*) *is a cooperative evolutionary pollution control equilibrium;*
- (e^*, I^*) *is a solution to variational inequality* (30)*;*
- *there exist* $\overline{\lambda}, \overline{\mu}, \overline{\tau} \in L^2([0, \overline{t}], \mathbb{R}^N_+)$, $\overline{\nu} \in L^2([0, \overline{t}])_+$ *satisfying conditions* (i)–(iii) *of Theorem* 9.

4.3 An Example

In this section, we present a small numerical example. In the time interval $[0, \overline{t}] = [\frac{1}{2}, \frac{3}{2}]$, we consider three countries (labeled as 1, 2, and 3, respectively) characterized by the functions:

$$R_i(t, e^i) = -\frac{1}{2}(e^i(t))^2 + 300\, e^i(t),$$

$$C_i(t, I_i) = 75(I_i(t))^2,$$

$$D_i(t, e, I) = (e^1(t) - \gamma_1(t)I_1(t) + e^2(t) - \gamma_2(t)I_2(t) + e^3(t) - \gamma_3(t)I_3(t))^2,$$

for $i = 1, 2, 3$. Moreover, we set: $\gamma_1(t) = \frac{1}{2}t$, $\gamma_2(t) = 2t$, $\gamma_3(t) = t$, $E_1(t) = 50t + 1$, $E_2(t) = 30t$, $E_3(t) = 25t + 2$.

The set of feasible solutions is given by

$$\mathbb{K} = \Big\{(e, I) \in L^2\Big(\Big[\frac{1}{2}, \frac{3}{2}\Big], \mathbb{R}^6\Big) : e^i(t) \geq 0,\ \ I_i(t) \geq 0, i = 1, 2, 3, \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big];$$
$$e^i(t) - \gamma_i(t)I_i(t) \geq 0, i = 1, 2, 3, \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big];$$
$$\sum_{i=1}^{3} \Big(e^i(t) - \gamma_i(t)I_i(t) - E_i(t)\Big) \leq 0 \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big]\Big\}.$$

Thus, the cooperative pollution control problem is described by the evolutionary variational inequality:

$$\sum_{i=1}^{3} \Bigg(\int_{\frac{1}{2}}^{\frac{3}{2}} -\frac{\partial W(t, e^*, I^*)}{\partial e^i}(e^i(t) - e^{*i}(t))dt$$
$$+ \int_{\frac{1}{2}}^{\frac{3}{2}} -\frac{\partial W(t, e^*, I^*)}{\partial I_i}(I_i(t) - I_i^*(t))dt \Bigg) \geq 0,$$

$\forall (e, I) \in K,$

where

$$-\frac{\partial W(t,e,I)}{\partial e^1} = 7e^1(t)+6e^2(t)+6e^3(t)-t(3I_1(t)+12I_2(t)+6I_3(t))-300,$$
$$-\frac{\partial W(t,e,I)}{\partial e^2} = 6e^1(t)+7e^2(t)+6e^3(t)-t(3I_1(t)+12I_2(t)+6I_3(t))-300,$$
$$-\frac{\partial W(t,e,I)}{\partial e^3} = 6e^1(t)+6e^2(t)+7e^3(t)-t(3I_1(t)+12I_2(t)+6I_3)-300,$$
$$-\frac{\partial W(t,e,I)}{\partial I_1}=150I_1(t)-3t\Big(e^1(t)+e^2(t)+e^3(t)-\frac{1}{2}tI_1(t)-2tI_2(t)-tI_3(t)\Big),$$
$$-\frac{\partial W(t,e,I)}{\partial I_2}=150I_2(t)-12t\Big(e^1(t)+e^2(t)+e^3(t)-\frac{1}{2}tI_1(t)-2tI_2(t)-tI_3(t)\Big),$$
$$-\frac{\partial W(t,e,I)}{\partial I_3}=150I_3(t)-6t\Big(e^1(t)+e^2(t)+e^3(t)-\frac{1}{2}tI_1(t)-2tI_2(t)-tI_3(t)\Big).$$

We point out that the above problem admits solutions; see [32]. We also observe that the problem is equivalent to the following one

$$\sum_{i=1}^{3}\Bigg(-\frac{\partial W(t,e^*,I^*)}{\partial e^i}(e^i(t)-e^{*i}(t))-\frac{\partial W(t,e^*,I^*)}{\partial I_i}(I_i(t)-I_i^*(t))\Bigg)\geq 0,$$
$$\forall (e,I)\in K \text{ a.e. in } \left[\frac{1}{2},\frac{3}{2}\right]. \tag{32}$$

We now compute the exact solution of (32) applying the direct method as in [31]. To this end, we exploit the constraints and derive the values of some variables. We first introduce the slack variables $h_1, h_2, z \in L^2_+\left(\left[\frac{1}{2},\frac{3}{2}\right]\right)$ such that

$$e^i(t)-\gamma_i(t)I_i(t)=h_i(t),\ i=1,2, \text{ a.e. in } \left[\frac{1}{2},\frac{3}{2}\right],$$
$$\sum_{i=1}^{3}\Big(e^i(t)-\gamma_i(t)I_i(t)\Big)=\sum_{i=1}^{3}E_i(t)-z(t) \text{ a.e. in } \left[\frac{1}{2},\frac{3}{2}\right].$$

Now, we set

$$I_1(t)=\frac{e^1(t)-h_1(t)}{\gamma_1(t)},\quad I_2(t)=\frac{e^2(t)-h_2(t)}{\gamma_2(t)}, \tag{33}$$
$$e_3(t)=\sum_{i=1}^{3}E_i(t)+\gamma_3(t)I_3(t)-z(t)-h_1(t)-h_2(t), \tag{34}$$

and

$$\tilde{\mathbb{K}} = \Big\{(e^1, e^2, I_3, h_1, h_2, z) \in L^2\Big(\Big[\frac{1}{2}, \frac{3}{2}\Big], \mathbb{R}^6\Big) :$$
$$e^1(t), e^2(t), I_3(t), h_1(t), h_2(t), z(t) \geq 0 \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big],$$
$$\frac{e^1(t) - h_1(t)}{\gamma_1(t)} \geq 0, \frac{e^2(t) - h_2(t)}{\gamma_2(t)} \geq 0 \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big],$$
$$\sum_{i=1}^{3} E_i(t) + \gamma_3(t) I_3(t) - z(t) - h_1(t) - h_2(t) \geq 0, \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big]\Big\}.$$

Therefore, using (33) and (34), variational inequality (32) becomes:

$$\Big(\frac{(e_1^*(t) - 300)t^2 - 600h_1^*(t) + 600e_1^*(t)}{t^2}\Big)(e^1(t) - e^{*1}(t))$$
$$+\Big(\frac{945t^3 + (-9z^*(t) + 2e_2^*(t) - 573)t^2 - 300h_1^*(t) + 300e_1^*(t)}{2t^2}\Big)(e^2(t) - e^{*2}(t))$$
$$+\Big((I_3^*(t) + 105)t^2 - (z^*(t) + h_1^*(t) + h_2^*(t) + 297)t + 150I_3^*(t)\Big)(I_3(t) - I_3^*(t))$$
$$+\Big(\frac{-(I_3^*(t) + 105)t^3 + (z^*(t) + h_1^*(t) + h_2^*(t) + 297)t^2 + 600h_1^*(t) - 600e_1^*(t)}{t^2}\Big)$$
$$(h_1(t) - h_1^*(t))$$
$$+\Big(\frac{-(2I_3^*(t) + 210)t^3 + (2z^*(t) + 2h_1^*(t) + 2h_2^*(t) + 594)t^2 + 75h_2^*(t) - 75e_2^*(t)}{2t^2}\Big)$$
$$(h_2(t) - h_2^*(t))$$
$$+\Big(-(I_3^*(t) + 735)t + 7z^*(t) + h_1^*(t) + h_2^*(t) + 279\Big)(z(t) - z^*(t)) \geq 0,$$
$$\forall (e^1, e^2, I_3, h_1, h_2, z) \in \tilde{K} \text{ a.e. in } \Big[\frac{1}{2}, \frac{3}{2}\Big].$$

We solve the system

$$\begin{cases} \Gamma_1 = (e_1^*(t) - 300)t^2 - 600h_1^*(t) + 600e_1^*(t) = 0 \\ \Gamma_2 = 945t^3 + (-9z^*(t) + 2e_2^*(t) - 573)t^2 - 300h_1^*(t) + 300e_1^*(t) = 0 \\ \Gamma_3 = (I_3^*(t) + 105)t^2 - (z^*(t) + h_1^*(t) + h_2^*(t) + 297)t + 150I_3^*(t) = 0 \\ \Gamma_4 = -(I_3^*(t)+105)t^3+(z^*(t)+h_1^*(t)+h_2^*(t)+297)t^2+600h_1^*(t)-600e_1^*(t)=0 \\ \Gamma_5 = -(2I_3^*(t)+210)t^3+(2z^*(t)+2h_1^*(t)+2h_2^*(t)+594)t^2+75h_2^*(t)-75e_2^*(t)=0 \\ \Gamma_6 = -(I_3^*(t) + 735)t + 7z^*(t) + h_1^*(t) + h_2^*(t) + 279 = 0, \end{cases}$$

and obtain the final solution

$$
\begin{cases}
e^{*1}(t) = e^{*2}(t) = e^{*3}(t) = \dfrac{6300t^2 + 30000}{21t^2 + 1900} \\
I_1^*(t) = \dfrac{1800t}{21t^2 + 1900} \\
I_2^*(t) = \dfrac{7200t}{21t^2 + 1900} \\
I_3^*(t) = \dfrac{3600t}{21t^2 + 1900} \\
h_1^*(t) = \dfrac{5400t^2 + 30000}{21t^2 + 1900} \\
h_2^*(t) = \dfrac{-8100t^2 + 30000}{21t^2 + 1900} \\
z^*(t) = \dfrac{2205t^3 + 63t^2 + 199500t - 84300}{21t^2 + 1900}.
\end{cases}
$$

Once the solution of the variational inequality is known, it is possible to compute Lagrange multipliers from system (ii)–(iii). Since $e^{*1}(t)$, $e^{*2}(t)$, $e^{*3}(t) > 0$, then $\overline{\lambda}_1(t) = \overline{\lambda}_2(t) = \overline{\lambda}_3(t) = 0$. Moreover, it results

$$
\begin{cases}
e^{*1}(t) - \gamma_1(t) I_1^*(t) = \dfrac{5400t^2 + 30000}{21t^2 + 1900} \\
e^{*2}(t) - \gamma_2(t) I_2^*(t) = \dfrac{-8100t^2 + 30000}{21t^2 + 1900} \\
e^{*3}(t) - \gamma_3(t) I_3^*(t) = \dfrac{2700t^2 + 30000}{21t^2 + 1900},
\end{cases}
$$

and from the last of (ii), we find that $\overline{\tau}_1(t) = \overline{\tau}_2(t) = \overline{\tau}_3(t) = 0$. Finally, from the first of (iii), since $\frac{\partial W(t,e^*,I^*)}{\partial e^i} = 0$, $i = 1, 2, 3$, we have that $\overline{\nu}(t) = 0$.

The results confirm the behaviors of emissions and investments at equilibrium (see Definition 5 and Theorem 9). In fact, the emissions e^{*1}, e^{*2}, e^{*3} are positive almost everywhere, so that the advantage for the coalition of countries, in case one of the country had the possibility to emit some more, equals the pollution threshold which is null. In addition, $I_1^*(t)$, $I_2^*(t)$, $I_3^*(t) > 0$, then $\overline{\mu}_1(t) = \overline{\mu}_2(t) = \overline{\mu}_3(t) = 0$ and from the second of (iii) we have $\frac{\partial W(t,e^*,I^*)}{\partial I_i} = 0$, $i = 1, 2, 3$. In this case, the coalition is indifferent to follow JI. In other words, countries take advantage from operating together and manage to balance marginal revenue and marginal investment so as to reach a profitable condition.

5 Walrasian Equilibrium Problem

In this section, we present the evolutionary competitive equilibrium for a Walrasian pure exchange economy; see the seminal paper [43]. We assume that data are time-dependent and, in order to have a more realistic model, the excess demand function depends on the current price and on previous events of the market (see [23]). Therefore, a memory term is introduced. During a period of time $[0, \bar{t}]$ we consider a pure exchange economy with $l > 1$ different commodities. At time t, at each commodity j we associate a nonnegative price $p^j(t)$, where

$$p^j : [0, \bar{t}] \to \mathbb{R}\,, \quad j = 1, \dots, l\,, \qquad p^j \in L^2([0, \bar{t}], \mathbb{R})\,.$$

Hence, the price vector $p = (p^1, p^2, \dots, p^l) \in L^2([0, \bar{t}], \mathbb{R}^l) = L$. Let us denote by z^j the aggregate excess demand function relative to the commodity j:

$$z^j : [0, \bar{t}] \times \mathbb{R}^l \to \mathbb{R},\ j = 1, \dots, l\,, \qquad (t, p) \to z^j(t, p)\,,$$

and by $z(t, p) = (z^1(t, p), \dots, z^l(t, p))$ the aggregate excess demand vector.

As usual in economy, we assume that z be homogeneous of degree zero in p, that is, for all p, $z(t, \alpha p) = z(t, p)$ with $\alpha > 0$ a.e. in $[0, \bar{t}]$. Because of homogeneity, the price may be normalized, so that they take values in the set:

$$S_0 := \{p \in L :\ p^j(t) \geq 0,\ j = 1, \dots, l,\ \sum_{j=1}^{l} p^j(t) = 1 \text{ a. e. in } [0, \bar{t}]\}\,.$$

In order to avoid some "free" commodity, it is convenient to fix a minimum price for each commodity j. In this model, it is convenient to fix, for each commodity j, a minimal price $\underline{p}^j(t)$ at the time t. We suppose that $\underline{p} : [0, \bar{t}] \to \mathbb{R}$ belongs to L and it is such that a. e. in $[0, \bar{t}]$: $\underline{p}^j(t) > 0$ and for all $j = 1, \dots, l$, $\underline{p}^j(t) < \dfrac{1}{l}$. Then the feasible set becomes:

$$S := \{p \in L :\ p^j(t) \geq \underline{p}^j(t),\ j = 1, \dots, l,\ \sum_{j=1}^{l} p^j(t) = 1 \text{ a. e. in } [0, \bar{t}]\}\,.$$

Since our aim is to provide a model closer to reality, for a Walrasian pure exchange equilibrium problem, we suppose that the price trend at time t be affected by the previous events of the market. So, we introduce the aggregate excess demand function with memory term:

$$Z : [0, \bar{t}] \times \mathbb{R}^l \to \mathbb{R}^l\,,$$

$$Z(t, p) = z(t, p) + \int_0^t I(t - s) p(s)\, ds\,,$$

where I is a nonnegative definite $l \times l$ matrix with entries in $L^2([0, \bar{t}], \mathbb{R})$. It is worth emphasizing the role of the matrix I. The entries of the matrix I represent the information of past trade of the market and they act on equilibrium solutions on the current time. Then, the new aggregate excess demand function takes into account a memory expressed in an integral form and it can also be interpreted as adjustment factors of prices. The meaning of the integral term is that it expresses the equilibrium distribution in which the commodity price incur at time t, and, hence, the effect of the previous situation on the present one. Moreover, the memory term is strictly connected with the concept of delay: the integral term represents the delay of the equilibrium solution, due to the previous equilibrium state.

We suppose that Z satisfies the Walras' law:

$$\langle Z(t, p), p(t) - \underline{p}(t)\rangle = 0 \quad \text{a. e. in } [0, \bar{t}] \quad \forall p \in S\,. \tag{35}$$

We require that the following growth condition holds: there exist $B \in L^2([0, \bar{t}])$ and $A \in L^\infty([0, \bar{t}])$ such that for $t \in [0, \bar{t}]$:

$$\|z(t, p)\| \leq A(t)\|p(t)\| + B(t)\,, \qquad \forall p \in S(t)\,,$$

where

$$S(t) := \{p(t) \in \mathbb{R}^l_+ : \; p(t) \geq \underline{p}^j(t)\,, \; \sum_{j=1}^{l} p^j(t) = 1\}\,.$$

Taking into account that matrix entries are in L^2, it is easy to prove that $\int_0^t I(t-s)p(s)ds$ is in L^2. The definition of a Walrasian equilibrium with memory term is now stated:

Definition 6 A price vector $\widehat{p} \in S$ is a dynamic Walrasian equilibrium vector for a pure exchange model with memory term if and only if

$$Z(t, \widehat{p}) \leq 0 \quad \text{a.e. in } [0, \bar{t}]\,.$$

We observe that, since Z satisfies the Walras' law, the equilibrium condition can be rewritten in the following way:

Definition 7 A price vector $\widehat{p} \in S$ is a dynamic Walrasian equilibrium vector for a pure exchange model with memory term, if and only if a. e. in $[0, \bar{t}]$:

$$Z^j(t, \widehat{p}) \begin{cases} \leq 0, & if \quad \widehat{p}^j(t) = \underline{p}^j(t), \\ = 0, & if \quad \widehat{p}^j(t) > \underline{p}^j(t)\,. \end{cases}$$

Now, we can characterize the equilibrium as a solution to an evolutionary variational inequality:

Theorem 12 *A price vector $\widehat{p} \in S$ is a dynamic Walrasian equilibrium with memory term, if and only if $\widehat{p}$ is a solution to the following evolutionary variational inequality:*

$$\langle Z(\widehat{p}), p - \widehat{p} \rangle_L \leq 0, \quad \forall p \in S\,. \tag{36}$$

Now, we can characterize a dynamic Walrasian price equilibrium with memory term by means of the Lagrangian multipliers. In particular, the following result holds:

Theorem 13 *$\widehat{p} \in S$ is a solution to the variational problem (36) if and only if there exist $\widehat{\alpha} \in L^2([0,\bar{t}], \mathbb{R}^l)$, $\widehat{\beta} \in L^2([0,\bar{t}], \mathbb{R})$ such that a. e. in $[0,\bar{t}]$ it results:*

(i) $\widehat{\alpha}^j(t) \geq 0, \ \forall j = 1, \ldots, l;$

(ii) $\widehat{\alpha}^j(t)\Big(\widehat{p}^j(t) - \underline{p}^j(t)\Big) = 0 \quad \forall j = 1, .., l;$

$$(iii) \quad \begin{cases} z(t, \widehat{p}(t)) + \displaystyle\int_0^t I(t-s)\widehat{p}(s)\,ds = -\widehat{\alpha}(t), \\ \\ \widehat{\beta}(t) = 0\,. \end{cases}$$

5.1 Example

During the trading session represented by the time interval $[0,\bar{t}]$, we consider an economy with two commodities, with typical commodity denoted by j, and two agents, with typical agent denoted by a. The typical agent has a demand function:

$$x_a^j(t) = \frac{\gamma(t)\sum_{j=1}^2 p_j(t) e_a^j(t)}{p_j(t)}, \quad j = 1, 2,$$

with $e_a^j(t)$, $j = 1, 2$ initial endowment and $\gamma(t) \in L^2([0,\bar{t}])$, $\gamma(t) \geq 0$ a.e. in $[0,\bar{t}]$. The excess aggregate demand is then given by

$$z^j(p) = \sum_{a=1}^2 x_a^j(t) - \sum_{a=1}^2 e_a^j(t), \quad j = 1, 2.$$

Now, we want to focus on the price formation for informed agents, where information is meant as memory of past trade and is represented as an integral term that leads to the price adjustment. Thus, in order to study the effective behavior of the excess aggregate demand function, we introduce a memory term of the form

$$\int_0^t I(t-s)p(s)ds = \beta(t)\int_0^t \mathrm{e}^{-\alpha(t-s)}p(s)ds,$$

namely an exponentially weighted average of past prices, given by an exponentially distributed adjustment. Here $\alpha(t), \beta(t) \in L^2([0, \bar{t}])$ represent the duration and the intensity of memory, respectively.

Exponential decay, the decrease at a rate proportional to its value, is a feature that appears in many fields to describe the decay of a perturbation. In such a way it is quite natural to consider a decay of memory with an exponential form. Of course other functions could be considered.

The effective excess aggregate demand is:

$$Z^j(t,p) = z^j(t,p) + \beta(t)\int_0^t \mathrm{e}^{-\alpha(t-s)}p^j(s)ds, \quad j = 1, 2,$$

where $\alpha(t), \beta(t) \in L^2([0, \bar{t}])$ and $\alpha(t), \beta(t) > 0$ a.e. in $[0, \bar{t}]$.

We pose: $A = \sum_{a=1}^2 e_a^1(t)$, $B = \sum_{a=1}^2 e_a^2(t)$ and $D = \frac{1}{\alpha}(\mathrm{e}^{-\alpha t} - 1)$, where we suppose that $A - B - \beta D > 0,\ 2B + \beta D > 0$, and we fix the minimum prices:

$$\underline{p}^1(t) = \frac{-2B - \beta D + \sqrt{(\beta D)^2 + 4AB}}{2(A - B - 3\beta D)},$$

$$\underline{p}^2(t) = \frac{2A - \beta D + \sqrt{(\beta D)^2 + 4AB}}{2(A - B - 3\beta D)}.$$

Thus, we are led to consider the following variational inequality:

$$\int_0^{\bar{t}} \Big[Z^1(t,\widehat{p})(p^1(t) - \widehat{p}^1(t)) + Z^2(t,\widehat{p})(p^2(t) - \widehat{p}^2(t))\Big]dt \le 0, \forall p \in S,$$

where

$$S = \{p \in L^2([0,\bar{t}], \mathbb{R}^2) : \underline{p}(t) \le p(t),\, p^1(t) + p^2(t) = 1, \text{ a.e. in } [0, T]\}.$$

In virtue of the continuity of solutions we are entitled to solve

$$Z^1(t,\widehat{p})(p^1(t) - \widehat{p}^1(t)) + Z^2(t,\widehat{p})(p^2(t) - \widehat{p}^2(t)) \le 0, \forall p \in S(t), \forall t \in [0, \bar{t}].$$

By applying the direct method as in [31], we have to equate the aggregate excess demands of consumers

$$Z^1(t,\widehat{p}) - Z^2(t,\widehat{p}) = 0$$

with $\widehat{p}^2(t) = 1 - \widehat{p}^1(t)$ and $\underline{p}^1(t) < \widehat{p}^1(t) < 1$, namely

$$0 = \sum_{a=1}^{2} \Big[(x_a^1(t) - e_a^1(t)) - (x_a^2(t) - e_a^2(t))$$
$$+\beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s) ds - \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^2(s) ds \Big]$$
$$= \sum_{a=1}^{2} \Big[\Big(\frac{\gamma(t) \sum_{j=1}^2 \widehat{p}^j(t) e_a^j(t)}{\widehat{p}^1(t)} - e_a^1(t) \Big) - \Big(\frac{\gamma(t) \sum_{j=1}^2 \widehat{p}^j(t) e_a^j(t)}{1 - \widehat{p}^1(t)} - e_a^2(t) \Big)$$
$$+\beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s) ds - \beta(t) \int_0^t \mathrm{e}^{-\alpha(t-s)} (1 - \widehat{p}^1(s)) ds \Big].$$

After some steps, we find

$$\sum_{a=1}^{2} (2\gamma(t) - 1)(e_a^1(t) - e_a^2(t)) = \gamma(t) \sum_{a=1}^{2} \frac{e_a^1(t)}{1 - \widehat{p}^1(t)} - \gamma(t) \sum_{a=1}^{2} \frac{e_a^2(t)}{\widehat{p}^1(t)}$$
$$-2\beta \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s) ds + \frac{\beta}{\alpha} (\mathrm{e}^{-\alpha t} - 1).$$

Setting $\gamma(t) = 1$, we are led to solve the equation

$$\Big(A - B + 2\beta \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s) ds - \frac{\beta}{\alpha} (\mathrm{e}^{-\alpha t} - 1) \Big) (\widehat{p}^1(t))^2$$
$$+ \Big(2B - 2\beta \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s) ds + \frac{\beta}{\alpha} (\mathrm{e}^{-\alpha t} - 1) \Big) \widehat{p}^1(t) - B = 0,$$

whose solution is

$$\widehat{p}^1(t) = \frac{-2B + 2\beta C - \beta D + \sqrt{(2\beta C - \beta D)^2 + 4AB}}{2(A - B + 2\beta C - \beta D)}$$

where $C = \int_0^t \mathrm{e}^{-\alpha(t-s)} \widehat{p}^1(s) ds$ is in turn a solution to the equation

$$16\beta^2 C^4 + (16\beta A - 32\beta^2 D - 16\beta B) C^3$$
$$+(20\beta^2 D^2 - 16\beta A D - 8AB + 4B^2 + 32\beta B D + 4A^2) C^2$$
$$+(-20\beta B D^2 - 8B^2 D + 8ABD - 4\beta^2 D^3 + 4\beta A D^2) C$$
$$+4\beta B D^3 + 4B^2 D^2 - 4ABD^2 = 0.$$

Finally, it is easy to verify that if $A - B + 2\beta C - \beta D > 0$ then $\underline{p}^1(t) < \widehat{p}^1(t) < 1$. Then, we found the following equilibrium price for a pure exchange economy with memory term:

$$\widehat{p}^{\,1}(t) = \frac{-2B + 2\beta C - \beta D + \sqrt{(2\beta C - \beta D)^2 + 4AB}}{2(A - B + 2\beta C - \beta D)}$$

$$\widehat{p}^{\,2}(t) = \frac{2A + 2\beta C - \beta D + \sqrt{(2\beta C - \beta D)^2 + 4AB}}{2(A - B + 2\beta C - \beta D)}.$$

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Meir–Keeler Sequential Contractions and Applications

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Abstract Some fixed point results are given for a class of Meir–Keeler sequential contractions acting on relational metric spaces. The connections with a related statement in Turinici [MDMFPT, Paper-3-3, Pim, Iaşi, 2016] are also being discussed. Finally, an application of the obtained facts to integral equations theory is given.

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1 Introduction

Let X be a nonempty set. Call the subset Y of X, *almost singleton* (in short: *asingleton*) provided $[y_1, y_2 \in Y$ implies $y_1 = y_2]$; and *singleton* if, in addition, Y is nonempty; note that in this case $Y = \{y\}$, for some $y \in X$. Further, let $d : X \times X \to R_+ := [0, \infty[$ be a *metric* over X; the couple (X, d) will be termed a *metric space*. Finally, let $T \in \mathscr{F}(X)$ be a selfmap of X. [Here, for each couple A, B of nonempty sets, $\mathscr{F}(A, B)$ stands for the class of all functions from A to B; when $A = B$, we write $\mathscr{F}(A)$ in place of $\mathscr{F}(A, A)$]. Denote $\text{Fix}(T) = \{x \in X; x = Tx\}$; each point of this set is referred to as *fixed* under T. In the metrical fixed point theory, such points are to be determined according to the context below, comparable with the one described in Rus [55, Ch 2, Sect 2.2]:

(pic-0) We say that T is *fix-asingleton*, if $\text{Fix}(T)$ is an asingleton; and *fix-singleton*, if $\text{Fix}(T)$ is a singleton.

(pic-1) We say that $x \in X$ is a *semi-Picard point* (modulo (d, T)) when $(T^n x; n \geq 0)$ is d-*asymptotic* ($\lim_n d(T^n x, T^{n+1} x) = 0$). If this property holds for all $x \in X$, we say that T is a *semi-Picard operator* (modulo d).

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(pic-2) We say that $x \in X$ is a *Picard point* (modulo (d, T)) when $(T^n x; n \geq 0)$ is d-Cauchy. If this property holds for all $x \in X$, we say that T is a *Picard operator* (modulo d).

(pic-3) We say that $x \in X$ is a *strongly Picard point* (modulo (d, T)) when $(T^n x; n \geq 0)$ is d-convergent with $\lim_n (T^n x) \in \mathrm{Fix}(T)$. If this property holds for all $x \in X$, we say that T is a *strongly Picard operator* (modulo d).

In this perspective, a basic answer to the posed question is the 1922 one due to Banach [5]. Given $\alpha \geq 0$, let us say that T is *Banach* $(d; \alpha)$*-contractive*, provided

(B-contr) $d(Tx, Ty) \leq \alpha d(x, y)$, for all $x, y \in X$.

Theorem 1 *Suppose that T is Banach $(d; \alpha)$-contractive, for some $\alpha \in [0, 1[$. In addition, let X be d-complete. Then,*

(11-a) T is fix-singleton: $\mathrm{Fix}(T) = \{z\}$, for some $z \in X$
(11-b) T is a strongly Picard operator (modulo d);
precisely, $T^n x \xrightarrow{d} z$ as $n \to \infty$, for each $x \in X$.

This result—referred to as *Banach's contraction principle*—found a multitude of applications in operator equations theory; so, it was the subject of many extensions. The most general ones have the (set) *implicit* form

(imp-set) $(d(x, Tx), d(x, y), d(x, Ty), d(Tx, y), d(Tx, Ty), d(y, Ty)) \in \mathscr{M}$, for all $x, y \in X, x \leq y$;

where $\mathscr{M} \subseteq R^6_+$ is a (nonempty) subset, and $(\leq)$ is a *quasi-order* (i.e.; reflexive transitive relation) over X. In particular, when $\mathscr{M}$ is the zero-section of a certain function $F : R^6_+ \to R$, i.e.

$$\mathscr{M} = \{(t_1, t_2, t_3, t_4, t_5, t_6) \in R^6_+; F(t_1, t_2, t_3, t_4, t_5, t_6) \leq 0\},$$

the implicit contractive condition above has the functional form:

(imp-fct) $F(d(x, Tx), d(x, y), d(x, Ty), d(Tx, y), d(Tx, Ty), d(y, Ty)) \leq 0$, for all $x, y \in X, x \leq y$.

On the other hand, when the function F appearing here admits the explicit form

$$F(t_1, t_2, t_3, t_4, t_5, t_6) = t_5 - G(t_1, t_2, t_3, t_4, t_6), (t_1, t_2, t_3, t_4, t_5, t_6) \in R^6_+,$$

(where $G : R^5_+ \to R_+$ is a function), one gets the *explicit* functional version of this (functional) contraction

(exp-fct) $d(Tx, Ty) \leq G(d(x, Tx), d(x, y), d(x, Ty), d(Tx, y), d(y, Ty))$, for all $x, y \in X, x \leq y$.

In particular, when $(\leq)$ is the *trivial* quasi-order of X, some outstanding explicit results have been established in Boyd and Wong [9], Reich [50], and Matkowski [41]; see also the survey paper by Rhoades [51]. And, for the implicit functional version above, certain technical aspects have been considered by Leader [36] and

Turinici [63]. On the other hand, when $(\leq)$ is *antisymmetric* (hence, a *(partial) order* on X), an appropriate extension of Matkowski's fixed point theorem was obtained in the 1986 paper by Turinici [64]; two decades later, this result has been re-discovered—at the level of Banach contractive maps—by Ran and Reurings [49]; see also Nieto and Rodriguez-Lopez [47]. Finally, an extension—to the same framework—of Leader's contribution was performed in Agarwal et al. [3]; and, since then, the number of such papers increased rapidly.

A basic particular case of the implicit contractive property above is

(imp-set-2) $(d(Tx, Ty), d(x, y)) \in \mathscr{M}$, for all $x, y \in X$, $x \leq y$;

where $\mathscr{M} \subseteq R_+^2$ is a (nonempty) subset. The classical example in this direction (under the same trivial relation setting) is due to Meir and Keeler [43]; further refinements of their method were proposed by Matkowski [42] and Cirić [13].

Now, some asymptotic extensions of these techniques were carried out in Geraghty [20, 21] Leader and Hoyle [38], Kirk [35], Proinov [48], and Suzuki [59, 60]; see also Abtahi [1]. It is our aim in the following to give a refinement of these facts, by means of certain techniques involving iterative couples and geometric/asymptotic Meir–Keeler relations. Note that further extensions of such developments to common fixed point results as in Jha et al. [27] are possible, by following the lines in Abtahi et al. [2]; we shall discuss these facts elsewhere.

2 Preliminaries

Throughout this exposition, the axiomatic system in use is Zermelo–Fraenkel's (abbreviated: ZF), as described by Cohen [14, Ch 2]. The notations and basic facts to be considered are standard; some important ones are discussed below.

(a) Let X be a nonempty set. By a *relation* over X, we mean any (nonempty) part $\mathscr{R}$ of $X \times X$; then, $(X, \mathscr{R})$ will be referred to as a *relational structure*. Note that $\mathscr{R}$ may be regarded as a mapping between X and $\exp[X]$ (=the class of all subsets in X). In fact, let us write $(x, y) \in \mathscr{R}$ as $x\mathscr{R}y$; and put, for $x \in X$,

$X(x, \mathscr{R}) = \{y \in X; x\mathscr{R}y\}$ (the *section* of $\mathscr{R}$ through x);

then, the desired mapping representation is $(\mathscr{R}(x) = X(x, \mathscr{R}); x \in X)$. A basic example of such object is

$\mathscr{I} = \{(x, x); x \in X\}$ [the *identical relation* over X].

Given the relations $\mathscr{R}$, $\mathscr{S}$ over X, define their *product* $\mathscr{R} \circ \mathscr{S}$ as

$(x, z) \in \mathscr{R} \circ \mathscr{S}$, if there exists $y \in X$ with $(x, y) \in \mathscr{R}$, $(y, z) \in \mathscr{S}$.

Also, for each relation $\mathscr{R}$ on X, denote

$\mathscr{R}^{-1} = \{(x, y) \in X \times X; (y, x) \in \mathscr{R}\}$ (the *inverse* of $\mathscr{R}$).

Finally, given the relations $\mathscr{R}$ and $\mathscr{S}$ over X, let us say that $\mathscr{R}$ is *coarser* than $\mathscr{S}$ (or, equivalently: $\mathscr{S}$ is *finer* than $\mathscr{R}$), provided

$\mathscr{R} \subseteq \mathscr{S}$; i.e., $x\mathscr{R}y$ implies $x\mathscr{S}y$.

Given a relation $\mathscr{R}$ on X, the following properties are to be discussed here:

(P1) $\mathscr{R}$ is *reflexive*: $\mathscr{I} \subseteq \mathscr{R}$
(P2) $\mathscr{R}$ is *irreflexive*: $\mathscr{R} \cap \mathscr{I} = \emptyset$
(P3) $\mathscr{R}$ is *transitive*: $\mathscr{R} \circ \mathscr{R} \subseteq \mathscr{R}$
(P4) $\mathscr{R}$ is *symmetric*: $\mathscr{R}^{-1} = \mathscr{R}$
(P5) $\mathscr{R}$ is *antisymmetric*: $\mathscr{R}^{-1} \cap \mathscr{R} \subseteq \mathscr{I}$.

This yields the classes of relations to be used; the following ones are important for our developments:

(C0) $\mathscr{R}$ is *amorphous* (i.e., it has no specific properties)
(C1) $\mathscr{R}$ is a *quasi-order* (reflexive and transitive)
(C2) $\mathscr{R}$ is a *strict order* (irreflexive and transitive)
(C3) $\mathscr{R}$ is an *equivalence* (reflexive, transitive, symmetric)
(C4) $\mathscr{R}$ is a *(partial) order* (reflexive, transitive, antisymmetric)
(C5) $\mathscr{R}$ is the *trivial* relation (i.e., $\mathscr{R} = X \times X$).

