Approximation Algorithms for the Minimum Power Partial Cover Problem

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Abstract. In this paper, we study the minimum power partial cover problem (MinPowerPartCov). Suppose X is a set of points and $\mathcal{S}$ is a set of sensors on the plane, each sensor can adjust its power, the covering range of a sensor s with power $p(s)$ is a disk centered at s which has radius $r(s)$ satisfying $p(s) = c \cdot r(s)^\alpha$. Given an integer $k \leq |X|$, the MinPowerPartCov problem is to determine the power assignment on each sensor such that at least k points are covered and the total power consumption is the minimum. We present an approximation algorithm with approximation ratio 3^α, using a local ratio method, which coincides with the best known ratio for the minimum power (full) cover problem. Compared with the paper [9] which studies the MinPowerPartCov problem for $\alpha = 2$, our ratio improves their ratio from $12 + \varepsilon$ to 9.

Keywords: Power · Partial cover · Approximation algorithm · Local ratio

1 Introduction

With the rapid development of wireless sensor networks (WSNs), intensive studies on WSNs have emerged, especially on the coverage problem. In a coverage problem, the most basic requirement is to keep all points under monitoring. In a typical WSN, the service area of a sensor is a disk centered at the sensor whose radius is determined by the power of the sensor. A typical relation between the power $p(s)$ of sensor s and the radius $r(s)$ of its service area is

$$p(s) = c \cdot r(s)^\alpha, \tag{1}$$

where c and $\alpha \geq 1$ are some constants (α is usually called the *attenuation facor*). So, the greater power a sensor possesses, the larger service it can provide. In other words, the consumption of energy and the quality of service are two conflicting factors. The question is how to balance these two conflicting factors by adjusting power at the sensors so that the desired service can be accomplished using the minimum total power. This question is motivated by the intention to extend the lifetime of WSN under limited energy supply, and we call it the *minimum power coverage* problem (MinPowerCov).

D.-Z. Du et al. (Eds.): AAIM 2019, LNCS 11640, pp. 179–191, 2019.
https://doi.org/10.1007/978-3-030-27195-4_17

In the real world, it is often too costly to satisfy the covering requirement of every point. So, it is beneficial to study the *minimum power partial coverage* problem (MinPowerPartCov), in which it is sufficient to cover at least k (instead of all) points. The problem is motivated by the purpose of further saving energy while keeping an acceptable quality of service.

The MinPowerPartCov problem can be viewed as a special case of the *minimum weight partial set cover* problem (MinWPSC). Given a set E of elements, a collection of sets $\mathcal{S}$, a weight function $w : \mathcal{S} \mapsto \mathbb{R}^+$, and an integer $k \leq |E|$, the MinWPSC problem is to find the minimum weight sub-collection of sets $\mathcal{F} \subseteq \mathcal{S}$ such that at least k elements are covered by $\mathcal{F}$, i.e., $|\bigcup_{S\in\mathcal{F}} S| \geq k$ and $w(\mathcal{F}) = \sum_{S\in\mathcal{F}} w(S)$ is minimum. Notice that in a MinPowerParCov problem, the power at a sensor can be discretized by assuming that there is a point on the boundary of the disk supported by the assigned power. We call such a disk as a *canonical disk*. So, if we associate with each sensor $|X|$ canonical disks, where X is the set of points, each disk corresponds to the set of points contained in it, and the weight of the disk equals the power supporting the disk which is determined by Eq. (1), then the MinPowerParCov problem is reduced to the MinWPSC problem.

It is known that the MinWPSC problem has an f-approximation [2], where f is the maximum frequency of an element, that is, the maximum number of sets containing a common element. For the MinWPSC problem obtained by the above reduction from a MinPowerParCov problem, f equals the number of sensors, which is too large to be a good approximation factor. So, the main purpose of this paper is to explore geometric properties of the MinPowerParCov problem to obtain a better approximation.

1.1 Related Works

The *minimum weight set cover* problem (MinWSC) is a classic combinatorial problem. It is well-known that MinSC admits approximation ratio $H(\Delta)$ [7,14], where $H(\Delta) = 1 + \frac{1}{2} + ... + \frac{1}{\Delta}$ is the Harmonic number and Δ denotes the size of the largest set. It is also known that a simple LP-rounding algorithm can achieve an approximation ratio of f, where f is the maximum number of sets containing a common element (see for example Chapter 12 of the book [23]).

