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Nonlinear Analysis and Boundary Value Problems

NABVP 2018, Santiago de Compostela, Spain, September 4–7

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Preface

This book consists of contributions presented at the International Conference on Nonlinear Analysis and Boundary Value Problems, held in Santiago de Compostela, Spain, from September 4 to 7, 2018, and is dedicated to Prof. Juan J. Nieto, on the occasion of his 60th birthday. The conference was organized by the Nonlinear Differential Equations Group at the University of Santiago de Compostela.

The book comprises 17 contributions that cover a wide variety of topics linked to Prof. Nieto's scientific work, ranging from differential, difference, and fractional equations to epidemiological models and dynamical systems and their applications. It is primarily intended for researchers involved in nonlinear analysis and boundary value problems in a broad sense.

Radu Precup presents a variational analogue of Krasnoselskii's cone compression–expansion fixed-point theorem, based on Ekeland's principle. He also includes a general scheme of applications to semilinear equations, making use of Mikhlin's variational theory on positive linear operators.

Aurelian Cernea studies a certain second-order evolution inclusion defined by a family of linear closed operators that is the generator for an evolution system of operators and by a set-valued map with nonconvex values in a separable Banach space. In his work, results are provided concerning the differentiability of mild solutions with respect to the initial conditions of the problem considered. The results may be interpreted as extensions to a special class of second-order differential inclusions of the classical Bendixson-Picard–Lindelöf theorem concerning the differentiability of the maximal flow of a differential equation.

Antonio Pumariño, José A. Rodríguez, and Enrique Vigil describe the attractors for a two-parameter family of two-dimensional piecewise affine maps using measure theory. These piecewise affine maps arise when studying the unfolding of homoclinic tangencies for a certain class of three-dimensional diffeomorphisms. They also prove the existence, for each natural number n, of an open set of parameters in which the respective transformation exhibits at least $2n$ two-dimensional strange attractors.

Vladimir E. Fedorov, Anna S. Avilovich and Lidiya V. Borel study initial problems for semilinear differential equations in Banach spaces with fractional Caputo derivative. They apply abstract results to the research into initial boundary value problems for a class of time-fractional order partial differential equations.

Feliz Minhós and Infeliz Coxe consider a system of *n*th-order differential equations with full nonlinearities coupled with two-point boundary conditions. They provide the solvability of such systems by using a Nagumo condition, lower and upper solutions, and Leray–Schauder degree theory. Moreover, they present two applications: to a Lorentz-Lagrangian system, for $n=2$, and to a stationary system of Korteweg-de Vries equations, for $n=3$.

Marina V. Plekhanova and Guzel D. Baybulatova deal with the Cauchy problem for two types of semilinear fractional differential equation in Banach spaces depending on the lower order fractional Caputo derivatives. They provide sufficient conditions for the existence of a unique solution in both cases and illustrate their abstracts results with two examples: a modified Oskolkov–Benjamin–Bona–Mahony–Burgers nonlinear equation with time-fractional derivatives and a non-linear system of partial differential equations not solvable with respect to the highest time-fractional derivative.

Bouchra Ben Amma, Said Melliani, and Lalla Saadia Chadli consider an intuitionistic fuzzy partial hyperbolic differential equation with integral boundary conditions. They obtain an existence and uniqueness result by means of Banach fixed-point theorem and present a procedure to solve some kinds of intuitionistic fuzzy partial hyperbolic differential equation. Several illustrative examples are also presented.

Olga Rozanova and Marko Turzynsky consider a model used for the description of the dynamics of the atmosphere of a rotating planet. Their main result proves that taking into account a small correction due to centrifugal force, which is usually neglected in the literature, drastically changes the stability properties of a specific class of vortices.

Jose S. Cánovas deals with the chaotic properties of the two-periodic Ricker model. In particular, he computes the topological entropy and shows the parameter region where the dynamics is chaotic for this model.

Alberto Cabada and Kadda Maazouz prove the existence, uniqueness, and location of solutions for implicit fractional differential equations involving the Hadamard fractional derivative in Banach spaces. The linear equation has been solved by means of the invertibility of the differential operator. Such an inverse operator is characterized by the kernel of a suitable integral operator. Its qualitative properties concerning the sign and boundedness properties allow the application of the Banach contraction principle and the deduction of the existence and uniqueness of the solution of the considered problem.

Francisco J. Fernández, Aurea Martínez, and Lino J. Alvarez-Vázquez formulate a suitable system of nonlinear partial differential equations to model a technique of artificial circulation for oxygenating eutrophic water bodies subject to quality problems. Then, they use fixed-point theory to prove the existence of at least one solution for the considered problem in an appropriate functional space.

Jiří Kadlec and Petr Nečesal deal with a boundary value problem for a second-order differential equation with a non-local boundary condition in integral form. Their main results describe the structure of the Fučík spectrum for this problem as a pair of regular curves.

Peter Tomiczek considers a second-order differential equation of Duffing type and uses a variational approach to prove the existence of at least one periodic solution. For it, he introduces a suitable function that satisfies the so-called Palais–Smale condition.

Sebastián Buedo-Fernández, Daniel Cao Labora, and Rosana Rodríguez-López present improvements in some comparison results for the periodic boundary value problem related to a first-order differential equation perturbed by a functional term. The comparison results presented cover many cases such as differential equations with delay, differential equations with maxima, and integrodifferential equations. The authors also analyze the interesting case of functional perturbation with piecewise constant arguments.

Gaber M. Bahaa and Delfim F. M. Torres investigate optimal control problems (OCP) for fractional systems involving fractional-time derivatives on time scales. In their analysis, the fractional-time derivatives and integrals are those of Riemann–Liouville. They consider a fractional OCP with a performance index given as a delta-integral function of both state and control variables, with time evolving on an arbitrarily given time scale. Interpreting the Euler–Lagrange first-order optimality condition with an adjoint problem, defined by means of right Riemann–Liouville fractional delta derivatives, they obtain an optimality system for the considered fractional OCP. For that, the authors prove new fractional integration by parts formulas on time scales.

Alberto Cabada and Lucía López-Somoza show several properties of the Green's functions related to various boundary value problems of arbitrary even order. As a consequence, they write the expression of the Green's functions related to the general differential operator of order $2n$ coupled to Neumann, Dirichlet, and mixed boundary conditions as a linear combination of the Green's functions corresponding to periodic conditions on a different interval. This allows to ensure the constant sign of various related Green's functions and to describe the spectrum of the considered differential operator with a given boundary condition as the union of several spectrums of the same operator with different boundary conditions.

Abdelkader Moulay and Abdelghani Ouahab consider an abstract evolution equation with random parameter. They introduce the notion of stabilization with respect to the random parameter and fractional integral-feedback. More precisely, they study the well-posedness and polynomial stabilization result for a random evolution equation with fractional integral-feedback. Finally, they show some applications to random heat and wave equations with fractional integral-feedback and bounded damping.

This volume would not have been possible without the help of various people who contributed in different ways. First of all, we would like to thank the authors themselves for submitting their work to this issue. Special thanks go to the referees who agreed to take part in this process: their comments and suggestions have led to improvements in most of the contributions.

We would also like to express our gratitude to Francesca Ferrari from Springer for her attention and constant support at every step in the editorial process. Moreover, we want to express our thanks for the financial support provided by Xunta de Galicia and Deputación de A Coruña (Spain). We are also grateful to the Faculty of Mathematics of the University of Santiago de Compostela for their support with respect to the conference location and the facilities available during the conference.

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A Variational Analogue of Krasnoselskii's Cone Fixed Point Theory

Radu Precup

Abstract Based on Ekeland's principle, a variational analogue of Krasnoselskii's cone compression-expansion fixed point theorem is presented. A general scheme of applications to semilinear equations making use of Mikhlin's variational theory on positive linear operators is included.

Keywords Critical point · Fixed point · Cone · Variational principle · Semilinear operator equation · Positive solution

Mathematics Subject Classification 47J30

1 Introduction

One of the most useful methods for the localization of positive solutions to nonlinear boundary value problems and to prove the existence of multiple positive solutions is Krasnoselskii's cone fixed point theorem [4–6]. There are known several versions of this result that we present shortly.

Let X be a Banach space, $K \subset X$ a cone and r, R two numbers with $0 < r < R$. Denote

$$K_r = \{u \in K : \|u\| \leq r\}, \quad \partial K_r = \{u \in K : \|u\| = r\},$$

and consider the conical shell

$$K_{rR} = \{u \in K : r \leq \|u\| \leq R\}.$$

Let $N : K_{rR} \to K$ be a continuous and compact mapping and consider the fixed point equation

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$$u = N(u), \quad u \in K_{rR}.$$

The original Krasnoselskii's cone fixed point theorem makes use of the strict order relation $<$ in X, with $u < v$ if $v - u \in K \setminus \{0\}$:

Theorem 1 (Order version) *The mapping N has a fixed point in K_{rR} if it satisfies one of the following conditions:*

(a) *$N(u) \nless u$ for $u \in \partial K_r$ and $N(u) \ngtr u$ for $u \in \partial K_R$ (compression condition);*

(b) *$N(u) \ngtr u$ for $u \in \partial K_r$ and $N(u) \nless u$ for $u \in \partial K_R$ (expansion condition).*

Some other versions are the following ones:

Theorem 2 (Norm version) *The mapping N has a fixed point in K_{rR} if it satisfies one of the following conditions:*

(a) *$\|N(u)\| > \|u\|$ for $u \in \partial K_r$ and $\|N(u)\| < \|u\|$ for $u \in \partial K_R$ (compression condition);*

(b) *$\|N(u)\| < \|u\|$ for $u \in \partial K_r$ and $\|N(u)\| > \|u\|$ for $u \in \partial K_R$ (expansion condition).*

Theorem 3 (Homotopy version) *The mapping N has a fixed point in K_{rR} if it satisfies one of the following conditions:*

(a) *$N(u) \neq \mu u$ for $u \in \partial K_r$, $\mu < 1$, $N(u) \neq \mu u$ for $u \in \partial K_R$, $\mu > 1$ and $\inf_{u \in \partial K_r} \|N(u)\| > 0$ (compression condition);*

(b) *$N(u) \neq \mu u$ for $u \in \partial K_r$, $\mu > 1$, $N(u) \neq \mu u$ for $u \in \partial K_R$, $\mu < 1$ and $\inf_{u \in K_R} \|N(u)\| > 0$ (expansion condition).*

In many cases, the fixed point equation has a variational structure in the sense that it is equivalent to the problem of finding critical points of a certain functional $F : X \to \mathbb{R}$, that is to the equation

$$F'(u) = 0, \quad u \in K_{rR}. \tag{1}$$

This clearly happens if X is a Hilbert space identified to its dual and

$$N(u) = u - F'(u),$$

when the **critical points** of F coincide with the **fixed points** of N.

A simple example is given by the following two-points boundary value problem for Newton's second law of motion,

$$mu'' + f(t, u) = 0, \quad t \in [0, T] \tag{2}$$
$$u(0) = u(T) = 0.$$

This can be expressed as a fixed point problem for the integral operator $N : C[0,T] \to C[0,T]$,

$$N(u)(t) = \int_0^T G(t,s) f(s,u(s))\,ds,$$

where G is the Green's function of the differential operator $-mu''$ under the conditions $u(0) = u(T) = 0$, and also as a critical point problem related to the functional $F : H_0^1(0,T) \to \mathbb{R}$,

$$F(u) = \int_0^T \left(\frac{m}{2}u'(t)^2 - g(t,u(t))\right) dt \quad \text{where} \quad g(t,u) = \int_0^u f(t,y)\,dy,$$

for which (1) holds. Physically, $F(u)$ is the total energy (kinetic + potential), and the kinetic energy (energy of motion) $(m/2)\int_0^T u'(t)^2\,dt$ introduces the so called "energetic" norm in the function space $H_0^1(0,T)$,

$$\|u\| = \left(\int_0^T u'(t)^2\,dt\right)^{1/2}.$$

Thus, a localization of a solution/state u in terms of the energetic norm automatically gives bounds of the kinetic energy.

Compared to the fixed point approach, the variational methods have as benefice, the use of the energy functional allowing to obtain characterizations of solutions as extrema or saddle points. In addition, some specific techniques such as Ekeland's variational principle and deformation lemmas [17] are available. In this paper we only deal with the direct variational method which exclusively uses Ekeland's variational principle [18].

Lemma 1 (Ekeland's principle—strong form) *Let D be a complete metric space with metric d, and let $F : D \to \mathbb{R}$ be lower semicontinuous and bounded from below. Then for any $\varepsilon, \delta > 0$, and any $w \in D$ with*

$$F(w) \le \inf_D F + \varepsilon,$$

there is an element $u \in D$ such that

$$F(u) \le F(w), \qquad d(w,u) \le \delta$$

and

$$F(u) \le F(v) + \frac{\varepsilon}{\delta} d(u,v) \quad \text{for all } v \in D.$$

As a consequence, one has the following weak form of Ekeland's variational principle:

Lemma 2 (Ekeland's principle—weak form) *Let D be a complete metric space with metric d, and let $F : D \to \mathbb{R}$ be lower semicontinuous and bounded from below. Then for each $\varepsilon > 0$, there is $u \in D$ such that*

$$F(u) \le \inf_D F + \varepsilon$$

and

$$F(u) \le F(v) + \varepsilon d(u, v) \quad \textit{for all } v \in D.$$

2 Variational Analogue of the Compression Krasnoselskii's Cone Fixed Point Theorem

In what follows, for simplicity, we only consider the case where X is a Hilbert space, with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, and we identify X to its dual.

Theorem 4 *Let $F \in C^1(X)$ be bounded from bellow on K_{rR}, $I - F'$ be continuous and compact on K_{rR}, and let the positivity condition*

$$\left(I - F'\right)(K_{rR}) \subset K \tag{3}$$

be satisfied. If

$$\begin{aligned} F'(u) + \lambda u &\neq 0 \ \ \textit{for all} \ \ u \in \partial K_R, \quad \lambda > 0, \\ F'(u) + \lambda u &\neq 0 \ \ \textit{for all} \ \ u \in \partial K_r, \quad \lambda < 0, \end{aligned} \tag{4}$$

and

$$\inf_{u \in \partial K_r} \left\| \left(I - F'\right)(u) \right\| > 0, \tag{5}$$

then there exists $u \in K_{rR}$ such that

$$F(u) = \inf_{K_{rR}} F \quad \textit{and} \quad F'(u) = 0.$$

Proof Step 1: Applying Ekeland's variational principle—weak form to F, on K_{rR}, with $\varepsilon = 1/n$, gives $u_n \in K_{rR}$ such that

$$F(u_n) \le \inf_{K_{rR}} F + \frac{1}{n}, \tag{6}$$

$$F(u_n) \le F(v) + \frac{1}{n} \|u_n - v\| \quad \text{for all } v \in K_{rR}. \tag{7}$$

Obviously, from (6), (u_n) is a minimizing sequence for F on K_{rR}, i.e., $F(u_n) \to \inf_{K_{rR}} F$ as $n \to \infty$. Next using (7) we shall estimate $F'(u_n)$. To this aim we approach u_n making suitable choices of v in K_{rR}. The choices of v in (7) depend on the location in the conical shell of each element u_n. The following cases are possible:

(a) $r < \|u_n\| < R$; or $\|u_n\| = R$ and $\langle F'(u_n), u_n \rangle > 0$; or $\|u_n\| = r$ and $\langle F'(u_n), u_n \rangle < 0$;

(b) $\|u_n\| = R$ and $\langle F'(u_n), u_n \rangle \leq 0$;

(c) $\|u_n\| = r$ and $\langle F'(u_n), u_n \rangle \geq 0$.

Case (a): If u_n is in case (a) then we may choose v of the form

$$v = u_n - tF'(u_n),$$

with $t > 0$ sufficiently small. Indeed, for $t \in (0, 1)$, one has

$$v = (1-t) u_n + t \left(u_n - F'(u_n) \right),$$

which, due to the positivity condition (3), belongs to K. In case that $r < \|u_n\| < R$, we also have $v \in K_{rR}$ for small enough t. If $\|u_n\| = R$ and $\langle F'(u_n), u_n \rangle > 0$, then from

$$\begin{aligned} \|v\|^2 &= \|u_n\|^2 + t^2 \left\| F'(u_n) \right\|^2 - 2t \langle F'(u_n), u_n \rangle \\ &= t^2 \left\| F'(u_n) \right\|^2 - 2t \langle F'(u_n), u_n \rangle + R^2, \end{aligned}$$

we derive that $\|v\| \leq R$ for $0 < t \leq 2 \langle F'(u_n), u_n \rangle / \left\| F'(u_n) \right\|^2$. Hence $v \in K_{rR}$ for every sufficiently small $t > 0$. The same happens if $\|u_n\| = r$ and $\langle F'(u_n), u_n \rangle < 0$. Replacing v in (7) gives

$$F\left(u_n - tF'(u_n)\right) - F(u_n) \geq -\frac{t}{n} \left\| F'(u_n) \right\|.$$

From the definition of the Fréchet derivative one has

$$F\left(u_n - tF'(u_n)\right) - F(u_n) = \langle F'(u_n), -tF'(u_n) \rangle + o(t).$$

Then

$$\langle F'(u_n), -tF'(u_n) \rangle + o(t) \geq -\frac{t}{n} \left\| F'(u_n) \right\|,$$

and dividing by t and letting $t \to 0$ gives

$$\left\| F'(u_n) \right\|^2 \leq \frac{1}{n} \left\| F'(u_n) \right\|,$$

or

$$\left\| F'\left(u_n\right)\right\| \leq \frac{1}{n}. \tag{8}$$

Case (b): Let $\varepsilon > 0$ and let

$$v = u_n - t\left(F'\left(u_n\right) + \lambda_n u_n + \varepsilon u_n\right),$$

where $t > 0$ and

$$\lambda_n = -\left\langle F'\left(u_n\right), u_n\right\rangle / R^2 \geq 0.$$

From $v = (1 - t - t\lambda_n - t\varepsilon)\, u_n + t\left(u_n - F'\left(u_n\right)\right)$ we see that $v \in K$ for small $t > 0$, while from

$$\left\langle F'\left(u_n\right) + \lambda_n u_n + \varepsilon u_n,\ u_n\right\rangle = \varepsilon R^2 > 0$$

and

$$\|v\|^2 = t^2\left\| F'\left(u_n\right) + \lambda_n u_n + \varepsilon u_n\right\|^2 - 2t\varepsilon R^2 + R^2,$$

we have $\|v\| \leq R,$ and finally that $v \in K_{rR}$ for small enough $t > 0$. Replacing v in (7) and proceeding as above we find

$$\left\langle F'\left(u_n\right), F'\left(u_n\right) + \lambda_n u_n + \varepsilon u_n\right\rangle \leq \frac{1}{n}\left\| F'\left(u_n\right) + \lambda_n u_n + \varepsilon u_n\right\|.$$

Letting $\varepsilon \to 0$ yields

$$\left\langle F'\left(u_n\right), F'\left(u_n\right) + \lambda_n u_n\right\rangle \leq \frac{1}{n}\left\| F'\left(u_n\right) + \lambda_n u_n\right\|,$$

and since $\left\langle F'\left(u_n\right) + \lambda_n u_n, u_n\right\rangle = 0,$

$$\left\| F'\left(u_n\right) + \lambda_n u_n\right\| \leq \frac{1}{n}. \tag{9}$$

Case (c) is analogous and leads to the same inequality (9), where now $\lambda_n \leq 0$.

Step 2: Passing if necessary to a subsequence, we may assume without lost of generality that all the terms of the minimizing sequence (u_n) are either in case (a), or in case (b), or in case (c). Then in view of (8) and (9), the minimizing sequence is in one of the following situations:

(a) $F'\left(u_n\right) \to 0$;

(b) $F'\left(u_n\right) + \lambda_n u_n \to 0$, where $\|u_n\| = R$ and $\lambda_n = -\left\langle F'\left(u_n\right), u_n\right\rangle / R^2 \geq 0$ for all n;

(c) $F'\left(u_n\right) + \lambda_n u_n \to 0$, where $\|u_n\| = r$ and $\lambda_n = -\left\langle F'\left(u_n\right), u_n\right\rangle / r^2 \leq 0$ for all n.

Step 3: Since u_n and $F'(u_n) = u_n - N(u_n)$ are bounded sequences, we may assume (passing if necessary again to a subsequence) that (λ_n) converges to some λ, where $\lambda \geq 0$ in case (b), and $\lambda \leq 0$ in case (c). Also, using the compactness of N, the above convergences lead to a convergent subsequence $u_n \to u$. We show this for the cases (b) and (c). To this aim we denote $v_n = F'(u_n) + \lambda_n u_n$. Then $(1 + \lambda_n) u_n = v_n + N(u_n)$ and since $v_n \to 0$ and N is compact, the sequence $v_n + N(u_n)$ is compact. If $1 + \lambda \neq 0$, this clearly implies the compactness of the sequence u_n. The situation $1 + \lambda = 0$ is only possible in case (c), but is excluded by hypothesis (5).

Finally, passing to limit we obtain one of the following situations:

(a) $F'(u) = 0$;

(b) $F'(u) + \lambda u = 0$, where $\|u\| = R$ and $\lambda \geq 0$;

(c) $F'(u) + \lambda u = 0$, where $\|u\| = r$ and $\lambda \leq 0$.

The cases $\lambda > 0$ in (b) and $\lambda < 0$ in (c) being excluded by the compression boundary conditions (4), it remains that in all cases $F'(u) = 0$, which finishes the proof. □

3 Variational Analogue of the Expansion Krasnoselskii's Cone Fixed Point Theorem

In this section, we give a variational analogue of Krasnoselskii's fixed point theorem of expansion. Recall that for proving Krasnoselskii's fixed point theorem of expansion it suffices to pass from the operator N satisfying the expansion conditions, to the operator $\widetilde{N} : K_{rR} \to K$,

$$\widetilde{N}(u) = \frac{1}{\theta(u)} N(\theta(u) u),$$

where

$$\theta(u) = \frac{R + r}{\|u\|} - 1.$$

It is easy to check that $\theta(u) u \in K_{rR}$ for every $u \in K_{rR}$, $\|\theta(u) u\| = r$ if $\|u\| = R$, $\|\theta(u) u\| = R$ if $\|u\| = r$, and that for $\widetilde{N}$ the compression conditions hold. We shall use the same idea in order to prove the variational analogue of the expansion fixed point result. More exactly, we shall pass from the functional F, assumed to be bounded from above on K_{rR}, to the functional

$$H(u) = -F(\theta(u) u), \quad u \in X \setminus \{0\},$$

bounded from below on K_{rR}. We shall need the following result about the Fréchet derivative of the new functional.

Lemma 3 *One has* $H \in C^1(X \setminus \{0\})$ *and*

$$H'(u) = F'(\theta(u)u) + A(u),$$

where

$$A(u) = \frac{R+r}{\|u\|}\left[\frac{\langle F'(\theta(u)u), u\rangle}{\|u\|^2}u - F'(\theta(u)u)\right].$$

Proof We first compute the derivative of the mapping $u/\|u\|$ in direction v. By definition,

$$\begin{aligned}\left\langle\left(\frac{u}{\|u\|}\right)', v\right\rangle &= \lim_{t\to 0+}\frac{1}{t}\left(\frac{u+tv}{\|u+tv\|} - \frac{u}{\|u\|}\right)\\ &= \frac{1}{\|u\|^2}\lim_{t\to 0+}\frac{1}{t}\left(\|u\|(u+tv) - \|u+tv\|u\right)\\ &= \frac{1}{\|u\|^2}\lim_{t\to 0+}\frac{1}{t}\left(\|u\| - \|u+tv\|\right)u + \frac{v}{\|u\|}.\end{aligned}$$

Furthermore

$$\lim_{t\to 0+}\frac{1}{t}\left(\|u\| - \|u+tv\|\right)u = \lim_{t\to 0+}\frac{1}{t}\frac{\|u\|^2 - \|u+tv\|^2}{\|u\| + \|u+tv\|}u = -\frac{\langle u, v\rangle}{\|u\|}u.$$

Hence

$$\left\langle\left(\frac{u}{\|u\|}\right)', v\right\rangle = -\frac{\langle u, v\rangle}{\|u\|^3}u + \frac{v}{\|u\|}.$$

Next,

$$\begin{aligned}\left\langle(\theta(u)u)', v\right\rangle &= (R+r)\left[-\frac{\langle u, v\rangle}{\|u\|^3}u + \frac{v}{\|u\|}\right] - v\\ &= \theta(u)v - \frac{R+r}{\|u\|^3}\langle u, v\rangle u.\end{aligned}$$

Finally, using the formula for computing the derivative of the composition of two mappings, we obtain

$$\begin{aligned}\left\langle H'(u), v\right\rangle &= -\left\langle F'(\theta(u)u), \left\langle(\theta(u)u)', v\right\rangle\right\rangle\\ &= -\left\langle F'(\theta(u)u), \theta(u)v - \frac{R+r}{\|u\|^3}\langle u, v\rangle u\right\rangle.\end{aligned}$$

Therefore

$$H'(u) = -\theta(u) F'(\theta(u) u) + \frac{R+r}{\|u\|^3} \left\langle F'(\theta(u) u),\ u \right\rangle u$$
$$= F'(\theta(u) u) + \frac{R+r}{\|u\|} \left[\frac{\left\langle F'(\theta(u) u),\ u \right\rangle}{\|u\|^2} u - F'(\theta(u) u) \right],$$

as claimed. □

Theorem 5 *Let $F \in C^1(X)$ be bounded from above on K_{rR}, $I - F'$ be compact on K_{rR}, the positivity condition*

$$\left(I - F'\right)(K_{rR}) \subset K$$

be satisfied, and assume in addition that for every two sequences u_n and v_n in K_{rR},

$$u_n - v_n \to 0 \ \text{ implies (via subsequences) } N(u_n) - N(v_n) \to 0. \qquad (10)$$

If

$$F'(u) + \lambda u \neq 0 \quad \text{for all} \quad u \in \partial K_R, \quad \lambda < 0, \qquad (11)$$
$$F'(u) + \lambda u \neq 0 \quad \text{for all} \quad u \in \partial K_r, \quad \lambda > 0,$$

and

$$\inf_{u \in \partial K_R} \left\| \left(I - F'\right)(u) \right\| > 0,$$

then there exists $u \in K_{rR}$ such that

$$F(u) = \sup_{K_{rR}} F \quad \text{and} \quad F'(u) = 0.$$

Proof Notice a useful property of A, namely

$$\langle A(u), u \rangle = 0 \ \text{ for every } \ u \in X \setminus \{0\}.$$

Let us fix a minimizing sequence (v_n) of H in K_{rR}, with

$$H(v_n) \leq \inf_{K_{rR}} H + \frac{1}{n^2}.$$

For each $n \geq 1$, consider the functional

$$G_n(u) = H(u) - \langle A(v_n), u \rangle, \quad u \in X \setminus \{0\}.$$

One has

$$G_n'(u) = H'(u) - A(v_n) = F'(\theta(u)u) + A(u) - A(v_n), \quad u \in X \setminus \{0\}.$$

Now apply Ekeland's variational principle—strong form to G_n on K_{rR}, with the given point v_n and for $\varepsilon = n^{-2}, \ \delta = n^{-1}$. Hence, there exists $u_n \in K_{rR}$ such that

$$\|u_n - v_n\| \le \delta = \frac{1}{n},$$

$$G_n(u_n) \le G_n(v_n) = H(v_n) \le \inf_{K_{rR}} H + \frac{1}{n^2},$$

$$\begin{aligned} G_n(u_n) &\le G_n(v) + \frac{\varepsilon}{\delta}\|u_n - v\| \\ &= G_n(v) + \frac{1}{n}\|u_n - v\| \quad \text{for all } v \in K_{rR}. \end{aligned} \tag{12}$$

First we show that, passing if necessary to a subsequence, we may assume that

$$A(u_n) - A(v_n) \to 0 \quad \text{as } n \to \infty. \tag{13}$$

Indeed, since u_n and v_n are bounded and N is compact on K_{rR}, passing to a subsequence we have that

$$\|u_n\| \to l_1, \quad \|v_n\| \to l_2, \quad N(\theta(u_n)u_n) \to w_1, \quad N(\theta(v_n)v_n) \to w_2.$$

From $|\|u_n\| - \|v_n\|| \le \|u_n - v_n\| \le 1/n$, we find that $l_1 = l_2 =: l$. Then $\theta(u_n)u_n - \theta(v_n)v_n \to 0$, and from (10), we deduce that $w_1 = w_2 =: w$. It remains to prove that $\alpha_n \to 0$, where

$$\begin{aligned} \alpha_n :&= \frac{\langle F'(\theta(u_n)u_n), u_n\rangle}{\|u_n\|^2}u_n - F'(\theta(u_n)u_n) \\ &- \frac{\langle F'(\theta(v_n)v_n), v_n\rangle}{\|v_n\|^2}v_n + F'(\theta(v_n)v_n). \end{aligned}$$

One has

$$\alpha_n = N(\theta(u_n)u_n) - N(\theta(v_n)v_n) - \frac{\langle N(\theta(u_n)u_n), u_n\rangle}{\|u_n\|^2}u_n + \frac{\langle N(\theta(v_n)v_n), v_n\rangle}{\|v_n\|^2}v_n$$

and since $N(\theta(u_n)u_n) - N(\theta(v_n)v_n) \to 0$, it remains to show that

$$\beta_n := \frac{\langle N\left(\theta\left(u_n\right) u_n\right), u_n\rangle}{\|u_n\|^2} u_n - \frac{\langle N\left(\theta\left(v_n\right) v_n\right), v_n\rangle}{\|v_n\|^2} v_n \to 0.$$

This, via

$$\beta_n = \frac{\langle N\left(\theta\left(u_n\right) u_n\right), u_n\rangle}{\|u_n\|^2}\left(u_n - v_n\right) + \left[\frac{\langle N\left(\theta\left(u_n\right) u_n\right), u_n\rangle}{\|u_n\|^2} - \frac{\langle N\left(\theta\left(v_n\right) v_n\right), v_n\rangle}{\|v_n\|^2}\right] v_n$$

reduces to

$$\frac{\langle N\left(\theta\left(u_n\right) u_n\right), u_n\rangle}{\|u_n\|^2} - \frac{\langle N\left(\theta\left(v_n\right) v_n\right), v_n\rangle}{\|v_n\|^2} \to 0.$$

But this immediately follows if again we pass to a subsequence in order to assume the weak convergence $u_n \rightharpoonup u,\ v_n \rightharpoonup u$. Thus (13) is proved.

Next, as in the proof of Theorem 4, we discuss several cases depending on the location of each element of the minimizing sequence u_n.

(a) In each one of the cases: $r < \|u_n\| < R$; $\|u_n\| = R$ and $\left\langle F'\left(\theta\left(u_n\right) u_n\right), u_n\right\rangle > 0$; $\|u_n\| = r$ and $\left\langle F'\left(\theta\left(u_n\right) u_n\right), u_n\right\rangle < 0$, we may apply (12) to the element

$$v = u_n - tF'\left(\theta\left(u_n\right) u_n\right)$$

which belongs to K_{rR} for all sufficiently small $t > 0$. Replacing into (12), dividing by t and then letting t go to zero it yields

$$\left\langle G_n'\left(u_n\right), F'\left(\theta\left(u_n\right) u_n\right)\right\rangle \le \frac{1}{n}\left\|F'\left(\theta_n\left(u_n\right) u_n\right)\right\|,$$

or equivalently

$$\left\|F'\left(\theta_n\left(u_n\right) u_n\right)\right\|^2 + \left\langle A\left(u_n\right) - A\left(v_n\right), F'\left(\theta\left(u_n\right) u_n\right)\right\rangle \le \frac{1}{n}\left\|F'\left(\theta_n\left(u_n\right) u_n\right)\right\|.$$

This implies

$$\left\|F'\left(\theta_n\left(u_n\right) u_n\right)\right\| \le \frac{1}{n} + \left\|A\left(u_n\right) - A\left(v_n\right)\right\|. \tag{14}$$

(b) Assume that $\|u_n\| = R$ and $\left\langle F'\left(\theta\left(u_n\right) u_n\right), u_n\right\rangle \le 0$. Then we choose

$$v = u_n - t\left(F'\left(\theta\left(u_n\right) u_n\right) + \lambda_n u_n + \varepsilon u_n\right),$$

where $\varepsilon > 0$ and $\lambda_n = -\left\langle F'\left(\theta\left(u_n\right) u_n\right), u_n\right\rangle / R^2 \ge 0$. We deduce

$$\left\|F'\left(\theta\left(u_n\right) u_n\right) + \lambda_n u_n\right\| \le \frac{1}{n} + \left\|A\left(u_n\right) - A\left(v_n\right)\right\|. \tag{15}$$

(c) Similarly, if $\|u_n\| = r$ and $\left\langle F'\left(\theta\left(u_n\right)u_n\right), u_n\right\rangle \geq 0$, we derive inequality (15), where this time $\lambda_n \leq 0$.

If there is a subsequence of u_n whose elements are all in case (a), then from (14) and (13) we have

$$F'\left(\theta\left(u_n\right)u_n\right) \to 0.$$

If there is a subsequence whose elements are all in case (b), or all in case (c), then

$$F'\left(\theta\left(u_n\right)u_n\right) + \lambda_n u_n \to 0.$$

As in the proof of Theorem 4 we may assume that $u_n \to u$ for some $u \in K_{rR}$. Then $v_n \to u$, and passing to the limit we obtain: $F'\left(\theta\left(u\right)u\right) = 0$; or $F'\left(\theta\left(u\right)u\right) + \lambda u = 0$ with $\|u\| = R$ and $\lambda \geq 0$; or $F'\left(\theta\left(u\right)u\right) + \lambda u = 0$ with $\|u\| = r$ and $\lambda \leq 0$. Denote $\overline{u} = \theta\left(u\right)u$. Then

$$F'\left(\overline{u}\right) = 0; \quad \text{or}$$

$$F'\left(\overline{u}\right) + \mu\overline{u} = 0 \quad \text{with} \quad \|\overline{u}\| = r \quad \text{and} \quad \mu = \lambda R/r \geq 0; \quad \text{or}$$

$$F'\left(\overline{u}\right) + \mu\overline{u} = 0 \quad \text{with} \quad \|\overline{u}\| = R \quad \text{and} \quad \mu = \lambda r/R \leq 0.$$

The case $\mu \neq 0$ being excluded by the expansion conditions (11), it remains that in any case, $F'\left(\overline{u}\right) = 0$.

Finally, from

$$H\left(u_n\right) = G_n\left(u_n\right) + \left\langle A\left(v_n\right), u_n\right\rangle \leq \inf_{K_{rR}} H + \frac{1}{n^2} + \left\langle A\left(v_n\right), u_n\right\rangle,$$

and $\langle Av_n), u_n\rangle \to \langle A\left(u\right), u\rangle = 0$, we have $H\left(u\right) = \inf_{K_{rR}} H$, that is $F\left(\overline{u}\right) = \sup_{K_{rR}} F$. □

Remark 1 Most of the assumptions of Theorems 4 and 5 may be expressed in terms of operator N, as in Theorem 3. There is however an additional hypothesis of Theorems 4 and 5, namely the representation of the operator N under the form $N = I - F'$, with some functional F bounded from below or from above on K_{rR}. As a result, we have a stronger conclusion: the existence of a fixed point of N which is an extremum point of the functional F.

4 A General Scheme of Application to Semilinear Equations

Krasnoselskii's cone fixed point theorem have been applied to numerous classes of boundary value problems. Also, in the last years, there have been given applications of critical point results in conical shells [1–3, 7, 11–13]. Thus, a natural question is which are the essential proprieties that allow applicability of this technique. We now present a general scheme of applicability of the variational analogue of Krasnoselskii's theorem, by which we give an answer to that question. To this aim, we use Mikhlin's variational theory for positive symmetric linear operators [8].

Consider a semilinear equation of the form

$$Lu = J'(u), \tag{16}$$

where $L : D(L) \subset H \to H$ is a positive symmetric densely defined linear operator in the Hilbert space H with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, while the nonlinear term is the Fréchet derivative of a C^1 functional $J : H \to \mathbb{R}$.

Recall that the operator L is said to be *symmetric* if $(Lu, v) = (u, Lv)$ for every $u, v \in D(L)$, and *positive* if there exists a constant $c > 0$ such that

$$(Lu, u) \geq c^2 |u|^2 \quad \text{for every } u \in D(L).$$

For such a linear operator, we endow the dense linear subspace $D(L)$ of H with the bilinear functional

$$\langle u, v\rangle := (Lu, v) \quad (u, v \in D(L)).$$

The completion of $(D(L), \langle\cdot,\cdot\rangle)$ is denoted by X and is called the *energetic space* of L. By the construction, $D(L) \subset X \subset H$ with dense inclusions. We use the same symbol $\langle\cdot,\cdot\rangle$ to denote the extended inner product on X. The corresponding norm $\|u\| = \sqrt{\langle u, u\rangle}$ is called the *energetic norm* associated to L. If $u \in D(L)$, then in view of the positivity of L, one has the *Poincaré inequality*

$$|u| \leq c^{-1} \|u\| \quad \text{for every } u \in D(L).$$

By density the above inequality extends to the whole X. Let X' be the dual space of X. If we identify the dual H' with H, via Riesz's representation theorem, then from $X \subset H$, we have $H \subset X'$. We attach to the operator L the following problem

$$Lu = f, \quad u \in X, \tag{17}$$

where $f \in X'$. By a *weak solution* of the problem we mean an element $u \in X$ with

$$\langle u, v\rangle = (f, v) \quad \text{for every } v \in X,$$

where the notation (f, v) stands for the value of the functional f on the element v. In case that $f \in H$, then (f, v) is the inner product in H of f and v. Notice that, if the weak solution u belongs to $D(L)$, then it is a classical solution of the problem. Using Riesz's representation theorem and the Poincaré inequality one has that for every $f \in X'$ there exists a unique weak solution $u \in X$ of the problem (17). Thus we may speak about the inverse of L, as the operator $L^{-1} : X' \to X$ attaching to each $f \in X'$, the unique weak solution $u \in X$ of the corresponding Eq. (17). Thus,

$$\left\langle L^{-1}f, v\right\rangle = (f, v) \quad \text{for all } v \in X.$$

Note that the operator L^{-1} is an isometry between X' and X.

We look for weak solutions of (16), namely, for $u \in X$ such that

$$\langle u, v\rangle = \left(J'(u), v\right) \quad \text{for all } v \in X,$$

that is for solutions of the fixed point equation

$$u = L^{-1}J'(u), \quad u \in X.$$

We associate to the Eq. (16) the *energy functional*

$$F : X \to \mathbb{R}, \quad F(u) = \frac{1}{2}\|u\|^2 - J(u).$$

One can check that $F \in C^1(X)$,

$$F'(u) = Lu - J'(u) \quad (u \in X),$$

and, if we identify X' to X via L^{-1}, one has

$$F' = I - N, \quad \text{where} \quad N = L^{-1}J'.$$

Our first hypothesis is a compactness condition:

(H1) The embedding $X \subset H$ is compact.

Next we consider a cone K_0 in H, the partial order relation $\leq$ in H induced by K_0, and we assume that

$$(u, v) \geq 0 \quad \text{for every } u, v \in K_0,$$

or equivalently, that the norm of H is monotone. In addition assume that

(H2) J is bounded on every bounded subset of H; J' is positive and increasing on H with respect to the order, i.e.,

$$0 \leq u \leq v \quad \text{implies} \quad 0 \leq J'(u) \leq J'(v).$$

Also consider a cone K_1 in H with

$$L^{-1}(K_0) \subset K_1 \subset K_0 \tag{18}$$

and assume that the following conditions are satisfied:

(H3) $J'(K_1 \cap X) \subset K_1$, and there exists $\varphi \in K_0 \setminus \{0\}$ such that for every $u \in K_1$,

$$\left\| L^{-1}u \right\| \varphi \leq L^{-1}u \quad \text{(Harnack type inequality);}$$

(H4) There is an element $\psi \in K_0 \setminus \{0\}$ such that for every $u \in K_1 \cap X$,

$$u \leq \|u\| \psi.$$

Now, define a subcone of $K_1 \cap X$,

$$K := \{u \in K_1 \cap X : \|u\| \varphi \leq u\}.$$

Note that the cone K does not reduce to the origin. To show this, let σ be any element of $K_1 \setminus \{0\}$. For example, such an element is $L^{-1}(\varphi)$. Then $L^{-1}\sigma \neq 0$, and using (18) and (H3), $L^{-1}\sigma \in K_1 \cap X$, and $\left\| L^{-1}\sigma \right\| \varphi \leq L^{-1}\sigma$, that is $L^{-1}\sigma \in K \setminus \{0\}$.

Theorem 6 *Assume that* (H1)–(H4) *hold. If for two positive numbers* α *and* β *with* $\alpha \neq \beta$, *the following conditions are satisfied*

$$\begin{aligned} \left(J'(\alpha\psi), \psi\right) &\leq \alpha, \\ \beta &\leq \left(J'(\beta\varphi), \varphi\right), \end{aligned} \tag{19}$$

then Eq. (16) has a weak solution $u \in K_{rR}$, *an extremum point of F in* K_{rR}, *where* $r = \min\{\alpha, \beta\}$ *and* $R = \max\{\alpha, \beta\}$.

Proof We shall apply Theorem 4 in case that $\beta < \alpha$, and Theorem 5 if $\alpha < \beta$.

First note that N maps K into K. Indeed, if $u \in K$, then $u \in K_1 \cap X$ and by (H3), $J'(u) \in K_1$ and $\left\| L^{-1}J'(u) \right\| \varphi \leq L^{-1}J'(u)$, which means that $N(u) = L^{-1}J'(u) \in K$.

Clearly the operator N is continuous. Furthermore, K_{rR} being bounded in X, it is relatively compact in H by (H1), and the continuity of J' from H to H guarantees that $J'(K_{rR})$ is relatively compact in H. Then $N(K_{rR}) = L^{-1}J'(K_{rR})$ is relatively compact in X. Hence N is continuous and compact from K_{rR} to K. Also, the additional compactness condition (10) is satisfied as a consequence of (H1). Indeed, if

$$u_n, v_n \in K_{rR} \text{ and } u_n - v_n \to 0 \text{ in } X,$$

then in view of (H1), passing if necessary to subsequences, we may assume that u_n and v_n converge in H to some element u. Then $J'(u_n)$, $J'(v_n)$ converge in H to $J'(u)$, and next $N(u_n)$, $N(v_n)$ converge in X to $N(u)$. Consequently,

$$N(u_n) - N(v_n) \to 0 \text{ in } X,$$

which proves (10).

The functional J being bounded on the bounded set K_R, one has that F is bounded from above and from below on K_{rR}.

Next we check the boundary conditions. Let $u \in \partial K_\alpha,\ \lambda > 0$, and assume that $F'(u) + \lambda u = 0$. Then $N(u) = (1+\lambda)u$. From (H4), $0 \le u \le \alpha\psi$, and the monotonicity of J' yields to $0 \le J'(u) \le J'(\alpha\psi)$. Then

$$\begin{aligned} \alpha^2 &< (1+\lambda)\alpha^2 = (1+\lambda)\|u\|^2 = \langle N(u), u\rangle \\ &= \left\langle L^{-1}J'(u), u\right\rangle = \left(J'(u), u\right) \le \left(J'(\alpha\psi), \alpha\psi\right) \\ &= \alpha\left(J'(\alpha\psi), \psi\right). \end{aligned}$$

Hence $\alpha < \left(J'(\alpha\psi), \psi\right)$, which is in contradiction with (19). Next, assume that $u \in \partial K_\beta,\ \lambda < 0$, and $F'(u) + \lambda u = 0$, that is $N(u) = (1+\lambda)u$. Since both u and $N(u)$ are in K_0, this equality is possible only if $1+\lambda \ge 0$. Hence $0 \le 1+\lambda < 1$. Consequently,

$$\beta^2 > (1+\lambda)\beta^2 = \left(J'(u), u\right),$$

and since $u \ge \|u\|\varphi = \beta\varphi$ gives $J'(u) \ge J'(\beta\varphi) \ge 0$, and the norm in H is monotone,

$$\left(J'(u), u\right) \ge \left(J'(\beta\varphi), \beta\varphi\right).$$

Hence we derive $\beta > \left(J'(\beta\varphi), \varphi\right)$, contrary to our hypothesis.

It remains to show that $\inf_{u\in\partial K_\beta}\|N(u)\| > 0$. Assume the contrary. Then there is a sequence $u_n \in \partial K_\beta$ with $N(u_n) \to 0$ in X and also in H. From $u_n \ge \beta\varphi \ge 0$, we obtain that $N(u_n) \ge N(\beta\varphi) \ge 0$. Passing to limit as $n \to \infty$ yields $N(\beta\varphi) = 0$, whence $J'(\beta\varphi) = 0$ which makes impossible the second inequality in (19). □

Remark 2 The inequality $L^{-1}(K_0) \subset K_0$ can be seen as a weak maximum principle, while by the use of a second cone K_1, the Harnack inequality is not required on the whole cone K_0, but only on its subcone K_1. This is useful in applications as shown by the following Example 2.

Example 1 In the simple case of problem (2), for $m = 1$, we have $H = L^2(0,1)$, $X = H_0^1(0,1)$ with inner product $\langle u, v\rangle = \int_0^1 u'v'$, $K_0 = K_1$ is the cone of positive functions in $L^2(0,1)$,

$$\psi = 1 \quad \text{and} \quad \varphi = \eta\chi_{[a,b]},$$

where $\eta = \min\{a, 1-b\}$, $0 < a < b < 1$ and $\chi_{[a,b]}$ is the characteristic function of the interval $[a,b]$. Here $J'(u)(t) = f(t, u(t))$, where $f \ge 0$ on $[0,1]\times\mathbb{R}_+$ and f is increasing in the second variable on $\mathbb{R}_+$. Then, condition (19) reduces to

$$\int_0^1 f(t,\alpha)\,dt \le \alpha,$$

$$\beta \le \eta\int_a^b f(t,\eta\beta)\,dt.$$

For more general examples we may consider semilinear boundary value problems with a linear part of the form

$$Lu = \sum_{k=0}^{m} (-1)^k \frac{d^k}{dt^k} \left[p_k (t) \frac{d^k u}{dt^k} \right] + q (t) u.$$

Such an example is considered in the paper [3]:

Example 2 Consider the boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), & 0 < t < 1 \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

Here $H = L^2(0, 1)$, $Lu = d^4/dt^4$, $X = \left\{u \in H^2(0, 1) : \ u(0) = u'(0) = 0\right\}$,

$$\langle u, v\rangle = \int_0^1 u''v''dt, \quad J(u) = \int_0^1 \int_0^{u(t)} f(t, s)dsdt, \quad J'(u) = f(\cdot, u),$$

$$K_0 = \left\{u \in L^2(0, 1) : \ u \geq 0\right\}, \quad K_1 = \{u \in K_0 : \ u \text{ - nondecreasing}\},$$

$$\psi = \frac{2}{3}t^{\frac{3}{2}}, \quad \varphi = \frac{\sqrt{2}}{6}(1 - t) t^3 \quad \text{(see [3])}.$$

Assuming that f is nonnegative and nondecreasing in each of its variables on $[0, 1] \times \mathbb{R}_+$, inequalities (19) reduce to

$$\int_0^1 \psi (t) f (t, \psi (t) \alpha) dt \leq \alpha,$$

$$\beta \leq \int_0^1 \varphi (t) f (t, \varphi (t) \beta) dt.$$

The application of the general scheme to concrete classes of boundary value problems mainly depends on the possibility to obtain a Harnack type inequality in terms of the energetic norm, as required by the abstract condition (H3). In many cases, including elliptic boundary value problems, Harnack type inequalities are only known with respect to a norm different from the energetic one. This was the reason in [9] to consider conical shells defined by two norms. Even more general, for the definition of conical shells, one may consider functionals which are not norms any more, like in [14]. Of course, in such situations, the conditions required on the shell boundary have to be adapted accordingly.

Finally, we mention that analogous results in conical shells, of mountain pass type, can be found in [9, 10]. For some extensions to Banach spaces and related topics we refer the reader to the papers [7, 15, 16].

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Differentiability Properties of Solutions of a Second-Order Evolution Inclusion

Aurelian Cernea

Abstract We study a certain second-order evolution inclusion defined by a family of linear closed operators which is the generator for an evolution system of operators and by a set-valued map with nonconvex values in a separable Banach space. We provide results concerning the differentiability of mild solutions with respect to the initial conditions of the problem considered. Certain variational inclusions are obtained in terms of set-valued derivatives defined by the contingent cone, the quasitangent cone and Clarke's tangent cone. Our results may be interpreted as extensions to a special class of second-order differential inclusions of the classical Bendixson–Picard–Lindelőf theorem concerning the differentiability of the maximal flow of a differential equation.

Keywords Differential inclusion · Mild solution · Tangent cone

1 Introduction

An outstanding result in the theory of differential equations is the classical Bendixson–Picard–Lindelőf theorem. This result states that the maximal flow of a differential equation is differentiable with respect to initial conditions and its derivatives verify the variational equation. Such kind of results are very useful in Control Theory, especially if we want to obtain necessary optimality conditions.

Bendixson–Picard–Lindelőf theorem has been generalized in various ways to differential inclusions by considering several variational inclusions and proving corresponding theorems that extend Bendixson–Picard–Lindelőf theorem [1, 6, 7] etc.

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The present paper is concerned with second-order evolution inclusions of the form

$$x'' \in A(t)x + F(t,x), \quad x(0) = x_0, \quad x'(0) = x_1, \tag{1}$$

where $F : [0,T] \times X \to \mathscr{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, x_1 \in X$ and $\{A(t)\}_{t\geq 0}$ is a family of linear closed operators from X into X which is the generator for an evolution system of operators $\{\mathscr{U}(t,s)\}_{t,s\in[0,T]}$.

The general framework of evolution operators $\{A(t)\}_{t\geq 0}$ that define problem (1) has been developed by Kozak [15] and improved by Henriquez [13]. In several recent papers [2–5, 8–10, 13, 14] existence results and qualitative properties of solutions for problem (1) have been obtained by using, mainly, fixed point techniques.

The aim of this paper is to extend the results concerning the differentiability of solutions of differential inclusions with respect to initial conditions to the mild solutions of problem (1). The results we extend known as the contingent, the intermediate (quasitangent) and the circatangent variational inclusion are obtained in the "classical case" of first-order differential inclusions. For these results and for a complete discussion on this topic we refer to [1].

The proofs of our results follow by a similar approach to the classical case of differential inclusions [1] and use a recent result [9] concerning the existence of mild solutions of problem (1).

The results in this paper may be interpreted as an extension of the results in [6] obtained for second-order differential inclusions defined by cosine family of operators to the more general problem (1).

The paper is organized as follows: in Sect. 2 we present preliminary results to be used in the next section and in Sect. 3 we prove our main results.

2 Preliminaries

In this short section we recall some basic notations and concepts concerning differential inclusions and nonsmooth analysis.

Let Y be a normed space, $X \subset Y$ and $x \in \overline{X}$ (the closure of X).

From the multitude of the tangent cones in the literature (e.g., [1]) we recall only the contingent, the quasitangent and Clarke's tangent cones, defined, respectively by

$$K_x X = \{v \in Y; \quad \exists s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\},$$
$$Q_x X = \{v \in Y; \quad \forall s_m \to 0+,\ \exists v_m \to v :\ x + s_m v_m \in X\},$$
$$C_x X = \{v \in Y; \forall (x_m, s_m) \to (x, 0+),\ x_m \in X,\ \exists y_m \in X :\ \frac{y_m - x_m}{s_m} \to v\}.$$

This cones are related as follows: $C_x X \subset Q_x X \subset K_x X$.

Corresponding to each type of tangent cone, say $\tau_x X$, one may introduce a *set-valued directional derivative* of a multifunction $G(\cdot) : X \subset Y \to \mathscr{P}(Y)$ (in particular of a single-valued mapping) at a point $(x, y) \in Graph(G)$ as follows

$$\tau_y G(x; v) = \{w \in Y; \quad (v, w) \in \tau_{(x,y)} Graph(G)\}, \quad v \in \tau_x X.$$

Let denote by I the interval $[0, T]$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$. Denote by $\mathscr{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I, by $\mathscr{P}(X)$ the family of all nonempty subsets of X and by $\mathscr{B}(X)$ the family of all Borel subsets of X. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}, \quad d^*(A, B) = \sup\{d(a, B); a \in A\},$$

where $d(x, B) = \inf_{y \in B} d(x, y)$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x(.) : I \to X$ endowed with the norm $|x(.)|_C = \sup_{t \in I} |x(t)|$, by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x(.) : I \to X$ endowed with the norm $|x(.)|_1 = \int_I |x(t)| \mathrm{d}t$ and by $B(X)$ the Banach space of linear bounded operators on X.

In what follows $\{A(t)\}_{t \geq 0}$ is a family of linear closed operators from X into X which is the generator for an evolution system of operators $\{\mathscr{U}(t, s)\}_{t,s \in I}$. By hypothesis the domain of $A(t)$, $D(A(t))$ is dense in X and is independent of t.

Definition 1 ([13, 15]) A family of bounded linear operators $\mathscr{U}(t, s) : X \to X$, $(t, s) \in \Delta := \{(t, s) \in I \times I; s \leq t\}$ is called an evolution operator of the equation

$$x''(t) = A(t)x(t) \tag{2}$$

if

(i) For any $x \in X$, the map $(t, s) \to \mathscr{U}(t, s)x$ is continuously differentiable and

(a) $\mathscr{U}(t, t) = 0$, $t \in I$.

(b) If $t \in I$, $x \in X$ then $\frac{\partial}{\partial t}\mathscr{U}(t, s)x|_{t=s} = x$ and $\frac{\partial}{\partial s}\mathscr{U}(t, s)x|_{t=s} = -x$.