(b) A basic example of relational structure is to be constructed as below. Let

$N = \{0, 1, 2, \ldots\}$, where $(0 = \emptyset, 1 = \{0\}, 2 = \{0, 1\}, \ldots)$

denote the set of *natural* numbers. Technically speaking, the basic (algebraic and order) structures over N may be obtained by means of the *(immediate) successor* function $\text{suc} : N \to N$, and the *Peano properties* (deductible in our system (ZF)):

(pea-1) $0 \in N$ and $0 \notin \text{suc}(N)$
(pea-2) $\text{suc}(.)$ is injective ($\text{suc}(n) = \text{suc}(m)$ implies $n = m$)
(pea-3) if $M \subseteq N$ fulfills $[0 \in M]$ and $[\text{suc}(M) \subseteq M]$, then $M = N$.

(Note that, in the absence of our axiomatic setting, these properties become the well-known *Peano axioms*, as described in Halmos [22, Ch 12]; we do not give details). In fact, starting from these properties, one may construct, in a recurrent way, an *addition* $(a, b) \mapsto a + b$ over N, according to

$(\forall m \in N)$: $m + 0 = m$; $m + \text{suc}(n) = \text{suc}(m + n)$.

This, in turn, makes possible the introduction of a relation $(\leq)$ over N, as

$(m, n \in N)$: $m \leq n$ iff $m + p = n$, for some $p \in N$.

Concerning the properties of this structure, the most important one writes

$(N, \leq)$ is well ordered:
any (nonempty) subset of N has a first element;

[hence (in particular), $(N, \leq)$ is (partially) ordered]; this tells us that *inductive* reasonings and constructions are allowed here. Denote, for $a, b \in N$, $a \leq b$,

$$N(a, \leq) = \{n \in N; a \leq n\}, N(a, >) = \{n \in N; a > n\};$$
$$N[a, b] = \{n \in N; a \leq n \leq b\}, N[a, b[= \{n \in N; a \leq n < b\};$$

the second one is referred to as an *initial interval* (in N) induced by a. Any set P with $P \sim N$ (in the sense: there exists a bijection from P to N) will be referred to as *effectively denumerable*. In addition, given some natural number $n \geq 1$, any set Q with $N(n, >) \sim Q$ will be said to be *n-finite*; when n is generic here, we say that Q is *finite*. Finally, the (nonempty) set Y is called (at most) *denumerable*, iff it is either effectively denumerable or finite.

Let X be a nonempty set. By a *sequence* in X, we mean any mapping $x : N \to X$, where $N := \{0, 1, \ldots\}$ is the set of *natural* numbers. For simplicity reasons, it will be useful to denote it as $(x(n); n \geq 0)$, or $(x_n; n \geq 0)$; moreover, when no confusion can arise, we further simplify this notation as $(x(n))$ or (x_n), respectively. Also, any sequence $(y_n := x_{i(n)}; n \geq 0)$ with

$(i(n); n \geq 0)$ is *strictly ascending* [hence: $i(n) \to \infty$ as $n \to \infty$],

will be referred to as a *subsequence* of $(x_n; n \geq 0)$. Note that, under such a convention, the relation "subsequence of" is transitive; i.e.,

(z_n) = subsequence of (y_n) and (y_n) = subsequence of (x_n)
imply (z_n) = subsequence of (x_n).

The above construction allows us to introduce the *powers* of a relation $\mathscr{R}$ as

$$\mathscr{R}^0 = \mathscr{I}, \mathscr{R}^{n+1} = \mathscr{R}^n \circ \mathscr{R}, n \in N.$$

The following properties will be useful in the sequel:

$$\mathscr{R}^{m+n} = \mathscr{R}^m \circ \mathscr{R}^n, (\mathscr{R}^m)^n = \mathscr{R}^{mn}, \forall m, n \in N.$$

Under these conventions, the transitivity of $\mathscr{R}$ writes: $\mathscr{R}^2 \subseteq \mathscr{R}$. This suggests us to introduce the following extension of the underlying properties

(P6) $\mathscr{R}$ is *k-transitive* (where $k \geq 2$); i.e., $\mathscr{R}^k \subseteq \mathscr{R}$
(P7) $\mathscr{R}$ is *finitely transitive*; i.e., $\mathscr{R}$ is k-transitive for some $k \geq 2$
(P8) $\mathscr{R}$ is *locally finitely transitive*; i.e.,
for each (effectively) denumerable subset Y of X, there exists $k = k(Y) \geq 2$, such that the restriction to Y of $\mathscr{R}$ is k-transitive.

(c) Remember that an outstanding part of (ZF) is the *Axiom of Choice* (abbreviated: AC), which, in a convenient manner, may be written as

(AC) For each couple (J, X) of nonempty sets and each function
$F : J \to \exp(X)$, there exists a (selective) function
$f : J \to X$, with $f(\nu) \in F(\nu)$, for each $\nu \in J$.

(Here, $\exp(X)$ stands for the class of all nonempty subsets in X). Sometimes, when the ambient set X is endowed with denumerable type structures, the existence of such a selective function (over $J = N$) may be determined by using a weaker form of (AC), referred to as: *Dependent Choice* principle (in short: DC). Call the relation $\mathscr{R}$ over X, *proper* when

$(X(x, \mathscr{R}) =)\mathscr{R}(x)$ is nonempty, for each $x \in X$.

Then, $\mathscr{R}$ is to be viewed as a mapping between X and $\exp(X)$; and the couple $(X, \mathscr{R})$ will be referred to as a *proper relational structure*. Further, given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; \mathscr{R})$-*iterative*, provided

$x_0 = a$, and $x_n \mathscr{R} x_{n+1}$ (i.e., $x_{n+1} \in \mathscr{R}(x_n)$), for all n.

Proposition 1 *Let the relational structure $(X, \mathscr{R})$ be proper. Then, for each $a \in X$ there is at least an $(a; \mathscr{R})$-iterative sequence in X.*

This principle—proposed, independently, by Bernays [6] and Tarski [62]—is deductible from (AC), but not conversely; cf. Wolk [70]. Moreover, by the developments in Moskhovakis [45, Ch 8], and Schechter [58, Ch 6], the *reduced system* (ZF-AC+DC) is comprehensive enough so as to cover the "usual" mathematics; see also Moore [44, Appendix 2].

Let $(\mathscr{R}_n; n \geq 0)$ be a sequence of relations on X. Given $a \in X$, let us say that the sequence $(x_n; n \geq 0)$ in X is $(a; (\mathscr{R}_n; n \geq 0))$-*iterative*, provided

$x_0 = a$, and $x_n \mathscr{R}_n x_{n+1}$ (i.e., $x_{n+1} \in \mathscr{R}_n(x_n)$), for all n.

The following *Diagonal Dependent Choice* principle (in short: DDC) is available.

Proposition 2 *Let $(\mathscr{R}_n; n \geq 0)$ be a sequence of proper relations on X. Then, for each $a \in X$ there exists at least one $(a; (\mathscr{R}_n; n \geq 0))$-iterative sequence in X.*

Clearly, (DDC) includes (DC), to which it reduces when $(\mathscr{R}_n; n \geq 0)$ is constant. The reciprocal of this is also true. In fact, letting the premises of (DDC) hold, put $P = N \times X$; and let $\mathscr{S}$ be the relation over P introduced as

$\mathscr{S}(i, x) = \{i + 1\} \times \mathscr{R}_i(x), \ (i, x) \in P.$

It will suffice applying (DC) to $(P, \mathscr{S})$ and $b := (0, a) \in P$ to get the conclusion in our statement; we do not give details.

Summing up, (DDC) is provable in (ZF-AC+DC). This is valid as well for its variant, referred to as: the *Selected Dependent Choice* principle (in short: SDC).

Proposition 3 *Let the map $F : N \to \exp(X)$ and the relation $\mathscr{R}$ over X fulfill*

$(\forall n \in N)$: $\mathscr{R}(x) \cap F(n+1) \neq \emptyset$, for all $x \in F(n)$.

Then, for each $a \in F(0)$ there exists a sequence $(x(n); n \geq 0)$ in X, with

$x(0) = a, \ x(n) \in F(n), \ x(n+1) \in \mathscr{R}(x(n)), \ \forall n.$

As before, (SDC) $\Longrightarrow$ (DC) ($\Longleftrightarrow$ (DDC)); just take $(F(n) = X; n \in N)$. But, the reciprocal is also true, in the sense: (DDC) $\Longrightarrow$ (SDC). This follows from

Proof (Proposition 3) Let the premises of (SDC) be true. Define a sequence of relations $(\mathscr{R}_n; n \geq 0)$ over X as: for each $n \in N$,

$$\mathscr{R}_n(x) = \mathscr{R}(x) \cap F(n+1), \text{ if } x \in F(n),$$
$$\mathscr{R}_n(x) = \{x\}, \text{ otherwise } (x \in X \setminus F(n)).$$

Clearly, $\mathscr{R}_n$ is proper, for all $n \in N$; whence (DDC) applies to these data. So, for the starting $a \in F(0)$, there exists an $(a; (R_n; n \geq 0))$-iterative sequence $(x(n); n \geq 0)$ in X. Combining with the very definition above, it follows that conclusion in our statement is holding.

In particular, when $\mathscr{R} = X \times X$, the regularity condition imposed in (SDC) holds. The corresponding variant of underlying statement is just (AC(N)) (=the *Denumerable Axiom of Choice*). Precisely, we have

Proposition 4 *Let $F : N \to \exp(X)$ be a function. Then, for each $a \in F(0)$ there exists a function $f : N \to X$ with $f(0) = a$ and $f(n) \in F(n)$, $\forall n \in N$.*

As a consequence of the above facts, (DC) $\Longrightarrow$ (AC(N)) in (ZF-AC). A direct verification of this is obtainable by taking $A = N \times X$ and introducing the relation $\mathscr{R}$ over it, according to:

$$\mathscr{R}(n, x) = \{n+1\} \times F(n+1), \ n \in N, x \in X;$$

we do not give details. The reciprocal of this inclusion is not true; see, for instance, Moskhovakis [45, Ch 8, Sect 8.25].

(d) In what follows, the concepts of convergence and Cauchy structure are introduced; and some basic facts about these are given.

Let X be a nonempty set; and $\mathscr{S}(X)$ stand for the class of all sequences (x_n) in X. By a (sequential) *convergence structure* on X we mean any part $\mathscr{C}$ of $\mathscr{S}(X) \times X$, with the properties (cf. Kasahara [31]):

(conv-1) ($\mathscr{C}$ is *hereditary*)
$((x_n); x) \in \mathscr{C} \Longrightarrow ((y_n); x) \in \mathscr{C}$, for each subsequence (y_n) of (x_n)
(conv-2) ($\mathscr{C}$ is *reflexive*)
$(\forall u \in X)$: the constant sequence $(x_n = u; n \geq 0)$ fulfills $((x_n); u) \in \mathscr{C}$.

For (x_n) in $\mathscr{S}(X)$ and $x \in X$, we write $((x_n); x) \in \mathscr{C}$ as $x_n \xrightarrow{\mathscr{C}} x$; this reads:

(x_n), $\mathscr{C}$-converges to x (also referred to as: x is the *$\mathscr{C}$-limit* of (x_n)).

The set of all such x is denoted $\lim_n(x_n)$; when it is nonempty, we say that (x_n) is *$\mathscr{C}$-convergent*. The following condition is to be optionally considered here:

(conv-3) $\mathscr{C}$ is *separated*: $\lim_n(x_n)$ is an asingleton, for each sequence (x_n);

when it holds, $x_n \xrightarrow{\mathscr{C}} z$ will be also written as $\lim_n(x_n) = z$.

Further, let us say that the subset $\mathscr{H} \subseteq \mathscr{S}(X)$ is a (sequential) *Cauchy structure* on X, provided (cf. Turinici [65])

(Cauchy-1) ($\mathscr{H}$ is *hereditary*)
$(x_n) \in \mathscr{H}$ implies $(y_n) \in \mathscr{H}$, for each subsequence $(y_n; n \geq 0)$ of $(x_n; n \geq 0)$
(Cauchy-2) ($\mathscr{H}$ is *reflexive*)
$(\forall u \in X)$: the constant sequence $(x_n = u; n \geq 0)$ fulfills $(x_n) \in \mathscr{H}$;

each element of $\mathscr{H}$ will be referred to as a $\mathscr{H}$*-Cauchy* sequence in X.

Finally, the couple $(\mathscr{C}, \mathscr{H})$ will be referred to as a *conv-Cauchy structure* on X. The natural conditions about $(\mathscr{C}, \mathscr{H})$ to be (optionally) considered here are

(CC-1) $(\mathscr{C}, \mathscr{H})$ is *regular*:
each $\mathscr{C}$-convergent sequence in X is $\mathscr{H}$-Cauchy
(CC-2) $(\mathscr{C}, \mathscr{H})$ is *complete*:
each $\mathscr{H}$-Cauchy sequence in X is $\mathscr{C}$-convergent.

A standard way of introducing such structures is as follows. By a *pseudometric* over X we shall mean any map $d : X \times X \to R_+$. Fix such an object, with

(r-s) d is *reflexive sufficient* ($x = y \Longleftrightarrow d(x, y) = 0$)
(tri) d is *triangular* ($d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$)
(sym) d is *symmetric* ($d(x, y) = d(y, x)$, for all $x, y \in X$).

Then, d is called a *metric* on X; and (X, d) will be referred to as a *metric space*.

Given the sequence (x_n) in X and the point $x \in X$, we say that (x_n), d*-converges* to x (written as: $x_n \xrightarrow{d} x$) provided $d(x_n, x) \to 0$ as $n \to \infty$; i.e.,

$\forall \varepsilon > 0, \exists i = i(\varepsilon)$: $i \leq n \Longrightarrow d(x_n, x) < \varepsilon$;
or, equivalently: $[\forall \varepsilon > 0, \exists i = i(\varepsilon)$: $i \leq n \Longrightarrow d(x_n, x) \leq \varepsilon]$.

By this very definition, we have the hereditary and reflexive properties:

(d-conv-1) $((\xrightarrow{d})$ is hereditary)
$x_n \xrightarrow{d} x$ implies $y_n \xrightarrow{d} x$, for each subsequence (y_n) of (x_n)
(d-conv-2) $((\xrightarrow{d})$ is reflexive)
$(\forall u \in X)$: the constant sequence $(x_n = u; n \geq 0)$ fulfills $x_n \xrightarrow{d} u$;

so that $(\xrightarrow{d})$ is a sequential convergence on X. The set of all such limit points of (x_n) will be denoted $\lim_n(x_n)$; if it is nonempty, then (x_n) is called d*-convergent*. Finally, note that (by the imposed conditions)

$(\xrightarrow{d})$ is separated (referred to as: d is *separated*):
$\lim_n(x_n)$ is an asingleton, for each sequence (x_n) in X.

Concerning these developments, the following auxiliary statement is useful.

Proposition 5 *The mapping* $(x, y) \mapsto d(x, y)$ *is d-Lipschitz, in the sense*

(25-1) $|d(x, y) - d(u, v)| \leq d(x, u) + d(y, v)$, $\forall (x, y), (u, v) \in X \times X$.

As a consequence, this map is d-continuous:

(25-2) $x_n \xrightarrow{d} x$, $y_n \xrightarrow{d} y$ *imply* $d(x_n, y_n) \to d(x, y)$.

The proof is immediate, by the involved concepts; we do not give details. Some extensions of these facts are possible under the lines discussed in the 2001 PhD Thesis by Hitzler [24, Ch 1, Sect 1.2].

Further, call the sequence (x_n), *d-Cauchy* when it satisfies $d(x_m, x_n) \to 0$ as $m, n \to \infty$, $m < n$; i.e.,

$\forall \varepsilon > 0, \exists j = j(\varepsilon)$: $j \leq m < n \Longrightarrow d(x_m, x_n) < \varepsilon$;
or, equivalently: $[\forall \varepsilon > 0, \exists j = j(\varepsilon)$: $j \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon]$;

the class of all these will be denoted as $Cauchy(d)$. As before, we have the hereditary and reflexive properties

(d-Cauchy-1) ($Cauchy(d)$ is hereditary)
(x_n) is d-Cauchy implies (y_n) is d-Cauchy, for each subsequence (y_n) of (x_n)
(d-Cauchy-2) ($Cauchy(d)$ is reflexive)
$(\forall u \in X)$: the constant sequence $(x_n = u; n \geq 0)$ is d-Cauchy;

hence, $Cauchy(d)$ is a Cauchy structure on X.

Finally—according to the general setting—call the couple $((\xrightarrow{d}), Cauchy(d))$, a *conv-Cauchy structure* induced by d. The following regularity conditions about this structure are to be considered

(CC-1) d is *regular*: each d-convergent sequence in X is d-Cauchy
(CC-2) d is *complete*: each d-Cauchy sequence in X is d-convergent.

The former of these holds in our setting; but the latter one is not in general true.

A weakened form of the d-Cauchy concept we just exposed is the following. Let $(x_n; n \geq 0)$ be a sequence; we call it *d-asymptotic*, provided

(d-asy) $\lim_n d(x_n, x_{n+1}) = 0$; i.e., for each $\varepsilon > 0$, there exists $r(\varepsilon) \in N$ such that $r(\varepsilon) \leq n \Longrightarrow d(x_n, x_{n+1}) < \varepsilon$.

Clearly, each d-Cauchy sequence is d-asymptotic too; but the reciprocal is not in general true.

(e) Finally, some convergence properties of real sequences are discussed.

For each sequence (r_n) in R, and each element $r \in R$, denote

$\lim_n r_n = r+$ (written as: $r_n \to r+$), when $r_n \to r$ and $(r_n > r, \forall n)$;
$\lim_n r_n = r++$ (written as: $r_n \to r++$), when $r_n \to r$ and $(r_n > r, \forall\forall n)$.

Here, a property $\pi(n)$ depending on $n \in N$ is said to hold for *almost all n* (written: $(\pi(n); \forall\forall n)$), provided

there exists $h = h(\pi) \in N$, such that ($\pi(n)$ holds, for all $n \geq h$).

Proposition 6 *Let the sequence* $(r_n; n \geq 0)$ *in* R *and the point* $\varepsilon \in R$ *be such that* $r_n \to \varepsilon+$. *Then, there exists a subsequence* $(r_n^* := r_{i(n)}; n \geq 0)$ *of* $(r_n; n \geq 0)$ *with*

$(r_n^*; n \geq 0)$ *is (strictly) descending and* $r_n^* \to \varepsilon+$.

Proof Put $i(0) = 0$. As $\varepsilon < r_{i(0)}$ and $r_n \to \varepsilon+$, we have that

$A(i(0)) := \{n > i(0); r_n < r_{i(0)}\}$ is not empty;
hence, $i(1) := \min(A(i(0)))$ is an element of it, and $r_{i(1)} < r_{i(0)}$.

Likewise, as $\varepsilon < r_{i(1)}$ and $r_n \to \varepsilon+$, we have that

$A(i(1)) := \{n > i(1); r_n < r_{i(1)}\}$ is not empty;
hence, $i(2) := \min(A(i(1)))$ is an element of it, and $r_{i(2)} < r_{i(1)}$.

This procedure may continue indefinitely; and yields (without any choice technique) a strictly ascending rank sequence $(i(n); n \geq 0)$ (hence, $i(n) \to \infty$ as $n \to \infty$), for which the attached subsequence $(r_n^* := r_{i(n)}; n \geq 0)$ of $(r_n; n \geq 0)$ fulfills

$r_{n+1}^* < r_n^*$, for all n; hence, (r_n^*) is (strictly) descending.

On the other hand, by this very subsequence property,

$(r_n^* > \varepsilon, \forall n)$, and $\lim_n r_n^* = \lim_n r_n = \varepsilon$.

Putting these together, we get the desired fact.

A bi-dimensional counterpart of these facts may be given along the lines below. Let $\pi(t, s)$ (where $t, s \in R$) be a logical property involving couples or real numbers. Given the couple of real sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$, call the subsequences $(t_n^*; n \geq 0)$ of (t_n) and $(s_n^*; n \geq 0)$ of (s_n), *compatible* when

$(t_n^* = t_{i(n)}; n \geq 0)$, and $(s_n^* = s_{i(n)}; n \geq 0)$,
for the same strictly ascending rank sequence $(i(n); n \geq 0)$.

Proposition 7 *Let the couple of real sequences* $(t_n; n \geq 0)$, $(s_n; n \geq 0)$ *and the couple of real numbers* (a, b) *be such that*

$t_n \to a+$, $s_n \to b+$ *as* $n \to \infty$ *and* ($\pi(t_n, s_n)$ *is true*, $\forall n$).

There exists then a compatible couple of subsequences $(t_n^*; n \geq 0)$ *of* $(t_n; n \geq 0)$ *and* $(s_n^*; n \geq 0)$ *of* $(s_n; n \geq 0)$ *respectively, with*

(27-1) $(t_n^*; n \geq 0)$ *and* $(s_n^*; n \geq 0)$ *are (strictly) descending*
(27-2) ($t_n^* \to a+$, $s_n^* \to b+$, *as* $n \to \infty$), *and* ($\pi(t_n^*, s_n^*)$ *holds, for all* n).

Proof By the preceding statement, the sequence $(t_n; n \geq 0)$ admits a subsequence $(T_n := t_{i(n)}; n \geq 0)$, with

$(T_n; n \geq 0)$ is (strictly) descending, and ($T_n \to a+$, as $n \to \infty$).

Denote $(S_n := s_{i(n)}; n \geq 0)$; clearly,

$(S_n; n \geq 0)$ is a subsequence of $(s_n; n \geq 0)$ with $S_n \to b+$ as $n \to \infty$.

Moreover, by this very construction [$\pi(T_n, S_n)$ holds, for all n]. Again by that statement, there exists a subsequence $(s_n^* := S_{j(n)} = s_{i(j(n))}; n \geq 0)$ of $(S_n; n \geq 0)$ (hence, of $(s_n; n \geq 0)$ as well), with

$$(s_n^*; n \geq 0) \text{ is (strictly) descending, and } (s_n^* \to b+, \text{ as } n \to \infty).$$

Denote further $(t_n^* := T_{j(n)} = t_{i(j(n))}; n \geq 0)$; this is a subsequence of $(T_n; n \geq 0)$ (hence, of $(t_n; n \geq 0)$ as well), with

$$(t_n^*; n \geq 0) \text{ is (strictly) descending, and } (t_n^* \to a+, \text{ as } n \to \infty).$$

Finally, by this very construction (and a previous relation)

$$\pi(t_n^*, s_n^*) \text{ holds, for all } n.$$

Summing up, the couple of subsequences $(t_n^*; n \geq 0)$ and $(s_n^*; n \geq 0)$ has all needed properties; and the conclusion follows.

3 Scattered Sets

In what follows, some technical questions concerning scattered sets of natural numbers are discussed.

(a) Let U be a subset of N with $\sup(U) = \infty$; it may be represented as $U = \{u_n; n \geq 0\}$ where $(u_n; n \geq 0)$ is strictly ascending. As a consequence,

$$U(p, <) = \{u \in U; p < u\} \text{ is nonempty, for each } p \in N.$$

Let $\Sigma_U \in \mathscr{F}(U)$ stand for the *immediate successor* function attached to U, as

$$\Sigma_U(p) = \min U(p, <),\ p \in U.$$

Given $h \in N(1, \leq)$, let us say that U is a *h-arithmetic-progression* (in short: *h-aprogression*), when

$$|p - \Sigma_U(p)| = h, \forall p \in U; \text{ or, equivalently: } (u_{n+1} - u_n = h, \text{ for all } n);$$

in this case, one has the representation

$$U = u_0 + Nh := \{u_0 + nh; n \geq 0\}, \text{ where } u_0 = \min(U).$$

Returning to the general setting, let $k \in N(1, \leq)$ be arbitrary fixed; we say that U is *k-scattered*, if

$$|p - \Sigma_U(p)| \leq k, \forall p \in U; \text{ or, equivalently: } (u_{n+1} - u_n \leq k, \text{ for all } n).$$

For practical reasons, a particular class of such subsets will be considered. Call the subset M of N, $(1, \infty)$-*chain* provided

$$\inf(M) = \min(M) = 1, \sup(M) = \infty.$$

In this case, $M = \{a_n; n \geq 0\}$, where the sequence $(a_n; n \geq 0)$ in N is 1-starting ($a_0 = 1$) and strictly ascending ($i < j$ implies $a_i < a_j$); hence, M is effectively denumerable. Given $k \in N(1, \leq)$, we have that M is *k-scattered*, provided

(M-sca) $a_{n+1} - a_n \leq k$, for all n.

The class of all such $k \in N(1, \leq)$ (if any) will be denoted as sca(M); referred to as the *scattered domain* of M. Clearly,

sca(M) is hereditary: $k \in \text{sca}(M), h \geq k \Longrightarrow h \in \text{sca}(M)$.

When sca(M) is nonempty, we say that M is *scattered.*

A good characterization of this property may be given as below. Call the $(1, \infty)$-chain M of N, *h-admissible* (for some $h \geq 1$), provided

(M-adm) for each $n \in N$ there exists $r \in M$ with $n < r \leq n + h$.

When $h \geq 1$ is generic in this convention, we say that M is *admissible*.

Proposition 8 *We have, over the class of all* $(1, \infty)$*-chains M of N,*

(31-1) $(\forall k \geq 1)$*: M is k-scattered implies M is k-admissible*
(31-2) $(\forall k \geq 1)$*: M is k-admissible implies M is k-scattered.*

Hence [for each $(1, \infty)$*-chain M of N]:*

(31-3) M is scattered iff M is admissible.

Proof

(i) Suppose that the $(1, \infty)$-chain $M = \{a_n; n \geq 0\}$ (where $(a_n; n \geq 0)$ is 1-starting, strictly ascending) appears as k-scattered; we have to prove that M is k-admissible. The property is clear for $n = 0$; so, without loss, one may assume that $n \geq 1 = a_0$. Then, there exists a uniquely determined rank $i \in N$ with

$(a_{i+1} - a_i \leq k$ and$)$ $a_i \leq n < a_{i+1}$;

and this, along with $[a_{i+1} \leq a_i + k \leq n + k]$, proves the desired fact, with $r = a_{i+1}$.

(ii) Suppose that the $(1, \infty)$-chain $M = \{a_n; n \geq 0\}$ (where $(a_n; n \geq 0)$ is 1-starting, strictly ascending) appears as k-admissible; we have to prove that M is k-scattered. Let $i \in N$ be arbitrary fixed. From the admissible property, there exists $j \in N$ such that (with $r = a_j \in M$)

$a_i < r = a_j \leq a_i + k.$

Combining with the strict increasing property of (a_n) yields

$a_i < a_{i+1} \leq a_j \leq a_i + k$; whence $a_{i+1} - a_i \leq k.$

(iii) Evident.

Now, a natural question is to determine whether any $(1, \infty)$-chain of N is scattered. A negative answer to this is available by the characterization of scattered sets given by the statement above.

Example 1 Let $(\varphi(n); n \geq 0)$ be a sequence in R_+ with $\lim_n \varphi(n) = \infty$; and M be a $(1, \infty)$-chain of N; hence, $M = \{a_n; n \geq 0\}$, where $(a_n; n \geq 0\}$ is a 1-starting strictly ascending sequence in $N(1, \leq)$. Suppose that the following condition holds:

$a_{n+1} - a_n \geq \varphi(n)$, for almost all $n \in N$.

We claim that $\text{sca}(M) = \emptyset$; i.e., M is not scattered. In fact, suppose that M is k-scattered, for some $k \in N(1, \leq)$; i.e.,

$a_{n+1} - a_n \leq k$, for all $n \in N$.

This, along with the choice of our sequence, gives

$k \geq \varphi(n)$, for almost all $n \in N$.

Passing to limit as $n \to \infty$ in this relation gives $k = \infty$; contradiction. Hence, M is not scattered; as claimed.

Returning to the scattering concept, it is clear that (for the arbitrary fixed $k \geq 1$)

if the $(1, \infty)$-chain $M = \{a_n; n \geq 0\}$ of N is k-progression
$(a_{n+1} - a_n = k, \forall n)$, then M is k-scattered.

A slight extension of this fact may be given along the lines below.

Proposition 9 *Let the* $(1, \infty)$*-chain* M *of* N *be such that*

there exists a h-progression $(y_n; n \geq 0)$ *in* $N(1, \leq)$*,*
(where $h \geq 1$*), with* $\{y_n; n \geq 0\} \subseteq M$*.*

Then, M *is k-scattered, where* $k = \max\{y_0, h\}$*.*

Proof Let $M = \{a_n; n \geq 0\}$ be the representation of our $(1, \infty)$-chain, where $(a_n; n \geq 0\}$ is a 1-starting strictly ascending sequence in $N(1, \leq)$. Further, denote $P = \{y_n; n \geq 0\}$. As $P \subseteq M$, there exists a strictly ascending sequence of ranks $(i(n); n \geq 0)$, such that $(y_n = a_{i(n)}; n \geq 0)$. Let $r \in N$ be an arbitrary rank. Two situations occur.

Case 1 Suppose that

$a_r \geq y_0 = a_{i(0)}$; whence $r \geq i(0)$.

As $(i(n); n \geq 0)$ is strictly ascending, we must have $\lim_n i(n) = \infty$; so that there must be a uniquely determined rank n, with

$i(n) \leq r < i(n+1)$; whence $y_n = a_{i(n)} \leq a_r < y_{n+1} = a_{i(n+1)}$.

Since a_{r+1} is the immediate successor of a_r in M, we must have

$y_n \le a_r < a_{r+1} \le y_{n+1}$; hence, $a_{r+1} - a_r \le y_{n+1} - y_n = h$.

Case 2 Suppose that

$a_r < y_0 = a_{i(0)}$; whence $r < i(0)$.

As a_{r+1} is the immediate successor of a_r in M, we must have

$0 \le a_r < a_{r+1} \le y_0$; so that $a_{r+1} - a_r \le y_0 - 0 = y_0$.

Putting these together yields

$a_{r+1} - a_r \le k := \max\{y_0, h\}$, for each $r \in N$; and we are done.

Given the $(1, \infty)$-chain M of N, denote

$\mathrm{reg}(M) = \{(a, b) \in N(1, \le)^2; a + Nb \subseteq M\}$;

referred to as: the *regularity domain* of M. If $\mathrm{reg}(M) \ne \emptyset$, then M will be called *regular*; note that, by the preceding statement, M is then scattered.

Concerning this aspect, we may ask whether the reciprocal inclusion is true:

(for each $(1, \infty)$-chain M of N): M is scattered implies M is regular.

This is a difficult question; some partial aspects of it were described by Wagstaff Jr [68] and Dubickas [18].

(b) A *uniform* type version of these concepts may be introduced along the lines below. Let $\mathscr{S} \subseteq N \times N$ be a (nonempty) relation over N. Given $h \ge 1$, let us say that $\mathscr{S}$ is *h-uniformly-scattered* (in short: *h-uscattered*), provided

$\forall m, n \in N(<), \exists p \in N\colon n < p \le n + h, m\mathscr{S}p.$

Here, for simplicity reasons, we denoted

$N(<) = \{(m, n) \in N \times N; m < n\}$; (the *graph* of $(<)$ over N).

If $h \ge 1$ is generic in this convention, we say that $\mathscr{S}$ is *uniformly scattered* (in short: *uscattered*). A functional way of expressing the previous convention is the following. Let $\mathscr{S}$ be a h-uscattered relation over N (where $h \ge 1$); this writes:

$\forall (m, n) \in N(<)\colon \mathscr{L}(m, n) = \{q \in N[1, h]; m\mathscr{S}(n + q)\}$ is nonempty;

where (cf. a previous convention) $N[1, h] = \{n \in N; 1 \le n \le h\}$; we then say that $\mathscr{L} : N(<) \to \exp(N[1, h])$ is the *associated* to $(\mathscr{S}, h)$ multivalued function. Let $L : N(<) \to N[1, h]$ stand for its selection

$L(m, n) = \min \mathscr{L}(m, n)$, $(m, n) \in N(<)$;

it will be referred to as the *associated to* $(\mathscr{S}, h)$ *univalued function*. By the very definition above, one has

$m\mathscr{S}(n + L(m, n))$, for each $(m, n) \in N(<)$.

Clearly, no choice techniques were used here; hence, this construction is valid over the strongly reduced system (ZF-AC).

Concerning some concrete classes of uscattered relations, the natural setting to solve this question is that characterized as

$\mathscr{S}$ is telescopic: $n\mathscr{S}(n+1)$, for all $n \geq 0$
$\mathscr{S}$ is translation invariant:
$(m, n) \in \mathscr{S}$ implies $(m+j, n+j) \in \mathscr{S}, \forall j \in N$.

Then, all reasonings above reduce to the corresponding ones involving the section $\mathscr{S}(0) := \{n \in N; 0\mathscr{S}n\}$.

Proposition 10 *Suppose that $\mathscr{S}$ is telescopic and translation invariant. Then,*

(33-1) $(\forall h \geq 1)$: $\mathscr{S}(0)$ is h-scattered implies $\mathscr{S}$ is h-uscattered
(33-2) $\mathscr{S}(0)$ is scattered implies $\mathscr{S}$ is uscattered.

Proof It will suffice verifying the first part. Let $m, n \in N$ be such that $m < n$. As $\mathscr{S}(0)$ is h-scattered, it is h-admissible as well; so, there exists $r \in \mathscr{S}(0)$ with $n-m < r \leq n-m+h$. Denote $p = m+r$. By the preceding relation, $n < p \leq n+h$. In addition, as $\mathscr{S}$ is translation invariant and $0\mathscr{S}r$, we get $m\mathscr{S}(m+r)$; i.e., $m\mathscr{R}p$.

(c) In what follows, a useful application of these concepts is discussed.

Let (X, d) be a metric space. Define, for each couple of subsets $A, B \in \exp(X)$,

$$d(A, B) = \inf\{d(a, b); a \in A, b \in B\} \text{ (the } gap \text{ between } A \text{ and } B).$$

Further, denote for each subset $C \in \exp(X)$

$$\text{diam}(C) = \sup\{d(x, y); x, y \in C\} \text{ (the } diameter \text{ of } C).$$

Proposition 11 *For each couple $A, B \in \exp(X)$ and each $(a, b) \in A \times B$,*

$$d(A, B) \geq d(a, b) - diam(A) - diam(B).$$

Proof Let (A, B) and (a, b) be as before. From the triangle inequality

$$d(a, b) \leq d(a, x) + d(x, y) + d(y, b) \leq d(x, y) + \text{diam}(A) + \text{diam}(B),$$
for all $(x, y) \in A \times B$.

Passing to infimum with respect to $(x, y) \in A \times B$, we are done.

Let $\mathscr{R}$ be a relation over X; the triplet $(X, d, \mathscr{R})$ will be termed a *relational metric space*. Further, take some natural number $k \geq 1$. Given the sequence (x_n) in X, call it *$(\mathscr{R}, k)$-uniformly-scattered* (in short: *$(\mathscr{R}, k)$-uscattered*), provided

$$\forall (m, n) \in N(<), \exists p \in N, \text{ such that } n < p \leq n+k \text{ and } x_m \mathscr{R} x_p.$$

When $k \geq 1$ is generic here, we say that (x_n) is *$\mathscr{R}$-uniformly-scattered* (in short: *$\mathscr{R}$-uscattered*).