For the *minimum weight partial set cover* problem (MinWPSC), Slavík [21] obtained an $H(\min\{\lceil k\rceil, \Delta\})$-approximation using a greedy strategy, Bar-Yehuda [2] obtained an f-approximation using local ratio method, Gandhi [10] also obtained f approximation using a primal-dual method. Very recently, Inamdar *et al.* [13] designed an LP-rounding algorithm, obtaining approximation ratio $2\beta + 2$, where β is the integrality gap for the natural linear program of the minimum weight (full) set cover problem.

For the *geometric minimum weight set cover* problem, much better approximation factors can be achieved. Using a partition and shifting method, Hochbaum *et al.* [12] obtained a PTAS for the minimum unit disk cover problem in which the disks are uniform and there are no prefixed locations for the disks. For the minimum disk cover problem in which disks may have different sizes, Mustafa *et al.* [15] designed a PTAS using a local search method. This PTAS was generalized by Roy *et al.* [20] to non-piercing regions including pseudo-disks. These are results for the cardinality version of the geometric set cover problem. Considering weight, Varadarajan [22] presented a clever quasi-uniform sampling technique, which was improved by Chan *et al.* [8], yielding a constant approximation for the minimum weight disk cover problem. This constant approximation was generalized by Bansal *et al.* [4] for the minimum weight disk multi-cover problem in which every point has to be covered multiple times. Using a separator framework, Mustafa *et al.* [16] obtained a quasi-PTAS for the minimum weight disk cover problem.

To our knowledge, there are two papers studying the *geometric minimum partial set cover* problem. The first paper is [10], in which Gandhi *et al.* presented a PTAS for the minimum (cardinality) partial unit disk cover problem using a partition and shifting method. Notice that this result only works for the case when the centers of the disks are not prefixed. Another paper is due to Inamdar *et al.* [13], in which a $(2\beta + 2)$-approximation was obtained for the *general* minimum weight partial set cover problem, where β is the integrality gap of the natural linear program for the minimum weight (full) set cover problem. As a consequence, for those geometric set cover problems (including the disk cover problem) in which β is a constant, the approximation ratio for the partial version is also a constant (but the constant is large).

Recently, there are a lot of works studying the *minimum power multi-cover* problem (MinPowerMC), in which every point p is associated with a covering requirement cr_p, and the goal is to find a power assignment with the minimum total power such that every point p is covered by at least cr_p disks. Let cr_{max} be the maximum number of times that a point requires to be covered. Using a local ratio method, Bar-Yehuda *et al.* [3] presented a $3^\alpha \cdot cr_{max}$-approximation algorithm. The dependence on $cr_{\max}$ was removed by Bhowmick *et al.* [5], achieving an approximation ratio of $4 \cdot (27\sqrt{2})^\alpha$. This result was further generalized to any metric space in [6], the approximation ratio is at most $2 \cdot (16 \cdot 9)^\alpha$. For the minimum power (*single*) cover problem, the best known ratio is 3^α (as a consequence of [3] and the fact $cr_{\max} = 1$ in this case).

There is only one paper [9] studying the minimum power *partial* (single) cover problem (MinPowerPartCov), and the study is on the special case when $\alpha = 2$. The approximation ratio obtained in [9] is $(12+\varepsilon)$, where ε is an arbitrary constant greater than zero, by a reduction to a prize-collecting coverage problem.

1.2 Contribution

In this paper, we show that the MinPowerPartCov problem can be approximated within factor 3^α, which coincides with the best known ratio for the MinPowerCov problem (the full version of the minimum power coverage problem). When applied to the case when $\alpha = 2$, our ratio is 9, which is better than $12 + \varepsilon$ obtained in [9].

Our algorithm is inspired by the local ratio method used in [3] to study the MinPowerCov problem. New ideas have to be explored to surmount the difficulty.

2 The Problem and a Preprocessing

The problem studied in this paper is formally defined as follows.

Definition 1 (Minimum Power Partial Cover (MinPowerPartCov)). *Suppose X is a set of n points and S is a set of m sensors on the plane, k is an integer satisfying $0 \leq k \leq n$. A point $x \in X$ is covered by a sensor $s \in S$ with power $p(s)$ if x belongs to the disk supported by $p(s)$, that is $x \in Disk(s, r(s))$, where $Disk(s, r(s))$ is the disk centered at s whose radius $r(s)$ is determined by $p(s)$ through equation $p(s) = c \cdot r(s)^\alpha$. A point is covered by a power assignment $p : S \mapsto \mathbb{R}^+$ if it is covered by some disk supported by p. The goal of MinPowerPartCov is to find a power assignment p covering at least k points such that the total power $\sum_{s \in S} p(s)$ is as small as possible.*

