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On Approximation Algorithm for the Edge Metric Dimension Problem

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Abstract. In this paper, we study the edge metric dimension problem (EMDP). We establish a potential function and give a corresponding greedy algorithm with approximation ratio $1 + \ln n + \ln(\log_2 n)$, where n is the number of vertices in the graph G.

Keywords: Edge metric generator · Edge metric dimension · Approximation algorithms · Submodular function

1 Introduction

The concepts of metric generators (originally called locating sets) and the concepts of metric dimension (originally called the location number) were introduced by Slater in [17] in connection with uniquely determining the position of an intruder in a network. Harary and Melter [11] discovered the same concepts independently.

We now recall the definition of the metric dimension. Let $G = (V, E)$ be a simple connected undirected graph. A vertex $v \in V$ is called to *resolve* or *distinguish* a pair of vertices $u, w \in V$ if $d(v, u) \neq d(v, w)$, where $d(\cdot, \cdot)$ denotes the distance between two vertices in G. A *metric generator* of G is a subset $V' \subseteq V$ such that for each pair $u, w \in V$ there exists some vertex $v \in V'$ that distinguishes u and w. The minimum cardinality of a metric generator is called the *metric dimension* of G, denoted by $\dim(G)$.

The metric dimension problem (MDP) has been widely investigated from the graph theoretical point of view. Cáceres et al. [3] studied the metric dimension of cartesian products $G \square H$, and proved that the metric dimension of $G \square G$ was tied in a strong sense to the minimum order of a so-called doubly resolving set in G. They established bounds on $G \square H$ for many examples of G and H. Chartrand et al. [7] studied resolvability in graphs and the metric dimension of a graph. It was shown that $\dim(H) \leq \dim(H \square K_2) \leq \dim(H) + 1$ for every connected graph H. Moreover, it was shown that for every positive real number ε, there exists a

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connected graph G and a connected induced subgraph H of G such that $\frac{\dim(G)}{\dim(H)} \leq \varepsilon$. Saputro et al. [16] studied the metric dimension of regular bipartite graphs, and determined the metric dimension of k-regular bipartite graphs $G(n, n)$ where $k = n - 1$ or $k = n - 2$. Chappell et al. [6] studied relationships between metric dimension, partition dimension, diameter, and other graph parameters. They constructed "universal examples" of graphs with given partition dimension, and they used these to provide bounds on various graph parameters based on metric and partition dimensions. They formed a construction showing that for all integers α and β with $3 \leq \alpha \leq \beta + 1$ there exists a graph G with partition dimension α and β. Cáceres et al. [5] studied the metric dimension of infinite locally finite graphs, i.e. those infinite graphs such that all its vertices have finite degree. They gave some necessary conditions for an infinite graph to have finite metric dimension and characterized infinite trees with finite metric dimension.

So far only a few papers have discussed the computational complexity issues of the MDP. The NP-hardness of the MDP was mentioned by Garey and Johnson [10]. An explicit reduction from the 3-SAT problem was given by Khuller et al. [14]. They also obtained for the Metric Dimension problem a $(2\ln(n)+\Theta(1))$-approximation algorithm based on the well-known greedy algorithm for the Set Cover problem and showed that the MDP is polynomial-time solvable for trees. Beerliova et al. [1] showed that the MDP (which they call the Network Verification problem) cannot be approximated within a factor of $O(\log(n))$ unless $P = NP$. Hauptmann et al. [12] gave a $(1+\ln(|V|)+\ln(\log_2(|V|)))$-approximation algorithm for the MDP in graphs.

The concept of a doubly resolving set of a graph G was introduced by Caceres et al. [4]. We say vertices u, v of the graph G *doubly resolve* vertices x, y of G, if $d(u, x) - d(u, y) \neq d(v, x) - d(v, y)$. A vertex set S is called a *doubly resolving set* of G if every two distinct vertices of G are doubly resolved by some two vertices of S.

Kratica et al. [15] proved that the minimal doubly resolving sets problem is NP-hard. Chen et al. [8] designed an $(1 + o(1)) \ln n$-approximation algorithm for the weighted minimum doubly resolving set problem.