To get an appropriate interpretation of this concept, denote by $\mathscr{S}$ the *trace* of $\mathscr{R}$ over this sequence

$(m, n \in N)$: $m\mathscr{S}n$ iff $x_m\mathscr{R}x_n$

Then, evidently,

(x_n) is $(\mathscr{R}, k)$-uscattered iff $\mathscr{S}$ is k-uscattered.

By a previous result, this happens when

(adm-1) $\mathscr{S}$ is telescopic: $n\mathscr{S}(n+1)$, for all $n \geq 0$
(adm-2) $\mathscr{S}$ is translation invariant: $m\mathscr{S}n$ implies $(m+j)\mathscr{S}(n+j)$, $\forall j \in N$
(adm-3) $S(0) := \{n \in N; 0\mathscr{S}n\}$ is k-scattered:
for each $n \in N$ there exists $r \in S(0)$ with $n < r \leq n + k$.

These, in terms of our initial sequence (x_n), mean (respectively)

(tr-asc-1) (x_n) is $\mathscr{R}$-ascending: $x_n\mathscr{R}x_{n+1}$, for each $n \in N$
(tr-asc-2) (x_n) is $\mathscr{R}$-translated: $x_m\mathscr{R}x_n$ implies $x_{m+j}\mathscr{R}x_{n+j}$, for each $j \in N$
(tr-asc-3) (x_n) is $(\mathscr{R}, x_0; k)$-scattered:
for each $n \in N$ there exists $r \in N$ with $x_0\mathscr{R}x_r$ and $n < r \leq n + k$.

However, these are but an illustration of the introduced general concept; so, we do not use them in the sequel.

The following auxiliary fact is available.

Proposition 12 *Suppose that the $\mathscr{R}$-ascending sequence $(x_n; n \geq 0)$ in X fulfills*

$(x_n; n \geq 0)$ *is* $(\mathscr{R}, h)$*-uscattered, for some* $h \geq 1$.

Then, there exists a mapping $L : N(<) \to N[1, h]$*, such that*

$x_m\mathscr{R}x_{n+L(m,n)}$, *for each* $(m, n) \in N(<)$.

Proof By definition, the associated relation $\mathscr{S}$ over N is h-admissible. From a previous observation, there exists a mapping $L : N(<) \to N[1, h]$, with

$m\mathscr{S}(n + L(m, n))$, for each $(m, n) \in N(<)$.

This, according to the definition of $\mathscr{S}$, gives us all desired facts.

We have now all ingredients to formulate the announced statement. Let us say that the subset Θ of R^0_+ is $(>)$*-cofinal* in R^0_+, when:

for each $\varepsilon \in R^0_+$, there exists $\theta \in \Theta$ with $\varepsilon > \theta$.

Letting $(x_n; n \geq 0)$ be a sequence in X, denote, for $n, k \in N$,

$x[n, n + k] = \{x_n, \ldots, x_{n+k}\}$, $\Delta_n(k) = \text{diam}(x[n, n + k])$.

Given the subset Y of X and the rank $h \geq 0$, let us say that the sequence $(x_n; n \geq 0)$ in X is h*-nearly* in Y, provided:

$x_n \in Y$, for all $n \geq h$; i.e., $\{x_h, x_{h+1}, \ldots\} \subseteq Y$.

When $h = 0$, this convention means: $(x_n; n \geq 0)$ is in Y; and if $h \geq 0$ is generic, the resulting property will be referred to as: $(x_n; n \geq 0)$ is *nearly* in Y. Finally, given the sequence $(r_n; n \geq 0)$ in R_+ and the point $r \in R_+$, let us write

$$r_n \to r++ \; (r_n \to r--) \text{ if } r_n \to r \text{ and } r_n > r \; (r_n < r), \text{ for almost all } n \geq 0.$$

Here, a property $\pi(n)$ depending on $n \in N$ is said to hold for *almost all n* [and we write this as: $(\pi(n); \forall\forall n)$], provided

$$\text{there exists } h = h(\pi) \in N, \text{ such that } (\pi(n) \text{ holds, for all } n \geq h).$$

Proposition 13 *Let the sequence $(x_n; n \geq 0)$ in X and the natural number $h \geq 1$ be such that*

(36-i) *(x_n) is d-asymptotic and $(\mathscr{R}, h)$-uscattered*
(36-ii) *(x_n) is not d-Cauchy.*

Further, let the subset Θ of R^0_+ be $(>)$-cofinal in R^0_+. There exist then a number $\gamma \in \Theta$, a rank $j(\gamma, h) \geq 1$, and a triple of rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$, $(p(j); j \geq 0)$, with

(36-1) $j \leq m(j) < n(j) < p(j) \leq n(j) + h$, $x_{m(j)} \mathscr{R} x_{p(j)}$, $\forall j \geq 0$
(36-2) $j \leq m(j) < m(j) + 2h < n(j)$, *for all* $j \geq j(\gamma, h)$
(36-3) $d(x_{m(j)}, x_{n(j)-1}) \leq \gamma + \Delta_{m(j)}(3h)$, $\forall j \geq j(\gamma, h)$
(36-4) *for each* $s, t \in N[0, 2h]$, *the sequence*
$(V_j(s, t) := d(x_{m(j)+s}, x_{p(j)+t}); j \geq 0)$ *(in R_+)*
is $j(\gamma, h)$-nearly in $]\gamma, \infty[$, with $V_j(s, t) \to \gamma++$ *as* $j \to \infty$.

Proof As $(x_n; n \geq 0)$ is d-asymptotic, we must have

$$\Delta_n(k) := \text{diam}(x[n, n+k]) \to 0, \text{ as } n \to \infty, \text{ for each } k \in N.$$

By definition, the d-Cauchy property of our sequence writes:

$$\forall \varepsilon \in R^0_+, \exists a = a(\varepsilon): a \leq m < n \Longrightarrow d(x_m, x_n) \leq \varepsilon.$$

As Θ is a $(>)$-cofinal part in R^0_+, this property may be also written as

$$\forall \theta \in \Theta, \exists \alpha = \alpha(\theta): \alpha \leq m < n \Longrightarrow d(x_m, x_n) \leq \theta.$$

The negation of this property means: there exists $\beta \in \Theta$, such that

$$A(j) := \{(m, n) \in N \times N; j \leq m < n, d(x_m, x_n) > \beta\} \neq \emptyset, \forall j \geq 0.$$

Let the number $\gamma \in \Theta$ be given according to

$$\beta > 3\gamma \text{ (possible, since } \Theta \text{ is } (>)\text{-cofinal in } R^0_+).$$

By the d-asymptotic property of (x_n), there must be $j(\gamma, h) \geq 1$, with

$$\Delta_n(3h) = \text{diam}(x[n, n+3h]) < \gamma, \text{ for all } n \geq j(\gamma, h).$$

For each $j \geq j(\gamma, h)$ and each $(m, n) \in A(j)$, we have (by an auxiliary fact)

$d(x[m, m+3h], x[n, n+3h]) \geq$
$d(x_m, x_n) - \text{diam}(x[m, m+3h]) - \text{diam}(x[n, n+3h]) \geq \beta - 2\gamma > \gamma;$

which tells us that

$B(j) := \{(m, n) \in N \times N;\ j \leq m < n, d(x[m, m+3h], x[n, n+3h]) > \gamma\}$
is nonempty, for all $j \geq j(\gamma, h)$.

For technical reasons, we complete the above convention as

$B(j) := A(j)$, for each $j < j(\gamma, h)$.

Having this precise, denote for each $j \geq 0$

$m(j) = \min \text{Dom}(B(j)), n(j) = \min B(j)(m(j))$.

By this very convention,

(pro-1) $j \leq m(j) < n(j)$, for all $j \geq 0$
(pro-2) $d(x[m(j), m(j)+3h], x[n(j), n(j)+3h]) > \gamma, \forall j \geq j(\gamma, h)$.

On the other hand, as $(x_n; n \geq 0)$ is $(\mathscr{R}, h)$-uscattered, there must be an associated function $L : N(<) \to N[1, h]$, such that

$x_m \mathscr{R} x_{n+L(m,n)}$, for each $(m, n) \in N(<)$.

Denoting, for simplicity $(p(j) = n(j) + L(m(j), n(j));\ j \geq 0)$, it results that

(pro–3) $(m(j) <) n(j) < p(j) \leq n(j) + h, x_{m(j)} \mathscr{R} x_{p(j)}, \forall j$.

We claim that the rank-sequences $(m(j); j \geq 0)$, $(n(j); j \geq 0)$ and $(p(j); j \geq 0)$ fulfill all conclusions in the statement.

(i) For the moment, (36-1) is fulfilled.
(ii) We claim that

$\forall j \geq j(\gamma, h), \forall (m, n) \in B(j)$: $m + 2h < n$;

and this, along with $(m(j), n(j)) \in B(j)$, established the conclusion (36-2). In fact, suppose by contradiction that

there exist $j \geq j(\gamma, h)$, $(m, n) \in B(j)$, with $n \leq m + 2h$.

Then,

$m < n < n + h \leq m + 3h$; so that
$x[m, m+3h] \cap x[n, n+h] = x[n, n+h] \neq \emptyset$.

On the other hand, as $(m, n) \in B(j)$,

$d(x[m, m+3h], x[n, n+h]) \geq d(x[m, m+3h], x[n, n+3h]) > \gamma > 0;$
whence $x[m, m+3h] \cap x[n, n+h] = \emptyset$.

The obtained relations are, however, contradictory. Hence, the working hypothesis cannot be true; and our claim follows.

(iii) Let $j \geq j(\gamma, h)$ be arbitrary fixed. By definition,

$$(m(j) < n(j) \text{ and}) \; d(x[m(j), m(j)+3h], x[n(j), n(j)+3h]) > \gamma.$$

Moreover, as $n(j)$ is minimal with respect to (pro-2), we derive

$$\text{(pro-4)} \;\; d(x[m(j), m(j)+3h], x[n(j)-1, n(j)-1+3h]) \leq \gamma.$$

But, in view of

$$n(j) \leq n(j)-1+3h \leq n(j)+3h,$$

and (pro-2), we have an evaluation like

$$\text{(pro-5)} \;\; d(x[m(j), m(j)+3h], x[n(j), n(j)-1+3h]) \geq$$
$$d(x[m(j), m(j)+3h], x[n(j), n(j)+3h]) > \gamma.$$

So, by simply combining with (pro-4) gives

$$d(x[m(j), m(j)+3h], x_{n(j)-1}) \leq \gamma;$$

wherefrom (taking an auxiliary fact into account)

$$d(x_{m(j)}, x_{n(j)-1}) \leq d(x[m(j), m(j)+3h], x_{n(j)-1}) +$$
$$\text{diam}(x[m(j), m(j)+3h]) \leq \gamma + \Delta_{m(j)}(3h), \forall j \geq j(\gamma, h),$$

and (36-3) is proved.

(iv) Let $s, t \in N[0, 2h]$ be arbitrary fixed. Further, take some $j \geq j(\gamma, h)$. From

$$m(j) \leq m(j)+s \leq m(j)+2h \leq m(j)+3h,$$
$$n(j) < p(j) \leq p(j)+t \leq n(j)+h+t \leq n(j)+3h,$$

we must have (for the precise ranks)

$$d(x_{m(j)+s}, x_{p(j)+t}) \geq d(x[m(j), m(j)+3h], x[n(j), n(j)+3h]) > \gamma;$$

so that the first half of our conclusion holds. For the second half, two cases occur.

Case 1 Assume that $s = t = 0$. By the very definition of these ranks,

$$n(j)-1 < p(j) \leq n(j)+h \leq n(j)-1+2h, \forall j \geq j(\gamma, h).$$

Combining with our preceding stage yields

$$\gamma < d(x_{m(j)}, x_{p(j)}) \leq d(x_{m(j)}, x_{n(j)-1}) + d(x_{n(j)-1}, x_{p(j)})$$
$$\leq \gamma + \Delta_{m(j)}(3h) + \Delta_{n(j)-1}(2h), \forall j \geq j(\gamma, h);$$

and the particular case of (36-4) (our second half) results by a limit process.

Case 2 For the remaining alternatives, we have by the Lipschitz property of $d(.,.)$,

$$|d(x_{m(j)}, x_{p(j)}) - d(x_{m(j)+s}, x_{p(j)+t}|$$
$$\leq d(x_{m(j)}, x_{m(j)+s}) + d(x_{p(j)}, x_{p(j)+t}), \forall j \geq j(\gamma, h);$$

so that (for the same ranks)

$$|d(x_{m(j)}, x_{p(j)}) - d(x_{m(j)+s}, x_{n(j)+t}| \le \Delta_{m(j)}(2h) + \Delta_{p(j)}(2h).$$

This gives us the general part of (36-4) (our second half), by simply passing to limit as $j \to \infty$. The proof is complete.

In particular, when $\Theta = R_+^0$ and $\mathscr{R} = X \times X$ (the trivial relation over X) the obtained statement covers the one in Khan et al. [32]; so, it is natural that this result be referred to as Khan–Swaleh–Sessa–Proposition (in short: KSS-Proposition). Further aspects may be found in Reich [50].

4 Meir–Keeler Relations

Let $\Omega \subseteq R_+^0 \times R_+^0$ be a relation over R_+^0; as a rule, we write $(t, s) \in \Omega$ as $t\Omega s$. The following global property upon this object is considered

(u-diag) Ω is *upper diagonal*: $t\Omega s$ implies $t < s$.

Denote the class of all upper diagonal relations as udiag(R_+^0). Our exposition below is essentially related to this basic condition.

To begin with, let us consider the global properties

(1-decr) Ω is *first variable decreasing*:
$t_1, t_2, s \in R_+^0$, $t_1 \ge t_2$ and $t_1\Omega s$ imply $t_2\Omega s$
(2-incr) Ω is *second variable increasing*:
$t, s_1, s_2 \in R_+^0$, $s_1 \le s_2$ and $t\Omega s_1$ imply $t\Omega s_2$.

Then, define the sequential condition below (for upper diagonal relations)

(M-ad) Ω in *Matkowski admissible*:
$(t_n; n \ge 0)$ in R_+^0 and $(t_{n+1}\Omega t_n, \forall n)$ imply $\lim_n t_n = 0$.

To discuss it, the local geometric conditions involving udiag(R_+^0) are in effect:

(g-mk) Ω has the *geometric Meir–Keeler property*:
$\forall \varepsilon > 0, \exists \delta > 0$: $t\Omega s, \varepsilon < s < \varepsilon + \delta \Longrightarrow t \le \varepsilon$
(g-bila-s) Ω is *geometric bilateral separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall (t, s)$: $t, s \in]\beta - \gamma, \beta + \gamma[\Longrightarrow (t, s) \notin \Omega$
(g-left-s) Ω is *geometric left separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall t$: $t \in]\beta - \gamma, \beta[\Longrightarrow (t, \beta) \notin \Omega$.

Remark 1 It is worth noting that, by the upper diagonal property, the geometric Meir–Keeler property is equivalent with

(g-mk-c) Ω has the *complete geometric Meir–Keeler property*:
$\forall \varepsilon > 0, \exists \delta > 0$: $t\Omega s, s < \varepsilon + \delta \Longrightarrow t \le \varepsilon$.

Since the verification is immediate, we do not give details.

The former of these local conditions—related to the developments in Meir and Keeler [43]—is strongly related to the Matkowski admissible property we just introduced. Precisely, the following auxiliary fact is available.

Proposition 14 *Under these conditions, one has in (ZF-AC+DC):*

(41-1) (for each $\Omega \in udiag(R^0_+)$):
Ω is geometric Meir–Keeler implies Ω is Matkowski admissible
(41-2) (for each first variable decreasing $\Omega \in udiag(R^0_+)$):
Ω is Matkowski admissible implies Ω is geometric Meir–Keeler.

Hence, summing up

(41-3) (for each first variable decreasing $\Omega \in udiag(R^0_+)$):
Ω is geometric Meir–Keeler iff Ω is Matkowski admissible.

Proof Three basic stages must be passed.

(i) Suppose that $\Omega \in$ udiag(R^0_+) is geometric Meir–Keeler; we have to establish that Ω is Matkowski admissible. Let $(t_n; n \geq 0)$ be a sequence in R^0_+, fulfilling $(t_{n+1}\Omega t_n$, for all $n)$. By the upper diagonal property, we get

$(t_{n+1} < t_n$, for all $n)$; i.e., (t_n) is strictly descending.

As a consequence, $\tau = \lim_n t_n$ exists in R_+; with, in addition: $(t_n > \tau, \forall n)$. Assume by contradiction that $\tau > 0$; and let $\sigma > 0$ be the number assured by the geometric Meir–Keeler property. By definition, there exists an index $n(\sigma)$, with

$(t_{n+1}\Omega t_n$ and$)$ $\tau < t_n < \tau + \sigma$, for all $n \geq n(\sigma)$.

This, by the quoted property, gives (for the same ranks)

$\tau < t_{n+1} \leq \tau$; contradiction.

Hence, necessarily, $\tau = 0$; and the conclusion follows.

(ii) Suppose that the first variable decreasing $\Omega \in$ udiag(R^0_+) is Matkowski admissible; we have to establish that Ω is geometric Meir–Keeler. Suppose by contradiction that this is not true; i.e. (for some $\varepsilon > 0$)

$H(\delta) := \{(t, s) \in \Omega; \varepsilon < s < \varepsilon + \delta, t > \varepsilon\}$ is nonempty, for each $\delta > 0$.

Taking a zero converging sequence $(\delta_n; n \geq 0)$ in R^0_+, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $((t_n, s_n); n \geq 0)$ in $R^0_+ \times R^0_+$, so as

$(\forall n)$: (t_n, s_n) is an element of $H(\delta_n)$;

or, equivalently (by definition and upper diagonal property)

$(t_n\Omega s_n$ and$)$ $\varepsilon < t_n < s_n < \varepsilon + \delta_n$, for all n.

Note that, as a direct consequence,

$(t_n\Omega s_n$, for all $n)$, and $t_n \to \varepsilon+$, $s_n \to \varepsilon+$, as $n \to \infty$.

Put $i(0) = 0$. As $\varepsilon < t_{i(0)}$ and $s_n \to \varepsilon+$ as $n \to \infty$, we have that

$A(i(0)) := \{n > i(0); s_n < t_{i(0)}\}$ is not empty;
hence, $i(1) := \min(A(i(0)))$ is an element of it, and $s_{i(1)} < t_{i(0)}$;
wherefrom, $s_{i(1)} \Omega s_{i(0)}$ (as Ω is first variable decreasing).

Likewise, as $\varepsilon < t_{i(1)}$ and $s_n \to \varepsilon+$ as $n \to \infty$, we have that

$A(i(1)) := \{n > i(1); s_n < t_{i(1)}\}$ is not empty;
hence, $i(2) := \min(A(i(1)))$ is an element of it, and $s_{i(2)} < t_{i(1)}$;
wherefrom, $s_{i(2)} \Omega s_{i(1)}$ (as Ω is first variable decreasing).

This procedure may continue indefinitely and yields (without any choice technique) a strictly ascending rank sequence $(i(n); n \geq 0)$ in N for which the attached subsequence $(r_n := s_{i(n)}; n \geq 0)$ of $(s_n; n \geq 0)$ fulfills

$r_{n+1} \Omega r_n$, for all n; whence $r_n \to 0$ (as Ω is Matkowski admissible).

On the other hand, by our subsequence property,

$(r_n > \varepsilon, \forall n)$ and $\lim_n r_n = \lim_n s_n = \varepsilon$; that is: $r_n \to \varepsilon+$.

The obtained relation is in contradiction with the previous one. Hence, the working condition cannot be true; and we are done.

(iii) Evident, by the above.

In the following, sufficient (sequential) conditions are given for the properties appearing in our (geometric) concepts above. Given the upper diagonal relation Ω over R^0_+, let us introduce the (asymptotic type) conventions

(a-mk) Ω is *asymptotic Meir–Keeler*:
there are no strictly descending sequences (t_n) and (s_n) in R^0_+ and no elements ε in R^0_+, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$
(a-bila-s) Ω is *asymptotic bilateral separable*:
there are no sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R^0_+ and no elements $\varepsilon \in R^0_+$, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon, s_n \to \varepsilon)$
(a-left-s) Ω is *asymptotic left separable*:
there are no strictly ascending sequences (t_n) in R^0_+
and no elements β in R^0_+, with $((t_n, \beta) \in \Omega, \forall n)$ and $(t_n \to \beta-)$.

Remark 2 Concerning the bilateral concept above, let us consider the condition

(a-bila-s-str) Ω is *strongly asymptotic bilateral separable*:
there are no sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R^0_+ and no elements $\varepsilon \in R^0_+$, with $((t_n, s_n) \in \Omega, \forall n)$ and $\liminf_n t_n \geq \varepsilon$, $\limsup_n s_n \leq \varepsilon$.

Clearly,

(Ω is strongly asymptotic bilateral separable) implies
(Ω is asymptotic bilateral separable).

But, the reciprocal inclusion

(Ω is asymptotic bilateral separable) implies
(Ω is strongly asymptotic bilateral separable)

is also true; whence these two conditions are equivalent to each other. In fact, suppose that Ω is not strongly asymptotic bilateral separable: there are sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R_+^0 and elements $\varepsilon \in R_+^0$, with

$((t_n, s_n) \in \Omega$ (hence, $t_n < s_n$), $\forall n)$ and $\liminf_n t_n \geq \varepsilon$, $\limsup_n s_n \leq \varepsilon$.

By these relations, we have

$\varepsilon \leq \liminf_n t_n \leq \limsup_n t_n \leq \limsup_n s_n \leq \varepsilon$,
$\varepsilon \leq \liminf_n t_n \leq \liminf_n s_n \leq \limsup_n s_n \leq \varepsilon$;
wherefrom $t_n \to \varepsilon$, $s_n \to \varepsilon$, as $n \to \infty$.

This last relation, added to the preceding ones, yields a contradiction with respect to Ω being asymptotic bilateral separable; so that our assertion is proved.

Remark 3 Concerning the left concept above, let us consider the condition

(a-left-s-str) Ω is *strongly asymptotic left separable*:
there are no sequences (t_n) in R_+^0
and no elements β in R_+^0, with $((t_n, \beta) \in \Omega, \forall n)$ and $(t_n \to \beta)$.

Clearly,

(for each $\Omega \in \text{udiag}(R_+^0)$):
(Ω is strongly asymptotic left separable) implies
(Ω is asymptotic left separable).

The converse inclusion is also valid in the class of upper diagonal relations; that is,

(for each $\Omega \in \text{udiag}(R_+^0)$):
(Ω is asymptotic left separable) implies
(Ω is strongly asymptotic left separable).

In fact, suppose that $\Omega \in \text{udiag}(R_+^0)$ is not strongly asymptotic left separable:

there is a sequence (t_n) in R_+^0 and an element β in R_+^0, with
$((t_n, \beta) \in \Omega, \forall n)$ and $t_n \to \beta$.

By the upper diagonal property of Ω,

$t_n < \beta$, for all n; so that $t_n \to \beta-$.

By an auxiliary fact, there exists a strictly ascending subsequence $(t_n^* = t_{i(n)}; n \geq 0)$ of $(t_n; n \geq 0)$, with

$((t_n^*, \beta) \in \Omega, \forall n)$ and $t_n^* \to \beta-$.

This, however, contradicts the asymptotic left separated property of Ω; and therefore, our claim is proved.

Returning to the general setting above, the relationships with the corresponding geometric type notions are described in the auxiliary statement below.

Proposition 15 *The following generic relationships are valid (for an arbitrary upper diagonal relation $\Omega \subseteq R^0_+ \times R^0_+$), in the reduced system (ZF-AC+DC):*

(42-1) geometric Meir–Keeler is equivalent with asymptotic Meir–Keeler
(42-2) geometric bilateral separable is equivalent with asymptotic bilateral separable (hence, with strongly asymptotic bilateral separable)
(42-3) geometric left separable is equivalent with asymptotic left separable (hence, with strongly asymptotic left separable).

Proof There are three steps to be passed.

(i-1) Let $\Omega \in \text{udiag}(R^0_+)$ be a geometric Meir–Keeler relation; but—contrary to the conclusion—assume that Ω does not have the asymptotic Meir–Keeler property:

there are two strictly descending sequences (t_n) and (s_n) in R^0_+ and an element ε in R^0_+, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$.

Let $\delta > 0$ be the number given by the geometric Meir–Keeler property of Ω. By definition, there exists a (common) rank $n(\delta)$, such that

$n \geq n(\delta)$ implies $\varepsilon < t_n < \varepsilon + \delta, \varepsilon < s_n < \varepsilon + \delta$.

From the second relation, we must have (by the hypothesis about Ω) $t_n \leq \varepsilon$, for all $n \geq n(\delta)$. This, however, contradicts the first relation above. Hence, Ω is asymptotic Meir–Keeler; as asserted.

(i-2) Let $\Omega \in \text{udiag}(R^0_+)$ be an asymptotic Meir–Keeler relation; but—contrary to the conclusion—assume that Ω does not have the geometric Meir–Keeler property; i.e. (for some $\varepsilon > 0$)

$H(\delta) := \{(t, s) \in \Omega; s < \varepsilon + \delta, t > \varepsilon\} \neq \emptyset$, for each $\delta > 0$.

Taking a zero converging sequence $(\delta_n; n \geq 0)$ in R^0_+, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $(t_n, s_n); n \geq 0)$ in $R^0_+ \times R^0_+$, so as

$(\forall n)$: (t_n, s_n) is an element of $H(\delta_n)$;

or, equivalently (by definition and upper diagonal property)

$((t_n, s_n) \in \Omega$ and) $\varepsilon < t_n < s_n < \varepsilon + \delta_n$, for all n.

Note that, as a direct consequence,

$(t_n \Omega s_n$, for all $n)$, and $t_n \to \varepsilon+, s_n \to \varepsilon+$, as $n \to \infty$.

By a previous result, there exists a compatible couple of subsequences $(t_n^* := t_{i(n)}; n \geq 0)$ of $(t_n; n \geq 0)$ and $(s_n^* := s_{i(n)}; n \geq 0)$ of $(s_n; n \geq 0)$, with

$(t_n^* \Omega s_n^*, \forall n)$; (t_n^*), (s_n^*) are strictly descending; $t_n^* \to \varepsilon+$ and $s_n^* \to \varepsilon+$.

This, however, is in contradiction with respect to the posed hypothesis upon Ω; wherefrom, our assertion follows.

(ii-1) Let $\Omega \in \text{udiag}(R_+^0)$ be a geometric bilateral separable relation; we have to establish that Ω is asymptotic bilateral separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is,

there are two sequences $(t_n; n \geq 0)$ and $(s_n; n \geq 0)$ in R_+^0 and an element $\varepsilon \in R_+^0$, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon, s_n \to \varepsilon)$.

Let $\delta > 0$ be the number given by the geometric left separable property of Ω. By definition, there exists a (common) rank $n(\delta)$, such that

$n \geq n(\delta)$ implies $\varepsilon - \delta < t_n < \varepsilon + \delta, \varepsilon - \delta < s_n < \varepsilon + \delta$.

This along with $[t_n \Omega s_n, \forall n \geq n(\delta)]$ contradicts the geometric bilateral separable property of Ω. Hence, Ω is asymptotic bilateral separable.

(ii-2) Let $\Omega \in \text{udiag}(R_+^0)$ be an asymptotic bilateral separable relation; we have to establish that Ω is geometric bilateral separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is (for some $\varepsilon > 0$),

$K(\delta) := \{(t, s) \in \Omega; t, s \in]\varepsilon - \delta, \varepsilon + \delta[\} \neq \emptyset$, for each $\delta \in]0, \varepsilon[$.

Taking a strictly ascending sequence $(\delta_n; n \geq 0)$ in $]0, \varepsilon[$ with $\delta_n \to 0$, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $((t_n, s_n); n \geq 0)$ in Ω, so as

$(\forall n)$: (t_n, s_n) is an element of $K(\delta_n)$;

or, equivalently (by the very definition above)

$(\forall n)$: $(t_n, s_n) \in \Omega$ and $t_n, s_n \in]\varepsilon - \delta_n, \varepsilon + \delta_n[$.

As a consequence of the latter, we must have $(t_n \to \varepsilon, s_n \to \varepsilon)$; and this, along with the former, contradicts the imposed hypothesis. Hence, necessarily, Ω is geometric bilateral separable.

(iii-1) Let $\Omega \in \text{udiag}(R_+^0)$ be a geometric left separable relation; we have to establish that Ω is asymptotic left separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is,

there is a strictly ascending sequence $(t_n; n \geq 0)$ and an element $\varepsilon \in R_+^0$, with $((t_n, \varepsilon) \in \Omega, \forall n)$ and $(t_n \to \varepsilon-)$.

Let $\delta > 0$ be the number given by the geometric left separable property of Ω. By definition, there exists a rank $n(\delta)$, such that

$n \geq n(\delta)$ implies $\varepsilon - \delta < t_n < \varepsilon$.

This along with $[t_n \Omega \varepsilon, \forall n \geq n(\delta)]$ contradicts the geometric left separable property of Ω. Hence, Ω is asymptotic bilateral separable.

(iii-2) Let $\Omega \in \text{udiag}(R^0_+)$ be an asymptotic left separable relation; we have to establish that Ω is geometric left separable. Suppose—contrary to this conclusion—that Ω is not endowed with such a property; that is (for some $\beta > 0$),

$$K(\gamma) := \{t \in]\beta - \gamma, \beta[; (t, \beta) \in \Omega\} \neq \emptyset, \text{ for each } \gamma \in]0, \beta[.$$

Taking a strictly ascending sequence $(\gamma_n; n \geq 0)$ in $]0, \beta[$ with $\gamma_n \to 0+$, we get by the Denumerable Axiom of Choice (AC(N)) [deductible, as precise, in (ZF-AC+DC)], a sequence $(t_n; n \geq 0)$ in R^0_+, so as

$(\forall n)$: t_n is an element of $K(\gamma_n)$;

or, equivalently (by the very definition above)

$(\forall n)$: $\beta - \gamma_n < t_n < \beta$, and $(t_n, \beta) \in \Omega$.

By the former part, we must have $t_n \to \beta-$; and this, along with an auxiliary fact, tells us that there exists a subsequence $(t^*_n := t_{i(n)}; n \geq 0)$ such that

$(t^*_n; n \geq 0)$ is strictly ascending, and $t^*_n \to \beta-$.

On the other hand, by the latter part of our previous relation, $[(t^*_n, \beta) \in \Omega, \forall n]$. This, however, contradicts the asymptotic left separable property. Hence, Ω is geometric left separable.

In the following, some basic examples of (upper diagonal) Matkowski admissible and geometric Meir–Keeler relations are given. The general scheme of constructing these may described along the lines below.

Let $R(\pm\infty) := R \cup \{-\infty, \infty\}$ stand for the set of all *extended real numbers*. For each relation Ω over R^0_+, let us associate a function $\xi : R^0_+ \times R^0_+ \to R(\pm\infty)$, as

$\xi(t, s) = 0$, if $(t, s) \in \Omega$; $\xi(t, s) = -\infty$, if $(t, s) \notin \Omega$.

It will be referred to as the *function* generated by Ω; clearly,

$(t, s) \in \Omega$ iff $\xi(t, s) \geq 0$.

Conversely, given a function $\xi : R^0_+ \times R^0_+ \to R(\pm\infty)$, we may associate it a relation Ω over R^0_+ as

$\Omega = \{(t, s) \in R^0_+ \times R^0_+; \xi(t, s) \geq 0\}$ (in short: $\Omega = [\xi \geq 0]$);
referred to as: the *the positive section* of ξ.

Note that the correspondence between the function ξ and its associated relation $[\xi \geq 0]$ is not injective; because, for the function $\eta := \lambda\xi$ (where $\lambda > 0$), its associated relation $[\eta \geq 0]$ is identical with the relation $[\xi \geq 0]$ attached to ξ.

Now, call the function $\xi : R^0_+ \times R^0_+ \to R(\pm\infty)$, *upper diagonal* provided:

(u-diag-fct) $\xi(t,s) \geq 0$ implies $t < s$.

Note that all subsequent constructions are being considered within this setting. In particular, the following basic property (condition) for upper diagonal functions ξ is considered:

(M-ad-fct) ξ in *Matkowski admissible*:
$(t_n; n \geq 0)$ in R^0_+ and $(\xi(t_{n+1}, t_n) \geq 0, \forall n)$ imply $\lim_n t_n = 0$.

The following geometric conditions involving our functions are—in particular—useful for discussing this property

(g-mk-fct) ξ is *geometric Meir–Keeler*:
$\forall \varepsilon > 0, \exists \delta > 0$: $\xi(t,s) \geq 0, \varepsilon < s < \varepsilon + \delta \Longrightarrow t \leq \varepsilon$
(g-bila-s-fct) ξ is *geometric bilateral separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall (t,s)$: $t, s \in]\beta - \gamma, \beta + \gamma[\Longrightarrow \xi(t,s) < 0$
(g-left-s-fct) ξ is *geometric left separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall t$: $t \in]\beta - \gamma, \beta[\Longrightarrow \xi(t, \beta) < 0$.

The relationships between the former geometric condition and the Matkowski one attached to upper diagonal functions are nothing else than a simple translation of the previous ones involving upper diagonal relations; we do not give details.

Summing up, any concept (like the ones above) about (upper diagonal) relations over R^0_+ may be written as a concept about (upper diagonal) functions in the class $\mathscr{F}(R^0_+ \times R^0_+, R(\pm\infty))$. For the rest of our exposition, it will be convenient working with relations over R^0_+, and not with functions in $\mathscr{F}(R^0_+ \times R^0_+, R(\pm\infty))$; this, however, is but a methodology question.

We may now pass to the description of some basic objects in this area.

Part-Case (I) Let $\mathscr{F}(re)(R^0_+, R)$ stand for the subclass of all $\varphi \in \mathscr{F}(R^0_+, R)$ with

φ is *regressive*: $\varphi(t) < t$, for all $t > 0$.

Call $\varphi \in \mathscr{F}(re)(R^0_+, R)$, *Meir–Keeler admissible* if

(mk-adm) $\forall \gamma > 0, \exists \beta > 0, \forall t$: $\gamma < t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma$;
or, equivalently: $[\forall \gamma > 0, \exists \beta > 0, \forall t$: $0 < t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma]$.

Some important examples of such functions may be given along the lines below. For any $\varphi \in \mathscr{F}(re)(R^0_+, R)$ and any $s \in R^0_+$, put

$\Lambda^+\varphi(s) = \inf_{0<\varepsilon<s} \Phi(s+)(\varepsilon)$; where $\Phi(s+)(\varepsilon) = \sup \varphi(]s, s+\varepsilon[)$
$\Lambda^-\varphi(s) = \inf_{0<\varepsilon<s} \Phi(s-)(\varepsilon)$; where $\Phi(s-)(\varepsilon) = \sup \varphi(]s-\varepsilon, s[)$
$\Lambda^\pm\varphi(s) = \inf_{0<\varepsilon<s} \Phi(s\pm)(\varepsilon)$; where $\Phi(s\pm)(\varepsilon) = \sup \varphi(]s-\varepsilon, s+\varepsilon[)$.