In an optimal solution, we may assume that for any sensor s, there is at least one point that is on the boundary of the disk $Disk(s, p(s))$, since otherwise we may reduce $p(s)$ to cover the same set of points, resulting in a lower power. Therefore, at most mn disks need to be considered. We denote the set of such disks by $\mathcal{D}$. In the following, denote by $(X, \mathcal{D}, k)$ an instance of the MinPowerPartCov problem, and use $opt(X, \mathcal{D}, k)$ to denote the optimal power for the instance $(X, \mathcal{D}, k)$. To simplify the notation, we use D to represent both a disk in $\mathcal{D}$ and the set of points covered by D, and use $r(D)$ and $p(D)$ to denote the radius and the power of disk D, where $p(D) = c \cdot r(D)^\alpha$. For a set of disks $\mathcal{D}$, we shall use $\mathcal{C}(\mathcal{D}) = \bigcup_{D \in \mathcal{D}} D$ to denote the set of points covered by $\mathcal{D}$.

In order to control the approximation factor of our algorithm, we need a preprocessing step: guessing the maximum power of a sensor (or equivalently, the radius of a maximum disk) in an optimal solution. Suppose $D_{\max} \in \mathcal{D}$ is the guessed disk. Denote by $\mathcal{D}_{\leq r(D_{\max})}$ the subset of disks of $\mathcal{D}$ whose radii are no greater than the radius of $D_{\max}$ (excluding $D_{\max}$), and denote by $(X \setminus D_{\max}, \mathcal{D}_{\leq r(D_{\max})}, k - |D_{\max}|)$ the *residual instance* after guessing $D_{\max}$. The following lemma is obvious.

Lemma 1. *Suppose $D_{\max}$ is the correctly guessed disk with the maximum power in an optimal solution of instance $(X, \mathcal{D}, k)$. Then*

$$opt(X, \mathcal{D}, k) = opt(X \setminus D_{\max}, \mathcal{D}_{\leq r(D_{\max})}, k - |D_{\max}|) + p(D_{\max}).$$

3 A Local Ratio Algorithm

In this section, we first present an algorithm for the MinPowerPartCov problem on the instance $(X \setminus D_{\max}, \mathcal{D}_{\leq r(D_{\max})}, k - |D_{\max}|)$. And then show how to make use of it to find a power assignment for the original MinPowerPartCov problem.

3.1 Algorithm After the Preprocessing

For simplicity of notation in this section, we still use $(X, \mathcal{D}, k)$ to denote the residual instance, assuming that every disk in $\mathcal{D}$ has radius at most $r(D_{\max})$.

The algorithm consists of three steps.

(i) In the first step, a local ratio method is employed to find a *minimal partial cover* $\bar{\mathcal{D}}$, that is, $\bar{\mathcal{D}}$ covers at least k points, while for any disk $D \in \bar{\mathcal{D}}$, the number of points covered by $\bar{\mathcal{D}} - \{D\}$ is strictly less than k.

(ii) Before going into the second step, remove a disk D_{rmv} from $\bar{\mathcal{D}}$ which is chosen in the last call of the local ratio method in the first step. Then, in the second step, a *maximal independent set of disks* $\mathcal{I} \subseteq \bar{\mathcal{D}} \setminus \{D_{rmv}\}$ is computed in a greedy manner, that is, disks in $\mathcal{I}$ are mutually disjoint, while any disk $D \in \bar{\mathcal{D}} \setminus \{D_{rmv}\}$ which is not picked into $\mathcal{I}$ intersects some disk in $\mathcal{I}$.

(iii) In the third step, every disk in $\mathcal{I}$ has its radius enlarged three times. Such set of disks together with $\{D_{\max}, D_{rmv}\}$ are the output of the algorithm.

The first step is accomplished by Algorithm 1, in which the MinPowerPartCov instance $(X, \mathcal{D}, k)$ is viewed as an instance of the minimum weight partial set cover problem, where X serves as the set of elements to be covered, $\mathcal{D}$ serves as the collection of sets, and the weight of each $D \in \mathcal{D}$ is $p(D)$. The local ratio method was first proposed by Bar-Yehuda and Even in [1]. The idea is to recursively peel off a special weight from the original weight. If the problem with the special weight admits an α-approximation, then one can assemble an α-approximate solution for the problem with respect to the original weight. In this paper, the special weight peeled off in each iteration (denoted by $\bar{p}$) is proportional to the number of uncovered points of a disk, and then the disks of residual weight zero are put into $\bar{\mathcal{D}}$.

Algorithm 1. $LR(X, \mathcal{D}, p, k)$.