The edge metric dimension is a variant of the metric dimension. We now recall the definition of the edge metric dimension. For any $v \in V$ and $e = uw \in E$, we use $d(e, v) = \min\{d(u, v), d(w, v)\}$ to denote the distance between the vertex v and the edge e. We say that two distinct edges $e_1, e_2 \in E$ are distinguished by the vertex $v \in V$ if $d(v, e_1) \neq d(v, e_2)$. A subset $S \subseteq V$ is said to be an *edge metric generator* of G if every two distinct edges of G can be distinguished by some vertex in S. An *edge metric basis* of G is an edge metric generator of G of the minimum cardinality and its cardinality is called the *edge metric dimension*, denoted by $\dim_e(G)$.

Kelenc et al. [13] proved that computing the edge metric dimension of connected graphs is NP-hard. As a response to an open problem presented in [13], Zhu et al. [18] considered the maximum edge metric dimension problem on graphs. Zubrilina [19] classified the graphs on n vertices for which $\dim_e(G) = n - 1$ and showed that $\frac{\dim_e(G)}{\dim(G)}$ is not bounded from above (here dim(G) is the standard metric dimension of G). They computed $\dim_e(G\Box P_m)$ and $\dim_e(G+K_1)$. Zubrilina [20] discussed the edge metric dimension of the random graph $G(n,p)$ and obtained $\dim_e(G(n,p)) = (1+o(1))\frac{4\log(n)}{\log(\frac{1}{q})}$, where $q = 1-2p(1-p)^2(2-p)$. In this paper, we discuss the edge metric dimension problem.

The paper is organized as follows: In Sect. 2, we construct a normalized, monotone increasing and submodular potential function and give a greedy algorithm for the edge metric dimension problem. In Sect. 3, we show that the algorithm presented in this paper has approximation ratio $1+\ln n+\ln(\log_2 n)$, where n is the number of vertices in the graph G.

2 Approximation Algorithm

Throughout this paper we assume that the graph $G=(V,E)$ is simple connected and undirected. In this section, we first construct a potential function and study the properties of the potential function. Then we give a greedy algorithm for the edge metric dimension of G.

Definition 2.1. Let Γ be a subset of V. We define the *equivalence relation* $\equiv_\Gamma$ for E as follows: for edges $e_1, e_2 \in E$,

$$e_1 \equiv_\Gamma e_2 \Longleftrightarrow d(e_1,w) = d(e_2,w) \quad \forall w \in \Gamma.$$

Definition 2.2. Let Γ be a subset of V and $\{E_1, E_2, \ldots, E_k\}$ be the set of equivalence classes of $\equiv_\Gamma$ for E. We call the value $H(\Gamma) = \log_2(\prod_{i=1}^k |E_i|!)$ the *entropy* of Γ.

For any $v \in V$, let

$$\Delta_v H(\Gamma) := H(\Gamma) - H(\Gamma \cup \{v\}).$$

It is direct to see that any equivalence class of $\equiv_\Gamma$ is either an equivalent class of $\equiv_{\Gamma\cup\{v\}}$ or a union of several equivalence classes of $\equiv_{\Gamma\cup\{v\}}$.

Lemma 2.3. *Let Γ be a subset of V and $v \in V$. Then $\Delta_v H(\Gamma) = 0$ if each equivalence class of $\equiv_\Gamma$ is one of $\equiv_{(\Gamma\cup\{v\})}$; and $\Delta_v H(\Gamma) > 0$ otherwise.*

Lemma 2.4. *Let Γ be a subset of V. Then Γ is an edge metric generator of G if and only if $H(\Gamma) = 0$.*

Proof. Observe that each of the two assertions is equivalent with the assertion that every equivalent class of $\equiv_\Gamma$ is a singleton. The result follows. □

Lemma 2.5. *For any two sets $\Gamma_0 \subseteq \Gamma_1 \subseteq V$ and any vertex $v \in V\backslash\Gamma_1$, we have*

$$\Delta_v H(\Gamma_0) \geq \Delta_v H(\Gamma_1). \tag{1}$$

Proof. If $\Gamma_0 = \Gamma_1$, then the lemma holds. If $\Gamma_0 \subset \Gamma_1$, we divide the proof into two cases: case 1, the vertex v partitions each equivalence class of $\equiv_{\Gamma_0}$ into at most two equivalence classes; case 2, the vertex v partitions some equivalence class of $\equiv_{\Gamma_0}$ into at least three equivalence classes.