From the regressive property of φ, these limit quantities fulfill

$$(-\infty \leq)\ \Lambda^+\varphi(s),\ \Lambda^-\varphi(s) \leq \Lambda^\pm\varphi(s) \leq s,\ \forall s \in R^0_+.$$

but the case of these limits having infinite values cannot be avoided.

The following auxiliary fact will be useful.

Proposition 16 *Let $\varphi \in \mathscr{F}(re)(R^0_+, R)$ and $s \in R^0_+$ be arbitrary fixed. Then,*

(43-1) $\limsup_n(\varphi(t_n)) \le \Lambda^+\varphi(s)$, *for each sequence (t_n) in R^0_+ with $t_n \to s+$*
(43-2) $\limsup_n(\varphi(t_n)) \le \Lambda^-\varphi(s)$, *for each sequence (t_n) in R^0_+ with $t_n \to s-$*
(43-3) $\limsup_n(\varphi(t_n)) \le \Lambda^\pm\varphi(s)$, *for each sequence (t_n) in R^0_+ with $t_n \to s$.*

Proof

(i) Given $\varepsilon \in]0, s[$, there exists a rank $p(\varepsilon) \ge 0$ such that $s < t_n < s + \varepsilon$, for all $n \ge p(\varepsilon)$; hence

$$\limsup_n(\varphi(t_n)) \le \sup\{\varphi(t_n); n \ge p(\varepsilon)\} \le \Phi(s+)(\varepsilon).$$

It suffices taking the infimum over $\varepsilon > 0$ in this relation to get the desired fact.

(ii) The proof mimics the preceding one; so, we omit it.

(iii) Given $\varepsilon \in]0, s[$, there exists a rank $p(\varepsilon) \ge 0$ such that $s - \varepsilon < t_n < s + \varepsilon$, for all $n \ge p(\varepsilon)$; hence

$$\limsup_n(\varphi(t_n)) \le \sup\{\varphi(t_n); n \ge p(\varepsilon)\} \le \Phi(s\pm)(\varepsilon).$$

Taking the infimum over $\varepsilon > 0$ in this relation, we get the desired conclusion.

Call $\varphi \in \mathscr{F}(re)(R^0_+, R)$, *Boyd–Wong admissible* [9], if

(bw-adm) $\Lambda^+\varphi(s) < s$, for all $s > 0$.

In particular, $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Boyd–Wong admissible provided it is *upper semicontinuous at the right* on R^0_+:

$$\Lambda^+\varphi(s) \le \varphi(s), \text{ for each } s \in R^0_+.$$

This, e.g., is fulfilled when φ is *continuous at the right* on R^0_+; for, in such a case,

$$\Lambda^+\varphi(s) = \varphi(s), \text{ for each } s \in R^0_+.$$

On the other hand, $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Boyd–Wong admissible when

φ is *strongly Boyd–Wong admissible*: $\Lambda^\pm\varphi(s) < s$, $\forall s \in R^0_+$.

Further, let $\mathscr{F}(re, in)(R^0_+, R)$ stand for the class of all $\varphi \in \mathscr{F}(re)(R^0_+, R)$, with

φ is increasing on R^0_+ ($0 < t_1 \le t_2$ implies $\varphi(t_1) \le \varphi(t_2)$).

Then, let us say that $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$ is *Matkowski admissible* [41], provided

(m-adm) $(\forall t > 0)$: $\lim_n \varphi^n(t) = 0$, as long as $(\varphi^n(t); n \ge 0)$ exists.

Here, as usual, for each $t > 0$,

$$\varphi^0(t) = t, \varphi^1(t) = \varphi(t), \dots, \varphi^{n+1}(t) = \varphi(\varphi^n(t)), n \ge 1.$$

Note that the obtained class of functions is distinct from the above introduced one, as simple examples show.

Remark 4 Under these conventions,

(BW-mk) each Boyd–Wong admissible function in $\mathscr{F}(re)(R^0_+, R)$ is Meir–Keeler admissible
(M-mk) each Matkowski admissible function in $\mathscr{F}(re, in)(R^0_+, R)$ is Meir–Keeler admissible.

The verification of this is as follows.

(i) (cf. Boyd and Wong [9]). Suppose that $\varphi \in \mathscr{F}(re)(R^0_+, R)$ is Boyd–Wong admissible, and fix $\gamma > 0$; hence $\Lambda^+\varphi(\gamma) < \gamma$. By definition, there exists $\beta = \beta(\gamma) > 0$ with $[\gamma < t < \gamma + \beta$ implies $\varphi(t) < \gamma]$; proving that φ is Meir–Keeler admissible.
(ii) (cf. Jachymski [25]). Assume that $\varphi \in \mathscr{F}(re, in)(R^0_+, R)$ is Matkowski admissible. If the underlying property fails, then (for some $\gamma > 0$):

$$\forall\beta > 0, \exists t \in]\gamma, \gamma + \beta[, \text{ such that } \varphi(t) \geq \gamma.$$

Combining with the increasing property of φ, one gets

$(\forall t > \gamma)$: $\varphi(t) \geq \gamma$ [whence (by induction): $\varphi^n(t) \geq \gamma$, for each n].

Fixing some $t > \gamma$ and passing to limit as $n \to \infty$, one derives $0 \geq \gamma$; contradiction. This ends the argument.

Having these precise, take a function $\varphi \in \mathscr{F}(re)(R^0_+, R)$ and define the associated relation $\Omega := \Omega[\varphi]$ over R^0_+, as

$(t, s \in R^0_+)$: $(t, s) \in \Omega$ iff $t \leq \varphi(s)$.

Clearly, Ω is upper diagonal. In fact, let $t, s \in R^0_+$ be such that $t\Omega s$; i.e., $t \leq \varphi(s)$. As φ is regressive, one has $\varphi(s) < s$; and this yields $t < s$, whence the conclusion follows. Further properties of this relation are deductible from

Proposition 17 *Let the function $\varphi \in \mathscr{F}(re)(R^0_+, R)$ be given, and $\Omega := \Omega[\varphi]$ stand for the associated upper diagonal relation over R^0_+. Then,*

(44-1) Ω is geometric Meir–Keeler when φ is Meir–Keeler admissible
(44-2) Ω is asymptotic Meir–Keeler when φ is Boyd–Wong admissible
(44-3) Ω is asymptotic bilateral separable when φ is strongly Boyd–Wong admissible (see above).

Proof

(i) Let $\varepsilon > 0$ be given; and $\delta > 0$ be the number associated to it, via Meir–Keeler admissible property for φ. Given $t, s \in R^0_+$ with $t\Omega s$, $\varepsilon < s < \varepsilon + \delta$ we have $[t \leq \varphi(s), \varepsilon < s < \varepsilon + \delta]$. This, according to the underlying property of φ, gives $\varphi(s) \leq \varepsilon$ [hence, $t \leq \varepsilon$]; wherefrom: Ω has the geometric Meir–Keeler property.

(ii) Suppose by contradiction that Ω is not asymptotic Meir–Keeler; i.e.,

there exist strictly descending sequences (t_n) and (s_n) in R_+^0
and elements ε in R_+^0, with
$((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$.

This, by the definition of Ω, yields

$(t_n \le \varphi(s_n), \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$

Passing to lim sup as $n \to \infty$, one derives $\varepsilon \le \Lambda^+\varphi(\varepsilon) < \varepsilon$; contradiction. Hence, our working assumption is not acceptable; and conclusion follows.

(iii) Suppose by contradiction that Ω is not asymptotic bilateral separable; i.e.,

there exist sequences $(t_n; n \ge 0)$ and $(s_n; n \ge 0)$ in R_+^0
and elements $\varepsilon \in R_+^0$, with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon, s_n \to \varepsilon)$.

This, by the definition of Ω, yields

$(t_n \le \varphi(s_n), \forall n)$ and $(t_n \to \varepsilon, s_n \to \varepsilon)$

Passing to lim sup as $n \to \infty$, one derives $\varepsilon \le \Lambda^\pm\varphi(\varepsilon) < \varepsilon$; contradiction. Hence, our working assumption is not acceptable; and conclusion follows.

Part-Case (II) Let $(\psi, \varphi) \in \mathscr{F}^2(R_+^0, R) := \mathscr{F}(R_+^0, R) \times \mathscr{F}(R_+^0, R)$ be a couple of functions endowed with

(norm) (ψ, φ) is *normal*:
ψ is increasing and φ is *strictly positive* $[\varphi(t) > 0, \forall t > 0]$.

(This concept may be related to the one introduced by Rhoades [52]; see also Dutta and Choudhury [19]). Then, define the relation $\Omega = \Omega[\psi, \varphi]$ in $R_+^0 \times R_+^0$, as

$(t, s) \in \Omega$ iff $\psi(t) \le \psi(s) - \varphi(s)$.

We claim that, necessarily, Ω is upper diagonal. In fact, let $t, s \in R_+^0$ be such that

$(t, s) \in \Omega$; i.e., $\psi(t) \le \psi(s) - \varphi(s)$.

By the strict positivity of φ, one gets $\psi(t) < \psi(s)$; and this, along with the increasing property of ψ, shows that $t < s$; whence the conclusion follows. Further properties of this relation are available under certain supplementary conditions about the normal couple (ψ, φ), like below:

(as-pos) φ is *asymptotic positive*:
for each strictly descending sequence $(t_n; n \ge 0)$ in R_+^0 and each $\varepsilon > 0$ with $t_n \to \varepsilon+$, we must have $\limsup_n(\varphi(t_n)) > 0$
(bd-osc) ψ is *φ-bounded oscillating*:
for each sequence $(t_n; n \ge 0)$ in R_+^0 and each $\varepsilon > 0$ with $t_n \to \varepsilon$, we have $\limsup_n(\varphi(t_n)) > \psi(\varepsilon + 0) - \psi(\varepsilon - 0)$.

Proposition 18 *Let (ψ, φ) be a normal couple of functions in $\mathscr{F}^2(R_+^0, R)$; and $\Omega := \Omega[\psi, \varphi]$ be the associated upper diagonal relation. Then,*

(45-1) if φ is asymptotic positive, then Ω is asymptotic Meir–Keeler
(45-2) if ψ is φ-bounded oscillating, then the associated relation Ω is asymptotic bilateral separable.

Proof

(i) Suppose by contradiction that Ω is not asymptotic Meir–Keeler:

there exist strictly descending sequences (t_n) and (s_n) in R_+^0
and elements ε in R_+^0 with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon+, s_n \to \varepsilon+)$.

By the former of these, we get

$(0 <)\varphi(s_n) \leq \psi(s_n) - \psi(t_n), \forall n.$

Passing to limit as $n \to \infty$, and noting that $\lim_n \psi(s_n) = \lim_n \psi(t_n) = \psi(\varepsilon + 0)$, one gets $\lim_n \varphi(t_n) = 0$; in contradiction with the asymptotic positivity of φ. So, necessarily, Ω has the asymptotic Meir–Keeler property; as claimed.

(ii) Suppose by contradiction that Ω is not asymptotic bilateral separable; i.e.,

there exist sequences (t_n) and (s_n) in R_+^0 and elements ε in R_+^0,
with $((t_n, s_n) \in \Omega, \forall n)$ and $(t_n \to \varepsilon, s_n \to \varepsilon)$.

By the former of these, we get

$(0 <)\varphi(s_n) \leq \psi(s_n) - \psi(t_n), \forall n.$

Passing to lim sup as $n \to \infty$ yields $\limsup_n \varphi(s_n) \leq \psi(\varepsilon + 0) - \psi(\varepsilon - 0)$; in contradiction with ψ being φ-bounded oscillating. This tells us that Ω is asymptotic bilateral separable; as claimed.

In the following, some basic (and useful) particular choices for the couple (ψ, φ) above are to be discussed.

Part-Case (II-a) The construction in the preceding step (involving a certain $\chi \in \mathscr{F}(re)(R_+^0, R)$) is nothing else than a particular case of this one, corresponding to the choice

$\psi(t) = t, \varphi(t) = t - \chi(t), t \in R_+^0.$

Since the verification is immediate, we do not give details.

Part-Case (II-b) Let $\lambda : R_+^0 \to]1, \infty[$ and $\mu : R_+^0 \to]0, 1[$ be a couple of functions, with λ=increasing. Define a relation $\Omega := \Omega[[\lambda, \mu]]$ over R_+^0 as

$t\Omega s$ iff $\lambda(t) \leq [\lambda(s)]^{\mu(s)}$.

This will be referred to as the *Jleli–Samet relation* attached to $\lambda(.)$ and $\mu(.)$. (The proposed conventions come from the developments in Jleli and Samet [28], corresponding to $\mu(.)$=constant). By a direct calculation, it is evident that

$t\Omega s$ iff $t\Omega[\psi, \varphi]s$; where $\psi(t) = \log[\log(\lambda(t))], \varphi(t) = -\log(\mu(t)), t > 0.$

Hence, this construction is entirely reducible to the standard one in this series. Further aspects may be found in Suzuki and Vetro [61].

Part-Case (II-c) Let $\psi \in \mathscr{F}(R_+^0, R)$ and $\Delta \in \mathscr{F}(R)$ be a couple of functions. The following regularity condition involving these objects will be considered here

(BV-c) (ψ, Δ) is a *Bari–Vetro couple*:
ψ is increasing and Δ is regressive ($\Delta(r) < r$, for all $r \in R$).

In this case, by definition,

$\varphi(t) := \psi(t) - \Delta(\psi(t)) > 0$, for all $t > 0$;

so that (ψ, φ) is a normal couple of functions in $\mathscr{F}(R_+^0, R)$. Let $\Omega := \Omega[\psi, \Delta]$ be the (associated) *Bari–Vetro relation* over R_+^0, introduced as

$t\Omega s$ iff $\psi(t) \leq \Delta(\psi(s))$.

(This convention is related to the developments in Di Bari and Vetro [16]). From (BV-c), Ω is an upper diagonal relation over R_+^0. It is natural then to ask: under which extra assumptions about our data we have that Ω is an asymptotic Meir–Keeler relation. The simplest one may be written as

(a-reg) Δ is *asymptotic regressive*:
for each descending sequence (r_n) in R and each $\alpha \in R$ with $r_n \to \alpha$,
we have that $\liminf_n \Delta(r_n) < \alpha$.

Note that, by the non-strict character of the descending property above, one has

Δ is asymptotic regressive implies Δ is regressive.

Proposition 19 *Let the functions ($\psi \in \mathscr{F}(R_+^0, R)$, $\Delta \in \mathscr{F}(R)$) be such that*

(ψ, Δ) *is an asymptotic Bari-Vetro couple; i.e.,*
ψ *is increasing and* Δ *is asymptotic regressive.*

Then,
(46-1) *The above defined function* φ *is asymptotic positive*
(46-2) *The associated relation* Ω *is upper diagonal, and asymptotic Meir–Keeler (hence, geometric Meir–Keeler).*

Proof There are two steps to be passed.

(i) Let the strictly descending sequence $(t_n; n \geq 0)$ in R_+^0 and the number $\varepsilon > 0$ be such that $t_n \to \varepsilon+$; we must derive that $\limsup_n(\varphi(t_n)) > 0$. Denote, for simplicity

$(r_n = \psi(t_n), n \geq 0)$; $\alpha = \psi(\varepsilon + 0)$.

By the imposed conditions (and ψ=increasing)

(r_n) is descending and $r_n \to \alpha$ as $n \to \infty$.

In this case,

$\limsup_n \varphi(t_n) = \limsup_n [r_n - \Delta(r_n)] = \alpha - \liminf_n \Delta(r_n) > 0$;

hence the claim.

(ii) The assertion follows at once from (ψ, φ) being a normal couple with (φ=asymptotic positive), and a previous remark involving these objects. However, for completeness reasons, we provide an argument for this.

Step ii-1 Let $t, s > 0$ be such that

$t\Omega s$; i.e., $\psi(t) \le \Delta(\psi(s))$.

As Δ is regressive,

$\psi(t) < \psi(s)$; whence $t < s$ (in view of ψ=increasing);

so that Ω is upper diagonal.

Step ii-2 Suppose by contradiction that there exists a couple of strictly descending sequences (t_n) and (s_n) in R^0_+, and a number $\varepsilon > 0$, with

$t_n \to \varepsilon+$, $s_n \to \varepsilon+$, and $t_n \Omega s_n$ [i.e., $\psi(t_n) \le \Delta(\psi(s_n))$], for each n.

From the increasing property of ψ, one has (under $\alpha := \psi(\varepsilon + 0)$)

$(u_n := \psi(t_n))$ and $(v_n := \psi(s_n))$ are descending sequences in R, with
$u_n \to \alpha$, $v_n \to \alpha$, as $n \to \infty$;

so, passing to $\liminf$ as $n \to \infty$ in the relation above [i.e., $u_n \le \Delta(v_n)$, $\forall n$], one gets (via Δ=asymptotic regressive)

$\alpha = \liminf_n u_n \le \liminf_n \Delta(v_n) < \alpha$; contradiction.

Hence, our working assumption is not acceptable; and the conclusion follows.

In particular, when ψ and Δ are continuous, our statement reduces to the one in Jachymski [26].

Part-Case (III) Let $\chi : R^0_+ \to R$ be an increasing function. We say that $\xi : R^0_+ \times R^0_+ \to R$ is a χ-*inf-simulation function*, if

(is-1) ξ is χ-*diagonal*: $\xi(t, s) < \chi(s) - \chi(t)$, for each $t, s > 0$
(is-2) ξ is *inf-asymptotic negative*:
for each couple of strictly descending sequences (t_n) and (s_n) in R^0_+ and each $\varepsilon > 0$ with $(t_n \to \varepsilon+, s_n \to \varepsilon+)$, we have $\liminf_n \xi(t_n, s_n) < 0$.

If the increasing function χ is generic in this convention, we say that ξ is an *inf-simulation function*.

The usefulness of this concept for the developments above is assured by the following auxiliary fact.

Proposition 20 *Suppose that $\xi : R^0_+ \times R^0_+ \to R$ is an inf-simulation function. Then, the associated relation $\Omega := \Omega[\xi]$ on R^0_+, introduced as*

$$(t, s \in R^0_+)\text{: } t\Omega s \text{ iff } \xi(t, s) \geq 0$$

is upper diagonal and asymptotic (hence, geometric) Meir–Keeler.

Proof By definition, there exists an increasing function $\chi : R^0_+ \to R$ such that ξ is a χ-inf-simulation function. There are two steps to verify.

(i) Let $t, s > 0$ be such that

$$t\Omega s\text{; i.e., } \xi(t, s) \geq 0.$$

By the χ-diagonal property, we get

$$\chi(t) - \chi(s) < -\xi(t, s) \leq 0\text{; whence } \chi(t) < \chi(s).$$

This, along with χ=increasing, yields $t < s$; and the first assertion follows.

(ii) Suppose (by contradiction) that there exist a couple of strictly descending sequences (t_n) and (s_n) in R^0_+ and a number $\varepsilon > 0$, such that

$$t_n\Omega s_n \text{ (i.e., } \xi(t_n, s_n) \geq 0\text{), for all } n\text{, and } (t_n \to \varepsilon+, s_n \to \varepsilon+).$$

By the former of these relations, we have $\liminf_n \xi(t_n, s_n) \geq 0$; in contradiction with the inf-asymptotic negative property of ξ. Hence, our working assumption is not acceptable; where the second assertion follows as well.

Remark 5 Let us say that $\xi : R^0_+ \times R^0_+ \to R$ is a *simulation function*, if (cf. Khojasteh et al. [34])

(s-1) ξ is *diagonal*: $\xi(t, s) < s - t$, for each $t, s > 0$
(s-2) ξ is *asymptotic negative*:
for each couple of sequences (t_n) and (s_n) in R^0_+ and each $\varepsilon > 0$
with $\lim_n t_n = \lim_n s_n = \varepsilon$, we have $\limsup_n \xi(t_n, s_n) < 0$.

Clearly, each simulation function is an inf-simulation one (under $\chi(t) = t$, $t \in R^0_+$); but the reciprocal is not in general true. This is shown from the example below.

Let $\chi : R^0_+ \to R$ be an increasing discontinuous function; hence,

$$\alpha := \sup\{\chi(t+0) - \chi(t-0); t \in R^0_+\} > 0.$$

Let $\beta \in]0, \alpha[$ be fixed; note that, by the very definition above,

there exists $\theta > 0$ such that $\chi(\theta + 0) - \chi(\theta - 0) \geq \beta$.

Now, given the couple (χ, β), define a function $\xi : R^0_+ \times R^0_+$ as

$$\xi(t, s) = -\beta + \chi(s) - \chi(t),\ t, s > 0.$$

We claim that ξ is an inf-simulation function, but not a simulation one.

(i) In fact, ξ is χ-diagonal, as $\beta > 0$.

(ii) On the other hand, ξ is inf-asymptotic negative. To verify this, let the couple of strictly descending sequences (t_n) and (s_n) in R_+^0 and the number $\varepsilon > 0$ be such that $(t_n \to \varepsilon+, s_n \to \varepsilon+)$. Then, by definition

$$\lim_n \xi(t_n, s_n) = -\beta + \chi(\varepsilon + 0) - \chi(\varepsilon + 0) = -\beta < 0;$$

and the first half of our assertion follows.

(iii) Finally, we claim that ξ is not asymptotic negative. For, let the couple of sequences (t_n) and (s_n) in R_+^0 and the number $\theta > 0$ (described as before) be such that $t_n \to \theta-$, $s_n \to \theta+$. Then, by definition

$$\lim_n \xi(t_n, s_n) = -\beta + \chi(\theta + 0) - \chi(\theta - 0) \geq 0;$$

hence the second half of our assertion follows as well.

Part-Case (IV) Let $\chi : R_+^0 \to R$ be an increasing function. We say that $\eta : R_+^0 \times R_+^0 \to R$ is a *χ–inf-manageable function*, if

(imf-1) η is *χ-diagonal*: $\eta(t, s) < \chi(s) - \chi(t)$, for each $t, s > 0$
(imf-2) η is *inf-asymptotic subunitary*:
for each couple of strictly descending sequences (t_n) and (s_n) in R_+^0 and each $\varepsilon > 0$ with $(t_n \to \varepsilon+, s_n \to \varepsilon+)$, we have $\liminf_n[t_n + \eta(t_n, s_n)]/s_n < 1$.

If the increasing function χ is generic in this convention, we say that η is a *inf-manageable function*.

Proposition 21 *Suppose that $\eta : R_+^0 \times R_+^0 \to R$ is an inf-manageable function. Then, the associated relation $\Omega := \Omega[[\eta]]$ on R_+^0, introduced as*

$$(t, s \in R_+^0)\text{: } t\Omega s \text{ iff } \eta(t, s) \geq 0$$

is upper diagonal and asymptotic (hence, geometric) Meir–Keeler.

Proof By definition, there exists an increasing function $\chi : R_+^0 \to R$ such that η is a χ-inf-manageable function. As before, there are two steps to be passed.

(i) Let $t, s > 0$ be such that

$$t\Omega s\text{; i.e., } \eta(t, s) \geq 0.$$

By the χ-diagonal property,

$$\chi(t) - \chi(s) < -\eta(t, s) \leq 0\text{; whence } \chi(t) < \chi(s).$$

Combining with χ=increasing yields $t < s$; so that Ω is upper diagonal.

(ii) Suppose by contradiction that there exists a couple of strictly descending sequences (t_n) and (s_n) in R_+^0 and a number $\varepsilon > 0$, with

$$t_n \to \varepsilon+, s_n \to \varepsilon+, \text{ and } t_n\Omega s_n \text{ (i.e., } \eta(t_n, s_n) \geq 0\text{), for each } n.$$

Then,

$$\liminf_n[t_n + \eta(t_n, s_n)]/s_n \geq \liminf_n[t_n/s_n] = 1;$$

in contradiction with η being inf-asymptotic subunitary. Hence, Ω has the asymptotic Meir–Keeler property; and, from this, we are done.

Remark 6 An indirect proof of the result above is to be obtained by means of inclusion (valid over the class $\mathscr{F}(R_+^0 \times R_+^0, R)$)

η is inf-manageable implies η is inf-simulation.

In fact, suppose that

η is inf-manageable;
whence η is χ-inf-manageable, for some increasing $\chi : R_+^0 \to R$.

By this very choice, it is clear that $\eta(.,.)$ appears as χ-diagonal. So, it remains to establish that

η is *inf-asymptotic negative*:
for each couple of strictly descending sequences (t_n) and (s_n) in R_+^0 and each $\varepsilon > 0$ with $(t_n \to \varepsilon+, s_n \to \varepsilon+)$, we have $\liminf_n \eta(t_n, s_n) < 0$.

In fact, let (t_n), (s_n) and $\varepsilon > 0$ be as in the premise above. As η is inf-asymptotic subunitary, we must have

$\liminf_n[t_n/s_n + \eta(t_n, s_n)/s_n] < 1$;

and this yields (in a direct way)

$\lambda := \liminf_n \eta(t_n, s_n)/s_n \leq$
$\liminf_n[(t_n/s_n + \eta(t_n, s_n)/s_n)] + \lim_n[-t_n/s_n] < 0.$

Let $\alpha \in]\lambda, 0[$ be arbitrary fixed. By this very relation, we derive

for each $p \in N$, there exists $i(p) \geq p$, with $\eta(t_{i(p)}, s_{i(p)})/s_{i(p)} < \alpha$;

and this yields (by the properties of (s_n), combined with $\alpha < 0$)

for each $p \in N$, there exists $i(p) \geq p$, with $\eta(t_{i(p)}, s_{i(p)}) < \alpha s_{i(p)} < \alpha\varepsilon$.

This gives $[\liminf_n \eta(t_n, s_n) \leq \alpha\varepsilon < 0]$; and proves the desired fact.

Remark 7 According to Du and Khojasteh [17], let us say that $\eta : R_+^0 \times R_+^0 \to R$ is a *manageable function*, if

(mf-1) η is *diagonal*: $\eta(t, s) < s - t$, for each $t, s > 0$
(mf-2) η is *asymptotic subunitary*:
for each bounded sequence (t_n) in R_+^0, and each descending sequence (s_n) in R_+^0 we have $\limsup_n[t_n + \eta(t_n, s_n)]/s_n < 1$.

We claim that each manageable function is inf-manageable. This is verified along the steps below.

(i) Clearly, η is χ-diagonal, where $(\chi(t) = t, t \in R_+^0)$.

(ii) Let the couple of strictly descending sequences (t_n) and (s_n) in R_+^0 and the number $\varepsilon > 0$ be such that $(t_n \to \varepsilon+, s_n \to \varepsilon+)$. Clearly, $(t_n; n \geq 0)$ is a bounded sequence; and then, by the imposed hypothesis

$$\liminf_n [t_n + \eta(t_n, s_n)]/s_n \leq \limsup_n [t_n + \eta(t_n, s_n)]/s_n < 1;$$

so that η is inf-asymptotic subunitary. This proves our claim.

Remark 8 Note that many other relations of this type—including the ones in Argoubi et al. [4], Du and Khojasteh [17], Jachymski [26], Karapinar et al. [30], Khojasteh and Rakočević [33], Lim [39], Nastasi and Vetro [46], Roldán et al. [54], to quote only a few—are ultimately reducible to the simpler ones $\Omega[\chi]$ and $\Omega[\psi, \varphi]$ for appropriate functions χ and (ψ, φ). However, none of these techniques can handle the contractive conditions appearing in Khan et al. [32], Roldán and Shahzad [53], or Turinici [63]. A way of avoiding these troubles is of dimensional nature; further aspects will be delineated in a separate paper.

5 Statement of the Problem

Let X be a nonempty set. Take a *metric* $d : X \times X \to R_+$ on X; and let $\mathscr{R} \subseteq X \times X$ be a *relation* over X; the triple $(X, d, \mathscr{R})$ will be said to be a *relational metric space*. Further, let $T \in \mathscr{F}(X)$ be a selfmap of X. In the following, sufficient conditions are given for the existence and/or uniqueness of elements in $\mathrm{Fix}(T)$.

Concerning the uniqueness question, call the subset Y of X, *$\mathscr{R}$-asingleton* if $[y_1, y_2 \in Y, y_1 \mathscr{R} y_2]$ imply $y_1 = y_2$. Then, let us introduce the condition

T is *fix-$\mathscr{R}$-asingleton*: $\mathrm{Fix}(T)$ is $\mathscr{R}$-asingleton.

This yields the strategy to be followed in concrete cases; we do not give details.

Passing to the existence question, the metrical way of solving it is by means of local and global (metrical) conditions involving our data.

(5-I) By an *iterative couple* attached to T, we mean any couple $(x_0; (x_n))$, where x_0 is a point in X and $(x_n = T^n x_0; n \geq 0)$ is the associated iterative sequence. Then, we say that the iterative couple $(x_0; (x_n))$ is $\mathscr{R}$-ascending, provided

(x_n) is $\mathscr{R}$-ascending $(x_n \mathscr{R} x_{n+1}, \forall n)$.

The class of all $\mathscr{R}$-ascending iterative couples will be denoted as $\mathrm{icouA}(T)$.

Further, let $(x_0; (x_n))$ be a $\mathscr{R}$-ascending iterative couple attached to T. Given the natural number $k \geq 1$, let us say that $(x_0; (x_n))$ is *$(\mathscr{R}, k)$-uscattered*, provided

$\forall (m, n) \in N(<), \exists p \in N$, such that $n < p \leq n + k$ and $x_m \mathscr{R} x_p$.

When $k \geq 1$ is generic, we say that $(x_0; (x_n))$ is *$\mathscr{R}$-uscattered*. The class of all ($\mathscr{R}$-ascending, $\mathscr{R}$-uscattered) iterative couples $(x_0; (x_n))$ will be denoted as $\mathrm{icouAUS}(T)$.

Now, let us consider the conditions

(a-reg) T is *ascending regular*: icouA(T) is nonempty
(a-usc-reg) T is *ascending-uscattered regular*: icouAUS(T) is nonempty.

Concerning these conditions, two basic problems occur.

(Pro-1) The first question is that of obtaining $\mathscr{R}$-ascending iterative couples by starting from certain conditions upon T. For example, this is available under

(s-pro) T is *$\mathscr{R}$-semi-progressive* $[X(T, \mathscr{R}) := \{x \in X; x\mathscr{R}Tx\} \neq \emptyset]$,
(incr) T is *$\mathscr{R}$-increasing* $[x\mathscr{R}y \Longrightarrow Tx\mathscr{R}Ty]$.

As the verification is immediate, we do not give details.

(Pro-2) The second problem is that of getting ($\mathscr{R}$-ascending, $\mathscr{R}$-uscattered) iterative couples by starting from $\mathscr{R}$-ascending iterative couples. So, assume that

T is ascending regular (icouA(T) is nonempty);

note that, necessarily, T is $\mathscr{R}$-semi-progressive. Denote, for each $x \in X(T, \mathscr{R})$,

spec$(x) = \{i \in N(1, \leq); x\mathscr{R}T^i x\}$.

This will be referred to as the *spectrum* of x (modulo $(\mathscr{R}, T)$). Clearly, $1 \in$ spec(x); but, the alternative spec$(x) = \{1\}$ cannot be avoided.

The following particular answer to the underlying question is available. Letting the $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$, define the property

(t-asc) $(x_0; (x_n))$ is translation ascending:
$x_i\mathscr{R}x_j$ implies $x_{i+s}\mathscr{R}x_{j+s}$, for all $s \in N$.

Clearly, this holds when T is $\mathscr{R}$-increasing.

Proposition 22 *Let the $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$ be translation ascending. In addition, suppose that spec(x_0) is scattered. Then, necessarily, $(x_0; (x_n))$ is $\mathscr{R}$-uscattered.*

Proof Fix $(m, n) \in N(<)$. By hypothesis, there exists $k \in N(1, \leq)$, with

spec$(x_0) = \{i \in N(1, \leq); x_0\mathscr{R}x_i\}$ is k-scattered; wherefrom
there exists $r \in$ spec(x_0) (hence, $x_0\mathscr{R}x_r$), with $n - m < r \leq n - m + k$.

In this case, under the notation $p = m + r$, we have

$n < p \leq n + k$ and $x_m\mathscr{R}x_{m+r}$ (i.e., $x_m\mathscr{R}x_p$);

if we remember that $(x_0; (x_n))$ is translation ascending. Hence, the iterative couple $(x_0; (x_n))$ is $\mathscr{R}$-uscattered; as claimed.

In the following, some other example of such objects is given. The following auxiliary fact is our starting point.

Proposition 23 *Let the $\mathscr{R}$-ascending sequence $(z_n; n \geq 0)$ in X and the number $h \in N(2, \leq)$ be such that (under the notation $Z := \{z_n; n \geq 0\}$)*

$\mathscr{R}_Z := \mathscr{R} \cap (Z \times Z)$ *is h-transitive:* $\mathscr{R}_Z^h \subseteq \mathscr{R}_Z$.

Then, necessarily,

$(\forall r \geq 0)$*:* $[(z_i, z_{i+1+r(h-1)}) \in \mathscr{R},\ \forall i \geq 0]$.

Proof We shall use the induction with respect to r. First, by the $\mathscr{R}$-ascending property of our sequence,

$(z_i, z_{i+1}) \in \mathscr{R}_Z \subseteq \mathscr{R}, \forall i \geq 0$; whence the case of $r = 0$ holds.

This, by definition (and the h-transitive hypothesis), yields

$(z_i, z_{i+h}) \in \mathscr{R}_Z^h \subseteq \mathscr{R}_Z \subseteq \mathscr{R}, \forall i \geq 0$;

hence, the case of $r = 1$ holds too. Suppose that the underlying property holds for $r \in \{0, \dots, s\}$, where $s \geq 1$; we claim that it holds as well for $r = s + 1$. In fact, let $i \geq 0$ be arbitrary fixed. Again by the $\mathscr{R}$-increasing property of our sequence,

$(z_{i+1+s(h-1)}, z_{i+1+(s+1)(h-1)}) \in \mathscr{R}_Z^{h-1}$;

so that by the inductive hypothesis (and properties of relational product)

$(z_i, z_{i+1+(s+1)(h-1)}) \in \mathscr{R}_Z \circ \mathscr{R}_Z^{h-1} = \mathscr{R}_Z^h \subseteq \mathscr{R}_Z \subseteq \mathscr{R}$;

hence the claim. The proof is thereby complete.

Given the $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$, call it $(\mathscr{R}, h)$*-transitive* (where $h \in N(2, \leq)$), when

the restriction of $\mathscr{R}$ to the T-orbit $X_0 := \{x_n; n \geq 0\}$ is h-transitive.

When $h \in N(2, \leq)$ is generic in this convention, we then say that the $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$ is $\mathscr{R}$*-transitive*.

The following answer to the posed question is now available.

Proposition 24 *Let the* $\mathscr{R}$*-ascending iterative couple* $(x_0; (x_n))$ *be* $\mathscr{R}$*-transitive. Then, necessarily,* $(x_0; (x_n))$ *is* $\mathscr{R}$*-uscattered.*

Proof By definition, there exists $h \in N(2, \leq)$ such that $(x_0; (x_n))$ is $(\mathscr{R}, h)$-transitive. Let $(m, n) \in N(<)$ be arbitrary fixed. By the preceding result,

$(z_m, z_{m+1+r(h-1)}) \in \mathscr{R}, \forall r \geq 0$.