Input: A set of points X, a set of disks $\mathcal{D}$, a weight function $p : \mathcal{D} \mapsto \mathbb{R}^+$, a covering requirement k.

Output: A *minimal* subset of disks $\bar{\mathcal{D}}$ covering at least k points.

1: If $k = 0$, then return $\bar{\mathcal{D}} \leftarrow \emptyset$
2: $\gamma \leftarrow \min_{D \in \mathcal{D}} p(D)/|X \cap D|$
3: $\bar{p}(D) \leftarrow \gamma \cdot |X \cap D|$ for each $D \in \mathcal{D}$
4: $p(D) \leftarrow p(D) - \bar{p}(D)$ for each $D \in \mathcal{D}$
5: $\mathcal{D}_{=0} \leftarrow \{D \in \mathcal{D} : p(D) = 0\}$
6: $X \leftarrow X \setminus \mathcal{C}(\mathcal{D}_{=0})$, $\mathcal{D} \leftarrow \mathcal{D} \setminus \mathcal{D}_{=0}$, $k \leftarrow \max\{0, k - |\mathcal{C}(\mathcal{D}_{=0})|\}$
7: $\bar{\mathcal{D}}' \leftarrow LR(X, \mathcal{D}, p, k)$
8: Let $\bar{\mathcal{D}}_{=0}$ be a minimal subset of $\mathcal{D}_{=0}$ such that $\bar{\mathcal{D}}' \cup \bar{\mathcal{D}}_{=0}$ covers at least k points.
9: Return $\bar{\mathcal{D}} \leftarrow \bar{\mathcal{D}}' \cup \bar{\mathcal{D}}_{=0}$

Algorithm 1 is in fact a function which will be recursively called. In the algorithm, after peeling off a special weight $\bar{p}$, we use $\mathcal{D}_{=0}$ to denote the set of disks with residual weight $p - \bar{p}$ being zero. Since taking disks of zero cost seems to be a free meal, we take all of them *temporarily* and consider the residual instance, the goal of which is to satisfy the residual covering requirement using the residual disks. Line 6 of the algorithm is to construct the residual instance. Having found a *minimal* solution $\bar{\mathcal{D}}'$ to the residual instance, the algorithm adds a *minimal* subset of disks of $\mathcal{D}_{=0}$, denoted as $\bar{\mathcal{D}}_{=0}$, into $\bar{\mathcal{D}}'$ to cover at least k points. This step is to guarantee that the resulting set of disks $\bar{\mathcal{D}}$ is *minimal*, which is very crucial to the control of the approximation factor.

Suppose the function LR is called $t+1$ times. Denote by $\bar{\mathcal{D}}^{(i)}$, $p^{(i)}$, $\bar{p}^{(i)}$ etc. those objects at the end of the i-th calling of function LR. Then we have the following relations.

(i) $X^{(0)} = X$, $\mathcal{D}^{(0)} = \mathcal{D}$, $p^{(0)} = p$, and $k^{(0)} = k$.

(ii) For $i = 1, \ldots, t$,

$$\gamma^{(i)} = \min\{p^{(i-1)}(D)/|X^{(i-1)} \cap D|\} \text{ for each } D \in \mathcal{D}^{(i-1)}$$

$$\bar{p}^{(i)}(D) = \gamma^{(i)} \cdot |X^{(i-1)} \cap D| \text{ for each } D \in \mathcal{D}^{(i-1)} \tag{2}$$

$$p^{(i)}(D) = p^{(i-1)}(D) - \bar{p}^{(i)}(D) \text{ for each } D \in \mathcal{D}^{(i-1)} \tag{3}$$

$$\mathcal{D}^{(i)}_{=0} = \{D \in \mathcal{D}^{(i-1)} : p^{(i)}(D) = 0\}$$

$$X^{(i)} = X^{(i-1)} \setminus \mathcal{C}(\mathcal{D}^{(i)}_{=0})$$

$$\mathcal{D}^{(i)} = \mathcal{D}^{(i-1)} \setminus \mathcal{D}^{(i)}_{=0}$$

$$k^{(i)} = \max\{0, k^{(i-1)} - |\mathcal{C}^{(i-1)}(\mathcal{D}^{(i)}_{=0})|\} \tag{4}$$

Here $\mathcal{C}^{(i-1)}(\mathcal{D}^{(i)}_{=0}) = \mathcal{C}(\mathcal{D}^{(i)}_{=0}) \cap X^{(i-1)}$. As a consequence of the above relations,

$$k^{(i)} = \max\{0, k - |\mathcal{C}(\bigcup_{j=1}^{i} \mathcal{D}^{(j)}_{=0})|\}. \tag{5}$$

It should be noticed that in expressions (4) and (5), except for $i = t$, the value of $k^{(i)}$ equals the second term.