(ii) If $(t, s) \in \Delta$, then $\frac{\partial}{\partial s}\mathscr{U}(t, s)x \in D(A(t))$, the map $(t, s) \to \mathscr{U}(t, s)x$ is of class C^2 and

(a) $\frac{\partial^2}{\partial t^2}\mathscr{U}(t, s)x \equiv A(t)\mathscr{U}(t, s)x$.

(b) $\frac{\partial^2}{\partial s^2}\mathscr{U}(t, s)x \equiv \mathscr{U}(t, s)A(t)x$.

(c) $\frac{\partial^2}{\partial s \partial t}\mathscr{U}(t, s)x|_{t=s} = 0$.

(iii) If $(t, s) \in \Delta$, then there exist $\frac{\partial^3}{\partial t^2 \partial s}\mathscr{U}(t, s)x$, $\frac{\partial^3}{\partial s^2 \partial t}\mathscr{U}(t, s)x$ and

(a) $\frac{\partial^3}{\partial t^2 \partial s}\mathscr{U}(t, s)x \equiv A(t)\frac{\partial}{\partial s}\mathscr{U}(t, s)x$ and the map $(t, s) \to A(t)\frac{\partial}{\partial s}\mathscr{U}(t, s)x$ is continuous.

(b) $\frac{\partial^3}{\partial s^2 \partial t}\mathscr{U}(t, s)x \equiv \frac{\partial}{\partial t}\mathscr{U}(t, s)A(s)x$.

As an example for Eq. (2) one may consider the problem (e.g., [13])

$$\frac{\partial^2 z}{\partial t^2}(t,\tau) = \frac{\partial^2 z}{\partial \tau^2}(t,\tau) + a(t)\frac{\partial z}{\partial t}(t,\tau), \quad t \in [0,T], \tau \in [0,2\pi],$$

$$z(t,0) = z(t,\pi) = 0, \quad \frac{\partial z}{\partial \tau}(t,0) = \frac{\partial z}{\partial \tau}(t,2\pi),\ t \in [0,T],$$

where $a(.) : I \to \mathbf{R}$ is a continuous function. This problem is modeled in the space $X = L^2(\mathbf{R}, \mathbf{C})$ of 2π-periodic 2-integrable functions from $\mathbf{R}$ to $\mathbf{C}$, $A_1 z = \frac{d^2 z(\tau)}{d\tau^2}$ with domain $H^2(\mathbf{R}, \mathbf{C})$ the Sobolev space of 2π-periodic functions whose derivatives belong to $L^2(\mathbf{R}, \mathbf{C})$. It is well known that A_1 is the infinitesimal generator of strongly continuous cosine functions $C(t)$ on X. Moreover, A_1 has discrete spectrum; namely the spectrum of A_1 consists of eigenvalues $-n^2$, $n \in \mathbf{Z}$ with associated eigenvectors $z_n(\tau) = \frac{1}{\sqrt{2\pi}} e^{in\tau}$, $n \in \mathbf{N}$. The set z_n, $n \in \mathbf{N}$ is an orthonormal basis of X. In particular, $A_1 z = \sum_{n \in \mathbf{Z}} -n^2 < z, z_n > z_n$, $z \in D(A_1)$. The cosine function is given by $C(t)z = \sum_{n \in \mathbf{Z}} \cos(nt) < z, z_n > z_n$ with the associated sine function $S(t)z = t < z, z_0 > z_0 + \sum_{n \in \mathbf{Z}^*} \frac{\sin(nt)}{n} < z, z_n > z_n$.

For $t \in I$ define the operator $A_2(t)z = a(t)\frac{dz(\tau)}{d\tau}$ with domain $D(A_2(t)) = H^1(\mathbf{R}, \mathbf{C})$. Set $A(t) = A_1 + A_2(t)$. It has been proved in [13] that this family generates an evolution operator as in Definition 1.

Definition 2 A continuous mapping $x(.) \in C(I, X)$ is called a mild solution of problem (1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that

$$f(t) \in F(t, x(t)) \quad a.e.\,(I), \tag{3}$$

$$x(t) = -\frac{\partial}{\partial s}\mathcal{U}(t,0)x_0 + \mathcal{U}(t,0)y_0 + \int_0^t \mathcal{U}(t,s)f(s)ds,\ t \in I. \tag{4}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1) if $f(.)$ verifies (3) and $x(.)$ is defined by (4).

We shall use the following notations for the solution sets of (1).

$$\mathcal{S}(x_0, x_1) = \{(x(.), f(.));\ (x(.), f(.)) \text{ is a trajectory-selection pair of (1)}\}. \tag{5}$$

In what follows $z_0, z_1 \in X$, $g(.) \in L^1(I, X)$ and $z(.) \in C(I, X)$ is a mild solution of the Cauchy problem

$$z'' = A(t)z + g(t) \quad z(0) = z_0, \quad z'(0) = z_1. \tag{6}$$

Hypothesis. (i) There exists an evolution operator $\{\mathcal{U}(t,s)\}_{t,s \in I}$ associated to the family $\{A(t)\}_{t \ge 0}$.

(ii) There exist $M, M_0 \ge 0$ such that $|\mathcal{U}(t,s)|_{B(X)} \le M$, $|\frac{\partial}{\partial s}\mathcal{U}(t,s)| \le M_0$, for all $(t,s) \in \Delta$.

(iii) $F(.,.) : I \times X \to \mathscr{P}(X)$ has nonempty closed values and for every $x \in X$, $F(., x)$ is measurable.

(iv) There exists $L(.) \in L^1(I, (0, \infty))$ such that for almost all $t \in I$, $F(t, .)$ is $L(t)$-Lipschitz on in the sense that

$$d_H(F(t, x_1), F(t, x_2)) \le L(t)|x_1 - x_2| \quad \forall \; x_1, x_2 \in X,$$

(v) The function $t \to \gamma(t) := d(g(t), F(t, z(t))$ is integrable on I.

Set $m(t) = e^{M \int_0^t L(u)du}$, $t \in I$. The next result [9] is an extension of Filippov's theorem concerning the existence of solutions to a Lipschitzian differential inclusion [12] to second-order differential inclusions of the form (1).

Theorem 1 ([9]) *Consider* $\delta \ge 0$*, assume that Hypothesis is satisfied and let* $\eta(t) = m(t)(\delta + M \int_0^t \gamma(s)ds)$.

Then for any $z_0, z_1 \in X$ *with* $M_0|x_0 - z_0| + M|x_1 - z_1| \le \delta$ *and any* $\varepsilon > 0$ *there exists* $(x(.), f(.)) \in \mathscr{S}(x_0, x_1)$ *such that*

$$|x(t) - z(t)| \le \eta(t) + \varepsilon M t m(t) \quad \forall t \in I,$$

$$|f(t) - g(t)| \le L(t)(\eta(t) + \varepsilon M t m(t)) + \gamma(t) + \varepsilon \quad a.e. \; (I).$$

3 The Main Results

Let $(z(.), g(.))$ be a trajectory-selection pair of problem (6). Our intention is to "linearize" (1) along $(z(.), g(.))$ by replacing it by several second-order variational inclusions.

First, we consider the quasitangent variational inclusion

$$\begin{cases} w''(t) \in A(t)w(t) + Q_{g(t)}(F(t, .))(z(t); w(t)) \quad a.e. \; (I) \\ w(0) = w_0, \quad w'(0) = w_1, \end{cases} \tag{7}$$

where $w_0, w_1 \in X$. In the next theorem we consider the solution map $\mathscr{S}(., .)$ as a set valued map from $X \times X$ into $C(I, X) \times L^1(I, X)$.

Theorem 2 *Assume that Hypothesis is satisfied.*

Then, for any $w_0, w_1 \in X$ *and any trajectory-selection pair* (w, p) *of the linearized inclusion (7) one has*

$$(w, p) \in Q_{(z,g)}\mathscr{S}((z(0), z'(0); (w_0, w_1)).$$

Proof Take $w_0, w_1 \in X$ and let $(w, p) \in C(I, X) \times L^1(I, X)$ be a trajectory-selection pair of (7). Taking into account the definition of the quasitangent derivative and the fact that $F(t, .)$ is Lipschitz, for almost all $t \in I$, we have

$$\lim_{h\to 0+} d\left(p(t), \frac{F(t,z(t)+hw(t))-g(t)}{h}\right)=0. \tag{8}$$

At the same time, since $g(t)\in F(t,z(t))$ a.e. (I), from Hypothesis, for all enough small $h>0$ and for almost all $t\in I$, we have

$$d(g(t)+hp(t),F(t,z(t)+hw(t)))\le h(|p(t)|+L(t)|w(t)|).$$

The function $t\to d(g(t)+hp(t),F(t,z(t)+hw(t)))$ is measurable; thus, with Lebesgue's dominated convergence theorem we deduce

$$\int_0^T d(g(t)+hp(t),F(t,z(t)+hw(t)))=o(h), \tag{9}$$

where $\lim_{h\to 0+}\frac{o(h)}{h}=0$.

We apply Theorem 1 with $\varepsilon=h^2$ and by (9) we obtain the existence of $M\ge 0$ and of trajectory-selection pairs $(z_h(.),g_h(.))$ of the second-order differential inclusion in (1) such that

$$|z_h-z-hw|_C+|g_h-g-h\pi|_1\le M(o(h)+h^2),$$
$$z_h(0)=z(0)+hw_0,\quad z_h'(0)=z'(0)+hw_1,$$

which implies

$$\lim_{h\to 0+}\frac{z_h-z}{h}=w \quad \text{in} \quad C(I,X),$$
$$\lim_{h\to 0+}\frac{g_h-g}{h_n}=\pi \quad \text{in} \quad L^1(I,X).$$

Therefore

$$\lim_{h\to 0+} d_{C\times L}\left((w,p),\frac{\mathscr{S}((z(0)+hw_0,z'(0)+hw_1))-(z,g)}{h}\right)=0$$

and the proof is complete. □

The next variational inclusion is defined by the Clarke directional derivative of the set-valued map $F(t,.)$, i.e., the so called circatangent variational inclusion

$$\begin{cases} w''(t)\in A(t)w(t)+C_{g(t)}(F(t,.))(z(t);w(t)) \quad a.e.\ (I)\\ w(0)=w_0,\quad w'(0)=w_1,\end{cases} \tag{10}$$

Once again, in what follows we consider the solution map $\mathscr{S}(.,.)$ as a set valued map from $X\times X$ into $C(I,X)\times L^1(I,X)$.

Theorem 3 *Assume that Hypothesis is satisfied.*

Then, for any $w_0, w_1 \in X$ and any trajectory-selection pair (w, p) of the linearized inclusion (10) one has

$$(w, p) \in C_{(z,g)}\mathscr{S}((z(0), z'(0); (w_0, w_1)).$$

Proof Take $w_0, w_1 \in X$, let $(w, p) \in C(I, X) \times L^1(I, X)$ be a trajectory-selection pair of (10), let (z_n, g_n) be a sequence of trajectory-selection pairs of (1) that converges to $(z, g) \in C(I, X) \times L^1(I, X)$ and let $h_n \to 0+$. Then there exists a subsequence $g_j(.) := g_{n_j}(.)$ such that

$$\lim_{j\to\infty} g_j(t) = g(t) \quad a.e.\ (I) \tag{11}$$

Set $\lambda_j := h_{n_j}$. From the definition of the Clarke directional derivative and from (10), for almost all $t \in I$ we have

$$\lim_{j\to\infty} d\left(p(t), \frac{F(t, z_j(t) + \lambda_j w(t)) - g_j(t)}{\lambda_j}\right) = 0. \tag{12}$$

Since $g_j(t) \in F(t, z_j(t))$ a.e. (I), for almost all $t \in I$, we get

$$d(g_j(t) + \lambda_j p(t), F(t, z_j(t) + \lambda_j w(t))) \le \lambda_j(|p(t)| + L(t)|w(t)|).$$

Lebesgue's dominated convergence theorem together with the last inequality implies

$$\int_0^T d(g_j(t) + \lambda_j p(t), F(t, z_j(t) + \lambda_j w(t))) = o(\lambda_j), \tag{13}$$

with $\lim_{j\to\infty} \frac{o(\lambda_j)}{\lambda_j} = 0$.

We apply Theorem 1 with $\varepsilon = \lambda_j^2$ and by (13) we find the existence of $M \ge 0$ and of trajectory-selection pairs $(\overline{z}_j(.), \overline{g}_j(.))$ of the second-order differential inclusion in (1) satisfying

$$|\overline{z}_j - z_j - \lambda_j w|_C + |\overline{g}_j - g_j - \lambda_j \pi|_1 \le M(o(\lambda_j) + \lambda_j^2),$$
$$\overline{z}_j(0) = z(0) + \lambda_j w_0, \quad \overline{z}_j'(0) = z'(0) + \lambda_j w_1.$$

Therefore,

$$\lim_{j\to\infty} \frac{\overline{z}_j - z}{\lambda_j} = w \quad \text{in} \quad C(I, X),$$
$$\lim_{j\to\infty} \frac{\overline{g}_j - g}{\lambda_j} = \pi \quad \text{in} \quad L^1(I, X),$$

which completes the proof. □

Our last approach concerns the contingent variational inclusion

$$\begin{cases} w''(t) \in A(t)w(t) + \overline{co}K_{g(t)}(F(t,.))(z(t);w(t)) \quad a.e.\ (I) \\ w(0) = w_0, \quad w'(0) = w_1, \end{cases} \tag{14}$$

This time, we consider the solution map $\mathscr{S}(.,.)$ as a set valued map from $X \times X$ into $C(I, X) \times L^\infty(I, X)$, with $L^\infty(I, X)$ supplied with the weak-* topology.

Theorem 4 *Assume that Hypothesis is satisfied.*
Then for any $w_0, w_1 \in X$ one has

$$K_{(z,g)}\mathscr{S}((z(0), z'(0); (w_0, w_1)) \subset$$
$$\{(w, p); \quad (w, p) \text{ is a trajectory-selection pair of } (14)\}.$$

Proof Take $w_0, w_1 \in X$ and let $(w, p) \in K_{(z,g)}\mathscr{S}((z(0), z'(0); (w_0, w_1))$. According to the definition of the contingent derivative there exist $h_n \to 0+$, $u_n \to w_0$, $v_n \to w_1$, $w_n(.) \to w(.)$ in $C(I, X)$, $p_n(.) \to p(.)$ in weak-* topology of $L^\infty(I, X)$ and $c > 0$ such that

$$\begin{aligned} &|p_n(t)| \le c \quad a.e.\ (I), \\ &g(t) + h_n p_n(t) \in F(t, z(t) + h_n w_n(t)) \quad a.e.\ (I), \\ &w_n(0) = u_n, \quad w_n'(0) = v_n. \end{aligned} \tag{15}$$

Therefore,

$$\begin{aligned} &w_n(.) \quad \text{converges pointwise to} \quad w(.) \\ &p_n(.) \quad \text{converges weak in } L^1(I, X) \text{ to} \quad p(.) \end{aligned} \tag{16}$$

We apply Mazur's theorem (e.g., [11]) and we find that there exists

$$\pi_m(t) = \sum_{q=m}^{\infty} b_m^q p_q(t)$$

$\pi_m(.) \to p(.)$ (strong) in $L^1(I, X)$, where $b_m^q \ge 0$, $\sum_{q=m}^{\infty} b_m^q = 1$ and for any m, $b_m^q \ne 0$ for a finite number of q.

Hence, a subsequence (again denoted) $\pi_m(.)$ converges la $p(.)$ a.e.. From (15) for any q and for almost all $t \in I$

$$p_q(t) \in \frac{1}{h_q}(F(t, z(t) + h_q w_q(t)) - g(t)) \cap cB.$$

Let $t \in I$ be such that $v_m(t) \to p(t)$ and $g(t) \in F(t, z(t))$. Fix $n \ge 1$ and $\varepsilon > 0$. From (15) there exists m such that $h_q \le 1/n$ and $|w_q(t) - w(t)| \le 1/n$ for any $q \ge m$.

If, we denote

$$a(y,h) := \frac{1}{h}(F(t,z(t)+hy) - g(t)) \cap cB$$

then

$$\pi_m(t) \in co(\cup_{h\in(0,\frac{1}{n}],y\in B(w(t),\frac{1}{n})} a(y,h))$$

and if $m \to \infty$, we get

$$p(t) \in \overline{co}(\cup_{h\in(0,\frac{1}{n}],y\in B(w(t),\frac{1}{n})} a(y,h)).$$

Since, $a(y,h) \subset cB$, we deduce that

$$p(t) \in \overline{co} \cap_{\varepsilon>0,n\geq 1} (\cup_{h\in(0,\frac{1}{n}],y\in B(w(t),\frac{1}{n})} a(y,h) + \varepsilon B).$$

On the other hand,

$$\cap_{\varepsilon>0,n\geq 1}(\cup_{h\in(0,\frac{1}{n}],y\in B(w(t),\frac{1}{n})} a(y,h) + \epsilon B) \subset K_{g(t)}F(t,.)(z(t);w(t))$$

which completes the proof. □

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How to Analytically Prove the Existence of Strange Attractors Using Measure Theory

Antonio Pumariño, José A. Rodríguez and Enrique Vigil

Abstract We describe the attractors for a two-parameter family of two-dimensional piecewise affine maps using measure theory. These piecewise affine maps arise when studying the unfolding of homoclinic tangencies for certain class of three dimensional diffeomorphisms. We also prove the existence, for each natural number n, of an open set of parameters in which the respective transformation exhibits at least 2^n two-dimensional strange attractors.

Keywords Piecewise affine maps · Strange attractors · Invariant measures

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1 Introduction

For dissipative dynamics, chaos is usually defined as the existence of *strange attractors*, see Definition 1.1 below. Chaotic transformations displaying this kind of attractors were often numerically observed. However, the first analytic proof on the persistence (existence with positive probability) of a strange attractor was given by Benedicks and Carleson in the Hénon family scenario, see [1].

The term strange attractor was first used by Ruelle and Takens, see [15], to suggest that turbulent behaviour in fluids might be caused by the presence of attractors displaying high sensibility with respect to the initial conditions. One of the most relevant dynamics of this type was earlier observed by Lorenz, see [4], when studying his famous quadratic three dimensional vector field. The first proof on the existence of strange attractors in the Lorenz family was given by Tucker thirty six years later, see [19].

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In view of the above mentioned results it seems difficult to give analytic proofs on the existence of strange attractors.

In this paper we shall consider *Expanding Baker Maps* (*EBMs* for short). These maps were early considered in [10] where a family $\mathbb{F} = \{\Psi_{a,b} : (a, b) \in \mathcal{P}\}$ defined on certain triangle $\mathcal{T}$ was introduced. Roughly speaking an *EBM* is a two-dimensional piecewise linear map which firstly folds (perhaps several times) its domain of definition and after that expands the folded domain. Throughout this article, we shall briefly give the details leading to the description of the attractors exhibited by the family of maps $\mathbb{F}$. These attractors, see Theorem 1.2, will be strange attractors. Moreover, we shall also deduce that such an attractor exists inside any compact invariant domain with non empty interior. Therefore the proof on the existence of strange attractors turns easy just by looking for compact invariant domain with non empty interior for the respective *EBM*. Before going on, let us introduce the definition of strange attractor as well as the main result of this paper.

Definition 1.1 By an **attractor** for a transformation f defined in a compact manifold M, we mean an f-invariant and transitive set A whose stable set

$$W^s(A) = \left\{z \in M : d\left(f^n(z), A\right) \to 0 \text{ as } n \to \infty\right\}$$

has non-empty interior. An attractor is said to be **strange** if there exists a dense orbit $\{f^n(z_1) : n \geq 0\}$ displaying exponential growth of the derivative: There exists some constant $c > 0$ and a unit vector $\mathbf{v}$ such that, for every $n \geq 0$,

$$\|Df^n(z_1)\mathbf{v}\| \geq e^{cn},$$

where $\|\cdot\|$ stands for the norm of a vector.

From the above definition, it follows that there always exists a dense orbit on A with at least one positive Lyapounov exponent: There exists a dense orbit $\{f^n(z_1) : n \geq 0\}$ and a unit vector $\mathbf{v}$ for which

$$\limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(z_1)\mathbf{v}\| \geq c > 0.$$

We shall say that a strange attractor is two-dimensional (from now on, 2-D strange attractor) if it contains a dense orbit with two positive Lyapounov exponents. We also observe that our definition of 2-D strange attractor is stronger than the usual one which only requires (see, for instance, [14]) that the sum of the Lyapounov exponents is positive. As we shall observe, see Theorem 1.2, any attractor of every map $\Psi_{a,b}$ is a two-dimensional strange attractor.

Before going into the results, let us describe where the family $\mathbb{F}$ comes from. It is known that homoclinic scenarios have been a noteworthy threshold to the appearance of chaotic dynamics. In the two-dimensional case a lot of chaotic behaviour has been observed when a family of two-dimensional diffeomorphisms unfolds a homoclinic tangency. In this setting the existence of infinitely many sinks, see [6], the existence of persistent strange attractors, see [5], or even the existence of infinitely many strange attractors, see [3], has been proved. All the above results depend on a common tool: The existence of families of limit return maps describing the behavior of a sufficiently large power of the diffeomorphism in a neighborhood of the homoclinic orbit, see for instance [7] for details.

In [17] the author studies certain unfoldings of homoclinic tangencies for three-dimensional diffeomorphisms. Dynamical properties of the associated limit return maps are described in [13, 14]. Later, in [8] the notion of *EBMs* arises as piecewise linear models for these limit return maps. The study of these *EBMs* is carried on in [9–11]. The family $\mathbb{F}$ will be introduced in Sect. 2. The main objective of this paper is to characterize the attractors exhibited by $\Psi_{a,b}$ for every $(a, b) \in \mathcal{P}$.

Theorem 1.2 *For every $(a, b) \in \mathcal{P}$, there exists a finite family $\mathcal{A}_{a,b}$ of 2-D strange attractors for $\Psi_{a,b}$ satisfying:*

(i) *If A is an attractor for $\Psi_{a,b}$, then $A \in \mathcal{A}_{a,b}$.*
(ii) *If $A \in \mathcal{A}_{a,b}$, then there exists an ergodic absolutely continuous invariant measure μ for $\Psi_{a,b}$ supported on A.*
(iii) *For every $A \in \mathcal{A}_{a,b}$ there exists a natural number p for which A can be decomposed $A = X_0 \cup X_1 \cup \cdots \cup X_{p-1}$ in such a way that $\Psi_{a,b}(X_i) = X_{i+1 \ mod \ p}$. The measure μ supported on A is mixing (up to the eventual period p) from which the map $\Psi^p_{a,b}$ is topologically mixing on every X_i.*
(iv) *If $A \in \mathcal{A}_{a,b}$, then A traps almost every point in $W^s(A)$; i.e., for almost every point $x \in W^s(A)$, there exists $j \in \mathbb{N}$ with $\Psi^j_{a,b}(x) \in A$. Moreover, the set $\bigcup_{A \in \mathcal{A}_{a,b}} W^s(A)$ covers a full Lebesgue measure set of $\mathcal{T}$.*
(v) *If U is a compact $\Psi_{a,b}$-invariant set with non-empty interior then there exists $A \in \mathcal{A}_{a,b}$ such that $A \subset U$.*

As a consequence of Theorem 1.2 we also are able to prove the following result.

Corollary 1.3 *For every $n \in \mathbb{N}$ there exists a set of parameters $\mathcal{P}^n$ with non-empty interior such that if $(a, b) \in \mathcal{P}^n$, then $\Psi_{a,b}$ has at least 2^n 2-D strange attractors.*

The complete details of the proof of Theorem 1.2 can be found in [12]. Here we present not only a more direct proof of this result but also a more detailed description of the proof of Corollary 1.3.

2 The Family $\mathbb{F}$ of Expanding Baker Maps

In [10] a two-parameter family $\mathbb{F} = \{\Psi_{a,b} : (a, b) \in \mathcal{P}\}$ of *EBMs* was introduced. Each map $\Psi_{a,b}$ is defined on the triangle $\mathcal{T} \subset \mathbb{R}^2$ with vertices $(0, 0)$, $(2, 0)$ and $(1, 1)$; i.e.

$$\mathcal{T} = \{(x, y) \in \mathbb{R}^2 :\ 0 \le x \le 1,\ 0 \le y \le x\} \cup \{(x, y) \in \mathbb{R}^2 :\ 1 \le x \le 2,\ 0 \le y \le 2 - x\}$$

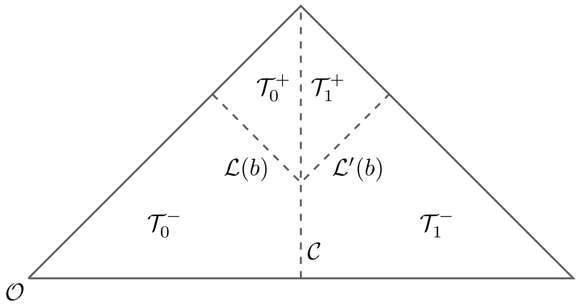

Fig. 1 The smoothness domains for a map in $\mathbb{F}$

The set of parameters $\mathcal{P}$ is given by

$$\mathcal{P} = \{(a, b) \in (1, 2] \times [1, 2] : \ ab \leq 2\}. \tag{1}$$

Splitting $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ with

$$\mathcal{T}_0 = \{(x, y) \in \mathcal{T} : \ 0 \leq x \leq 1\}, \mathcal{T}_1 = \{(x, y) \in \mathcal{T} : \ 1 \leq x \leq 2\} \ ,$$

the maps $\Psi_{a,b}$ are defined by

$$\Psi_{a,b}(x, y) = \begin{cases} (ax, ay) & , \text{if } (x, y) \in \mathcal{T}_0^- \\ (a(b - y), a(b - x)) & , \text{if } (x, y) \in \mathcal{T}_0^+ \\ (a(2 - x), ay) & , \text{if } (x, y) \in \mathcal{T}_1^- \\ (a(b - y), a(b - 2 + x)) & , \text{if } (x, y) \in \mathcal{T}_1^+ \end{cases}$$

where (Fig. 1)

$$\begin{aligned} \mathcal{T}_0^- &= \{(x, y) \in \mathcal{T}_0 : \ x + y \leq b\} \ , \\ \mathcal{T}_0^+ &= \{(x, y) \in \mathcal{T}_0 : \ x + y \geq b\} \ , \\ \mathcal{T}_1^- &= \{(x, y) \in \mathcal{T}_1 : \ x - y \geq 2 - b\} \ , \\ \mathcal{T}_1^+ &= \{(x, y) \in \mathcal{T}_1 : \ x - y \leq 2 - b\} \ . \end{aligned}$$

It is not hard to see that any map $\Psi_{a,b}$ is the composition of three maps: The first one, which does not depend on the parameters, folds the triangle $\mathcal{T}_1$ onto the triangle $\mathcal{T}_0$. The second one is only defined on $\mathcal{T}_0$ and folds $\mathcal{T}_0^+$ onto $\mathcal{T}_0^-$. Finally, the third one expands the folded region $\mathcal{T}_0^-$ under the action of the linear map given by the matrix

$$B_a = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

We also point out that the critical set (the set in which $\Psi_{a,b}$ in not smooth) is given by the union of three straight segments

$$\mathcal{C} = \{(x, y) \in \mathcal{T} : \ x = 1\} \ ,$$

$$\mathcal{L}(b) = \{(x, y) \in \mathcal{T} : \ x \leq 1 \text{ and } x + y = b\}$$

and the symmetric $\mathcal{L}'(b)$ of $\mathcal{L}(b)$ with respect to $\mathcal{C}$.

It is easy to see that the domain $\mathcal{T}$ is invariant by $\Psi_{a,b}$, for every $(a, b) \in \mathcal{P}$, where $\mathcal{P}$ is the set of parameters given in (1).

3 Preliminary Results

The proof of Theorem 1.2 is based on the results given in [2, 16, 18]. Of course, the first step for proving Theorem 1.2 is to check that our maps $\Psi_{a,b}$ satisfy the assumptions of these three papers. After that, the following results can be stated.

Proposition 3.1 *For every $(a, b) \in \mathcal{P}$, there exist absolutely continuous invariant measures for $\Psi_{a,b}$. Moreover:*

(i) Each one of these a.c.i.m.'s is a convex combination of a fixed, finite collection of ergodic ones.
(ii) For every ergodic measure μ of $\Psi_{a,b}$, there exists a natural number p and a decomposition of the support of μ, $supp(\mu) = X_0 \cup X_1 \cup \cdots \cup X_{p-1}$ with $\Psi_{a,b}(X_i) = X_{i+1 \ mod \ p}$ such that the map $\Psi^p_{a,b}$ is topologically mixing on any X_i.

Proposition 3.2 *For every $(a, b) \in \mathcal{P}$, there exist finitely many absolutely continuous ergodic probability measures $\mu_1, \mu_2, \ldots \mu_l$, for $\Psi_{a,b}$. Moreover, the basin of each measure μ_i is a non-empty open set modulo sets with null Lebesgue measure, and the union $\bigcup_{i=1}^{l} int(Basin(\mu_i))$ has full Lebesgue measure in $\mathcal{T}$, where $int(A)$ stands for the interior of a set A.*

Proposition 3.3 *For every $(a, b) \in \mathcal{P}$ and for every a.c.i.m. μ of $\Psi_{a,b}$ the interior of the support of μ has full μ-measure. Moreover, each a.c.i.m. μ of $\Psi_{a,b}$ is finite.*

Proposition 3.1 follows from the results of Buzzi given in [2] and it is essential for proving the third statement in Theorem 1.2. In view of Proposition 3.2, it seems convenient to choose the set of attractors announced in Theorem 1.2 as being

$$\mathcal{A}_{a,b} = \{supp(\mu_i) : i = 1, \ldots l\}$$

where $supp(\mu)$ stands for the support of a measure μ. Statements (i), (ii) and (iv) in Theorem 1.2 strongly depend on Proposition 3.2. This result is a direct consequence of the ones given in [18]. In this sense, we recall that the basin of an a.c.i.m. μ for $\Psi_{a,b}$ is given by

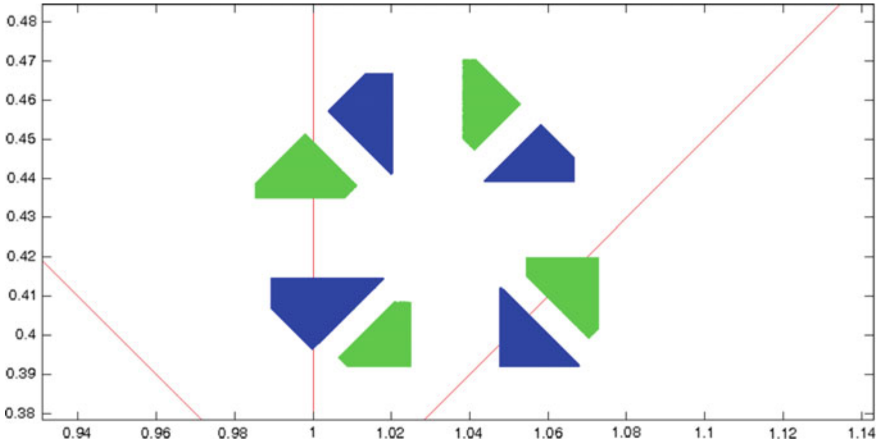

Fig. 2 Two numerically obtained attractors for $\Psi_{a,b}$ when $a = 1.12$ and $b = 1.35$

$$Basin(\mu_i) = \left\{ x \in \mathcal{T} : \frac{1}{n} \sum_{j=0}^{n-1} \delta_{\Psi^j_{a,b}(x)} \to \mu_i \ weakly \right\}. \tag{2}$$

The fact that each one of the elements of $\mathcal{A}_{a,b}$ is a 2-D strange attractor uses Proposition 3.3. This result comes from the ones given in [16]. We point out here that each attractor $A = supp(\mu_i)$ in $\mathcal{A}_{a,b}$ **traps** every point of $Basin(\mu_i)$, the basin of the measure μ_i. The key to obtain this is the fact that the interior of $supp(\mu_i)$ is a set with, according to [16] (see also Proposition 3.3), full μ_i-measure. All these arguments are crucial to prove statements (iv) and (v) in Theorem 1.2.

In order to illustrate the second statement of Proposition 3.1, we show in Fig. 2 two numerically obtained attractors (recall that each one of these attractors coincides with the support of an ergodic absolutely continuous invariant measure) for the map $\Psi_{a,b}$ when $a = 1.12$ and $b = 1.35$. The respective map displays two different non-connected attractors, each one of them formed by 4 connected pieces which are dynamically defined by $\Psi_{a,b}$.

4 A Sketch of the Proof of Theorem 1.2

Fix $(a, b) \in \mathcal{P}$. From Proposition 3.2 there exists a finite number of absolutely continuous invariant ergodic probabilities $\mu_1, \mu_2, \ldots, \mu_l$ for $\Psi_{a,b}$. Let us denote by $\mathcal{Z}_i$ the support of μ_i, $i = 1, \ldots, l$. From now on, let us consider the family of subsets of $\mathcal{T}$ given by

$$\mathcal{A}_{a,b} = \{\mathcal{Z}_i : i = 1, \ldots l\}. \tag{3}$$

Lemma 4.1 *For every $i = 1, \dots, l$ it holds that:*

(i) The sets $int(\mathcal{Z}_i)$ trap every point in $Basin(\mu_i)$; i.e., for every $x \in Basin(\mu_i)$, there exists some $j \in \mathbb{N}$ such that $\Psi^j_{a,b}(x) \in int(\mathcal{Z}_i)$.
(ii) The sets $\mathcal{Z}_i$ are 2-D strange attractors for $\Psi_{a,b}$.

Proof Let x be any point in $Basin(\mu_i)$. Then, see (2), we have that

$$\frac{1}{n}\sum_{j=0}^{n-1} \delta_{\Psi^j_{a,b}(x)} \to \mu_i$$

weakly. Since μ_i is an a.c.i.m. for $\Psi_{a,b}$, Proposition 3.3 implies that $\mu_i(int(\mathcal{Z}_i)) = 1$ and hence $int(\mathcal{Z}_i)$ are sets of continuity of μ_i. Therefore, applying Portmanteau's Theorem,

$$\frac{1}{n}\sum_{j=0}^{n-1} \chi_{int(\mathcal{Z}_i)}(\Psi^j_{a,b}(x)) = \frac{1}{n}\sum_{j=0}^{n-1} \delta_{\Psi^j_{a,b}(x)}(int(\mathcal{Z}_i)) \to 1.$$

This fact clearly implies the existence of $j \in \mathbb{N}$ such that $\Psi^j_{a,b}(x) \in int(\mathcal{Z}_i)$.

To demonstrate the second statement we begin by observing that, by the definition of support, it is clear that $\mathcal{Z}_i$ are compact sets. Since μ_i is invariant by $\Psi_{a,b}$ it follows that each $\mathcal{Z}_i$ is (forward) invariant for $\Psi_{a,b}$.

Now it is enough to demonstrate the existence of a dense orbit in $\mathcal{Z}_i$ not visiting the critical set $\mathcal{C}_{a,b}$ of $\Psi_{a,b}$. Let us denote by

$$\widetilde{\mathcal{C}}_{a,b} = \{x \in \mathcal{T} : \Psi^j_{a,b}(x) \in \mathcal{C}_{a,b} \text{ for some } j \in \mathbb{N}\}$$

and by *Leb* the Lebesgue measure in $\mathbb{R}^2$. Since $Leb(\widetilde{\mathcal{C}}_{a,b}) = 0$ and μ_i is absolutely continuous with respect to *Leb* it follows that $\mu_i(\widetilde{\mathcal{C}}_{a,b}) = 0$.

On the other hand, in the same way as Lemma 4 in [13] was proved, we may demonstrate that there exists a set $\mathcal{S}$ with $\mu_i(\mathcal{S}) = 1$, in such a way that, if x_0 belongs to $\mathcal{S}$, then its $\Psi_{a,b}$-orbit is dense in $\mathcal{Z}_i$. Then, the result is proved. ∎

Now, let us start by proving the fifth statement of Theorem 1.2. Let U be any $\Psi_{a,b}$-invariant compact set with non-empty interior.

According to Proposition 3.2, there exists $i \in \{1, \dots, l\}$ with $int(U) \cap int(Basin(\mu_i)) \neq \emptyset$. Therefore, the set $int(U) \cap Basin(\mu_i)$ must contain an open subset V. Let x be a point in V. Since $V \subset Basin(\mu_i)$, from Lemma 4.1 we know that there exists a natural number j such that $\Psi^j_{a,b}(x) \in int(\mathcal{Z}_i)$. Let W be a neighbourhood of $\Psi^j_{a,b}(x)$ contained in $\mathcal{Z}_i$. Since $\Psi^j_{a,b}$ is continuous, there exists an open neighbourhood V_x (which may be assumed to be contained in V) such that $\Psi^j_{a,b}(V_x) \subset \mathcal{Z}_i$. On the other hand, since U is invariant we deduce that $\Psi^j_{a,b}(V_x)$ is also contained in U. From this fact, using that $\Psi_{a,b}$ is transitive on $\mathcal{Z}_i$, we easily deduce that $\mathcal{Z}_i \subset U$.

To prove the first statement we observe that if A is an attractor for $\Psi_{a,b}$, then from Proposition 3.2, there exists $i \in \{1, \ldots, l\}$ such that the set $W^s(A) \cup Basin(\mu_i)$ has non-empty interior. From Lemma 4.1 it is clear that $A = \mathcal{Z}_i$.

From the definition of $\mathcal{A}_{a,b}$ given at (3) the second statement of Theorem 1.2 is obvious.

The third statement of Theorem 1.2 is a direct consequence of the definition of $\mathcal{A}_{a,b}$, the second statement of Propositions 3.1 and 3.2.

To prove the fourth statement let us choose some attractor $A \in \mathcal{A}_{a,b}$. From Lemma 4.1 we know that $Basin(\mu_i) \subset W^s(\mathcal{Z}_i)$ where $A = \mathcal{Z}_i = supp(\mu_i)$. Moreover, also from Lemma 4.1, we also know that if $x \in Basin(\mu_i)$, then there exists some natural number j for which $\Psi^j_{a,b}(x) \in A$. Therefore, the result is proved using Proposition 3.2 to deduce that the set $W^s(\mathcal{Z}_i) \setminus Basin(\mu_i)$ has null Lebesgue measure. Hence, $\bigcup_{i=1}^{l} W^s(\mathcal{Z}_i)$ covers a full Lebesgue measure set of $\mathcal{T}$.

5 Proof of Corollary 1.3

Let us start by recovering from [10] the definition of *renormalizable EBM*.

Definition 5.1 Let Γ be a map defined in certain domain $\mathcal{K}$. We said that $\mathcal{D} \subset \mathcal{K}$ is a **restrictive domain** if $\mathcal{D} \neq \mathcal{K}$ and there exists $k = k(\mathcal{D}) \in \mathbb{N}$ such that

(i) $\Gamma^j(\mathcal{D}) \cap \mathcal{D} = \emptyset$ for every $j = 1, \ldots, k-1$,
(ii) $\Gamma^k(\mathcal{D}) \subset \mathcal{D}$.

Definition 5.2 An *EBM* Γ defined on certain domain $\mathcal{K}$ is said to be **renormalizable** if there exists a restrictive domain $\mathcal{D}$ (with an associated natural number $k = k(\mathcal{D})$) such that $\Gamma^k_{|\mathcal{D}}$ is, up to an affine change in coordinates, an *EBM* defined on $\mathcal{K}$.

Definition 5.3 Let Γ be a *renormalizable EBM* with restrictive domain $\mathcal{D}$ (with an associated natural number $k = k(\mathcal{D})$). Let us denote $\Gamma_1 = \Gamma^k_{|\mathcal{D}}$. If Γ_1 is a *renormalizable EBM*, we call Γ **twice renormalizable EBM**. In this way **n renormalizable EBMs**, for every $n \in \mathbb{N}$, or even **infinitely renormalizable EBMs** may also be defined.

In [11] a subset $\mathcal{P}_3 \subset \mathcal{P}$ of parameters was constructed in such a way that, if $(a, b) \in \mathcal{P}_3$, then $\Psi_{a,b}$ is renormalizable in $\mathbb{F}$. In this particular case, this means that, for every $(a, b) \in \mathcal{P}_3$, the restriction of $\Psi^4_{a,b}$ to each one of two different restrictive domains (denoted by $\Delta = \Delta_{a,b}$ and $\Pi = \Pi_{a,b}$) is conjugate by means of an affine change in coordinates to an *EBM* which belongs to the family $\mathbb{F}$.

More precisely, see Theorem 3.5 in [11], there exist two (renormalization) operators

$$H_\Delta : (a, b) \in \mathcal{P}_3 \mapsto H_\Delta(a, b) = \left(a^4, \frac{-2 + b + ab}{a^2(1 + a - ab)}\right) \in \mathcal{P}$$

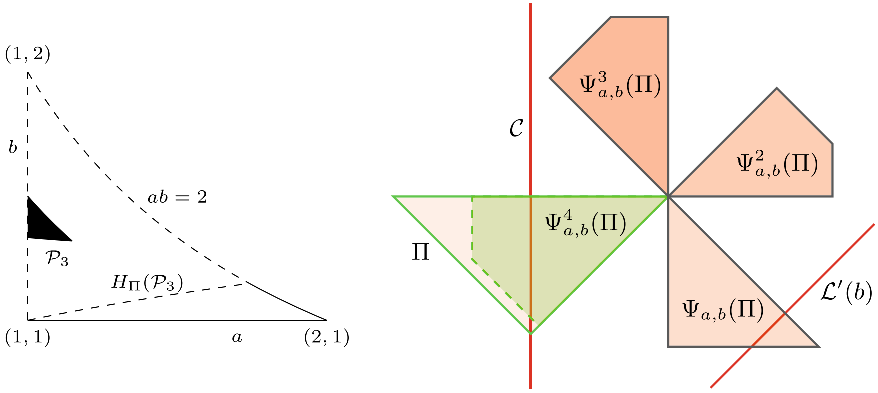

Fig. 3 The action of H_Π on $\mathcal{P}_3$ and the first four iterates by $\Psi_{a,b}$ of the restrictive domain Π

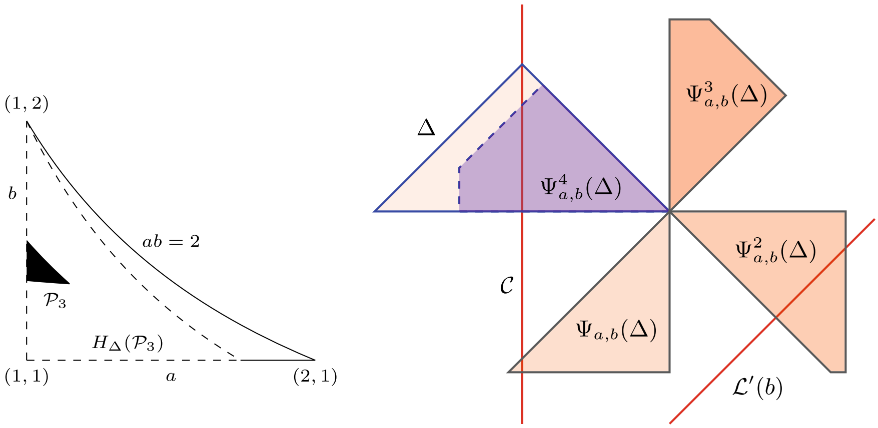

Fig. 4 The action of H_Δ on $\mathcal{P}_3$ and the first four iterates by $\Psi_{a,b}$ of the restrictive domain Δ

and

$$H_\Pi : (a, b) \in \mathcal{P}_3 \mapsto H_\Pi(a, b) = \left(a^4, \frac{-2 + b + ab}{a(1 + a - ab)}\right) \in \mathcal{P}$$

such that $\Psi^4_{a,b}$ restricted to Δ (respectively to Π) is conjugate by means of an affine change in coordinates to $\Psi_{H_\Delta(a,b)}$ (respectively to $\Psi_{H_\Pi(a,b)}$). In Fig. 3 we show the domain Π in which $\Psi^4_{a,b}$ is conjugate to $\Psi_{H_\Pi(a,b)}$. Observe that, in particular, this implies that Π is invariant by $\Psi^4_{a,b}$ or, in other words, the union of Π and its first three iterates by $\Psi_{a,b}$ form a compact set with non empty interior invariant for $\Psi_{a,b}$. According to the fifth statement of Theorem 1.2, there exists a strange attractor contained in this invariant set. In Fig. 4 we also show the same situation for the case of Δ.

The main properties of the renormalization operators H_Δ and H_Π have been described in Proposition 6 and Proposition 7 in [11], respectively. In particular, as was done in Sect. 3.4 in [11], if we define the set of parameters

$$\overline{\mathcal{P}} = \{(a, b) \in [1, 2] \times [1, 2) : \ ab < 2\}$$

then both H_Δ and H_Π are expanding diffeomorphisms on any small enough neighbourhood of $\overline{\mathcal{P}}$ and they have a unique fixed point $P^* = (1, \sqrt{2})$. This fixed point is a global repeller for H_Δ and H_Π and furthermore, see Remarks 3.6 and 3.7 in [11], it also follows that $\mathcal{P}_3 \subset H_\Delta(\mathcal{P}_3) \subset \mathcal{P}$ and $\mathcal{P}_3 \subset H_\Pi(\mathcal{P}_3) \subset \mathcal{P}$, see also the left hand side representations in Figs. 3 and 4.

Hence, there exist chains

$$A_0 \supsetneq A_1 \supsetneq A_2 \supsetneq \cdots \supsetneq A_n \supsetneq \cdots$$

and

$$A'_0 \supsetneq A'_1 \supsetneq A'_2 \supsetneq \cdots \supsetneq A'_n \supsetneq \cdots$$

of subsets of $\mathcal{P}$ such that

1. $A_0 = H_\Delta(\mathcal{P}_3)$, $A'_0 = H_\Pi(\mathcal{P}_3)$, $A_1 = A'_1 = \mathcal{P}_3$.
2. $H_\Delta(A_n) = A_{n-1}$ and $H_\Pi(A'_n) = A'_{n-1}$ for every $n \in \mathbb{N}$.
3. $\bigcap_{n\in\mathbb{N}} closure(A_n) = \bigcap_{n\in\mathbb{N}} closure(A'_n) = \{P^*\}$.

This means that, according to Definition 5.3, the map $\Psi_{a,b}$ is n-times renormalizable (using n-times the operator H_Δ) whenever the parameter (a, b) belongs to A_n and $\Psi_{a,b}$ is n-times renormalizable (using n-times the operator H_Π) if (a, b) belongs to A'_n.

For every $n \in \mathbb{N}$, we denote by

$$\mathcal{P}^n = A_n \cap A'_n. \tag{4}$$

Then $\Psi_{a,b}$ is n-times renormalizable in 2^n restrictive domains with pairwise disjoint interiors whenever $(a, b) \in \mathcal{P}^n$. Let us denote these restrictive domains by $\{\mathcal{U}_{n,j}\}_{j=1}^{2^n}$ and observe that, for every $j = 1, \ldots, 2^n$, it holds that $\Psi_{a,b}^{4^n}(\mathcal{U}_{n,j}) \subset \mathcal{U}_{n,j}$. In Fig. 5 we represent these sets $\mathcal{U}_{n,j}$ for the case $n = 2$.

Thus, if we define, for every $j = 1, \ldots, 2^n$, the sets

$$\mathcal{V}_{n,j} = \bigcup_{i=0}^{4^n-1} \Psi_{a,b}^i(\mathcal{U}_{n,j})$$

it follows that $\{\mathcal{V}_{n,j}\}_{j=1}^{2^n}$ is a family of 2^n compact Ψ-invariant sets with pairwise disjoint interiors. Therefore Corollary 1.3 easily follows from the fifth statement of Theorem 1.2.

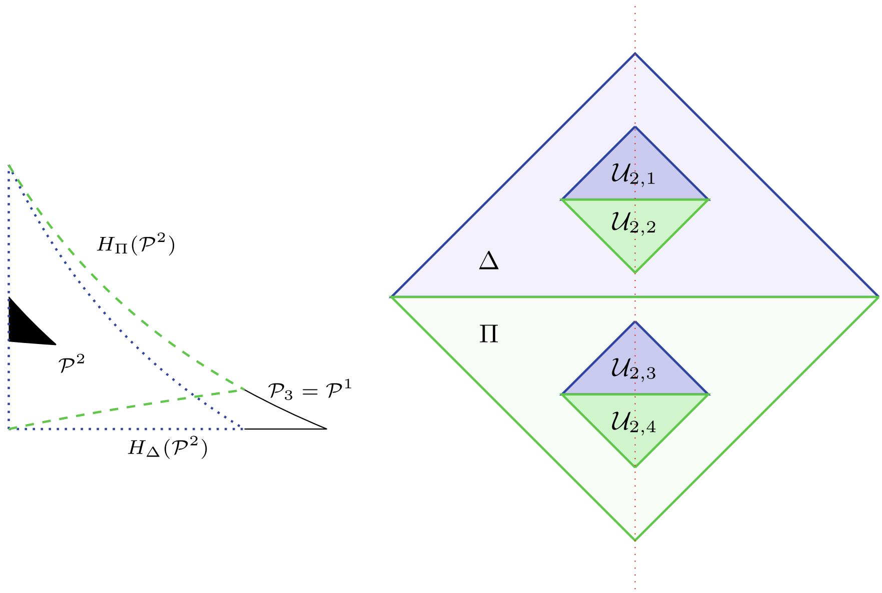

Fig. 5 The renormalization process at second stage: The restrictive domains $\mathcal{U}_{2,j}, j = 1, \ldots, 4$

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Initial Problems for Semilinear Degenerate Evolution Equations of Fractional Order in the Sectorial Case

Vladimir E. Fedorov, Anna S. Avilovich and Lidiya V. Borel

Abstract Initial problems for semilinear differential equations in Banach spaces with fractional Caputo derivative are studied. Firstly the unique solvability of the Cauchy problem for the semilinear equation solved with respect to the fractional derivative is researched, when the linear operator in the equation generates a resolving family of operators which is analytic in a sector. Then equation with degenerate operator at the Caputo derivative is considered in the case of the generation of an analytic in a sector degenerate resolving family of operators by the linear part of the equation. The unique solvability sufficient conditions for the Cauchy problem and for the Showalter–Sidorov problem are found. Abstract results are applied to the research of initial boundary value problems for a class of time-fractional order partial differential equations.

Keywords Fractional differential equation · Caputo derivative · Initial problem · Degenerate evolution equation · Sectorial pair of operators · Analytic resolving family of operators

1 Introduction

Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, $L, M \in \mathscr{C}l(\mathscr{X}; \mathscr{Y})$ (i. e. they are linear and closed, have dense domains $D_L, D_M \subset \mathscr{X}$ and act into $\mathscr{Y}$). Consider the evolution equation

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$$D_t^\alpha Lx(t) = Mx(t) + N(t, x(t), x^{(1)}(t), \dots, x^{(m-1)}(t)), \quad t \in [t_0, T), \tag{1}$$

where $\alpha > 0$, D_t^α is the fractional Caputo derivative, $N : U \to \mathscr{Y}$ is nonlinear operator, $U \subset \mathbf{R} \times \mathscr{X}^m$, $T > t_0$. The equation is supposed to be degenerate, i.e. $\ker L \neq \{0\}$. In this work the unique solvability of the Cauchy problem

$$x^{(k)}(0) = x_k, \quad k = 0, 1, \dots, m-1, \tag{2}$$

and of the Showalter–Sidorov problem

$$L(x^{(k)}(0) - x_k) = 0, \quad k = 0, 1, \dots, m-1, \tag{3}$$

for Eq. (1) is studied. The conditions on L, M ensuring the existence of analytic in a sector resolving operators family of equation

$$D_t^\alpha Lu(t) = Mu(t) \tag{4}$$

are obtained in [4]. Here these conditions is used for the proof of the unique solution existence to problems (1), (2) and (1), (3).

The conditions of the existence of degenerate operators family for (4), which is analytic in the complex plane, cut along the negative semiaxis, were obtained in [2]. That conditions were applied to investigation of the inhomogeneous equation in [3], to research of the Sobolev equations system of time fractional order, to study of the equations system of the fractional Kelvin–Voigt viscoelastic bodies dynamics in [5], to research of various nonlinear degenerate evolution equations of time fractional and highest order and optimal control problems for them in the papers of Plekhanova [9–11]. Resolving operators families for various classes of degenerate fractional equations in sequentially complete locally convex spaces are studied in numerous works of Kostić (see [7, 8] and their bibliographies).

In Sect. 2 the unique solvability is researched for the Cauchy problem to the semilinear equation (1) with $\mathscr{X} = \mathscr{Y}$, $L = I$, $A \in \mathscr{A}^\alpha(a_0, \theta_0)$ [1]. In the third section the existence of a unique solution to problems (1), (2) and (1), (3) is proved. In Sect. 4 obtained abstract results are applied to the study of initial boundary value problems unique solvability for a class of time-fractional order partial differential equations.

2 Nondegenerate Equation

In this section we study the existence and the uniqueness of the Cauchy problem classical solutions to equations, solved with respect to the fractional derivative.

2.1 Linear Equation

Denote with $\beta > 0$ $g_\beta(t) = t^{\beta-1}/\Gamma(\beta)$ for $t > 0$,

$$J_t^\beta h(t) := (g_\beta * h)(t) := \int_0^t g_\beta(t-s)h(s)ds = \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)ds.$$

Let $\alpha > 0$, m is the smallest integer, greater than or equal to α, D_t^m is the usual derivative of the order $m \in \mathbf{N}$, D_t^α is the fractional Caputo derivative, i.e. $D_t^\alpha f(t) := D_t^m J_t^{m-\alpha}\left(f(t) - \sum\limits_{k=0}^{m-1} f^{(k)}(0)g_{k+1}(t)\right)$.