As $m+1 \leq n$, there exists at least one $r \in N(1, \leq)$ with $n < m+1+r(h-1) \leq n+h$. It will suffice putting $p = m + 1 + r(h - 1)$ to end the reasoning.

Note, finally, that when $\mathscr{R}$ is transitive on X, then

$\text{spec}(x) = N(1, \leq)$, for all $x \in X(T, \mathscr{R})$;

so, the whole theory above becomes trivial.

(5-II) Having these precise, we may now pass to the essential part of our developments. Given the $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$, one of the following alternative holds:

(Alt-1) The $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$ is *telescopic*, in the sense

the sequence (x_n) is *telescopic*: there exists $h \geq 0$, such that $d(x_h, x_{h+1}) = 0$.

By the very definition of our sequence, one derives

$x_h = x_n$, for all $n \geq h$; whence $z := x_h$ is an element of $\text{Fix}(T)$.

As a consequence, this case is completely clarified from the fixed point perspective.

(Alt-2) The $\mathscr{R}$-ascending iterative couple $(x_0; (x_n))$ is *non-telescopic*, in the sense

the sequence (x_n) is non-telescopic: $d(x_n, x_{n+1}) > 0$, $\forall n$.

We then say that the iterative couple $(x_0; (x_n))$ is *(a-nt)-admissible*. This is the effective case when the posed problem is to be solved.

Under the precise framework, let us list the specific directions under which the proposed problem is to be considered.

- **(rpo-1)** We say that the (a-nt)-admissible iterative couple $(x_0; (x_n))$ attached to T is *semi-Picard* (modulo $(d, \mathscr{R}; T)$) when the ($\mathscr{R}$-ascending non-telescopic) sequence (x_n) (in X) is d-asymptotic ($\lim_n d(x_n, x_{n+1}) = 0$)
- **(rpo-2)** We say that the (a-nt)-admissible iterative couple $(x_0; (x_n))$ attached to T is *Picard* (modulo $(d, \mathscr{R}; T)$) when the ($\mathscr{R}$-ascending non-telescopic) sequence (x_n) (in X) is d-Cauchy
- **(rpo-3)** We say that the (a-nt)-admissible iterative couple $(x_0; (x_n))$ attached to T is *strongly Picard* (modulo $(d, \mathscr{R}; T)$) when the ($\mathscr{R}$-ascending non-telescopic) sequence (x_n) (in X) is d-convergent and $\lim_n(x_n) \in \text{Fix}(T)$
- **(rpo-4)** We say that the (a-nt)-admissible iterative couple $(x_0; (x_n))$ attached to T is *semi-Bellman Picard* (modulo $(d, \mathscr{R}; T)$) when the ($\mathscr{R}$-ascending non-telescopic) sequence (x_n) (in X) is d-convergent and $(x_n; n \geq 0)\mathscr{R}\mathscr{R} \lim_n(x_n) \in \text{Fix}(T)$.

Here, if $(z_n; n \geq 0)$ is a sequence in X and z is an element in X, we defined

$(z_n; n \geq 0)\mathscr{R}z$ iff $z_n\mathscr{R}z$, for all n
$(z_n; n \geq 0)\mathscr{R}\mathscr{R}z$ iff $(w_n; n \geq 0)\mathscr{R}z$, for some subsequence (w_n) of (z_n).

In particular, when T is $\mathscr{R}$-semi-progressive and $\mathscr{R}$-increasing, these conventions are comparable with the ones in Turinici [66], which, in case of $\mathscr{R} = X \times X$, reduce to the ones in Rus [55, Ch 2, Sect 2.2]; because, in this setting, $X(T, \mathscr{R}) = X$.

Sufficient conditions for such properties are being founded on *ascending orbital full* concepts (in short: (a-o-f)-concepts). Call the sequence $(z_n; n \geq 0)$ in X,

$\mathscr{R}$*-ascending*, if $z_i\mathscr{R}z_{i+1}$ for all $i \geq 0$
T*-orbital*, when $(z_n = T^n x; n \geq 0)$, for some $x \in X$
full, if $n \mapsto z_n$ is injective ($i \neq j$ implies $z_i \neq z_j$);

the intersections of these are just the precise concepts.

(reg-1) Call X, *(a-o-f,d)-complete* provided (for each (a-o-f)-sequence) d-Cauchy $\Longrightarrow$ d-convergent

(reg-2) Let us say that T is *(a-o-f,d)-continuous*, if [(z_n)=(a-o-f)-sequence and $z_n \xrightarrow{d} z$] imply $Tz_n \xrightarrow{d} Tz$

(reg-3) Call $\mathscr{R}$, *(a-o-f,d)-almost-selfclosed* when [(z_n)=(a-o-f)-sequence and $z_n \xrightarrow{d} z$] imply $(z_n; n \geq 0)\mathscr{R}\mathscr{R}z$.

(5-III) To solve our posed problem along the precise directions, the metrical contractive technique will be used; it is strongly connected with certain Meir–Keeler conditions [43] upon the considered data. Denote, for $x, y \in X$

$$Q_1(x,y) = d(x,Tx),\ Q_2(x,y) = d(x,y),$$
$$Q_3(x,y) = d(x,Ty),\ Q_4(x,y) = d(Tx,y),$$
$$Q_5(x,y) = d(Tx,Ty),\ Q_6(x,y) = d(y,Ty),$$
$$\mathscr{Q}(x,y) = (Q_1(x,y), Q_2(x,y), Q_3(x,y), Q_4(x,y), Q_5(x,y), Q_6(x,y)),$$
$$K_1(x,y) = d(x,T^2x), K_2(x,y) = d(Tx,T^2x), \mathscr{K}(x,y) = (K_1(x,y), K_2(x,y)).$$

Further, let us construct the family of functions [for $x, y \in X$]

$$P_0(x,y) = Q_5(x,y) = d(Tx,Ty),$$
$$P_1(x,y) = (1/2)[Q_3(x,y) + Q_4(x,y)],$$
$$P_2(x,y) = (1/2)[K_1(x,y) + Q_4(x,y)],$$
$$M_0(x,y) = \min\{Q_1(x,y), Q_2(x,y), Q_5(x,y), Q_6(x,y), K_2(x,y)\},$$
$$M_1(x,y) = \max\{Q_1(x,y), Q_6(x,y)\},$$
$$M_2(x,y) = \max\{Q_1(x,y), Q_2(x,y), Q_6(x,y)\}.$$

(5-III-1) Having this precise, let $P = P(T)$ be a map in $\mathscr{F}(X \times X, R_+)$. For example, one may take

$$P(x,y) = \Theta(\mathscr{Q}(x,y), \mathscr{K}(x,y)),\ x, y \in X;$$

where $\Theta : R_+^6 \times R_+^2 \to R_+$ is a map; but this is not the only possible choice. We say that T is *Meir–Keeler* $(d, \mathscr{R}; P)$*-contractive* if

(mk-1) for each $x, y \in X$ with $x\mathscr{R}y$ and $P(x,y) > 0$, we have $P_0(x,y) < P(x,y)$;
referred to as: T is *strictly contractive* (modulo $(d, \mathscr{R}; P)$)
(mk-2) $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X$:
$x\mathscr{R}y$ and $\varepsilon < P(x,y) < \varepsilon + \delta$ imply $P_0(x,y) \leq \varepsilon$;
referred to as: T has the *Meir–Keeler property* (modulo $(d, \mathscr{R}; P)$).

In particular, when $\mathscr{R} = X \times X$, these concepts are comparable with the ones introduced by Meir and Keeler [43] and Matkowski [42]; see also Ćirić [13].

Remark 9 By the former of these conditions, the Meir–Keeler property (modulo $(d, \mathscr{R}; P)$) of T writes

(mk-3) $\forall \varepsilon > 0, \exists \delta > 0, \forall x, y \in X$:
$x\mathscr{R}y$ and $0 < P_0(x,y) < P(x,y) < \varepsilon + \delta$ imply $P_0(x,y) \leq \varepsilon$.

(5-III-2) A geometric version of the above concept may be given along the lines below. Remember that the relation $\Omega \subseteq R_+^0 \times R_+^0$ is called *upper diagonal*, provided

(u-diag) $(t, s) \in \Omega$ implies $t < s$;

the class of all these will be denoted as udiag(R_+^0). Further, let us introduce the conditions (over the class udiag(R_+^0))

(g-mk) Ω is *geometric Meir–Keeler*:
$\forall \varepsilon > 0, \exists \delta > 0, \forall (t, s)$: $t\Omega s, \varepsilon < s < \varepsilon + \delta \Longrightarrow t \leq \varepsilon$
(g-bila-s) Ω is *geometric bilateral separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall (t, s)$: $t, s \in]\beta - \gamma, \beta + \gamma[\Longrightarrow (t, s) \notin \Omega$
(g-left-s) Ω is *geometric left separable*:
$\forall \beta > 0, \exists \gamma \in]0, \beta[, \forall t$: $t \in]\beta - \gamma, \beta[\Longrightarrow (t, \beta) \notin \Omega$.

Now, given the mapping $P = P(T) : X \times X \to R_+$ and the relation $\Omega \subseteq R_+^0 \times R_+^0$, let us say that T is $(d, \mathscr{R}; P; \Omega)$*-contractive*, provided

(Om-contr) $(P_0(x, y), P(x, y)) \in \Omega$,
for all $x, y \in X$ with $x\mathscr{R}y$ and $P_0(x, y), P(x, y) > 0$.

Proposition 25 *Suppose that T is $(d, \mathscr{R}; P; \Omega)$-contractive where the relation $\Omega \subseteq R_+^0 \times R_+^0$ is upper diagonal and geometric Meir–Keeler. Then, T is Meir–Keeler $(d, \mathscr{R}; P)$-contractive.*

Proof

(i) Let $x, y \in X$ be such that $x\mathscr{R}y$ and $P(x, y) > 0$. If $P_0(x, y) = 0$, all is clear. Suppose now that $P_0(x, y) > 0$. As a consequence of this,

$$(t, s) \in \Omega; \text{ where } t := P_0(x, y), s := P(x, y).$$

Combining with the upper diagonal property of Ω, one gets $t < s$; i.e., $P_0(x, y) < P(x, y)$. Summing up, T is strictly contractive (modulo $(d, \mathscr{R}; P)$).

(ii): Let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number assured by the geometric Meir–Keeler property of Ω. Further, let $x, y \in X$ be such that $x\mathscr{R}y$ and $\varepsilon < s := P(x, y) < \varepsilon + \delta$. As before, if $P_0(x, y) = 0$, all is clear. Suppose now that $P_0(x, y) > 0$. By definition, we must have

$$(t, s) \in \Omega; \text{ where } t := P_0(x, y), s := P(x, y);$$

and this along with $\varepsilon < s < \varepsilon + \delta$ gives (by the geometric Meir–Keeler property of Ω), $t \leq \varepsilon$; i.e., $P_0(x, y) \leq \varepsilon$. Putting these together, it follows that T has the Meir–Keeler property (modulo $(d, \mathscr{R}; P)$). The proof is complete.

(5-III-3) In the following, a converse result is formulated. Given the mapping $P : X \times X \to R_+$, let $\Omega = \Omega[d, \mathscr{R}; P; T]$ stand for the associated relation over R_+^0:

$\Omega = \{(P_0(x, y), P(x, y)); x\mathscr{R}y, P_0(x, y), P(x, y) > 0\}$;
or, in other words:
$(t, s) \in \Omega$ iff $t = P_0(x, y)$, $s = P(x, y)$, where
$x\mathscr{R}y$ and $P_0(x, y), P(x, y) > 0$.

Proposition 26 *Under these conventions, we have*

(55-1) If T is Meir–Keeler $(d, \mathscr{R}; P)$-contractive, then the attached relation $\Omega = \Omega[d, \mathscr{R}; P; T]$ is upper diagonal and geometric Meir–Keeler
(55-2) T is Meir–Keeler $(d, \mathscr{R}; P)$-contractive if and only if the attached relation $\Omega = \Omega[d, \mathscr{R}; P; T]$ is upper diagonal and geometric Meir–Keeler.

Proof

(i) Suppose that T is Meir–Keeler $(d, \mathscr{R}; P)$-contractive. There are two steps to be passed.
(i-1) Let $(t, s) \in R^0_+ \times R^0_+$ be such that $(t, s) \in \Omega$; hence (by definition)

$$t = P_0(x, y), s = P(x, y), \text{ where } x\mathscr{R}y \text{ and } P_0(x, y), P(x, y) > 0.$$

From the strict contractive property of T, we must have $P_0(x, y) < P(x, y)$; or, equivalently, $t < s$, which shows that Ω is upper diagonal.
(i-2) Let $\varepsilon > 0$ be arbitrary fixed; and $\delta > 0$ be the number associated by the Meir–Keeler property of T. Further, let $(t, s) \in R^0_+ \times R^0_+$ be such that $(t, s) \in \Omega$ and $\varepsilon < s < \varepsilon + \delta$; hence (see above)

$$t = P_0(x, y), s = P(x, y), \text{ where } x\mathscr{R}y \text{ and } P_0(x, y), P(x, y) > 0;$$

so that (by definition):

$$x\mathscr{R}y, P_0(x, y) > 0, \text{ and } \varepsilon < P(x, y) < \varepsilon + \delta.$$

By the underlying Meir–Keeler-property for T, we get

$$P_0(x, y) \le \varepsilon; \text{ i.e. (under our notation): } t \le \varepsilon;$$

so that Ω has the geometric Meir–Keeler property.
(ii) Suppose that the associated relation $\Omega = \Omega[d, \mathscr{R}; P; T]$ over R^0_+ is upper diagonal and has the geometric Meir–Keeler property. By the very definition of this object, T is $(d, \mathscr{R}; P; \Omega)$-contractive. Combining with the preceding result, one derives that T appears as Meir–Keeler $(d, \mathscr{R}; P)$-contractive.

As a consequence of this, it follows that the Meir–Keeler $(d, \mathscr{R}; P)$-contractive properties of T are finally reducible to the upper diagonal and geometric Meir–Keeler properties for the associated relation $\Omega[d, \mathscr{R}; P; T]$.

Concerning this aspect, remember that various examples of such objects were treated in a previous place. There are three cases to discuss.

Case I Given $\varphi \in \mathscr{F}(R^0_+, R)$, we say that the mapping T is *Boyd–Wong $(d, \mathscr{R}; P; \varphi)$-contractive*, if

$$P_0(x, y) \le \varphi(P(x, y)), \forall x, y \in X, x\mathscr{R}y, P(x, y) > 0.$$

The specific classes of such functions φ are founded on Meir–Keeler admissible concepts. Let $\mathscr{F}(re)(R_+^0, R)$ stand for the class of all $\varphi \in \mathscr{F}(R_+^0, R)$ with

φ is *regressive*: $\varphi(t) < t, \forall t > 0$.

We say that $\varphi \in \mathscr{F}(re)(R_+)$ is *Meir–Keeler admissible*, if

$\forall \gamma > 0, \exists \beta > 0, \forall t: \gamma < t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma$;
or, equivalently: $[\forall \gamma > 0, \exists \beta > 0, \forall t: 0 < t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma]$.

The usefulness of this concept is to be judged from

Proposition 27 *Let $\varphi \in \mathscr{F}(re)(R_+)$ be Meir–Keeler admissible (see above). Then, the following inclusion is true in (ZF-AC+DC):*

If T is Boyd–Wong $(d, \mathscr{R}; P; \varphi)$-contractive, then
T is Meir–Keeler $(d, \mathscr{R}; P)$-contractive.

Proof Let $\Omega := \Omega[\varphi]$ be the relation over R_+^0 introduced as

$(t, s \in R_+)$: $t\Omega s$ iff $t \leq \varphi(s)$.

By a previous fact, Ω is upper diagonal and geometric Meir–Keeler. This along with T being $(d, \mathscr{R}; P; \Omega)$-contractive yields (see above) the written conclusion.

Case II Given the functional couple $(\psi, \varphi) \in \mathscr{F}^2(R_+^0, R)$, let us say that the mapping T is *Rhoades* $(d, \mathscr{R}; P; \psi, \varphi)$*-contractive*, provided

$\psi(P_0(x, y)) \leq \psi(P(x, y)) - \varphi(P(x, y))$,
for all $x, y \in X$, with $x\mathscr{R}y$ and $P_0(x, y), P(x, y) > 0$.

To discuss it, remember that some compatible properties of the couple (ψ, φ) were introduced. First, let us assume that

(comp-1) (ψ, φ) is *normal*:
ψ is increasing, and φ is *strictly positive* $(\varphi(t) > 0, \forall t > 0)$.

Further, let us introduce the condition upon (ψ, φ)

(comp-2) φ is *asymptotic positive*:
for each strictly descending sequence
$(t_n; n \geq 0)$ in R_+^0 and each $\varepsilon > 0$ with $t_n \to \varepsilon+$,
we must have $\limsup_n(\varphi(t_n)) > 0$.

Proposition 28 *Suppose that (ψ, φ) is normal and φ is asymptotic positive. Then, the inclusion below holds in (ZF-AC+DC)*

T is Rhoades $(d, \mathscr{R}; P; \psi, \varphi)$-contractive implies
T is Meir–Keeler $(d, \mathscr{R}; P)$-contractive.

Proof Let $\Omega := \Omega[\psi, \varphi]$ be the associated relation over R_+^0

$(t, s \in R_+^0)$: $t\Omega s$ iff $\psi(t) \leq \psi(s) - \varphi(s)$.

First, by the normality of (ψ, φ), Ω is upper diagonal. Second, the asymptotic positivity of φ gives (by a previous result) that Ω is asymptotic Meir–Keeler; hence geometric Meir–Keeler as well. This along with T being $(d, \mathscr{R}; P; \Omega)$-contractive yields (see above) the written conclusion.

Case III Denote, for $x, y \in X$,

$$\mathscr{M}_1(x, y) = (Q_1(x, y), Q_6(x, y)),$$
$$M_1(x, y) = \max \mathscr{M}_1(x, y) = \max\{Q_1(x, y), Q_6(x, y)\},$$
$$\mathscr{M}_2(x, y) = (Q_1(x, y), Q_2(x, y), Q_6(x, y)),$$
$$M_2(x, y) = \max \mathscr{M}_2(x, y) = \max\{Q_1(x, y), Q_2(x, y), Q_6(x, y)\}.$$

Given the couple of maps $g \in \mathscr{F}(R_+)$, $H \in \mathscr{F}(R^3_+, R_+)$, let us say that the mapping T is *Khan* $(d, \mathscr{R}; \mathscr{M}_2; g, H)$*-contractive*, provided

(K-con) $g(P_0(x, y)) \le g(M_2(x, y)) - H(\mathscr{M}_2(x, y)), \forall x, y \in X, x\mathscr{R}y.$

The class of functions (g, H) appearing here may be described as follows. Let $k \ge 1$ be a natural number. According to Khan et al. [32], we say that $G \in \mathscr{F}(R^k_+, R_+)$ is an *altering function*, in case

(alter-1) G is increasing in each variable
(alter-2) G is reflexive sufficient: $(t_1 = \ldots = t_k = 0)$ iff $G(t_1, \ldots, t_k) = 0$.

The class of all such functions will be denoted $\mathscr{F}(alt)(R^k_+, R_+)$. Note that, given $G \in \mathscr{F}(alt)(R^3_+, R_+)$, the associated function $(g(t) = G(t, t, t); t \in R_+)$ is an element of $\mathscr{F}(alt)(R_+)$. Moreover, by our previous notations,

$G(\mathscr{M}_2(x, y)) \le g(M_2(x, y)), \forall x, y \in X.$

Proposition 29 *Suppose that the mapping T is Khan $(d, \mathscr{R}; \mathscr{M}_2; g, H)$-contractive, where $g \in \mathscr{F}(alt)(R_+)$ and $H \in \mathscr{F}(alt)(R^3_+, R_+)$. Then, necessarily, T is Meir–Keeler $(d, \mathscr{R}; M_2)$-contractive, in (ZF-AC+DC).*

Proof The verification consists of two stages.

(i) Assume by contradiction that T is not strictly contractive (modulo $(d, \mathscr{R}; M_2)$): there exist $x, y \in X$ such that $x\mathscr{R}y$, $M_2(x, y) > 0$ and $P_0(x, y) \ge M_2(x, y)$. By the contractive condition (and g=increasing),

$g(M_2(x, y)) \le g(P_0(x, y)) \le g(M_2(x, y)) - H(\mathscr{M}_2(x, y));$
wherefrom $H(\mathscr{M}_2(x, y)) = 0$.

This, along with $H \in \mathscr{F}(alt)(R^3_+, R_+)$, yields $\mathscr{M}_2(x, y) = 0$; hence $M_2(x, y) = 0$; in contradiction with the posed hypothesis.

(ii) Assume by contradiction that T does not have the Meir–Keeler property (modulo $(d, \mathscr{R}; M_2)$): there exists $\varepsilon > 0$, so that

$C(\delta) := \{(u, v) \in X \times X; u\mathscr{R}v, \varepsilon < M_2(u, v) < \varepsilon + \delta,\ P_0(u, v) > \varepsilon\}$
is a nonempty subset of $X \times X$, for each $\delta > 0$.

Taking a zero converging sequence (δ_n) in R_+^0, we get by the Denumerable Axiom of Choice (deductible in (ZF-AC+DC)), a sequence $((x_n, y_n); n \geq 0)$ in $X \times X$, with the property

$(\forall n)$: $(x_n, y_n) \in C(\delta_n)$; that is (by definition and preceding step), $x_n \mathscr{R} y_n$, $\varepsilon < P_0(x_n, y_n) < M_2(x_n, y_n) < \varepsilon + \delta_n$;

note that, as a direct consequence of this,

$P_0(x_n, y_n) \to \varepsilon+$ and $M_2(x_n, y_n) \to \varepsilon+$, as $n \to \infty$.

By the contractive condition, we get for all n,

$H(\mathscr{M}_2(x_n, y_n)) \leq g(M_2(x_n, y_n)) - g(P_0(x_n, y_n))$;

and this (via g=increasing) gives (by the above)

$(0 \leq) \limsup_n H(\mathscr{M}_2(x_n, y_n)) \leq g(\varepsilon + 0) - g(\varepsilon + 0) = 0$;
so, $\lim_n H(\mathscr{M}_2(x_n, y_n)) = 0$.

On the other hand, by the very construction of our sequence $((x_n, y_n); n \geq 0)$, there must be some index $i \in \{1, 2, 6\}$ such that

$\varepsilon < Q_i(x_n, y_n) < \varepsilon + \delta_n$, for infinitely many n.

Without loss, one may assume that $i = 1$. Combining with (H=increasing in all variables) yields

$H(\mathscr{M}_2(x_n, y_n)) \geq H(\varepsilon, 0, 0)$, for infinitely many n;
wherefrom $\lim_n H(\mathscr{M}_2(x_n, y_n)) \geq H(\varepsilon, 0, 0) > 0$;

in contradiction with the limit property above. Consequently, our working assumption is not acceptable; wherefrom, T does have the Meir–Keeler property (modulo $(d, \mathscr{R}; M_2)$). The proof is complete.

Note that similar conclusions may be derived for the pair $(\mathscr{M}_1, M_1)$; we do not give further details.

6 Main Result

Let X be a nonempty set. Take a metric $d(.,.)$ and a relation $\mathscr{R}$ on X; the triple $(X, d, \mathscr{R})$ will be then referred to as a *relational metric space*. Further, take some selfmap $T \in \mathscr{F}(X)$, and put $\text{Fix}(T) = \{z \in X; z = Tz\}$; each point of it will be referred to as *fixed* with respect to T. As precise, we have to determine conditions assuring us that $\text{Fix}(T)$ is nonempty.

(6-I) By an *iterative couple* attached to T, we mean any couple $(x_0; (x_n))$, where x_0 is a point in X and $(x_n = T^n x_0; n \geq 0)$ is the associated iterative sequence. Then, we say that the iterative couple $(x_0; (x_n))$ is $\mathscr{R}$-ascending, provided

(x_n) is $\mathscr{R}$-ascending $(x_n \mathscr{R} x_{n+1}, \forall n)$.

The class of all such $\mathscr{R}$-ascending iterative couples will be denoted as icouA(T).

Further, let $(x_0; (x_n))$ be a $\mathscr{R}$-ascending iterative couple attached to T. Given the natural number $k \geq 1$, let us say that $(x_0; (x_n))$ is *$(\mathscr{R}, k)$-uscattered*, provided

$$\forall (m, n) \in N(<), \exists p \in N, \text{ such that } n < p \leq n + k \text{ and } x_m \mathscr{R} x_p.$$

When $k \geq 1$ is generic here, we say that $(x_0; (x_n))$ is *$\mathscr{R}$-uscattered*. The class of all ($\mathscr{R}$-ascending, $\mathscr{R}$-uscattered) iterative couples $(x_0; (x_n))$ will be denoted as icouAUS(T).

In the following, the basic condition to be posed is

(a-usc-reg) T is *ascending-uscattered regular*: icouAUS(T) is nonempty.

As a consequence of this, there exists an iterative couple $(x_0; (x_n))$, with

(adm-1) $(x_0; (x_n))$ is $\mathscr{R}$-ascending
(adm-2) $(x_0; (x_n))$ is $\mathscr{R}$-uscattered.

In addition (cf. a previous discussion) one may suppose that

(adm-3) $(x_0; (x_n))$ is non-telescopic ($d(x_n, x_{n+1}) > 0, \forall n$).

We then say that the iterative couple $(x_0; (x_n))$ is *(a-us-nt)-admissible*.

The specific directions under which the posed problem is to be solved were already listed; as precise, these are based on local/global regularity conditions involving the (a-us-nt)-admissible iterative couples we just introduced. On the other hand, the metrical tools of our investigations consist in geometric contractive conditions upon T, involving the couple $(d, \mathscr{R})$, a mapping $P = P(T) : X \times X \to R_+$, and a relation $\Omega \subseteq R^0_+ \times R^0_+$. Precisely, denote, for $x, y \in X$

$Q_1(x, y) = d(x, Tx)$, $Q_2(x, y) = d(x, y)$,
$Q_3(x, y) = d(x, Ty)$, $Q_4(x, y) = d(Tx, y)$,
$Q_5(x, y) = d(Tx, Ty)$, $Q_6(x, y) = d(y, Ty)$,
$\mathscr{Q}(x, y) = (Q_1(x, y), Q_2(x, y), Q_3(x, y), Q_4(x, y), Q_5(x, y), Q_6(x, y))$,
$K_1(x, y) = d(x, T^2x)$, $K_2(x, y) = d(Tx, T^2x)$, $\mathscr{K}(x, y) = (K_1(x, y), K_2(x, y))$.

Then, let us construct the family of functions [for $x, y \in X$]

$P_0(x, y) = Q_5(x, y) = d(Tx, Ty)$,
$P_1(x, y) = (1/2)[Q_3(x, y) + Q_4(x, y)]$,
$P_2(x, y) = (1/2)[K_1(x, y) + Q_4(x, y)]$,
$M_0(x, y) = \min\{Q_1(x, y), Q_2(x, y), Q_5(x, y), Q_6(x, y), K_2(x, y)\}$,
$M_1(x, y) = \max\{Q_1(x, y), Q_6(x, y)\}$,
$M_2(x, y) = \max\{Q_1(x, y), Q_2(x, y), Q_6(x, y)\}$.

Usually, the mapping $P(., .)$ above is of the form

$$P = \Theta(\mathscr{Q}, \mathscr{K}); \text{i.e., } P(x, y) = \Theta(\mathscr{Q}(x, y), \mathscr{K}(x, y)), x, y \in X;$$

where $\Theta : R^6_+ \times R^2_+ \to R_+$ fulfills certain mild conditions; but, this does not exhaust the class of all these. Likewise, the relation Ω is upper diagonal (see above).

(6-II) The second condition upon our data is of starting type; and writes

(posi) $(P; M_0)$ is *positive*: for each $x, y \in X$ with $(x\mathscr{R}y, x\mathscr{R}Tx, Tx\mathscr{R}T^2x)$, we have $[M_0(x, y) > 0$ implies $P(x, y) > 0]$.

This condition allows us applying the contractive conditions (to be introduced below) at each stage of the proof.

(6-III) The third group of conditions allows us to get a d-asymptotic property and full property for the (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ to be considered. It may be formulated as

(o-bd) $(P; M_1)$ is *orbitally bounded* over each (a-us-nt)-admissible iterative couple $(x_0; (x_n))$: $P(x_n, x_{n+1}) \leq M_1(x_n, x_{n+1}), \forall n$.

Some concrete versions of it will be discussed a bit further.

(6-IV) The fourth group of conditions has, as objective, a deduction of d-Cauchy properties for the full d-asymptotic (a-us-nt)-admissible iterative couples $(x_0; (x_n))$ above. Some preliminaries are needed. Let the (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ be d-asymptotic but not d-Cauchy. According to a previous auxiliary fact, there exist a natural number $h \geq 1$, a number $\varepsilon \in R_+^0$, a rank $J := j(\varepsilon, h) \geq 1$, and a sequence $((m(j), n(j)); j \geq 0)$ in $N \times N$, such that

(aqua-1) $j \leq m(j) < n(j), x_{m(j)}\mathscr{R}x_{n(j)}, \forall j \geq 0$
(aqua-2) for each $s, t \in N(0, 2h)$,
the sequence $(V_j(s, t) := d(x_{m(j)+s}, x_{n(j)+t}); j \geq 0)$ (in R_+)
is J-nearly in $]\varepsilon, \infty[$, with $V_j(s, t) \to \varepsilon++$ as $j \to \infty$.

By definition, $[h; \varepsilon, J; ((m(j), n(j)); j \geq 0)]$ will be referred to as the *associated to* $(x_0; (x_n))$ *quadruple*.

Having these precise, we may now formulate the announced condition:

P is *orbitally small* over each full, d-asymptotic, d-non-Cauchy, and (a-us-nt)-admissible iterative couple $(x_0; (x_n))$:
for each associated to $(x_0; (x_n))$ quadruple $[h; \varepsilon, J; ((m(j), n(j)); j \geq 0)]$, we have $\limsup_j P(x_{m(j)}, x_{n(j)}) \leq \varepsilon$.

As before, some concrete realizations of it will be given a bit further.

(6-V) Finally, the fifth group of conditions—referred to as *orbitally normal ones*—assures us the desired fixed point property for the limit of each full d-convergent (a-us-nt)-admissible couple $(x_0; (x_n))$. These may be written as

(o-nor-1) P is *orbitally singular asymptotic* over each full d-convergent (a-us-nt)-admissible couple $(x_0; (x_n))$:
whenever $x_n \xrightarrow{d} z$ and $d(z, Tz) > 0$, we have $\liminf_n P(u_n, z) < d(z, Tz)$, for each subsequence $(u_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$
(o-nor-2) P is *orbitally regular asymptotic* over each full d-convergent (a-us-nt)-admissible couple $(x_0; (x_n))$:
whenever $x_n \xrightarrow{d} z$ and $d(z, Tz) > 0$, we have $P(u_n, z) \to d(z, Tz)$, for each subsequence $(u_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$

(o-nor-3) P is *orbitally strongly regular asymptotic* over each full d-convergent (a-us-nt)-admissible couple $(x_0; (x_n))$:
whenever $x_n \xrightarrow{d} z$ and $d(z, Tz) > 0$, we have $P(u_n, z) \to\to d(z, Tz)$, for each subsequence $(u_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$.

Here, given the sequence $(r_n; n \geq 0)$ in R and the point $r \in R$, we denoted

$r_n \to\to r$, if $r_n \to r$ and there exists a subsequence
$(s_n = r_{i(n)}; n \geq 0)$ of $(r_n; n \geq 0)$ such that $[s_n = r, \forall n \geq 0]$.

The main (fixed point) result of this exposition (referred to as *Geometric Meir–Keeler theorem*; in short: (MK-g)) may be stated as below.

Theorem 2 *Suppose that T is ascending-uscattered regular as well as $(d, \mathscr{R}; P; \Omega)$-contractive, for some mapping $P = P(T) : X \times X \to R_+$ and some relation $\Omega \subseteq R_+^0 \times R_+^0$ with the properties:*

(61-i) $(P; M_0)$ is positive
(61-ii) Ω is upper diagonal and geometric Meir–Keeler.

In addition, let X be (a-o-f,d)-complete; and take an (a-us-nt)-admissible iterative couple $(x_0; (x_n))$, with (in addition)

(61-iii) $(P; M_1)$ is orbitally bounded at $(x_0; (x_n))$
(61-iv) P is orbitally small at $(x_0; (x_n))$, whenever this iterative couple is d-asymptotic and d-non-Cauchy.

Then,
(61-a) *$(x_0; (x_n))$ is Picard (modulo $(d, \mathscr{R}; T)$)*
(61-b) *$(x_0; (x_n))$ is strongly Picard (modulo $(d, \mathscr{R}; T)$) provided the following extra condition holds*

(61-b-1) T is (a-o-f,d)-continuous

(61-c) *$(x_0; (x_n))$ is Bellman Picard (modulo $(d, \mathscr{R}; T)$), whenever $\mathscr{R}$ is (a-o-f,d)-almost-selfclosed and one of the following extra conditions holds:*

(61-c-1) P is orbitally singular asymptotic over $(x_0; (x_n))$
(61-c-2) P is orbitally regular asymptotic over $(x_0; (x_n))$, and Ω is asymptotic bilateral separable
(61-c-3) P is orbitally strongly regular asymptotic over $(x_0; (x_n))$, and Ω is asymptotic left separable.

Proof There are some steps to be passed.

Step 1 We firstly show that, under these conditions, the (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ is full and d-asymptotic. Denote, for simplicity,

$(r_n := d(x_n, x_{n+1}); n \geq 0)$; hence (by hypothesis), $(r_n > 0, \forall n)$.

Let $n \geq 0$ be arbitrary fixed. According to definition,

$Q_1(x_n, x_{n+1}) = r_n > 0$, $Q_2(x_n, x_{n+1}) = r_n > 0$,
$Q_3(x_n, x_{n+1}) = d(x_n, x_{n+2})$, $Q_4(x_n, x_{n+1}) = d(x_{n+1}, x_{n+1}) = 0$,
$Q_5(x_n, x_{n+1}) = r_{n+1} > 0$, $Q_6(x_n, x_{n+1}) = r_{n+1} > 0$,
$K_1(x_n, x_{n+1}) = d(x_n, x_{n+2})$, $K_2(x_n, x_{n+1}) = r_{n+1} > 0$;

and this yields

$P_0(x_n, x_{n+1}) = Q_5(x_n, x_{n+1}) = r_{n+1}$,
$P_1(x_n, x_{n+1}) = (1/2)d(x_n, x_{n+2})$, $P_2(x_n, x_{n+1}) = (1/2)d(x_n, x_{n+2})$,
$M_0(x_n, x_{n+1}) = \min\{r_n, r_{n+1}\} > 0$,
$M_1(x_n, x_{n+1}) = \max\{r_n, r_{n+1}\}$, $M_2(x_n, x_{n+1}) = \max\{r_n, r_{n+1}\}$.