(iii) $k^{(t)} = 0$, $\bar{\mathcal{D}}^{(t+1)} = \emptyset$. And for $i = t, t-1, \ldots, 1$,

$$\bar{\mathcal{D}}^{(i)} = \bar{\mathcal{D}}^{(i+1)} \cup \bar{\mathcal{D}}^{(i)}_{=0}.$$

As a consequence

$$\bar{\mathcal{D}}^{(i)} = \bigcup_{j=i}^{t} \bar{\mathcal{D}}^{(j)}_{=0} \subseteq \bigcup_{j=i}^{t} \mathcal{D}^{(j)}_{=0}. \tag{6}$$

The above relation can be illustrated by the following figure.

Remark 1. If a disk D has its weight reduced to zero in the i-th call of LR, that is, if $p^{(i-1)}(D) > 0$ and $p^{(i)}(D) = 0$, then D does not play roles in the deeper calls of LR. In this case, we may view $p^{(j)}(D) = \bar{p}^{(j)}(D) = 0$ for any j with $i+1 \leq j \leq t$. By such a point of view, for any $0 \leq i \leq t$, we may extend the definition of functions $p^{(i)}$ and $\bar{p}^{(i)}$ on any disk $D \in \mathcal{D}$.

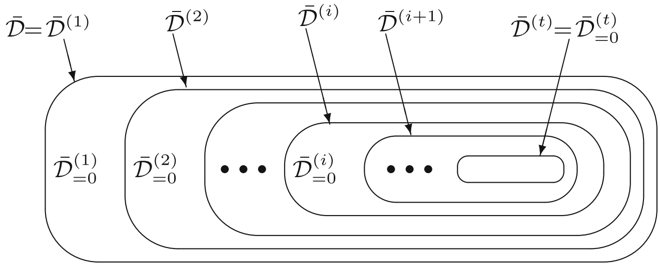

Fig. 1. Illustration for the structure of $\bar{\mathcal{D}}^{(i)}$.

Lemma 2. *For any* $i = 1, \ldots, t+1$*, the set* $\bar{\mathcal{D}}^{(i)}$ *is a* minimal *set of disks covering* $k^{(i-1)}$ *points of* $X^{(i-1)}$*.*

Proof. We prove the lemma by a backward induction on i. The base step when $i = t+1$ is obvious, since $k^{(t)} = 0$ and $\bar{\mathcal{D}}^{(t+1)} = \emptyset$.

For the induction step, suppose $i \leq t$ and $\bar{\mathcal{D}}^{(i+1)}$ is a minimal set of disks covering $k^{(i)}$ points of $X^{(i)}$. By expression (4) and the remark below it, we have

$$k^{(i-1)} = k^{(i)} + |\mathcal{C}^{(i-1)}(\mathcal{D}^{(i)}_{=0})|. \tag{7}$$

So $\bar{\mathcal{D}}^{(i+1)} \cup \mathcal{D}^{(i)}_{=0}$ can cover $k^{(i-1)}$ elements of $X^{(i-1)}$, which implies that a minimal subset $\bar{\mathcal{D}}^{(i)}_{=0} \subseteq \mathcal{D}^{(i)}_{=0}$ exists such that $\bar{\mathcal{D}}^{(i+1)} \cup \bar{\mathcal{D}}^{(i)}_{=0}$ can cover $k^{(i-1)}$ elements of $X^{(i-1)}$ (Fig. 1).

What remains to show is that $\bar{\mathcal{D}}^{(i+1)} \cup \bar{\mathcal{D}}^{(i)}_{=0}$ is minimal. By line 8 of Algorithm 1, no disk in $\bar{\mathcal{D}}^{(i)}_{=0}$ can be removed without violating the covering requirement $k^{(i-1)}$. For any disk $D \in \bar{\mathcal{D}}^{(i+1)}$, by the minimality of $\bar{\mathcal{D}}^{(i+1)}$, we have $|\mathcal{C}^{(i)}(\bar{\mathcal{D}}^{(i+1)} \setminus \{D\})| < k^{(i)}$. Then by (7), we have $|\mathcal{C}^{(i-1)}\big((\bar{\mathcal{D}}^{(i+1)} \setminus \{D\}) \cup \bar{\mathcal{D}}^{(i)}_{=0}\big)| < k^{(i-1)}$. The minimality of $\bar{\mathcal{D}}^{(i)}$ is proved. □

The second step is realized by Algorithm 2. Given a set of disks $\mathcal{D}$, Algorithm 2 finds a maximal independent set of disks by recursively choosing the disk with the maximum radius and deleting those disks intersecting it.