Let $\overline{\mathbf{R}}_+ = \mathbf{R}_+ \cup \{0\}$, $\mathscr{Z}$ is a Banach space, $\mathscr{C}l(\mathscr{Z})$ is the set of linear, closed and densely defined in $\mathscr{Z}$ operators, which are acting into $\mathscr{Z}$. For $A \in \mathscr{C}l(\mathscr{Z})$, $\alpha > 0$ consider the equation

$$D_t^\alpha z(t) = Az(t)\,, \quad t \in \overline{\mathbf{R}}_+. \tag{5}$$

A function $z \in C(\overline{\mathbf{R}}_+; D_A) \cap C^{m-1}(\overline{\mathbf{R}}_+; \mathscr{Z})$, for which

$$g_{m-\alpha} * \left(z - \sum_{k=0}^{m-1} z^{(k)}(0)g_{k+1}\right) \in C^m(\overline{\mathbf{R}}_+; \mathscr{Z}),$$

is called a solution of Eq. (5), if for all $t \in \overline{\mathbf{R}}_+$ equality (5) holds. Here D_A is the Banach space with the graph norm of the closed operator A.

Denote by $\mathscr{L}(\mathscr{Z})$ the Banach space of linear continuous on the space $\mathscr{Z}$ operators. A set of operators $\{Z(t) \in \mathscr{L}(\mathscr{Z}) : t \in \overline{\mathbf{R}}_+\}$ is called a resolving operators family for Eq. (5), if

(i) $Z(\cdot)$ is strongly continuous on $\overline{\mathbf{R}}_+$, $Z(0) = I$;

(ii) for all $t \in \overline{\mathbf{R}}_+$ $Z(t)[D_A] \subset D_A$, $Z(t)Az_0 = AZ(t)z_0$ for any $z_0 \in D_A$;

(iii) for every $z_0 \in D_A$ the function $Z(t)z_0$ is a solution of the Cauchy problem $z(0) = z_0$, $z^{(k)}(0) = 0$, $k = 1, 2, \ldots, m-1$, for Eq. (5).

In the terminology of [1] an operator $A \in \mathscr{C}l(\mathscr{Z})$ belongs to $\mathscr{A}^\alpha(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, if there exists a resolving operators family $\{Z(t) \in \mathscr{L}(\mathscr{Z}) : t \in \overline{\mathbf{R}}_+\}$ for Eq. (5), which admits an analytic extension to the sector $\Sigma_{\theta_0} := \{t \in \mathbf{C} : |\arg t| < \theta_0 - \pi/2, t \neq 0\}$ and for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $C(\theta, a)$ such that for all $t \in \Sigma_\theta$ $\|Z(t)\|_{\mathscr{L}(\mathscr{Z})} \leq C(\theta, a)e^{a\mathrm{Re}t}$. According to Theorem 2.14 [1] (see also more general Theorem I.2.1 [12]) for $\alpha \in (0, 2)$ $A \in \mathscr{A}^\alpha(\theta_0, a_0)$ if and only if the following conditions are valid:

(i) for every $\lambda \in S_{\theta_0,a_0} := \{\mu \in \mathbf{C} : |\arg(\mu - a_0)| < \theta_0, \mu \neq a_0\}$ the inclusion $\lambda^\alpha \in \rho(A) := \{\mu \in \mathbf{C} : (\mu I - A)^{-1} \in \mathscr{L}(\mathscr{Z})\}$ is true;

(ii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists a constant $K = K(\theta, a) > 0$, such that for every $\lambda \in S_{\theta,a}$

$$\|(\lambda^\alpha I - A)^{-1}\|_{\mathscr{L}(\mathscr{Z})} \le \frac{K(\theta, a)}{|\lambda^{\alpha-1}(\lambda - a)|}.$$

We will write $A \in \mathscr{A}_\alpha(\theta_0, a_0)$ for some $\alpha > 0$, if $A \in \mathscr{C}l(\mathscr{Z})$ and conditions (i), (ii) are satisfied. So $\mathscr{A}_\alpha(\theta_0, a_0) = \mathscr{A}^\alpha(\theta_0, a_0)$ for $\alpha \in (0, 2)$.

Lemma 1 *Let* $\alpha > 0$, $A \in \mathscr{A}_\alpha(\theta_0, a_0)$, $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm = \{\mu \in \mathbf{C} : \mu = a + re^{\pm i\theta},\ r \in (\delta, \infty)\}$, $\Gamma_0 = \{\mu \in \mathbf{C} : \mu = a + \delta e^{i\varphi},\ \varphi \in (-\theta, \theta)\}$ *for* $\theta \in (\pi/2, \theta_0)$, $a > a_0$, $\delta > 0$. *Then the families of operators*

$$\left\{ Z_{\alpha,\beta}(t) = \frac{1}{2\pi i} \int\limits_\Gamma \mu^{\alpha-\beta}(\mu^\alpha I - A)^{-1} e^{\mu t} d\mu \in \mathscr{L}(\mathscr{Z}) : t \in \mathbf{R}_+ \right\}, \quad \beta \in \mathbf{R},$$

admit analytic extensions to the sector Σ_{θ_0}. *For any* $\theta \in (\pi/2, \theta_0)$, $a > a_0$ *there exists such* $C_\beta = C_\beta(\theta, a)$, *that for each* $t \in \Sigma_\theta$

$$\|Z_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{Z})} \le C_\beta(\theta, a) e^{a\mathrm{Re}t}(|t|^{-1} + a)^{1-\beta}, \quad \beta \le 1, \tag{6}$$

$$\|Z_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{Z})} \le C_\beta(\theta, a) e^{a\mathrm{Re}t}|t|^{\beta-1}, \quad \beta > 1. \tag{7}$$

Besides,

$$\frac{d^k}{dt^k} Z_{\alpha,\beta} = Z_{\alpha,\beta-k}, \quad k \in \mathbf{N}, \tag{8}$$

$$\lim_{t \to 0+} Z_{\alpha,\beta}(t) = 0, \quad \beta > 1. \tag{9}$$

Proof For $\varepsilon \in (0, \theta - \pi/2)$, $t \in \Sigma_{\theta-\varepsilon}$, $\mu \in \Gamma_\pm$ we have

$$\mathrm{Re}(\mu t) = a\mathrm{Re}t + r|t|\cos(\arg t \pm \theta) \le a\mathrm{Re}t - r|t|\sin\varepsilon,$$

and in the case $\mu \in \Gamma_0$ $\mathrm{Re}(\mu t) = a\mathrm{Re}t + \delta|t|\cos(\arg t \pm \varphi)$, therefore, for $\beta \le 1$

$$\|Z_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{Z})} \le \frac{Ke^{a\mathrm{Re}t}}{\pi} \int\limits_\delta^\infty \frac{(r+a)^{1-\beta}}{r} e^{-r|t|\sin\varepsilon} dr + \frac{Ke^{\delta|t| + a\mathrm{Re}t}(\delta + a)^{1-\beta}\theta}{\pi}.$$

At $\beta > 1$ the analogous estimate has the form

$$\|Z_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{Z})} \le \frac{Ke^{a\mathrm{Re}t}c^{1-\beta}}{\pi} \int\limits_\delta^\infty r^{-\beta} e^{-r|t|\sin\varepsilon} dr + \frac{Ke^{\delta|t| + a\mathrm{Re}t}c^{1-\beta}\delta^{1-\beta}\theta}{\pi}.$$

Here the inequality $|\mu| \ge c|\mu - a|$ is used, which is valid with some $c = c(\theta, a) > 0$ for all $\mu \in \Gamma$. Thus, for every $\beta \in \mathbf{R}$ the corresponding integral converges uniformly with respect to t on every compact subset of the sector Σ_θ. Consequently, the integral defines an analytic function with respect to t in the sector.

Take $\delta = |t|^{-1}$, then for $\beta \le 1$

$$\left\| Z_{\alpha,\beta}(t) \right\|_{\mathscr{L}(\mathscr{Z})} \le \frac{Ke^{a\mathrm{Re}t}}{\pi} \int\limits_1^\infty \frac{(r|t|^{-1}+a)^{1-\beta}}{r} e^{-r\sin\varepsilon} dr + \frac{Ke^{1+a\mathrm{Re}t}(|t|^{-1}+a)^{1-\beta}\theta}{\pi} \le$$

$$\le \frac{Ke^{a\mathrm{Re}t}(|t|^{-1}+a)^{1-\beta}}{\pi} \int\limits_1^\infty r^{-\beta} e^{-r\sin\varepsilon} dr + \frac{Ke^{1+a\mathrm{Re}t}(|t|^{-1}+a)^{1-\beta}\theta}{\pi}.$$

Hence, inequalities (6) are true with

$$C_\beta(\theta, a) = \frac{K\left(a, \frac{\theta+\theta_0}{2}\right)}{\pi} \int\limits_1^\infty r^{-\beta} e^{-r\sin\varepsilon} dr + \frac{K\left(a, \frac{\theta+\theta_0}{2}\right)\theta e}{\pi},$$

if to take $\varepsilon = \min\left\{\frac{\theta-\pi/2}{2}, \frac{\theta_0-\theta}{2}\right\}$.

At $\delta = |t|^{-1}$, $\beta > 1$, arguing as in the proof of the analyticity $Z_{\alpha,\beta}$, we obtain that

$$\left\| Z_{\alpha,\beta}(t) \right\|_{\mathscr{L}(\mathscr{Z})} \le \frac{Ke^{a\mathrm{Re}t}c^{1-\beta}|t|^{\beta-1}}{\pi} \int\limits_1^\infty r^{-\beta} e^{-r\sin\varepsilon} dr + \frac{Ke^{1+a\mathrm{Re}t}c^{1-\beta}|t|^{\beta-1}\theta}{\pi},$$

which implies (7) with

$$C_\beta(\theta, a) = \frac{K\left(a, \frac{\theta+\theta_0}{2}\right)c\left(a, \frac{\theta+\theta_0}{2}\right)^{1-\beta}}{\pi} \int\limits_1^\infty r^{-\beta} e^{-r\sin\varepsilon} dr +$$

$$+ \frac{K\left(a, \frac{\theta+\theta_0}{2}\right)c\left(a, \frac{\theta+\theta_0}{2}\right)^{1-\beta}\theta e}{\pi}, \quad \varepsilon = \min\left\{\frac{\theta-\pi/2}{2}, \frac{\theta_0-\theta}{2}\right\}.$$

Inequalities (7) imply (9). Equalities (8) is obvious due to the analyticity of $Z_{\alpha,\beta}$. □

Corollary 1 *Let* $\alpha > 0$, $A \in \mathscr{A}_\alpha(\theta_0, a_0)$. *Then* $s\text{-}\lim\limits_{t\to 0+} Z_{\alpha,1}(t) = I$.

Proof We have

$$I = \frac{1}{2\pi i} \int\limits_\Gamma \frac{e^{\mu t}}{\mu} d\mu$$

with the same contour Γ, as before. Then at $z_0 \in D_A$

$$Z_{\alpha,1}(t)z_0 - z_0 = \frac{1}{2\pi i}\int\limits_{\Gamma}\left(\mu^\alpha(\mu^\alpha I - A)^{-1} - I\right)\frac{e^{\mu t}}{\mu}z_0 d\mu =$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma}\mu^{-1}(\mu^\alpha I - A)^{-1}Az_0 e^{\mu t}d\mu = Z_{\alpha,\alpha+1}Az_0 \to 0$$

as $t \to 0+$ due to (7). The inequality (6) at $\beta = 1$ implies that $Z_{\alpha,1}(t)z_0$ tends to z_0 as $t \to 0+$ for any $z_0 \in \mathscr{Z}$, since D_A is dense in the space $\mathscr{Z}$. □

Remark 1 It is known that the resolving operators family for Eq. (5) is $\{Z_{\alpha,1}(t) : t \in \overline{\mathbf{R}}_+\}$, where $Z_{\alpha,1}(0) = I$ [1].

Consider the Cauchy problem

$$z^{(k)}(0) = z_k, \quad k = 0, 1, \dots, m-1, \tag{10}$$

for the inhomogeneous equation

$$D_t^\alpha z(t) = Az(t) + f(t), \quad t \in [0, T), \tag{11}$$

where $A \in \mathscr{A}_\alpha(\theta_0, a_0)$ in a Banach space $\mathscr{Z}$, $f : [0, T) \to \mathscr{Z}$ for a given $T > 0$. A solution of problem (10), (11) is a function $z \in C([0, T); D_A) \cap C^{m-1}([0, T); \mathscr{Z})$, such that $g_{m-\alpha} * \left(z - \sum\limits_{k=0}^{m-1} z^{(k)}(0)g_{k+1}\right) \in C^m([0, T); \mathscr{Z})$ and equalities (10) and (11) for all $t \in [0, T)$ are satisfied.

Theorem 1 *Let $\alpha > 0$, $A \in \mathscr{A}_\alpha(\theta_0, a_0)$, $f \in C([0, T); D_A)$. Then for any $z_k \in D_A$, $k = 0, 1, \dots, m-1$, there exists a unique solution of problem* (10), (11)*. It has the form*

$$z(t) = \sum_{k=0}^{m-1} Z_{\alpha,k+1}(t)z_k + \int\limits_0^t Z_{\alpha,\alpha}(t-s)f(s)ds.$$

Proof In [4, Remark 2] it is proved, that $\sum\limits_{k=0}^{m-1} Z_{\alpha,k+1}(t)z_k$ at $z_k \in D_A$, $k = 0, \dots, m-1$, is a solution of Cauchy problem (10) to the homogeneous equation $D_t^\alpha z(t) = Az(t)$.

Denote $Z_f(t) := \int\limits_0^t Z_{\alpha,\alpha}(t-s)f(s)ds$. Due to (9) $Z_{\alpha,\alpha}^{(k)}(0) = Z_{\alpha,\alpha-k}(0) = 0$ for all $\alpha > 1$, $k = 0, 1, \dots, m-2$, since $\alpha - k > 1$. Hence, for $\alpha > 0$, $k = 0, 1, \dots, m-1$

$$Z_f^{(k)}(t) = \int\limits_0^t Z_{\alpha,\alpha}^{(k)}(t-s)f(s)ds,$$

and $Z_f^{(k)}(0) = 0$ at $k = 0, 1, \dots, m-2$. Note also, that due to (6) at some $\theta \in (\pi/2, \theta_0)$, $a > a_0$

$$\|Z_f^{(m-1)}(t)\| \le t \max_{s\in[0,t]} \|Z_{\alpha,\alpha}^{(m-1)}(s)\|_{\mathscr{L}(\mathscr{Z})} \max_{s\in[0,T/2]} \|f(s)\|_{\mathscr{Z}} \le$$

$$\le C_{\alpha-m+1}(\theta, a)e^{at}t^{\alpha+1-m}\,(1+at)^{m-\alpha}\,\|f\|_{C([0,T/2];\mathscr{Z})} \to 0 \quad \text{as} \quad t \to 0+\,.$$

Therefore $Z_f \in C^{m-1}([0, T); \mathscr{Z})$.

We have $\mathrm{im} Z_{\alpha,\alpha}(t) \subset D_A$ at $t > 0$, since

$$AZ_{\alpha,\alpha}(t) = \frac{1}{2\pi i}\int_\Gamma (A - \mu^\alpha I + \mu^\alpha I)(\mu^\alpha I - A)^{-1}e^{\mu t}d\mu = Z_{\alpha,0}(t).$$

Moreover,

$$AZ_f(t) = \int_0^t AZ_{\alpha,\alpha}(t-s)f(s)ds = \int_0^t Z_{\alpha,\alpha}(t-s)Af(s)ds = Z_{Af}(t).$$

Thus, $Z_f \in C([0, T); D_A)$.

By $\widehat{f}$ denote the Laplace transform of a function f. We define $Z_{\alpha,\beta}$ and f by zero outside $[0, T)$ with finite $T > 0$, then $Z_f = Z_{\alpha,\alpha} * f$ is a convolution, hence, $\widehat{Z_f} = \widehat{Z_{\alpha,\alpha}}\widehat{f} = R_{\lambda^\alpha}(A)\widehat{f}$ on $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda > a_0\}$, because it is easy to show, that for $\mathrm{Re}\lambda > a_0$, $\beta > 0$ $\widehat{Z_{\alpha,\beta}}(\lambda) = \lambda^{\alpha-\beta}R_{\lambda^\alpha}(A)$. Using the formula of the Laplace transformation for the Caputo derivative [1], we obtain the equality

$$\widehat{D^\alpha Z_f} = \lambda^\alpha \widehat{Z_f} - \sum_{l=0}^{m-1}\lambda^{\alpha-l-1}Z_f^{(l)}(0) = AR_{\lambda^\alpha}(A)\widehat{f} + \widehat{f} = A\widehat{Z_f} + \widehat{f} = \widehat{AZ_f} + \widehat{f}.$$

Acting on the both sides of this equality by the inverse Laplace transform, obtain that Z_f is a solution of problem (10) with $z_k = 0$, $k = 0, 1, \dots, m-1$, for Eq. (11), and $\sum_{k=0}^{m-1} Z_{\alpha,k+1}(t)z_k + Z_f(t)$ is a solution of problem (10), (11) with arbitrary z_k, $k = 0, 1, \dots, m-1$.

If there exist two solutions $\tilde{z}_1$ and $\tilde{z}_2$ of problem (10), (11), then their difference $z = \tilde{z}_1 - \tilde{z}_2$ is a solution of problem (10) with $z_k = 0$, $k = 0, 1, \dots, m-1$, for the equation $D_t^\alpha z(t) = Az(t)$. Acting by the Laplace transform on this equation, obtain $\widehat{D^\alpha z} = \lambda^\alpha\widehat{z} = A\widehat{z}$, hence, $\widehat{z} \equiv 0$ on $\{\lambda \in \mathbf{C} : \mathrm{Re}\lambda > a_0\}$, since $A \in \mathscr{A}_\alpha(\theta_0, a_0)$. Thus, $z(t) \equiv 0$ due to the uniqueness of the inverse Laplace transform. □

2.2 Semilinear Equation

Let U be an open set in $\mathbf{R} \times D_A \times \mathscr{Z}^{m-1}$, an operator $B : U \to \mathscr{Z}$, generally speaking, is nonlinear, $z_k \in \mathscr{Z}$, $k = 0, 1, \ldots, m-1$. Here $m - 1 < \alpha \le m \in \mathbf{N}$. Consider the Cauchy problem

$$z^{(k)}(t_0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{12}$$

for the semilinear equation

$$D_t^\alpha z(t) = Az(t) + B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t)). \tag{13}$$

Denote $\tilde{g}_\beta(t) = g_\beta(t - t_0)$ at $\beta > 0$, $t > t_0$. A solution of problem (12), (13) on a segment $[t_0, t_1]$ is a function $z \in C([t_0, t_1]; D_A) \cap C^{m-1}([t_0, t_1]; \mathscr{Z})$, such that conditions (12) are satisfied, $g_{m-\alpha} * \left(z - \sum\limits_{k=0}^{m-1} z^{(k)}(t_0)\tilde{g}_{k+1}\right) \in C^m([t_0, t_1]; \mathscr{Z})$, at $t \in [t_0, t_1]$ $(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t)) \in U$ and equality (13) is valid.

Lemma 2 *Let* $\alpha > 0$, $A \in \mathscr{A}_\alpha(\theta_0, a_0)$, $z_k \in D_A$, $k = 0, 1, \ldots, m-1$, *U be an open set in* $\mathbf{R} \times D_A \times \mathscr{Z}^{m-1}$, $B \in C(U; D_A)$, $(t_0, z_0, z_1, \ldots, z_{m-1}) \in U$. *Then a function* $z \in C([t_0, t_1]; D_A) \cap C^{m-1}([t_0, t_1]; \mathscr{Z})$ *is a solution of problem* (12), (13) *on a segment* $[t_0, t_1]$, *if and only if for every* $t \in [t_0, t_1]$

$$z(t) = \sum_{k=0}^{m-1} Z_{\alpha,k+1}(t - t_0)z_k + \int_{t_0}^{t} Z_{\alpha,\alpha}(t - s)B(s, z(s), z^{(1)}(s), \ldots, z^{(m-1)}(s))ds. \tag{14}$$

Proof If z is a solution of problem (12), (13), then the mapping

$$t \to B(t, z(t), z^{(1)}(t), \ldots, z^{(m-1)}(t))$$

acts from $[t_0, t_1]$ into D_A continuously. From Theorem 1 it follows, that the equality (14) is valid.

Let $z \in C([t_0, t_1]; D_A) \cap C^{m-1}([t_0, t_1]; \mathscr{Z})$ satisfies (14), then it can be shown directly as in the proof of Theorem 1, that z is a solution of problem (12), (13). □

Further the line over a symbol will mean an ordered set of m elements with indices from zero till $m - 1$, for example, $\overline{x} = (x_0, x_1, \ldots, x_{m-1})$. Denote $S_\delta(\overline{x}) = \{\overline{y} \in D_A \times \mathscr{Z}^{m-1} : \|y_0 - x_0\|_{D_A} \le \delta,\ \|y_k - x_k\|_{\mathscr{Z}} \le \delta,\ k = 1, 2, \ldots, m-1\}$. A mapping $B : U \to D_A$ is called locally Lipschitzian in $\overline{x}$, if for every $(t, \overline{x}) \in U$ there exist such $\delta > 0$, $l > 0$, that $[t - \delta, t + \delta] \times S_\delta(\overline{x}) \subset U$, and for all elements $(s, \overline{y}), (s, \overline{v}) \in [t - \delta, t + \delta] \times S_\delta(\overline{x})$ the inequality

$$\|B(s,\overline{y})-B(s,\overline{v})\|_{D_A} \le l\|y_0-v_0\|_{D_A}+l\sum_{k=1}^{m-1}\|y_k-v_k\|_{\mathscr{X}}$$

is fulfilled.

Theorem 2 *Let* $\alpha>0,\ A\in\mathscr{A}_\alpha(\theta_0,a_0),\ z_k\in D_A,\ k=0,1,\dots,m-1,\ U$ *be an open set in* $\mathbf{R}\times D_A\times\mathscr{X}^{m-1}$, $(t_0,z_0,z_1,\dots,z_{m-1})\in U$, *a mapping* $B\in C(U;D_A)$ *is locally Lipschitzian with respect to* $\overline{x}$. *Then there exists such* $t_1>t_0$, *that problem* (12), (13) *has a unique solution on the segment* $[t_0,t_1]$.

Proof Due to Lemma 2 it is sufficient to show, that Eq. (14) has a unique solution $z\in C([t_0,t_1];D_A)\cap C^{m-1}([t_0,t_1];\mathscr{X})$ for some $t_1>t_0$.

Choose such $\tau>0$ and $\delta>0$, that $V=[t_0,t_0+\tau]\times S_\delta(\overline{z})\subset U$, where $\overline{z}$ consists of the initial vectors $z_0,\ z_1,\ \dots,\ z_{m-1}\in D_A$ from conditions (12). Denote by $\mathscr{S}$ the set of functions $y\in C([t_0,t_0+\tau];D_A)\cap C^{m-1}([t_0,t_0+\tau];\mathscr{X})$, such that $\|y(t)-z_0\|_{D_A}\le\delta,\ \|y^{(k)}(t)-z_k\|_{\mathscr{X}}\le\delta$ at $t_0\le t\le t_0+\tau,\ k=1,2,\dots,m-1$. Then $\mathscr{S}$ is a complete metric space with the metrics

$$d(y,v):=\sup_{t\in[t_0,t_0+\tau]}\|y(t)-v(t)\|_{D_A}+\sum_{k=1}^{m-1}\sup_{t\in[t_0,t_0+\tau]}\|y^{(k)}(t)-v^{(k)}(t)\|_{\mathscr{X}}.$$

Define the operator

$$G(y)(t):=\sum_{k=0}^{m-1}Z_{\alpha,k+1}(t-t_0)z_k+\int_{t_0}^{t}Z_{\alpha,\alpha}(t-s)B(s,y(s),\dots,y^{(m-1)}(s))\,ds$$

for $t\in[t_0,t_0+\tau]$ and prove that the operator G is the contraction on the metric space $\mathscr{S}$ for sufficiently small $\tau>0$.

Indeed, denote $K:=\max\limits_{t\in[t_0,t_0+\tau]}\|B(t,\overline{z})\|_{D_A}$,

$$d(y,\overline{z}):=\sup_{t\in[t_0,t_0+\tau]}\|y(t)-z_0\|_{D_A}+\sum_{k=1}^{m-1}\sup_{t\in[t_0,t_0+\tau]}\|y^{(k)}(t)-z_k\|_{\mathscr{X}}.$$

At $n=0,1,\dots,m-1$ due to (8) and the proof of Theorem 1

$$G^{(n)}(y)=\sum_{k=0}^{m-1}Z_{\alpha,k+1-n}(t-t_0)z_k+\int_{t_0}^{t}Z_{\alpha,\alpha-n}(t-s)B(s,y(s),\dots,y^{(m-1)}(s))ds.$$

Then for some $\theta\in(\pi/2,\theta_0)$, $a>a_0$, at sufficiently small $\tau>0$ and $t\in[t_0,t_0+\tau]$

$$\|G(y)(t)-z_0\|_{D_A} \le \left\|Z_{\alpha,1}(t-t_0)z_0-z_0\right\|_{D_A}+\left\|\sum_{k=1}^{m-1}Z_{\alpha,k+1}(t-t_0)z_k\right\|_{D_A}+$$

$$+\left\|\int_{t_0}^{t}Z_{\alpha,\alpha}(t-s)(B(s,\overline{y}(s))-B(s,\overline{z}))ds\right\|_{D_A}+\left\|\int_{t_0}^{t}Z_{\alpha,\alpha}(t-s)B(s,\overline{z})ds\right\|_{D_A}\le$$

$$\le\left\|Z_{\alpha,1}(t-t_0)z_0-z_0\right\|_{D_A}+\left\|\sum_{k=1}^{m-1}Z_{\alpha,k+1}(t-t_0)z_k\right\|_{D_A}+$$

$$+(ld(y,\overline{z})+K)\left\|\int_{t_0}^{t}Z_{\alpha,\alpha}(t-s)ds\right\|_{\mathscr{L}(\mathscr{Z})}\le\frac{\delta}{2}+(lm\delta+K)Ce^{a\tau}\tau^{\alpha}\le\delta,$$

$$\|G^{(n)}(y)(t)-z_n\|_{\mathscr{Z}}\le\left\|\sum_{k=0,k\neq n}^{m-1}Z_{\alpha,k+1-n}(t-t_0)z_k\right\|_{\mathscr{Z}}+$$

$$+\left\|Z_{\alpha,1}(t-t_0)z_n-z_n\right\|_{\mathscr{Z}}+(ld(y,\overline{z})+K)\left\|\int_{t_0}^{t}Z_{\alpha,\alpha-n}(t-s)ds\right\|_{\mathscr{L}(\mathscr{Z})}\le$$

$$\le\frac{\delta}{2}+(lm\delta+K)Ce^{a\tau}\tau^{\alpha-n}\le\delta,\quad n=1,2,\ldots,m-1,$$

due to (6), (7) and Corollary 1. Here the equality $AZ_{\alpha,\beta}(t)=Z_{\alpha,\beta}(t)A$ on D_A is used. Consequently, $G:\mathscr{S}\to\mathscr{S}$.

Besides, for small τ and for every $t\in[t_0,t_0+\tau]$, $n=1,2,\ldots,m-1$, $y,v\in\mathscr{S}$

$$\|G^{(n)}(y)(t)-G^{(n)}(v)(t)\|_{\mathscr{Z}}=\left\|\int_{t_0}^{t}Z_{\alpha,\alpha-n}(t-s)(B(s,\overline{y}(s))-B(s,\overline{v}))ds\right\|_{\mathscr{Z}}\le$$

$$\le Ce^{a\tau}\tau^{\alpha-n}l\left(\sup_{t\in[t_0,t_0+\tau]}\|y(t)-v(t)\|_{D_A}+\sum_{k=1}^{m-1}\sup_{t\in[t_0,t_0+\tau]}\|y^{(k)}(t)-v^{(k)}(t)\|_{\mathscr{Z}}\right)\le$$

$$\le\frac{d(y,v)}{2m},$$

$$\|G(y)(t)-G(v)(t)\|_{D_A}\le Ce^{a\tau}\tau^{\alpha}l\left(\sup_{t\in[t_0,t_0+\tau]}\|y(t)-v(t)\|_{D_A}+\right.$$

$$+\sum_{k=1}^{m-1} \sup_{t\in[t_0,t_0+\tau]} \|y^{(k)}(t) - v^{(k)}(t)\|_{\mathscr{X}} \Bigg) \le \frac{d(y,v)}{2m}.$$

Therefore, $d(G(y), G(v)) \le d(y, v)/2$ and the operator G has a unique fixed point in $\mathscr{S}$. It is a solution of problem (12), (13) on the segment $[t_0, t_1]$ with $t_1 = t_0 + \tau$. □

3 Degenerate Equations

The unique solvability issues for initial problems to differential equations in Banach spaces with degenerate linear operator at the Caputo derivative are studied in this section. Such equations are often called degenerate evolution equations or Sobolev type equations.

3.1 Resolving Operators Families

Here some results of [4] are given for using in further considerations.

Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, operators $L, M \in \mathscr{C}l(\mathscr{X}; \mathscr{Y})$, $\ker L \ne \{0\}$. The set of points $\mu \in \mathbf{C}$, such that the operator $\mu L - M : D_L \cap D_M \to \mathscr{Y}$ is injective, and $(\mu L - M)^{-1}L \in \mathscr{L}(\mathscr{X})$, $L(\mu L - M)^{-1} \in \mathscr{L}(\mathscr{Y})$, is called as L-resolvent set $\rho^L(M)$ of operator M. Denote $R^L_\mu(M) = (\mu L - M)^{-1}L$, $L^L_\mu(M) = L(\mu L - M)^{-1}$. By $\mathscr{L}(\mathscr{X}; \mathscr{Y})$ the Banach space of linear and continuous operators from $\mathscr{X}$ into $\mathscr{Y}$ will be denoted.

Definition 1 Let $\alpha > 0$. We say that a pair of operators (L, M) belongs to the class $\mathscr{H}_\alpha(\theta_0, a_0)$, if

(i) there exist $\theta_0 \in (\pi/2, \pi)$ and $a_0 \ge 0$, such that for all $\lambda \in S_{\theta_0,a_0}$ inclusion $\lambda^\alpha \in \rho^L(M)$ is valid;

(ii) for any $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exist a constant $K = K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$

$$\max\left\{\|R^L_{\lambda^\alpha}(M)\|_{\mathscr{L}(\mathscr{X})}, \|L^L_{\lambda^\alpha}(M)\|_{\mathscr{L}(\mathscr{Y})}\right\} \le \frac{K(\theta, a)}{|\lambda^{\alpha-1}(\lambda - a)|}.$$

Remark 2 If there exists the inverse operator $L^{-1} \in \mathscr{L}(\mathscr{Y}; \mathscr{X})$, then $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, if and only if $L^{-1}M \in \mathscr{A}_\alpha(\theta_0, a_0)$ and $ML^{-1} \in \mathscr{A}_\alpha(\theta_0, a_0)$.

It is not difficult to show that $\ker R^L_\mu(M) = \ker L$, $\mathrm{im} R^L_\mu(M)$, $\ker L^L_\mu(M)$, $\mathrm{im} L^L_\mu(M)$ do not depend on $\mu \in \rho^L(M)$, if $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. Denote $\ker R^L_\mu(M) = \mathscr{X}^0$, $\ker L^L_\mu(M) = \mathscr{Y}^0$. By $\mathscr{X}^1$ ($\mathscr{Y}^1$) we denote the closure of the linear subspace

$\mathrm{im}R^L_\mu(M)$ ($\mathrm{im}L^L_\mu(M)$). By L_k the restriction of operator L on $D_{L_k} := D_L \cap \mathscr{X}^k$ is denoted, and M_k is a restriction of M on $D_{M_k} := D_M \cap \mathscr{X}^k$, $k = 0, 1$.

The next assertion can be proved similar to Lemma 1.

Lemma 3 *Let $\alpha > 0$, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$. Then the families of operators*

$$\left\{X_{\alpha,\beta}(t) := \frac{1}{2\pi i}\int_\Gamma \mu^{\alpha-\beta}(\mu^\alpha L - M)^{-1}Le^{\mu t}d\mu \in \mathscr{L}(\mathscr{X}) : t \in \mathbf{R}_+\right\},\ \beta \in \mathbf{R},$$

$$\left\{Y_{\alpha,\beta}(t) := \frac{1}{2\pi i}\int_\Gamma \mu^{\alpha-\beta}L(\mu^\alpha L - M)^{-1}e^{\mu t}d\mu \in \mathscr{L}(\mathscr{Y}) : t \in \mathbf{R}_+\right\},\ \beta \in \mathbf{R},$$

admit analytic extensions to the sector Σ_{θ_0}. For any $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists such $C_\beta = C_\beta(\theta, a)$, that for each $t \in \Sigma_\theta$

$$\|X_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{X})} \le C_\beta(\theta, a)e^{a\mathrm{Re}t}(|t|^{-1} + a)^{1-\beta}, \quad \beta \le 1,$$

$$\|X_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{X})} \le C_\beta(\theta, a)e^{a\mathrm{Re}t}|t|^{\beta-1}, \quad \beta > 1,$$

$$\|Y_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{Y})} \le C_\beta(\theta, a)e^{a\mathrm{Re}t}(|t|^{-1} + a)^{1-\beta}, \quad \beta \le 1,$$

$$\|Y_{\alpha,\beta}(t)\|_{\mathscr{L}(\mathscr{Y})} \le C_\beta(\theta, a)e^{a\mathrm{Re}t}|t|^{\beta-1}, \quad \beta > 1.$$

Besides,

$$\frac{d^k}{dt^k}X_{\alpha,\beta} = X_{\alpha,\beta-k}, \quad \frac{d^k}{dt^k}Y_{\alpha,\beta} = Y_{\alpha,\beta-k}, \quad k \in \mathbf{N},$$

$$\lim_{t\to 0+} X_{\alpha,\beta}(t) = 0, \quad \lim_{t\to 0+} Y_{\alpha,\beta}(t) = 0, \quad \beta > 1.$$

Theorem 3 ([4]) *Let a Banach space $\mathscr{X}$ ($\mathscr{Y}$) be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. Then $\mathscr{X} = \mathscr{X}^0 \oplus \mathscr{X}^1$ ($\mathscr{Y} = \mathscr{Y}^0 \oplus \mathscr{Y}^1$).*

Denote by P (Q) the projection on $\mathscr{X}^1$ ($\mathscr{Y}^1$) along the subspace $\mathscr{X}^0$ ($\mathscr{Y}^0$).

Corollary 2 ([4]) *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. Then*
(i) $P = s\text{-}\lim\limits_{n\to\infty} nR^L_n(M)$, $Q = s\text{-}\lim\limits_{n\to\infty} nL^L_n(M)$;
(ii) $L_0 = 0$, $M_0 \in \mathscr{C}l(\mathscr{X}^0; \mathscr{Y}^0)$;
(iii) $L_1, M_1 \in \mathscr{C}l(\mathscr{X}^1; \mathscr{Y}^1)$;
(iv) *there exist operators* $L_1^{-1} \in \mathscr{C}l(\mathscr{Y}^1; \mathscr{X}^1)$, $M_0^{-1} \in \mathscr{L}(\mathscr{Y}^0; \mathscr{X}^0)$.

Introduce the notations $S := L_1^{-1}M_1 : D_S \to \mathscr{X}^1$, $D_S := \{x \in D_{M_1} : M_1x \in \mathrm{im}L_1\}$; $T := M_1L_1^{-1} : D_T \to \mathscr{Y}^1$, $D_T := \{y \in \mathrm{im}L_1 : L_1^{-1}y \in D_{M_1}\}$.

Lemma 4 ([4]) *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. Then*
(i) *$S \in \mathscr{C}l(\mathscr{X}^1)$, if $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$;*
(ii) *$T \in \mathscr{C}l(\mathscr{Y}^1)$, if $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$.*

Denote $X^1_{\alpha,\beta}(t) = X_{\alpha,\beta}(t)|_{\mathscr{X}^1}$, $Y^1_{\alpha,\beta}(t) = Y_{\alpha,\beta}(t)|_{\mathscr{Y}^1}$ for $t > 0$.

Lemma 5 ([4]) *Let Banach space $\mathscr{X}$ ($\mathscr{Y}$) be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, then for every $t > 0$*

$$X_{\alpha,\beta}(t) = X^1_{\alpha,\beta}(t)P, \quad \mathscr{X}^0 \subset \ker X_{\alpha,\beta}(t), \quad \operatorname{im} X_{\alpha,\beta}(t) \subset \mathscr{X}^1$$

$$\left(Y_{\alpha,\beta}(t) = Y^1_{\alpha,\beta}(t)Q, \quad \mathscr{Y}^0 \subset \ker Y_{\alpha,\beta}(t), \quad \operatorname{im} Y_{\alpha,\beta}(t) \subset \mathscr{Y}^1\right).$$

Corollary 3 ([4]) *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $\alpha > 0$, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$.*

(i) *If $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, then there exists an operator $s\text{-}\lim\limits_{t\to 0+} X_{\alpha,1}(t) := X_{\alpha,1}(0) = P$.*

(ii) *If $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, then there exists an operator $s\text{-}\lim\limits_{t\to 0+} Y_{\alpha,1}(t) := Y_{\alpha,1}(0) = Q$.*

In the proof of this assertion we used Corollary 1.

Theorem 4 ([4]) *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $\alpha \in (0, 2)$, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$.*

(i) *If $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, then $S \in \mathscr{A}^\alpha(\theta_0, a_0)$ and $\{X^1_{\alpha,1}(t) \in \mathscr{L}(\mathscr{X}^1) : t \in \overline{\mathbf{R}}_+\}$ is the resolving operators family for the equation $D^\alpha_t x(t) = Sx(t)$.*

(ii) *If $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, then $T \in \mathscr{A}^\alpha(\theta_0, a_0)$ and $\{Y^1_{\alpha,1}(t) \in \mathscr{L}(\mathscr{Y}^1) : t \in \overline{\mathbf{R}}_+\}$ is the resolving operators family for the equation $D^\alpha_t y(t) = Ty(t)$.*

Corollary 4 *Let Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $\alpha\theta_0 > \pi$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$. Then operator M is $(L, 0)$-bounded* [2, 3].

Proof Corollary 2 and the condition $\alpha\theta_0 > \pi$ imply that for all $|\mu| > a_0$ operator

$$(\mu L - M)^{-1} = (\mu L_1 - M_1)^{-1}Q - M_0^{-1}(I - Q) =$$

$$= L_1^{-1}L^{L_1}_\mu(M_1)Q - M_0^{-1}(I - Q) = L_1^{-1}L^{L_1}_{\lambda^\alpha}(M_1)Q - M_0^{-1}(I - Q)$$

is continuous as difference of the two continuous operators. Here $\lambda^\alpha = \mu$, $\lambda \in S_{\theta_0,a_0}$. □

3.2 Degenerate Inhomogeneous Linear Equation

Consider the degenerate inhomogeneous equation

$$D_t^\alpha Lx(t) = Mx(t) + f(t), \quad t \in [0, T), \tag{15}$$

with a given $f : [0, T) \to \mathscr{Y}$. Its solution is a function $x \in C([0, T); D_M)$, such that $Lx \in C^{m-1}([0, T); \mathscr{Y})$, $g_{m-\alpha} * \left(Lx - \sum\limits_{k=0}^{m-1} (Lx)^{(k)}(0) g_{k+1} \right) \in C^m([0, T); \mathscr{Y})$, and for all $t \in [0, T)$ equality (15) is fulfilled. A solution of the Cauchy problem

$$x^{(k)}(0) = x_k, \; k = 0, 1, \ldots, m-1, \tag{16}$$

for Eq. (15) is a solution of the equation, such that $x \in C^{m-1}([0, T); \mathscr{X})$ and conditions (16) are satisfied.

Theorem 5 *Let $\alpha > 0$, Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$. Suppose that $f : [0, T) \to \mathscr{Y}^0 \dot{+} L_1 \left[D_{L_1^{-1} M_1} \right]$, $L_1^{-1} Q f \in C\big([0, T); D_{L_1^{-1} M_1}\big)$, $(I - Q) f \in C^{m-1}([0, T); \mathscr{Y})$, $x_k \in D_M$, $P x_k \in D_{L_1^{-1} M_1}$, $k = 0, 1, \ldots, m-1$, equalities*

$$D_t^k|_{t=0} M_0^{-1} (I - Q) f(t) = -(I - P) x_k, \quad k = 0, 1, \ldots, m-1, \tag{17}$$

are valid. Then there exists a unique solution of problem (15), (16), and it has form

$$x(t) = \sum_{k=0}^{m-1} X_{\alpha,k+1}(t) x_k + \int_0^t X_{\alpha,\alpha}(t-s) L_1^{-1} Q f(s) ds - M_0^{-1} (I - Q) f(t). \tag{18}$$

Proof Put $w(t) := (I - P)x(t)$, $v(t) := P x(t)$. By virtue of Corollary 2 Eq. (15) can be reduced to the system of the two equations $0 = w(t) + M_0^{-1}(I - Q) f(t)$, and

$$D_t^\alpha v(t) = S v(t) + g(t), \quad S := L_1^{-1} M_1, \quad g(t) := L_1^{-1} Q f(t). \tag{19}$$

Therefore $w(t) = -M_0^{-1}(I - Q) f(t)$, and for the satisfying of Cauchy conditions (16) it is necessary the fulfillment of (17). The operator $S \in \mathscr{A}_\alpha(\theta_0, a_0)$, therefore Theorem 1 implies the existence of a unique solution of the Cauchy problem $v^{(k)}(0) = P x_k$, $k = 0, 1, \ldots, m-1$, for Eq. (19). It has form

$$v(t) = \frac{1}{2\pi i}\sum_{k=0}^{m-1}\int_{\Gamma}\mu^{\alpha-k-1}(\mu^\alpha I - S)^{-1}e^{\mu t}d\mu Px_k +$$
$$+\frac{1}{2\pi i}\int_0^t\int_{\Gamma}(\mu^\alpha I - S)^{-1}e^{\mu(t-s)}d\mu g(s)ds =$$
$$= \frac{1}{2\pi i}\sum_{k=0}^{m-1}\int_{\Gamma}\mu^{\alpha-k-1}(\mu^\alpha L_1 - M_1)^{-1}L_1e^{\mu t}d\mu Px_k +$$
$$+\frac{1}{2\pi i}\int_0^t\int_{\Gamma}(\mu^\alpha L_1 - M_1)^{-1}L_1e^{\mu(t-s)}d\mu g(s)ds =$$
$$= \frac{1}{2\pi i}\sum_{k=0}^{m-1}\int_{\Gamma}\mu^{\alpha-k-1}(\mu^\alpha L - M)^{-1}Le^{\mu t}d\mu x_k +$$
$$+\frac{1}{2\pi i}\int_0^t\int_{\Gamma}(\mu^\alpha L - M)^{-1}Le^{\mu(t-s)}d\mu g(s)ds,$$

since $L(I-P)=0$, the operator $(\lambda L_0 - M_0)^{-1} = -M_0^{-1}$ exists for every $\lambda \in \mathbf{C}$. It remains to note, that

$$Lx(t) = \sum_{k=0}^{m-1}Y_{\alpha,k+1}(t)Lx_k + \int_0^t Y_{\alpha,\alpha}(t-s)Qf(s)ds$$

satisfies the definition of the problem solution. □

Theorem 6 *Let* $\alpha > 0$, *Banach spaces* $\mathscr{X}$, $\mathscr{Y}$ *be reflexive,* $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ *or* $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$. *Suppose that* $f : [0, T) \to \mathscr{Y}^0\dot{+}L_1[D_{M_1}]$, $Qf \in C\big([0,T); D_{M_1L_1^{-1}}\big)$, $(I-Q)f \in C^{m-1}([0,T); \mathscr{Y})$, $x_k \in D_M$, $Px_k \in D_L$, $k = 0, 1, \dots, m-1$, *equalities* (17) *are valid. Then there exists a unique solution of problem* (15), (16), *and it has form* (18).

Proof If $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, instead of Eq. (19) we obtain

$$D_t^\alpha z(t) = Tz(t) + h(t), \quad T := M_1L_1^{-1}, \quad h(t) := Qf(t), \tag{20}$$

where $z(t) := L_1v(t) = L_1Px(t)$. We have $T \in \mathscr{A}_\alpha(\theta_0, a_0)$, and due to Theorem 1 there exists a unique solution of the Cauchy problem $z^{(k)}(0) = L_1Px_k \in D_T$, $k = 0, 1, \dots, m-1$, for Eq. (20). The solution has the form

$$z(t) = \frac{1}{2\pi i}\sum_{k=0}^{m-1}\int_{\Gamma}\mu^{\alpha-k-1}(\mu^\alpha I - T)^{-1}e^{\mu t}d\mu L_1Px_k +$$
$$+\frac{1}{2\pi i}\int_0^t\int_{\Gamma}(\mu^\alpha I - T)^{-1}e^{\mu(t-s)}d\mu h(s)ds =$$
$$= \frac{1}{2\pi i}\sum_{k=0}^{m-1}\int_{\Gamma}\mu^{\alpha-k-1}L_1(\mu^\alpha L_1 - M_1)^{-1}e^{\mu t}d\mu L_1Px_k +$$
$$+\frac{1}{2\pi i}\int_0^t\int_{\Gamma}L_1(\mu^\alpha L_1 - M_1)^{-1}e^{\mu(t-s)}d\mu Qf(s)ds,$$

therefore,

$$v(t) = L_1^{-1}z(t) = \frac{1}{2\pi i}\sum_{k=0}^{m-1}\int_{\Gamma}\mu^{\alpha-k-1}(\mu^\alpha L - M)^{-1}Le^{\mu t}d\mu x_k +$$
$$+\frac{1}{2\pi i}\int_0^t\int_{\Gamma}(\mu^\alpha L - M)^{-1}Le^{\mu(t-s)}d\mu L_1^{-1}Qf(s)ds.$$

The function $w(t)$ is the same as in the previous proof. □

Consider the so-called Showalter–Sidorov problem

$$(Lx)^{(k)}(0) = y_k, \quad k = 0, 1, \dots, m-1, \tag{21}$$

which is natural for weakly degenerate evolution equations, when the degeneracy subspace coincide with $\ker L$.

Theorem 7 *Let Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $\alpha > 0$, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$. Suppose that $f : [0, T) \to \mathscr{Y}^0 \dot{+} L_1 \left[D_{L_1^{-1}M_1}\right]$, $L_1^{-1}Qf \in C\big([0, T); D_{L_1^{-1}M_1}\big)$, $(I - Q)f \in C^{m-1}([0, T); \mathscr{Y})$, $y_k \in \operatorname{im}L$, $L_1^{-1}y_k \in D_{L_1^{-1}M_1}$ at $k = 0, 1, \dots, m-1$. Then there exists a unique solution of problem* (15), (21)*, and it has form* (18).

Theorem 8 *Let Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $\alpha > 0$, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$. Suppose that $f : [0, T) \to \mathscr{Y}^0 \dot{+} L_1 [D_{M_1}]$, $Qf \in C\big([0, T); D_{M_1L_1^{-1}}\big)$, $(I - Q)f \in C^{m-1}([0, T); \mathscr{Y})$, $y_k \in D_{M_1L_1^{-1}}$, $k = 0, \dots, m-1$. Then there exists a unique solution of problem* (15), (21)*, and it has form* (18).

The proofs is similar to the previous one. The feature of conditions (21) is such that in initial moment it doesn't imply restrictions on the projection v of Eq. (15) solution ant its derivatives, since $L_0 = 0$. Therefore, concordance conditions (17) fulfillment are not needed.

Remark 3 It can be shown, that in the case of the reflexivity of Banach spaces $\mathscr{X}$ and $\mathscr{Y}$ for $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$ conditions (21) are equivalent to the conditions $(Px)^{(k)}(0) = L_1^{-1}y_k$, $k = 0, 1, \dots, m-1$. Recall that $\operatorname{im}L \subset \mathscr{X}^1$.

3.3 Degenerate Semilinear Equations

Let $0 \le r \le m-1 < \alpha \le m \in \mathbf{N}$, $U \subset \mathbf{R} \times D_M \times \mathscr{X}^r$ be an open set, $N : U \to \mathscr{Y}$ be a nonlinear mapping. Consider the degenerate semilinear equation

$$D_t^\alpha Lx(t) = Mx(t) + N(t, x(t), x^{(1)}(t), \dots, x^{(r)}(t)). \tag{22}$$

A function $x \in C([t_0, t_1]; D_M) \cap C^r([t_0, t_1]; \mathscr{X})$ is called a solution of Eq. (22) on a segment $[t_0, t_1]$, if $Lx \in C^{m-1}([t_0, t_1]; \mathscr{Y})$, $g_{m-\alpha} * \left(Lx - \sum\limits_{k=0}^{m-1} (Lx)^{(k)}(0) g_{k+1} \right) \in C^m([t_0, t_1]; \mathscr{Y})$, for all $t \in [t_0, t_1]$ $(t, x(t), x^{(1)}(t), \dots, x^{(r)}(t)) \subset U$, and equality (22) is valid. A solution of the problem

$$(Lx)^{(k)}(0) = y_k, \quad k = 0, 1, \dots, m-1, \tag{23}$$

for Eq. (22) is a solution of the equation, such that conditions (23) are satisfied.

Theorem 9 *Let $\alpha > 0$, Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, $0 \le 2r \le m-1$, U be open in the space $\mathbf{R} \times D_M \times \mathscr{X}^r$, $y_k \in \operatorname{im} L$, $k = 0, 1, \dots, m-1$, $(t_0, L_1^{-1} y_0, L_1^{-1} y_1, \dots, L_1^{-1} y_r) \in U \cap \left(\mathbf{R} \times \left(D_{L_1^{-1} M_1}\right)^{r+1}\right)$, $L_1^{-1} y_k \in D_{L_1^{-1} M_1}$ at $k = r+1, r+2, \dots, m-1$, the set $V := U \cap \left(\mathbf{R} \times D_{L_1^{-1} M_1} \times \left(\mathscr{X}^1\right)^r\right)$ be open in the space $\mathbf{R} \times D_{L_1^{-1} M_1} \times \left(\mathscr{X}^1\right)^r$, $N : U \to \mathscr{Y}$, $\operatorname{im} QN \subset \operatorname{im} L_1$. Suppose that for every $(t, z_0, z_1, \dots, z_r) \in U$, such that $(t, Pz_0, Pz_1, \dots, Pz_r) \in V$, we have $N(t, z_0, \dots, z_r) = N_1(t, Pz_0, \dots, Pz_r)$ with some $N_1 \in C\left(U \cap \left(\mathbf{R} \times \left(\mathscr{X}^1\right)^{r+1}\right); \mathscr{Y}\right)$, and $L_1^{-1} Q N_1 \in C\left(V; D_{L_1^{-1} M_1}\right)$ is locally Lipschitzian in $\overline{z}$. Then there exists such $t_1 > t_0$, that problem (22), (23) has a unique solution on the segment $[t_0, t_1]$.*

Proof Put $w(t) := (I - P)x(t)$, $v(t) := Px(t)$, as before. If $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, then due to Corollary 2 Eq. (22) can be reduced locally to the system of two equations

$$0 = w(t) + M_0^{-1}(I - Q) N_1(t, v(t), v^{(1)}(t), \dots, v^{(r)}(t)), \tag{24}$$

$$D_t^\alpha v(t) = Sv(t) + L_1^{-1} Q N_1(t, v(t), v^{(1)}(t), \dots, v^{(r)}(t)). \tag{25}$$

Remark 3 means that the Cauchy conditions are set for the function v only. Since the operator $S = L_1^{-1} M_1 \in \mathscr{A}_\alpha(\theta_0, a_0)$, Theorem 2 with $\mathscr{Z} = \mathscr{X}^1$, $A = S$, $B = L_1^{-1} Q N_1$, $U = V \times (\mathscr{X}^1)^{m-1-r}$ implies the existence on some segment $[t_0, t_1]$ of a unique solution of the Cauchy problem $v^{(k)}(0) = L_1^{-1} y_k$, $k = 0, 1, \dots, m-1$, to Eq. (25). Moreover,

$$v(t) = \sum_{k=0}^{m-1} X_{\alpha,k+1}(t - t_0) L_1^{-1} y_k + \int_{t_0}^{t} X_{\alpha,\alpha}(t - s) L_1^{-1} Q N_1(s, v(s), v^{(1)}(s), \dots, v^{(r)}(s)) ds,$$

therefore,

$$Lx(t) = \sum_{k=0}^{m-1} Y_{\alpha,k+1}(t - t_0) y_k + \int_{t_0}^{t} Y_{\alpha,\alpha}(t - s) Q N_1(s, v(s), v^{(1)}(s), \dots, v^{(r)}(s)) ds$$

satisfies the conditions of the problem solution.

Equation (24) implies that $w(t) = -M_0^{-1}(I - Q)N_1(t, v(t), v^{(1)}(t), \dots, v^{(r)}(t))$. This function satisfies the smoothness conditions of the solution definition because $2r \le m - 1$. □

Remark 4 From the proof of Theorem 9 it follows that for the existence of a solution of the Cauchy problem for Eq. (22) the condition

$$w(t) = -M_0^{-1}(I - Q)N_1(t, v(t), v^{(1)}(t), \dots, v^{(r)}(t)) \in C^r([t_0, t_1]; \mathscr{X})$$

is necessary. Therefore, we need a solution $v \in C^{2r}([t_0, t_1]; \mathscr{X})$ of the Cauchy problem to Eq. (25). In other words, it is necessary to have a solution v of additional smoothness, if $2r > m - 1$.