By the relation involving M_0,

$P(x_n, x_{n+1}) > 0$ (if we remember that $(P; M_0)$ is positive).

Moreover, as $x_n \mathscr{R} x_{n+1}$, the contractive property applies to (x_n, x_{n+1}); and yields

(contr) $r_{n+1} \Omega P(x_n, x_{n+1}), \forall n$.

Combining with $(P; M_1)$ being orbitally bounded yields

$P(x_n, x_{n+1}) \leq M_1(x_n, x_{n+1}) = \max\{r_n, r_{n+1}\}$,

This, along with (contr), gives (via Ω=upper diagonal)

(P-ev) $r_{n+1} < P(x_n, x_{n+1}) \leq \max\{r_n, r_{n+1}\}$.

From the inequality between extremal terms, one derives

$(r_{n+1} < r_n, \forall n)$; i.e., (r_n) is strictly descending.

Note that, as a first consequence of this,

(x_n) is full: $i < j$ implies $x_i \neq x_j$ (whence $d(x_i, x_j) > 0$).

In fact, suppose by contradiction that

there exists $i, j \in N$ with $i < j$, $x_i = x_j$.

Then, by definition

$x_{i+1} = x_{j+1}$; so that $r_i = r_j$;

in contradiction with $r_i > r_j$; and the assertion follows. As a second consequence,

$r := \lim_n r_n$ exists in R_+; with, in addition, $(r_n > r, \forall n)$.

Suppose by contradiction that $r > 0$; and let $\delta = \delta(r) > 0$ be the number given by the Meir–Keeler property of Ω. By definition, there exists $m = m(\delta) \geq 1$ such that

$n \geq m$ implies $r < r_n < r + \delta$.

Let $n \geq m$ be fixed in the sequel. From (P-ev),

$r < r_{n+1} < P(x_n, x_{n+1}) \leq r_n < r + \delta$.

This, combined with (contr), yields (by the underlying property of Ω)

$(r <) r_{n+1} \leq r$; contradiction.

Hence, $r = 0$; and this tells us that the (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ is, in addition, d-asymptotic.

Step 2 We prove that under the imposed conditions, the full d-asymptotic (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ is d-Cauchy. Suppose by contradiction that this is not true:

$(x_0; (x_n))$ is (full, d-asymptotic and) d-non-Cauchy.

By a previous auxiliary fact, there exist a natural number $h \geq 1$, a number $\varepsilon \in R^0_+$, a rank $J := j(\varepsilon, h) \geq 1$, and a sequence $((m(j), n(j)); j \geq 0)$ in $N \times N$, with

(aqua-1) $j \leq m(j) < n(j)$, $x_{m(j)} \mathscr{R} x_{n(j)}$, $\forall j \geq 0$
(aqua-2) for each $s, t \in N[0, 2h]$,
the sequence $(V_j(s,t) := d(x_{m(j)+s}, x_{n(j)+t}); j \geq 0)$ (in R_+)
is J-nearly in $]\varepsilon, \infty[$, with $V_j(s,t) \to \varepsilon++$ as $j \to \infty$;

by definition, $[h, \varepsilon, J; ((m(j), n(j)); j \geq 0)]$ will be referred to as the *associated to* $(x_0; (x_n))$ *quadruple*. By these properties (and the choice of our iterative couple $(x_0; (x_n))$), we have for each $j \geq J$

$$Q_1(x_{m(j)}, x_{n(j)}) = r_{m(j)} > 0, \ Q_2(x_{m(j)}, x_{n(j)}) = V_j(0,0) > \varepsilon,$$
$$Q_3(x_{m(j)}, x_{n(j)}) = V_j(0,1) > \varepsilon, \ Q_4(x_{m(j)}, x_{n(j)}) = V_j(1,0) > \varepsilon,$$
$$Q_5(x_{m(j)}, x_{n(j)}) = V_j(1,1) > \varepsilon, \ Q_6(x_{m(j)}, x_{n(j)}) = r_{n(j)} > 0,$$
$$K_1(x_{m(j)}, x_{n(j)}) = d(x_{m(j)}, x_{m(j)+2}) > 0,$$
$$K_2(x_{m(j)}, x_{n(j)}) = r_{m(j)+1} > 0;$$

and this yields (again for all $j \geq J$)

$M_0(x_{m(j)}, x_{n(j)}) > 0$; hence, $P(x_{m(j)}, x_{n(j)}) > 0$,

if we remember that (P, M_0) is positive. Putting these together yields

$x_{m(j)} \mathscr{R} x_{n(j)})$, $P_0(x_{m(j)}, x_{n(j)}) > \varepsilon$, $P(x_{m(j)}, x_{n(j)}) > 0$, for all $j \geq J$;

which shows that the contractive condition applies to the precise data, for all $j \geq J$. This along with the convention

$\alpha_j = P_0(x_{m(j)}, x_{n(j)})$, $\beta_j = P(x_{m(j)}, x_{n(j)})$, $j \geq J$,

tells us that we have

(contr) $A_j \Omega B_j, \forall j \geq 0$;

where, for simplicity, we denoted

$(A_j = \alpha_{j+J}; j \geq 0)$, $(B_j = \beta_{j+J}; j \geq 0)$.

As Ω is upper diagonal, we derive $[A_j < B_j, \forall j \geq 0]$. This by the limit properties above gives (via P=orbitally small)

$\lim_j A_j = \varepsilon+, \lim_j B_j = \varepsilon+.$

By a previous auxiliary fact, there exists a couple of strictly descending subsequences $(A_j^* := A_{q(j)}; j \geq 0)$ and $(B_j^* := B_{q(j)}; j \geq 0)$ of $(A_j; j \geq 0)$ and $(B_j; j \geq 0)$ respectively, so that

$(A_j^* \Omega B_j^*, \forall j \geq 0)$, and $(A_j^* \to \varepsilon+, B_j^* \to \varepsilon+)$.

This, however, is in contradiction with Ω being asymptotic Meir–Keeler; or, equivalently: geometric Meir–Keeler. Hence, our initial d-non-Cauchy hypothesis about the iterative couple $(x_0; (x_n))$ is not acceptable; and our assertion follows.

Step 3 We prove that under the extra conditions above, the full d-Cauchy (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ yields a fixed point of T.

As X is (a-o-f,d)-complete, $x_n \overset{d}{\longrightarrow} z$, for some (uniquely determined) $z \in X$. There are two cases to discuss.

Case 3a Suppose that T is (a-o-f,d)-continuous. Then $y_n := Tx_n \overset{d}{\longrightarrow} Tz$ as $n \to \infty$. On the other hand, $(y_n = x_{n+1}; n \geq 0)$ is a subsequence of $(x_n; n \geq 0)$; whence $y_n \overset{d}{\longrightarrow} z$; and this gives (as d is separated), $z = Tz$.

Case 3b Suppose that $\mathscr{R}$ is (a-o-f,d)-almost-selfclosed.

By the full property of $(Tx_n = x_{n+1}; n \geq 0)$,

$E := \{n \in N; Tx_n = Tz\}$ is an asingleton;

so that the following separation property holds:

(sepa) $\exists k = k(z) \geq 0$, such that $n \geq k$ implies
$Q_5(x_n, z) = d(Tx_n, Tz) > 0$; hence, $Q_2(x_n, z) = d(x_n, z) > 0$.

On the other hand, by the non-telescopic property of (x_n),

$Q_1(x_n, z) = d(x_n, Tx_n) > 0, K_2(x_n, z) = d(Tx_n, T^2x_n) > 0, \forall n.$

Suppose by contradiction that

(pos) $b := d(z, Tz) > 0$ [whence $Q_6(x_n, z) = b > 0, \forall n$].

We show that this is not compatible with any of the orbital conditions upon P.

From the preceding observations, we have, $\forall n \geq k$,

$M_0(x_n, z) > 0$; hence, $P(x_n, z) > 0$ (as $(P; M_0)$ is positive).

Further, as $\mathscr{R}$ is (a-o-f,d)-almost-selfclosed, we must have

$(x_n; n \geq 0)\mathscr{R}\mathscr{R}z$; i.e., there exists a subsequence $(u_n = x_{i(n)}; n \geq 0)$
of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$.

Note that, since $(i(n))$ is divergent, one may arrange for $(i(n) \geq k, \forall n)$; so, combining with the preceding relation,

$u_n \mathscr{R} z, d(Tu_n, Tz) > 0$, and $P(u_n, z) > 0, \forall n$.

The contractive condition is therefore applicable $((u_n, z); n \geq 0)$; and yields

(Om-contr) $d(Tu_n, Tz) \Omega P(u_n, z), \forall n \geq 0$.

Moreover, from the d-Cauchy and convergence properties, one gets (taking a metrical property of $d(.,.)$ into account)

$$d(u_n, z),\ d(Tu_n, z),\ d(T^2 u_, z) \to 0;$$
$$d(u_n, Tu_n),\ d(Tu_n, T^2 u_n),\ d(u_n, T^2 u_n) \to 0;$$
$$d(u_n, Tz),\ d(Tu_n, Tz),\ d(T^2 u_n, Tz) \to b.$$

There are several sub-cases to be analyzed.

Alter 1 Assume that P is orbitally singular asymptotic. By the contractive property and Ω being upper-diagonal,

(str-in) $d(Tu_n, Tz) < P(u_n, z)$, for all n,

Passing to (inferior) limit as $n \to \infty$ gives

$$b = \liminf_n d(Tu_n, Tz) \leq \liminf_n P(u_n, z) < b;$$

a contradiction. Hence, this alternative is not acceptable.

Alter 2 Suppose that P is orbitally regular asymptotic, and Ω is asymptotic bilateral separable. From the above limit relations (and the choice of P)

$$\lim_n d(Tu_n, Tz) = \lim_n P(u_n, z) = b.$$

This, along with the contractive condition, yields a contradiction with respect to the choice of Ω. Hence, this alternative is non-acceptable too.

Alter 3 Finally, assume that P is orbitally strongly regular asymptotic and Ω is asymptotic left separable. According to this property (involving P) there must be a subsequence $(w_n := u_{j(n)}; n \geq 0)$ of $(u_n; n \geq 0)$, such that (in addition)

$P(w_n, z) = b(> 0)$, for all $n \geq 0$.

From the imposed contractive condition, we get

$t_n \Omega b$ (hence, $t_n < b$), $\forall n$; where $(t_n = d(Tw_n, Tz); n \geq 0)$.

On the other hand, from the limit relations above, we have $t_n \to \beta$; whence $t_n \to \beta-$. Taking an auxiliary fact into account, there exists a subsequence $(t_n^* = t_{g(n)}; n \geq 0)$ of $(t_n; n \geq 0)$, with

(t_n^*) is strictly ascending and $t_n^* \Omega b, \forall n$.

The obtained relation contradicts the choice of Ω; so, the posed alternative is again non-acceptable for us.

Summing up, the working hypothesis $d(z, Tz) > 0$ cannot be true; so that $z = Tz$. The proof is complete.

Note that coincidence type versions of these facts are available, by means of related techniques in Roldán et al. [54]. On the other hand, all these developments may be extended to quasi-metric structures, under the lines in Nastasi and Vetro [46]. Further aspects may be found in Leader [37]; see also Turinici [67].

7 Particular Aspects

Let $(X, d, \mathscr{R})$ be a relational metric space; and T be a selfmap of X. As precise, we have to determine appropriate conditions under which $\mathrm{Fix}(T)$ is nonempty. The specific directions for solving this problem were already listed. Sufficient conditions for getting such properties are being founded on the (orbital type) positivity, boundedness, small and normal concepts we just introduced. Finally, the specific contractive properties to be used have been described; and the main result incorporating all these is the already formulated one. It is our aim in the sequel to expose certain particular cases of it, with some technical relevance. To do this, remember that for each $x, y \in X$, we defined the (basic) maps

$$Q_1(x, y) = d(x, Tx),\ Q_2(x, y) = d(x, y),$$
$$Q_3(x, y) = d(x, Ty),\ Q_4(x, y) = d(Tx, y),$$
$$Q_5(x, y) = d(Tx, Ty),\ Q_6(x, y) = d(y, Ty),$$
$$\mathscr{Q}(x, y) = (Q_1(x, y), Q_2(x, y), Q_3(x, y), Q_4(x, y), Q_5(x, y), Q_6(x, y)),$$
$$K_1(x, y) = d(x, T^2x),\ K_2(x, y) = d(Tx, T^2x),\ \mathscr{K}(x, y) = (K_1(x, y), K_2(x, y)).$$

By taking elementary order/algebraic combinations between these, one gets a lot of functions to be used in our reasonings:

$$P_0 = Q_5,\ P_1 = (1/2)[Q_3 + Q_4],$$
$$P_2 = (1/2)[K_1 + Q_4],\ M_0 = \min\{Q_1, Q_2, Q_5, Q_6, K_2\},$$
$$M_1 = \max\{Q_1, Q_6\},\ M_2 = \max\{Q_1, Q_2, Q_6\};$$

or, explicitly (for $x, y \in X$)

$$P_0(x, y) = d(Tx, Ty),\ P_1(x, y) = (1/2)[d(x, Ty) + d(Tx, y)],$$
$$P_2(x, y) = (1/2)[d(x, T^2x) + d(Tx, y)],$$
$$M_0(x, y) = \min\{d(x, Tx), d(x, y), d(Tx, Ty), d(y, Ty), d(Tx, T^2x)\},$$
$$M_1(x, y) = \max\{d(x, Tx), d(y, Ty)\},$$
$$M_2(x, y) = \max\{d(x, Tx), d(x, y), d(y, Ty)\}.$$

Then, by means of (further) intricate order/algebraic operations, we may define some other functions of this type; the following ones will be taken as concrete examples in our developments. Let us introduce the diagonal type subset of R_+^2

$$\Delta = \{(\xi, \eta) \in R_+ \times R_+^0; \xi \leq \eta\}.$$

This set is composed of a "singular" and "regular" part, expressed as

$$\Delta_s = \{(\xi, \eta) \in \Delta; \xi < \eta\},$$
$$\Delta_r = \{(\xi, \eta) \in \Delta; \xi = \eta\} = \{(\zeta, \zeta); \zeta \in R_+^0\}.$$

For each $(\xi, \eta) \in \Delta$, let us introduce the map $B := B[\xi, \eta] : X \times X \to R_+$, as

$$B = Q_6(\xi + Q_1)/(\eta + Q_2); \text{ or, explicitly (for } x, y \in X)$$
$$B(x, y) = d(y, Ty)[\xi + d(x, Tx)]/[\eta + d(x, y)].$$

Further, let us define

B_1=one of the maps $B[\xi, \eta]$ with $(\xi, \eta) \in \Delta_s$,
B_2=one of the maps $B[\xi, \eta]$ with $(\xi, \eta) \in \Delta_r$; or, equivalently:
B_2=one of the maps $B[\zeta, \zeta]$ with $\zeta \in R^0_+$.

The reason of splitting these maps will become clear later. Finally, given $(\alpha, \beta) \in \Delta$, let us introduce the map $B_3 := B_3[\alpha, \beta] : X \times X \to R_+$, according to

$$B_3 = K_2(\alpha + Q_4)/(\beta + Q_2); \text{ or, explicitly (for } x, y \in X):$$
$$B_3(x, y) = d(Tx, T^2x)[\alpha + d(Tx, y)]/[\beta + d(x, y)].$$

Having these precise, fix in the following the couples $(\xi, \eta) \in \Delta_s$, $(\zeta, \zeta) \in \Delta_r$, $(\alpha, \beta) \in \Delta$; and (according to the previous conventions), denote

$\mathscr{A} = \{Q_1, Q_2, Q_4, Q_5, Q_6, K_2\}$,
$\mathscr{A}_+ = \{Q_1, Q_2, Q_5, Q_6, K_2\}$, $\mathscr{A}_- = \{Q_4\}$,
$\mathscr{B} = \{P_1, P_2, B_1, B_2, B_3\}$, $\mathscr{B}_+ = \{B_1, B_2\}$, $\mathscr{B}_- = \{P_1, P_2, B_3\}$,
$\mathscr{H} = \mathscr{A} \cup \mathscr{B}$, $\mathscr{H}_+ = \mathscr{A}_+ \cup \mathscr{B}_+ = \{Q_1, Q_2, Q_5, Q_6, K_2, B_1, B_2\}$,
$\mathscr{H}_- = \mathscr{A}_- \cup \mathscr{B}_- = \{Q_4, P_1, P_2, B_3\}$.

For each (nonempty) subset $\Upsilon \in \exp(\mathscr{H})$, let $\max(\Upsilon) \in \mathscr{F}(X \times X, R_+)$ be the mapping defined as

$$\max(\Upsilon)(x, y) = \max\{G(x, y); G \in \Upsilon\}, x, y \in X;$$

clearly, there are $\text{card}[\exp(\mathscr{H})] = 2^{11} - 1 = 2047$ subsets of this type. Denote also

$$\exp(\mathscr{H}; \mathscr{H}_+) = \{\Upsilon \in \exp(\mathscr{H}); \Upsilon \cap \mathscr{H}_+ \neq \emptyset\}.$$

Note that any $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ may be written as

$$\Upsilon = \Upsilon_+ \cup \Upsilon_-, \text{ where } \Upsilon_+ \in \exp(\mathscr{H}_+), \Upsilon_- \in \exp[\mathscr{H}_-];$$

so, there are $(2^7 - 1)2^4 = 2032$ subsets of this type.

Technically speaking, the admissible maps $P : X \times X \to R_+$ to be considered are of the form

$$P = \max(\Upsilon); \text{ where, } \Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+).$$

So, it remains to establish of to what extent is this functional family compatible with the regularity conditions required by our main result.

(I) The first of these properties is positivity. Precisely, we have to establish that (with $P : X \times X \to R_+$ as before)

(posi) $(P; M_0)$ is *positive*: for each $x, y \in X$ with $(x\mathscr{R}y, x\mathscr{R}Tx, Tx\mathscr{R}T^2x)$, we have $[M_0(x, y) > 0$ implies $P(x, y) > 0]$.

The following positivity result is valid.

Proposition 30 *All maps* $P = \max(\Upsilon)$*, where* $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ *fulfill the property* $(P; M_0)$*=positive.*

Proof Clearly, all maps $P \in \mathscr{H}_+$ have the property in question. This conclusion extends to all maps $P = \max(\Upsilon)$ where $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$; so that we are done.

Remark 10 In the case when

(a-sym) $\mathscr{R}$ is antisymmetric: $x\mathscr{R}y$ and $y\mathscr{R}x$ imply $x = y$

the positivity condition includes $P_1 = (1/2)[Q_3 + Q_4]$ as well; that is:

$(x\mathscr{R}y, x\mathscr{R}Tx, Tx\mathscr{R}T^2x)$ and $M_0(x, y) > 0$ imply $P_1(x, y) > 0$.

In fact, suppose—under these premises—that

$P_1(x, y) = 0$; that is: $x = Ty$ and $Tx = y$.

As a consequence of these, $x = T^2x$; whence $T^2x\mathscr{R}Tx$. Combining with $Tx\mathscr{R}T^2x$ yields (as $\mathscr{R}$=antisymmetric)

$Tx = T^2x$; in contradiction with $d(Tx, T^2x) \geq M_0(x, y) > 0$.

Hence, $P_1(x, y) > 0$; and conclusion follows.

(II) The next property to be checked is orbital boundedness, which writes

(o-bd) $(P; M_1)$ is *orbitally bounded* over each (a-us-nt)-admissible iterative couple $(x_0; (x_n))$: $P(x_n, x_{n+1}) \leq M_1(x_n, x_{n+1})$, $\forall n$.

This, in particular holds whenever (letting P be as before)

(o-bd-a) $(P; M_1)$ is *almost orbitally bounded*:
for each $x \in X$ with $(x\mathscr{R}Tx, Tx\mathscr{R}T^2x)$, we have $P(x, Tx) \leq M_1(x, Tx)$.

Proposition 31 *All maps* $P = \max(\Upsilon)$*, where* $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ *fulfill*

(72-1) $(P; M_1)$ *is almost orbitally bounded; whence*
(72-2) (P, M_1)*=orbitally bounded at each iterative couple* $(x_0; (x_n))$.

Proof It will suffice verifying the first part. Given the arbitrary fixed point $x \in X$ with $x\mathscr{R}Tx$, $Tx\mathscr{R}T^2x$, we have

$Q_i(x, Tx) \leq d(x, Tx) \leq M_1(x, Tx)$, $i \in \{1, 2, 4\}$,
$Q_j(x, Tx) = d(Tx, T^2x) \leq M_1(x, Tx)$, $j \in \{5, 6\}$,
$K_2(x, Tx) = d(Tx, T^2x) \leq M_1(x, Tx)$,
$P_j(x, Tx) = (1/2)d(x, T^2x) \leq M_1(x, Tx)$, $j \in \{1, 2\}$,
$B(x, Tx) = d(Tx, T^2x)[\xi + d(x, Tx)]/[\eta + d(x, Tx)] \leq$
$d(Tx, T^2x) \leq M_1(x, Tx)$, $\forall(\xi, \eta) \in \Delta$,
$B_3(x, Tx) = d(Tx, T^2x)(\alpha + 0)/(\beta + d(x, Tx)) \leq$
$d(Tx, T^2x) \leq M_1(x, Tx)$, $\forall(\alpha, \beta) \in \Delta$;

and this, along with any map B_1 or B_2 having the form $B = B[\xi, \eta]$ where $(\xi, \eta) \in \Delta$, tells us that each $P \in \mathscr{H}$ fulfills the written property. Combining with the structure of $\exp(\mathscr{H}; \mathscr{H}_+)$, we are done.

(III) Passing to the orbitally small property, let the full d-asymptotic, d-non-Cauchy (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ be given. According to a previous auxiliary fact, there exist a number $h \in N(1, \leq)$, a number $\varepsilon \in R^0_+$, a rank $J := j(\varepsilon, h) \geq 1$, and a sequence $((m(j), n(j)); j \geq 0)$ in $N \times N$, such that

(aq-1) $j \leq m(j) < n(j), x_{m(j)} \mathscr{R} x_{n(j)}, \forall j \geq 0$
(aq-2) for each $s, t \in N[0, 2h]$,
the sequence $(V_j(s, t) := d(x_{m(j)+s}, x_{n(j)+t}); j \geq 0)$ (in R_+)
is J-nearly in $]\varepsilon, \infty[$, with $V_j(s, t) \to \varepsilon + +$ as $j \to \infty$.

By definition, $[h, \varepsilon, J; ((m(j), n(j)); j \geq 0)]$ will be referred to as the *associated to* $(x_0; (x_n))$ *quadruple*. Note that putting $(r_n = d(x_n, x_{n+1}); n \geq 0)$, we have

(aq-3) $\lim_j Q_1(x_{m(j)}, x_{n(j)}) = \lim_j r_{m(j)} = 0$,
$\lim_j Q_2(x_{m(j)}, x_{n(j)}) = \lim_j V_j(0, 0) = \varepsilon$,
$\lim_j Q_3(x_{m(j)}, x_{n(j)}) = \lim_j V_j(0, 1) = \varepsilon$,
$\lim_j Q_4(x_{m(j)}, x_{n(j)}) = \lim_j V_j(1, 0) = \varepsilon$,
$\lim_j Q_5(x_{m(j)}, x_{n(j)}) = \lim_j V_j(1, 1) = \varepsilon$,
$\lim_j Q_6(x_{m(j)}, x_{n(j)}) = \lim_j r_{n(j)} = 0$,
$\lim_j K_1(x_{m(j)}, x_{n(j)}) = \lim_j d(x_{m(j)}, x_{m(j)+2}) = 0$,
$\lim_j K_2(x_{m(j)}, x_{n(j)}) = \lim_j r_{m(j)+1} = 0$.

Having these precise, we have to establish that (with $P : X \times X \to R_+$ as before)

P is *orbitally small* over each full d-asymptotic, d-non-Cauchy, (a-us-nt)-admissible iterative couple $(x_0; (x_n))$: for each associated to $(x_0; (x_n))$ quadruple $[h, \varepsilon, J; ((m(j), n(j)); j \geq 0)]$, $\limsup_j P(x_{m(j)}, x_{n(j)}) \leq \varepsilon$.

An appropriate situation when such a property holds is defined as

$P = \Theta(\mathscr{Q}, \mathscr{K})$ (i.e., $P(x, y) = \Theta(\mathscr{Q}(x, y), \mathscr{K}(x, y)), x, y \in X$),
where $\Theta : R^6_+ \times R^2_+ \to R_+$ is a function.

Proposition 32 *Suppose that* Θ *is a continuous function with*

$\Theta(t(\varepsilon), s(\varepsilon)) \leq \varepsilon$, *for all* $\varepsilon > 0$,
where $(t(\varepsilon) = (0, \varepsilon, \varepsilon, \varepsilon, \varepsilon, 0); \varepsilon > 0)$, $(s(\varepsilon) = (0, 0); \varepsilon > 0)$.

Then,
(73-1) *All composed maps* $P = \Theta(\mathscr{Q}, \mathscr{K})$ *are orbitally small.*
As a consequence of this,
(73-2) *All maps* $P = \max(\Upsilon)$, *where* $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ *are orbitally small.*

Proof

(i) Let the full d-asymptotic, d-non-Cauchy, (a-us-nt)-admissible iterative couple $(x_0; (x_n))$ be given; and $[h, \varepsilon, J; ((m(j), n(j)); j \geq 0)]$ be an associated to $(x_0; (x_n))$ quadruple. By the above limit properties we have (cf. our notations)

$$\lim_j \mathscr{Q}(x_{m(j)}, x_{n(j)}) = (0, \varepsilon, \varepsilon, \varepsilon, \varepsilon, 0) = t(\varepsilon),$$
$$\lim_j \mathscr{K}(x_{m(j)}, x_{n(j)}) = (0, 0) = s(\varepsilon).$$

This, along with the choice of P, yields (by the continuity of Θ)

$$\limsup_j P(x_{m(j)}, x_{n(j)}) = \lim_j P(x_{m(j)}, x_{n(j)}) = \Theta(t(\varepsilon), s(\varepsilon)) \leq \varepsilon;$$

and the proof is complete.

(ii) Clearly, all maps in $\mathscr{H}$ are composed ones, in the sense

$$(Q_i = \Theta[Q_i](\mathscr{Q}, \mathscr{K}), i \in \{1, 2, 4, 5, 6\}), K_2 = \Theta[K_2](\mathscr{Q}, \mathscr{K}),$$
$$(P_k = \Theta[P_k](\mathscr{Q}, \mathscr{K}), 1 \leq k \leq 2), (B_i = \Theta[B_i](\mathscr{Q}, \mathscr{K}), 1 \leq i \leq 3);$$

where the corresponding functions $(\Theta[Q_i]; i \in \{1, 2, 4, 5, 6\})$, $\Theta[K_2]$, $(\Theta[P_k]; 1 \leq k \leq 2)$, $(\Theta[B_i]; 1 \leq i \leq 3)$ in the class $\mathscr{F}(R^6_+ \times R^4_+, R_+)$ are expressed as: for each $t = (t_1, t_2, t_3, t_4, t_5, t_6) \in R^6_+$, and each $s = (s_1, s_2) \in R^2_+$,

$$\Theta[Q_i](t, s) = t_i, i \in \{1, 2, 4, 5, 6\}, \Theta[K_2](t, s) = s_2,$$
$$\Theta[P_1](t, s) = (1/2)(t_3 + t_4), \Theta[P_2](t, s) = (1/2)(s_1 + t_4),$$
$$\Theta[B_1](t, s) = t_6(\xi + t_1)/(\eta + t_2), \Theta[B_2](t, s) = t_6(\zeta + t_1)/(\zeta + t_2),$$
$$\Theta[B_3](t, s) = s_2(\alpha + t_4)/(\beta + t_2).$$

In this case, letting $\varepsilon > 0$, we have

$$\Theta[Q_i](t(\varepsilon), s(\varepsilon)) \leq \varepsilon, i \in \{1, 2, 4, 5, 6\}, \Theta[K_2](t(\varepsilon), s(\varepsilon)) \leq \varepsilon,$$
$$\Theta[P_1](t(\varepsilon), s(\varepsilon)) = (1/2)(\varepsilon + \varepsilon) = \varepsilon,$$
$$\Theta[P_2](t(\varepsilon), s(\varepsilon)) = (1/2)(0 + \varepsilon) = \varepsilon/2 \leq \varepsilon,$$
$$\Theta[B_1](t(\varepsilon), s(\varepsilon)) = 0(\xi + 0)/(\eta + \varepsilon) = 0 \leq \varepsilon,$$
$$\Theta[B_2](t(\varepsilon), s(\varepsilon)) = 0(\zeta + 0)/(\zeta + \varepsilon) = 0 \leq \varepsilon$$
$$\Theta[B_3](t(\varepsilon), s(\varepsilon)) = 0(\alpha + \varepsilon)/(\beta + \varepsilon) \leq \varepsilon.$$

This completes the argument.

(IV) Concerning the orbital asymptotic properties, remember that these may be written as (letting $P : X \times X \to R_+$ be as before)

(o-nor-1) P is *orbitally singular asymptotic* over each full d-Cauchy (a-us-nt)-admissible couple $(x_0; (x_n))$:
whenever $x_n \overset{d}{\longrightarrow} z$ and $d(z, Tz) > 0$, we have $\liminf_n P(u_n, z) < d(z, Tz)$, for each subsequence $(u_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$
(o-nor-2) P is *orbitally regular asymptotic* over each full d-Cauchy (a-us-nt)-admissible couple $(x_0; (x_n))$:
whenever $x_n \overset{d}{\longrightarrow} z$ and $d(z, Tz) > 0$, we have $P(u_n, z) \to d(z, Tz)$, for each subsequence $(u_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$
(o-nor-3) P is *orbitally strongly regular asymptotic* over each full d-Cauchy (a-us-nt)-admissible couple $(x_0; (x_n))$:

whenever $x_n \xrightarrow{d} z$ and $d(z, Tz) > 0$, we have $P(u_n, z) \to\to d(z, Tz)$, for each subsequence $(u_n; n \geq 0)$ of $(x_n; n \geq 0)$ with $(u_n; n \geq 0)\mathscr{R}z$.

In this direction, the following synthetic answer is available.

Proposition 33 *Under the above conventions,*

(74-1) *Each (admissible) map $P = \max(\Upsilon)$, where $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ fulfills $\{Q_5, Q_6, B_2\} \cap \Upsilon = \emptyset$ is orbitally singular asymptotic*

(74-2) *Each (admissible) map $P = \max(\Upsilon)$, where $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ fulfills $\{Q_5, Q_6, B_2\} \cap \Upsilon \neq \emptyset$ is orbitally regular asymptotic*

(74-3) *Each (admissible) map $P = \max(\Upsilon)$, where $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ fulfills $\{Q_5, B_2\} \cap \Upsilon = \emptyset$ and $Q_6 \in \Upsilon$ is orbitally strongly regular asymptotic.*

Proof There are three steps to be passed.

Step 1 First, we have to discuss the orbital asymptotic properties of the maps $P \in \mathscr{A} \cup \{P_1, P_2\}$ at each full d-Cauchy (a-us-nt)-admissible couple $(x_0; (x_n))$. Let the point $z \in X$ be such that $x_n \xrightarrow{d} z$ and $b := d(z, Tz) > 0$; then, let the subsequence $(u_n = x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$ be taken so as $(u_n; n \geq 0)\mathscr{R}z$. From the d-Cauchy and convergence properties one gets (taking a metrical property of $d(.,.)$ into account) the limit properties (as $n \to \infty$)

$$d(u_n, z),\ d(Tu_n, z),\ d(T^2u_n, z) \to 0,$$
$$d(u_n, Tu_n),\ d(Tu_n, T^2u_n),\ d(u_n, T^2u_n) \to 0,$$
$$d(u_n, Tz),\ d(Tu_n, Tz),\ d(T^2u_n, Tz) \to b.$$

This, by definition, gives (as $n \to \infty$)

$$Q_1(u_n, z) = d(u_n, Tu_n) \to 0,\ Q_2(u_n, z) = d(u_n, z) \to 0$$
$$Q_3(u_n, z) = d(u_n, Tz) \to b,\ Q_4(u_n, z) = d(Tu_n, z) \to 0$$
$$Q_5(u_n, z) = d(Tu_n, Tz) \to b,\ Q_6(u_n, z) = d(z, Tz) = b,$$
$$K_1(u_n, z) = d(u_n, T^2u_n) \to 0,\ K_2(u_n, z) = d(Tu_n, T^2u_n) \to 0;$$

wherefrom (according to involved constructions)

$$P_1(u_n, z) \to b/2,\ P_2(u_n, z) \to 0, \text{ as } n \to \infty.$$

As a consequence,

(o-sa) any mapping $P \in \{Q_1, Q_2, Q_4, K_2, P_1, P_2\}$
is orbitally singular asymptotic
(o-ra) the mapping $P = Q_5$ is orbitally regular asymptotic
(o-sra) the mapping $P = Q_6$ is orbitally strongly regular asymptotic.

Step 2 Second, we discuss the orbital asymptotic properties of the maps $Q \in \{B_1, B_2, B_3\}$ at each full d-Cauchy (a-us-nt)-admissible couple $(x_0; (x_n))$. Let the point $z \in X$ be such that $x_n \xrightarrow{d} z$ and $b := d(z, Tz) > 0$; then, let the subsequence $(u_n = x_{i(n)}; n \geq 0)$ of $(x_n; n \geq 0)$ be taken so as $(u_n; n \geq 0)\mathscr{R}z$. By definition, we have for $B = B[\xi\ \eta]$, where $(\xi, \eta) \in \Delta$,

$B(u_n, z) = b[\xi + d(u_n, Tu_n)]/[\eta + d(u_n, z)], \forall n$; so, $\lim_n B(u_n, z) = b\xi/\eta$.

This, along with the above conventions, means

$\lim_n B_1(u_n, z) = b\xi/\eta < b$; $\lim_n B_2(x_n, z) = b\zeta/\zeta = b$.

On the other hand,

$B_3(u_n, z) = d(Tu_n, T^2 u_n)[\alpha + d(Tu_n, z)]/[\beta + d(u_n, z)], \forall n$;
so, $\lim_n B_3(x_n, z) = 0 < b$.

Putting these together yields

(o-sa-1) each mapping $P \in \{B_1, B_3\}$ is orbitally singular asymptotic
(o-ra-1) the mapping $P = B_2$ is orbitally regular asymptotic.

Step 3 By the above discussion and limit definition, our conclusion follows.

Now, by simply combining these with our main result, one gets the following *rational* type fixed point statement (referred to as *Rational Function geometric theorem*; in short: (G-ra)).

Theorem 3 *Suppose that the selfmap T is ascending-uscattered regular as well as $(d, \mathscr{R}; \max(\Upsilon); \Omega)$-contractive, for some subset $\Upsilon \in \exp(\mathscr{H})$ and some relation $\Omega \subseteq R^0_+ \times R^0_+$ with*

(71-i) $\Upsilon \in \exp(\mathscr{H}; \mathscr{H}_+)$ (see above)
(71-ii) Ω is upper diagonal and geometric Meir–Keeler.

In addition, let X be (a-o-f,d)-complete; and take an (a-us-nt)-admissible iterative couple $(x_0; (x_n))$.