Algorithm 2. $IS(\mathcal{D})$.

Input: A set of disks $\mathcal{D}$.
Output: A maximal independent set of disks $\mathcal{I}$.

```
I ← ∅
while D ≠ ∅ do
    D' ← arg max_{D∈D} r(D)
    I ← I ∪ {D'}
    N ← the set of disks of D that intersect D'
    D ← D \ N
end while
Return I
```

Algorithm 3 combines the above two algorithms to compute a feasible solution $\mathcal{M}$ to the residual instance. We use $c(D)$ and $r(D)$ to denote the center and the radius of disk D, respectively. So, $Disk(c(D), 3r(D))$ represents the disk with center $c(D)$ and radius $3r(D)$ (which a disk obtained from D by enlarging its radius by three times). Notice that $\mathcal{M}$ is not a subset of $\mathcal{D}$. Before calling Algorithm 2, a disk D_{rmv} is deleted from $\bar{\mathcal{D}}$, where D_{rmv} belongs to the set of disks added in the deepest call of LR. This is to control the approximation ratio which will be clear in the latter proofs.

Algorithm 3. $Cov(X, \mathcal{D}, k)$

Input: A residual instance $(X, \mathcal{D}, k)$.
Output: a set of disks $\mathcal{M}$ covering at least k points.

1: $\bar{\mathcal{D}} \leftarrow LR(X, \mathcal{D}, k)$
2: $D_{rmv} \leftarrow$ an arbitrary disk in $\bar{\mathcal{D}}^{(t)}_{=0}$ where t is the last call of LR
3: $\mathcal{I} \leftarrow \text{IS}(\bar{\mathcal{D}} \setminus \{D_{rmv}\})$
4: $\mathcal{M} \leftarrow \{Disk(c(D), 3r(D)) \colon D \in \mathcal{I}\} \cup \{D_{rmv}\}$
5: Return $\mathcal{M}$

The next theorem shows that Algorithm 3 computes a feasible solution to the residual instance.

Theorem 1. *The set of disks $\mathcal{M}$ computed by Algorithm 3 covers at least k points.*

Proof. The set of disks in $\bar{\mathcal{D}}$ computed in line 1 of the algorithm cover at least k points. For any point x which is covered by $\bar{\mathcal{D}}$, if x is covered by D_{rmv} or any disk in $\mathcal{I}$, then it is also covered by $\mathcal{M}$. Otherwise, x is covered by a disk D which is removed in line 6 of Algorithm 2. This disk D is removed because it intersects a disk $D' \in \mathcal{I}$. Because of the greedy choice of disk D' in line 3 of Algorithm 2, we have $r(D) \leq r(D')$. Hence $d(x, c(D')) \leq d(x, c(D)) + d(c(D), c(D')) \leq r(D) + (r(D) + r(D')) \leq 3r(D')$, where $d(\cdot,\cdot)$ denotes the Euclidean distance. So, x is covered by $disk(c(D'), 3r(D')) \in \mathcal{M}$. □

The following lemma is a key lemma towards the analysis of the approximation ratio.

Lemma 3. *Suppose $\mathcal{D}^*$ is an optimal solution for $(X, \mathcal{D}, k)$. Then the independent set of disks $\mathcal{I}$ output by Algorithm 2 satisfies $p(\mathcal{I}) \leq p(\mathcal{D}^*)$.*

Proof. We prove

$$p^{(i)}(\mathcal{I}) \leq p^{(i)}(\mathcal{D}^*) \tag{8}$$

by a backward induction on $i = t, t-1, \ldots, 0$. Since $p^{(0)} = p$, what is required by the lemma is exactly $p^{(0)}(\mathcal{I}) \leq p^{(0)}(\mathcal{D}^*)$.

For the base step, we have $p^{(t)}(\mathcal{I}) = 0$ because every disk $D \in \mathcal{I} \subseteq \bar{\mathcal{D}}^{(1)}$ belongs to some $\bar{\mathcal{D}}^{(j)}_{=0}$ (by (6)) and thus $p^{(t)}(D) = 0$. So (8) holds for $i = t$.