As in the linear case, consider another type conditions on L_1 and M_1, corresponding to assertion (ii) of Theorem 4. For this aim denote

$$W := \big\{(t, L_1x_0, L_1x_1, \dots, L_1x_r) \in \mathbf{R} \times (\mathscr{Y}^1)^{r+1} : \\ (t, x_0, x_1, \dots, x_r) \in U,\ x_0 \in D_{L_1} \cap D_{M_1},\ x_k \in D_{L_1},\ k = 1, 2, \dots, m - 1\big\},$$

$QN_1 \circ L_1^{-1}$ is the mapping with the correspondence law

$$(t, z_0, z_1, \dots, z_r) \to QN_1(t, L_1^{-1}z_0, L_1^{-1}z_1, \dots, L_1^{-1}z_r).$$

Theorem 10 *Let $\alpha > 0$, Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, $0 \le 2r \le m - 1$, U be open in the space $\mathbf{R} \times D_M \times \mathscr{X}^r$, W be open in $\mathbf{R} \times D_{M_1L_1^{-1}} \times \left(\mathscr{Y}^1\right)^r$, $(t_0, y_0, y_1, \dots, y_r) \in W \cap \left(\mathbf{R} \times \left(D_{M_1L_1^{-1}}\right)^{r+1}\right)$, $y_k \in D_{M_1L_1^{-1}}$, $k = r + 1, r + 2, \dots, m - 1$, $N : U \to \mathscr{Y}$. Suppose that for every element $(t, z_0, \dots, z_r) \in U$, such that $(t, L_1Pz_0, \dots, L_1Pz_r) \in W$, we have $N(t, z_0, \dots, z_r) = N_1(t, Pz_0, \dots, Pz_r)$ with some $N_1 \in C\big(U \cap \left(\mathbf{R} \times \left(\mathscr{X}^1\right)^{r+1}\right); \mathscr{Y}\big)$, and $QN_1 \circ L_1^{-1} \in C\big(W; D_{M_1L_1^{-1}}\big)$ is locally Lipschitzian in $\overline{z}$. Then there exists such $t_1 > t_0$, that problem (22), (23) has a unique solution on the segment $[t_0, t_1]$.*

Proof Put $\eta(t) := L_1v(t) = L_1Px(t)$, then from Eq. (22) it follows that

$$D_t^\alpha \eta(t) = T\eta(t) + QN_1(t, L_1^{-1}\eta(t), (L_1^{-1}\eta)^{(1)}(t), \dots, (L_1^{-1}\eta)^{(r)}(t)), \tag{26}$$

$\eta^{(k)}(t_0) = y_k$, $k = 0, 1, \dots, m - 1$. Operator $T = M_1L_1^{-1} \in \mathscr{A}_\alpha(\theta_0, a_0)$, hence Theorem 2 with $\mathscr{Z} = \mathscr{Y}^1$, $A = T$, $U = W \times \left(\mathscr{Y}^1\right)^{m-1-r}$, $B = QN_1 \circ L_1^{-1}$ implies the existence on some segment $[t_0, t_1]$ of a unique solution of the Cauchy problem for Eq. (26). Besides,

$$\eta(t) = \sum_{k=0}^{m-1} Y_{\alpha,k+1}(t-t_0)LPx_k +$$
$$+\int_{t_0}^{t} Y_{\alpha,\alpha}(t-s)QN_1(s, L_1^{-1}\eta(s), L_1^{-1}\eta^{(1)}(s), \dots, L_1^{-1}\eta^{(m-1)}(s))ds.$$

Thus, we have the same form of v and w as in the previous proof. □

Consider the semilinear equation with $r = 0$:

$$D_t^\alpha Lx(t) = Mx(t) + N(t, x(t)), \tag{27}$$

where $N : U \to \mathscr{Y}$, U is an open subset of $\mathbf{R} \times D_M$. A function x is called a solution of the Cauchy problem

$$x^{(k)}(0) = x_k, \quad k = 0, 1, \dots, m-1, \tag{28}$$

for Eq. (27), if it is a solution of the equation, such that $x \in C^{m-1}([t_0, t_1]; \mathscr{X})$ and conditions (28) are fulfilled.

Theorem 11 *Let $\alpha > 0$, Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$ or $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$, U be open in $\mathbf{R} \times D_M$, $N : U \to \mathscr{Y}$, $\mathrm{im}QN \subset \mathrm{im}L_1$, $V := U \cap \big(\mathbf{R} \times D_{L_1^{-1}M_1}\big)$ be open in $\mathbf{R} \times D_{L_1^{-1}M_1}$. Suppose also that for every $(t, z_0) \in U$, such that $(t, Pz_0) \in V$, we have $N(t, z_0) = N_1(t, Pz_0)$ with some $N_1 \in C\big(U \cap (\mathbf{R} \times \mathscr{X}^1); \mathscr{Y}\big)$, and $L_1^{-1}QN_1 \in C\big(V; D_{L_1^{-1}M_1}\big)$ is locally Lipschitzian in $\overline{z}$, $(t_0, x_0) \in U \cap \big(\mathbf{R} \times D_{L_1^{-1}M_1}\big)$, $x_k \in D_{L_1^{-1}M_1}$, $k = 1, 2, \dots, m-1$, for the solution of the Cauchy problem (28) to the equation $D_t^\alpha v(t) = Sv(t) + L_1^{-1}QN_1(t, v(t))$ the conditions*

$$D_t^k|_{t=t_0} M_0^{-1}(I-Q)N_1(t, v(t)) = -(I-P)x_k, \quad k = 0, 1, \dots, m-1, \tag{29}$$

is satisfied. Then there exists such $t_1 > t_0$, that problem (27), (28) has a unique solution on the segment $[t_0, t_1]$.

Proof Theorem 2 implies the existence on some segment $[t_0, t_1]$ of a unique solution of the Cauchy problem for the equation $D_t^\alpha v(t) = Sv(t) + L_1^{-1}QN_1(t, v(t))$. Unlike the previous proof $w(t) = -M_0^{-1}(I-Q)N_1(t, v(t)) \in C^{m-1}([t_0, t_1]; \mathscr{X})$. For the satisfying of Cauchy conditions (28) it is necessary the fulfillment of (29). □

Now let $W := \{(t, L_1x) \in \mathbf{R} \times \mathscr{Y}^1 : (t, x) \in U, \ x \in D_{L_1} \cap D_{M_1}\}$, $QN_1 \circ L_1^{-1}$ is the mapping with the correspondence law $(t, z) \to QN_1(t, L_1^{-1}z)$.

Theorem 12 *Let $\alpha > 0$, Banach spaces $\mathscr{X}$, $\mathscr{Y}$ be reflexive, $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$, $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$ or $M_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$, U be open in $\mathbf{R} \times D_M$, $N : U \to \mathscr{Y}$, W be an open set in $\mathbf{R} \times D_{M_1L_1^{-1}}$. Suppose also that for every element $(t, z_0) \in U$, such that $(t, L_1Pz_0) \in W$, we have $N(t, z_0) = N_1(t, Pz_0)$ with some operator $N_1 \in C\big(U \cap \big(\mathbf{R} \times \mathscr{X}^1\big); \mathscr{Y}\big)$, and $QN_1 \circ L_1^{-1} \in C\big(W; D_{M_1L_1^{-1}}\big)$ is locally Lipschitzian* *in* $\overline{z}$, $(t_0, x_0) \in U \cap \left(\mathbf{R} \times \left(D_{M_1} \cap D_{L_1}\right)\right)$, $x_k \in D_{M_1} \cap D_{L_1}$, $k = 1, 2, \dots, m-1$, *for the solution of the Cauchy problem to the equation* $D_t^\alpha z(t) = Tz(t) + N_1(t, L_1^{-1} z(t))$ *the conditions*

$$D_t^k|_{t=t_0} M_0^{-1}(I - Q)N_1(t, L_1^{-1}z(t)) = -(I - P)x_k, \quad k = 0, 1, \dots, m-1,$$

is satisfied. Then there exists such $t_1 > t_0$, *that problem* (27), (28) *has a unique solution on the segment* $[t_0, t_1]$.

Proof Due to Theorem 2 there exists a unique solution of the Cauchy problem for equation $D_t^\alpha z(t) = Tz(t) + N_1(t, L_1^{-1}z(t))$ on a segment $[t_0, t_1]$. The rest of the arguments is similar to the previous proof. □

4 Application to a Class of Initial-Boundary Value Problems

Let $P_n(\lambda) = \sum\limits_{i=0}^{n} c_i \lambda^i$, $Q_l(\lambda) = \sum\limits_{j=0}^{l} d_j \lambda^j$ such that $c_i, d_j \in \mathbf{R}$, $i = 0, 1, \dots, n$, $j = 0, 1, \dots, l$, $c_n \neq 0$, $d_l \neq 0$, $n < l$, $\Omega \subset \mathbf{R}^d$ be a bounded domain with a smooth boundary $\partial\Omega$, $\Delta = \sum\limits_{k=1}^{d} \frac{\partial^2}{\partial s_k^2}$ is the Laplace operator. At $\alpha \in (1, 2)$ consider the initial-boundary value problem

$$D_t^\alpha P_n(\Delta)u(s,t) = Q_l(\Delta)u(x,t) + g\left(s, P_n(\Delta)u(s,t), \frac{\partial P_n(\Delta)u}{\partial t}(s,t)\right), \ (s,t) \in \Omega \times \mathbf{R}, \tag{30}$$

$$\Delta^j u(s,t) = 0, \quad j = 0, 1, \dots, l-1, \quad (s,t) \in \partial\Omega \times \mathbf{R}, \tag{31}$$

$$P_n(\Delta)u(s,0) = u_0(s), \quad \frac{\partial P_n(\Delta)u}{\partial t}(s,0) = u_1(s), \quad s \in \Omega, \tag{32}$$

within the framework of problem (22), (23). For this aim define operator $A \in \mathscr{C}l(L_2(\Omega))$, acting on its domain $D_A = H_0^2(\Omega) = \left\{u \in H^2(\Omega) : u(s) = 0, \ s \in \partial\Omega\right\}$ as $Au = \Delta u$. Choose integer number $j_0 > d/4$, $\mathscr{Y} = H^{2j_0}(\Omega)$,

$$\mathscr{X} = H_0^{2(n+j_0)}(\Omega) := \left\{u \in H^{2(n+j_0)}(\Omega) : \Delta^k u(x) = 0, \ k = 0, 1, \dots, n-1, \ x \in \partial\Omega\right\},$$

$$L = P_n(A) \in \mathscr{L}(\mathscr{X}; \mathscr{Y}), \ \ M = Q_l(A) \in \mathscr{C}l(\mathscr{X}; \mathscr{Y}), \ \ D_M = H_0^{2(l+j_0)}(\Omega).$$

By $\{\varphi_k : k \in \mathbf{N}\}$ denote the orthonormal in the inner product $\langle\cdot,\cdot\rangle$ of $L_2(\Omega)$ eigenfunctions of the operator A, that corresponding to the eigenvalues $\{\lambda_k : k \in \mathbf{N}\}$, numbered in the non-increasing order with taking into account their multiplicities.

In contrast to results of [3] (Theorems 5 and 6) we will consider the case $l > n$ here.

Theorem 13 *Let $l > n$, $(-1)^{l-n}(d_l/c_n) < 0$, spectrum $\sigma(A)$ does not contain common roots of polynomials $P_n(\lambda)$ and $Q_l(\lambda)$. Then the operator $L_1 : \mathscr{U}^1 \to \mathscr{V}^1$ is a homeomorphism, and for $\alpha \in [1, 2)$ there exist $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, such that $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$. If, moreover, $\max\limits_{P_n(\lambda_k)\neq 0} \mathrm{Re}(Q_l(\lambda_k)/P_n(\lambda_k)) < 1$, then $(L, M) \in \mathscr{H}_\alpha(\theta_0, a_0)$ in the case $\alpha \in (0, 1)$.*

Proof In the case $j_0 = 0$ such statement is proved in [4]. In the case $j_0 \in \mathbf{N}$ that proof can be transferred here word for word. □

Remark 5 It can be prove easy that under the conditions of Theorem 13 projection P has the form $P = \sum\limits_{P_n(\lambda_k)\neq 0} \langle \cdot, \varphi_k \rangle_{L_2(\Omega)} \varphi_k$.

Theorem 14 *Let $\alpha \in (1, 2)$, $l > n$, $(-1)^{l-n}(d_l/c_n) < 0$, spectrum $\sigma(A)$ do not contain common roots of polynomials $P_n(\lambda)$ and $Q_l(\lambda)$, $g \in C^\infty(\Omega \times \mathbf{R}^2; \mathbf{R})$, $u_k = P_n(\Delta)x_k$ for some $x_k \in H_0^{2(n+j_0)}(\Omega)$, $k = 0, 1$. Then for some $t_1 > t_0$ there exists a unique solution of problem (30)–(32) on $[t_0, t_1]$.*

Proof The nonlinear operator $N(z_0, z_1) := g(\cdot, P_n(A)z_0, P_n(A)z_1)$ acts from $\mathscr{X}^2$ into $\mathscr{Y}$ due to [6], since $P_n(A)z_k \in H^{2j_0}(\Omega)$, $k = 0, 1$, the function g acts smoothly from $\left(H^{2j_0}(\Omega)\right)^2$ into $H^{2j_0}(\Omega)$ and $2j_0 > d/2$. Remark 5 implies the equality $N(z_0, z_1) = N(Pz_0, Pz_1)$. □

Let $\alpha \in (1, 2)$, $P_1(\lambda) = 1 + \lambda$, $Q_2(\lambda) = \lambda + 2\lambda^2$, $\Omega = (0, \pi)$, $d = 1$, $j_0 = 1$. Then $\lambda_k = -k^2$, $\varphi_k(x) = \sin kx$, $k \in \mathbf{N}$, problem (30)–(32) has the form

$$D_t^\alpha (u + u_{xx}) = u_{xx} + 2u_{xxxx} + g(u + u_{xx}, u_t + u_{xxt}), \quad (x, t) \in (0, \pi) \times \mathbf{R},$$

$$u(0, t) = u(\pi, t) = u_{xx}(0, t) = u_{xx}(\pi, t) = 0, \quad t \in \mathbf{R},$$

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in (0, \pi).$$

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Solvability for nth Order Coupled Systems with Full Nonlinearities

Feliz Minhós and Infeliz Coxe

Abstract In this article, we consider nth order coupled systems with full nonlinearities, and two-point boundary conditions. The arguments used are based on lower and upper solutions method, together with Leray-Schauder's degree theory. They provide general methods and techniques to ensure the solvability of such systems, and, moreover, to localize the solutions and some of its derivatives. In this way, the paper generalizes the results existent in the literature. Two applications are presented: to some Lorentz-Lagrangian systems, for $n = 2$, and, for $n = 3$, to stationary coupled system of Korteweg-de Vries equations, with damping and forced terms.

Keywords Higher order coupled systems · Nagumo-type conditions · Coupled lower and upper solutions · Lorentz-Lagrangian systems · Korteweg-de Vries coupled equations

2010 Mathematics Subject Classification 34B15 · 34L30 · 47H11 · 34A34

1 Introduction

In this paper we consider the nth order coupled system composed of the fully coupled differential equations

$$\begin{cases} u^{(n)}(t) = f\left(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(n-1)}(t)\right) \\ v^{(n)}(t) = g\left(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(n-1)}(t)\right) \end{cases} \tag{1}$$

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with $f, g : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ continuous functions, and the boundary conditions

$$\begin{cases} u^{(i)}(0) = A_i, u^{(n-2)}(1) = B, i = 0, 1, \ldots, n-2, \\ v^{(i)}(0) = C_i, v^{(n-2)}(1) = D, \end{cases} \tag{2}$$

for $A_i, B, C_i, D \in \mathbb{R}$.

Coupled systems of nonlinear boundary value problems of second and higher order with ordinary differential equations have received, in these last years, a great deal of attention in the literature, by means of different methods and several types of arguments. For recent trends in this field, we recommend interested readers to [1, 11, 15, 21, 23, 28–30, 32, 33], and the references therein.

Our arguments are focused on general methods and techniques to ensure the solvability of such systems, and, moreover, to localize the solutions and some of the derivatives. In this way, the paper generalizes the results existent in the literature. For example, in [22], the authors consider the system

$$\begin{cases} -u'''(t) = f(t, v(t), v'(t)) \\ -v'''(t) = h(t, u(t), u'(t)) \\ u(0) = u'(0) = 0, u'(1) = \alpha u'(\eta) \\ v(0) = v'(0) = 0, v'(1) = \alpha v'(\eta), \end{cases}$$

with non-negative continuous functions $f, h \in C\left([0, 1] \times [0, +\infty)^2, [0, +\infty)\right)$ verifying adequate superlinear and sublinear conditions near 0 and $+\infty$, $0 < \eta < 1$ and the parameter α such that $1 < \alpha < \frac{1}{\eta}$. Applying the Guo–Krasnosel'skiĭ theorem on expansion-compression cones, and defining an adequate cone, to overcome the dependence on the first derivatives, it is proved the existence of a positive and increasing solution of the system.

In [24], it is studied the existence of solutions for a system of bending elastic beam equations

$$\begin{cases} u''''(t) = f(t, u(t), v(t), u'(t), v'(t)), t \in (0, 1), \\ v''''(t) = g(t, u(t), v(t), u'(t), v'(t)), t \in (0, 1), \\ u(0) = u(1) = u''(0) = u''(1) = 0, \\ v(0) = v(1) = v''(0) = v''(1) = 0, \end{cases}$$

via the fixed point index theory, assuming sufficient conditions, some of them of the lipschitzian type.

In [19, 20], the authors present the system

$$\begin{cases} u^{(4)}(t) = f\left(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)\right) \\ v^{(4)}(t) = h\left(t, u(t), u'(t), u''(t), u'''(t), v(t), v'(t), v''(t), v'''(t)\right) \end{cases}$$

with $f, h : [0, 1] \times \mathbb{R}^8 \to \mathbb{R}$ some L^1-Carathéodory functions, together with the boundary conditions

$$\begin{cases} u(0) = u'(0) = u''(0) = u''(1) = 0 \\ v(0) = v'(0) = v''(0) = v''(1) = 0, \end{cases}$$

and prove its solvability applying Green's functions, with integral operators theory and Schauder's fixed point.

In [31], it is considered the nth-order nonlinear boundary value problem

$$\begin{cases} u^{(n)}(t) + f(t, u(t), v(t)) = 0, 0 < t < 1, \\ v^{(n)}(t) + g(t, u(t), v(t)) = 0, 0 < t < 1, \\ u^{(i)}(0) = u(1) = 0, i = 0, 1, \ldots, n-2, \\ v^{(i)}(0) = v(1) = 0, i = 0, 1, \ldots, n-2, \end{cases}$$

where $n \geq 2$ and $f, g \in C([0, 1] \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, $(\mathbb{R}^+ := [0, \infty))$. Based on *a priori* estimates, achieved by Jensen's integral inequality, fixed point index theory and assumptions on the nonlinearities, formulated in terms of spectral radii of associated linear integral operators, it is proved the existence of, at least, one positive solution.

This type of coupled systems cover some classical systems of differential equations, as, for instance, Lorenz-Lagrangian systems [3, 26], and Korteweg-de Vries (KdV) coupled equations [6, 8–10, 12, 17, 18, 27], and have a huge variety of applications, such as, in solitary waves theory [5, 7, 14], the study of the bending of elastic beams [2, 13, 16, 25], among others.

Motivated by the above works, we present a technique for coupled higher order systems that, to the best of our knowledge, is new in the literature, and opens the possibility of new types of models. Our method applies a new Nagumo-type condition for coupled equations, with adequate growth conditions on the nonlinearities, to obtain not only the existence of a solution but also some data about the location of the unknown functions and their derivatives, given by lower and upper solutions method. The existence tool will be given by a homotopic problem and Leray-Schauder topological degree theory. Moreover, this paper contains two applications for higher order coupled systems. The first one, for n even, $n = 2$, to a family of Lorentz-Lagrangian systems, and the second one, for $n = 3$, to some stationary coupled system of Korteweg-de Vries equations with damping and forced terms.

The paper is organized in this way: Sect. 2 contains the functional framework, definitions, and some *a priori* estimations given by Nagumo-type conditions. The main result in Sect. 3 is based on some growth assumptions on the nonlinearities. Last two sections contain the applications: Sect. 4, deals with some Lorentz-Lagrangian systems, and Sect. 5 with a coupled system of KdV equations.

2 Definitions and Preliminaries

Let $E := C^{n-1}[0, 1]$ be a Banach space equipped with the norm $\|\cdot\|_{C^{n-1}}$, defined by

$$\|w\|_{C^{n-1}} := \max\left\{\|w\|, \ldots, \|w^{(n-1)}\|\right\},$$

where

$$\|y\| := \max_{t\in[0,\,1]} |y(t)|$$

and the product space $E^2 := \left(C^{n-1}[0,\ 1]\right)^2$ with the norm

$$\|(u,v)\|_{E^2} = \max\left\{\|u\|_{C^{n-1}}, \|v\|_{C^{n-1}}\right\}.$$

Throughout the paper we apply the following relation:

$$\begin{aligned} &(x_0,\ldots,x_{n-1},y_0,\ldots,y_{n-1}) \leq (z_0,\ldots,z_{n-1},w_0,\ldots,w_{n-1}) \\ \Longleftrightarrow\ & x_i \leq z_i \wedge y_i \leq w_i, \forall x_i, y_i, z_i, w_i \in \mathbb{R}, i = 0,1,\ldots,n-1. \end{aligned}$$

For some functions $\gamma_j^i, \delta_j^i \in C[0,\ 1]$, for $j = 1, 2$, and $i = 0, 1, \ldots, n-2$, such that

$$\gamma_j^i(t) \leq \delta_j^i(t), \forall t \in [0, 1],$$

define the set

$$S := S_{\gamma_j^i,\delta_j^i} = \left\{ \begin{array}{c} (t,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1}) \in [0,1]\times\mathbb{R}^{2n} : \gamma_1^i(t) \leq u_i \leq \delta_1^i(t), \\ \gamma_2^i(t) \leq v_i \leq \delta_2^i(t), i = 0,1,\ldots,n-2 \end{array} \right\}. \tag{3}$$

To control the growth of the $(n-1)$ derivatives we need Nagumo-type conditions:

Definition 1 The continuous functions $f, g : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ satisfy Nagumo-type conditions relative to the set S, if there are positive continuous functions $\phi_i : [0, +\infty[\to]0, +\infty[$, $i = 1, 2$, such that

$$|f(t,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1})| \leq \phi_1(|u_{n-1}|) \tag{4}$$

and

$$|g(t,u_0,\ldots,u_{n-1},v_0,\ldots,v_{n-1})| \leq \phi_2(|v_{n-1}|), \tag{5}$$

for $(t, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}) \in S$, with

$$\int_0^{+\infty} \frac{s}{\phi_i(s)} ds = +\infty, i = 1, 2. \tag{6}$$

Lemma 2 *Let $f, g : [0, 1] \times \mathbb{R}^{2n} \to \mathbb{R}$ be continuous functions satisfying a Nagumo type condition relative to the set S.*

Then there are $N_1, N_2 > 0$ such that, for every solution (u, v) of (1)–(2) with $\left(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(n-1)}(t)\right) \in S$,

$$\left\|u^{(n-1)}\right\| < N_1, \tag{7}$$

and

$$\left\| v^{(n-1)} \right\| < N_2. \tag{8}$$

Proof Let (u, v) be a solution of (1) such that

$$\left(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(n-1)}(t)\right) \in S.$$

For $r > B - A_{n-2}$, consider $N_1, N_2 > r$ such that

$$\int_r^{N_1} \frac{s}{\phi_1(s)} ds > \max_{t\in[0,1]} \delta_1^{n-2}(t) - \min_{t\in[0,1]} \gamma_1^{n-2}(t), \tag{9}$$

and

$$\int_r^{N_2} \frac{s}{\phi_2(s)} ds > \max_{t\in[0,1]} \delta_2^{n-2}(t) - \min_{t\in[0,1]} \gamma_2^{n-2}(t). \tag{10}$$

If $\left|u^{(n-1)}\right| \le r, \forall t \in [0,1]$, then this part of the proof is finished, as $\left\|u^{(n-1)}\right\| \le r < N_1$.

On the other hand if $\left|u^{(n-1)}(t)\right| \ge r, \forall t \in [0,1]$, we obtain the following contradiction for the case where $u^{(n-1)}(t) > r$,

$$r > B - A_{n-2} = \int_0^1 u^{(n-1)}(t)\, dt \ge \int_0^1 r dt = r.$$

If $u^{(n-1)}(t) < -r$ the contradiction is analogous.

By (2) and the Mean Value Theorem, there is $t_0 \in]0,1[$ such that $u^{(n-1)}(t_0) > r$, $t_2 \in]0,1[$, $t_2 < t_0$, with $u^{(n-1)}(t_2) = r$ and $u^{(n-1)}(t) > r, \forall t \in]t_2, t_0]$.

Then, by (1), (5) and (10),

$$\begin{aligned}
\int_{u^{(n-1)}(t_2)}^{u^{(n-1)}(t_0)} \frac{s}{\phi_1(s)} ds &= \int_{t_2}^{t_0} \frac{u^{(n-1)}(s)}{\phi_1\left(u^{(n-1)}(s)\right)} u^{(n)}(s)\, ds \\
&\le \int_{t_2}^{t_0} \frac{u^{(n-1)}(s)}{\phi_1\left(u^{(n-1)}(s)\right)} \left| f\begin{pmatrix} s, u(s), \ldots, u^{(n-1)}(s), \\ v(s), \ldots, v^{(n-1)}(s) \end{pmatrix} \right| ds \\
&\le \int_{t_2}^{t_0} u^{(n-1)}(s)\, ds = u^{(n-2)}(t_0) - u^{(n-2)}(t_2)
\end{aligned}$$

$$\leq \max_{t\in[0,1]} \delta_1^{n-2}(t) - \min_{t\in[0,1]} \gamma_1^{n-2}(t) < \int_r^{N_1} \frac{s}{\phi_1(s)} ds.$$

By the arbitrariness of t_0 related to the values where $u^{(n-1)}(t_0) > r$, we have

$$u^{(n-1)}(t) < N_1, \forall t \in [0,1].$$

For $t_2 > t_0$ with $u^{(n-1)}(t_2) = r$ and $u^{(n-1)}(t) > r, \forall t \in [t_0, t_2[$, the arguments are similar.

In the case where $u^{(n-1)}(t) < -r$ the technique is analogous and, therefore, $\left\| u^{(n-1)} \right\| \leq N_1$.

Applying the same method as above, it can be proved, by (5) and (10), that $\left\| v^{(n-1)} \right\| \leq N_2$. ∎

Lower and upper functions will be defined as a pair, as follows:

Definition 3 For $A_i, B, C_i, D \in \mathbb{R}$, $i = 0, 1, \ldots, n-2$, the functions $(\alpha_1, \alpha_2) \in E^2$ are coupled lower solutions of (1)–(2) if

$$\begin{aligned} &\alpha_1^{(n)}(t) \geq f\left(t, \alpha_1(t), \ldots, \alpha_1^{(n-1)}(t), \alpha_2(t), \ldots, \alpha_2^{(n-2)}(t), v_{n-1}\right), \\ &\quad \text{for } t \in [0,1] \text{ and } v_{n-1} \in \mathbb{R}, \\ &\alpha_2^{(n)}(t) \geq g\left(t, \alpha_1(t), \ldots, \alpha_1^{(n-2)}(t), u_{n-1}, \alpha_2(t), \ldots, \alpha_2^{(n-1)}(t)\right), \\ &\quad \text{for } t \in [0,1] \text{ and } u_{n-1} \in \mathbb{R}, \end{aligned} \tag{11}$$

with

$$\begin{aligned} \alpha_1^{(i)}(0) \leq A_i, \alpha_1^{(n-2)}(1) \leq B, \\ \alpha_2^{(i)}(0) \leq C_i, \alpha_2^{(n-2)}(1) \leq D. \end{aligned} \tag{12}$$

The functions $(\beta_1, \beta_2) \in E^2$ are coupled upper solutions of (1)–(2) if they verify the reversed inequalities.

3 Main Result

The main theorem is an existence and localization result, meaning that, it provides not only the existence, but also some data about the localization of the unknown functions and their derivatives:

Theorem 4 *Let $f, g : [0,1] \times \mathbb{R}^{2n} \to \mathbb{R}$ be continuous functions. Suppose that there are coupled lower and upper solutions of (1)–(2), (α_1, α_2) and (β_1, β_2), respectively, such that*

$$\left(\alpha_1^{(n-2)}(t), \alpha_2^{(n-2)}(t)\right) \le \left(\beta_1^{(n-2)}(t), \beta_2^{(n-2)}(t)\right), \forall t \in [0,1].$$

Assume that f and g verify the Nagumo conditions relative to the set $S_{\alpha^{(i)},\beta^{(i)}}$, $i = 0, 1, \ldots, n-2$, and the growth conditions

$$\begin{aligned} & f\left(t, \alpha_1(t), \ldots, \alpha_1^{(n-3)}(t), u_{n-2}, u_{n-1}, \alpha_2(t), \ldots, \alpha_2^{(n-2)}(t), v_{n-1}\right) \\ & \ge f(t, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}) \\ & \ge f\left(t, \beta_1(t), \ldots, \beta_1^{(n-3)}(t), u_{n-2}, u_{n-1}, \beta_2(t), \ldots, \beta_2^{(n-2)}(t), v_{n-1}\right), \end{aligned} \tag{13}$$

for $\alpha_1^{(i)}(t) \le u_i \le \beta_1^{(i)}(t)$, $i = 0, 1, \ldots, n-3$, $\alpha_2^{(j)}(t) \le v_j \le \beta_2^{(j)}(t)$, $j = 0, 1, \ldots, n-2$, and $(t, u_{n-2}, u_{n-1}, v_{n-1}) \in [0,1] \times \mathbb{R}^3$,

$$\begin{aligned} & g\left(t, \alpha_1(t), \ldots, \alpha_1^{(n-2)}(t), u_{n-1}, \alpha_2(t), \ldots, \alpha_2^{(n-3)}(t), v_{n-2}, v_{n-1}\right) \\ & \ge g(t, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}) \\ & \ge g\left(t, \beta_1(t), \ldots, \beta_1^{(n-2)}(t), u_{n-1}, \beta_2(t), \ldots, \beta_2^{(n-3)}(t), v_{n-2}, v_{n-1}\right), \end{aligned} \tag{14}$$

for $\alpha_1^{(j)}(t) \le u_j \le \beta_1^{(j)}(t)$, $j = 0, 1, \ldots, n-2$, $\alpha_2^{(i)}(t) \le v_i \le \beta_2^{(i)}(t)$, $i = 0, 1, \ldots, n-3$, and $(t, u_{n-1}, v_{n-2}, v_{n-1}) \in [0,1] \times \mathbb{R}^3$.

Then problem (1), (2) has, at least, a solution $(u, v) \in E^2$, such that,

$$\begin{aligned} & \alpha_1^{(i)}(t) \le u^{(i)}(t) \le \beta_1^{(i)}(t), \\ & \alpha_2^{(i)}(t) \le v^{(i)}(t) \le \beta_2^{(i)}(t), \text{ for } i = 0, 1, \ldots, n-2, \forall t \in [0,1]. \end{aligned}$$

Remark 5 If $\alpha_1^{(n-2)}(t) \le \beta_1^{(n-2)}(t)$ for $t \in [0,1]$, then by integration in $[0,t]$, (2) and (12),

$$\alpha_1^{(i)}(t) \le \beta_1^{(i)}(t), \text{ for } i = 0, 1, \ldots, n-3, \text{ and } t \in [0,1].$$

Analogously from $\alpha_2^{(n-2)}(t) \le \beta_2^{(n-2)}(t), \forall t \in [0,1]$, by integration in $[0,t]$, then

$$\alpha_2^{(i)}(t) \le \beta_2^{(i)}(t), \text{ for } i = 0, 1, \ldots, n-3, \text{ and } t \in [0,1].$$

Proof Define the continuous functions, for $i = 0, 1, \ldots, n-2$, $j = 1, 2$,

$$\delta_{j,i}(t, w_i) = \begin{cases} \beta_j^{(i)} & \text{if } w_i > \beta_j^{(i)} \\ w_i & \text{if } \alpha_j^{(i)} \le w_i \le \beta_j^{(i)}, \\ \alpha_j^{(i)} & \text{if } w_i < \alpha_j^{(i)}. \end{cases}$$

For $\lambda \in [0,1]$, consider the homotopic problem composed by the equations

$$\begin{cases} u^{(n)}(t) = \lambda f\begin{pmatrix} t, \delta_{1,0}(t,u(t)), \ldots, \delta_{1,n-2}\left(t, u^{(n-2)}(t)\right), u^{(n-1)}(t), \\ \delta_{2,0}(t,v(t)), \ldots, \delta_{2,n-2}\left(t, v^{(n-2)}(t)\right), v^{(n-1)}(t) \end{pmatrix} \\ \qquad +u^{(n-2)}(t) - \lambda\delta_{1,n-2}\left(t, u^{(n-2)}(t)\right), \\ v^{(n)}(t) = \lambda g\begin{pmatrix} t, \delta_{1,0}(t,u(t)), \ldots, \delta_{1,n-2}\left(t, u^{(n-2)}(t)\right), u^{(n-1)}(t), \\ \delta_{2,0}(t,v(t)), \ldots, \delta_{2,n-2}\left(t, v^{(n-2)}(t)\right), v^{(n-1)}(t) \end{pmatrix} \\ \qquad +v^{(n-2)}(t) - \lambda\delta_{2,n-2}\left(t, v^{(n-2)}(t)\right), \end{cases} \tag{15}$$

for $t \in [0, 1]$, together with boundary conditions

$$\begin{cases} u^{(i)}(0) = \lambda A_i, u^{(n-2)}(1) = \lambda B, i = 0, 1, \ldots, n-2, \\ v^{(i)}(0) = \lambda C_i, v^{(n-2)}(1) = \lambda D. \end{cases} \tag{16}$$

Take $r_1, r_2 > 0$ such that, for $u^{(n-1)}(t), v^{(n-1)}(t), \in \mathbb{R}$,

$$-r_j < \alpha_j^{(n-2)}(t) \le \beta_j^{(n-2)}(t) \le r_j,\ j = 1, 2,\ \text{for } t \in [0,1], \tag{17}$$

$$|A_{n-2}| < r_1, |B| < r_1, |C_{n-2}| < r_2, |D| < r_2, \tag{18}$$

$$f\left(t, \alpha_1(t), \ldots, \alpha_1^{(n-2)}(t), 0, \alpha_2(t), \ldots, \alpha_2^{(n-2)}(t), v^{(n-1)}(t)\right) - r_1 - \alpha_1^{(n-2)}(t) < 0, \tag{19}$$

$$f\left(t, \beta_1(t), \ldots, \beta_1^{(n-2)}(t), 0, \beta_2(t), \ldots, \beta_2^{(n-2)}(t), v^{(n-1)}(t)\right) + r_1 - \beta_1^{(n-2)}(t) > 0, \tag{20}$$

$$g\left(t, \alpha_1(t), \ldots, \alpha_1^{(n-2)}(t), u^{(n-1)}(t), \alpha_2(t), \ldots, \alpha_2^{(n-2)}(t), 0\right) - r_2 - \alpha_2^{(n-2)}(t) < 0, \tag{21}$$

$$g\left(t, \beta_1(t), \ldots, \beta_1^{(n-2)}(t), u^{(n-1)}(t), \beta_2(t), \ldots, \beta_2^{(n-2)}(t), 0\right) + r_2 - \beta_2^{(n-2)}(t) > 0. \tag{22}$$

For clearness the proof will follow several steps:

Step 1: *Every solution* (u, v)*of* (15), (16) *verifies,*

$$\left|u^{(n-2)}(t)\right| < r_1, \left|v^{(n-2)}(t)\right| < r_2,$$

$$\left|u^{(i)}(t)\right| < r_1 + \sum_{k=i}^{n-3} |A_k| := r_1^i,$$

$$\left|v^{(i)}(t)\right| < r_2 + \sum_{k=i}^{n-3} |C_k| := r_2^i,$$

for $i = 0, 1, \ldots, n-3$, *independently of* $\lambda \in [0, 1]$.

Suppose, by contradiction, that the first inequality is not verified for $i = n-2$. Then there is a solution $u(t)$ of (15), (2) and $t \in [0, 1]$ such that $\left|u^{(n-2)}(t)\right| \geq r_1$, that is,

$$u^{(n-2)}(t) \geq r_1 \text{ or } u^{(n-2)}(t) \leq -r_1.$$

In the first case define

$$\max_{t\in[0,1]} u^{(n-2)}(t) := u^{(n-2)}(t_0) \geq r_1.$$

As, by (18), $t_0 \neq 0$ and $t_0 \neq 1$, then $t_0 \in \left]0, 1\right[$, $u^{(n-1)}(t_0) = 0$ and $u^{(n)}(t_0) \leq 0$.
Therefore, for $\lambda \in \left]0, 1\right]$, it is obtained, by (15), (17) and (20), the following contradiction,

$$\begin{aligned}
0 &\geq u^{(n)}(t_0) \\
&= \lambda f\begin{pmatrix} t_0, \delta_{1,0}(t_0, u(t_0)), \ldots, \delta_{1,n-2}\left(t_0, u^{(n-2)}(t_0)\right), 0, \\ \delta_{2,0}(t_0, v(t_0)), \ldots, \delta_{2,n-2}\left(t_0, v^{(n-2)}(t_0)\right), v^{(n-1)}(t_0) \end{pmatrix} \\
&\quad + u^{(n-2)}(t_0) - \lambda\beta_1^{(n-2)}(t_0) \\
&\geq \lambda\begin{bmatrix} f\left(t_0, \beta_1(t_0), \ldots, \beta_1^{(n-2)}(t_0), 0, \beta_2(t_0), \ldots, \beta_2^{(n-2)}(t_0), v^{(n-1)}(t_0)\right) \\ + r_1 - \beta_1^{(n-2)}(t_0) \end{bmatrix} \\
&> 0
\end{aligned}$$

For $\lambda = 0$ the contradiction is given by

$$0 \geq u^{(n)}(t_0) = u^{(n-2)}(t_0) \geq r_1 > 0.$$

Following similar arguments, it can be proved that

$$u^{(n-2)}(t) \geq -r_1, \forall t \in [0, 1],$$

and, therefore,

$$\left|u^{(n-2)}(t)\right| \leq r_1, \forall \in [0, 1].$$

As

$$\int_0^t u^{(n-2)}(s)\, ds = u^{(n-3)}(t) - \lambda A_{n-3}$$

and

$$-r_1 \leq \int_0^t u^{(n-2)}(s)\, ds \leq r_1,$$

therefore

$$\left|u^{(n-3)}(t)\right| < r_1 + |A_{n-3}|, \forall t \in [0,1].$$

By iteration of this type of arguments we have

$$\left|u^{(i)}(t)\right| < r_1 + \sum_{k=i}^{n-3} |A_k|, \forall t \in [0,1],$$

for $i = 0, 1, \ldots, n-3$, independently of $\lambda \in [0,1]$.

By the technique above it can be obtained that $\left|v^{(n-2)}(t)\right| < r_2$, and

$$\left|v^{(i)}(t)\right| < r_2 + \sum_{k=i}^{n-3} |C_k|, \forall t \in [0,1],$$

for $i = 0, 1, \ldots, n-3$, independently of $\lambda \in [0,1]$.

Step 2: *Every solution* (u, v) *of* (15), (16) *satisfies* $\left\|u^{(n-1)}\right\| < N_1$, *and* $\left\|v^{(n-1)}\right\| < N_2$, *independently of* $\lambda \in [0,1]$.

For $\lambda \in [0,1]$ define the functions

$$\begin{aligned} &F_\lambda(t, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}) := \\ &\lambda f\begin{pmatrix} t, \delta_{1,0}(t,u_0), \ldots, \delta_{1,n-2}(t,u_{n-2}), u_{n-1}, \delta_{2,0}(t,v_0), \\ \ldots, \delta_{2,n-2}(t,v_{n-2}), v_{n-1} \end{pmatrix} \\ &+ u_{n-2} - \lambda\delta_{1,n-2}(t,u_{n-2}) \end{aligned} \tag{23}$$

and

$$\begin{aligned} &G_\lambda(t, u_0, \ldots, u_{n-1}, v_0, \ldots, v_{n-1}) := \\ &\lambda g\begin{pmatrix} t, \delta_{1,0}(t,u_0), \ldots, \delta_{1,n-2}(t,u_{n-2}), u_{n-1}, \delta_{2,0}(t,v_0), \\ \ldots, \delta_{2,n-2}(t,v_{n-2}), v_{n-1} \end{pmatrix} \\ &+ v_{n-2} - \lambda\delta_{2,n-2}(t,v_{n-2}). \end{aligned} \tag{24}$$

The functions F_λ and G_λ verify the Nagumo conditions (4)–(6), as

$$\begin{aligned} |F_\lambda| &\le \left| f\begin{pmatrix} t, \delta_{1,0}(t,u_0), \ldots, \delta_{1,n-2}(t,u_{n-2}), u_{n-1}, \\ \delta_{2,0}(t,v_0), \ldots, \delta_{2,n-2}(t,v_{n-2}), v_{n-1} \end{pmatrix} \right| \\ &\quad + |u_{n-2}| + \left|\delta_{1,n-2}(t,u_{n-2})\right| \\ &\le \phi_1(|u_{n-1}|) + 2r_1, \end{aligned}$$

$$|G_\lambda| \le \left| g\begin{pmatrix} t, \delta_{1,0}(t, u_0), \ldots, \delta_{1,n-2}(t, u_{n-2}), u_{n-1}, \\ \delta_{2,0}(t, v_0), \ldots, \delta_{2,n-2}(t, v_{n-2}), v_{n-1} \end{pmatrix} \right|$$
$$+ |v_{n-2}| + \left|\delta_{2,n-2}(t, v_{n-2})\right|$$
$$\le \phi_2(|v_{n-1}|) + 2r_2,$$

and

$$\int_0^{+\infty} \frac{s}{\phi_1(s) + 2r_1} ds = +\infty, \int_0^{+\infty} \frac{s}{\phi_2(s) + 2r_2} ds = +\infty.$$

By Step 1, and applying Lemma 2, with, for $j = 1, 2$,

$$\gamma_j^{n-2}(t) \equiv -r_j, \delta_j^{n-2}(t) \equiv r_j,$$
$$\gamma_j^{i}(t) \equiv -r_j^i, \delta_j^{i}(t) \equiv r_j^i,$$

for $i = 0, 1, \ldots, n-3$, there are $N_1, N_2 > 0$ such that

$$\left\|u^{(n-1)}\right\| < N_1 \text{ and } \left\|v^{(n-1)}\right\| < N_2.$$

Step 3: *Problem* (15), (16) *has, at least, a solution for* $\lambda = 1$.

Define the operators

$$\mathcal{L} : \left(C^n([0, 1])\right)^2 \subseteq E^2 \to (C([0, 1]))^2 \times \mathbb{R}^{2n}$$

given by

$$\mathcal{L}(u, v) = \begin{pmatrix} u^{(n)}(t), v^{(n)}(t), u(0), \ldots, u^{(n-2)}(0), u^{(n-2)}(1), \\ v(0), \ldots, v^{(n-2)}(0), v^{(n-2)}(1) \end{pmatrix},$$

and $\mathcal{N}_\lambda : \left(C^{n-1}([0, 1])\right)^2 \to (C([0, 1]))^2 \times \mathbb{R}^{2n}$, given by

$$\mathcal{N}_\lambda(u, v) = \begin{pmatrix} F_\lambda\left(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(n-1)}(t)\right), \\ G_\lambda\left(t, u(t), \ldots, u^{(n-1)}(t), v(t), \ldots, v^{(n-1)}(t)\right), \\ \lambda A_1, \ldots, \lambda A_{n-2}, \lambda B, \lambda C_1, \ldots, \lambda C_{n-2}, \lambda D \end{pmatrix},$$

where F_λ and G_λ are defined in (23) and (24), respectively.

As $\mathcal{L}^{-1}$ is compact then it can be defined the completely continuous operator $\mathcal{T}_\lambda : \left(\left(C^{n-1}([0, 1])\right)^2, \mathbb{R}\right) \to \left(\left(C^{n-1}([0, 1])\right)^2, \mathbb{R}\right)$ given by

$$\mathcal{T}_\lambda(u, v) = \mathcal{L}^{-1}\mathcal{N}_\lambda(u, v).$$

Consider

$$\rho = \max\left\{N_1, N_2, r_j^i, \text{ for } j=1,2, i=0,1,\ldots,n-3,\right\},$$

where r_j^i, N_1, N_2, are given in Steps 1 and 2, respectively, and define the set

$$\Omega = \left\{(u,v) \in E^2 : \|(u,v)\|_{E^2} < \rho + 1\right\}.$$

Therefore the degree $d\left(I - \mathcal{T}_\lambda, \Omega, (0,0)\right)$ is well defined for every $\lambda \in [0,1]$, and by the invariance under homotopy,

$$d\left(I - \mathcal{T}_0, \Omega, (0,0)\right) = d\left(I - \mathcal{T}_1, \Omega, (0,0)\right).$$

The equation $\mathcal{T}_0(u,v) = (u,v)$ is equivalent to the homogeneous problem

$$\begin{cases} u^{(n)}(t) - u^{(n-2)}(t) = 0 \\ v^{(n)}(t) - v^{(n-2)}(t) = 0 \\ u^{(i)}(0) = u^{(n-2)}(1) = 0, \\ v^{(i)}(0) = v^{(n-2)}(1) = 0, i = 0,1,\ldots,n-2, \end{cases}$$

which admits only the trivial solution.

Then, by degree theory, $d\left(I - \mathcal{T}_0, \Omega\,(0,0)\right) = \pm 1$, and so the equation

$$(u,v) = \mathcal{T}_1(u,v)$$

has at least one solution. That is, by Step 1, the problem composed by Eq. (15), and the boundary conditions (16) has at least a solution $(u_1(t), v_1(t))$ in Ω.

Step 4: *This solution* $(u_1(t), v_1(t))$ *is a solution of* (1), (2).

To prove this assertion it will be enough, by Steps 1 and 2, to show that

$$\alpha_1^{(i)}(t) \le u_1^{(i)}(t) \le \beta_1^{(i)}(t),$$

and

$$\alpha_2^{(i)}(t) \le v_1^{(i)}(t) \le \beta_2^{(i)}(t), \forall t \in [0,1], i = 0,1,\ldots,n-2.$$

Suppose, by contradiction, that there exists $t \in [0,1]$ such that

$$u_1^{(n-2)}(t) > \beta_1^{(n-2)}(t),$$

and define

$$\max_{t\in[0,1]}\left[u_1^{(n-2)}(t) - \beta_1^{(n-2)}(t)\right] := u_1^{(n-2)}(t_1) - \beta_1^{(n-2)}(t_1) > 0. \tag{25}$$

As, by (2) and Definition 3, $t_1 \neq 0$ and $t_1 \neq 1$, then $t_1 \in]0,1[$, $u_1^{(n-1)}(t_1) = \beta_1^{(n-1)}(t_1)$ and

$$u_1^{(n)}(t_1) \leq \beta_1^{(n)}(t_1). \tag{26}$$

So, by (13), (25), Definition 3, Steps 1 and 2, we obtained the following contradiction

$$\begin{aligned}
0 \geq\ & u_1^{(n)}(t_1) - \beta_1^{(n)}(t_1) = \\
& f\begin{pmatrix} t_1, \delta_{1,0}(t_1, u_1(t_1)), \ldots, \delta_{1,n-2}\left(t_1, u_1^{(n-2)}(t_1)\right), u_1^{(n-1)}(t_1), \\ \delta_{2,0}(t_1, v_1(t_1)), \ldots, \delta_{2,n-2}\left(t_1, v_1^{(n-2)}(t_1)\right), v_1^{(n-1)}(t_1) \end{pmatrix} \\
& + u_1^{(n-2)}(t_1) - \delta_{1,n-2}\left(t_1, u_1^{(n-2)}(t_1)\right) - \beta_1^{(n)}(t_1) \\
\geq\ & f(t_1, \beta_1(t_1), \ldots, \beta_1^{(n-1)}(t_1), \beta_2(t_1), \ldots, \beta_2^{(n-2)}(t_1), v^{(n-1)}(t)) \\
& + u_1^{(n-2)}(t_1) - \beta_1^{(n-2)}(t_1) - \beta_1^{(n)}(t_1) \\
\geq\ & u_1^{(n-2)}(t_1) - \beta_1^{(n-2)}(t_1) > 0.
\end{aligned}$$

Therefore,

$$u_1^{(n-2)}(t) \leq \beta_1^{(n-2)}(t), \forall t \in [0,1].$$

Applying the same argument, it can be justified that $\alpha_1^{(n-2)}(t) \leq u_1^{(n-2)}(t)$, for $t \in [0,1]$.

Integrating in $[0,t]$ the inequalities

$$\alpha_1^{(n-2)}(t) \leq u_1^{(n-2)}(t) \leq \beta_1^{(n-2)}(t),$$

we have, for the first one,

$$\begin{aligned}
\alpha_1^{(n-3)}(t) - A_{n-3} \leq \alpha_1^{(n-3)}(t) - \alpha_1^{(n-3)}(0) = \int_0^t \alpha_1^{(n-2)}(s)\,ds \leq \int_0^t u_1^{(n-2)}(s)\,ds \\
= u_1^{(n-3)}(t) - u_1^{(n-3)}(0) = u_1^{(n-3)}(t) - A_{n-3}, \forall t \in [0,1],
\end{aligned}$$

and therefore

$$\alpha_1^{(n-3)}(t) \leq u_1^{(n-3)}(t), \forall t \in [0,1].$$

By similar technique, we get

$$\alpha_1^{(i)}(t) \leq u_1^{(i)}(t), \forall t \in [0,1], i = 0, 1, \ldots, n-2.$$

In an analogous way, we can prove that

$$u_1^{(i)}(t) \leq \beta_1^{(i)}(t), \text{ for } i = 0, 1, \ldots, n-2,$$

and

$$\alpha_2^{(i)}(t) \leq v_1^{(i)}(t) \leq \beta_2^{(i)}(t), \forall t \in [0, 1], i = 0, 1, \ldots, n-2. \qquad \blacksquare$$

4 Lorentz-Lagrangian System Model

This model was presented by Voigt in 1887, and adopted later by Lorentz in 1904, and by Poincaré in 1906. Lorentz-Lagrangian systems have many analogies with classical Lagrangian systems $q'' + V(q) = 0$, for which the results of existence of periodic and homoclinic solutions were established through a variety of methods.

In [3], the author presents, as example, a system of the Lorentz-Lagrangian type, modelling the motion of a particle in a rotating potential in a frame, that moves with the potential.

Based on the ideas of [3], we consider the Lorentz-Lagrangian system:

$$\begin{cases} u''(t) + k(v(t) - u(t)) - 2(k-1)^2 \frac{u(t)}{(1+u^2(t))^2} = 0, \\ v''(t) - v(t) - k\left(v'(t) - u(t)\right) = 0, \end{cases} \tag{27}$$

with $k > 1$ a parameter, together with the boundary conditions

$$\begin{aligned} u(0) = 0, u(1) = 1 \\ v(0) = 0, v(1) = 1. \end{aligned} \tag{28}$$

The system above is a particular case of problem (1), (2), with $n = 2$,

$$f(t, u_0, u_1, v_0, v_1) = -k(v_0 - u_0) + 2(k-1)^2 \frac{u_0}{\left(1 + u_0^2\right)^2}$$

and

$$g(t, u_0, u_1, v_0, v_1) = v_0 + k(v_1 - u_0).$$

Moreover the functions

$$\begin{aligned} \alpha_1(t) &= -t, \quad \beta_1(t) = t \\ \alpha_2(t) &= -t \quad \beta_2(t) = t \end{aligned}$$

are lower and upper solutions of problem (27), (28), respectively, for $k > 1$, and the nonlinearities f and g satisfy the growth conditions (13) and (14).

These functions verify the Nagumo conditions (4) and (6) with

$$\phi_1(u_1) \equiv 2k + \frac{4}{5}(k-1)^2$$

and

$$\phi_2(v_1) \equiv 1 + 2k(|v_1| + 1),$$

with

$$-t \le u_0 \le t, -t \le v_0 \le t, \text{ for } t \in [0, 1].$$

Therefore, by Theorem 4, there is a solution (u, v) of problem (27), (28) for $k > 1$ such that

$$\begin{aligned} &-t \le u(t) \le t \\ &-t \le v(t) \le t, \forall t \in [0, 1]. \end{aligned}$$

5 Coupled System of Two Korteweg-De Vries (KdV) Equations

The Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0,$$

models the unidirectional propagation of water waves with small amplitude lying in a channel [18]. It was first introduced by Boussinesq and then reformulated by Diederik Korteweg and Gustav de Vries.

In [27], it is studied the coupled KdV equations

$$\begin{cases} u_t - \frac{1}{2}(7-3\alpha)u_{xxx} - u_x u - uv_x - \frac{1}{2}(1-\alpha)vu_x + \frac{1}{2}(1+\alpha)vu_x = 0 \\ \\ v_t + v_{xxx} + u_x v + uv_x + vu_x + \frac{1}{2}(1+\alpha)uv_x - \frac{1}{2}(1-\alpha)uu_x = 0, \end{cases} \tag{29}$$

with $\alpha^2 = 5$, which are a new model for describing two-layer fluids with different dispersion relations.