Then,

(71-a) $(x_0; (x_n))$ *is Picard (modulo $(d, \mathscr{R}; T)$)*

(71-b) $(x_0; (x_n))$ *is strongly Picard (modulo $(d, \mathscr{R}; T)$) provided the following extra condition holds:*

(71-b-1) T is (a-o-f,d)-continuous

(71-c) $(x_0; (x_n))$ *is Bellman Picard (modulo $(d, \mathscr{R}; T)$) whenever $\mathscr{R}$ is (a-o-f,d)-almost-selfclosed and one of the following extra conditions holds:*

(71-c-1) $\{Q_5, Q_6, B_2\} \cap \Upsilon = \emptyset$
(71-c-2) $\{Q_5, Q_6, B_2\} \cap \Upsilon \neq \emptyset$ and Ω is asymptotic bilateral separable
(71-c-3) $\{Q_5, B_2\} \cap \Upsilon = \emptyset$, $Q_6 \in \Upsilon$, and Ω is asymptotic left separable.

Some particular cases of this result may be described as follows.

Case 1 Suppose that $\mathscr{R}$ is the trivial relation over X. Then, our particular main result includes in a direct way the basic ones in Boyd and Wong [9], Matkowski [41] and Leader [36]; see also Yadava et al. [71].

Case 2 Suppose that $\mathscr{R}$ is a partial order on X. Then, our particular main result includes the related statements in Agarwal et al. [3] when $\Upsilon = \{Q_1, Q_2, Q_6, P_1\}$, the ones in Cabrera et al. [10] when $\Upsilon = \{Q_2, B_2\}$, as well as the ones in Choudhury and Kundu [12] when $\Upsilon = \{Q_1, Q_6\}$. Further aspects may be given in Harjani et al. [23], Saluja et al. [56], or Chandok et al. [11]; see also Wardowski [69].

Finally, it is worth noting that, by the used techniques, our particular fixed point statement does not include the ones in Berzig [7] or Samet and Turinici [57]; because, in the quoted results, the ambient relation $\mathscr{R}$ is *amorphous*; i.e., it has no further properties at all. However, if one starts from a technical version of the present developments—involving a deduction of d-Cauchy property by avoiding the d-asymptotic stage—this inclusion holds; we do not give details. Some other aspects will be developed elsewhere.

8 Application (Integral Equations)

Let $n \in N_1 := N(1, \leq)$ be a natural number; and $R^n = \{(\xi_1, \ldots, \xi_n); \xi_1, \ldots, \xi_n \in R\}$ be the standard n-dimensional vector space endowed with the usual norm

$$||(\xi_1, \ldots, \xi_n)|| = \max\{|\xi_i|; 1 \leq i \leq n\}, (\xi_1, \ldots, \xi_n) \in R^n,$$

and the standard order

$$(\xi_1, \ldots, \xi_n) \leq (\eta_1, \ldots, \eta_n) \text{ iff } \xi_i \leq \eta_i, i \in \{1, \ldots, n\}.$$

Then, let X_n stand for the class of all continuous applications $x : R_+ \to R^n$. Clearly, X_n is a vectorial space with respect to the operations: for each $x, y \in X_n$, $\lambda \in R$,

$$(x + y)(t) = x(t) + y(t), (\lambda x)(t) = \lambda x(t), t \in R_+.$$

Concerning its topological structure, let us define for each $i \in N_1$ a seminorm $q_i : X_n \to R_+$ as

$$q_i(x) = \sup(||x(t)||; 0 \leq t \leq i), x \in X_n.$$

The family of seminorms $Q := (q_i; i \in N_1)$ has the properties

(asc) Q is ascending: $q_i \leq q_j$ when $i \leq j$
(suf) Q is sufficient: $q_i(x) = 0, \forall i \in N_1$ implies $x = 0$.

As a consequence of this, the structure (X_n, Q) appears as a metrizable complete locally convex space. Let also $(\leq)$ stand for the usual order

$$(x, y \in X_n): x \leq y \text{ iff } x(t) \leq y(t), t \in R_+.$$

Finally, denote by X_0 the class of all continuous applications $a : R_+ \to R_+$. Clearly, X_0 is but a *convex cone* in X_1; precisely,

(cc-1) $a, b \in X_0$ implies $a + b \in X_0$
(cc-2) $\lambda \in R_+, a \in X_0$ imply $\lambda a \in X_0$.

Further, it will be useful to introduce the mapping $x \mapsto ||x||$ from X_n to X_0 as

$$||x||(t) = ||x(t)||, t \in R_+, x \in X_n.$$

Having these precise, let $x \mapsto k(x)$ be an application from X_n to X_n, and $x^0 \in R^n$, a given vector. We may consider the integro-functional equation

(IFE) $x(t) = x^0 + \int_0^t k(x)(s)ds, t \in R_+;$

as well as the integro-functional inequality

(IFI) $x(t) \le x^0 + \int_0^t k(x)(s)ds, t \in R_+.$

Note that, by defining the integro-functional operator $T : X_n \to X_n$, as

$Tx(t) = x^0 + \int_0^t k(x)(s)ds, t \in R_+, x \in X_n,$

the above equation/inequality may be abstractly written as

$x = Tx$ (respectively, $x \le Tx$).

This tells us that, for solving such problems, one may use the general results in our preceding sections. This may be done under the lines below.

(L1) Let $g \in X_0$ be a given function. Define a (generalized) map $||.||_g : X_n \to R_+ \cup \{\infty\}$ as: for each $x \in X_n$,

$||x||_g = \inf\{\lambda \ge 0; ||x|| \le \lambda g\}$, if $\{\lambda \ge 0; ||x|| \le \lambda g\} \ne \emptyset$
$||x|_g = \infty$, if $\{\lambda \ge 0; ||x|| \le \lambda g\} = \emptyset$.

Then, let $d_g : X_n \times X_n \to R_+ \cup \{\infty\}$ be the associated (generalized) pseudometric

$d_g(x, y) = ||x - y||_g, x, y \in X_n.$

It is not hard to see that (according to Luxemburg [40] and Jung [29])

(g-Bs) $(X_n, ||.||_g)$ is a generalized Banach space
(g-ms) (X_n, d_g) is a complete generalized metric space.

In addition, denote

$(X_n)_g = (x \in X_n; ||x||_g < \infty)$ (the *component* of $0 \in X_n$).

By the observations above,

(g-Bs-0) $((X_n)_g, ||.||_g)$ is a (standard) Banach space
(g-ms-0) $((X_n)_g, d_g)$ is a complete metric space.

These conclusions remain valid for all translates $X_u := u + (X_n)_g$, where $u \in X_n$.

(L2) Let $\varphi : R_+ \to R_+$ be a function; we say that it is *regressive*, provided

$(\varphi(0) = 0)$, and $(\varphi(t) < t, \forall t > 0)$;

the class of all these functions will be denoted as $\mathscr{F}(re)(R_+)$. Let also $\mathscr{F}(re, in)(R_+)$ stand for the class of all increasing $\varphi \in \mathscr{F}(re)(R_+)$. Given some $\varphi \in \mathscr{F}(re, in)(R_+)$, call it *Matkowski admissible*, provided

$\varphi^n(t) \to 0$ as $n \to \infty$, for each $t \ge 0$.

(Here, φ^n is the n-th iterate of φ, for all $n \in N$). In particular, the Matkowski admissible property holds whenever $\varphi \in \mathscr{F}(re, in)(R_+)$ fulfills

φ is *Meir–Keeler admissible*:
$\forall\gamma > 0, \exists\beta > 0, (\forall t)$: $\gamma < t < \gamma + \beta \Longrightarrow \varphi(t) \leq \gamma$.

On the other hand, the Meir–Keeler admissible property for $\varphi \in \mathscr{F}(re, in)(R_+)$ is obtainable under (see above)

φ is *Boyd–Wong admissible*: $(\Lambda^+\varphi(s) =)\varphi(s + 0) < s$, for all $s > 0$.

This tells us that the class of Matkowski admissible functions in $\mathscr{F}(re, in)(R_+)$ is large enough; we do not give further details.

In the following, an existence, uniqueness, and approximation result for the solutions of integro-functional equation (IFE) will be established, by means of successive approximation method. To this end, the following conditions will be used.

(I) Suppose that there exist a (nonzero) function $g \in X_0$, a mapping $x \mapsto h(x)$ from X_0 to itself, and a function $\varphi \in \mathscr{F}(R_+)$ such that

(I-1) $(x, y \in X_n, x \leq y), a \in X_0, ||x - y|| \leq a$ imply $||k(x) - k(y)|| \leq h(a)$
(I-2) $\int_0^t h(g\tau)(s)ds \leq \varphi(\tau)g(t), \tau > 0, t \in R_+$
(I-3) φ is regressive, increasing, and Matkowski admissible.

Under these circumstances, the attached operator T is contractive, in the sense

whenever $x, y \in X_n$ with $x \leq y$ and $\tau > 0$ satisfy
$||x - y||_g \leq \tau$, then $||Tx - Ty||_g \leq \varphi(\tau)$.

In fact, let $x, y \in X_n$ and $\tau > 0$ be as in this premise. According to the definition of metric $||.||_g$, we have $||x - y|| \leq g\tau$; and then, by the posed hypotheses,

$||Tx(t) - Ty(t)|| \leq \int_0^t ||k(x)(s) - k(y)(s)||ds$
$\leq \int_0^t h(g\tau)(s)ds \leq \varphi(\tau)g(t), t \in R_+$; that is: $||Tx - Ty||_g \leq \varphi(\tau)$.

(II) Suppose that the set $X_n(T)$ of all $u \in X_n$ with

$u(t) \leq x^0 + \int_0^t k(u)(s)ds, t \in R_+$, and
$||u(t) - x^0 - \int_0^t k(u)(s)ds|| \leq \mu g(t), t \in R_+$, for some $\mu > 0$,

is a nonempty part of X_n.

In this case, it is clear that

$(\forall u \in X_n(T))$: u=solution of (IFI) with $||u - Tu||_g < \infty$.

(III) Suppose that the following local increasing condition for $k(.)$ is valid:

$(x, y \in X_n, x \leq y, ||x - y||_g < \infty)$ imply $k(x)(t) \leq k(y)(t), t \in R_+$.

Clearly, under such an assumption, we have a corresponding increasing property for the associated operator:

$(x, y \in X_n, x \leq y, ||x - y||_g < \infty)$ imply $Tx \leq Ty$.

This, however, does not give us a global increasing property of the underlying operator, as it can be directly seen.

We may now pass to the announced statement.

Theorem 4 *Suppose that the posed conditions hold. Then, fix $u_0 \in X_n(T)$; and let $(u_p := T^p u_0;\ p \geq 0)$ be its iterative sequence, described as*

$$u_{p+1}(t) = x^0 + \int_0^t k(u_p)(s)ds,\ t \in R_+,\ p \geq 0.$$

Then,

(81-a) *the iterative sequence $(u_p;\ p \geq 0)$ is $||.||_g$-convergent in $u_0 + (X_n)_g$; hence, in particular, $\rho := \sup\{d_g(u_p, u_q);\ p, q \geq 0\}$ exists in R_+*

(81-b) *the associated d_g-limit $z = \lim_p u_p$ is the unique solution of the integro-functional equation (IF) in $u_0 + (X_n)_g$ up to comparison:*

$$z, w = \text{solution of (IF) in } u_0 + (X_n)_g, \text{ and } z \leq w \text{ imply } z = w$$

(81-c) *the iterative process $(u_p;\ p \geq 0)$ converges to its limit z according to the error approximation formula*

$$d_g(u_p, z) \leq \varphi^p(\rho),\ p \geq 0.$$

Proof Roughly speaking, the existence part of this statement follows at once from our previous developments; however, for completeness reasons, we shall provide it, with some modifications.

Step 1 Let $x, y \in u_0 + (X_n)_g$ and $\tau > 0$ be a couple of elements with $x \leq y$, $d_g(x, y) \leq \tau$. We claim that the following relation holds

(it-comp) $(\forall p)$: $d_g(T^p x, T^p y) \leq \varphi^p(\tau)$.

To this end, the induction argument with respect to p will be used. The case $p = 0$ is clear. Suppose that (it-comp) is clear, for each $p \in \{0, \ldots, q\}$, where $q \geq 0$; we claim that it holds as well for $p = q + 1$. This, however, is evident in view of our contractive condition and increasing property of T; so, the claim is proved.

Step 2 As a direct consequence of the obtained fact,

$\lim_p d_g(u_p, u_{p+1}) = 0$; whence (u_p) is d_g-asymptotic.

Step 3 We now establish that (u_p) is a d_g-Cauchy sequence. Let $\varepsilon > 0$ be arbitrary fixed. By the previous observation,

there exists $p = p(\varepsilon)$ such that $d_g(u_i, u_{i+1}) < \varepsilon - \varphi(\varepsilon)$, for all $i \geq p$.

We claim that

(dg-C) $(\forall r \geq 1)$: $d_g(u_i, u_{i+r}) < \varepsilon$, for all $i \geq p$;

and, from this, the d_g-Cauchy property of (u_p) follows. To this end, an induction argument with respect to r will be performed. The case $r = 1$ is clear, by the very choice of our index p. Suppose now that (dg-C) holds for all $r \in \{1, \ldots, s\}$, where

$s \geq 1$; we claim that it holds for $r = s + 1$. In fact, given $i \geq p$, we have by induction hypothesis

$$d_g(u_{i+1}, u_{i+s+1}) \leq \varphi(\varepsilon) \text{ (cf. the contractive property);}$$

and this, along with the preceding relation, yields

$$d_g(u_i, u_{i+s+1}) \leq d_g(u_i, u_{i+1}) + d_g(u_{i+1}, u_{i+s+1}) < \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon;$$

and our assertion follows.

Step 4 By the completeness of $u_0+(X_n)_g$ (and self-closeness of $(\leq)$ over the same) $z := \lim_p(u_p)$ exists in $u_0 + (X_n)_g$, with $u_p \leq z$, $\forall p$. Again by the contractive condition, we have, under the convention $(\sigma_p := d_g(u_p, z);\ p \geq 0)$,

$$d_g(u_{p+1}, Tz) \leq \varphi(\sigma_p) \leq \sigma_p, \forall p; \text{ whence } Tz \in u_0 + (X_n)_g.$$

Passing to limit as $p \to \infty$ yields $\lim_p d_p(u_p, Tz) = 0$; This, and $\lim_p d_g(u_p, z) = 0$, yields $z = Tz$ (as d_g is separated on $u_p+(X_n)_g$); i.e., $z \in u_0+(X_n)_g$ is a solution of our integro-functional equation (IF).

Step 5 Let $w \in u_0 + (X_n)_g$ be another solution of (IF) with $z \leq w$. By a preceding step, we have

$$d_g(z, w) = d_g(T^p z, T^p w) \leq \varphi^p(\tau), \forall p, \text{ where } \tau = d_g(z, w).$$

Passing to limit as $p \to \infty$ gives $d_g(z, w) = 0$; that is, $z = w$. Hence, (IF) has a unique solution over $u_0 + (X_n)_g$.

Step 6 As $z = \lim_p u_p$, we must have

$$d(u_0, z) \leq \tau, \text{ where } \tau := \sup\{d_g(u_p, u_q);\ p, q \geq 0\}.$$

This, again by a preceding step, yields

$$d_g(u_p, z) = d_g(T^p u_0, T^p z) \leq \varphi^p(\tau), \forall p;$$

and the final conclusion follows. The proof is complete.

Remark 11 Concerning the first group of conditions, note that, under

(I-1a) $h(.)$ is increasing ($a, b \in X_0, a \leq b$ imply $h(a) \leq h(b)$)

then, (I-1) is equivalent with

(I-1b) $||k(x) - k(y)|| \leq h(||x - y||), x, y \in X_n, x \leq y, ||x - y||_g < \infty$.

On the other hand, whenever

(I-2a) $h(.)$ is homogeneous: $h(\tau a) = \tau h(a), \tau > 0, a \in X_0$,
(I-3a) φ is linear: $\varphi(t) = \lambda t, t \in R_+$, for some $\lambda \in [0, 1[$,

then, (I-2)+(I-3) are equivalent with

(I-4) $\int_0^t h(g)(s)ds \leq \lambda g(t), t \in R_+$;

which allows us to determine the admissible functions $g \in X_0$ for the problem (IFE) under discussion. For example, in the particular case of

(I-5) $h(g) \leq Lg$, for all $g \in X_0$,

unde $L \in X_0$ is positive, a solution in X_0 of (I-4) is

(I-4a) $g(t) = \exp((1/\lambda) \int_0^t L(s)ds)$, $t \in R_+$

which leads us to norms $||.||_g$ or metrics d_g of the form introduced by Bielecki [8] and Corduneanu [15].

Remark 12 Concerning the second group of conditions, note that, under

(II-2a) $||x^0|| + \int_0^t k(0)(s)|| \leq \mu g(t)$, $t \in R_+$, for some $\mu > 0$,

it follows that $0 \in X_0$ fulfills this requirement; so that, under the remaining hypotheses, our existence and uniqueness result holds.

Finally, we stress that, technically speaking, a solution in X_n of the integro-functional equation (IFE) is to be determined by means of a solution in X_n of the integro-functional inequality (IFI). But, the reverse way is also valid: a solution in X_n of the integro-functional inequality (IFI) is ultimately obtainable by means of a solution in X_n of the integro-functional equation (IFE). This means that, for a large class of Gronwall–Bellman inequalities, a solution of these is to be obtained by means of attached equation. Further aspects may be found in Turinici [64].

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An Extended Multidimensional Half-Discrete Hardy–Hilbert-Type Inequality with a General Homogeneous Kernel

Bicheng Yang

Abstract By the use of the weight functions, the transfer formula, and the technique of real analysis, an extended multidimensional half-discrete Hardy–Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of a published result. Moreover, the equivalent forms, a few particular cases, and the operator expressions with some examples are considered.

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1 Introduction

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), g(y) \geq 0$, $f \in L^p(\mathbf{R}_+)$, $g \in L^q(\mathbf{R}_+)$,

$$||f||_p = (\int_0^\infty f^p(x)dx)^{\frac{1}{p}} > 0,$$

$||g||_q > 0$, then we have the following Hardy–Hilbert's integral inequality (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible. Assuming that $a_m, b_n \geq 0$, $a = \{a_m\}_{m=1}^\infty \in l^p$, $b = \{b_n\}_{n=1}^\infty \in l^q$,

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$$||a||_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0,$$

$||b||_q > 0$, we still have the following discrete variant of the above inequality with the same best constant factor $\frac{\pi}{\sin(\pi/p)}$:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{2}$$

Inequalities (1) and (2) are important in analysis and its applications (cf. [1–6]).

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [7] gave an extension of (1) at $p = q = 2$. In 2009–2011, Yang [3, 4] gave some extensions of (1) and (2) as follows:

Assuming that $\lambda_1, \lambda_2 \in \mathbf{R} = (-\infty, \infty)$, $\lambda_1 + \lambda_2 = \lambda$, $k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$, such that

$$k_\lambda(tx, ty) = t^{-\lambda}k_\lambda(x, y)\ (t, x, y > 0),$$

if $k(\lambda_1) = \int_0^\infty k_\lambda(t, 1)t^{\lambda_1-1}dt \in \mathbf{R}_+ = (0, \infty)$, $\phi(x) = x^{p(1-\lambda_1)-1}$, $\psi(y) = y^{q(1-\lambda_2)-1}$, $f(x)$, $g(y) \ge 0$, satisfying

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{f;\ ||f||_{p,\phi} := \{\int_0^\infty \phi(x)|f(x)|^p dx\}^{\frac{1}{p}} < \infty\right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+)$, $||f||_{p,\phi}$, $||g||_{q,\psi} > 0$, then we have

$$\int_0^\infty\int_0^\infty k_\lambda(x, y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{3}$$

where the constant factor $k(\lambda_1)$ is the best possible. Moreover, if $k_\lambda(x, y)$ is finite and $k_\lambda(x, y)x^{\lambda_1-1}(k_\lambda(x, y)y^{\lambda_2-1})$ is decreasing with respect to $x > 0 (y > 0)$, then for $a_m, b_n \ge 0$,

$$a \in l_{p,\phi} = \left\{a;\ ||a||_{p,\phi} := \{\sum_{n=1}^{\infty}\phi(n)|a_n|^p\}^{\frac{1}{p}} < \infty\right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}$, $||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}k_\lambda(m, n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{4}$$

where the constant factor $k(\lambda_1)$ is still the best possible. Clearly, for $\lambda = 1, k_1(x, y) = \frac{1}{x+y}, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3) reduces to (1), while (4) reduces to (2). Some other results including multidimensional integral and discrete Hilbert-type inequalities are provided by [8–25]. Kato [26–30] also published some other type of inequalities and operators.

About the topic of half-discrete Hilbert-type inequalities with the non-homogeneous kernels, Hardy et al. provided a few results in Theorem 351 of [1]. But they did not prove that the constant factors are the best possible. However, Yang [31] gave a result with the kernel $\frac{1}{(1+nx)^{\lambda}}$ by introducing an interval variable and proved that the constant factor is the best possible. In 2011 Yang [32] gave the following half-discrete Hardy–Hilbert's inequality with the best possible constant factor $B(\lambda_1, \lambda_2)$:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} \frac{a_n}{(x+n)^{\lambda}} dx < B(\lambda_1, \lambda_2) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{5}$$

where $\lambda_1, \lambda_2 > 0, 0 \le \lambda_2 \le 1, \lambda_1 + \lambda_2 = \lambda$,

$$B(u, v) = \int_0^{\infty} \frac{1}{(1+t)^{u+v}} t^{u-1} dt \, (u, v > 0)$$

is the beta function. Zhong et al. [33–38] investigated several half-discrete Hilbert-type inequalities with particular kernels.

Using the weight functions and the techniques of discrete and integral Hilbert-type inequalities with some additional conditions on the kernel, a half-discrete Hilbert-type inequality with a general homogeneous kernel of degree $-\lambda \in \mathbf{R}$ and a best constant factor $k(\lambda_1)$ is obtained as follows:

$$\int_0^{\infty} f(x) \sum_{n=1}^{\infty} k_{\lambda}(x, n) a_n dx < k(\lambda_1) ||f||_{p,\phi} ||a||_{q,\psi}, \tag{6}$$

which is an extension of (5) (see Yang and Chen [39]). A half-discrete Hilbert-type inequality with a general non-homogeneous kernel and a best constant factor is given by Yang [40]. In 2013–2014, Micheal and Yang [41, 42] gave two multidimensional half-discrete Hilbert-type inequalities with the particular non-homogeneous kernels.

Remark 1

(1) Many different kinds of discrete, half-discrete and integral Hilbert-type inequalities with applications are presented in recent 20 years. Special attention is given to new results proved during 2009–2014. Included are many generalizations, extensions, and refinements of discrete, half-discrete, and integral Hilbert-type inequalities involving many special functions such as beta, gamma,

hypergeometric, trigonometric, hyperbolic, zeta, Bernoulli functions, Bernoulli numbers, and Euler constant

(2) In his six books, Yang [3–6, 43, 44] presented many new results on Hilbert-type operators with general homogeneous kernels of degree of real numbers and two pairs of conjugate exponents as well as the related inequalities. These research monographs contained recent developments of discrete, half-discrete, and integral types of operators and inequalities with proofs, examples and applications.

In this chapter, by the use of the weight functions, the transfer formula, and the technique of real analysis, an extended multidimensional half-discrete Hardy–Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of (6). Moreover, the equivalent forms, a few particular cases, and the operator expressions with some examples are considered.

2 Some Lemmas

If $\mu_i^{(k)} > 0$ $(k = 1, \ldots, i_0; i = 1, \ldots, m)$, $\upsilon_j^{(l)} > 0$ $(l = 1, \ldots, j_0; j=1, \ldots, n)$, then we set

$$V_n^{(l)} \; : \; = \sum_{j=1}^{n} \upsilon_j^{(l)} \; (l = 1, \ldots, j_0), \tag{7}$$

$$V_n = (V_n^{(1)}, \ldots, V_n^{(j_0)}) \; (n \in \mathbf{N} = \{1,2, \ldots\}). \tag{8}$$

We also set functions $\mu_i(t) := \mu_m^{(i)}, t \in (m-1, m]$ $(m \in \mathbf{N})$; $\upsilon_j(t) := \upsilon_n^{(j)}$, $t \in (n-1, n]$ $(n \in \mathbf{N})$, and

$$U_i(x) : = \int_0^x \mu_i(t)dt \; (i = 1, \ldots, i_0), \tag{9}$$

$$V_j(y) : = \int_0^y \upsilon_j(t)dt \; (j = 1, \ldots, j_0), \tag{10}$$

$$U(x) : = (U_1(x), \ldots, U_{i_0}(x)),$$

$$V(y) : = (V_1(y), \ldots, V_{j_0}(y)) \; (x, y \geq 0). \tag{11}$$

It follows that $V_j(n) = V_n^{(j)}$ $(j = 1, \ldots, j_0; n \in \mathbf{N})$, and for $x \in (m-1, m)$, $U_i'(x) = \mu_i(x) = \mu_m^{(i)}$ $(i = 1, \ldots, i_0; m \in \mathbf{N})$; for $y \in (n-1, n)$, $V_j'(y) = \upsilon_j(y) = \upsilon_n^{(j)}$ $(j = 1, \ldots, j_0; n \in \mathbf{N})$. We still have

$$dU(x) = \sum_{i=1}^{i_0} \mu_i(x)dx \; (x \in \mathbf{R}_+^{i_0}), dV(y) = \sum_{j=1}^{j_0} \upsilon_j(y)dy \; (y \in \mathbf{R}_+^{j_0}).$$

Lemma 1 (cf. [45]) *Suppose that* $g(t)(>0)$ *is decreasing in* $\mathbf{R}_+$ *and strictly decreasing in* $[n_0, \infty)$ $(n_0 \in \mathbf{N})$, *satisfying* $\int_0^\infty g(t)dt \in \mathbf{R}_+$. *We have*

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)dt. \tag{12}$$

Lemma 2 *If* $i_0 \in \mathbf{N}, \alpha, M > 0, \Psi(u)$ *is a non-negative measurable function in* $(0, 1]$, *and*

$$D_M := \left\{ x \in \mathbf{R}_+^{i_0}; u = \sum_{i=1}^{i_0} \left(\frac{x_i}{M} \right)^\alpha \leq 1 \right\}, \tag{13}$$

then we have the following transfer formula (cf. [6]):

$$\int \cdots \int_{D_M} \Psi \left(\sum_{i=1}^{i_0} \left(\frac{x_i}{M} \right)^\alpha \right) dx_1 \cdots dx_{i_0}$$
$$= \frac{M^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \int_0^1 \Psi(u) u^{\frac{i_0}{\alpha}-1} du. \tag{14}$$

Lemma 3 *For* $i_0, j_0 \in \mathbf{N}, \upsilon_n^{(l)} \geq \upsilon_{n+1}^{(l)}$ $(n \in \mathbf{N}; l = 1, \ldots, j_0)$, $\alpha, \beta, \varepsilon > 0$,

$$b = \min_{1 \leq i \leq i_0, 1 \leq j \leq j_0} \{\mu_1^{(i)}, \upsilon_1^{(j)}\},$$

$$[1, \infty)^{i_0} := \{x \in \mathbf{R}_+^{i_0}; x_i \geq 1 \, (i = 1, \ldots, i_0)\},$$

we have

$$\int_{[1,\infty)^{i_0}} ||U(x)||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_k(x) dx \leq \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha} \alpha^{i_0-1} \Gamma(\frac{i_0}{\alpha})}, \tag{15}$$

$$\sum_n ||V_n||_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \leq \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\beta} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + O(1), \tag{16}$$

where $\sum_n = \sum_{n_{j_0}}^\infty \cdots \sum_{n_1}^\infty$,

$$||x||_\alpha = (\sum_{i=1}^{i_0} x_i^\alpha)^{\frac{1}{\alpha}} \; (x \in \mathbf{R}_+^{i_0}), \; ||y||_\beta = (\sum_{j=1}^{j_0} y_j^\beta)^{\frac{1}{\beta}} \; (y \in \mathbf{R}_+^{j_0}).$$

Proof Setting $v = U(x)$, we find

$$\int_{[1,\infty)^{i_0}} ||U(x)||_\alpha^{-i_0-\varepsilon} \prod_{k=1}^{i_0} \mu_k(x)dx$$
$$= \int_{\{v\in\mathbf{R}_+^{i_0};v_i\geq\mu_1^{(i)}\}} \frac{dv}{||v||_\alpha^{i_0+\varepsilon}} \leq \int_{[b,\infty)^{i_0}} \frac{dv}{||v||_\alpha^{i_0+\varepsilon}}.$$

For $M > bi_0^{1/\alpha}$, by (14), it follows that

$$\int_{[b,\infty)^{i_0}} \frac{dv}{||v||_\alpha^{i_0+\varepsilon}}$$
$$= \lim_{M\to\infty} \int_{\{v\in\mathbf{R}_+^{i_0};\frac{b^\alpha i_0}{M^\alpha}<\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^\alpha\leq 1\}} \frac{dv_1\cdots dv_{i_0}}{\{M[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^\alpha]^{\frac{1}{\alpha}}\}^{i_0+\varepsilon}}$$
$$= \lim_{M\to\infty} \frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})} \int_{b^\alpha i_0/M^\alpha}^1 \frac{u^{\frac{i_0}{\alpha}-1}}{(Mu^{1/\alpha})^{i_0+\varepsilon}}du$$
$$= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}.$$

Then we have (15).

We have

$$\sum_n ||V_n||_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \leq H_0 + \sum_{i=1}^{j_0} H_i,$$

$$H_0 := \sum_{\{n\in\mathbf{N}^{j_0};n_j\geq 2\}} ||V_n||_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)},$$

$$H_i := \sum_{\{n\in\mathbf{N}^{j_0};n_i=1;n_j\geq 1(j\neq i)\}} ||V_n||_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(l)}.$$

By (12) and the way of obtaining (15), we find

$$0 < H_0 = \sum_{\{n\in\mathbf{N}^{j_0};n_j\geq 2\}} \int_{\{y\in\mathbf{N}^{j_0};n_j-1\leq y_j<n_j\}} ||V(n)||_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_n^{(l)}dy$$

$$< \sum_{\{n\in\mathbf{N}^{j_0};n_j\geq 2\}} \int_{\{y\in\mathbf{R}_+^{j_0};n_j-1\leq y_j<n_j\}} ||V(y)||_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_l(y)dy$$

$$= \int_{[1,\infty)^{j_0}} ||V(y)||_\beta^{-j_0-\varepsilon} \prod_{l=1}^{j_0} \upsilon_l(y)dy \; (v = V(y))$$

$$\leq \int_{[b,\infty)^{j_0}} ||v||_\beta^{-j_0-\varepsilon} dv = \frac{\Gamma^{i_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\alpha} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})}.$$

Without loss of generality, we estimate H_{j_0}. If $j_0 = 1$, then we find

$$0 < H_{j_0} = (\upsilon_1^{(1)})^{-1-\varepsilon} \upsilon_1^{(1)} = (\upsilon_1^{(1)})^{-\varepsilon} < \infty;$$

if $j_0 \geq 2$, then we find

$$H_{j_0} = \sum_{\{n\in\mathbf{N}^{j_0-1}\}} \int_{\{y\in\mathbf{R}_+^{j_0-1};n_j-1<y_j\leq n_j\}} \frac{\upsilon_1^{(j_0)} \prod_{l=1}^{j_0-1} \upsilon_l(y)dy}{[(\upsilon_1^{(j_0)})^\beta + \sum_{j=1}^{j_0-1} (V_n^{(j)})^\beta]^{\frac{j_0+\varepsilon}{\beta}}}$$

$$\leq \upsilon_1^{(j_0)} \int_{\mathbf{R}_+^{j_0-1}} \frac{\prod_{l=1}^{j_0-1} \upsilon_l(y)}{[(\upsilon_1^{(j_0)})^\beta + \sum_{j=1}^{j_0-1} (V^{(j)}(y))^\beta]^{\frac{1}{\beta}(j_0+\varepsilon)}} dy.$$

Setting $v = V(y) = (V_1(y), \ldots, V_{j_0-1}(y))$, by (14), we have

$$0 < H_{j_0} \leq \upsilon_1^{(j_0)} \int_{\mathbf{R}_+^{j_0-1}} \frac{1}{[(\upsilon_1^{(j_0)})^\beta + \sum_{j=1}^{j_0-1} v_j^\beta]^{\frac{1}{\beta}(j_0+\varepsilon)}} dv$$

$$= \upsilon_1^{(j_0)} \lim_{M\to\infty} \frac{M^{j_0-1}\Gamma^{j_0-1}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0-1}{\beta})} \int_0^1 \frac{u^{\frac{j_0-1}{\beta}-1} du}{[(\upsilon_1^{(j_0)})^\beta + M^\beta u]^{\frac{1}{\beta}(j_0+\varepsilon)}}$$

$$\overset{t=\frac{M^\beta u}{(\upsilon_1^{(j_0)})^\beta}}{=} (\upsilon_1^{(j_0)})^{-\varepsilon} \frac{\Gamma^{j_0-1}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0-1}{\beta})} \int_0^\infty \frac{t^{\frac{j_0-1}{\beta}-1}}{(1+t)^{\frac{1}{\beta}(j_0+\varepsilon)}} dt$$

$$= (\upsilon_1^{(j_0)})^{-\varepsilon} \frac{\Gamma^{j_0-1}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0-1}{\beta})} B(\frac{j_0-1}{\beta}, \frac{1+\varepsilon}{\beta}) < \infty.$$

Hence, we have

$$\sum_n ||V_n||_\beta^{-j_0-\varepsilon} \prod_{k=1}^{j_0} \upsilon_n^{(k)} \le \frac{\Gamma^{i_0}(\frac{1}{\beta})}{\varepsilon b^\varepsilon j_0^{\varepsilon/\alpha} \beta^{j_0-1} \Gamma(\frac{j_0}{\beta})} + \sum_{i=1}^{j_0} O_i(1),$$

namely, (16) follows.

The lemma is proved.