Case 1. Since $\Delta_v H(\Gamma_0) = H(\Gamma_0) - H(\Gamma_0 \cup \{v\} = \log_2 \frac{|\prod_{\Gamma_0}|}{|\prod_{\Gamma_0 \cup \{v\}}|}$, it suffices to show

$$\frac{|\prod_{\Gamma_0}|}{|\prod_{\Gamma_0 \cup \{v\}}|} \geq \frac{|\prod_{\Gamma_1}|}{|\prod_{\Gamma_1 \cup \{v\}}|}. \tag{2}$$

Write $S = \Gamma_1 \setminus \Gamma_0$. Let $\{E_1, E_2, \cdots, E_k\}$ be the equivalence classes of $\equiv_{\Gamma_0}$, $\{A_1, A_2, \cdots, A_n\}$ the equivalence classes of $\equiv_{\Gamma_0 \cup \{v\}}$ and $\{B_1, B_2, \cdots, B_t\}$ the equivalence classes of $\equiv_{\Gamma_1}$. By the comments above Lemma 2.3 and the assumption, for each i, $E_i = A_{i_1} \cup A_{i_2}$ and E_i is a union of some $B_{i_1}, \cdots, B_{i_t}$. Without loss of generality, assume $t = 2$. Let $F_i = A_{i_1} \cap B_{i_1}$, $H_i = A_{i_2} \cap B_{i_1}$, $C_i = (E_i \cap A_{i_1}) \backslash F_i$, $D_i = (E_i \cap A_{i_2}) \backslash H_i$. Let $|F_i| = f_i$, $|H_i| = h_i$, $|C_i| = c_i$, $|D_i| = d_i$. Then $|E_i| = f_i + h_i + c_i + d_i$. Since $\binom{f_i+c_i}{f_i+h_i+c_i+d_i} \geq \binom{f_i}{h_i+f_i}\binom{c_i}{c_i+d_i}$, we have

$$\prod_{i=0}^{k} \binom{f_i + c_i}{f_i + h_i + c_i + d_i} \geq \prod_{i=0}^{k} \binom{f_i}{f_i + h_i}\binom{c_i}{c_i + d_i},$$

i.e.

$$\prod_{i=0}^{k} (\frac{(f_i + h_i + c_i + d_i)!}{(f_i + c_i)!(h_i + d_i)!} \geq \prod_{i=0}^{k} (\frac{(f_i + h_i)!(c_i + d_i)!}{(f_i)!(h_i)!(c_i)!(d_i)!}).$$

Thus

$$\frac{|\prod_{\Gamma_0}|}{|\prod_{\Gamma_0 \cup \{v\}}|} \geq \frac{|\prod_{\Gamma_1}|}{|\prod_{\Gamma_1 \cup \{v\}}|}.$$

Case 2. Assume that the vertex v partitions each E_j into k_j equivalence classes, where $j = 1, 2, \ldots, m$. Let $k = \max_j\{k_j\}$. Then by assumption, $k \geq 3$. For $1 \leq j \leq m$, there exist the vertices x_1, x_2,..., x_k such that x_1 divides E_j into E_{j_1} and $E_j \setminus E_{j_1}$, vertex x_2 divides $E_j \setminus E_{j_1}$ into E_{j_2} and $E_j \setminus (E_{j_1} \cup E_{j_2})$, ..., vertex x_k divides $E_j \setminus (E_{j_1} \cup E_{j_2} \cup \ldots \cup E_{j_{k_j-1}})$ into $E_{j_{k_j}}$ and $\emptyset$. Then by the argument in Case 1, we have