It can be observed in [17] that, for the case of constant boundary and initial conditions, various types of steady and transient solutions were derived.

Based on the above model (29), we consider a particular case of a stationary coupled system of the KdV equations with damping and forced terms:

$$\begin{cases} u'''(t) = m(t) - 0.01\,(u(t) + v(t))\,\left|u'(t)\right| - u(t) \\ v'''(t) = n(t) - 0.01\,(u(t) + v(t))\,\left|v'(t)\right| - v(t), \end{cases} \tag{30}$$

with $m, n : \mathbb{R} \to \mathbb{R}$ continuous functions, together with the boundary conditions

$$\begin{array}{l} u(0) = u'(0) = 0,\, u'(1) = 7 \\ v(0) = v'(0) = 0,\, v'(1) = 7. \end{array} \tag{31}$$

The functions

$$\begin{array}{l} \alpha_1(t) = \alpha_2(t) = t^3 - 5t^2, \\ \beta_1(t) = \beta_2(t) = -t^3 + 5t^2 + t \end{array}$$

are lower and upper solutions of problem (30), (31), if the forcing terms $m(t)$ and $n(t)$ are bounded from above by

$$6 + 0.02\left(t^3 - 5t^2\right)\left|3t^2 - 10t\right| + \left(t^3 - 5t^2\right), \text{ for } t \in [0, 1], \tag{32}$$

and below by

$$-6 + 0.02\left(-t^3 + 5t^2 + t\right)\left|-3t^2 + 10t\right| + \left(-t^3 + 5t^2 + t\right), \forall t \in [0, 1]. \tag{33}$$

It can be easily seen that (30), (31) is a particular case of problem (1), (2), with $n = 3$,

$$f(t, u_0, u_1, u_2, v_0, v_1, v_2) = m(t) - 0.01\,(u_0 + v_0)\,|u_1| - u_0$$

and

$$g(t, u_0, u_1, u_2, v_0, v_1, v_2) = n(t) - 0.01\,(u_0 + v_0)\,|v_1| - v_0.$$

These functions verify trivially the Nagumo conditions (4)–(6), as they have no dependence on the second derivatives. Moreover they satisfy the growth conditions (13) and (14), therefore, by Theorem 4, for functions m and n in the strip bounded by (32) and (33), there is a solution (u, v) of problem (30), (31) such that

$$\begin{array}{l} t^3 - 5t^2 \le u(t) \le -t^3 + 5t^2 + t, \\ t^3 - 5t^2 \le v(t) \le -t^3 + 5t^2 + t, \forall t \in [0, 1]. \end{array}$$

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Semilinear Equations in Banach Spaces with Lower Fractional Derivatives

Marina V. Plekhanova and Guzel D. Baybulatova

Abstract In the first part of the work we find conditions of the unique classical solution existence for the Cauchy problem to solved with respect to the highest fractional Caputo derivative semilinear fractional order equation with nonlinear operator, depending on the lower Caputo derivatives. Abstract result is applied to study of an initial-boundary value problem to a modified Oskolkov–Benjamin–Bona–Mahony–Burgers nonlinear equation with time-fractional derivatives. In the second part of the work the unique solvability of the generalized Showalter–Sidorov problem for semilinear fractional order equation with degenerate linear operator at the highest-order Caputo derivative is researched. The nonlinear operator, generally speaking, depends on the lower fractional Caputo derivatives. Here the result on the unique solvability of the Cauchy problem to equation, solved with respect to the highest Caputo derivative, is used also. The abstract result from the second part of the work is demonstrated on an example of an initial-boundary value problem to a nonlinear system of partial differential equations, not solvable with respect to the highest time-fractional derivative.

Keywords Fractional differential equation · Caputo derivative · Degenerate evolution equation · Initial value problem

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1 Introduction

Let $\mathscr{X}$, $\mathscr{Y}$ be Banach spaces, $L \in \mathscr{L}(\mathscr{X};\mathscr{Y})$ (linear and continuous operator from $\mathscr{X}$ in $\mathscr{Y}$), $M \in \mathscr{C}l(\mathscr{X};\mathscr{Y})$ (linear closed operator with dense domain D_M in the space $\mathscr{X}$ and with image in $\mathscr{Y}$), $n \in \mathbf{N}$, $X \subset \mathbf{R} \times \mathscr{X}^n$, $N : X \to \mathscr{Y}$ is nonlinear operator. Consider the equation of fractional order

$$D_t^{\alpha} Lx(t) = Mx(t) + N(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots, D_t^{\alpha_n}x(t)), \tag{1}$$

where $D_t^{\alpha}, D_t^{\alpha_1}, D_t^{\alpha_2}, \dots, D_t^{\alpha_n}$ are the fractional Caputo derivatives, $m-1 < \alpha \leq m \in \mathbf{N}$, $0 \leq \alpha_1 < \alpha_2 < \cdots < \alpha_n \leq m-1$. The equation is supposed to be degenerate, i.e. $\ker L \neq \{0\}$.

Similar equations of integer order are often found among non-classical equations of mathematical physics [5, 6, 8, 9, 15]. Interest in problems for fractional order equations is associated with a lot of results of the successful application of fractional calculus in different areas, in particular, in the systems of equations that describe the motion of viscoelastic fluids, the theory of semiconductors, the motion along fractal structures, etc. (see [16, 22] and many others).

The unique solvability of initial problems for linear degenerate fractional order equations was considered by many authors [1, 2, 4, 10–14, 20]. In contrast to the papers [17–19] on semilinear degenerate fractional order equations with nonlinear operator, depending on lower derivatives of integer orders, in the present work lower derivatives have fractional order, generally speaking.

Firstly we find conditions of the unique classical solution existence for the Cauchy problem to Eq. (1), solved with respect to the highest fractional Caputo derivative ($\mathscr{X} = \mathscr{Y}$, L is the identical operator). Corresponding abstract result is applied to study of an initial-boundary value problem to a modified Oskolkov–Benjamin–Bona–Mahony–Burgers nonlinear equation with time-fractional derivatives.

In the second part of the work the unique solvability of the generalized Showalter–Sidorov problem

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, \dots, m-1 \tag{2}$$

for degenerate semilinear fractional order Eq. (1) is researched. (The projection P on the complement $\mathscr{X}^1$ of the degeneracy subspace will be defined further.) Here the result on the unique solvability of the Cauchy problem to Eq. (1), solved with respect to the highest Caputo derivative, is used also. The conditions of the theorem on the existence of a unique classical solution of problem (1), (2) is demonstrated on the example of an initial-boundary value problem to a nonlinear system of partial differential equations, not solvable with respect to the highest time-fractional derivative.

2 Equations Solved with Respect to the Highest Derivative

2.1 Linear Equation

Let $\mathscr{Z}$ be a Banach space. Denote $g_\delta(t) = \Gamma(\delta)^{-1}t^{\delta-1}$, $J_t^\delta h(t) = (g_\delta * h)(t) = \int\limits_0^t g_\delta(t-s)h(s)ds$ for $\delta > 0$, $t > 0$, $\tilde{g}_\delta(t) = (t-t_0)^{\delta-1}/\Gamma(\delta)$. Let $m-1 < \alpha \leq m \in \mathbf{N}$, D_t^m is the usual derivative of the order $m \in \mathbf{N}$, J_t^0 is the identical operator. The Caputo derivative of a function h is (see [1, p. 11])

$$D_t^\alpha h(t) = D_t^m J_t^{m-\alpha}\left(h(t) - \sum_{k=0}^{m-1} h^{(k)}(t_0)\tilde{g}_{k+1}(t)\right), \quad t \geq t_0.$$

Consider the Cauchy problem

$$z^{(k)}(t_0) = z_k, \quad k = 0, 1, \ldots, m-1, \tag{3}$$

for the inhomogeneous differential equation

$$D_t^\alpha z(t) = Az(t) + f(t), \quad t \in [t_0, T], \tag{4}$$

where $A \in \mathscr{L}(\mathscr{Z}) = \mathscr{L}(\mathscr{Z};\mathscr{Z})$, the function $f : [t_0, T] \to \mathscr{Z}$ is given for $T > t_0$.

A solution of problem (3), (4) is a function $z \in C^{m-1}([t_0, T];\mathscr{Z})$, such that

$$g_{m-\alpha} * \left(z - \sum_{k=0}^{m-1} z^{(k)}(t_0)\tilde{g}_{k+1}\right) \in C^m([t_0, T];\mathscr{Z})$$

and equalities (3), (4) are true.

For $\alpha, \beta > 0$ denote the Mittag-Leffler function $E_{\alpha,\beta}(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)}$.

Theorem 1 ([17]) *Let $A \in \mathscr{L}(\mathscr{Z})$, $f \in C([t_0, T];\mathscr{Z})$. Then for any $z_k \in \mathscr{Z}$, $k = 0, 1, \ldots, m-1$, there exists a unique solution of problem* (3), (4)*. Moreover, it has the form*

$$z(t) = \sum_{k=0}^{m-1}(t-t_0)^k E_{\alpha,k+1}(A(t-t_0)^\alpha)z_k + \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)f(s)ds. \tag{5}$$

2.2 Semilinear Equation

Let $m-1<\alpha\leq m\in\mathbf{N}$, $n\in\mathbf{N}$, Z be an open set in $\mathbf{R}\times\mathscr{Z}^n$, an operator $B:Z\to\mathscr{Z}$ be nonlinear, $z_k\in\mathscr{Z}$, $k=0,1,\dots,m-1$. Consider the Cauchy problem (3) for the nonlinear differential equation

$$D_t^\alpha z(t)=Az(t)+B(t,D_t^{\alpha_1}z(t),D_t^{\alpha_2}z(t),\dots,D_t^{\alpha_n}z(t)) \tag{6}$$

where $0\leq\alpha_1<\alpha_2<\cdots<\alpha_n\leq m-1$.

A solution of problem (3), (6) on a segment $t\in[t_0,t_1]$ is a such function $z\in C^{m-1}([t_0,t_1];\mathscr{Z})$, that $g_{m-\alpha}*\left(z-\sum\limits_{k=1}^{m-1}z^{(k)}(t_0)\tilde{g}_{k+1}\right)\in C^m([t_0,t_1];\mathscr{Z})$, for all $t\in[t_0,t_1]$ $(t,D_t^{\alpha_1}z(t),D_t^{\alpha_2}z(t),\dots,D_t^{\alpha_n}z(t))\in Z$, equalities (3) and (6) for all $t\in[t_0,t_1]$ are valid.

Lemma 1 *Let $l-1<\beta\leq l\in\mathbf{N}$. Then*

$$\exists C>0\quad\forall h\in C^l([t_0,t_1];\mathscr{Z})\quad\|D_t^\beta h\|_{C([t_0,t_1];\mathscr{Z})}\leq C\|h\|_{C^l([t_0,t_1];\mathscr{Z})}.$$

Proof For the function $f(t)=h(t)-\sum\limits_{k=0}^{l-1}h^{(k)}(t_0)\tilde{g}_{k+1}(t)$ we have $f^{(k)}(t_0)=0$, $k=0,1,\dots,l-1$. So

$$\|D_t^lJ_t^{l-\beta}f\|_{C([t_0,t_1];\mathscr{Z})}=\left\|D_t^l\int\limits_0^{t-t_0}\frac{s^{l-\beta-1}f(t-s)}{\Gamma(l-\beta)}ds\right\|_{C([t_0,t_1];\mathscr{Z})}=$$

$$=\|J_t^{l-\beta}f^{(l)}\|_{C([t_0,t_1];\mathscr{Z})}\leq\frac{(t_1-t_0)^{l-\beta}}{\Gamma(l-\beta+1)}\|f^{(l)}\|_{C([t_0,t_1];\mathscr{Z})}\leq C\|h\|_{C^l([t_0,t_1];\mathscr{Z})}.$$

□

Lemma 2 *Let $A\in\mathscr{L}(\mathscr{Z})$, $B\in C(Z;\mathscr{Z})$. Then function $z\in C^{m-1}([t_0,t_1];\mathscr{Z})$ is a solution of problem* (3), (6)*, if and only if for all $t\in[t_0,t_1]$*

$$z(t)=\sum_{k=0}^{m-1}(t-t_0)^kE_{\alpha,k+1}(A(t-t_0)^\alpha)z_k+$$

$$+\int\limits_{t_0}^t(t-s)^{\alpha-1}E_{\alpha,\alpha}(A(t-s)^\alpha)B(s,D_t^{\alpha_1}z(s),D_t^{\alpha_2}z(s),\dots,D_t^{\alpha_n}z(s))ds. \tag{7}$$

Proof Let $z\in C^{m-1}([t_0,t_1];\mathscr{Z})$ be a solution of problem (3), (6), then

$$t \to B(t, D_t^{\alpha_1} z(t), D_t^{\alpha_2} z(t), \dots, D_t^{\alpha_n} z(t))$$

is the continuous mapping from $[t_0, t_1]$ into $\mathscr{Z}$ by Lemma 1, since $\alpha_n \le m - 1$. Due to Theorem 1 the solution satisfies Eq. (7).

Let $z \in C^{m-1}([t_0, t_1]; \mathscr{Z})$ satisfies Eq. (7), then arguing as in the proof of Theorem 1 (see [17]) we obtain directly that the function z is a solution of problem (3), (6). □

The bar over a symbol will mean an ordered set of n elements with indexes from 1 to n, for example, $\bar{x} = (x_1, x_2, \dots, x_n)$. Let $S_\delta(\bar{x}) = \{\bar{y} \in \mathscr{Z}^n : \|y_k - x_k\|_{\mathscr{Z}} \le \delta, k = 1, 2, \dots, n\}$. A mapping $B : Z \to \mathscr{Z}$ is called locally Lipshitz continuous in z, if for every $(t, \bar{x}) \in Z$, there exist $\delta > 0$ and $l > 0$, for which $[t - \delta, t + \delta] \times S_\delta(\bar{x}) \subset Z$ and for all $(s, \bar{y}), (s, \bar{v}) \in [t - \delta, t + \delta] \times S_\delta(\bar{x})$

$$\|B(s, \bar{y}) - B(s, \bar{v})\|_{\mathscr{Z}} \le l \sum_{k=1}^{n} \|y_k - v_k\|_{\mathscr{Z}}.$$

Using the initial data $z_0, z_1, \dots, z_{m-1}$ from (3), define the Taylor polynomial

$$\tilde{z}(t) = z_0 + z_1(t - t_0) + \dots + \frac{z_{m-1}}{(m-1)!}(t - t_0)^{m-1}.$$

Theorem 2 *Suppose that $A \in \mathscr{L}(\mathscr{Z})$, a set Z is open in $\mathbf{R} \times \mathscr{Z}^n$, and the mapping $B \in C(Z; \mathscr{Z})$ is locally Lipschitz continuous in z, $z_k \in \mathscr{Z}$, $k = 0, 1, \dots, m - 1$, such that $(t_0, D_t^{\alpha_1}|_{t=t_0}\tilde{z}(t), D_t^{\alpha_2}|_{t=t_0}\tilde{z}(t), \dots, D_t^{\alpha_n}|_{t=t_0}\tilde{z}(t)) \in Z$. Then there exists $t_1 > t_0$, such that problem (3), (6) has a unique solution on $[t_0, t_1]$.*

Proof By Lemma 2, it suffices to prove that Eq. (7) has a unique solution $z \in C^{m-1}([t_0, t_1]; \mathscr{Z})$ for some $t_1 > t_0$.

We can choose $\tau > 0$ and $\delta > 0$ so that $V = [t_0, t_0 + \tau] \times S_\delta(\bar{z}) \subset Z$. Denote by S the set of all functions $y \in C^{m-1}([t_0, t_0 + \tau]; \mathscr{Z})$ such that for all $t \in [t_0, t_0 + \tau]$ we have $\|y^{(k)}(t) - z_k\|_{\mathscr{Z}} \le \delta$, $k = 0, 1, \dots, m - 1$. Endow S with the metric $d(y, v) = \sum\limits_{k=0}^{m-1} \sup\limits_{t \in [t_0, t_0+\tau]} \|y^{(k)}(t) - v^{(k)}(t)\|_{\mathscr{Z}}$. It is obvious, that S is complete metric space and $\tilde{z} \in S$ for sufficiently small $\tau > 0$.

Consider for all $t \in [t_0, t_0 + \tau]$

$$G(y)(t) = \sum_{k=0}^{m-1} (t - t_0)^k E_{\alpha,k+1}(A(t - t_0)^\alpha) z_k +$$

$$+ \int_{t_0}^{t} (t - s)^{\alpha-1} E_{\alpha,\alpha}(A(t - s)^\alpha) B(s, D_t^{\alpha_1} y(s), D_t^{\alpha_2} y(s), \dots, D_t^{\alpha_n} y(s)) ds$$

and let show that operator G maps S into itself, if $\tau > 0$ is sufficiently small, and it is a contraction of S. Indeed, for $r = 0, 1, \dots, m-1$

$$G^{(r)}(y) = \sum_{k=0}^{r-1}(t-t_0)^{\alpha+k-r} A E_{\alpha,\alpha+k+1-r}(A(t-t_0)^\alpha) z_k +$$

$$+ \sum_{k=r}^{m-1}(t-t_0)^{k-r} E_{\alpha,k+1-r}(A(t-t_0)^\alpha) z_k +$$

$$+ \int_{t_0}^{t} (t-s)^{\alpha-r-1} E_{\alpha,\alpha-r}(A(t-s)^\alpha) B(s, D_t^{\alpha_1} y(s), D_t^{\alpha_2} y(s), \dots, D_t^{\alpha_n} y(s)) ds.$$

Denote $K = \max_{t\in[t_0,t_0+\tau]} \|B(t, D_t^{\alpha_1}\tilde{z}(t), D_t^{\alpha_2}\tilde{z}(t), \dots, D_t^{\alpha_n}\tilde{z}(t))\|_{\mathscr{Z}}$. In according to Lemma 1, for $y \in S$

$$\|B(t, D_t^{\alpha_1} y(t), D_t^{\alpha_2} y(t), \dots, D_t^{\alpha_n} y(t))\|_{\mathscr{Z}} \le$$

$$\le \|B(t, D_t^{\alpha_1} y(t), D_t^{\alpha_2} y(t), \dots, D_t^{\alpha_n} y(t)) - B(t, D_t^{\alpha_1}\tilde{z}(t), D_t^{\alpha_2}\tilde{z}(t), \dots, D_t^{\alpha_n}\tilde{z}(t))\|_{\mathscr{Z}} + K \le$$

$$\le l \sum_{k=1}^{n} \|D_t^{\alpha_k} y(t) - D_t^{\alpha_k}\tilde{z}(t)\|_{\mathscr{Z}} + K \le Cln \sum_{k=0}^{m-1} \sup_{t\in[t_0,t_0+\tau]} \|y^{(k)}(t) - \tilde{z}^{(k)}(t)\|_{\mathscr{Z}} + K \le$$

$$\le Cln \left(\sum_{k=0}^{m-1} \sup_{t\in[t_0,t_0+\tau]} \|y^{(k)}(t) - z_k\|_{\mathscr{Z}} + \sum_{k=0}^{m-1} \sup_{t\in[t_0,t_0+\tau]} \|\tilde{z}^{(k)}(t) - z_k\|_{\mathscr{Z}} \right) + K \le$$

$$\le 2Clmn\delta + K.$$

Then for all $t \in [t_0, t_0 + \tau]$

$$\|G^{(r)}(y)(t) - z_r\|_{\mathscr{Z}} \le \sum_{k=0}^{r-1} \tau^{\alpha+k-r} \|A\|_{\mathscr{L}(\mathscr{Z})} E_{\alpha,\alpha+k+1-r}(\|A\|_{\mathscr{L}(\mathscr{Z})}\tau^\alpha) \|z_k\|_{\mathscr{Z}} +$$

$$+ \left\| E_{\alpha,1}(A(t-t_0)^\alpha) z_r - z_r \right\|_{\mathscr{Z}} + \sum_{k=r+1}^{m-1} \tau^{k-r} E_{\alpha,k+1-r}(\|A\|_{\mathscr{L}(\mathscr{Z})}\tau^\alpha) \|z_k\|_{\mathscr{Z}} +$$

$$+ \frac{\tau^{\alpha-r}}{\alpha-r} E_{\alpha,\alpha-r}\left(\|A\|_{\mathscr{L}(\mathscr{Z})}\tau^\alpha\right)(2Clmn\delta + K) \le \delta$$

for sufficiently small τ. Therefore, $G : S \to S$.

Besides, for small τ and for all $t \in [t_0, t_0 + \tau]$, $r = 0, 1, \dots, m-1$, $y, v \in S$

$$\|G^{(r)}(y)(t) - G^{(r)}(v)(t)\|_{\mathscr{Z}} \leq$$

$$\leq \frac{\tau^{\alpha-r}}{\alpha - r} E_{\alpha,\alpha-r}(\|A\|_{\mathscr{L}(\mathscr{Z})}\tau^{\alpha}) Cln \sum_{k=0}^{m-1} \sup_{t\in[t_0,t_0+\tau]} \|y^{(k)}(t) - v^{(k)}(t)\|_{\mathscr{Z}} \leq \frac{d(y,v)}{2m}.$$

Therefore $d(G(y), G(v)) \leq d(y, v)/2$ and operator G has a unique fixed point in S, which is a solution of problem (3), (6) on $[t_0, t_0 + \tau]$. □

2.3 Application

Nonlinear surface waves that propagate along the direction of the axis, taking into account the viscosity and some other processes are modeled by the pseudoparabolic Oskolkov–Benjamin–Bona–Mahony–Burgers equation [3], we will consider some its modification.

Consider the initial-boundary value problem

$$\frac{\partial^k w}{\partial t^k}(x, t_0) = v_k(x), \quad k = 0, 1, \dots, m-1, \quad x \in (a, b), \tag{8}$$

$$w(a, t) = w(b, t), \quad \frac{\partial w}{\partial x}(a, t) = \frac{\partial w}{\partial x}(b, t), \quad t \geq t_0, \tag{9}$$

for the equation

$$D_t^{\alpha} w - D_t^{\alpha} w_{xx} = \beta w_{xx} + \gamma w_x - \left(D_t^{\alpha_1} w\right)^{\delta} \left(D_t^{\alpha_2} w_x\right)^{\varepsilon}, \quad x \in (a, b), \quad t \geq t_0, \tag{10}$$

where $a, b, \beta, \gamma, \delta, \varepsilon \in \mathbf{R}$, $a < b$, $m - 1 < \alpha \leq m \in \mathbf{N}$, $0 \leq \alpha_1 < \alpha_2 \leq m - 1$.

Define Banach spaces $\mathscr{X} = \{v : H^2(a, b) : v(a) = v(b),\ v'(a) = v'(b)\}$, $\mathscr{Y} = L_2(a, b)$, and operators on $\mathscr{X}$ $L = 1 - \frac{\partial^2}{\partial x^2}$, $M = \beta \frac{\partial^2}{\partial x^2} + \gamma \frac{\partial}{\partial x}$, $N(v_1, v_2) = -v_1^{\delta} v_{2x}^{\varepsilon}$,

$$v_{k0} = D_t^{\alpha_k}|_{t=t_0} \left[v_0(x) + (t - t_0)v_1(x) + \dots + \frac{(t - t_0)^{m-1}}{(m-1)!} v_{m-1}(x) \right], \quad k = 1, 2.$$

Theorem 3 *Let* $\delta, \varepsilon \geq 1$, $v_k \in \mathscr{X}$, $k = 0, 1, \dots, m-1$, $v_{10}, v_{20} > 0$. *Then for some* $t_1 > t_0$ *there exists a unique solution of problem* (8)–(10) *on the segment* $[t_0, t_1]$.

Proof It is clear that $1 + \lambda^2 \neq 0$ for λ from the set $\{2\pi k(b - a)^{-1} : k \in \mathbf{Z}\} \subset \mathbf{R}$, so L has a continuous inverse operator $L^{-1} : \mathscr{Y} \to \mathscr{X}$. Then Eq. (10) has the form $D_t^{\alpha} v(t) = Av(t) + B(D_t^{\alpha_1} v(t), D_t^{\alpha_2} v(t))$ where the operator $A = L^{-1}M : \mathscr{X} \to \mathscr{X}$ is continuous as composition of continuous operators, $B(v_1, v_2) = L^{-1}N(v_1, v_2)$.

Due to the Sobolev embedding theorem $\mathscr{X} \subset C^1[a, b]$, therefore, for any functions $v_{11}, v_{12}, v_{21}, v_{22}$ from a small neighbourhood of a point $(v_{10}, v_{20}) \in \mathscr{X}^2$, where we have

$$\|v_{11}^\delta v_{21x}^\varepsilon - v_{12}^\delta v_{22x}^\varepsilon\|_{L_2(a,b)}^2 \leq 2\|v_{11}^\delta v_{21x}^\varepsilon - v_{12}^\delta v_{21x}^\varepsilon\|_{L_2(a,b)}^2 + 2\|v_{12}^\delta v_{21x}^\varepsilon - v_{12}^\delta v_{22x}^\varepsilon\|_{L_2(a,b)}^2 \leq$$

$$\leq 2\delta^2 \|(\theta v_{11} + (1-\theta)v_{12})^{\delta-1} v_{21x}^\varepsilon (v_{11} - v_{12})\|_{L_2(a,b)}^2 +$$

$$+2\varepsilon^2 \|v_{12}^\delta (\theta v_{21x} + (1-\theta)v_{22x})^{\varepsilon-1}(v_{21x} - v_{22x})\|_{L_2(a,b)}^2 \leq$$

$$\leq l \left(\|v_{11} - v_{12}\|_{H^2(a,b)}^2 + \|v_{21} - v_{22}\|_{H^2(a,b)}^2 \right),$$

where $\theta \in [0, 1]$. Thus, operator $B : \mathscr{X}^2 \to \mathscr{X}$ is locally Lipschitz continuous in (v_1, v_2). By Theorem 2 we obtain the required. □

3 Degenerate Equations

Let $L \in \mathscr{L}(\mathscr{X}; \mathscr{Y})$, $M \in \mathscr{C}l(\mathscr{X}; \mathscr{Y})$, D_M is a domain of an operator M, endowed by the graph norm $\|\cdot\|_{D_M} = \|\cdot\|_{\mathscr{X}} + \|M\cdot\|_{\mathscr{Y}}$. Define L-resolvent set $\rho^L(M) = \{\mu \in \mathbf{C} : (\mu L - M)^{-1} \in \mathscr{L}(\mathscr{Y}; \mathscr{X})\}$ of an operator M and its L-spectrum $\sigma^L(M) = \mathbf{C}\backslash\rho^L(M)$, and denote $R_\mu^L(M) = (\mu L - M)^{-1}L$, $L_\mu^L = L(\mu L - M)^{-1}$.

An operator M is called (L, σ)-*bounded*, if

$$\exists a > 0 \quad \forall \mu \in \mathbf{C} \quad (|\mu| > a) \Rightarrow (\mu \in \rho^L(M)).$$

Lemma 3 ([21, p. 89, 90]) *Let an operator M be (L, σ)-bounded, $\gamma = \{\mu \in \mathbf{C} : |\mu| = r > a\}$. Then operators*

$$P = \frac{1}{2\pi i} \int_\gamma R_\mu^L(M)\, d\mu \in \mathscr{L}(\mathscr{X}), \quad Q = \frac{1}{2\pi i} \int_\gamma L_\mu^L(M)\, d\mu \in \mathscr{L}(\mathscr{Y})$$

are projections.

Put $\mathscr{X}^0 = \ker P$, $\mathscr{Y}^0 = \ker Q$; $\mathscr{X}^1 = \operatorname{im} P$, $\mathscr{Y}^1 = \operatorname{im} Q$. Denote by L_k (M_k) the restriction of the operator L (M) on $\mathscr{X}^k$ ($D_{M_k} = D_M \cap \mathscr{X}^k$), $k = 0, 1$.

Theorem 4 ([21, p. 90, 91]) *Let an operator M be (L, σ)-bounded. Then*
(i) $M_1 \in \mathscr{L}(\mathscr{X}^1; \mathscr{Y}^1)$*,* $M_0 \in \mathscr{C}l(\mathscr{X}^0; \mathscr{Y}^0)$*,* $L_k \in \mathscr{L}(\mathscr{X}^k; \mathscr{Y}^k)$*,* $k = 0, 1$;
(ii) *there exist operators* $M_0^{-1} \in \mathscr{L}(\mathscr{Y}^0; \mathscr{X}^0)$*,* $L_1^{-1} \in \mathscr{L}(\mathscr{Y}^1; \mathscr{X}^1)$.

Denote $\mathbf{N}_0 = \{0\} \cup \mathbf{N}$, $G = M_0^{-1}L_0$. For $p \in \mathbf{N}_0 = \mathbf{N} \cup \{0\}$ operator M is called (L, p)-bounded, if it is (L, σ)-bounded, $G^p \neq 0$, $G^{p+1} = 0$.

Lemma 4 ([17]) *Let $H \in \mathscr{L}(\mathscr{X})$ be a nilpotent operator of a power $p \in \mathbf{N}_0$, there exist $(D_t^\alpha H)^k g \in C([t_0, T]; \mathscr{X})$ for $k = 0, 1, \dots, p$. Then there exists a unique solution of equation*

$$D_t^\alpha Hx(t) = x(t) + g(t), \quad t \in [t_0, T]. \tag{11}$$

It has a form

$$x(t) = -\sum_{k=0}^{p} (D_t^\alpha H)^k g(t). \tag{12}$$

Let $n \in \mathbf{N}$, $X \subset \mathbf{R} \times \mathscr{X}^n$, $N : X \to \mathscr{Y}$ is nonlinear operator. As before $\alpha_1 < \alpha_2 < \dots < \alpha_n \leq m - 1$. Let $l - 1 < \alpha_n \leq l$. Consider the equation

$$D_t^\alpha Lx(t) = Mx(t) + N(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots, D_t^{\alpha_n}x(t)) + f(t). \tag{13}$$

Its solution on a segment $[t_0, t_1]$ is a function $x \in C([t_0, t_1]; D_M) \cap C^l([t_0, t_1]; \mathscr{X})$, such that $g_{m-\alpha} * \left(Lx(t) - \sum\limits_{k=0}^{m-1} (Lx)^{(k)}(t_0)\tilde{g}_{k+1}(t) \right) \in C^m([t_0, t_1]; \mathscr{X})$, for all $t \in [t_0, t_1]$ $\left(t, D_t^{\alpha_1}x(t), D_t^{\alpha_2}x(t), \dots, D_t^{\alpha_n}x(t)\right) \in X$, and equality (13) holds.

A solution of the generalized Showalter–Sidorov problem

$$(Px)^{(k)}(t_0) = x_k, \quad k = 0, 1, \dots, m - 1, \tag{14}$$

to Eq. (13) is a solution of the equation, such that conditions (14) are true.

Denote $V = X \cap (\mathbf{R} \times (\mathscr{X}^1)^n)$,

$$\tilde{x} = x_0 + \frac{x_1}{1!}(t - t_0) + \frac{x_2}{2!}(t - t_0)^2 + \dots + \frac{x_{m-1}}{(m-1)!}(t - t_0)^{m-1},$$

for x_k, $k = 0, 1, \dots, m - 1$, from conditions (14). Now the condition $\mathrm{im}N \subset \mathscr{Y}^1$ will be substantially used.

Theorem 5 *Let $p \in \mathbf{N}_0$, an operator M be (L, p)-bounded, X be open set in the space $\mathbf{R} \times \mathscr{X}^n$, V be open in the space $\mathbf{R} \times (\mathscr{X}^1)^n$, the mapping $N \in C(X; \mathscr{Y})$ be locally Lipshitz continuous in x, $\mathrm{im}N \subset \mathscr{Y}^1$, $f \in C([t_0, T]; \mathscr{Y})$ for some $T > t_0$, $(D_t^\alpha G)^k M_0^{-1}(I - Q)f \in C([t_0, T]; \mathscr{X})$, $x_k \in \mathscr{X}^1$, $k = 0, 1, \dots, m - 1$,*

$$\left(t_0, D_t^{\alpha_1}|_{t=t_0}\tilde{x}, D_t^{\alpha_2}|_{t=t_0}\tilde{x}, \dots, D_t^{\alpha_n}|_{t=t_0}\tilde{x}\right) \in X,$$

$$\left(t_0, D_t^{\alpha_1}|_{t=t_0}(\tilde{x} + w), D_t^{\alpha_2}|_{t=t_0}(\tilde{x} + w), \dots, D_t^{\alpha_n}|_{t=t_0}(\tilde{x} + w)\right) \in X,$$

where $w(t) = -\sum\limits_{k=0}^{p} (D_t^\alpha G)^k M_0^{-1}(I - Q)f(t)$. Then there exists $t_1 \in (t_0, T]$, such that problem (13), (14) has a unique solution on the segment $[t_0, t_1]$.

Proof By condition im$N \subset \mathscr{Y}^1$ we have $(I-Q)N \equiv 0$, $QN \equiv N$. Equation (13) after action of the operator $M_0^{-1}(I-Q)$ has a form $D_t^\alpha Gw(t) = w(t) + M_0^{-1}(I-Q)f(t)$, where $w(t) = (I-P)x(t)$. So by Lemma 4 the unique solution of this equation has the form

$$w(t) = -\sum_{k=0}^{p}(D_t^\alpha G)^k M_0^{-1}(I-Q)f(t).$$

Note that there exists derivatives $D_t^\alpha L(D_t^\alpha G)^k M_0^{-1}(I-Q)f \in C([t_0,T];\mathscr{Y})$ for $k = 0, 1, \ldots, p$, because

$$D_t^\alpha L(D_t^\alpha G)^k M_0^{-1}(I-Q)f = M_0 D_t^\alpha G(D_t^\alpha G)^k M_0^{-1}(I-Q)f =$$

$$= M_0(D_t^\alpha G)^{k+1}M_0^{-1}(I-Q)f,$$

and if $k = p$, then $(D_t^\alpha G)^{p+1} = (D_t^\alpha)^{p+1}G^{p+1} = 0$. Therefore, all the conditions from the definition of the solution for the obtained function w are satisfied.

It remains to prove the uniqueness of the solution for the problem

$$D_t^\alpha v(t) = S_1 v(t) + L_1^{-1}N(t, D_t^{\alpha_1}(v(t)+w(t)), \ldots, D_t^{\alpha_n}(v(t)+w(t))) + L_1^{-1}Qf(t),$$
$$v^{(k)}(t_0) = x_k,\ \ k = 0, 1, \ldots, m-1,$$

where $v(t) = Px(t)$, $S_1 \in \mathscr{L}(\mathscr{X}^1)$ due to Theorem 4. We have it from (13), (14) after the action of the operator $L_1^{-1}Q$. Here the operator

$$B(t, v_0, v_1, \ldots, v_n) = L_1^{-1}N(t, v_0 + D_t^{\alpha_1}w(t), \ldots, v_n + D_t^{\alpha_n}w(t)) + L_1^{-1}Qf(t)$$

is continuous in V and locally Lipschitz continuous in $\overline{v} = (v_0, v_1, \ldots, v_n)$, the element $\left(t_0, D_t^{\alpha_1}|_{t=t_0}\tilde{x}, D_t^{\alpha_2}|_{t=t_0}\tilde{x}, \ldots, D_t^{\alpha_n}|_{t=t_0}\tilde{x}\right) \in V$. By Theorem 2 we have get the proof. □

3.1 Example

Let $\Omega \subset \mathbf{R}^d$ be a bounded region with a smooth boundary $\partial\Omega$. Consider the initial-boundary value problem

$$\frac{\partial^k x_1}{\partial t^k}(s, t_0) = x_{1k}(s),\ \ k = 0, 1, \ldots, m-1,\ \ s \in \Omega, \tag{15}$$

$$x_i(s,t) = 0,\ \ (s,t) \in \partial\Omega \times [t_0, t_1],\ \ i = 1, 2, 3, \tag{16}$$

$$D_t^{\alpha}\triangle x_1 = x_1 + h_1\left(s, D_t^{\alpha_1}x_1, D_t^{\alpha_1}x_2, D_t^{\alpha_1}x_3, \dots, D_t^{\alpha_n}x_1, D_t^{\alpha_n}x_2, D_t^{\alpha_n}x_3\right) + f_1(s,t),$$
$$D_t^{\alpha}\triangle x_3 = x_2 + f_2(s,t), \quad (s,t) \in \Omega \times [t_0, t_1],$$
$$0 = \triangle x_3 + f_3(s,t), \quad (s,t) \in \Omega \times [t_0, t_1], \tag{17}$$

where $m-1 < \alpha \le m \in \mathbf{N}$, $0 \le \alpha_1 < \alpha_2 < \cdots < \alpha_n \le m-1$.

Denote by A the Laplace operator with domain $H_0^2(\Omega) = \{z \in H^2(\Omega) : z(s) = 0,\ s \in \partial\Omega\} \subset L_2(\Omega)$, $\{\varphi_k\}$ is orthonormal in $L_2(\Omega)$ system of its eigenfunctions, which correspond to the eigenvalues $\{\lambda_k\}$ of A, numbered in the ascending order taking into account their multiplicities.

Reduce problem (15)–(17) to (13), (14) by choosing of the spaces

$$\mathscr{X} = H_0^{2+2j}(\Omega) \times H^{2j}(\Omega) \times H_0^{2+2j}(\Omega), \quad \mathscr{Y} = (H^{2j}(\Omega))^3, \tag{18}$$

$j > \frac{d}{4} - 1$, $H_0^{2+2j}(\Omega) = \{z \in H^{2+2j}(\Omega) : z(s) = 0,\ s \in \partial\Omega\}$,

$$L = \begin{pmatrix} \triangle & 0 & 0 \\ 0 & 0 & \triangle \\ 0 & 0 & 0 \end{pmatrix} \in \mathscr{L}(\mathscr{X};\mathscr{Y}), \quad M = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \triangle \end{pmatrix} \in \mathscr{L}(\mathscr{X};\mathscr{Y}). \tag{19}$$

Lemma 5 *Let spaces are defined by* (18) *and operators have the form* (19). *Then the operator M is $(L,1)$-bounded and the projections have the form*

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{20}$$

Proof We have for $\mu \ne \lambda_k^{-1}$, $k \in \mathbf{N}$,

$$(\mu L - M)^{-1} = \sum_{k=1}^{\infty} \langle \cdot, \varphi_k \rangle \varphi_k \begin{pmatrix} (\mu\lambda_k - 1)^{-1} & 0 & 0 \\ 0 & -1 & -\mu \\ 0 & 0 & -\lambda_k^{-1} \end{pmatrix},$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L_2(\Omega)$. Therefore, at $|\mu| > |\lambda_1|^{-1}$ the operator $(\mu L - M)^{-1} : \mathscr{Y} \to \mathscr{X}$ is bounded,

$$R_\mu^L(M) = \sum_{k=1}^{\infty} \langle \cdot, \varphi_k \rangle \varphi_k \begin{pmatrix} \lambda_k(\mu\lambda_k - 1)^{-1} & 0 & 0 \\ 0 & 0 & -\lambda_k \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_\mu^L(M) = \sum_{k=1}^{\infty} \langle \cdot, \varphi_k \rangle \varphi_k \begin{pmatrix} \lambda_k(\mu\lambda_k - 1)^{-1} & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

These equalities imply the form of the projections (20), and, consequently, the equalities $\mathscr{X}^1 = H_0^{2+2j}(\Omega) \times \{0\} \times \{0\}$, $\mathscr{X}^0 = \{0\} \times H^{2j}(\Omega) \times H_0^{2+2j}(\Omega)$, $\mathscr{Y}^1 = H^{2j}(\Omega) \times \{0\} \times \{0\}$, $\mathscr{Y}^0 = \{0\} \times H^{2j}(\Omega) \times H^{2j}(\Omega)$, $G = \begin{pmatrix} 0 & \Delta \\ 0 & 0 \end{pmatrix}$. Thus, $G^2 = 0$ and the operator M is $(L, 1)$-bounded. □

Theorem 6 *Let $h_1 \in C^\infty(\Omega \times \mathbf{R}^{3n}; \mathbf{R})$, for some $T > t_0$ $f_i \in C([t_0, T]; H^{2j}(\Omega))$, $i = 1, 2, 3$, $D_t^\alpha f_3 \in C([t_0, T]; H^{2j}(\Omega))$, $x_{1k} \in H_0^{2+2j}(\Omega)$, $k = 0, 1, \dots, m-1$. Then for some $t_1 \in (t_0, T]$ there exists a unique solution of problem* (15)–(17) *on the segment* $[t_0, t_1]$.

Proof From the form of the projection P it follows, that conditions (15) define the generalized Showalter–Sidorov problem for the system of Eqs. (16), (17). The nonlinear operator $N(z_1, z_2, \dots, z_{3n}) = h_1(\cdot, z_1, z_2, \dots, z_{3n})$ due to Proposition B.1 [7] acts from $\mathscr{X}$ to $H^{2+2j}(\Omega)$, therefore, it acts into $H^{2j}(\Omega)$. From the form of the projection P it follows, that $\mathrm{im} N \subset \mathscr{Y}^1$, $x_{1k} \in \mathscr{X}^1$, $k = 0, 1, \dots, m-1$. Note that

$$Qf = f_1, \quad (I - Q)f = \begin{pmatrix} f_2 \\ f_3 \end{pmatrix}, \quad GM_0^{-1}(I - Q)f = \begin{pmatrix} f_3 \\ 0 \end{pmatrix}. \tag{21}$$

By Theorem 5 obtain the required result. □

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Integral Boundary Value Problem for Intuitionistic Fuzzy Partial Hyperbolic Differential Equations

Bouchra Ben Amma, Said Melliani and Lalla Saadia Chadli

Abstract The concept of intuitionistic fuzzy is introduced by Atanassov [5, 8]. It's a generalization of fuzzy theory introduced by Zadeh [55]. Few works on intuitionistic fuzzy differential equations till date after developing intuitionistic fuzzy set theory [17, 18, 37]. Intuitionistic partial differential equations are very rare, the concept of intuitionistic fuzzy partial differential equations was introduced by S. Melliani and L. S. Chadli in [38]. In this paper, we consider the boundary valued problems for intuitionistic fuzzy partial hyperbolic differential equations with integral boundary conditions. A new complete intuitionistic fuzzy metric space [39] is proposed to investigate the existence and uniqueness of intuitionistic fuzzy solutions for these problems using the theorem of fixed point. Also we have presented an useful procedure to solve intuitionistic fuzzy partial hyperbolic differential equations. Some illustrated examples for our results are given with some numerical simulations for α-cuts of the intuitionistic fuzzy solutions.

Keywords Integral boundary condition · Hyperbolic partial differential equations · Intuitionistic fuzzy solutions

1 Introduction

Generalizations of fuzzy sets theory [55] is considered to be one of intuitionistic fuzzy set (IFS). Later on Atanassov generalized the concept of fuzzy set and introduced the idea of intuitionistic fuzzy set [5, 8]. Atanassov [6] explored the concept of

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fuzzy set theory by intuitionistic fuzzy set (IFS) theory. Now-a-days, IFSs are being studied extensively and being used in different disciplines of Science and Technology. Amongst the all research works mainly on IFS we can include Atanassov [4, 7, 9–11], Atanassov and Gargov [12], Szmidt and Kacprzyk [52], Buhaescu [22, 23], Ban [13], Deschrijver and Kerre [27], Cornelis et al. [25], Gerstenkorn and Manko [29], Mahapatra and Roy [36], Adak et al. [1], Oscar Castillo and Patricia Melin [24] Melliani et al. [39], Sotir Sotirov et al. [51]. Further improvement of IFS theory, together with intuitionistic fuzzy geometry, intuitionistic fuzzy logic, intuitionistic fuzzy topology, an intuitionistic fuzzy approach to artificial intelligence, and intuitionistic fuzzy generalized nets can be found in [43], then they are very necessary and powerful tool in modeling imprecision, valuable applications of IFSs have been flourished in many different fields, medical diagnosis [26], drug selection [31], along pattern recognition [33], microelectronic fault analysis [50], weight assessment [53], and decision-making problems [32, 54].

The concept of intuitionistic fuzzy differential was first introduced by Melliani and Chadli [37]. The first step which included applicable definitions of intuitionistic fuzzy derivative and the intuitionistic fuzzy integral was followed by introducing intuitionistic fuzzy differential equations and establishing sufficient conditions for the existence of unique solutions to these equations [17, 18, 40, 41, 48, 49]. It is difficult to obtain exact solution for intuitionistic fuzzy differential equations and hence some applications of numerical methods such as the intuitionistic fuzzy Euler and Taylor methods, Runge–Kutta of order four, Runge–Kutta Gill, Variational iteration method, Adams–Bashforth, Adams–Moulton and Predictor-Corrector methods in intuitionistic fuzzy differential equations presented in [14–16, 19, 42, 44, 47].

On the other hand, the theory of partial differential equations has been emerging as an important area of investigation in recent years and has been developed very rapidly due to the fact that such equations find a wide range of applications modeling adequately many real processes observed in physics, chemistry, biology and engineering. Correspondingly, applications of the theory of partial differential equations to different areas were considered by many authors (see [6, 10, 21]). Introduction to fuzzy partial differential equations is presented by J. Buckley and T. Feuring in [21]. There are not too many papers on fuzzy partial differential equations [2, 28, 30, 45], but some basic results on fuzzy partial hyperbolic differential equations can be found in [3, 20, 34, 35]. Intuitionistic partial differential equations is very rare, the concept of intuitionistic fuzzy partial differential equations was introduced by S. Melliani and L. S. Chadli in [38].

Motivated and inspired by the above works, in this paper a new complete intuitionistic fuzzy metric space [39] is used to investigate the existence and uniqueness of intuitionistic fuzzy solutions by using the Banach fixed point theorem for the following boundary valued problems for intuitionistic fuzzy partial hyperbolic differential equations with integral boundary conditions:

$$\begin{cases} \frac{\partial^2 \langle u,v\rangle(x,y)}{\partial x \partial y} = f(x, y, \langle u, v\rangle(x, y)), \qquad (x, y) \in J_a \times J_b \\ \langle u, v\rangle(x, 0) + \int_0^b k_1(x)\langle u, v\rangle(x, y)dy = \eta_1(x),\ x \in J_a \\ \langle u, v\rangle(0, y) + \int_0^a k_2(y)\langle u, v\rangle(x, y)dx = \eta_2(y),\ y \in J_b \end{cases} \tag{1}$$

and

$$\begin{cases} \frac{\partial^2 \langle u,v\rangle(x,y)}{\partial x \partial y} = \big(h(x, y)\langle u, v\rangle(x, y)\big)_y + f(x, y, \langle u, v\rangle(x, y)),\ \ (x, y) \in J_a \times J_b \\ \langle u, v\rangle(x, 0) + \int_0^b k_1(x)\langle u, v\rangle(x, y)dy = \eta_1(x),\ x \in J_a \\ \langle u, v\rangle(0, y) + \int_0^a k_2(y)\langle u, v\rangle(x, y)dx = \eta_2(y),\ y \in J_b \end{cases} \tag{2}$$

where $f : J_a \times J_b \times IF_n \to IF_n$ is continuous, $h \in C(J_a \times J_b, \mathbb{R})$, $k_1 \in C\big(J_a, \mathbb{R}\big)$, $k_2 \in C\big(J_b, \mathbb{R}\big)$, $\eta_1 \in C\big(J_a, IF_n\big)$ and $\eta_2 \in C\big(J_b, IF_n\big)$ are given functions.

Combining the two aspects introduced, intuitionistic fuzzy mathematics and partial differential equations, we get intuitionistic fuzzy partial differential equations, which will be attract the interest of many researchers. Also we propose a method of steps, it can be useful to solve intuitionistic fuzzy partial hyperbolic differential equations. However, to the best of our knowledge, no result on intuitionistic fuzzy partial hyperbolic differential equations has been published before. Therefore, it is worthwhile mentioning that the intuitionistic fuzzy hyperbolic partial differential equations are studied for the first time in this paper.

The main contributions of this paper include: Introducing intuitionistic fuzzy partial hyperbolic differential equations with integral boundary conditions and defining their solution; further develop theoretical results on the existence, uniqueness of the solution; and using the level-set representation of intuitionistic fuzzy functions and defining the solution to an intuitionistic fuzzy partial hyperbolic differential equations problem through a corresponding parametric problem; and developing a numerical examples for computing intuitionistic fuzzy quantities, taking into account the full interaction between intuitionistic fuzzy variables. The development of some efficient examples for solving intuitionistic fuzzy partial hyperbolic differential equations is the subjects of our current work and will be presented elsewhere.

The remainder of the paper is arranged as follows: In Sect. 2, some basic definitions and results are brought. In Sect. 3 we gain the existence and uniqueness of intuitionistic fuzzy solution for the partial hyperbolic differential equations with integral boundary conditions. In Sect. 4 we propose an useful procedure to solve intuitionistic fuzzy partial hyperbolic differential equations. We present some examples to illustrate the applicability of the main results with some numerical simulations for α-cuts of the intuitionistic fuzzy solutions in Sect. 5 and finally conclusion is drawn in Sect. 6.

2 Preliminaries

Throughout this paper, $(\mathbb{R}^n, B(\mathbb{R}^n), \mu)$ denotes a complete finite measure space.

Let us $P_k(\mathbb{R}^n)$ the set of all nonempty compact convex subsets of $\mathbb{R}^n$.

We denote by

$$IF_n = \mathrm{IF}(\mathbb{R}^n) = \left\{\langle u, v\rangle \;:\; \mathbb{R}^n \to [0,1]^2, |\forall\, x \in \mathbb{R}^n 0 \le u(x) + v(x) \le 1\right\}$$

An element $\langle u, v\rangle$ of IF_n is said an intuitionistic fuzzy number if it satisfies the following conditions

(i) $\langle u, v\rangle$ is normal i.e there exists $x_0, x_1 \in \mathbb{R}^n$ such that $u(x_0) = 1$ and $v(x_1) = 1$.
(ii) u is fuzzy convex and v is fuzzy concave.
(iii) u is upper semi-continuous and v is lower semi-continuous
(iv) $supp\langle u, v\rangle = \mathrm{cl}\{x \in \mathbb{R}^n \;: |\; v(x) < 1\}$ is bounded.

So we denote the collection of all intuitionistic fuzzy number by IF_n.

For $\alpha \in [0, 1]$ and $\langle u, v\rangle \in IF_n$, the upper and lower α-cuts of $\langle u, v\rangle$ are defined by

$$[\langle u, v\rangle]^\alpha = \left\{x \in \mathbb{R}^n : v(x) \le 1 - \alpha\right\}$$

and

$$[\langle u, v\rangle]_\alpha = \left\{x \in \mathbb{R}^n : u(x) \ge \alpha\right\}$$

Remark 1 If $\langle u, v\rangle \in IF_n$, so we can see $[\langle u, v\rangle]_\alpha$ as $[u]^\alpha$ and $[\langle u, v\rangle]^\alpha$ as $[1 - v]^\alpha$ in the fuzzy case.

We define $0_{(1,0)} \in IF_n$ as

$$0_{(1,0)}(t) = \begin{cases} (1,0) & t = 0 \\ (0,1) & t \neq 0 \end{cases}$$

Let $\langle u, v\rangle, \langle u', v'\rangle \in IF_n$ and $\lambda \in \mathbb{R}$, we define the following operations by:

$$\Big(\langle u, v\rangle + \langle u', v'\rangle\Big)(z) = \Big(\sup_{z=x+y} \min\big(u(x), u'(y)\big), \inf_{z=x+y} \max\big(v(x), v'(y)\big)\Big)$$

$$\lambda\langle u, v\rangle = \begin{cases} \langle \lambda u, \lambda v\rangle & \text{if } \lambda \neq 0 \\ 0_{(1,0)} & \text{if } \lambda = 0 \end{cases}$$

For $\langle u, v\rangle, \langle z, w\rangle \in IF_n$ and $\lambda \in \mathbb{R}$, the addition and scalar-multiplication are defined as follows

$$\Big[\langle u,v\rangle+\langle z,w\rangle\Big]^{\alpha}=\Big[\langle u,v\rangle\Big]^{\alpha}+\Big[\langle z,w\rangle\Big]^{\alpha},\quad \Big[\lambda\langle z,w\rangle\Big]^{\alpha}=\lambda\Big[\langle z,w\rangle\Big]^{\alpha}$$
$$\Big[\langle u,v\rangle+\langle z,w\rangle\Big]_{\alpha}=\Big[\langle u,v\rangle\Big]_{\alpha}+\Big[\langle z,w\rangle\Big]_{\alpha},\quad \Big[\lambda\langle z,w\rangle\Big]_{\alpha}=\lambda\Big[\langle z,w\rangle\Big]_{\alpha}$$

Definition 1 Let $\langle u,v\rangle$ an element of IF_n and $\alpha\in[0,1]$, we define the following sets:

$$\Big[\langle u,v\rangle\Big]_l^{+}(\alpha)=\inf\{x\in\mathbb{R}^n \mid u(x)\geq\alpha\},\quad \Big[\langle u,v\rangle\Big]_r^{+}(\alpha)=\sup\{x\in\mathbb{R}^n \mid u(x)\geq\alpha\}$$
$$\Big[\langle u,v\rangle\Big]_l^{-}(\alpha)=\inf\{x\in\mathbb{R}^n \mid v(x)\leq 1-\alpha\},\quad \Big[\langle u,v\rangle\Big]_r^{-}(\alpha)=\sup\{x\in\mathbb{R}^n \mid v(x)\leq 1-\alpha\}$$

Remark 2

$$\Big[\langle u,v\rangle\Big]_{\alpha}=\Big[\Big[\langle u,v\rangle\Big]_l^{+}(\alpha),\Big[\langle u,v\rangle\Big]_r^{+}(\alpha)\Big]$$
$$\Big[\langle u,v\rangle\Big]^{\alpha}=\Big[\Big[\langle u,v\rangle\Big]_l^{-}(\alpha),\Big[\langle u,v\rangle\Big]_r^{-}(\alpha)\Big]$$

Proposition 1 ([39]) *For all* $\alpha,\beta\in[0,1]$ *and* $\langle u,v\rangle\in IF_n$

(i) $\Big[\langle u,v\rangle\Big]_{\alpha}\subset\Big[\langle u,v\rangle\Big]^{\alpha}$
(ii) $\Big[\langle u,v\rangle\Big]_{\alpha}$ *and* $\Big[\langle u,v\rangle\Big]^{\alpha}$ *are nonempty compact convex sets in* $\mathbb{R}^n$
(iii) if $\alpha\leq\beta$ *then* $\Big[\langle u,v\rangle\Big]_{\beta}\subset\Big[\langle u,v\rangle\Big]_{\alpha}$ *and* $\Big[\langle u,v\rangle\Big]^{\beta}\subset\Big[\langle u,v\rangle\Big]^{\alpha}$
(iv) If $\alpha_n\nearrow\alpha$ *then* $\Big[\langle u,v\rangle\Big]_{\alpha}=\bigcap_n\Big[\langle u,v\rangle\Big]_{\alpha_n}$ *and* $\Big[\langle u,v\rangle\Big]^{\alpha}=\bigcap_n\Big[\langle u,v\rangle\Big]^{\alpha_n}$

Let M any set and $\alpha\in[0,1]$ we denote by

$$M_{\alpha}=\Big\{x\in\mathbb{R}^n\ :\ u(x)\geq\alpha\Big\}\quad\text{and}\quad M^{\alpha}=\Big\{x\in\mathbb{R}^n\ :\ v(x)\leq 1-\alpha\Big\}$$

Lemma 1 ([39]) *Let* $\Big\{M_{\alpha},\ \alpha\in[0,1]\Big\}$ *and* $\Big\{M^{\alpha},\ \alpha\in[0,1]\Big\}$ *two families of subsets of* $\mathbb{R}^n$ *satisfies (i)–(iv) in Proposition 1, if u and v define by*

$$u(x)=\begin{cases}0 & if\ x\notin M_0\\ \sup\{\alpha\in[0,1]\ :\ x\in M_{\alpha}\} & if\ x\in M_0\end{cases}$$
$$v(x)=\begin{cases}1 & if\ x\notin M^0\\ 1-\sup\{\alpha\in[0,1]\ :\ x\in M^{\alpha}\} & if\ x\in M^0\end{cases}$$

Then $\langle u,v\rangle\in IF_n$.