Definition 1 If $i_0, j_0 \in \mathbf{N}, \alpha, \beta > 0, \lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda, k_\lambda(x, y)$ is a nonnegative homogeneous function of degree $-\lambda$, such that for any fixed $x > 0$, $k_\lambda(x, y)\frac{1}{y^{j_0-\lambda_2}}$ is decreasing with respect to $y \in \mathbf{R}_+$, and strictly decreasing in an interval $(b_x, \infty) \subset (0, \infty)$,

$$k(\lambda_1) = \int_0^\infty k_\lambda(u, 1) u^{\lambda_1 - 1} du \in \mathbf{R}_+,$$

then we define two weight functions $w(\lambda_1, n)$ $(n \in \mathbf{N}^{j_0})$ and $W(\lambda_2, x)$ $(x \in \mathbf{R}_+^{i_0})$ as follows:

$$w(\lambda_1, n) := \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) \frac{||V_n||_\beta^{\lambda_2}}{||U(x)||_\alpha^{i_0-\lambda_1}} \prod_{k=1}^{i_0} \mu_k(x) dx, \quad (17)$$

$$W(\lambda_2, x) := \sum_n k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) \frac{||U(x)||_\alpha^{\lambda_1}}{||V_n||_\beta^{j_0-\lambda_2}} \prod_{l=1}^{j_0} \upsilon_n^{(l)}. \quad (18)$$

Example 1 For $\lambda_1, \lambda_2 \in \mathbf{R}, \lambda_1 + \lambda_2 = \lambda,\ \lambda_1 + \eta > 0, 0 < \lambda_2 + \eta \le j_0$, we set

$$k_\lambda(x, y) = \frac{(\min\{x, y\})^\eta}{(\max\{x, y\})^{\lambda+\eta}} \ (x, y > 0).$$

Then for any fixed $x > 0$,

$$k_\lambda(x, y)\frac{1}{y^{j_0-\lambda_2}} = \begin{cases} \frac{1}{x^{\lambda+\eta} y^{j_0-\lambda_2-\eta}}, 0 < y < x \\ \frac{x^\eta}{y^{j_0+\lambda_1+\eta}}, y \ge x \end{cases}$$

is decreasing in $y \in \mathbf{R}_+$, and strictly decreasing in interval $([x] + 1, \infty) \subset (0, \infty)$. We still have

$$\begin{aligned} k(\lambda_1) &= \int_0^\infty \frac{(\min\{u, 1\})^\eta}{(\max\{u, 1\})^{\lambda+\eta}} \frac{1}{u^{1-\lambda_1}} du \\ &= \int_0^1 \frac{u^\eta}{u^{1-\lambda_1}} du + \int_1^\infty \frac{1}{u^{\lambda+\eta}} \frac{1}{u^{1-\lambda_1}} du \end{aligned}$$

$$= \frac{\lambda + 2\eta}{(\lambda_1 + \eta)(\lambda_2 + \eta)} \in \mathbf{R}_+.$$

Note 1 For $b, \beta > 0$, we have

$$\frac{d}{dy}(b + y^\beta)^{\frac{1}{\beta}} = (b + y^\beta)^{\frac{1}{\beta}-1} y^{\beta-1} > 0 \ (y > 0).$$

Hence, with regard to the assumptions of Definition 1, for $n_j - 1 < y_j < n_j$ ($j = 1, \ldots, j_0; n \in \mathbf{N}^{j_0}$), we have $||V(n)||_\beta > ||V(y)||_\beta$ and

$$\frac{k_\lambda(||U(x)||_\alpha, ||V(n)||_\beta)}{||V(n)||_\beta^{j_0-\lambda_2}} < \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)}{||V(y)||_\beta^{j_0-\lambda_2}};$$

for $n_j < y_j < n_j + 1 (j = 1, \ldots, j_0; n \in \mathbf{N}^{j_0})$, $\frac{\varepsilon}{q} > 0$, we have $||V(n)||_\beta < ||V(y)||_\beta$, and

$$\frac{k_\lambda(||U(x)||_\alpha, ||V_n||_\beta)}{||V_n||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} = \frac{k_\lambda(||U(x)||_\alpha, ||V(n)||_\beta)}{||V(n)||_\beta^{j_0-\lambda_2}} \frac{1}{||V(n)||_\beta^{\frac{\varepsilon}{q}}}$$
$$> \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)}{||V(y)||_\beta^{j_0-\lambda_2}} \frac{1}{||V(y)||_\beta^{\frac{\varepsilon}{q}}} = \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)}{||V(y)||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}}. \tag{19}$$

Lemma 4 *With regard to the assumptions of Definition 1, we have*

$$w(\lambda_1, n) \leq K_\alpha(\lambda_1) \ (n \in \mathbf{N}^{j_0}), \tag{20}$$
$$W(\lambda_2, x) < K_\beta(\lambda_1) \ (x \in \mathbf{R}_+^{i_0}), \tag{21}$$

where

$$K_\beta(\lambda_1) = \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} k(\lambda_1), K_\alpha(\lambda_1) = \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})} k(\lambda_1); \tag{22}$$

Proof Setting $v = \frac{U(x)}{||V_n||_\beta}$, since $U_k(\infty) \leq \infty$ ($k = 1, \ldots, i_0$), we find

$$w(\lambda_1, n) \leq \int_{\mathbf{R}_+^{i_0}} k_\lambda(||v||_\alpha, 1) \frac{1}{v^{i_0-\lambda_1}} dv.$$

By (14), it follows that

$$\begin{aligned}
&\int_{\mathbf{R}_+^{i_0}} k_\lambda(||v||_\alpha, 1)\frac{dv}{v^{i_0-\lambda_1}}\\
&= \lim_{M\to\infty}\int_{D_M}\frac{k_\lambda(M[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^\alpha]^{\frac{1}{\alpha}}, 1)dv_1\cdots dv_{i_0}}{\{M[\sum_{i=1}^{i_0}\left(\frac{v_i}{M}\right)^\alpha]^{\frac{1}{\alpha}}\}^{i_0-\lambda_1}}\\
&= \lim_{M\to\infty}\frac{M^{i_0}\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0}\Gamma(\frac{i_0}{\alpha})}\int_0^1\frac{k_\lambda(Mu^{\frac{1}{\alpha}}, 1)u^{\frac{i_0}{\alpha}-1}du}{(Mu^{1/\alpha})^{i_0-\lambda_1}}\\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\int_0^\infty k_\lambda(v, 1)v^{\lambda_1-1}dv\\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1).
\end{aligned}$$

Hence, we have (20).

By (12), (14) and Note 1, it follows that

$$\begin{aligned}
&W(\lambda_2, x)\\
&= \sum_n \int_{\{y\in\mathbf{R}_+^{j_0}; n_j-1<y_j\le n_j\}} k_\lambda(||U(x)||_\alpha, ||V(n)||_\beta)\frac{||U(x)||_\alpha^{\lambda_1}}{||V(n)||_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}\upsilon_l(y)dy\\
&< \sum_n \int_{\{y\in\mathbf{R}_+^{j_0}; n_j-1<y_j\le n_j\}} k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)\frac{||U(x)||_\alpha^{\lambda_1}}{||V(y)||_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}\upsilon_l(y)dy\\
&= \int_{\mathbf{R}_+^{j_0}} k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)\frac{||U(x)||_\alpha^{\lambda_1}}{||V(y)||_\beta^{j_0-\lambda_2}}\prod_{l=1}^{j_0}\upsilon_l(y)dy\ (v = V(y))\\
&\le \int_{\mathbf{R}_+^{j_0}} k_\lambda(||U(x)||_\alpha, ||v||_\beta)\frac{||U(x)||_\alpha^{\lambda_1}}{||v||_\beta^{j_0-\lambda_2}}dv.\\
&= \lim_{M\to\infty}\int_{\left\{v\in\mathbf{R}_+^{j_0}; \sum_{j=1}^{j_0}\left(\frac{v_j}{M}\right)^\alpha\le 1\right\}} k_\lambda(||U(x)||_\alpha, M[\sum_{j=1}^{j_0}(\frac{v_j}{M})^\beta]^{1/\beta})\\
&\quad\times\frac{||U(x)||_\alpha^{\lambda_1}}{\{M[\sum_{j=1}^{j_0}(\frac{v_j}{M})^\beta]^{\frac{1}{\beta}}\}^{j_0-\lambda_2}}dv_1\ldots dv_{j_0}
\end{aligned}$$

$$= \lim_{M\to\infty} \frac{M^{j_0}\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0}\Gamma(\frac{j_0}{\beta})} \int_0^1 k_\lambda(||U(x)||_\alpha, Mv^{1/\beta}) \frac{||U(x)||_\alpha^{\lambda_1} v^{\frac{j_0}{\beta}-1}}{(Mv^{\frac{1}{\beta}})^{j_0-\lambda_2}} dv$$

$$\overset{t=\frac{Mv^{1/\beta}}{||U(x)||_\alpha}}{=} \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \int_0^\infty k_\lambda(1,t) t^{\lambda_2-1} dt$$

$$= \frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} \int_0^\infty k_\lambda(v,1) v^{\lambda_1-1} dv.$$

Hence, we have (21).

The lemma is proved.

Note 2 If $U_k(\infty) = \infty \, (k = 1, \ldots, i_0)$, then we have $w(\lambda_1, n) = K_\alpha(\lambda_1) \quad (n \in \mathbf{N}^{j_0})$.

3 Main Results

For $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, setting functions

$$\Phi(x) := \frac{||U(x)||_\alpha^{p(i_0-\lambda_1)-i_0}}{(\prod_{k=1}^{i_0} \mu_k(x))^{p-1}} \quad (x \in \mathbf{R}_+^{i_0}),$$

$$\Psi(n) := \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0}}{(\prod_{l=1}^{j_0} \upsilon_n^{(l)})^{q-1}} \quad (n \in \mathbf{N}^{j_0}),$$

and the following normed spaces:

$$L_{p,\Phi}(\mathbf{R}_+^{i_0}) := \left\{ f = f(x); ||f||_{p,\Phi} := (\int_{\mathbf{R}_+^{i_0}} \Phi(x)|f(x)|^p dx)^{\frac{1}{p}} < \infty \right\},$$

$$l_{q,\Psi} := \left\{ b = \{b_n\}; ||b||_{q,\Psi} := (\sum_n \Psi(n)|b_n|^q)^{\frac{1}{q}} < \infty \right\},$$

$$L_{q,\Phi^{1-q}}(\mathbf{R}_+^{i_0}) := \left\{ g = g(x); ||g||_{q,\Phi^{1-q}} := (\int_{\mathbf{R}_+^{i_0}} \Phi^{1-q}(x)|g(x)|^q dx)^{\frac{1}{q}} < \infty \right\},$$

$$l_{p,\Psi^{1-p}} := \left\{ c = \{c_n\}; ||c||_{p,\Psi^{1-p}} := (\sum_n \Psi^{1-p}(n)|c_n|^p)^{\frac{1}{p}} < \infty \right\},$$

we have

Theorem 1 *With regard to the assumptions of Definition 1, if* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $f(x), b_n \geq 0$, $f = f(x) \in L_{p,\Phi}(\mathbf{R}_+^{i_0})$, $b = \{b_n\} \in l_{q,\Psi}$, $||f||_{p,\Phi}, ||b||_{q,\Psi} > 0$, *then we have the following equivalent inequalities:*

$$I := \sum_n b_n \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi} ||b||_{q,\Psi}, \tag{23}$$

$$J_1 := \left\{ \sum_n \frac{\prod_{j=1}^{j_0} \upsilon_n^{(j)}}{||V_n||_\beta^{j_0 - p\lambda_2}} \left[\int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx \right]^p \right\}^{\frac{1}{p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi}, \tag{24}$$

$$J_2 := \left\{ \int_{\mathbf{R}_+^{i_0}} \frac{\prod_{i=1}^{i_0} \mu_i(x)}{||U(x)||_\alpha^{i_0 - q\lambda_1}} \left[\sum_n k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) b_n \right]^q dx \right\}^{\frac{1}{q}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||b||_{q,\Psi} \tag{25}$$

where

$$K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0 - 1} \Gamma(\frac{j_0}{\beta})} \right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0 - 1} \Gamma(\frac{i_0}{\alpha})} \right]^{\frac{1}{q}} k(\lambda_1).$$

Proof By Hölder's inequality with weight (cf. [46]), we have

$$I = \sum_n \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) \left[\frac{||U(x)||_\alpha^{\frac{i_0 - \lambda_1}{q}} (\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{\frac{1}{p}} f(x)}{||V_n||_\beta^{\frac{j_0 - \lambda_2}{p}} (\prod_{i=1}^{i_0} \mu_k(x))^{\frac{1}{q}}} \right] \times \left[\frac{||V_n||_\beta^{\frac{j_0 - \lambda_2}{p}} (\prod_{i=1}^{i_0} \mu_k(x))^{\frac{1}{q}} b_n}{||U(x)||_\alpha^{\frac{i_0 - \lambda_1}{q}} (\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{\frac{1}{p}}} \right] dx$$

$$\leq \left[\int_{\mathbf{R}_+^{i_0}} W(\lambda_2, x) \frac{||U(x)||_\alpha^{p(i_0-\lambda_1)-i_0} f^p(x)}{(\prod_{i=1}^{i_0} \mu_i(x))^{p-1}} dx \right]^{\frac{1}{p}}$$

$$\times \left[\sum_n w(\lambda_1, n) \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0} b_n^q}{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})} \right]^{\frac{1}{q}}.$$

Then by (20) and (21), we have (23). We set

$$b_n := \frac{\prod_{j=1}^{j_0} \upsilon_n^{(j)}}{||V_n||_\beta^{j_0 - p\lambda_2}} \left[\int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx \right]^{p-1}, n \in \mathbf{N}^{j_0}.$$

Then we have $J_1 = ||b||_{q,\Psi}^{q-1}$. Since the right-hand side of (24) is finite, it follows $J_1 < \infty$. If $J_1 = 0$, then (24) is trivially valid; if $J_1 > 0$, then by (23), we have

$$||b||_{q,\Psi}^q = J_1^p = I < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi} ||b||_{q,\Psi},$$

$$||b||_{q,\Psi}^{q-1} = J_1 < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi},$$

namely, (24) follows.

On the other hand, assuming that (24) is valid, by Hölder's inequality (cf. [46]), we have

$$I = \sum_n \left[\frac{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{1/p}}{||V_n||_\beta^{(j_0/p)-\lambda_2}} \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx \right]$$

$$\times \frac{||V_n||_\beta^{(j_0/p)-\lambda_2}}{(\prod_{j=1}^{j_0} \upsilon_n^{(j)})^{1/p}} b_n \leq J_1 ||b||_{q,\Psi}. \tag{26}$$

Then by (24), we have (23), which is equivalent to (24).

In the same way, we still can find

$$I \leq ||f||_{p,\Phi} J_2, \tag{27}$$

and prove that (23) is equivalent to (25). Therefore, (23), (24), and (25) are equivalent.

The theorem is proved.

Theorem 2 *With regard to the assumptions of Theorem 1, if* $\upsilon_n^{(j)} \geq \upsilon_{n+1}^{(j)}$ $(n \in \mathbf{N})$, $U_i(\infty) = V_\infty^{(j)} = \infty$ $(i = 1, \ldots, i_0, j = 1, \ldots, j_0)$, *then the constant factor* $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ *in (23), (24), and (25) is the best possible.*

Proof For $\varepsilon > 0$, we set

$$\widetilde{f} = \widetilde{f}(x),\ \widetilde{f}(x) := \begin{cases} 0,\ x \in \mathbf{R}_+^{i_0} \backslash [1,\infty)^{i_0} \\ ||U(x)||_\alpha^{-i_0+\lambda_1-\frac{\varepsilon}{p}} \prod_{i=1}^{i_0} \mu_i(x),\ x \in [1,\infty)^{i_0} \end{cases},$$

$$\widetilde{b} = \{\widetilde{b}_n\},\ \widetilde{b}_n := ||V_n||_\beta^{-j_0+\lambda_2-\frac{\varepsilon}{q}} \prod_{j=1}^{j_0} \upsilon_n^{(j)}\ \ (n \in \mathbf{N}^{j_0}).$$

Then by (15) and (16), we obtain

$$\begin{aligned} & ||\widetilde{f}||_{p,\Phi}||\widetilde{b}||_{q,\Psi} \\ &= \left[\int_{\mathbf{R}_+^{i_0}} \frac{||U(x)||_\alpha^{p(i_0-\lambda_1)-i_0}\widetilde{f}^p(x)}{(\prod_{i=1}^{i_0}\mu_i(x))^{p-1}}dx\right]^{\frac{1}{p}} \left[\sum_n \frac{||V_n||_\beta^{q(j_0-\lambda_2)-j_0}\widetilde{b}_n^q}{(\prod_{j=1}^{j_0}\upsilon_n^{(j)})^{q-1}}\right]^{\frac{1}{q}} \\ &= \left(\int_{[1,\infty)^{i_0}} ||U(x)||_\alpha^{-i_0-\varepsilon}\prod_{i=1}^{i_0}\mu_i(x)dx\right)^{\frac{1}{p}} \left(\sum_n ||V_n||_\beta^{-j_0-\varepsilon}\prod_{j=1}^{j_0}\upsilon_n^{(j)}\right)^{\frac{1}{q}} \\ &\leq \frac{1}{\varepsilon}\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right)^{\frac{1}{p}} \left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})} + \varepsilon O(1)\right)^{\frac{1}{q}}. \end{aligned}$$

By (19), since $\upsilon_n^{(j)} \geq \upsilon_{n+1}^{(j)} = \upsilon_j(y)(n_j < y_j < n_j+1)$, we find

$$\begin{aligned} \widetilde{I} :&= \sum_n \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta)\widetilde{f}(x)\widetilde{b}_n dx \\ &= \sum_n \int_{[1,\infty)^{i_0}} \int_{\{y\in\mathbf{R}_+^{j_0};n_j\leq y_j<n_j+1\}} \frac{k_\lambda(||U(x)||_\alpha, ||V_n||_\beta)}{||U(x)||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}}||V_n||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \\ &\quad \times \prod_{j=1}^{j_0}\upsilon_j(y)\prod_{i=1}^{i_0}\mu_k(x)dydx \end{aligned}$$

$$> \int_{[1,\infty)^{i_0}} \sum_n \int_{\{y \in \mathbf{R}_+^{j_0}; n_j \le y_j < n_j+1\}} \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta)}{||U(x)||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} ||V(y)||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}}$$

$$\times \prod_{l=1}^{j_0} \upsilon_l(y) \prod_{k=1}^{i_0} \mu_k(x) dy dx$$

$$= \int_{[1,\infty)^{i_0}} \int_{[1,\infty)^{j_0}} \frac{k_\lambda(||U(x)||_\alpha, ||V(y)||_\beta) \prod_{l=1}^{j_0} \upsilon_l(y)}{||U(x)||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} ||V(y)||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} \prod_{k=1}^{i_0} \mu_k(x) dy dx.$$

Setting $u = U(x), v = V(y)$, $c := \max_{1 \le i \le i_0, 1 \le j \le j_0} \{\mu_1^{(i)}, \upsilon_1^{(j)}\}$, since $U_\infty^{(k)} = V_\infty^{(l)} = \infty$, we have

$$\widetilde{I} > \int_{[c,\infty)^{i_0}} \int_{[c,\infty)^{j_0}} \frac{k_\lambda(||u||_\alpha, ||v||_\beta)}{||u||_\alpha^{i_0-\lambda_1+\frac{\varepsilon}{p}} ||v||_\beta^{j_0-\lambda_2+\frac{\varepsilon}{q}}} dv du$$

$$= \int_{[c,\infty)^{i_0}} \int_{[c,\infty)^{j_0}} \frac{k_\lambda(M_1[\sum_{i=1}^{i_0}(\frac{x_i}{M_1})^\alpha]^{\frac{1}{\alpha}}, M_2[\sum_{j=1}^{j_0}(\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}}) dy dx}{\{M_1[\sum_{i=1}^{i_0}(\frac{x_i}{M_1})^\alpha]^{\frac{1}{\alpha}}\}^{i_0-\lambda_1+\frac{\varepsilon}{p}} \{M_2[\sum_{j=1}^{j_0}(\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}}\}^{j_0-\lambda_2+\frac{\varepsilon}{q}}}.$$

For $M_1 > c i_0^{1/\alpha}$, $M_2 > c j_0^{1/\beta}$, we put

$$\Psi_1(u) = \begin{cases} 0, 0 < u \le \frac{c^\alpha i_0}{M_1^\alpha}, \\ k_\lambda(M_1 u^{1/\alpha}, M_2[\sum_{j=1}^{j_0}(\frac{y_j}{M_2})^\beta]^{\frac{1}{\beta}}) \frac{1}{(M_1 u^{1/\alpha})^{i_0-\lambda_1}}, \frac{c^\alpha i_0}{M_1^\alpha} < u \le 1 \end{cases},$$

$$\Psi_2(v) = \begin{cases} 0, 0 < v \le \frac{c^\beta j_0}{M_2^\beta}, \\ k_\lambda(M_1 u^{1/\alpha}, M_2 v^{1/\beta}) \frac{1}{(M_2 v^{1/\beta})^{j_0-\lambda_2}}, \frac{c^\beta j_0}{M_2^\beta} < v \le 1 \end{cases},$$

By (14) twice, it follows that

$$\widetilde{I} > \lim_{M_1 \to \infty} \lim_{M_2 \to \infty} \frac{M_1^{i_0} \Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0} \Gamma(\frac{i_0}{\alpha})} \frac{M_2^{j_0} \Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0} \Gamma(\frac{j_0}{\beta})} \int_{c^\alpha i_0/M_1^\alpha}^1 u^{\frac{i_0}{\alpha}-1}$$

$$\times \left[\int_{c^\beta j_0/M_2^\beta}^1 \frac{k_\lambda(M_1 u^{\frac{1}{\alpha}}, M_2 v^{\frac{1}{\beta}}) v^{\frac{j_0}{\beta}-1}}{(M_1 u^{\frac{1}{\alpha}})^{i_0-\lambda_1+\frac{\varepsilon}{p}} (M_2 v^{\frac{1}{\beta}})^{j_0-\lambda_2+\frac{\varepsilon}{q}}} dv \right] du.$$

Setting $x = M_1 u^{\frac{1}{\alpha}}$, $y = M_2 v^{\frac{1}{\beta}}$ in the above, we find

$$
\begin{aligned}
\widetilde{I} &> \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})} \\
&\quad\times\int_{ci_0^{1/\alpha}}^{\infty} x^{\lambda_1-\frac{\varepsilon}{p}-1}\left(\int_{cj_0^{1/\beta}}^{\infty} k_\lambda(x,y)y^{\lambda_2-\frac{\varepsilon}{q}-1}dy\right)dx \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})} \\
&\quad\times\int_{ci_0^{1/\alpha}}^{\infty} x^{-\varepsilon-1}\left(\int_{0}^{x/cj_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})} \\
&\quad\times\left[\int_{ci_0^{1/\alpha}}^{\infty} x^{-\varepsilon-1}\left(\int_{0}^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx\right. \\
&\quad\left.+\int_{ci_0^{1/\alpha}}^{\infty} x^{-\varepsilon-1}\left(\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^{x/cj_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right)dx\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})} \\
&\quad\times\left[\frac{1}{\varepsilon(ci_0^{1/\alpha})^{\varepsilon}}\int_{0}^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right. \\
&\quad\left.+\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^{\infty}\left(\int_{cj_0^{1/\beta}v}^{\infty} x^{-\varepsilon-1}dx\right)k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right] \\
&= \frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\varepsilon\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})} \\
&\quad\times\left[\frac{1}{(ci_0^{1/\alpha})^{\varepsilon}}\int_{0}^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right. \\
&\quad\left.+\frac{1}{(cj_0^{1/\beta})^{\varepsilon}}\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^{\infty} k_\lambda(v,1)v^{\lambda_1-\frac{\varepsilon}{p}-1}dv\right].
\end{aligned}
$$

If there exists a constant $K \le K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$, such that (23) is valid when replacing $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ by K, then we have $\varepsilon\widetilde{I} < \varepsilon K||\widetilde{a}||_{p,\Phi}||\widetilde{b}||_{q,\Psi}$, namely,

$$\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{i_0-1}\Gamma(\frac{j_0}{\beta})}\left[\frac{1}{(ci_0^{1/\alpha})^\varepsilon}\int_0^{i_0^{1/\alpha}/j_0^{1/\beta}} k_\lambda(v,1)v^{\lambda_1+\frac{\varepsilon}{q}-1}dv\right.$$

$$\left.+\frac{1}{(cj_0^{1/\beta})^\varepsilon}\int_{i_0^{1/\alpha}/j_0^{1/\beta}}^\infty k_\lambda(v,1)v^{\lambda_1-\frac{\varepsilon}{p}-1}dv\right]$$

$$< K\left(\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{b^\varepsilon i_0^{\varepsilon/\alpha}\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right)^{\frac{1}{p}}\left(\frac{\Gamma^{j_0}(\frac{1}{\beta})}{b^\varepsilon j_0^{\varepsilon/\beta}\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}+\varepsilon O(1)\right)^{\frac{1}{q}}.$$

For $\varepsilon \to 0^+$, in view of Fatou lemma (cf. [47]), we find

$$\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}k(\lambda_1)$$

$$\le K\left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{p}}\left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{q}},$$

and then $K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1) \le K$. Hence, $K = K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)$ is the best possible constant factor of (23). The constant factor in (24) ((25)) is still the best possible, otherwise, we would reach a contradiction by (26) ((27)) that the constant factor in (23) is not the best possible.

The theorem is proved.

In particular, for $\mu_i(t) = 1$ $(i = 1,\ldots,i_0)$, $\upsilon_j^{(l)} = 1$ $(l = 1,\ldots,j_0;\, j = 1,\ldots,n)$ in Theorem 1-2, setting

$$\varphi(x) := ||x||_\alpha^{p(i_0-\lambda_1)-i_0}\ (x \in \mathbf{R}_+^{i_0}),\ \psi(n) := ||n||_\beta^{q(j_0-\lambda_2)-j_0}\ (n \in \mathbf{N}^{j_0}),$$

we have

Corollary 1 *With regard to the assumptions of Definition 1, if* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $f(x), b_n \ge 0$, $f = f(x) \in L_{p,\varphi}(\mathbf{R}_+^{i_0})$, $b = \{b_n\} \in l_{q,\psi}$, $||f||_{p,\varphi}, ||b||_{q,\psi} > 0$, *then we have the following equivalent inequalities:*

$$\sum_n b_n \int_{\mathbf{R}_+^{i_0}} k_\lambda(||x||_\alpha, ||n||_\beta)f(x)dx < K_\beta^{\frac{1}{p}}(\lambda_1)K_\alpha^{\frac{1}{q}}(\lambda_1)||f||_{p,\varphi}||b||_{q,\psi}, \tag{28}$$

$$\left\{\sum_n \frac{1}{||n||_\beta^{j_0-\lambda_2}}\left[\int_{\mathbf{R}_+^{i_0}} k_\lambda(||x||_\alpha, ||n||_\beta) f(x)dx\right]^p\right\}^{\frac{1}{p}}$$

$$< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)||f||_{p,\varphi}, \tag{29}$$

$$\left\{\int_{\mathbf{R}_+^{i_0}} \frac{1}{||x||_\alpha^{i_0-q\lambda_1}}\left[\sum_n k_\lambda(||x||_\alpha, ||n||_\beta) b_n\right]^q dx\right\}^{\frac{1}{q}} \tag{30}$$

$$< K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)||b||_{q,\psi} \tag{31}$$

where the constant factor

$$K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$$

$$= \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\beta^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1)$$

is the best possible.

Corollary 2 *With regard to the assumptions of Definition 1 (for* $i_0 = j_0 = 1$*), setting*

$$\Phi_1(x) := \frac{(U_1(x))^{p(1-\lambda_1)-1}}{(\mu_1(x))^{p-1}},$$

$$\Psi_1(n) := \frac{(V_n^{(1)})^{q(1-\lambda_2)-1}}{(\upsilon_n^{(1)})^{q-1}} \ (x \in \mathbf{R}_+, n \in \mathbf{N}),$$

if $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), b_n \geq 0, f = f(x) \in L_{p,\Phi_1}(\mathbf{R}_+), b = \{b_n\} \in l_{q,\Psi_1}, ||f||_{p,\Phi_1}, ||b||_{q,\Psi_1} > 0$, *then we have the following equivalent inequalities:*

$$\sum_{n=1}^\infty b_n \int_0^\infty k_\lambda(U_1(x), V_n^{(1)}) f(x)dx < k(\lambda_1)||f||_{p,\Phi_1}||b||_{q,\Psi_1}, \tag{32}$$

$$\left\{\sum_{n=1}^\infty \frac{\upsilon_n^{(1)}}{(V_n^{(1)})^{1-p\lambda_2}}\left[\int_0^\infty k_\lambda(U_1(x), V_n^{(1)}) f(x)dx\right]^p\right\}^{\frac{1}{p}} < k(\lambda_1)||f||_{p,\Phi_1}, \tag{33}$$

$$\left\{\int_0^\infty \frac{\mu_1(x)}{(U_1(x))^{1-q\lambda_1}} \left[\sum_{n=1}^\infty k_\lambda(U_1(x), V_n^{(1)}) b_n\right]^q dx\right\}^{\frac{1}{q}} < k(\lambda_1)||b||_{q,\Psi_1}. \quad (34)$$

Moreover, if $\upsilon_n^{(1)} \geq \upsilon_{n+1}^{(1)}$ $(n \in \mathbf{N})$, $U_1(\infty) = V_\infty^{(1)} = \infty$, then the constant factor $k(\lambda_1)$ is the best possible.

Remark 2 For $i_0 = j_0 = 1$ $(\mu_1(t) = 1, \upsilon_j^{(1)} = 1 (j = 1, \ldots, n))$, (28) ((32)) reduces to (6). Hence, (28) ((32)) is an extension of (6). So is (23).

4 Operator Expressions

With regard to the assumptions of Theorem 2, in view of

$$c_n := \frac{\prod_{j=1}^{j_0} \upsilon_n^{(j)}}{||V_n||_\beta^{j_0 - p\lambda_2}} \left[\int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx\right]^{p-1}, n \in \mathbf{N}^{j_0},$$

$$c = \{c_n\}, ||c||_{p,\Psi^{1-p}} = J_1 < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi} < \infty,$$

we can set the following definition:

Definition 2 Define a half-discrete multidimensional Hardy–Hilbert-type operator $T_1 : L_{p,\Phi}(\mathbf{R}_+^{i_0}) \to l_{p,\Psi^{1-p}}$ as follows: For any $f \in L_{p,\Phi}(\mathbf{R}_+^{i_0})$, there exists a unique representation $T_1 f = c \in l_{p,\Psi^{1-p}}$, satisfying

$$T_1 f(n) := \int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx \ (n \in \mathbf{N}^{j_0}). \quad (35)$$

For $b \in l_{q,\Psi}$, we define the following formal inner product of $T_1 f$ and b as follows:

$$(T_1 f, b) := \sum_n \left[\int_{\mathbf{R}_+^{i_0}} k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) f(x) dx\right] b_n. \quad (36)$$

Then by (23) and (24), we have the following equivalent inequalities:

$$(T_1 f, b) < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi} ||b||_{q,\Psi}, \quad (37)$$

$$||T_1 f||_{p,\Psi^{1-p}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) ||f||_{p,\Phi}. \quad (38)$$

It follows that T_1 is bounded with

$$||T_1|| := \sup_{f(\neq\theta)\in L_{p,\Phi}(\mathbf{R}_+^{i_0})} \frac{||T_1 f||_{p,\Psi^{1-p}}}{||f||_{p,\Phi}} \leq K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{39}$$

By Theorem 2, the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (38) is the best possible, we have

$$\begin{aligned} ||T_1|| &= K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \\ &= \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1). \end{aligned} \tag{40}$$

With regard to the assumptions of Theorem 2, in view of

$$g(x) \ : = \frac{\prod_{i=1}^{i_0} \mu_i(x)}{||U(x)||_\alpha^{i_0-q\lambda_1}} \left[\sum_n k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) b_n\right]^{q-1}, x \in \mathbf{R}_+^{i_0},$$

$$g = g(x), ||g||_{q,\Phi^{1-q}} = J_2 < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)||b||_{q,\Psi} < \infty,$$

we can set the following definition:

Definition 3 Define a half-discrete multidimensional Hardy–Hilbert-type operator $T_2 : l_{q,\Psi} \to L_{q,\Phi^{1-q}}(\mathbf{R}_+^{i_0})$ as follows: For any $b \in l_{q,\Psi}$, there exists a unique representation $T_2 b = g \in L_{q,\Phi^{1-q}}(\mathbf{R}_+^{i_0})$, satisfying

$$T_2 b(x) := \sum_n k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) b_n \ (x \in \mathbf{R}_+^{i_0}). \tag{41}$$

For $f \in L_{p,\Phi}(\mathbf{R}_+^{i_0})$, we define the following formal inner product of $T_2 b$ and f as follows:

$$(f, T_2 b) := \int_{\mathbf{R}_+^{i_0}} f(x) \left[\sum_n k_\lambda(||U(x)||_\alpha, ||V_n||_\beta) b_n\right] dx. \tag{42}$$

Then by (23) and (25), we have the following equivalent inequalities:

$$(f, T_2 b) < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)||f||_{p,\Phi}||b||_{q,\Psi}, \tag{43}$$

$$||T_2 b||_{q,\Phi^{1-q}} < K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)||b||_{q,\Psi}. \tag{44}$$

It follows that T_2 is bounded with

$$||T_2|| := \sup_{b(\neq\theta)\in l_{q,\Psi}} \frac{||T_2 b||_{q,\Phi^{1-q}}}{||b||_{q,\Psi}} \leq K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1). \tag{45}$$

By Theorem 2, the constant factor $K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1)$ in (44) is the best possible, we have

$$\begin{aligned} ||T_2|| &= K_\beta^{\frac{1}{p}}(\lambda_1) K_\alpha^{\frac{1}{q}}(\lambda_1) \\ &= \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} k(\lambda_1). \end{aligned} \tag{46}$$

Example 2

(i) In view of Example 1, by (40) and (46), for $k_\lambda(x, y) = \frac{(\min\{x,y\})^\eta}{(\max\{x,y\})^{\lambda+\eta}}$, we have

$$\begin{aligned} ||T_1|| = ||T_2|| &= \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \\ &\times \frac{\lambda + 2\eta}{(\lambda_1+\eta)(\lambda_2+\eta)}. \end{aligned}$$

(ii) For $k_\lambda(x, y) = \frac{1}{x^\lambda + y^\lambda}$ $(\lambda_1 > 0, 0 < \lambda_2 \leq j_0, \lambda_1 + \lambda_2 = \lambda)$, we find $k(\lambda_1) = \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}$, and then by (40) and (46), we have

$$\begin{aligned} ||T_1|| = ||T_2|| &= \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}} \\ &\times \frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}. \end{aligned}$$

(iii) For $k_\lambda(x, y) = \frac{\ln(x/y)}{x^\lambda - y^\lambda}$ $(\lambda_1 > 0, 0 < \lambda_2 \leq j_0, \lambda_1 + \lambda_2 = \lambda)$, we find $k(\lambda_1) = [\frac{\pi}{\lambda \sin(\frac{\pi\lambda_1}{\lambda})}]^2$, and then by (40) and (46), we have

$$||T_1|| = ||T_2|| = \left[\frac{\Gamma^{j_0}(\frac{1}{\beta})}{\beta^{j_0-1}\Gamma(\frac{j_0}{\beta})}\right]^{\frac{1}{p}} \left[\frac{\Gamma^{i_0}(\frac{1}{\alpha})}{\alpha^{i_0-1}\Gamma(\frac{i_0}{\alpha})}\right]^{\frac{1}{q}}$$

$$\times \left[\frac{\pi}{\lambda \sin(\frac{\pi \lambda_1}{\lambda})} \right]^2 .$$

Remark 3 For $0 < \lambda_1 + \eta \leq i_0, 0 < \lambda_2 + \eta \leq j_0$,

$$k_\lambda(x, y) = \frac{(\min\{x, y\})^\eta}{(\max\{x, y\})^{\lambda+\eta}} \ (x, y > 0),$$

(23) reduces to (23) in [48], which is an extension of (4) for $\mu_i = \upsilon_j = 1 \ (i, j \in \mathbf{N})$.

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