For the induction step, suppose (8) is true for some $i \le t$. We are going to prove

$$p^{(i-1)}(\mathcal{I}) \le p^{(i-1)}(\mathcal{D}^*). \tag{9}$$

By (3), inequality (9) is equivalent with

$$p^{(i)}(\mathcal{I}) + \bar{p}^{(i)}(\mathcal{I}) \le p^{(i)}(\mathcal{D}^*) + \bar{p}^{(i)}(\mathcal{D}^*).$$

Combining this with the induction hypothesis, it suffices to prove

$$\bar{p}^{(i)}(\mathcal{I}) \le \bar{p}^{(i)}(\mathcal{D}^*). \tag{10}$$

By (6) and Remark 1,

$$\text{for any disk } D \in \bar{\mathcal{D}}^{(1)} \setminus \bar{\mathcal{D}}^{(i)}, \text{we have } \bar{p}^{(i)}(D) = 0. \tag{11}$$

Combining this with (2) and the fact $\mathcal{I} \subseteq \bar{\mathcal{D}}^{(1)}$, we have

$$\begin{aligned}\bar{p}^{(i)}(\mathcal{I}) &= \sum_{D \in \mathcal{I}} \bar{p}^{(i)}(D) = \sum_{D \in \mathcal{I} \cap \bar{\mathcal{D}}^{(i)}} \gamma^{(i)} \cdot |X^{(i-1)} \cap D| \\ &= \sum_{x \in X^{(i-1)}} \gamma^{(i)} \cdot |\{D \in \mathcal{I} \cap \bar{\mathcal{D}}^{(i)} \colon x \in D\}|.\end{aligned}$$

Since no disks in $\mathcal{I}$ can intersect, we have

$$\sum_{x \in X^{(i-1)}} |\{D \in \mathcal{I} \cap \bar{\mathcal{D}}^{(i)} \colon x \in D\}| = |X^{(i-1)} \cap \mathcal{C}(\mathcal{I} \cap \bar{\mathcal{D}}^{(i)})|.$$

Since $D_{rmv} \notin \mathcal{I}$, we have $\mathcal{I} \cap \bar{\mathcal{D}}^{(i)} \subseteq \bar{\mathcal{D}}^{(i)} \setminus \{D_{rmv}\}$. Combining this with Lemma 2 and the observation that $D_{rmv} \in \bar{\mathcal{D}}^{(t)}_{=0} \subseteq \bar{\mathcal{D}}^{(i)}$, we have

$$|X^{(i-1)} \cap \mathcal{C}(\mathcal{I} \cap \bar{\mathcal{D}}^{(i)})| < k^{(i-1)}.$$

Hence,

$$\bar{p}^{(i)}(\mathcal{I}) \le \gamma^{(i)} k^{(i-1)} \tag{12}$$

On the other hand, because of (11),

$$\bar{p}^{(i)}(\mathcal{D}^*) = \sum_{D \in \mathcal{D}^*} \bar{p}^{(i)}(D) = \sum_{D \in \mathcal{D}^* \setminus \left(\bar{\mathcal{D}}^{(1)} \setminus \bar{\mathcal{D}}^{(i)}\right)} \gamma^{(i)} \cdot |X^{(i-1)} \cap D|.$$

Combining the facts

$$|\mathcal{C}(\mathcal{D}^*)| \ge k$$

$$\bar{\mathcal{D}}^{(1)} \setminus \bar{\mathcal{D}}^{(i)} \subseteq \bigcup_{j=1}^{i-1} \mathcal{D}^{(j)}_{=0} \text{ by (6), and}$$

$$k^{(i-1)} = \max\{0, k - |\mathcal{C}(\bigcup_{j=1}^{i-1} \mathcal{D}^{(j)}_{=0})|\} \text{ by (5),}$$

we have

$$\sum_{D\in\mathcal{D}^*\setminus\left(\bar{\mathcal{D}}^{(1)}\setminus\bar{\mathcal{D}}^{(i)}\right)} |X^{(i-1)}\cap D| \geq |X^{(i-1)}\cap \mathcal{C}(\mathcal{D}^*\setminus\left(\bar{\mathcal{D}}^{(1)}\setminus\bar{\mathcal{D}}^{(i)}\right))|$$

$$\geq \left|X^{(i-1)}\cap \mathcal{C}\left(\mathcal{D}^*\setminus\bigcup_{j=1}^{i-1}\mathcal{D}_{=0}^{(j)}\right)\right| \geq k^{(i-1)}.$$

Hence,

$$\bar{p}^{(i)}(\mathcal{D}^*)\geq \gamma^{(i)}k^{(i-1)}. \tag{13}$$

Then inequality (10) follows from (12) and (13), and the lemma is proved. □

The next theorem estimates the approximation effect of Algorithm 3.