Lemma 2 *Let I a dense subset of* $[0, 1]$*, if* $\Big[\langle u, v\rangle\Big]_\alpha = \Big[\langle u', v'\rangle\Big]_\alpha$ *and* $\Big[\langle u, v\rangle\Big]^\alpha = \Big[\langle u', v'\rangle\Big]^\alpha$*, for all* $\alpha \in I$ *then* $\langle u, v\rangle = \langle u', v'\rangle$*.*

On the space IF_n we will consider the following metric,

$$\begin{aligned} d_\infty^n\Big(\langle u, v\rangle, \langle z, w\rangle\Big) &= \frac{1}{4}\sup_{0<\alpha\leq 1}\Big\|\Big[\langle u, v\rangle\Big]_r^+(\alpha) - \Big[\langle z, w\rangle\Big]_r^+(\alpha)\Big\| \\ &+ \frac{1}{4}\sup_{0<\alpha\leq 1}\Big\|\Big[\langle u, v\rangle\Big]_l^+(\alpha) - \Big[\langle z, w\rangle\Big]_l^+(\alpha)\Big\| \\ &+ \frac{1}{4}\sup_{0<\alpha\leq 1}\Big\|\Big[\langle u, v\rangle\Big]_r^-(\alpha) - \Big[\langle z, w\rangle\Big]_r^-(\alpha)\Big\| \\ &+ \frac{1}{4}\sup_{0<\alpha\leq 1}\Big\|\Big[\langle u, v\rangle\Big]_l^-(\alpha) - \Big[\langle z, w\rangle\Big]_l^-(\alpha)\Big\| \end{aligned}$$

where $\|.\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$.

Theorem 1 ([39]) d_∞^n *define a metric on* IF_n*.*

Theorem 2 ([39]) *The metric space* (IF_n, d_∞^n) *is complete.*

Proof There exists $i_0 \leq n$ such that

$$d_\infty^n\left(<u, v>, <u', v'>\right) \leq \sqrt{n} d_\infty\left(<u, v>_{i_0}, <u', v'>_{i_0}\right)$$

Since d_∞ defined a complete topology in IF_1, then d_∞^n also is complete.

We denote by $C(J_a \times J_b, IF_n)$ the space of all continuous mappings defined over $J_a \times J_b$ into IF_n.

A standard proof applies to show that the metric space $\Big(C(J_a \times J_b, IF_n), D\Big)$ is complete. Where, the supermum metric D on $C(J_a \times J_b, IF_n)$ is defined by

$$D(\langle u, v\rangle, \langle u', v'\rangle) = \sup_{(t,s)\in J_a\times J_b} d_\infty^n\Big(\langle u, v\rangle(t, s), \langle u', v'\rangle(t, s)\Big)$$

Definition 2 A mapping $f : J_a \times J_b \to IF_n$ is called continuous at point $(t_0, s_0) \in J_a \times J_b$ provided for any arbitrary $\varepsilon > 0$, there exists an $\delta(\varepsilon)$ such that

$$d_\infty^n\Big(f(t, s), f(t_0, s_0)\Big) < \varepsilon$$

whenever $\max\{|t - t_0|, |s - s_0|\} < \delta(\varepsilon)$ for all $(t, s) \in J_a \times J_b$.

Definition 3 A mapping $f : J_a \times J_b \times IF_n \to IF_n$ is called continuous at point $(t_0, s_0, \langle u, v\rangle_0) \in J_a \times J_b \times IF_n$ provided for any arbitrary $\varepsilon > 0$, there exists an $\delta(\varepsilon)$ such that

$$d_\infty^n\Big(f(t, s, \langle u, v\rangle), f(t_0, s_0, \langle u, v\rangle_0)\Big) < \varepsilon$$

whenever $\max\{|t - t_0|, |s - s_0|\} < \delta(\varepsilon)$ and $d_\infty^n(\langle u, v\rangle, \langle u, v\rangle_0) < \delta(\varepsilon)$ for all $(t, s) \in J_a \times J_b$, $\langle u, v\rangle \in IF_n$.

Definition 4 we say that a mapping $f : J_a \times J_b \to IF_n$ is strongly measurable if for all $\alpha \in [0, 1]$ the set-valued mappings $f_\alpha : J_a \times J_b \to P_k(\mathbb{R}^n)$ defined by $f_\alpha(t, s) = [f(t, s)]_\alpha$ and $f^\alpha : J_a \times J_b \to P_k(\mathbb{R}^n)$ defined by $f^\alpha(t, s) = [f(t, s)]^\alpha$ are (Lebesgue) measurable, when $P_k(\mathbb{R}^n)$ is endowed with the topology generated by the Hausdorff metric d_H.

Where d_H is the Hausdorff metric defined in $P_k(\mathbb{R}^n)$ by $d_H([a, b][c, d]) = \max\{||a - c||; ||b - d||\}$.

Definition 5 $f : J_a \times J_b \to IF_n$ is called integrably bounded if there exists an integrable function $h : J_a \times J_b \to \mathbb{R}^n$ such that $\|y\| \le h(t)$ holds for any $y \in supp(f(t, s))$, $(t, s) \in J_a \times J_b$.

Theorem 3 *If $f : J_a \times J_b \to IF_n$ is strongly measurable and integrably bounded, then f is integrable.*

Definition 6 Suppose $f : J_a \times J_b \to IF_n$ is integrably bounded and strongly measurable for each $\alpha \in (0, 1]$ write

$$\begin{aligned}\left[\int_0^a \int_0^b f(t, s)dsdt\right]_\alpha &= \int_0^a \int_0^b \big[f(t, s)\big]_\alpha\, dsdt\\ &= \Big\{\int_0^a \int_0^b F(t, s)dsdt\\ &\qquad |F : J_a \times J_b \to \mathbb{R}^n \text{ is a measurable selection for } f_\alpha\Big\}.\\ \left[\int_0^a \int_0^b f(t, s)dsdt\right]^\alpha &= \int_0^a \int_0^b \big[f(t, s)\big]^\alpha\, dsdt\\ &= \Big\{\int_0^a \int_0^b F(t, s)dsdt\\ &\qquad |F : J_a \times J_b \to \mathbb{R}^n \text{ is a measurable selection for } f^\alpha\Big\}.\end{aligned}$$

If there exists $\langle u, v\rangle \in IF_n$ such that $[\langle u, v\rangle]^\alpha = \left[\int_0^a \int_0^b f(t, s)dsdt\right]^\alpha$ and $[\langle u, v\rangle]_\alpha = \left[\int_0^a \int_0^b f(t, s)dsdt\right]_\alpha$ $\forall\alpha \in (0, 1]$. Then f is called integrable on T, write $\langle u, v\rangle = \int_0^a \int_0^b f(t, s)dsdt$.

Let $\langle u, v\rangle$ and $\langle u', v'\rangle \in IF_1$, the Hukuhara difference is the intuitionistic fuzzy number $\langle z, w\rangle \in IF_1$, if it exists, such that

$$\langle u, v\rangle - \langle u', v'\rangle = \langle z, w\rangle \Longleftrightarrow \langle u, v\rangle = \langle u', v'\rangle + \langle z, w\rangle$$

Definition 7 Let $f : J_a \times J_b \to IF_n$. The intuitionistic fuzzy partial derivative of f with respect to x at the point $(t_0, s_0) \in J_a \times J_b$ is the intuitionistic fuzzy quantity $\frac{\partial f(t,s)}{\partial x} \in IF_n$ if there exists, such that for all $h > 0$ sufficiently small, the H-difference $f(t_0 + h, s_0) - f(t_0, s_0)$ exist in IF_n and the limit

$$\frac{\partial f(t,s)}{\partial x} = \lim_{h \to 0^+} \frac{f(t_0 + h, s_0) - f(t_0, s_0)}{h}$$

Here the limit is taken in the metric space (IF_n, d_∞^n). The intuitionistic fuzzy partial derivative of f with respect to y at the point $(t_0, s_0) \in J_a \times J_b$ is defined similarly.

3 Existence and Uniqueness

In this part of this section, we provide an existence and uniqueness result for the the following intuitionistic fuzzy hyperbolic partial differential equation with integral boundary conditions:

$$\begin{cases} \frac{\partial^2 \langle u,v\rangle(x,y)}{\partial x \partial y} = f(x, y, \langle u, v\rangle(x, y)), \qquad (x, y) \in J_a \times J_b \\ \langle u, v\rangle(x, 0) + \int_0^b k_1(x)\langle u, v\rangle(x, y)dy = \eta_1(x), \ x \in J_a \\ \langle u, v\rangle(0, y) + \int_0^a k_2(y)\langle u, v\rangle(x, y)dx = \eta_2(y), \ y \in J_b \end{cases} \tag{3}$$

Assume that $f : J_a \times J_b \times IF_n \to IF_n$ is continuous, $k_1 \in C(J_a, \mathbb{R})$, $k_2 \in C(J_b, \mathbb{R})$, $\eta_1 \in C(J_a, IF_n)$ and $\eta_2 \in C(J_b, IF_n)$ are given functions, where $J_a = [0, a]$ and $J_b = [0, b]$. In the second part of this section we consider the equation of the form

$$\begin{cases} \frac{\partial^2 \langle u,v\rangle(x,y)}{\partial x \partial y} = \big(h(x, y)\langle u, v\rangle(x, y)\big)_y + f(x, y, \langle u, v\rangle(x, y)), \ \ (x, y) \in J_a \times J_b \\ \langle u, v\rangle(x, 0) + \int_0^b k_1(x)\langle u, v\rangle(x, y)dy = \eta_1(x), \ x \in J_a \\ \langle u, v\rangle(0, y) + \int_0^a k_2(y)\langle u, v\rangle(x, y)dx = \eta_2(y), \ y \in J_b \end{cases} \tag{4}$$

where f, k_1, k_2, η_1 and η_2 are as in problem (3) and $h : J_a \times J_b \to \mathbb{R}$.

Definition 8 A function $\langle u, v\rangle \in C(J_a \times J_b, IF_n)$ is called a solution of the problem (3) if $\langle u, v\rangle$ satisfies the following integral equation

$$\langle u, v\rangle(x, y) = Q(x, y) - \int_0^b k_1(x)\langle u, v\rangle(x, y)dy - \int_0^a k_2(y)\langle u, v\rangle(x, y)dx - k_1(0)\int_0^b \int_0^a k_2(y)\langle u, v\rangle(x, y)dxdy + \int_0^x \int_0^y f\big(t, s, \langle u, v\rangle(t, s)\big)dsdt$$

where

$$Q(x, y) = \eta_1(x) + \eta_2(y) - \eta_1(0) + \eta_1(0)\int_0^b \eta_2(s)ds$$

for all $(x, y) \in J_a \times J_b$.

Set $k_1 = \sup_{t\in J_a} |k_1(t)|$ and $k_2 = \sup_{s\in J_b} |k_2(s)|$. By applying the fixed point theorem, we prove the following result.

Theorem 4 *Assume that*

1. *A mapping* $f : J_a \times J_b \times IF_n \to IF_n$ *is continuous,*
2. *for any pair* $\big(t, s, \langle u, v\rangle\big), \big(t, s, \langle u', v'\rangle\big) \in J_a \times J_b \times IF_n$, *we have*

$$d_\infty^n\Big(f\big(t, s, \langle u, v\rangle\big), f\big(t, s, \langle u', v'\rangle\big)\Big) \le K d_\infty^n\Big(\langle u, v\rangle, \langle u', v'\rangle\Big) \tag{5}$$

where $K > 0$ *is a given constant.*
Moreover, if $k_1 + k_2 + k_1k_2 + K < 1$, *then the problem (3) has an unique intuitionistic fuzzy solution in* $C(J_a \times J_b, IF_n)$.

Proof Transform the problem (3) into a fixed point problem. It is clear that the solutions of the problem (3) are fixed points of the operator $N : C(J_a \times J_b, IF_n) \longrightarrow C(J_a \times J_b, IF_n)$ defined by:

$$N(\langle u, v\rangle(x, y)) := Q(x, y) - \int_0^b k_1(t)\langle u, v\rangle(x, s)ds - \int_0^a k_2(s)\langle u, v\rangle(t, y)dt - k_1(0)\int_0^b \int_0^a k_2(s)\langle u, v\rangle(t, s)dtds + \int_0^x \int_0^y f\big(t, s, \langle u, v\rangle_{(t,s)}\big)dsdt$$

We shall show that N is a contraction operator. Indeed, consider $\langle u, v\rangle, \langle u', v'\rangle \in C(J_a \times J_b, IF_n)$ and $\alpha \in (0, 1]$, then

$$N(\langle u, v\rangle(x, y)) := Q(x, y) - \int_0^b k_1(t)\langle u, v\rangle(x, s)ds - \int_0^a k_2(s)\langle u, v\rangle(t, y)dt \\ - k_1(0)\int_0^b\int_0^a k_2(s)\langle u, v\rangle(t, s)dtds \\ + \int_0^x\int_0^y f\big(t, s, \langle u, v\rangle(t, s)\big)dsdt$$

and

$$N(\langle u', v'\rangle(x, y)) := Q(x, y) - \int_0^b k_1(t)\langle u', v'\rangle(x, s)ds - \int_0^a k_2(s)\langle u', v'\rangle(t, y)dt \\ - k_1(0)\int_0^b\int_0^a k_2(s)\langle u', v'\rangle(t, s)dtds \\ + \int_0^x\int_0^y f\big(t, s, \langle u', v'\rangle(t, s)\big)dsdt,$$

From the properties of supremum metric, we have the following inequality

$$\begin{aligned}
& d_\infty^n\big(N(\langle u, v\rangle)(s, t), N(\langle u', v'\rangle)(s, t)\big) \\
&\leq d_\infty^n\left(\int_0^b k_1(t)\langle u, v\rangle(x, s)ds, \int_0^b k_1(t)\langle u', v'\rangle(x, s)ds\right) \\
&\quad + d_\infty^n\left(\int_0^a k_2(s)\langle u, v\rangle(t, y)dt, \int_0^a k_2(s)\langle u', v'\rangle(t, y)dt\right) \\
&\quad + d_\infty^n\left(k_1(0)\int_0^b\int_0^a k_2(s)\langle u, v\rangle(t, s)dtds, k_1(0)\int_0^b\int_0^a k_2(s)\langle u', v'\rangle(t, s)dtds\right) \\
&\quad + d_\infty^n\left(\int_0^x\int_0^y f\,(t, s, \langle u, v\rangle(t, s))\,dsdt, \int_0^x\int_0^y f\,\big(t, s, \langle u', v'\rangle(t, s)\big)\,dsdt\right) \\
&\leq k_1\int_0^b d_\infty^n\Big(\langle u, v\rangle(x, s), \langle u', v'\rangle(x, s)\Big)ds \\
&\quad + k_2\int_0^a d_\infty^n\Big(\langle u, v\rangle(t, y), \langle u', v'\rangle(t, y)\Big)dt \\
&\quad + |k_1(0)|\sup_{s\in J_b}|k_2(s)|\int_0^b\int_0^a d_\infty^n\Big(\langle u, v\rangle(t, s), \langle u', v'\rangle(t, s)\Big)dtds \\
&\quad + \int_0^x\int_0^y d_\infty^n\Big(f\big(t, s, \langle u, v\rangle(t, s)\big), f\big(t, s, \langle u', v'\rangle(t, s)\big)\Big)dsdt \\
&\leq \big(k_1 b + k_2 a + k_1 k_2 ab\big)d_\infty^n\Big(\langle u, v\rangle(x, s), \langle u', v'\rangle(x, s)\Big)ds \\
&\quad + K\int_0^x\int_0^y d_\infty^n\Big(\langle u, v\rangle(t, s)\big), \langle u', v'\rangle(t, s)\big)\Big)dsdt \\
&\leq \big(k_1 b + k_2 a + k_1 k_2 ab + Kab\big)D(\langle u, v\rangle, \langle u', v'\rangle)
\end{aligned}$$

Hence for each $(t, s) \in J_a \times J_b$

$$D\big(N(\langle u, v\rangle), N(\langle u', v'\rangle)\big) \leq \big(k_1 + k_2 + k_1 k_2 + K\big)D\big(\langle u, v\rangle, \langle u', v'\rangle\big)$$

N is a contraction and thus, by Banach fixed point theorem, N has an unique fixed point, which is solution to (3).

Definition 9 A function $\langle u, v\rangle \in C(J_a \times J_b, IF_n)$ is called a solution of the problem (4) if $\langle u, v\rangle$ satisfies the following integral equation

$$\begin{aligned}\langle u, v\rangle(x, y) = Q(x, y) &- \int_0^b k_1(x)\langle u, v\rangle(x, y)dy \\ &- \int_0^a k_2(y)\langle u, v\rangle(x, y)dx + \int_0^x h(t, y)\langle u, v\rangle(t, y)dt \\ &- k_1(0)\int_0^b \int_0^a k_2(y)\langle u, v\rangle(x, y)dxdy \\ &+ \int_0^x \int_0^b h(t, 0)k_1(t)\langle u, v\rangle(t, y)dydt + \int_0^x \int_0^y f\big(t, s, \langle u, v\rangle(t, s)\big)dsdt\end{aligned}$$

where

$$Q(x, y) = \eta_1(x) + \eta_2(y) - \eta_1(0) + k_1(0)\int_0^b \eta_2(s)ds - \int_0^x h(t, 0)\eta_1(t)dt$$

for all $(x, y) \in J_a \times J_b$.

Let $k_1 = \sup_{t\in J_a} |k_1(t)|$, $k_2 = \sup_{s\in J_b} |k_2(s)|$ and $q = \sup_{(t,s)\in J_a\times J_b} |h(t, s)|$. By applying the fixed point theorem, we prove the following result.

Theorem 5 *Assume that*

1. *A mapping* $f : J_a \times J_b \times IF_n \to IF_n$ *is continuous,*
2. *for any pair* $\big(t, s, \langle u, v\rangle\big), \big(t, s, \langle u', v'\rangle\big) \in J_a \times J_b \times IF_n$*, we have*

$$d_\infty^n\Big(f\big(t, s, \langle u, v\rangle\big), f\big(t, s, \langle u', v'\rangle\big)\Big) \leq K d_\infty^n\Big(\langle u, v\rangle, \langle u', v'\rangle\Big) \tag{6}$$

where $K > 0$ *is a given constant.*
Moreover, if $(k_1 + 1)(k_2 + q + 1) + K < 2$*, then the problem* (4) *has an unique intuitionistic fuzzy solution in* $C(J_a \times J_b, IF_n)$*.*

Proof Transform the problem (4) into a fixed point problem. It is clear that the solutions of the problem (4) are fixed points of the operator $N : C(J_a \times J_b, IF_n) \longrightarrow C(J_a \times J_b, IF_n)$ defined by:

$$N(\langle u,v\rangle(x,y)) := Q(x,y) - \int_0^b k_1(x)\langle u,v\rangle(x,y)dy - \int_0^a k_2(y)\langle u,v\rangle(x,y)dx \\ + \int_0^x h(t,y)\langle u,v\rangle(t,y)dt - k_1(0)\int_0^b\int_0^a k_2(y)\langle u,v\rangle(x,y)dxdy \\ + \int_0^x\int_0^b h(t,0)k_1(t)\langle u,v\rangle(t,y)dydt + \int_0^x\int_0^y f\big(t,s,\langle u,v\rangle(t,s)\big)dsdt$$

We shall show that N is a contraction operator. Indeed, consider $\langle u,v\rangle$, $\langle u',v'\rangle \in C(J_a \times J_b, IF_n)$ and $\alpha \in (0,1]$, then

$$N(\langle u,v\rangle(x,y)) := Q(x,y) - \int_0^b k_1(x)\langle u,v\rangle(x,y)dy - \int_0^a k_2(y)\langle u,v\rangle(x,y)dx \\ + \int_0^x h(t,y)\langle u,v\rangle(t,y)dt - k_1(0)\int_0^b\int_0^a k_2(y)\langle u,v\rangle(x,y)dxdy \\ + \int_0^x\int_0^b h(t,0)k_1(t)\langle u,v\rangle(t,y)dydt + \int_0^x\int_0^y f\big(t,s,\langle u,v\rangle(t,s)\big)dsdt$$

and

$$N(\langle u',v'\rangle(x,y)) := Q(x,y) - \int_0^b k_1(x)\langle u',v'\rangle(x,y)dy - \int_0^a k_2(y)\langle u',v'\rangle(x,y)dx \\ + \int_0^x h(t,y)\langle u',v'\rangle(t,y)dt - k_1(0)\int_0^b\int_0^a k_2(y)\langle u',v'\rangle(x,y)dxdy \\ + \int_0^x\int_0^b h(t,0)k_1(t)\langle u',v'\rangle(t,y)dydt + \int_0^x\int_0^y f\big(t,s,\langle u',v'\rangle(t,s)\big)dsdt$$

From the properties of supremum metric, we have the following inequality

$$d_\infty^n\Big(N(\langle u,v\rangle)(s,t), N(\langle u',v'\rangle)(s,t)\Big) \\ \leq d_\infty^n\left(\int_0^b k_1(t)\langle u,v\rangle(x,s)ds, \int_0^b k_1(t)\langle u',v'\rangle(x,s)ds\right) \\ + d_\infty^n\left(\int_0^a k_2(s)\langle u,v\rangle(t,y)dt, \int_0^a k_2(s)\langle u',v'\rangle(t,y)dt\right) \\ + d_\infty^n\left(\int_0^x h(t,y)\langle u,v\rangle(t,y)dt, \int_0^x h(t,y)\langle u',v'\rangle(t,y)dt\right) \\ + d_\infty^n\left(k_1(0)\int_0^b\int_0^a k_2(s)\langle u,v\rangle(t,s)dtds, k_1(0)\int_0^b\int_0^a k_2(s)\langle u',v'\rangle(t,s)dtds\right) \\ + d_\infty^n\left(\int_0^x\int_0^b h(t,0)k_1(t)\langle u,v\rangle(t,y)dydt, \int_0^x\int_0^b h(t,0)k_1(t)\langle u',v'\rangle(t,y)dydt\right) \\ + d_\infty^n\left(\int_0^x\int_0^y f\,(t,s,\langle u,v\rangle(t,s))\,dsdt, \int_0^x\int_0^y f\,\big(t,s,\langle u',v'\rangle(t,s)\big)\,dsdt\right)$$

$$
\begin{aligned}
&\leq k_1 \int_0^b d_\infty^n\Big(\langle u,v\rangle(x,s), \langle u',v'\rangle(x,s)\Big)ds \\
&\quad + k_2 \int_0^a d_\infty^n\Big(\langle u,v\rangle(t,y), \langle u',v'\rangle(t,y)\Big)dt \\
&\quad + \sup_{(t,s)\in J_a\times J_b} |h(t,s)| \int_0^x d_\infty^n\Big(\langle u,v\rangle(t,y), \langle u',v'\rangle(t,y)\Big)dt \\
&\quad + |k_1(0)| \sup_{s\in J_b} |k_2(s)| \int_0^b \int_0^a d_\infty^n\Big(\langle u,v\rangle(t,s), \langle u',v'\rangle(t,s)\Big)dtds \\
&\quad + \sup_{x\in J_a} |k_1(x)| \sup_{(t,s)\in J_a\times J_b} |h(t,s)| \int_0^x \int_0^b d_\infty^n\Big(\langle u,v\rangle(t,y), \langle u',v'\rangle(t,y)\Big)dydt \\
&\quad + \int_0^x \int_0^y d_\infty^n\Big(f\big(t,s,\langle u,v\rangle(t,s)\big), f\big(t,s,\langle u',v'\rangle(t,s)\big)\Big)dsdt \\
&\leq \big(k_1 b + k_2 a + qa + k_1 k_2 ab + k_1 q\big) d_\infty^n\Big(\langle u,v\rangle(x,s), \langle u',v'\rangle(x,s)\Big)ds \\
&\quad + K \int_0^x \int_0^y d_\infty^n\Big(\langle u,v\rangle(t,s)\big), \langle u',v'\rangle(t,s)\big)\Big)dsdt \\
&\leq \big(k_1 b + k_2 a + qa + k_1 k_2 ab + k_1 q + Kab\big) D(\langle u,v\rangle, \langle u',v'\rangle)
\end{aligned}
$$

Hence for each $(t, s) \in J_a \times J_b$

$$
D\Big(N(\langle u,v\rangle), N(\langle u',v'\rangle)\Big) \leq \big(k_1 + k_2 + q + k_1 k_2 + k_1 q + K\big) D\big(\langle u,v\rangle, \langle u',v'\rangle\big)
$$

Since $(k_1 + 1)(k_2 + q + 1) + K < 2$ then $((k_1 + 1)(k_2 + q + 1) + K - 1) < 1$, then N is a contraction and thus, by Banach fixed point theorem, N has an unique fixed point, which is solution to (4).

4 Solving Intuitionistic Fuzzy PHDEs with Integral Boundary Conditions

We give an useful procedure to solve the following integral boundary value problem for an intuitionistic fuzzy partial hyperbolic differential equation:

$$
\begin{cases}
\frac{\partial^2 \langle u,v\rangle(x,y)}{\partial x \partial y} = f(x, y, \langle u,v\rangle(x,y)), & (x,y) \in J_a \times J_b \\
\langle u,v\rangle(x,0) + \int_0^b k_1(x)\langle u,v\rangle(x,y)dy = \eta_1(x), & x \in J_a \\
\langle u,v\rangle(0,y) + \int_0^a k_2(y)\langle u,v\rangle(x,y)dx = \eta_2(y), & y \in J_b
\end{cases}
\tag{7}
$$

where $f : J_a \times J_b \times IF_n \to IF_n$ is obtained by extension principle from a continuous function $F : J_a \times J_b \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$. Since

$$\big[f\big(t, s, \langle u, v\rangle\big)\big]_\alpha = f\big(t, s, [\langle u, v\rangle\big)]_\alpha)$$

$$\big[f\big(t, s, \langle u, v\rangle\big)\big]^\alpha = f\big(t, s, [\langle u, v\rangle\big)]^\alpha)$$

for all $\alpha \in [0, 1]$ and $\langle u, v\rangle \in IF_n$, we denote

$$\Big[\langle u, v\rangle(t, s)\Big]_\alpha = \Big[\Big[\langle u, v\rangle(t, s)\Big]_l^+(\alpha), \Big[\langle u, v\rangle(t, s)\Big]_r^+(\alpha)\Big],$$
$$\Big[\langle u, v\rangle(t, s)\Big]^\alpha = \Big[\Big[\langle u, v\rangle(t, s)\Big]_l^-(\alpha), \Big[\langle u, v\rangle(t, s)\Big]_r^-(\alpha)\Big]$$

$$\Big[\frac{\partial^2\langle u, v\rangle(t, s)}{\partial t\partial s}\Big]_\alpha = \Big[\Big[\frac{\partial^2\langle u, v\rangle(t, s)}{\partial t\partial s}\Big]_l^+(\alpha), \Big[\frac{\partial^2\langle u, v\rangle(t, s)}{\partial t\partial s}\Big]_r^+(\alpha)\Big]$$

$$\Big[\frac{\partial^2\langle u, v\rangle(t, s)}{\partial t\partial s}\Big]^\alpha = \Big[\Big[\frac{\partial^2\langle u, v\rangle(t, s)}{\partial t\partial s}\Big]_l^-(\alpha), \Big[\frac{\partial^2\langle u, v\rangle(t, s)}{\partial t\partial s}\Big]_r^-(\alpha)\Big]$$

$$\Big[\eta_1(t)\Big]_\alpha = \Big[\Big[\eta_1(t)\Big]_l^+(\alpha), \Big[\eta_1(t)\Big]_r^+(\alpha)\Big], \quad \Big[\eta_1(t)\Big]^\alpha = \Big[\Big[\eta_1(t)\Big]_l^-(\alpha), \Big[\eta_1(t)\Big]_r^-(\alpha)\Big]$$

$$\Big[\eta_2(s)\Big]_\alpha = \Big[\Big[\eta_2(s)\Big]_l^+(\alpha), \Big[\eta_2(s)\Big]_r^+(\alpha)\Big], \quad \Big[\eta_2(s)\Big]^\alpha = \Big[\Big[\eta_2(s)\Big]_l^-(\alpha), \Big[\eta_2(s)\Big]_r^-(\alpha)\Big]$$

and

$$\Big[f\big(t, s, \langle u, v\rangle\big)\Big]_\alpha$$
$$= \Big[f_l^+(t, s, [\langle u, v\rangle(t, s)]_l^+(\alpha), [\langle u, v\rangle(t, s)]_r^+(\alpha)), f_r^+\big(t, s, [\langle u, v\rangle(t, s)]_l^+(\alpha), [\langle u, v\rangle(t, s)]_r^+(\alpha)\big)\Big]$$

$$\Big[f(t, s, \langle u, v\rangle)\Big]^\alpha$$
$$= \Big[f_l^-(t, s, [\langle u, v\rangle(t, s)]_l^-(\alpha), [\langle u, v\rangle(t, s)]_r^-(\alpha)), f_r^-(t, s, [\langle u, v\rangle(t, s)]_l^-(\alpha), [\langle u, v\rangle(t, s)]_r^-(\alpha)\big)\Big]$$

Then, with this notations, problem (7) is transformed into the following parametrized partial differential system:

$$\begin{cases} \left[\frac{\partial^2 \langle u,v\rangle(t,s)}{\partial t \partial s}\right]_l^+(\alpha) & = f_l^+\Big(t, s, \Big[\langle u, v\rangle(t,s)\Big]_l^+(\alpha), \Big[\langle u, v\rangle(t,s)\Big]_r^+(\alpha)\Big), \quad (t,s) \in J_a \times J_b \\ \left[\frac{\partial^2 \langle u,v\rangle(t,s)}{\partial t \partial s}\right]_r^+(\alpha) & = f_r^+\Big(t, s, \Big[\langle u, v\rangle(t,s)\Big]_l^+(\alpha), \Big[\langle u, v\rangle(t,s)\Big]_r^+(\alpha)\Big), \quad (t,s) \in J_a \times J_b \\ \left[\frac{\partial^2 \langle u,v\rangle(t,s)}{\partial t \partial s}\right]_l^-(\alpha) & = f_l^-\Big(t, s, \Big[\langle u, v\rangle(t,s)\Big]_l^-(\alpha), \Big[\langle u, v\rangle(t,s)\Big]_r^-(\alpha)\Big), \quad (t,s) \in J_a \times J_b \\ \left[\frac{\partial^2 \langle u,v\rangle(t,s)}{\partial t \partial s}\right]_r^-(\alpha) & = f_r^-\Big(t, s, \Big[\langle u, v\rangle(t,s)\Big]_l^-(\alpha), \Big[\langle u, v\rangle(t,s)\Big]_r^-(\alpha)\Big), \quad (t,s) \in J_a \times J_b \end{cases} \tag{8}$$

with initial conditions

$$\begin{cases} [\langle u,v\rangle(t,0)]_l^+(\alpha) + \int_0^b k_1(t)[\langle u,v\rangle(t,s)]_l^+(\alpha)ds & = \Big[\eta_1(t)\Big]_l^+(\alpha), \quad t \in J_a \\ [\langle u,v\rangle(t,0)]_r^+(\alpha) + \int_0^b k_1(t)[\langle u,v\rangle(t,s)]_r^+(\alpha)ds & = \Big[\eta_1(t)\Big]_r^+(\alpha), \quad t \in J_a \\ [\langle u,v\rangle(t,0)]_l^-(\alpha) + \int_0^b k_1(t)[\langle u,v\rangle(t,s)]_l^-(\alpha)ds & = \Big[\eta_1(t)\Big]_l^-(\alpha), \quad t \in J_a \\ [\langle u,v\rangle(t,0)]_r^-(\alpha) + \int_0^b k_1(t)[\langle u,v\rangle(t,s)]_r^-(\alpha)ds & = \Big[\eta_1(t)\Big]_r^-(\alpha), \quad t \in J_a \\ [\langle u,v\rangle(0,s)]_l^+(\alpha) + \int_0^a k_2(s)[\langle u,v\rangle(t,s)]_l^+(\alpha)dt & = \Big[\eta_2(s)\Big]_l^+(\alpha), \quad s \in J_b \\ [\langle u,v\rangle(0,s)]_r^+(\alpha) + \int_0^a k_2(s)[\langle u,v\rangle(t,s)]_r^+(\alpha)dt & = \Big[\eta_2(s)\Big]_r^+(\alpha), \quad s \in J_b \\ [\langle u,v\rangle(0,s)]_l^-(\alpha) + \int_0^a k_2(s)[\langle u,v\rangle(t,s)]_l^-(\alpha)dt & = \Big[\eta_2(s)\Big]_l^-(\alpha), \quad s \in J_b \\ [\langle u,v\rangle(0,s)]_r^-(\alpha) + \int_0^a k_2(s)[\langle u,v\rangle(t,s)]_r^-(\alpha)dt & = \Big[\eta_2(s)\Big]_r^-(\alpha), \quad s \in J_b \end{cases} \tag{9}$$

1. We solve the system (8)–(9).
2. If $\Big[\langle u,v\rangle(t,s)\Big]_l^+(\alpha), \Big[\langle u,v\rangle(t,s)\Big]_r^+(\alpha), \Big[\langle u,v\rangle(t,s)\Big]_l^-(\alpha), \Big[\langle u,v\rangle(t,s)\Big]_r^-(\alpha)$ is the solution of system (8)–(9), then denote

   $$\Big[\Big[\langle u,v\rangle(t,s)\Big]_l^+(\alpha), \Big[\langle u,v\rangle(t,s)\Big]_r^+(\alpha)\Big] = M_\alpha, \quad \Big[\Big[\langle u,v\rangle(t,s)\Big]_l^-(\alpha), \Big[\langle u,v\rangle(t,s)\Big]_r^-(\alpha)\Big] = M^\alpha$$

   and

$$\left[\left[\frac{\partial^2\langle u,v\rangle(t,s)}{\partial t\partial s}\right]_l^+(\alpha),\left[\frac{\partial^2\langle u,v\rangle(t,s)}{\partial t\partial s}\right]_r^+(\alpha)\right]=M_\alpha^{'},$$

$$\left[\left[\frac{\partial^2\langle u,v\rangle(t,s)}{\partial t\partial s}\right]_l^-(\alpha),\left[\frac{\partial^2\langle u,v\rangle(t,s)}{\partial t\partial s}\right]_r^-(\alpha)\right]=M^{'\alpha}$$

ensure that (M_α, M^α) and $(M_\alpha^{'}, M^{'\alpha})$ verifying (i)–(iv) of Proposition 1.

3. After, by using the Lemma 1 we can construct the intuitionistic fuzzy solution $\langle u,v\rangle(t,s)\in IF_n$ for (7) such that

$$\left[\langle u,v\rangle(t,s)\right]_\alpha=\left[\left[\langle u,v\rangle(t,s)\right]_l^+(\alpha),\left[\langle u,v\rangle(t,s)\right]_r^+(\alpha)\right],$$

$$\left[\langle u,v\rangle(t,s)\right]^\alpha=\left[\left[\langle u,v\rangle(t,s)\right]_l^-(\alpha),\left[\langle u,v\rangle(t,s)\right]_r^-(\alpha)\right]$$

for all $\alpha\in[0,1]$.

Remark 3 The same procedure can be used to the this equation $\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}=\big(h(x,y)\langle u,v\rangle(x,y)\big)_y+f(x,y,\langle u,v\rangle(x,y)),\qquad (x,y)\in J_a=[0,a]\times J_b=[0,b]$ with integral boundary conditions.

5 Application

Example 1 Consider the following intuitionistic fuzzy hyperbolic equation

$$\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}=\frac{1}{15}Ce^{x+y}=f(x,y,C)\tag{10}$$

where $C\in IF_1$ is a triangular intuitionistic fuzzy number, $(x,y)\in[0,1]\times[0,1]$. And the integral boundary conditions are

$$\langle u,v\rangle(x,0)+\frac{1}{8}\int_0^1\langle u,v\rangle(x,y)dy=C\tfrac{1}{120}e^x(7+e),\tag{11}$$

$$\langle u,v\rangle(0,y)+\frac{1}{8}\int_0^1\langle u,v\rangle(x,y)dx=C\tfrac{1}{120}e^y(7+e).\tag{12}$$

Then the function $f:[0,1]\times[0,1]\times IF_1\to IF_1$ define by

$$f(x,y,\langle u,v\rangle)=\frac{1}{15}Ce^{x+y}$$

It implies

$$d_\infty\big(f(x,y,\langle u,v\rangle), f(x,y,\langle u',v'\rangle)\big) = 0$$

and conditions in Theorem 4 hold for any positive number K, like $K = \frac{e^2}{15}$, and $k_1 = k_2 = \frac{1}{8}$, $a = b = 1$. That follows all the conditions in the Theorem 4 hold. Therefore there exists an unique intuitionistic fuzzy solution of this problem.

We will find an intuitionistic fuzzy solution of this problem:

We apply the fuzzification in c, and supposed that the parametric form of corresponding intuitionistic fuzzy number C is

$$\Big[C\Big]_\alpha = \Big[C_l^+(\alpha), C_r^+(\alpha)\Big]$$
$$\Big[C\Big]^\alpha = \Big[C_l^-(\alpha), C_r^-(\alpha)\Big]$$

where is verify the conditions of Lemma 1.

Then the function $f : [0,1] \times [0,1] \times IF_1 \longrightarrow IF_1$ define by $f\big(x,y,C\big) = \frac{1}{15}Ce^{x+y}$ is obtained by extension principle from the function $F(x,y,c) = \frac{1}{15}ce^{x+y}$, $(x,y,c) \in [0,1] \times [0,1] \times \mathbb{R}$

$$\big[f\big]_\alpha = \Big[\frac{1}{15}Ce^{x+y}\Big]_\alpha = \Big[\frac{1}{15}e^{x+y}C_l^+(\alpha), \frac{1}{15}e^{x+y}C_r^+(\alpha)\Big]$$
$$\big[f\big]^\alpha = \Big[\frac{1}{15}Ce^{x+y}\Big]^\alpha = \Big[\frac{1}{15}e^{x+y}C_l^-(\alpha), \frac{1}{15}e^{x+y}C_r^-(\alpha)\Big]$$

If

$$\Big[\langle u,v\rangle(x,y)\Big]_\alpha = \Big[\Big[\langle u,v\rangle(x,y)\Big]_l^+(\alpha), \Big[\langle u,v\rangle(x,y)\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u,v\rangle(x,y)\Big]^\alpha = \Big[\Big[\langle u,v\rangle(x,y)\Big]_l^-(\alpha), \Big[\langle u,v\rangle(x,y)\Big]_r^-(\alpha)\Big]$$

Then

$$\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_\alpha = \Big[\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_l^+(\alpha), \Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_r^+(\alpha)\Big]$$
$$\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]^\alpha = \Big[\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_l^-(\alpha), \Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_r^-(\alpha)\Big]$$

Therefore, we have to solve the following partial hyperbolic differential equations:

$$\begin{cases} \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_l^+(\alpha) &= \frac{1}{15}e^{x+y}C_l^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_r^+(\alpha) &= \frac{1}{15}e^{x+y}C_r^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_l^-(\alpha) &= \frac{1}{15}e^{x+y}C_l^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_r^-(\alpha) &= \frac{1}{15}e^{x+y}C_r^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \end{cases} \tag{13}$$

with initial conditions

$$\begin{cases} [\langle u,v\rangle(x,0)]_l^+(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_l^+(\alpha)dy &= \frac{1}{120}e^x(7+e)C_l^+(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(x,0)]_r^+(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_r^+(\alpha)dy &= \frac{1}{120}e^x(7+e)C_r^+(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(x,0)]_l^-(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_l^-(\alpha)dy &= \frac{1}{120}e^x(7+e)C_l^-(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(x,0)]_r^-(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_r^-(\alpha)dy &= \frac{1}{120}e^x(7+e)C_r^-(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(0,y)]_l^+(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_l^+(\alpha)dx &= \frac{1}{120}e^y(7+e)C_l^+(\alpha), \quad y\in[0,1] \\ [\langle u,v\rangle(0,y)]_r^+(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_r^+(\alpha)dx &= \frac{1}{120}e^y(7+e)C_r^+(\alpha), \quad y\in[0,1] \\ [\langle u,v\rangle(0,y)]_l^-(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_l^-(\alpha)dx &= \frac{1}{120}e^y(7+e)C_l^-(\alpha), \quad y\in[0,1] \\ [\langle u,v\rangle(0,y)]_r^-(\alpha)+\frac{1}{8}\int_0^1[\langle u,v\rangle(x,y)]_r^-(\alpha)dx &= \frac{1}{120}e^y(7+e)C_r^-(\alpha), \quad y\in[0,1] \end{cases} \tag{14}$$

we get

$$\begin{cases} \left[\langle u,v\rangle(x,y)\right]_l^+(\alpha) &= \frac{1}{15}e^{x+y}C_l^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\langle u,v\rangle(x,y)\right]_r^+(\alpha) &= \frac{1}{15}e^{x+y}C_r^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\langle u,v\rangle(x,y)\right]_l^-(\alpha) &= \frac{1}{15}e^{x+y}C_l^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\langle u,v\rangle(x,y)\right]_r^-(\alpha) &= \frac{1}{15}e^{x+y}C_r^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \end{cases} \tag{15}$$

Therefore

$$\Big[\langle u,v\rangle(x,y)\Big]_\alpha = \left[\frac{1}{15}e^{x+y}C_l^+(\alpha), \frac{1}{15}e^{x+y}C_r^+(\alpha)\right]$$
$$\Big[\langle u,v\rangle(x,y)\Big]^\alpha = \left[\frac{1}{15}e^{x+y}C_l^-(\alpha), \frac{1}{15}e^{x+y}C_r^-(\alpha)\right]$$

Now we denote

$$\left[\frac{1}{15}e^{x+y}C_l^+(\alpha), \frac{1}{15}e^{x+y}C_r^+(\alpha)\right] = M_\alpha = M_\alpha^{'}, \ \left[\frac{1}{15}e^{x+y}C_l^-(\alpha), \frac{1}{15}e^{x+y}C_r^-(\alpha)\right] = M^\alpha = M^{'\alpha}$$

It easy to see that (M_α, M^α) verify (i)–(iv) of Proposition 1 and by using the Lemma 1 we can construct the intuitionistic fuzzy solution $\langle u,v\rangle(x,y) \in IF_1$ for (16)–(18) by the following form:

$$\Big[\langle u,v\rangle(x,y)\Big]_\alpha = \left[\frac{1}{15}e^{x+y}C_l^+(\alpha), \frac{1}{15}e^{x+y}C_r^+(\alpha)\right]$$
$$\Big[\langle u,v\rangle(x,y)\Big]^\alpha = \left[\frac{1}{15}e^{x+y}C_l^-(\alpha), \frac{1}{15}e^{x+y}C_r^-(\alpha)\right]$$

for every $\alpha \in [0, 1]$.

Therefore, $\langle u,v\rangle(x,y)$ is an intuitionistic fuzzy solution which also satisfies the initial conditions (17)–(18). This solution may be written

$$\langle u,v\rangle(x,y) = \frac{1}{15}Ce^{x+y}$$

Numerical simulations are used to obtain a graphical representation of the intuitionistic fuzzy solution. The membership and nonmembership functions of triangular intuitionistic fuzzy number C = (−1, 0, 1; −2, 0, 2) in Fig. 1.

By using numerical simulations by Matlab, we present the surface of intuitionistic fuzzy solution in Fig. 2 with triangular intuitionistic fuzzy number C = (−1, 0, 1; −2, 0, 2).

Example 2 Consider the following intuitionistic fuzzy hyperbolic equation

$$\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y} = -\frac{1}{4}\big(y\langle u,v\rangle(x,y)\big)_y + \frac{1}{2}\langle u,v\rangle(x,y) + C \tag{16}$$

where $C \in IF_1$ is a triangular intuitionistic fuzzy number, $(x,y) \in [0,1]\times[0,1]$. And the integral boundary conditions are

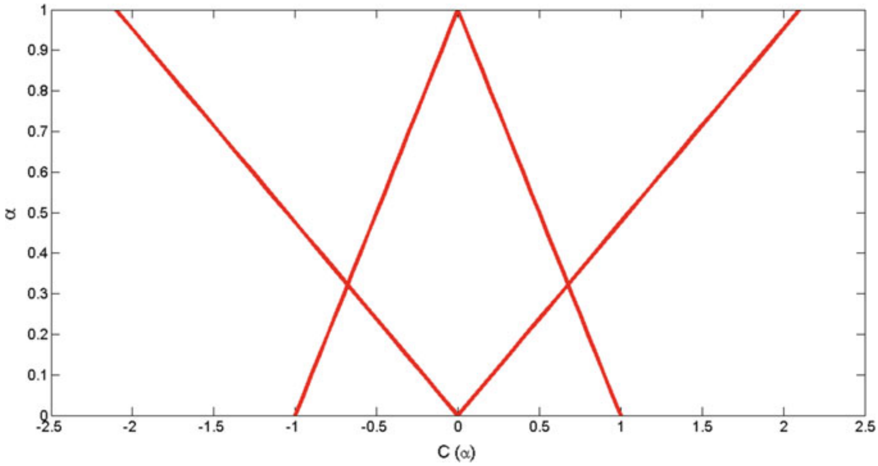

Fig. 1 C = (−1, 0, 1; −2, 0, 2)

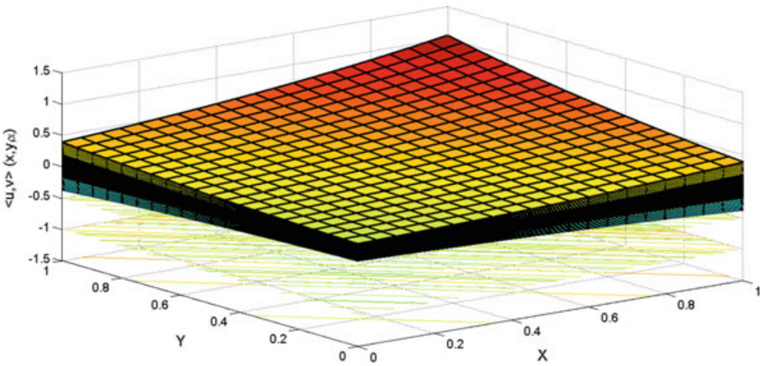

Fig. 2 The surface of intuitionistic fuzzy solution

$$\langle u, v\rangle(x, 0) + \frac{1}{10}\int_0^1 \langle u, v\rangle(x, y)dy = \tfrac{1}{20}xC, \tag{17}$$

$$\langle u, v\rangle(0, y) + \frac{1}{10}\int_0^1 \langle u, v\rangle(x, y)dx = \tfrac{1}{20}yC. \tag{18}$$

From (16), we have the function $f : [0, 1] \times [0, 1] \times IF_1 \to IF_1$ define by

$$f(x, y, \langle u, v\rangle(x, y)) = \frac{1}{2}\langle u, v\rangle(x, y) + C$$

satisfies assumptions 1. and 2. of the Theorem 5. Indeed, is easy to see that f is continuous.
Thus:

$$d_\infty\Big(f(x,y,\langle u,v\rangle(x,y)), f(x,y,\langle u',v'\rangle(x,y))\Big) \le \frac{1}{2} d_\infty\Big(\langle u,v\rangle(x,y), \langle u',v'\rangle(x,y)\Big)$$

and so f satisfy 2.

It is clear that the hypotheses are satisfied with an positive number $K = \frac{1}{2}$, $k_1 = k_2 = \frac{1}{10}$, $a = b = 1$ and $q(x,y) = -\frac{1}{4}y$, we have $\sup_{(x,y)\in[0,1]\times[0,1]} |q(x,y)| = \frac{1}{4}$. That follows all the conditions in the Theorem 5 hold. Therefore there exists an unique intuitionistic fuzzy solution of this problem.

We will find an intuitionistic fuzzy solution of this problem:

We apply the fuzzification in c, and supposed that the parametric form of corresponding intuitionistic fuzzy number C is

$$\Big[C\Big]_\alpha = \Big[C_l^+(\alpha), C_r^+(\alpha)\Big]$$
$$\Big[C\Big]^\alpha = \Big[C_l^-(\alpha), C_r^-(\alpha)\Big]$$

where is verify the conditions of Lemma 1.