$$\begin{aligned}
\Delta_v H(\Gamma_0) =& H(\Gamma_0) - H(\Gamma_0 \cup \{v\}) \\
=&(H(\Gamma_0) - H(\Gamma_0 \cup \{x_1\}) + (H(\Gamma_0 \cup \{x_1\}) - H(\Gamma_0 \cup \{x_1\} \cup \{x_2\})) \\
&+ \ldots + (H(\Gamma_0 \cup \{x_1\} \cup \{x_2\} + \ldots \cup \{x_{k-1}\}) - H(\Gamma_0 \cup \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_k\})) \\
\geq&(H(\Gamma_1) - H(\Gamma_1 \cup \{x_1\})) + (H(\Gamma_1 \cup \{x_1\}) - H(\Gamma_1 \cup \{x_1\} \cup \{x_2\})) \\
&+ \ldots + (H(\Gamma_1 \cup \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_{k-1}\}) - H(\Gamma_1 \cup \{x_1\} \cup \{x_2\} \cup \ldots \cup \{x_k\})) \\
=&H(\Gamma_1) - H(\Gamma_1 \cup \{v\}) \\
=&\Delta_v H(\Gamma_1).
\end{aligned}$$

□

Let $\mathbb{R}$ be the real number field. We define a function $f : 2^V \to \mathbb{R}$ by

$$f(\Gamma) = -H(\Gamma) + H(\emptyset) \quad \text{for } \Gamma \in 2^V.$$

Lemma 2.6. *The function f defined above is normalized, monotone increasing and submodular.*

Proof. It is easy to know that $f(\emptyset) = 0$, that is to say, the function f is normalized. By Lemma 2.3, f is monotone increasing. By Lemma 2.5 f is submodular. □

Based on the above lemmas, we give a greedy approximation algorithm for the EMDP.

Algorithm 1

Input: a simple connected undirected graph $G = (V, E)$.
Output: an edge metric generator of G.

1: Set $\Gamma \leftarrow \emptyset$.
2: **while** there exists a vertex $v \in V \setminus \Gamma$ such that $\Delta_v f(\Gamma) > 0$ **do**
3: select a vertex $v \in V \setminus \Gamma$, that maximizes $\Delta_v f(\Gamma)$.
4: $\Gamma \leftarrow \Gamma \cup \{v\}$.
5: **return** $\Gamma_g \leftarrow \Gamma$

3 Theoretical Analysis

To obtain the ratio of Algorithm 1. We first prove the following lemma.

Lemma 3.1. *Let $v_1, v_2, \cdots, v_k$ be the elements in Γ_g in the order of their selection into the set Γ_g. Denote $\Gamma_0 = \emptyset$ and $\Gamma_i = \{v_1, v_2, \cdots, v_i\}$, for $i = 1, \cdots, k$. Then for $i = 2, \cdots, k$, we have*

$$\Delta_{v_i} f(\Gamma_{i-1}) \geq 1.$$

Proof. By [2, Lemma 6], it is sufficient to prove $\Delta_{v_i} f(\Gamma_{i-1}) > 0$. Assume $\Delta_{v_i} f(\Gamma_{i-1}) = 0$ for some i ($2 \leq i \leq k$), for a contradiction. Then $H(\Gamma_{i-1} \cup \{v_i\}) = H(\Gamma_{i-1})$. By the greedy strategy, the vertex v_i can not be chosen in Γ_g. A contradiction. □

Theorem 3.2. *Algorithm 1 produces an approximate solution within a ratio $1 + \ln n + \ln(\log_2 n)$.*

Proof. Let Γ^* denote an optimal solution to the edge metric dimension problem. By Lemmas 2.6 and 3.1 and [9, Theorem 3.7], and since $f(\Gamma^*) = f(V) = \log_2(n!)$, the approximation ratio of Algorithm 1 is

$$\begin{aligned} & 1 + \ln(\frac{f(\Gamma^*)}{|\Gamma^*|}) \\ =& 1 + \ln(\frac{\log_2(n!)}{|\Gamma^*|}) \\ \leq& 1 + \ln(n \log_2 n) - \ln(|\Gamma^*|) \\ \leq& 1 + \ln(\log_2 n) + \ln n. \end{aligned}$$

□

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