Theorem 2. *Suppose $\mathcal{C}^*$ is an optimal solution on instance $(X,\mathcal{D},p,k)$, and $\mathcal{M}$ is the output of Algorithm 3. Then*

$$p(\mathcal{M})\leq 3^\alpha p(\mathcal{C}^*)+p(D_{rmv}).$$

Proof. For each disk $D\in\mathcal{M}\setminus\{D_{rmv}\}$, it comes from a disk $D'\in\mathcal{I}$ by expanding the radius by three times. Hence by (1), $p(D)=3^\alpha p(D')$. So $p(\mathcal{M})\leq 3^\alpha p(\mathcal{I})+p(D_{rmv})$, and the theorem follows from Lemma 3. □

By Theorem 2, the approximate effect is related with $p(D_{rmv})$. The reason why we should guess a disk $D_{\max}$ with the largest radius in an optimal solution is now clear: to control the term $p(D_{rmv})$ to be not too large. The algorithm combining the guessing technique is presented as follows.

3.2 The Whole Algorithm

Algorithm 4 is the whole algorithm for the MinPowerPartCov problem. It first guesses a disk $D_{\max}$ with the maximum radius in an optimal solution, takes it, and then uses Algorithm 3 on the residual instance. For a guessed disk D, the residual instance consists of all those disks $\mathcal{D}_{\leq r(D)}$ whose radii are no larger than $r(D)$ (excluding D itself), and the goal is to cover the remaining elements $X\setminus D$ beyond the remaining covering requirement $\max\{0,k-|D|\}$. The weight function, denoted as p_D, is determined by (1). If for a guessed disk D, Algorithm 3 does not return a feasible solution, then we regard the solution to have cost ∞. Algorithm 4 returns the best solution among all the guesses.

Algorithm 4. $MinPowerPartCov(X, \mathcal{D}, k, p)$

Input: A set of points X, a set of sensors $\mathcal{S}$, a covering requirement k.
Output: A power assignment p to cover at least k points.

1: Construct the set $\mathcal{D}$ of canonical disks, determine the weight of each disk by (1).
2: **for** $D \in \mathcal{D}$ **do**
3: $\quad \mathcal{M}_D \leftarrow Cov(X \setminus D, \mathcal{D}_{\leq r(D)}, p_D, \max\{0, k - |D|\})$
4: $\quad \mathcal{F}_D \leftarrow \mathcal{M}_D \cup \{D\}$
5: **end for**
6: $\widetilde{D} \leftarrow \arg\min_{D \in \mathcal{D}}\{p(\mathcal{F}_D)\}$
7: Return the power assignment corresponding to $\mathcal{F}_{\widetilde{D}}$

Theorem 3. *Algorithm 4 is an* 3^α*-approximation algorithm for the MinPowerPartCov problem.*

Proof. Suppose $D_{\max}$ is the disk with the maximum radius in an optimal solution. By Theorem 2 and the fact $p(D_{max,rmv}) \leq p(D_{max})$, we have

$$p(\mathcal{F}_{D_{\max}}) = p(\mathcal{M}_{D_{\max}}) + p(D_{\max}) \leq 3^\alpha p(\mathcal{C}^*_{D_{\max}}) + 2p(D_{\max})$$
$$\leq 3^\alpha \left(p(\mathcal{C}^*_{D_{\max}}) + p(D_{\max})\right) = 3^\alpha opt,$$

where *opt* is the optimal power. Since the set $\mathcal{F}_{\widetilde{D}}$ computed by Algorithm 4 satisfies $p(\mathcal{F}_{\widetilde{D}}) \leq p(\mathcal{F}_{D_{\max}})$, the theorem is proved. □

4 Conclusion

In this paper, we presented an approximation algorithm for the minimum power partial cover problem achieving approximation ratio 3^α, using a local ratio method. This ratio improves the ratio of $(12 + \varepsilon)$ in [9], and matches the best known ratio for the minimum power (full) cover problem in [3].

Recently, there are a lot of studies on the minimum power multi-cover problem [5,6]. A problem which deserves to be explored is the minimum power partial multi-cover problem (adding partial covering requirement). According to current studies on the minimum partial set multi-cover problem [17–19], it seems that studying the combination of multi-cover and partial cover in a general setting is very difficult. An interesting question is whether geometry can make the situation better?

Acknowledgment. This research is supported in part by NSFC (11771013, 61751303, 11531011) and the Zhejiang Provincial Natural Science Foundation of China (LD19A010001, LY19A010018).

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