Then the function $f : [0,1]\times[0,1]\times IF_1 \longrightarrow IF_1$ define by $f\big(x,y,\langle u,v\rangle(x,y)\big) = \langle u,v\rangle(x,y) + C$ is obtained by extension principle from the function $F(x,y,u) = u + c$, $(x,y,c) \in [0,1]\times[0,1]\times\mathbb{R}$

$$\Big[f\Big]_\alpha = \Big[\langle u,v\rangle(x,y) + C\Big]_\alpha = \Big[\Big[\langle u,v\rangle(x,y)\Big]_l^+(\alpha) + C_l^+(\alpha), \Big[\langle u,v\rangle(x,y)\Big]_l^+(\alpha) + C_r^+(\alpha)\Big]$$
$$\Big[f\Big]^\alpha = \Big[\langle u,v\rangle(x,y) + C\Big]^\alpha = \Big[\Big[\langle u,v\rangle(x,y)\Big]_l^+(\alpha) + C_l^-(\alpha), \Big[\langle u,v\rangle(x,y)\Big]_l^+(\alpha) + C_r^-(\alpha)\Big]$$

If

$$\Big[\langle u,v\rangle(x,y)\Big]_\alpha = \Big[\Big[\langle u,v\rangle(x,y)\Big]_l^+(\alpha), \Big[\langle u,v\rangle(x,y)\Big]_r^+(\alpha)\Big]$$
$$\Big[\langle u,v\rangle(x,y)\Big]^\alpha = \Big[\Big[\langle u,v\rangle(x,y)\Big]_l^-(\alpha), \Big[\langle u,v\rangle(x,y)\Big]_r^-(\alpha)\Big]$$

Then

$$\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_\alpha = \Big[\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_l^+(\alpha), \Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_r^+(\alpha)\Big]$$
$$\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]^\alpha = \Big[\Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_l^-(\alpha), \Big[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\Big]_r^-(\alpha)\Big]$$

Therefore, we have to solve the following partial hyperbolic differential equations:

$$\begin{cases} \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_l^+(\alpha) &= -\frac{1}{4}\left[\left(y\langle u,v\rangle(x,y)\right)_y\right]_l^+(\alpha) + \frac{1}{2}\left[\langle u,v\rangle(x,y)\right]_l^+(\alpha) + C_l^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_r^+(\alpha) &= -\frac{1}{4}\left[\left(y\langle u,v\rangle(x,y)\right)_y\right]_r^+(\alpha) + \frac{1}{2}\left[\langle u,v\rangle(x,y)\right]_r^+(\alpha) + C_r^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_l^-(\alpha) &= -\frac{1}{4}\left[\left(y\langle u,v\rangle(x,y)\right)_y\right]_l^-(\alpha) + \frac{1}{2}\left[\langle u,v\rangle(x,y)\right]_l^-(\alpha) + C_l^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\frac{\partial^2\langle u,v\rangle(x,y)}{\partial x\partial y}\right]_r^-(\alpha) &= -\frac{1}{4}\left[\left(y\langle u,v\rangle(x,y)\right)_y\right]_r^-(\alpha) + \frac{1}{2}\left[\langle u,v\rangle(x,y)\right]_r^-(\alpha) + C_r^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \end{cases} \tag{19}$$

with initial conditions

$$\begin{cases} [\langle u,v\rangle(x,0)]_l^+(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_l^+(\alpha)dy &= \frac{1}{20}xC_l^+(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(x,0)]_r^+(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_r^+(\alpha)dy &= \frac{1}{20}xC_r^+(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(x,0)]_l^-(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_l^-(\alpha)dy &= \frac{1}{20}xC_l^-(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(x,0)]_r^-(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_r^-(\alpha)dy &= \frac{1}{20}xC_r^-(\alpha), \quad x\in[0,1] \\ [\langle u,v\rangle(0,y)]_l^+(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_l^+(\alpha)dx &= \frac{1}{20}yC_l^+(\alpha), \quad y\in[0,1] \\ [\langle u,v\rangle(0,y)]_r^+(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_r^+(\alpha)dx &= \frac{1}{20}yC_r^+(\alpha), \quad y\in[0,1] \\ [\langle u,v\rangle(0,y)]_l^-(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_l^-(\alpha)dx &= \frac{1}{20}yC_l^-(\alpha), \quad y\in[0,1] \\ [\langle u,v\rangle(0,y)]_r^-(\alpha) + \frac{1}{10}\int_0^1[\langle u,v\rangle(x,y)]_r^-(\alpha)dx &= \frac{1}{20}yC_r^-(\alpha), \quad y\in[0,1] \end{cases} \tag{20}$$

we get

$$\begin{cases} \left[\langle u,v\rangle(x,y)\right]_l^+(\alpha) &= xyC_l^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\langle u,v\rangle(x,y)\right]_r^+(\alpha) &= xyC_r^+(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\langle u,v\rangle(x,y)\right]_l^-(\alpha) &= xyC_l^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \\ \left[\langle u,v\rangle(x,y)\right]_r^-(\alpha) &= xyC_r^-(\alpha), \quad (x,y)\in[0,1]\times[0,1] \end{cases} \tag{21}$$

Therefore

$$\Big[\langle u, v\rangle(x, y)\Big]_\alpha = \Big[xyC_l^+(\alpha), xyC_r^+(\alpha)\Big]$$
$$\Big[\langle u, v\rangle(x, y)\Big]^\alpha = \Big[xyC_l^-(\alpha), xyC_r^-(\alpha)\Big]$$

Now we denote

$$\Big[xyC_l^+(\alpha), xyC_r^+(\alpha)\Big] = M_\alpha, \quad \Big[xyC_l^-(\alpha), xyC_r^-(\alpha)\Big] = M^\alpha$$

and

$$\Big[C_l^+(\alpha), C_r^+(\alpha)\Big] = M_\alpha^{'}, \quad \Big[C_l^-(\alpha), C_r^-(\alpha)\Big] = M^{'\alpha}$$

It easy to see that (M_α, M^α) verify (i)–(iv) of Proposition 1 and by using the Lemma 1 we can construct the intuitionistic fuzzy solution $\langle u, v\rangle(x, y) \in IF_1$ for (16)–(18) by the following form:

$$\Big[\langle u, v\rangle(x, y)\Big]_\alpha = \Big[xyC_l^+(\alpha), xyC_r^+(\alpha)\Big]$$
$$\Big[\langle u, v\rangle(x, y)\Big]^\alpha = \Big[xyC_l^-(\alpha), xyC_r^-(\alpha)\Big]$$

for every $\alpha \in [0, 1]$.

Therefore, $\langle u, v\rangle(x, y)$ is an intuitionistic fuzzy solution which also satisfies the initial conditions (17)–(18). This solution may be written

$$\langle u, v\rangle(x, y) = xyC$$

Numerical simulations are used to obtain a graphical representation of the intuitionistic fuzzy solution. The membership and nonmembership functions of triangular intuitionistic fuzzy number C = (−1, 0, 1; −0.75, 0, 0.75) in Fig. 3.

By using numerical simulations by Matlab, we present the surface of intuitionistic fuzzy solution in Fig. 4 with triangular intuitionistic fuzzy number C = (−1, 0, 1; −0.75, 0, 0.75).

6 Conclusion

In this paper, we have obtained the existence and uniqueness result for a solution to intuitionistic fuzzy partial hyperbolic differential equations using Banach fixed point theorem. Also we have given an useful procedure to solve intuitionistic fuzzy partial

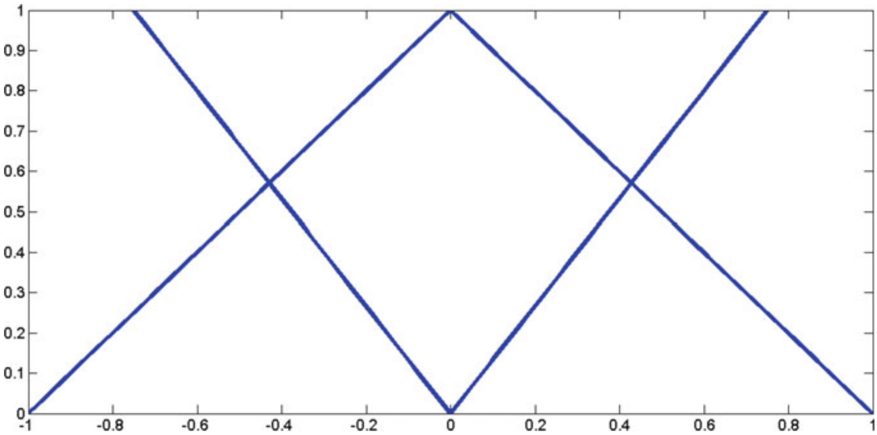

Fig. 3 C = (−1, 0, 1; −0.75, 0, 0.75)

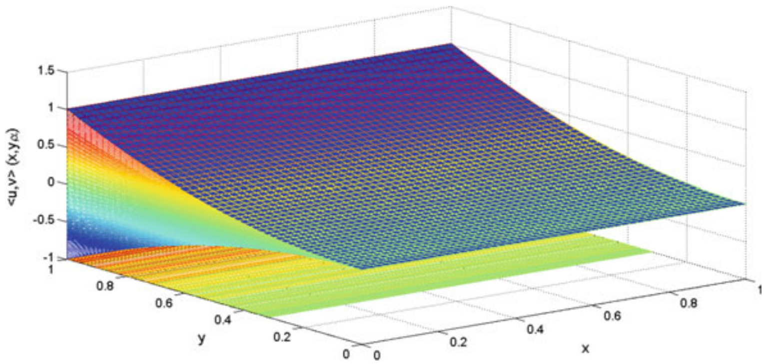

Fig. 4 The surface of intuitionistic fuzzy solution

hyperbolic differential equations. For future research we can apply these results on intuitionistic fuzzy partial functional differential equations.

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On the Periodic Ricker Equation

Jose S. Cánovas

Abstract Switching systems have recently used to model phenomena from Biology, Economy, Physics…They consist on the iteration of a finite number of maps. In this paper we consider periodic systems and analyze the dynamics of the periodic Ricker equation of period two.

Keywords Ricker equation · Topological entropy · Lyapunov exponents

1 Introduction

In this paper we consider nonautonomous difference equations

$$x_{n+1} = f_n(x_n),$$

where $f_n : X \to X$ are maps defined on a metric space (X, d). The difference equation generates a nonautonomous discrete system $(X, f_{1,\infty})$, where $f_{1,\infty} = (f_n)$ denotes the sequence of maps. Among them, periodic nonautonomous systems have been investigated by several authors recently (see e.g. [3, 9] or [4]) The interest for this kind of systems comes mainly from the existence of population models given by periodic difference equations. These models are motivated by changes in the environment (see [13]) and human activities like harvesting, fishing, plague control…(see [18]).

In a periodic difference equation there is a minimal $k \in \mathbb{N}$ such that $f_{n+k} = f_n$ for all $n \geq 1$. If $k = 1$, then we have an autonomous difference equation. Here we are interested in the case $k \geq 2$ and we will denote by $[f_1, \ldots, f_k]$ the sequence $f_{1,\infty}$. Given an initial condition $x \in X$, it is easy to see that its orbit, denoted by $Orb(x, f_{1,\infty})$, can be organized as

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$$Orb(x, [f_1, \ldots, f_k]) = Orb(x, f_k \circ \cdots \circ f_1) \cup \cdots \cup Orb((f_{k-1} \circ \cdots \circ f_1)(x), f_{k-1} \circ \cdots \circ f_1 \circ f_k)$$

and then its ω–limit set, which consists of the accumulation points of the orbit, satisfies

$$\omega(x, [f_1, \ldots, f_k]) = \omega(x, f_k \circ \cdots \circ f_1) \cup \cdots \cup \omega((f_{k-1} \circ \cdots \circ f_1)(x), f_{k-1} \circ \cdots \circ f_1 \circ f_k).$$

Therefore the dynamics of $[f_1, \ldots, f_k]$ can be related to that of the maps $f_k \circ \cdots \circ f_1, \ldots, f_{k-1} \circ \cdots \circ f_1 \circ f_k$, that is, it can be analyzed as the dynamics of k associated dynamical systems. Additionally, it is also easy to see that

$$(f_{i-1} \circ \cdots \circ f_1)(Orb(x, f_k \circ \cdots \circ f_1)) = Orb((f_{i-1} \circ \cdots \circ f_1)(x), f_{i-1} \circ \cdots \circ f_k \circ f_1 \circ \cdots \circ f_i)$$

for $i = 1, \ldots, k-1$, and therefore it is reasonable to expect that the dynamics of $f_k \circ \cdots \circ f_1$ is similar to that of $f_{i-1} \circ \cdots \circ f_k \circ f_1 \circ \cdots \circ f_i$, $i = 1, \ldots, k-1$. In general, the last assertion is not true; for instance the topological sequence entropy of two non-commuting maps can be different [5], but for many dynamical notions, say e.g. periodicity, topological entropy, attractors, this is the case. So, in many cases the dynamics of periodic systems can be derived from that of the map $f_k \circ \cdots \circ f_1$ and hence, in principle, it does not differ from the one-dimensional case. However, in practical examples, the composition operation used to form the k-iterated equations introduces some difficulties when one is interested in analyzing real models, which in general depend on one or several parameters.

So, in this paper, we will analyze the periodic Ricker model. The Ricker model with changes in the parameter has been introduced in [17]. Then several authors studied the global stability of the periodic Ricker model (see [22, 23] or [19]). Here, we are interested in the chaotic properties of the two-periodic Ricker model. Specifically, we compute the topological entropy and show the parameter region when the dynamics is chaotic.

The paper is organized as follows. We will present some basic tools which will be helpful to analyze the model and after that we will use them to obtain practical information on the model.

2 Some Useful Tools in One-Dimensional Dynamics

The aim of this section is to give some guidelines for the practical analysis of periodically switched sequences of interval maps. In our case, we are interested in piecewise monotone maps. Let $I = [a, b] \subset \mathbb{R}$ and consider $f : I \to I$ a continuous interval map for which there are $a = c_0 < c_1 < \cdots < c_i = b$ such that $f|_{[c_j, c_{j+1}]}$ is strictly monotone for $j = 0, 1, \ldots, i-1$. The points c_j, $j = 1, \ldots, i-1$, are called turning points of f. In addition (see [27]), we will assume that the map f is smooth enough, in our case C^3 and the turning points are non flat, that is, for x close to c_j, $j = 1, 2, \ldots, i-1$,

$$f(x) = \pm|\phi(x)|^{\beta_i} + f(c_j),$$

where ϕ is C^3, $\phi(c_j) = 0$ and $\beta_i > 0$.

Following [20], a metric attractor is a subset $A \subset [0, 1]$ such that $f(A) \subseteq A$, $O(A) = \{x : \omega(x, f) \subset A\}$ has positive Lebesgue measure, and there is no proper subset $A' \subsetneqq A$ with the same properties. The set $O(A)$ is called the basin of the attractor.

By [27], the regularity properties of f imply that there are three possibilities for its metric attractors, which can be of one of the following types:

(A1) A periodic orbit.
(A2) A minimal Cantor set containing at least one recurrent turning point. Recall that a set is minimal if it is the ω–limit set of all the orbits, and a point is recurrent if belongs to its ω–limit set.
(A3) A union of periodic intervals $J_1, \ldots, J_k$, such that $f^k(J_i) = J_i$ and $f^{j-i}(J_i) = J_j$, $1 \leq i < j \leq k$, and such that f^k is topologically mixing. Topologically mixing implies the existence of dense orbits on each periodic interval (under the iteration of f^k).

Moreover, if f has an attractor of type (A2) or (A3), then they must contain the orbit of a turning point, and therefore its number is bounded by the turning points. In addition, if the map has negative Schwarzian derivative (see [24, 26]), given by

$$S(f)(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2,$$

then the total number of attractors is bounded by that of turning points of f.[1]

Notice that if a finite orbit $P = \{x_0, \ldots, x_{n-1}\}$ is an attractor, then $|(f^n)'(x_0)| \leq 1$. So, if the orbit of a turning point is attracted by this periodic orbit, then the Lyapunov exponent (see [21]) at the turning point c_j, given by

$$\text{lyex}(c_j) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(f(c_j))| = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \log|f'(f^j(c_j))|,$$

must be smaller or equal than zero. Hence, the existence of positive Lyapunov exponents can be seen as an evidence of the existence of chaos.

Below, we introduce the notion of topological entropy which will be useful to define topological chaos. It was introduced in the setting of continuous maps on compact topological spaces by Adler, Konheim and McAndrew [1] and Bowen [8].[2] For continuous interval maps the definition reads as follows. Given $\varepsilon > 0$, we say

[1] Plus perhaps a fixed point at the boundary of the interval or a two-periodic orbit consisting of both endpoints of the interval.

[2] Dinaburg [15] gave simultaneously a Bowen like definition for continuous maps on a compact metric space.

that a set $E \subset I$ is (n, ε, f)–separated if for any $x, y \in E$, $x \neq y$, there exists $k \in \{0, 1, \ldots, n-1\}$ such that $|f^k(x) - f^k(y)| > \varepsilon$. Denote by $s(n, \varepsilon, f)$ the biggest cardinality of any maximal (n, ε, f)–separated set in I. Then the topological entropy of f is

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, f).$$

There is an equivalent definition using spanning sets (see [8]). For one-dimensional dynamics the topological entropy is an useful tool to check the dynamical complexity of a map because it is strongly connected with the notion of horseshoe (see [2, page 205]). We say that the map $f : I \to I$ has a k–horseshoe,[3] $k \in \mathbb{N}$, $k \geq 2$, if there are k disjoint subintervals J_i, $i = 1, \ldots, k$, such that $J_1 \cup \cdots \cup J_k \subseteq f(J_i)$, $i = 1, \ldots, k$. Then, a continuous interval maps have positive entropy if and only if there is a positive integer l such that f^l has a horseshoe.

The problem is that Bowen's definitions of topological entropy are not suitable for working with families of interval maps depending on parameters. Then, in practice, some algorithms are needed to make the computations.

For unimodal maps (see [7]) it is possible to make the computations by using the turning point. Let f be a unimodal map with maximum (turning point) at c. Let $k(f) = (k_1, k_2, k_3, \ldots)$ be its kneading sequence given by the rule

$$k_i = \begin{cases} R & \text{if } f^i(c) > c, \\ C & \text{if } f^i(c) = c, \\ L & \text{if } f^i(c) < c. \end{cases}$$

We fix that $L < C < R$. For two different unimodal maps f_1 and f_2, we fix their kneading sequences $k(f_1) = (k_n^1)$ and $k(f_2) = (k_n^2)$. We say that $k(f_1) \leq k(f_2)$ provided there is $m \in \mathbb{N}$ such that $k_i^1 = k_i^2$ for $i < m$ and either an even number of k_i^1's are equal to R and $k_m^1 < k_m^2$ or an odd number of k_i^1's are equal to R and $k_m^2 < k_m^1$. Then it is proved in [7] that if $k(f_1) \leq k(f_2)$, then $h(f_1) \leq h(f_2)$. In addition, if $k_m(f)$ denotes the first m symbols of $k(f)$, and $k_m(f_1) < k_m(f_2)$, then $h(f_1) \leq h(f_2)$.

The algorithm for computing the topological entropy is based in the fact that the tent family

$$g_k(x) = \begin{cases} kx & if\ x \in [0, 1/2], \\ -kx + k & if\ x \in [1/2, 1], \end{cases}$$

with $k \in [1, 2]$, holds that $h(g_k) = \log k$. The idea of the algorithm is to bound the topological entropy of a unimodal maps between the topological entropies of two tent maps. The algorithm is divided in four steps:

Step 1. Fix $\varepsilon > 0$ (fixed accuracy) and an integer n such that $\delta = 1/n < \varepsilon$.

Step 2. Find the least positive integer m such that $k_m(g_{1+i\delta})$, $0 \leq i \leq n$, are distinct kneading sequences.

[3] Since Smale's work (see [25]), horseshoes have been in the core of chaotic dynamics, describing what we could call random deterministic systems.

Step 3. Compute $k_m(f)$ for a fixed unimodal map f.
Step 4. Find r the largest integer such that $k_m(g_{1+r\delta}) < k_m(f)$. Hence $\log(1 + r\delta) \leq h(f) \leq \log(1 + (r + 2)\delta)$.

The algorithm is easily programmed. We usually use Mathematica, which has the advantage of computing the kneading invariants of tent maps without round off errors, improving in practice the accuracy of the method.

In the next section, we will show how to apply the above results and some extensions of them to the periodic Ricker family.

3 Periodic Ricker Family

Before studying the periodic Ricker family, we will show some facts on the autonomous case. The Ricker population model is given by the map $f(x) = xe^{r-x}$. This map is unimodal and its turning point $x_M = 1$, which is a maximum, can be obtained easily. The dynamics of f is contained in the invariant interval $[f^2(1), f(1)]$ and hence f can be considered as an interval map, whose dynamics can be given by the orbit of the turning point. In fact, its Schwarzian derivative is given by

$$S(f)(x) = -\frac{6 - 4x + x^2}{2(x-1)^2} < 0$$

for all $x \neq 1$. The bifurcation diagram, estimations of Lyapunov exponents and computation of topological entropy with accuracy 10^{-6} can be found in Fig. 1.

It is remarkable that we can have positive topological entropy, and hence topological chaos, while the attractor is a periodic orbit since the Lyapunov exponent is negative, and chaos is not physically observable since it is hidden in a set of zero Lebesgue measure.

Here we go back to the periodic models and for instance we consider a periodic sequence $[f_1, \ldots, f_k]$ where each map $f_i(x) = xe^{r_i - x}$, are given by the Ricker model. We consider the case $k = 2$, that is, the sequence of maps has periodicity two.

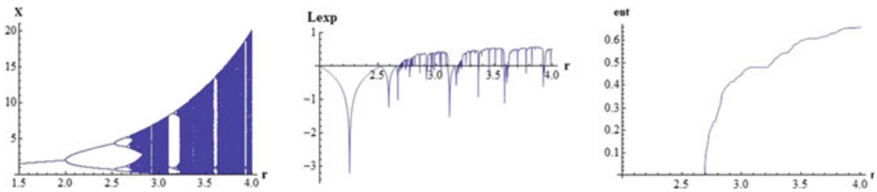

Fig. 1 The bifurcation diagram of Ricker map (left) is obtained by computing 50,000 points of the orbits of the turning points and plotting the last 500. The estimations of the Lyapunov exponent (center) is obtained with the last 25,000 points of the iteration. The topological entropy (right) is computed with accuracy 10^{-6}

In this case, the dynamics are given by the composition maps $f_2 \circ f_1$ and $f_1 \circ f_2$. From [14, 16] we know that topological entropy, which can be introduced as well for periodic switching systems, can be computed as

$$h([f_1, f_2]) = \frac{1}{2}h(f_1 \circ f_2) = \frac{1}{2}h(f_2 \circ f_1),$$

and hence we can take any of the two possible compositions. The turning points of $f_1 \circ f_2$ are computed by solving the equation

$$f_1'(f_2(x))f_2'(x) = 0,$$

obtaining the solution 1 and the equation $f_2(x) = 1$, which has a solution if $f_2(1) = e^{r_2-1} \geq 1$, which give us the condition

$$r_2 \geq 1.$$

We have to solve the equation

$$f_2(x) = xe^{r_2-x} = 1$$

numerically, for instance using the Newton method. Now, the turning point 1 is a minimum while the new turning points are maximum which have eventually the same orbit. The Schwarzian derivative

$$S(f_1 \circ f_2)(x) = (Sf_1)(f_2(x))f_2'(x)^2 + S(f_2)(x) < 0$$

and hence the multimodal map $f_1 \circ f_2$ has at most two metric attractors, characterized by the two essentially different turning points. Figure 2 shows several examples of bifurcation diagrams.

Additionally, in Fig. 3, we show the estimations of Lyapunov exponents of the orbits plotted in Fig. 2. We check that the existence of more than one attractor is given by different values of Lyapunov exponents at the turning points, as a consequence of the ergodic theorem (see e.g. [28]).

Finally, to conclude, we compute numerically the topological entropy. If $r_2 < 1$ we can use the previous algorithm because the map is unimodal, but if $r_2 \geq 1$ the map is trimodal and we proceed as in [11], by adapting an algorithm by Block and Keesling [6] for bimodal maps. We refer the reader to the references because the explanation of the algorithms is large. In Fig. 4, we show the values of topological entropy fixing r_1 to be the values 2.5 and 2, respectively. In Fig. 5 we show that the entropy can be zero for r_1 and r_2, the smallest value for which the topological entropy is positive is $\widetilde{r} = 2.6923688545$ with accuracy 10^{-11}, while the entropy of the composition of the maps can be positive, showing the existence of the so–called Parrondo's paradox [12], obtaining a similar result of that of logistic family [10].

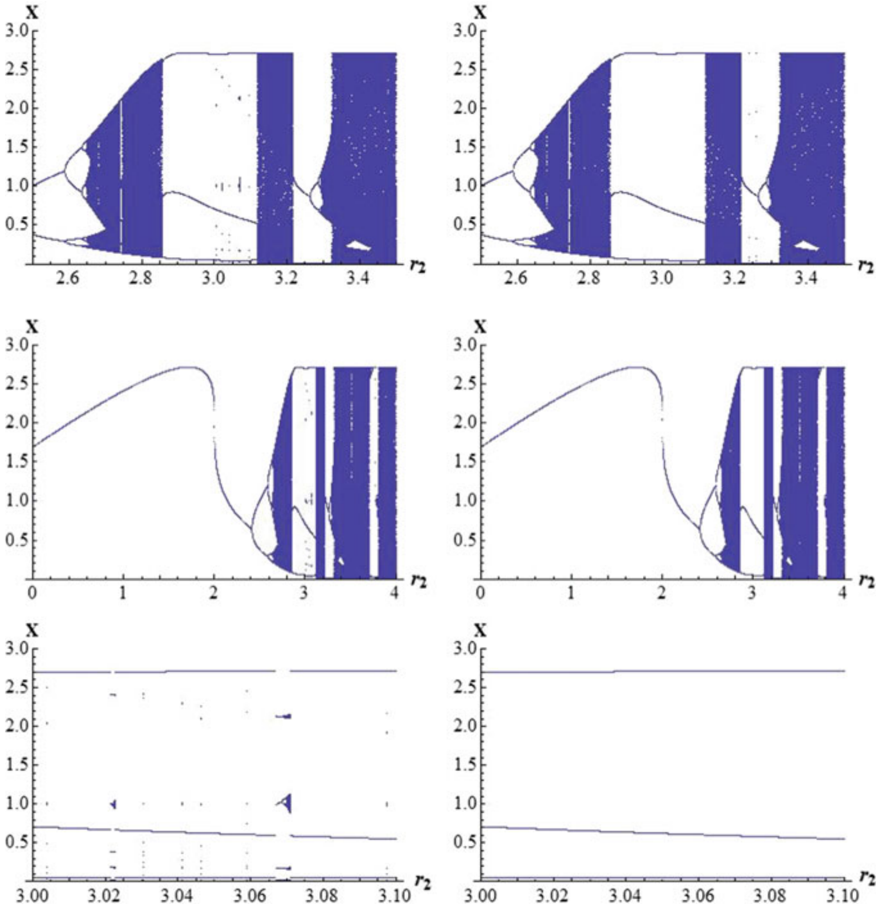

Fig. 2 We fix $r_1 = 2.5$ (top) and 2 (center) and show the bifurcation diagram for the turning point 1 on the left while on the right we take the other turning point, which is a maximum. For $r_1 = 2$ we make a zoom for values of r_2 in a neighborhood of 3.5. They are obtained computing 50,000 points and plotting the last 500

From our computations it seems that paradox "complex plus complex simple" is not possible for topological entropy.

As a summary, we have seen that increasing the turning points also increases the difficulties of analyzing the system. On the one hand, we have to make some computations of turning points numerically. On the other hand, now the map depends on two parameters, which increases the computation time and the way the results are presented. The difficulties increase when we consider longer periodic sequences of maps because both the computations and their graphical representations are more complicated.

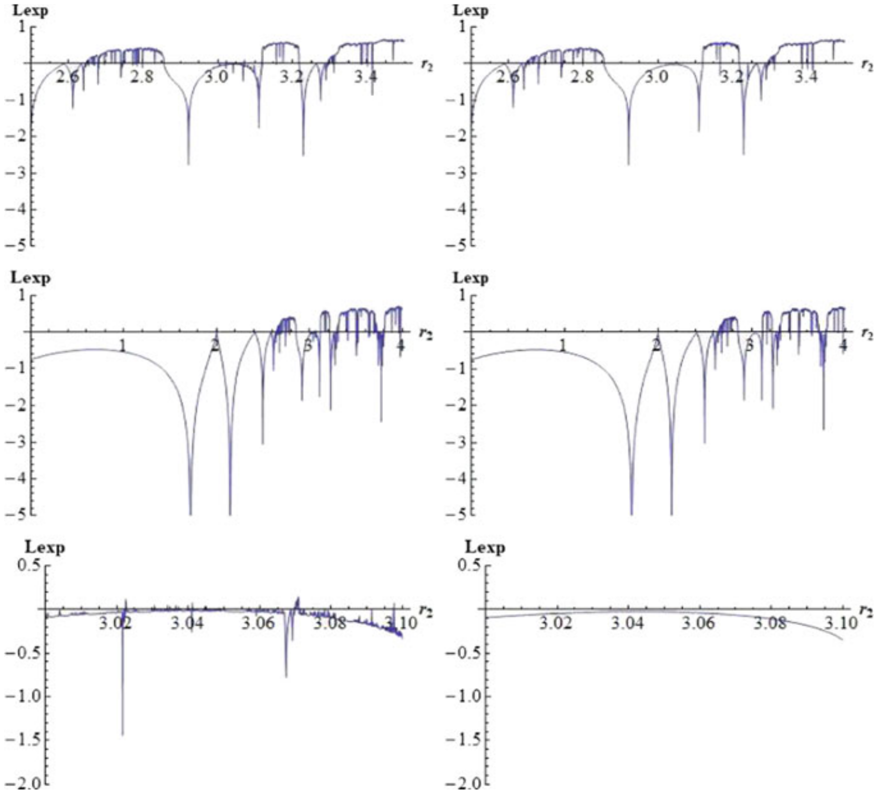

Fig. 3 Estimations of Lyapunov exponents related to Fig. 2, that is $r_1 = 2.5$ at the top and $r_1 = 2$ in the middle making a zoom down. We take the last 25,000 points of the orbits of turning points

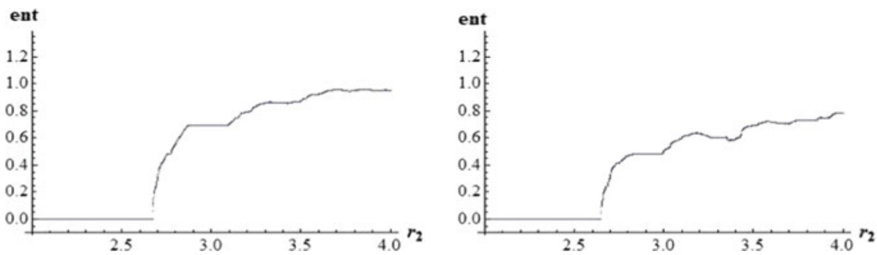

Fig. 4 Computations of topological entropy related to Fig. 2 with accuracy 10^{-6} for $r_1 = 2.5$ (left) and 2 (right)

Finally, it is showed that the paradox "simple plus simple complex" is possible for Ricker family when two maps are considered. However, it follows from our computations that we cannot observe if the paradox "complex plus complex simple" may happen. The same happened in [10] for the logistic family. So, it is an interesting

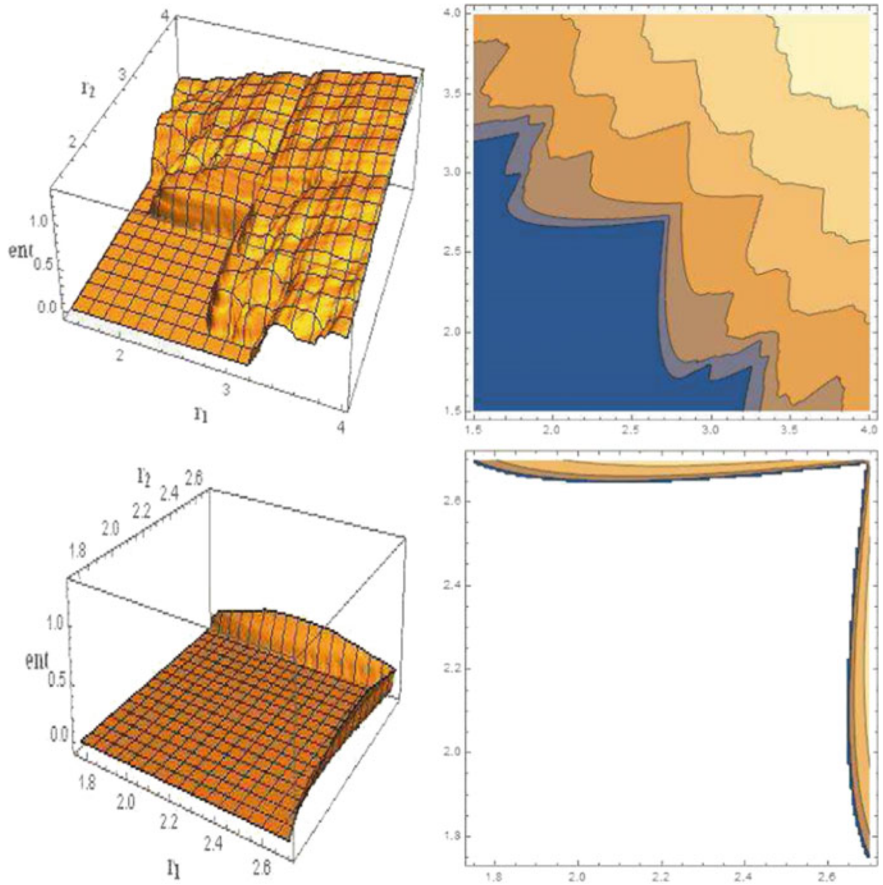

Fig. 5 Topological entropy (right) and level curves (right) when $r_1, r_2 \in [2, 4]$ (top) with accuracy 10^{-6}. Down, it is shown for values $r_1, r_2 \in [2, 2.7]$ with the same accuracy. For $r_1, r_2 \in [2, 2.7]$, the level curves shows the existence of positive entropy for values smaller than $\widetilde{r}$

question to check whether simplicity can be generated by combining two interval maps with positive topological entropy.

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The Stability of Vortices in Gas on the l-Plane: The Influence of Centrifugal Force

Olga Rozanova and Marko Turzynsky

Abstract We show that a small correction due to centrifugal force usually neglected in the l-plane model of atmosphere drastically influences on the stability of vortices. Namely, in the presence of the Coriolis force only there exists a wide range of parameter ensuring nonlinear stability of a vortex with uniform deformation. Taking into account the centrifugal force results in a disappearance of stable vortices in the above mentioned class of motions. We also prove that for the heat ratio $\gamma = 2$, corresponding to the one-atomic gas, the system of equations, describing the gas on the l-plane with the correction due to centrifugal force can be integrated in a special case.

Keywords Compressible 2D medium · Vortex motion · Stability · Rotation · Centrifugal force

1 Reduction of 3D Primitive Equations of the Atmosphere Dynamics to the l-Plane Model

We consider the system of gas dynamics, widely used for a description of the dynamics of atmosphere of a rotating planet, such as the Earth. We do not dwell on the processes of heat and moisture transfer and study the behavior of atmosphere in the frame of the model of ideal polytropic inviscid gas. The model consists of the following equations [15]

$$\rho(\partial_t \bar{U} + (\bar{U}, \nabla)\bar{U} + 2\,\bar{\omega} \times \bar{U} + \bar{\omega} \times (\bar{\omega} \times \bar{r}) + g\bar{e}_3) = -\nabla p, \tag{1}$$

$$\partial_t \rho + \operatorname{div}(\rho \bar{U}) = 0, \tag{2}$$

$$\partial_t S + (\bar{U}, \nabla S) = 0. \tag{3}$$

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for $\rho \geq 0$, $p \geq 0$, $\bar{U} = (U_1, U_2, U_3)$, S (density, pressure, velocity and entropy). The functions depend on time t and on point $x \in \mathbb{R}^3$, $\bar{e}_3 = (0, 0, 1)$ is the "upward" unit vector, g is the acceleration due to gravity (points in $-e_3$ direction to the center of planet), $\bar{\omega}$ is the velocity of angular rotation, $\bar{r}$ is the radius-vector of point in the coordinate system, associated with the center of planet. The state equation is

$$p = \rho^\gamma e^S, \qquad \gamma > 1. \tag{4}$$

System (1)–(3) describes a layer of air over a spherical planet, therefore for the processes of planetary scale one should use spherical coordinates. Nevertheless, for the process of so called middle scale it is possible to fix a point on the Earth surface and consider a coordinate system with the origin in these point. Thus, axes x_1 and x_2 are directed in the tangent plane along parallel and meridian, respectively, and the axis x_3 is directed out of the center of planet.

Let us compute the term $\bar{\omega} \times (\bar{\omega} \times \bar{r})$ for the point lying on the plane $x_3 = 0$. Thus, the coordinates of the radius-vector $\bar{r}$ are (x_1, x_2, R), where R is the radius of planet. If we denote θ the latitude of the origin of the rotating coordinate system and take into account that $\bar{\omega} = (0, \omega \cos\theta, \omega \sin\theta)$, $\omega = |\bar{\omega}|$, then

$$\bar{\omega} \times (\bar{\omega} \times \bar{r}) = R\omega^2(0, \sin\theta\, \cos\theta, -\cos^2\theta) - \omega^2(x_1, x_2 \sin^2\theta, x_2 \sin\theta \cos\theta).$$

The first vector in this sum is constant, the second one depends on the position of point on the plane $x_3 = 0$.

Since $\bar{\omega} \times (\bar{\omega} \times \bar{r}) = -\frac{1}{2}\nabla\omega^2|\bar{r}|^2$, traditionally they include this term in the geopotential $\Phi = gx_3 - \frac{1}{2}\omega^2|r|^2$ and turn the axis (x_2, x_3) such that the constant vector

$$R\omega^2(0, \sin\theta\, \cos\theta, -\cos^2\theta) - (0, 0, g)$$

in the new coordinate system $(x_1, \tilde{x}_2, \tilde{x}_3)$ is directed strictly vertically. In fact, we should turn the coordinate system by the angle ξ such that

$$\cos\xi(-R\omega^2 \sin\theta \cos\theta) - \sin\xi(-g + R\omega^2 \cos^2\theta) = 0,$$

therefore

$$\tan\xi = \frac{R\omega^2 \sin\theta \cos\theta}{g - R\omega^2 \cos^2\theta}.$$

We can see that $\xi \ll 1$, since $\frac{R\omega^2}{g} \ll 1$.

Thus, we change the spherical surface of planet to the geopotential surface and regard the surfaces of constant Φ as being true spheres. The horizontal component of apparent gravity is then identically zero (see the details in [15], Sect. 2.2.1).

If we are exactly at the origin, the term $\omega^2(x_1, x_2 \sin^2\theta, x_2 \sin\theta\cos\theta)$ is zero. However, in the general case

$$k = \cos\xi \sin^2\theta + \sin\xi \sin\theta\cos\theta \neq 0, \quad k \approx \sin^2\theta,$$

and this means that the centrifugal force still has a projection on the new horizontal coordinates $(x_1, \tilde{x}_2)$ as $(-\omega^2 x_1, -k\omega^2\tilde{x}_2)$. Of course, one can include these terms in the gradient of some new pressure, however this would change the state equation.

Let us repeat the standard procedure of deriving the l-plane model for relatively thin atmosphere, ignoring vertical processes (e.g. [15], Sect. 2.3.1). They define a plane tangent to the surface of the earth at a fixed latitude θ, and then use a Cartesian coordinate system (x_1, x_2) to describe motion on that plane (we will write x_2 instead of our previous $\tilde{x}_2$). For small excursions on the plane, they fix the projection of $\bar{\omega}$ in the direction of the local vertical, ignore the components of $\bar{\omega}$ in the horizontal direction, set the component of velocity $U_3 = 0$, and make the hydrostatic approximation, which means that the gravitational term is assumed to be balanced by the pressure gradient term. Thus the 3D vectorial Eq. (1) results in the following 2D vectorial equation:

$$\rho(\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathcal{L}\mathbf{u} - \delta\omega^2\mathcal{D}\mathbf{x}) + \nabla p = 0, \tag{5}$$

where $\mathbf{u}(t,x) = (U_1, U_2)$, $x \in \mathbb{R}^2$, $\mathbf{x}$ is a radius-vector of point, $\mathcal{L} = lL$, $L = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $l = 2\omega\sin\theta > 0$ is the Coriolis parameter, $\mathcal{D} = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}$, $k = \sin^2\theta$. Equations (2)–(4) are not changed formally, however, the density, pressure and entropy depend now only on $x \in \mathbb{R}^2$ and t. In the isentropic case the Eq. (3) is valid identically. The value of δ is zero if we do not consider the centrifugal term or one, otherwise.

The reduction to the 2D system for a thin layer of atmosphere can be obtained more strictly by averaging over the height, see [7]. The case $\gamma = 2$ corresponds to the 2D one-atomic gas ($\gamma = 2 = 1 + \frac{1}{n}$, where n is the dimension of space). Also it arises in the shallow water equations.

2 Motion with Uniform Deformation: Steady States and Their Stability

As is well known, the gas dynamics system has a special class of solutions, characterized be a linear profile of velocity, that is

$$\mathbf{u}(t,\mathbf{x}) = Q\mathbf{x}, \quad Q = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix}. \tag{6}$$

In [11] the system (5), (2), (3) was considered for $\delta = 0$ and it was shown that the components of matrix Q can be found as a part of solution of a certain quadraticallynonlinear matrix system of equations. For the case of arbitrary δ this system has the form

$$\dot{R} + RQ + Q^T R + (\gamma - 1)\mathbf{tr} QR = 0, \tag{7}$$
$$\dot{Q} + Q^2 + lLQ + 2c_0 R - \delta\omega^2 \mathcal{D} = 0, \tag{8}$$

where the symmetric matrix R describes distribution of density and pressure, $c_0 =$ const.

2.1 $\delta = 0$

System (7), (9) has an one-parametric family of nontrivial equilibria, depending on the parameter b_*:

$$a = d = 0,\ b = -c = b_*,\ A = C = \frac{b_*(b_* - l)}{2c_0},\ B = 0, \tag{9}$$

and a two-parametric family with parameters a_* and c_*

$$a = a_*,\ c = c_*,\ b = -\frac{a_*^2}{c_*},\ d = -a_*,\ A = \frac{lc_*}{2c_0},\ B = -\frac{la_*}{c_0},\ C = \frac{la_*^2}{2c_0 c_*}, \tag{10}$$

for $c_* = 0$ the latter equilibrium should be completed by

$$a = c = d = 0,\ b = b_*,\ A = B = 0,\ C = -\frac{1}{2c_0} l b_*, \tag{11}$$

with a parameter b_*.

Equilibrium (9) corresponds to the case of axisymmetric vortex. For $b_* \in (0, l)$ the motion is anticyclonic with a higher pressure in the center of vortex (anticyclonic high), for $b_* < 0$ the motion is cyclonic with a lower pressure in the center of vortex (cyclonic low), and at last, for $b_* > l$ the motion is anticyclonic with a higher pressure in the center (anticyclonic low).

Equilibria (10), (11) correspond to a shear flow along a straight line, the ratio $\frac{a_*}{c_*}$ gives the inclination of this line with respect to the coordinate axis.

Let us give a review of the results about stability of the equilibria.

Theorem 1 *Equilibrium* (9) *is unstable for any* $\gamma > 1$ *if*

$$\frac{b_*}{l} \in \mathbb{R} \backslash \Sigma,\ \textit{where}\ \Sigma = [K_-, K_+],\ K_\pm = \frac{1 \pm \sqrt{2}}{2}. \tag{12}$$

Moreover, for $\gamma > 4$ *the domain* Σ *can be replaced by* $\sigma = [k_-, k_+]$, $k_\pm = \frac{1 \pm \sqrt{\gamma/(\gamma-2)}}{2}$. *Thus,* $\sigma = [k_-, k_+] \to [0, 1]$ *as* $\gamma \to \infty$ *and the domain of instability enlarges.*

Proof Eigenvalues of matrix of linearization in the point (9) for the system (7), (8), written as 7 equations are the following:

$$\lambda_{1,2,3,4} = \pm\sqrt{2}\sqrt{-l\left(b_* + \frac{l}{4}\right) \pm \sqrt{\left(b_* + \frac{l}{2}\right)^2 \left(\frac{l^2}{4} + b_* l - (b_*)^2\right)}},$$

$$\lambda_{5,6} = \pm\sqrt{-(2(2-\gamma)b_*(b_* - l) + l^2)}, \qquad \lambda_7 = 0.$$

For $b_* < \frac{1-\sqrt{2}}{2}\,l$ and $b_* > \frac{1+\sqrt{2}}{2}\,l > l$ we get $\lambda_{3,4,5,6} = \pm\alpha \pm i\beta$, $\alpha \neq 0, \beta \neq 0$, therefore there exists an eigenvalue with a positive real part. Thus, the Lyapunov theorem implies instability of the equilibrium.

For $\gamma \in (1, 2]$ (this case was considered in [11]) the value $-(2(2-\gamma)b_*(b_* - l) + l^2) < 0$, then $\Re(\lambda_{1,2}) = 0$. For $\gamma > 2$ we have $-(2(2-\gamma)b_*(b_* - l) + l^2) > 0$ (the eigenvalues $\lambda_{1,2}$ are real, and one of them is positive) for

$$\frac{b_*}{l} < k_- \quad \text{or} \quad \frac{b_*}{l} > k_+. \tag{13}$$

It is easy to check that for $\gamma \in (2, 4)$ condition (12) is more restrictive than (13), whereas for $\gamma > 4$ condition (13) is more restrictive. Therefore the proof is over. □

One can readily check that if $b_* \in \left[\frac{1-\sqrt{2}}{2}\,l,\ \frac{1+\sqrt{2}}{2}\,l\right]$, then all eigenvalues have zero real part. Thus, for this range of parameters the linear analysis does not give an answer about stability of the equilibrium. Nevertheless, the following theorem holds.

Theorem 2 ([11]) *Equilibrium* (9) *is stable in the Lyapunov sense for any* $\gamma > 1$ *if*

$$\frac{b_*}{l} \in (0, 1).$$

The proof is based on the explicit construction of the Lyapunov function (see the details in [11]).

Very recently the following theorem was proved.

Theorem 3 ([14]) *For* $\gamma = 2$ *equilibrium* (9) *is stable in the Lyapunov sense for all*

$$b_* \in \left(\frac{1-\sqrt{2}}{2}\,l,\ \frac{1+\sqrt{2}}{2}\,l\right), \quad b_* \neq 0, \quad b_* \neq l,$$

and unstable otherwise.

The proof is quite technical and based on the use of the Lagrangian coordinates.

We do not dwell here on the study of stability of equilibria (10), (11), it is not trivial. The linear analysis implies that the matrix of linearization has a couple of eigenvalues for $\pm\frac{1}{c_*}\sqrt{c_* l((a_*^2+c_*^2)\gamma - c_* l)}$ for (10) and $\pm\sqrt{-b_* l(\gamma+1)}$ for (11), whereas the other eigenvalues have the zero real part. Thus, one can easily find the range of parameters guaranteeing instability. The proof of instability in the rest of cases is much more delicate.

2.2 $\delta = 1$, $s = 1$

To reduce computations and obtain analytical result, we dwell on a particular case $s = 1$. We have the following real equilibria:

$$a = d = 0,\ b = -c = b_*,\ A = C = \frac{(\omega - b_*)^2}{2c_0},\ B = 0, \tag{14}$$

$$a = a_*,\ c = c_*,\ b = \frac{\omega^2 - a_*^2}{c_*},\ d = -a_*,\ A = \frac{c_*\omega}{c_0},\ B = -\frac{2a_*\omega}{c_0},\ C = \frac{\omega(a_*^2 - \omega^2)}{c_* c_0}, \tag{15}$$

for $c_* = 0$ the later equilibrium should be completed by

$$a = \pm\omega,\ b = b_*,\ d = \mp\omega,\ A = 0,\ B = \mp\frac{2\omega^2}{c_0},\ C = -\frac{\omega b_*}{c_0}. \tag{16}$$

Thus, (14) and (15), (16) correspond to (9) and (10), (11) for $\delta = 0$.

One can easily see that the "anticyclonic high" domain of parameters disappears from (14). Thus, the situation is similar to the case $\delta = 0$ with $l = 0$ (the plane does not rotate).

Equilibria (15), (16) correspond to the saddle point of pressure.

Theorem 4 *Equilibrium* (14) *is unstable for all values of parameter* b_*.

Proof The eigenvalues of matrix of linearization for the system (7), (8) at the point (14) are the following:

$$\lambda_1 = 0,\ \lambda_{2,3} = \pm\sqrt{2(\gamma-2)}(\omega - b_*),\ \lambda_{4,5,6,7} = \pm\sqrt{-4b_*\omega \pm 2(b_*^2 - \omega^2)i}$$

Therefore for $b_* \neq \omega$ there is an eigenvalue with a positive real part. The only possibility to have the zero real part is $b_* = \omega$. Let us prove that this case is also unstable. First of all we notice that if $b_* = \omega$, than $A_* = C_* = 0$. If we fix $A = B = C = 0$, we will get a subset of solutions of the full system (7), (8), corresponding to $R = 0$, that is

$$\dot{Q} + Q^2 + 2\omega L Q - \omega^2 \mathcal{E} = 0,$$

where $\mathcal{E}$ is the identity matrix. We are going to find a small perturbation of equilibrium of this matrix Riccati equation which results in a finite time blow up. For this raison we first make a change $Q = \tilde{Q} - \omega L$, since $Q = -\omega L$ is the steady state, corresponding to the equilibrium point. For $\tilde{Q}$ we get the matrix Bernoulli equation

$$\dot{\tilde{Q}} = -\tilde{Q}^2 + \omega \tilde{Q} L - \omega L \tilde{Q}, \tag{17}$$

where the equilibrium is shifted to the zero point. Equation (17) can be reduced to the following linear equation for $U = \tilde{Q}^{-1}$ (see, e.g [5]):

$$\dot{U} = \omega(UL - LU) + E. \tag{18}$$

The initial condition $U(0)$ can be derived from $\tilde{Q}(0)$ provided $\det \tilde{Q}(0) \neq 0$. Equation (18) can be explicitly solved. Let us choose initial perturbations of the equilibrium $\tilde{Q} = 0$ as $\tilde{Q}_{11} = \tilde{Q}_{22} = 0$, $\tilde{Q}_{12} = \tilde{Q}_{21} = \varepsilon$. The components of $U(t)$ are polynomials with respect to t up to the second order (we skip the standard computations), nevertheless, the components $\tilde{Q}(t) = U^{-1}(t)$ contain $\det U(t)$ in the denominator. For our specific choice of perturbation $\det U(t) = \left(\frac{2\omega}{\varepsilon} + 1\right)^2 t - \frac{1}{\varepsilon^2}$ and the components of $\tilde{Q}(t)$ tend to infinity as $t \to (2\omega + \varepsilon)^{-2}$. Thus, the theorem is proved. □

For equilibria (15) and (16) the spectra of matrices of linearization contain three zeros and the quadriple

$$\pm \frac{1}{c_*} \sqrt{c_* w (\gamma z_* \pm \sqrt{z_*^2 \gamma^2 + 32(\gamma - 2) c_* w^2})}, \qquad z_* = a_*^2 + c_*^2 - w^2$$

(for (15)) and

$$\pm \sqrt{-b_* w \gamma \pm \omega \sqrt{b_*^2 \gamma^2 + 32(\gamma - 2) w^2}),$$

(for (16)). Thus, one can find conditions on the parameters ensuring instability of the equilibria. Let us notice that these conditions strongly depend on the heat ratio γ. For example, (16) is always unstable for $\gamma > 2$. The stability for the linearly neutral cases is not studied, nevertheless, it is no reason to expect stability for the saddle points of the pressure.

2.3 $\delta = 1$, $s = \sin\theta$

In the general case the equilibria of system (7), (8) can be found as roots of algebraic equations of a higher order. We do not list them all, nevertheless let us mention the following nontrivial family of equilibria depending only on one parameter c_*:

$$a = 0,\ b = z, c = c_*,\ d = 0,$$
$$A = \frac{1}{2c_0}(2c_* ws - zc_* + w^2),\ B = 0,\ C = \frac{1}{2c_0}(w^2 s^2 - 2wsz - zc_*),$$

where $z = z(c_*, w)$ is a root of the quadratic equation

$$c_* z^2 + (c_*^2 - w^2)z - c_* s^2 w^2 = 0.$$

This equation always has two real roots, therefore the properties of the equilibrium depend on the choice of the root. One equilibrium is a non-axisymmetric vortex ($z = -c_*$), another one is a saddle point ($z = \frac{w^2}{c_*}$). If we set $s = 1$, we get (14) and (15) for $a_* = 0$.

For this case analytical computations are very cumbersome, nevertheless one can still find the eigenvalues of the matrix of linearization and see that the spectrum always contains a quadriple $\pm\alpha \pm \beta i$ with a nonzero α, therefore, the equilibrium is unstable.

3 Study of System (7), (8) in the Lagrangian Coordinates

In the present section we use the Lagrangian coordinates to show that for $\gamma = 2$ and $s = 1$ the solution can be found in terms of elliptic integrals.

Let $F = (F_{ik})_{i,k=1\ldots2}$ be the matrix of transfer from the Lagrangian coordinates $\mathbf{w}$ to the Eulerian ones $\mathbf{x}$, i.e. $\mathbf{x}(t, \mathbf{w}) = F(t)\mathbf{w}$. This property is called the uniform deformation. Since $\mathbf{w} = F^{-1}\mathbf{x}$, then $\mathbf{u} = \dot{\mathbf{x}} = \dot{F}\mathbf{w} = \dot{F}F^{-1}\mathbf{x} = Q\mathbf{x}$. Thus, the condition of uniform deformation is equivalent to condition (6).

System (7), (8) can be written in the Lagrangian coordinates as

$$\ddot{F}_{ik} + \frac{\partial U}{\partial F_{ik}} + \sum_j \mathcal{L}_{ij}\dot{F}_{jk} - \delta\omega^2 \mathcal{D} F_{ik} = 0. \tag{19}$$

Here U is the internal energy of gas, $U = U_0(|\det F|)^{-\gamma} \det F$, $U_0 = \text{const} > 0$, F is nondegenerate, the derivatives are taken with respect to time.

For $l = \delta = 0$ system (19) was considered extensively in the literature both in 3D and 2D cases. It models an expansion of a gas ellipsoid (or elliptic cylinder in 2D case) into vacuum. The problem originates from the paper of L. V. Ovsyannikov [8], the state of art and respective references can be found in [3]. In [1, 4] the first integrals of the system (19) were found for $l = \delta = 0$. The integrals are the following:

$$\frac{1}{2}\sum_{i,k}(\dot{F}_{ik})^2 + U = E,$$

$$F_{11}\dot{F}_{21} + F_{12}\dot{F}_{22} - F_{21}\dot{F}_{11} - F_{22}\dot{F}_{12} = J,$$

$$F_{11}\dot F_{12} + F_{21}\dot F_{22} - F_{12}\dot F_{11} - F_{22}\dot F_{21} = K.$$

Moreover, for the case of one-atomic gas (for $x \in \mathbb{R}^2$ it corresponds to $\gamma = 2$) in [1] it was shown the existence of a supplemental first integral

$$G = \sum_{i,k} F_{ik}^2 = 2Et^2 + k_1 t + k_0, \tag{20}$$

where k_0 and k_1 are constants. It gives a possibility to find exact solution to system (19) in the 2D case. In [2] an oscillating regime of (19) were established.

3.1 First Integrals

Theorem 5 *System* (19) *has the integral of energy*

$$\tilde E = E - \frac{1}{2}\omega^2(F_{11}^2 + F_{12}^2 + k(F_{21}^2 + F_{22}^2)). \tag{21}$$

For $s = 1$ the system (19) *has three first integrals:*

$$\tilde E = E - \frac{1}{2}\omega^2 G, \quad A = J + \omega G, \quad B = K - 2\omega \det F. \tag{22}$$

For $\gamma = 2$, there exists a supplemental first integral

$$G = \sum_{i,k} F_{ik}^2 = 2(\tilde E + \omega A)t^2 + k_1 t + k_0, \tag{23}$$

where k_0 and k_1 are constants.

Proof To prove (21) we multiply every equation of system (19) by $\dot F_{ik}$ and add the results. We get

$$(\ddot F_{ik}\dot F + \tfrac{\partial U}{\partial F} + L\dot F + \omega^2 \mathcal{D} F)\dot F = \tilde E' = 0.$$

To prove the existence of integrals A and B from (22) with $k = 1$ we check that

$$-F_{11}\frac{\partial U}{\partial F_{21}} - F_{12}\frac{\partial U}{\partial F_{22}} + F_{21}\frac{\partial U}{\partial F_{11}} + F_{22}\frac{\partial U}{\partial F_{12}} = 0.$$

Indeed, we denote $\eta = \det F$ and get

$$-F_{11}\frac{\partial U}{\partial F_{21}} - F_{12}\frac{\partial U}{\partial F_{22}} + F_{21}\frac{\partial U}{\partial F_{11}} + F_{22}\frac{\partial U}{\partial F_{12}} =$$

$$(F_{11}F_{12} - F_{12}F_{11} + F_{21}F_{22} - F_{22}F_{21})U'_\eta = 0.$$

Then we differentiate J and take into account (19) to obtain

$$\dot{J} = F_{11}\ddot{F}_{21} + F_{12}\ddot{F}_{22} - \ddot{F}_{11}F_{21} - \ddot{F}_{12}F_{22} =$$

$$F_{11}\left(-l\dot{F}_{11} - \frac{\partial U}{\partial F_{21}} + \omega^2 F_{21}\right) + F_{12}\left(-l\dot{F}_{12} - \frac{\partial U}{\partial F_{22}} + \omega^2 F_{22}\right) -$$

$$F_{21}\left(l\dot{F}_{21} - \frac{\partial U}{\partial F_{11}} + \omega^2 F_{11}\right) - F_{22}\left(l\dot{F}_{22} - \frac{\partial U}{\partial F_{22}} + \omega^2 F_{12}\right) = -l\sum_{i,j} F_{ij}\dot{F}_{ij} = -\frac{l}{2}\dot{G}.$$

It implies $J + \frac{l}{2}G = \text{const}$ and (22), if we take into account that $l = 2\omega$.

The proof of existence of the integral B is analogous, since

$$-F_{11}\frac{\partial U}{\partial F_{12}} - F_{21}\frac{\partial U}{\partial F_{22}} + F_{12}\frac{\partial U}{\partial F_{11}} + F_{22}\frac{\partial U}{\partial F_{21}} = 0. \quad (24)$$

To prove (23) we notice that for $\gamma = 2$ by the Euler theorem $\sum\limits_{i,j} F_{ij}\frac{\partial U}{\partial F_{ij}} = 2U$, since $U = U_0(\det F)^{-1}$ is a homogeneous function of order -2.

Thus,

$$\frac{\ddot{G}}{2} = \sum_{i,j}(\dot{F}_{ij}^2 + F_{ij}\ddot{F}_{ij}) =$$

$$= \sum_{i,j}\dot{F}_{ij}^2 + F_{11}\left(l\dot{F}_{21} - \frac{\partial U}{\partial F_{11}} + \omega^2 F_{11}\right) + F_{12}\left(l\dot{F}_{22} - \frac{\partial U}{\partial F_{12}} + \omega^2 F_{12}\right) +$$

$$F_{21}\left(-l\dot{F}_{11} - \frac{\partial U}{\partial F_{21}} + \omega^2 F_{21}\right) + F_{22}\left(-l\dot{F}_{12} - \frac{\partial U}{\partial F_{22}} + \omega^2 F_{22}\right) =$$

$$(2E - 2U) + l(F_{11}\dot{F}_{21} + F_{12}\dot{F}_{22} - F_{21}\dot{F}_{11} - F_{22}\dot{F}_{12}) - \sum_{i,j} F_{ij}\frac{\partial U}{\partial F_{ij}} + \omega^2 G =$$

$$2E + lJ - w^2 G = 2\tilde{E} + lJ + 2\omega^2 G = 2(\tilde{E} + \omega A) = \text{const}.$$

It implies (23). □

Remark 1 We can see that (23) and (20) are similar, they are polynomials of order 2 with respect to t. For $\delta = 0$ the situation is absolutely different. Namely, as is shown in [14],

$$G = M\sin(lt + \phi_0) + \frac{4\tilde{E} + 2lA}{l^2},$$

with constant M and ϕ_0, it is periodic. One can see that the in the case $\delta = 1, k = 1$ the influence of rotation is neutralized.

3.2 Representation of Solution

Theorem 6 *For $\delta = 0$ and $k = 1$ the system (19) can be reduced to one ODE*

$$u'(t) = \pm\frac{1}{s^2(t)}\sqrt{f(u)} \tag{25}$$

and integrated. Here $f(u) = D - \frac{4U_0}{\sin 2u} - \frac{A^2+B^2+2AB\sin 2u}{\cos^2 2u}$, $D = \frac{8(\tilde{E}+\omega A)k_0 - k_1^2}{4} =$ const, $s^2(t) = G(t)$.

Proof The change of variables

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} \cos v & -\sin v \\ \sin v & \cos v \end{pmatrix} \begin{pmatrix} s\cos u & 0 \\ 0 & s\sin u \end{pmatrix} \begin{pmatrix} \cos w & -\sin w \\ \sin w & \cos w \end{pmatrix}$$

reduces the system of first integrals (22) to

$$\frac{s^2}{2}((u')^2 + \omega^2 + (v')^2 + 2\sin(2u)v'w' + (w')^2) + \frac{(s')^2}{2} + \frac{2U_0}{s^2\sin 2u} = \tilde{E}, \tag{26}$$

$$\left(v' + \frac{l}{2}\right) + \sin(2u)w' = \frac{A}{s^2}, \qquad \sin(2u)\left(v' + \frac{l}{2}\right) + w' = -\frac{B}{s^2}. \tag{27}$$

Further, $G = \sum_{i,k} F_{ik}^2 = s^2(t)$, therefore we can use the expression (23).

From (27) we obtain

$$v' = \frac{1}{s^2} \cdot \frac{A + B\sin 2u}{\cos^2 2u} - \frac{l}{2},$$

$$w' = -\frac{1}{s^2} \cdot \frac{B + A\sin 2u}{\cos^2 2u}.$$

Together with (26) it implies (25). □

Remark 2 The solution $u(t)$ of (25) contains elliptic integrals. Indeed, if we denote $r = \sin 2u$, then $\dot{u} = \frac{\dot{r}}{2\sqrt{1-r^2}}$ and $\cos^2 2u = 1 - r^2$. Thus, (25) can be reduced to $\frac{\dot{r}}{2} = \pm\frac{1}{s^2}\sqrt{\frac{f(r)}{r}}$, where $f(r)$ is a polynomial of the third order. Thus,

$$\int_0^t \frac{1}{s^2(\tau)} d\tau =$$

$$= \mp \frac{r_1}{\sqrt{-Dr_2(r_3-r_1)}} \left(F\left(\sqrt{\frac{r(r_3-r_1)}{r_3(r-r_1)}}, \sqrt{\frac{r_3(r_1-r_2)}{r_2(r_1-r_3)}} \right) - \Pi\left(\frac{r_3}{r_1-r_3}, \sqrt{\frac{r(r_3-r_1)}{r_3(r-r_1)}}, \sqrt{\frac{r_3(r_1-r_2)}{r_2(r_1-r_3)}} \right) \right),$$

where r_1, r_2, r_3 are the roots of equation $f(r) = 0$, F and Π are normal elliptic Legendre integrals of the first and third order in the Jacobi form. The roots are simple, otherwise among them there is one that corresponds to equilibrium and the solution does not exist in its neighborhood.

Remark 3 It is easy to check that if we set $s^2 = G = 2Et^2 + k_1 t + k_0$ and $l = \omega = 0$, then $D = \frac{8Ek_0 - k_1^2}{4}$. Thus, we get the result of [1], where the left-hand side of the formula is $\int \frac{dt}{2Et^2+k_1t+k_0} = \frac{2}{\beta} \arctan \frac{4Et+k_1}{\beta}$, with $\beta^2 = 4D > 0$.

Remark 4 If we know the solution to (25), we can go back to the Eulerian coordinates and find $\mathbf{u}$. Indeed, the functions v and w can be found from (27). Thus, the coefficients of the matrix F are known, $Q = \dot{F}F^{-1}$ and $\mathbf{u} = Q\mathbf{x}$.

Remark 5 If u tends to zero, then $\det F$ tends to zero and the components of Q tends to infinity. Since these components have sense of derivatives of the solution, then it implies a blow-up. Thus, for the model of vortex with uniform deformation the instability of an equilibrium in fact implies blow-up of derivatives. One can hypothesize that the same phenomenon holds for a localized vortex in a gas, i.e. the instability leads to formation of singularities (see [10]).

4 Discussion

We prove that if we take into account a small correction due to centrifugal force usually neglected in the l-plane model of atmosphere, we drastically change properties of a specific class of vortices. However, this does not mean that the generally accepted model is erroneous. It is well known that it is approximate and adequately describes only processes of relatively small scale. The model we are considering is also approximate, and an accurate description of the processes of the vortex dynamics of the scale of tropical cyclones can be carried out only within the framework of equations on a sphere.

In addition, vortex solutions with a uniform deformation, on which the difference in stability is manifested, have infinite energy. That is, they are nonphysical on the whole plane. To notice the effect described in this paper, it is necessary that the members $\mathcal{L}\mathbf{u}$ and $\omega^2\mathbf{x}$ (see (5)) have the same order. For this, the linear velocity profile must be maintained inside the vortex within a radius of the order of $\frac{l}{\omega^2}\mathbf{u}$ ≈ 100 km (the typical value of velocity $\mathbf{u}$ is about 10 m/s). This is not observed in nature, in fact, this radius is about 30 km. Near the equator, where the Coriolis parameter is small, stable atmospheric vortices are not observed. Perhaps the effect described in this paper may be one of the factors of their destabilization.

Solutions with a linear velocity profile may seem only an interesting mathematical object. However, it is not quite like that. In fact, solutions of this class have enormous applications in astrophysics (e.g. [2]), in the point blast theory [6], etc. For us, their application to geophysics is important. The motivation is the fact that, near the center of a large atmospheric vortex in its conservative stage, the structure of wind velocity is linear. There are attempts to use this fact to predict the motion of the atmospheric vortex [12, 13]. Note that the full structure of stationary vortices is very diverse [9].

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Results for Fractional Differential Equations with Integral Boundary Conditions Involving the Hadamard Derivative

Alberto Cabada and Kadda Maazouz

Abstract This paper deals with the existence, uniqueness and location of solutions of implicit fractional differential equations involving the Hadamard fractional derivative on Banach spaces. The results follow from the Banach contraction principle.

Keywords Fractional-order differential equation · Hadamard fractional derivative · Green's function · Integral equation · Banach fixed point theorem

AMS Subject Classification 26A33 · 34A08 · 34B15

1 Introduction

Recently a large amount of interest in differential equations of fractional order has been stimulated due to its various applications, especially in different fields of applied science like mechanics, biology, physics, control, porous media and engineering. Indeed, we can find numerous applications in, among others, [6, 10, 11, 16, 18, 22]. There has been significant development in the study of fractional differential equations in recent years; see the monographs of Abbas et al. [1], Baleanu et al. [5], Kilbas et al. [13], Lakshmikantham et al. [14], Miller and Ross [17], Podlubny [19], Samko et al. [20] and Zhou [23]. For some recent contributions on fractional differential equations, see [2, 3, 6, 7, 15] and references therein.

Dedicated in occasion of the 60-birthday of professor Juan José Nieto.

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Recently, considerable attention has been given to the existence of solutions of boundary value problems for fractional differential equations and integral equations with Hadamard fractional derivative (see [2, 8, 12, 15]).

The Green's function for boundary value problems of ordinary differential equations have been investigated in detail in several studies and monographs [9, 21].

Motivated by the above cited works, the purpose of this paper, is to establish existence, uniqueness and location results for the following class of implicit fractional differential equations:

$$^{H}D^{\alpha}y(t) = f(t, (\log(t))^{1-\alpha}y(t), ^{H}D^{\alpha}y(t)), \quad \text{for all } t \in J_1 := [1, b],\ 0 < \alpha \le 1, \tag{1}$$

$$y(a) + \lambda \int_a^b y(t)dt = y(b),\ 1 < a < b, \tag{2}$$

where $^{H}D^{\alpha}$ is the Hadamard fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $\lambda \in (0, +\infty)$.

In this paper, after a preliminary section, where the main definitions and properties are introduced, we present, in Sect. 3, an existence, uniqueness and location result of problem (1)–(2). Finally, in Sect. 4, an illustrative example is showed.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. First, we introduce the following space:

$$C_{\alpha}(J_1) = \{y : (1, b] \to \mathbb{R}, \quad y \in C((1, b]), \quad \text{there exists } \lim_{t \to 1^+} (\log(t))^{1-\alpha} y(t) \in \mathbb{R}\}.$$

It is not difficult to verify that $C_{\alpha}(J_1)$ is Banach space coupled with the norm

$$\|y\|_{\alpha} = \max_{t \in J_1}\{|(\log(t))^{1-\alpha} y(t)|\}.$$

Moreover it is immediate to verify that $C_{\alpha}(J_1) \subset L^1(J_1)$.

Now, we introduce the concept of Hadamard integral and derivative for $\alpha \in (0, 1]$. We fix the value of the starting point of the integral at 1. In [13, Sect. 2.7] the general definition for arbitrary positive starting point and any positive α is introduced together their main properties.

Definition 2.1 The Hadamard fractional integral of order $\alpha \in (0, 1]$ for a function $\varphi : [1, \infty) \longrightarrow \mathbb{R}$ is defined as

$$_H I^{\alpha}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{\varphi(s)}{s} ds,$$

where $\log(\cdot) = \log_e(\cdot)$, and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi) = \int_0^{+\infty} t^{\xi-1} e^{-t} dt, \quad \xi > 0.$$

On [13, Lemma 2.32] it is proved that if $\varphi \in L^p(J_1)$ for some $p \in [1, \infty]$, then $_H I^{\alpha}\varphi \in L^p(J_1)$.

Definition 2.2 The Hadamard derivative of fractional order $\alpha \in (0, 1]$ for a function $\varphi : [1, \infty) \longrightarrow \mathbb{R}$ is defined as

$$^H D^{\alpha}\varphi(t) = \frac{t}{\Gamma(1-\alpha)} \frac{d}{dt} \left(\int_1^t \left(\log \frac{t}{s}\right)^{-\alpha} \frac{\varphi(s)}{s} ds \right).$$

Lemma 2.1 ([13, Corollary 2.4]) *Let $\alpha \in (0, 1]$. The equality $^H D^{\alpha} y(t) = 0, \ t \in J_1$, holds if and only if*

$$y(t) = c_1 (\log t)^{\alpha-1}, \quad \textit{for all } t \in J_1 \ \textit{ and } c_1 \in \mathbb{R}.$$

For the existence results of problem (1)–(2) we need to prove some auxiliary lemmas. To this end, we introduce the following function:

$$F_{\alpha}(x) = \int_1^x (\log s)^{\alpha-1}\, ds, \qquad x \geq 1.$$

It is immediate to verify that F_{α} is continuous on $[1, \infty)$ for all $\alpha \in (0, 1]$.

Lemma 2.2 *Let $0 < \alpha < 1$, $1 < a < b$ and $h \in C(J_1)$ be a given function. Then the boundary value problem*

$$^H D^{\alpha} y(t) = h(t), \quad t \in J_1, \tag{3}$$

coupled to the functional condition (2) has a unique solution $u \in C_{\alpha}(J_1)$ and it is given by the following expression

$$y(t) = \int_1^b G(t, s) \frac{h(s)}{s} ds,$$

where $G(t, s)$ is the Green's function defined as follows:

If $1 \leqslant s \leqslant a$ *and* $1 \leqslant s \leqslant t \leq b$*:*

$$G(t,s)=\frac{1}{\Gamma(\alpha)}\left(\left(\log\frac{t}{s}\right)^{\alpha-1}+\frac{1}{\gamma}(\log t)^{\alpha-1}\left[\left(\log\frac{a}{s}\right)^{\alpha-1}-\left(\log\frac{b}{s}\right)^{\alpha-1}+\lambda s\Big[F_\alpha\Big(\frac{b}{s}\Big)-F_\alpha\Big(\frac{a}{s}\Big)\Big]\right]\right).$$

If $1 \leqslant t < s \leqslant a$*:*

$$G(t,s)=\frac{1}{\gamma\,\Gamma(\alpha)}(\log t)^{\alpha-1}\left[\left(\log\frac{a}{s}\right)^{\alpha-1}-\left(\log\frac{b}{s}\right)^{\alpha-1}+\lambda s\Big[F_\alpha\Big(\frac{b}{s}\Big)-F_\alpha\Big(\frac{a}{s}\Big)\Big]\right].$$

For $a \leqslant s \leqslant t \leqslant b$

$$G(t,s))=\frac{1}{\Gamma(\alpha)}\left(\left(\log\frac{t}{s}\right)^{\alpha-1}+\frac{1}{\gamma}(\log t)^{\alpha-1}\left[-\left(\log\frac{b}{s}\right)^{\alpha-1}+\lambda sF_\alpha\Big(\frac{b}{s}\Big)\right]\right).$$

and, if $a \leqslant t < s \leqslant b$*:*

$$G(t,s)=\frac{1}{\gamma\,\Gamma(\alpha)}(\log t)^{\alpha-1}\left[-\left(\log\frac{b}{s}\right)^{\alpha-1}+\lambda sF_\alpha\Big(\frac{b}{s}\Big)\right].$$

Here

$$\gamma=\Gamma(\alpha)\Big((\log b)^{\alpha-1}-(\log a)^{\alpha-1}-\lambda\,(F_\alpha(b)-F_\alpha(a))\Big)<0.$$

Proof First, let's see that this problem has, at most, one solution in $LC_\alpha(J_1)$. On the contrary, if we have y_1 and y_2 two different solutions of problem (3), (2), we have that $z := y_1 - y_2$ is a solution of ${}^H D^\alpha z(t) = 0,\ t \in J_1$. So, from Lemma 2.1, we have that there is a $c \in \mathbb{R}$ such that

$$z(t)=c(\log t)^{\alpha-1},\quad \textit{for all } t\in J_1.$$

In such a case, we have that the functional boundary condition (3) say us that if $c \neq 0$, then

$$0<\lambda\,(F(b)-F(a))=(\log(b)^{\alpha-1}-(\log(a))^{\alpha-1}<0,$$

which is a contradiction.

As a consequence, for every $c_1 \in \mathbb{R}$ we have, from [13, Property 2.38], that

$$\begin{aligned}y(t)&={}_HI^\alpha(h(t))+c_1\,(\log t)^{\alpha-1}\\&=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}h(s)\frac{ds}{s}+c_1\,(\log t)^{\alpha-1},\end{aligned}$$

satisfies Eq. (3).

As consequence, we must look for the unique $c_1 \in \mathbb{R}$, if it exists, for which the functional condition (2) is fulfilled.

Moreover, by using Fubini's Theorem, we deduce the following equalities

$$\begin{aligned}
\int_a^b y(s)ds &= \frac{1}{\Gamma(\alpha)} \int_a^b \left(\int_1^s \left(\log \frac{s}{\tau}\right)^{\alpha-1} \frac{h(\tau)}{\tau} d\tau \right) ds + c_1 \int_a^b (\log s)^{\alpha-1}\, ds \\
&= \frac{1}{\Gamma(\alpha)} \int_a^b \left(\int_1^s \left(\log \frac{s}{\tau}\right)^{\alpha-1} \frac{h(\tau)}{\tau} d\tau \right) ds + c_1\ (F_\alpha(b) - F_\alpha(a)) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^a \left(\int_a^b \left(\log \frac{s}{\tau}\right)^{\alpha-1} ds \right) \frac{h(\tau)}{\tau} d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^b \left(\int_\tau^b \left(\log \frac{s}{\tau}\right)^{\alpha-1} ds \right) \frac{h(\tau)}{\tau} d\tau + c_1\ (F_\alpha(b) - F_\alpha(a)) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^a \left(\int_{\frac{a}{\tau}}^{\frac{b}{\tau}} (\log l)^{\alpha-1}\, \tau dl \right) \frac{h(\tau)}{\tau} d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^b \left(\int_1^{\frac{b}{\tau}} (\log l)^{\alpha-1}\, \tau dl \right) \frac{h(\tau)}{\tau} d\tau + c_1\ (F_\alpha(b) - F_\alpha(a)) \\
&= \frac{1}{\Gamma(\alpha)} \int_1^a \left[F_\alpha\left(\frac{b}{\tau}\right) - F_\alpha\left(\frac{a}{\tau}\right) \right] h(\tau) d\tau \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^b F_\alpha\left(\frac{b}{\tau}\right) h(\tau) d\tau + c_1\ (F_\alpha(b) - F_\alpha(a)).
\end{aligned}$$

So, the boundary condition (2) is fullfilled if and only if

$$\begin{aligned}
&\int_1^a \left(\log \frac{a}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds - \int_1^b \left(\log \frac{b}{s}\right)^{\alpha-1} \frac{h(s)}{s} ds \\
&+\lambda \int_1^a s \left[F_\alpha\left(\frac{b}{\tau}\right) - F_\alpha\left(\frac{a}{\tau}\right) \right] \frac{h(s)}{s} ds + \lambda \int_a^b s F_\alpha\left(\frac{b}{s}\right) \frac{h(s)}{s} ds \\
= {}& c_1 \Gamma(\alpha) \Big((\log b)^{\alpha-1} - (\log a)^{\alpha-1} - \lambda\, (F_\alpha(b) - F_\alpha(a)) \Big),
\end{aligned}$$

that is

$$c_1 = \frac{1}{\gamma} \left[\int_1^b K_1(s) \frac{h(s)}{s} ds + \lambda \int_1^b K_2(s) \frac{h(s)}{s} ds \right],$$

with K_1 and K_2 defined as follows:

$$K_1(s) = \frac{1}{\Gamma(\alpha)} \begin{cases} -\left(\log \dfrac{b}{s}\right)^{\alpha-1} & a \leqslant s \leqslant b \\ \left(\log \dfrac{a}{s}\right)^{\alpha-1} - \left(\log \dfrac{b}{s}\right)^{\alpha-1} & 1 \leqslant s \leqslant a. \end{cases}$$

$$K_2(s) = \frac{1}{\Gamma(\alpha)} \begin{cases} s\Big[F_\alpha\Big(\frac{b}{s}\Big) - F_\alpha\Big(\frac{a}{s}\Big)\Big], & 1 \leqslant s \leqslant a \\ sF_\alpha\Big(\frac{b}{s}\Big), & a \leqslant s \leqslant b. \end{cases}$$

Therefore, the unique solution of (3), (2) is given by

$$y(t) = \int_1^b G(t,s)\frac{h(s)}{s}ds, \qquad t \in J_1,$$

with $G : J_1 \times ([1,a) \cup (a,b)) \backslash \{(t,t), t \in J_1\} \to \mathbb{R}$ defined in the enunciate of this lemma.

Remark 2.1 Now, we define $\tilde{G} : J_1 \to [0,\infty)$ as follows:
If $1 \leqslant s < a$:

$$\tilde{G}(s) = \frac{1}{\Gamma(\alpha)} \left| \frac{1}{\gamma}\left[\left(\log\frac{a}{s}\right)^{\alpha-1} - \left(\log\frac{b}{s}\right)^{\alpha-1} + \lambda s\left[F_\alpha\Big(\frac{b}{s}\Big) - F_\alpha\Big(\frac{a}{s}\Big)\right]\right]\right|.$$

and, if $a < s < b$:

$$\tilde{G}(s) = \left| \frac{1}{\gamma\,\Gamma(\alpha)}\left[-\left(\log\frac{b}{s}\right)^{\alpha-1} + \lambda s F_\alpha\Big(\frac{b}{s}\Big)\right]\right|.$$

It is immediate to verify that

$$\int_1^b |(\log(t))^{1-\alpha}\, G(t,s)|ds \leq \int_1^t \left(\log\frac{t}{s}\right)^{\alpha-1} (\log(t))^{1-\alpha} ds + \int_1^b \tilde{G}(s)ds.$$

So, by means of the change of variables $r = t/s$ it is easy to verify that

$$\int_1^b |(\log(t))^{1-\alpha}\, G(t,s)|ds \leq t\,F_\alpha(t) + \int_1^b \tilde{G}(s)ds \leq b\,F_\alpha(b) + \|\tilde{G}\|_1 =: \tilde{K}_b, \quad \text{for all } t \in J_1.$$

As a consequence of the previous results, we arrive at the following Lemma.

Lemma 2.3 *A function $y \in C_\alpha(J_1)$ is a solution of Problem (1)–(2) if and only if y is a solution of the integral equation*

$$y(t) = \int_1^b G(t,s)\frac{\varphi(s)}{s}ds, \quad t \in J_1, \tag{4}$$

where $G(t,s)$ is the Green's function defined in Lemma 2.2, and $\varphi \in C(J_1)$ satisfies the implicit functional equation

$$\varphi(s) = f(s, (\log(s))^{1-\alpha}\, y(s), \varphi(s)), \quad s \in J_1.$$

3 An Existence, Uniqueness and Location Result

Theorem 3.1 *Assume that*

(H1) $f : J_1 \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous.*
(H2) There exist constants $0 < l < 1$ *and* $0 < k$ *such that*

$$|f(t, u, v) - f(t, \overline{u}, \overline{v})| \leqslant k|u - \overline{u}| + l|v - \overline{v}|$$

for any $u, v, \overline{u}, \overline{v} \in \mathbb{R}$, *and* $t \in J_1$.

If $\sigma = \frac{k}{1-l}\tilde{K}_b < 1$, *then Problem (1)–(2) has a unique solution* $y \in C_\alpha(J_1)$.

Proof To transform Problem (1)–(2) into fixed point problem, we consider an operator $A : C_\alpha(J_1) \longrightarrow C_\alpha(J_1)$ defined by

$$A(y)(t) = \int_1^b G(t, s)\frac{\varphi(s)}{s}ds, \tag{5}$$

where $G(t, s)$ is the Green's function defined in Lemma 2.2, and $\varphi \in C(J_1)$ satisfies the implicit functional equation

$$\varphi(s) = f(s, (\log(s))^{1-\alpha}y(s), \varphi(s)). \tag{6}$$

It is important to note that (6) has, for every $y \in C_\alpha(J_1)$, a unique solution φ. This is deduced directly from the fact that f is a continuous function an $0 < l < 1$ on condition $(H2)$.

As a direct consequence of the Dominated Convergence Theorem, it is immediate to verify that $A(y) \in C_\alpha(J_1)$ for every $y \in C_\alpha(J_1)$.

Clearly, from Lemmas 2.2 and 2.3, the fixed points of A are the solutions of Problem (1)–(2). We shall show that A is a contraction on $C_\alpha(J_1)$.

Let $u, v \in C_\alpha(J_1)$. Then, for each $t \in J_1$, we have

$$(\log(t))^{1-\alpha}(Au)(t) - (\log(t))^{1-\alpha}(Av)(t) = (\log(t))^{1-\alpha}\int_a^b G(t, s)\big(\varphi(s) - \psi(s)\big)\frac{ds}{s},$$

where

$$\varphi(s) = f(s, (\log(s))^{1-\alpha}u(s), \varphi(s)),$$

$$\psi(s) = f(s, (\log(s))^{1-\alpha}v(s), \psi(s)),$$

and

$$|\varphi(s) - \psi(s)| \leqslant k(\log(s))^{1-\alpha}|u(s) - v(s)| + l|\varphi(s) - \psi(s)|.$$

Thus,

$$|\varphi(s) - \psi(s)| \leq \frac{k}{1-l}(\log(s))^{1-\alpha}|u(s) - v(s)|.$$

Then,

$$\begin{aligned}(\log(t))^{1-\alpha}\Big|(Au)(t)-(Av)(t)\Big| &\leqslant (\log(t))^{1-\alpha}\int_1^b \Big|G(t,s)\big(\varphi(s)-\psi(s)\big)\Big|\frac{ds}{s}\\ &\leqslant \frac{k}{1-l}(\log(t))^{1-\alpha}\int_1^b |G(t,s)|(\log(s))^{1-\alpha}|u(s)-v(s)|\frac{ds}{s}\\ &\leqslant \frac{k}{1-l}\tilde{K}_b\|u-v\|_\alpha.\end{aligned}$$

Hence

$$\|Au-Av\|_\alpha \leqslant \sigma\|u-v\|_\alpha. \tag{7}$$

Since $\sigma < 1$, operator A is a contraction and, by the Banach's fixed point theorem, problem (1)–(2) has a unique solution in $C_\alpha(J_1)$ and the result is proved.

Now we obtain a location of the unique solution of Problem (1)–(2).

Theorem 3.2 *Assume that the hypotheses of Theorem 3.1 are fulfilled. Then the unique solution y of Problem (1)–(2) satisfies that*

$$\|y\|_\alpha \leq \frac{f^*\tilde{K}_b}{1-l-k\tilde{K}_b}, \tag{8}$$

with $f^* = \sup_{t\in J_1}|f(t,0,0)|$.

Proof First, notice that, since $\sigma < 1$, we have that

$$0 < \delta \leq \frac{f^*\tilde{K}_b}{1-l-k\tilde{K}_b}$$

if and only if

$$0 < \delta \leq \frac{k\delta + f^*}{1-l}\tilde{K}_b.$$

Let $y \in C_\alpha(J_1)$ be the unique solution of Problem (1)–(2), which existence is ensured in Theorem 3.1.

From Lemma 2.3 and (5), we have that $y(t) = A\,y(t)$ for all $t \in J_1$. As consequence, for each $t \in J_1$, we have

$$(\log(t))^{1-\alpha}|y(t)| = \Big|\int_1^b \log(t))^{1-\alpha}G(t,s)\frac{\varphi(s)}{s}ds\Big| \leq \int_1^b |\log(t))^{1-\alpha}G(t,s)||\varphi(s)|\frac{ds}{s}.$$

By $(H2)$ we have

$$\begin{aligned}|\varphi(s)| &= |f(s, (\log(s))^{1-\alpha} y(s), \varphi(s))| \\ &\leqslant |f(s, (\log(s))^{1-\alpha} y(s), \varphi(s)) - f(s, 0, 0)| + |f(s, 0, 0)| \\ &\leqslant k(\log(s))^{1-\alpha} |y(s)| + l|\varphi(s)| + |f(s, 0, 0)|.\end{aligned}$$

Thus, for all $s \in J_1$ it is fulfilled that

$$|\varphi(s)| \leqslant \frac{k(\log(s))^{1-\alpha}|y(s)| + |f(s, 0, 0)|}{1 - l} \leqslant \frac{k\|y\|_\alpha + f^*}{1 - l},$$

and so

$$\begin{aligned}(\log(t))^{1-\alpha}|y(t)| &\leqslant \frac{k\|y\|_\alpha + f^*}{1 - l} \int_1^b |\log(t))^{1-\alpha} G(t, s)| \frac{ds}{s} \\ &\leqslant \frac{k|y\|_\alpha + f^*}{1 - l} \tilde{K}_b, \quad \text{for all } t \in J_1.\end{aligned}$$

As a consequence, we deduce that

$$\|y\|_\alpha \leqslant \frac{f^* \tilde{K}_b}{1 - l - k\tilde{K}_b},$$

and the proof is concluded.

Remark 3.1 Notice that if $f(t, 0, 0) = 0$ for all $t \in J_1$ we deduce from previous results that the unique solution of problem (1)–(2) is the identically zero one.

4 Example

In this section we give an example to illustrate the usefulness of our main results.

Example 4.1 Consider the boundary value problem

$$^H D^{\frac{1}{2}} y(t) = \frac{|(\log(t))^{1/2} y(t)| + |^H D^{\frac{1}{2}} y(t)|}{7(1 + |y(t)| + |^H D^{\frac{1}{2}} y(t)|)} + \sin t, \quad t \in J_1 = [1, 2], \tag{9}$$

$$y(3/2) + \frac{1}{2} \int_{3/2}^2 y(t)dt = y(2). \tag{10}$$

Problem (9)–(10) is a particular case of Problem (1)–(2), with $a = 3/2$, $b = 2$, $\lambda = 1/2$ and

$$f(t, u, v) = \frac{|u| + |v|}{7(1 + |u| + |v|)} + \sin t.$$

It is clear that f is continuous on $J_1 \times \mathbb{R} \times \mathbb{R}$ and $f^* = 1$.

Since

$$\left|\frac{\partial f}{\partial u}(t, u, v)\right| = \left|\frac{\partial f}{\partial v}(t, u, v)\right| = \frac{1}{7(1 + |u| + |v|)^2},$$

we have that condition $(H2)$ holds for

$$k = l = \frac{1}{7}.$$

By numerical approach, one can verify that $\tilde{K}_b \approx 5.55986$. As a consequence $\sigma = \tilde{K}_b/6 \approx 0.926643$, and Theorems 3.1 and 3.2 hold.

Thus, problem (9)–(10) has a unique solution y such that

$$\|y\|_{1/2} \leq \frac{f^* \tilde{K}_b}{1 - l - k\,\tilde{K}_b} \approx 88.4243.$$

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A Nonlinear Problem Related to Artificial Circulation in a Lake

Francisco J. Fernández, Aurea Martínez and Lino J. Alvarez-Vázquez

Abstract This work deals with artificial circulation as a shallow water aeration technique. Large waterbodies (for instance, lakes or reservoirs) get much of their oxygen from the atmosphere through diffusion processes. Artificial circulation increases water's oxygen by forcefully circulating the water to expose more of it to the atmosphere. Two techniques are the most common: air injection and mechanical mixing. The former has been analyzed from an ecological viewpoint in several works (see, for instance Haynes in Hydrobiologia 43:463–504, 1973 [1] and the references therein). However, in this work we will focus our attention on the latter that, as far as we know, has remained unaddressed in the mathematical literature. In this work we introduce the mathematical formulation of the environmental problem as a system of nonlinear partial differential equations more general than that studied by Martínez et al. (Math. Control Relat. Fields 8:277–313, 2018 [2]), and we prove the existence of solution using a fixed point technique. This work was supported by funding from project MTM2015-65570-P of Ministerio de Economía y Competitividad (Spain)/FEDER.

Keywords Nonlinear partial differential equation · Eutrophication

1 The Environmental Problem

Eutrophication is one of the most important problems of large masses of water and it is caused by high levels of pollutants that reach the waters. These pollutants come mainly from human activities and can cause an excessive phytoplankton growth that lead to undesirable effects like algal blooms. This abnormal growth of algae

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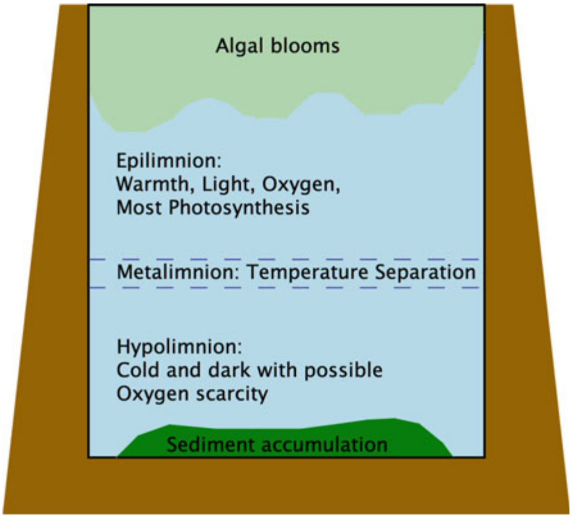

Fig. 1 Algal blooms caused by eutrophication and its consequences

directly affects the concentration of dissolved oxygen in the deeper layers, since the process of remineralization of organic detritus (which accumulate in the bottom due to the effects of sedimentation) consumes oxygen. In Fig. 1 we can see a schematic representation of the problem.

Artificial circulation is a management technique for oxygenating eutrophic water bodies subject to quality problems, such as loss of oxygen, sediment accumulation and algal blooms. It disrupts stratification and minimizes the development of stagnant zones that may be subject to water quality problems (low levels of oxygen). In our case we are interested in increase the dissolved oxygen concentration in the bottom layers. A flow pump takes water from the well aerated upper layers by means of a collector and injecting it into the poorly oxygenated bottom layers, setting up a circulation pattern that prevents stratification. Oxygen-poor water from the bottom is circulated to the surface, where oxygenation from the atmosphere and photosynthesis can occur. In Fig. 2 we can see the main idea of this technique.

2 Mathematical Formulation of the Problem

We consider a domain $\Omega \subset \mathbb{R}^3$ corresponding, for instance, to a lake. In order to promote the artificial circulation of water inside the domain, we suppose the existence of a set of N_{CT} pairs collector-injector $\{(C^k, T^k)\}_{k=1}^{N_{CT}} \subset \partial\Omega$ in such a way that each water collector is connected to its corresponding injector by a pipe with a pumping group. We assume a smooth enough boundary $\partial\Omega$, such that it can be split into three disjoint subsets $\partial\Omega = \Gamma_S \cup \Gamma_C \cup \Gamma_T \cup \Gamma_N$, where Γ_C corresponds to the part of the boundary where the water collectors are located, $\Gamma_C = \cup_{k=1}^{N_{CT}} C^k$, Γ_T corresponds to

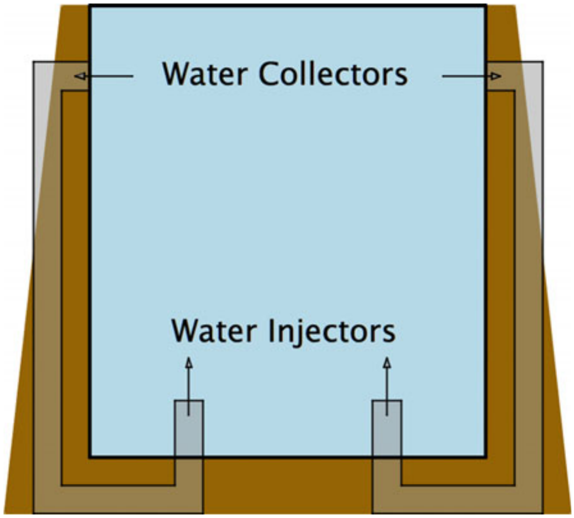

Fig. 2 Artificial circulation

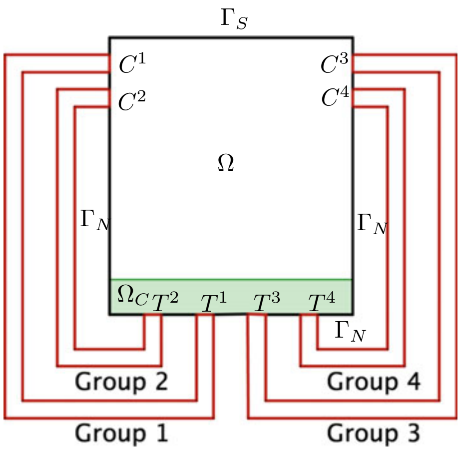

Fig. 3 Geometrical configuration of an example domain Ω with $N_{CT} = 4$ collector/injector pairs, showing the different boundary sections: Γ_S, $\Gamma_C = \cup_{k=1}^{4} C^k$, $\Gamma_T = \cup_{k=1}^{4} T^k$, Γ_N, Γ_S

the part of the boundary where the water injectors are located, $\Gamma_T = \cup_{k=1}^{N_{CT}} T^k$, Γ_S is the part of the boundary in contact with the air, and $\Gamma_N = \Gamma \setminus (\Gamma_C \cup \Gamma_T \cup \Gamma_S)$ corresponds to the rest of the boundary. We suppose the boundary $\partial\Omega$ regular enough to assure the existence of elements φ^k, $\widetilde{\varphi}^k \in H^{3/2}(\partial\Omega), k = 1, \ldots, N_{CT}$, satisfying the following assumptions (corresponding to suitable regularizations of the indicator functions of T^k and C^k, respectively):

- $\varphi^k(\mathbf{x})$, $\widetilde{\varphi}^k(\mathbf{x}) \geq 0$, a.e. $\mathbf{x} \in \partial\Omega$,
- $\varphi^k(\mathbf{x}) = 0$, a.e. $\mathbf{x} \in \partial\Omega \setminus T^k$, and $\int_{T^k} \varphi^k(\mathbf{x})\, d\gamma = \mu(T^k)$,
- $\widetilde{\varphi}^k(\mathbf{x}) = 0$, a.e. $\mathbf{x} \in \partial\Omega \setminus C^k$, and $\int_{C^k} \widetilde{\varphi}^k(\mathbf{x})\, d\gamma = \mu(C^k)$,

where $\mu(S)$ represents the measure of a generic set S (Fig. 3).

We denote by $g^k(t) \in H^1(0,T)$ the volumetric flow rate by pump k at each time t ($\mathrm{m^3\,s^{-1}}$), $k = 1, \ldots, N_{CT}$, where T (s) is the length of the time interval. For technical reasons, we suppose that $\mathbf{g} \in \mathcal{U}_{ad} = \{\mathbf{g} \in [H^1(0,T)]^{N_{CT}} : \ \|g^k(t)\|_{H^1(0,T)} \leq c_k, \ k = 1, \ldots, N_{CT}\}$, with $c_k > 0$, $k = 1, \ldots, N_{CT}$, are constants related with the technical limitations of the pumps. We also suppose that the flow rate acts over the system through a Dirichlet boundary condition on the water velocity:

$$\mathbf{v} = \boldsymbol{\phi}_{\mathbf{g}} \text{ on } (0,T) \times \partial\Omega, \tag{1}$$

where $\mathbf{v}$ ($\mathrm{m\,s^{-1}}$) is the water velocity and:

$$\boldsymbol{\phi}_{\mathbf{g}}(t,\mathbf{x}) = \sum_{k=1}^{N_{CT}} g^k(t) \left[\frac{\varphi^k(\mathbf{x})}{\mu(T^k)} - \frac{\widetilde{\varphi}^k(\mathbf{x})}{\mu(C^k)} \right] \mathbf{n}, \quad (t,\mathbf{x}) \in (0,T) \times \partial\Omega \tag{2}$$

is the Dirichlet condition for the hydrodynamic system. Thanks to the regularity of the functions $\mathbf{g}$ and $\{(\varphi^k, \widetilde{\varphi}^k)\}_{k=1}^{N_{CT}}$, we have that

$$\boldsymbol{\phi}_{\mathbf{g}} \in W^{1,2,2}(0,T; H^{3/2}(\partial\Omega), H^{3/2}(\partial\Omega)) \tag{3}$$

and also

$$\int_{\partial\Omega} \boldsymbol{\phi}_{\mathbf{g}} \cdot \mathbf{n} \, d\gamma = 0. \tag{4}$$

Thus, the water velocity $\mathbf{v}(\mathbf{x},t)$ ($\mathrm{m\,s^{-1}}$) is the solution of the following modified Navier-Stokes system with a Smagorinsky model of turbulence:

$$\begin{cases} \frac{\partial \mathbf{v}}{\partial t} + \nabla\mathbf{v}\,\mathbf{v} - \nabla \cdot \Xi(\mathbf{v}) + \nabla p = \alpha^0(\theta - \theta^0)\mathbf{a}_g & \text{in } \Omega \times (0,T), \\ \nabla \cdot \mathbf{v} = 0 \quad \text{in } \Omega \times (0,T), \\ \mathbf{v} = \boldsymbol{\phi}_{\mathbf{g}} \quad \text{on } \partial\Omega \times (0,T), \\ \mathbf{v}(0) = \mathbf{v}^0 \quad \text{in } \Omega, \end{cases} \tag{5}$$

where $\mathbf{a}_g$ ($\mathrm{m\,s^{-1}}$) is the gravity acceleration, $\alpha^0 = -\frac{1}{\rho}\frac{\partial\rho}{\partial\theta}$ ($\mathrm{K^{-1}}$) is the thermic expansion coefficient, $\mathbf{v}^0$ is the initial velocity, and $\boldsymbol{\phi}_{\mathbf{g}}$ is given by (2). The diffusion term $\Xi(\mathbf{v})$ is:

$$\Xi(\mathbf{v}) = \left.\frac{\partial D(e)}{\partial e}\right|_{e=e(\mathbf{v})}, \quad \text{with } e(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + \nabla\mathbf{v}^t\right), \tag{6}$$

where D is a potential function. For instance, in the particular case of the classical Navier-Stokes equations, $D(e) = \nu\,[e : e]$, with ν ($\mathrm{m^2\,s^{-1}}$) the kinematic viscosity of the water, and, consequently, $\Xi(\mathbf{v}) = 2\nu\, e(\mathbf{v})$. However, in our case, the Smagorinsky model, the potential function is defined as [3]:

$$D(e) = \nu\,[e:e] + \frac{2}{3}\nu_{tur}\,[e:e]^{3/2}\,, \tag{7}$$

where ν_{tur} (m^2) is the turbulent viscosity.

The water temperature, $\theta(\mathbf{x}, t)$ (K), is the solution of the following convection-diffusion partial differential equation with nonhomogeneous, nonlinear, mixed boundary conditions:

$$\begin{cases} \frac{\partial\theta}{\partial t} + \mathbf{v}\cdot\nabla\theta - \nabla\cdot(K\nabla\theta) = 0 \quad \text{in } \Omega\times(0,T), \\ \theta = \phi_\theta \quad \text{on } \Gamma_T\times(0,T), \\ K\frac{\partial\theta}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma_C\times(0,T), \\ K\frac{\partial\theta}{\partial\mathbf{n}} = b_1^N(\theta_N-\theta) \quad \text{on } \Gamma_N\times(0,T), \\ K\frac{\partial\theta}{\partial\mathbf{n}} = b_1^S(\theta_S-\theta) + b_2^S(T_r^4 - |\theta|^3\theta) \quad \text{on } \Gamma_S\times(0,T), \\ \theta(0) = \theta^0 \quad \text{in } \Omega, \end{cases} \tag{8}$$

where Dirichlet boundary condition ϕ_θ is given by expression:

$$\phi_\theta(\mathbf{x}, t) = \sum_{k=1}^{N_{CT}} \varphi^k(\mathbf{x}) \int_{-T}^{T} \rho_\epsilon(t-\epsilon-s)\,\gamma_\theta^k(s)\,ds \tag{9}$$

with, for each $k = 1, \ldots, N_{CT}$,

$$\gamma_\theta^k(s) = \begin{cases} \frac{1}{\mu(C^k)}\int_{C^k}\theta^0\,d\gamma & \text{if } s\le 0, \\ \frac{1}{\mu(C^k)}\int_{C^k}\theta(s)\,d\gamma & \text{if } s>0, \end{cases} \tag{10}$$

representing the mean temperature of water in the collector C_k, and with the weight function ρ_ϵ defined by:

$$\rho_\epsilon(t) = \begin{cases} \frac{c}{\epsilon}\exp\left(\frac{t^2}{t^2-\epsilon^2}\right) & \text{if } |t|<\epsilon, \\ 0 & \text{if } |t|\ge\epsilon, \end{cases} \tag{11}$$

for $c \in \mathbb{R}$ the positive constant satisfying the unitary condition:

$$\int_{\mathbb{R}} \rho_1(t)\,dt = 1. \tag{12}$$

In other words, we are assuming that the mean temperature of water at each injector T_k is a weighted average in time of the mean temperatures of water at its corresponding collector C_k. In order to obtain the mean temperature at each injector, we convolute the mean temperature at the collector with a smooth function with support in $(t-2\epsilon, t)$. In this way, we have that the temperature in the injector only depends on the mean temperature in the collector in the time interval $(t-2\epsilon, t)$. Parameter $0<\epsilon<T$ represents, in a certain sense, the technical characteristics of the pipeline that define

the stay time of water in the pipe. We also suppose that there is not heat transfer thought the walls of the pipelines (that is, they are isolated). Moreover,

- $\mathbf{n}$ is the unit outward normal vector to the boundary $\partial\Omega$.
- $K > 0$ ($\mathrm{m^2\,s^{-1}}$) is the thermal diffusivity of the fluid: $K = \frac{\alpha}{\rho c_p}$, where α ($\mathrm{W\,m^{-1}\,K^{-1}}$) is the thermal conductivity, ρ ($\mathrm{g\,m^{-3}}$) is the density, and c_p ($\mathrm{W\,s\,g^{-1}\,K^{-1}}$) is the specific heat capacity of water.
- $b_1^K \geq 0$ ($\mathrm{m\,s^{-1}}$), for $K = N, S$, are the coefficients related to convective heat transfer through the boundaries Γ_N and Γ_S, obtained from the relation $\rho\, c_p\, b_1^K = h^K$, where $h^K \geq 0$ ($\mathrm{W\,m^{-2}\,K^{-1}}$) are the convective heat transfer coefficients on each surface.
- $b_2^S > 0$ ($\mathrm{m\,s\,K^{-3}}$) is the coefficient related to radiative heat transfer through the boundary Γ_S, given by $b_2^S = \frac{\sigma_B\,\varepsilon}{\rho\, c_p}$, where σ_B ($\mathrm{W\,m^{-2}\,K^{-4}}$) is the Stefan-Boltzmann constant and ε is the emissivity.
- $\theta^0 \geq 0$ (K) is the initial temperature.
- θ_S, $\theta_N \geq 0$ (K) are the temperatures related to convection heat transfer on the surfaces Γ_S and Γ_N.
- $T_r \geq 0$ (K) is the radiation temperature on the surface Γ_S, derived from expression $\sigma_B\,\varepsilon\, T_r^4 = (1-a)R_{sw,net} + R_{lw,down}$, where a is the albedo, $R_{sw,net}$ ($\mathrm{W\,m^{-2}}$) denotes the net incident shortwave radiation on the surface Γ_R, and $R_{lw,down}$ ($\mathrm{W\,m^{-2}}$) denotes the downwelling longwave radiation.

We consider the following system for the eutrophication processes, based in a Michaelis Menten kinetics (see [4, 5] and the references therein)

$$\begin{cases} \frac{\partial u^i}{\partial t} + \mathbf{v}\cdot\nabla u^i - \nabla\cdot(\mu^i \nabla u^i) = A^i(\mathbf{x}, t, \theta, \mathbf{u}), \text{ in } \Omega\times(0,T), \\ u^i = \phi_{u^i}, \text{ on } \Gamma_T \times (0,T), \\ \mu^i \frac{\partial u^i}{\partial \mathbf{n}} = 0, \text{ on } (\Gamma_S \cup \Gamma_N \cup \Gamma_C)\times(0,T), \\ u^i(0) = u^{0,i}, \text{ in } \Omega, \; i = 1,\ldots,5, \end{cases} \tag{13}$$

where, for $i = 1,\ldots,5$,

$$\phi_{u^i} = \sum_{k=1}^{N_{CT}} \varphi^k(\mathbf{x}) \int_{-T}^{T} \rho_\epsilon(t-\epsilon-s)\gamma_{u^i}^k(s)\, ds, \tag{14}$$

and, for all $k = 1,\ldots,N_{CT}$, and $i = 1,\ldots,5$,

$$\gamma_{u^i}^k(s) = \begin{cases} \frac{1}{\mu(C^k)} \int_{C^k} u^{0,i}\, d\gamma, & \text{if } s \leq 0, \\ \frac{1}{\mu(C^k)} \int_{C^k} u^i(s)\, d\gamma, & \text{if } s > 0. \end{cases} \tag{15}$$

Finally, the reaction term $\mathbf{A} = (A^i) : \Omega \times (0,T) \times [\mathbb{R}_+]^6 \to \mathbb{R}^5$ is defined by the following expression:

$$\mathbf{A}(\mathbf{x},t,\theta,\mathbf{u}) = \begin{bmatrix} -\frac{C_{nc}L(\mathbf{x},t,\theta)}{K_N+|u^1|}u^1u^2 + C_{nc}K_r u^2 + C_{nc}K_{rd}D(\theta)u^4 \\ \frac{L(\mathbf{x},t,\theta)}{K_N+|u^1|}u^1u^2 - K_r u^2 - K_{mf}u^2 - \frac{K_z}{K_F+|u^2|}u^2u^3 \\ \frac{C_{fz}K_z}{K_F+|u^2|}u^2u^3 - K_{mz}u^3 \\ K_{mf}u^2 + K_{mz}u^3 - K_{rd}D(\theta)u^4 \\ \frac{C_{oc}L(\mathbf{x},t,\theta)}{K_N+|u^1|}u^1u^2 - C_{oc}K_r u^2 - C_{oc}K_{rd}D(\theta)u^4 \end{bmatrix}, \tag{16}$$

where:

- u^1 is the nutrient (nitrogen) (mg l^{-1}),
- u^2 is the phytoplankton (mgC/1),
- u^3 is the zooplankton (mgC/1),
- u^4 is the organic detritus (mgC/1),
- u^5 is the dissolved oxygen (mg l^{-1}),
- C_{oc} is the oxygen-carbon stoichiometric relation (mg/mgC),
- C_{nc} is the nitrogen-carbon stoichiometric relation (mg/mgC),
- C_{fz} is the zooplankton grazing efficiency factor,
- K_{rd} is the detritus regeneration rate (s^{-1}),
- K_r is the phytoplankton endogenous respiration rate (s^{-1}),
- K_{mf} is the phytoplankton death rate (s^{-1}),
- K_{mz} is the zooplankton death rate (including predation) (s^{-1}),
- K_z is the zooplankton predation (grazing) (s^{-1}),
- K_F is the phytoplankton half-saturation constant (mgC/1),
- K_N is the nitrogen half-saturation constant (mg l^{-1}),
- μ^i, $i = 1, \ldots, 5$, are the diffusion coefficients of each species ($\mathrm{m}^2\mathrm{s}^{-1}$),
- D is the thermic regeneration function for the organic detritus:

$$D(\theta) = \Theta^{\theta-\theta^0}, \tag{17}$$

being $\log(\Theta)$ (K^{-1}) the thermic regeneration constant for the reference temperature θ^0 (K). In order to simplify the mathematical analysis of the state equations we will consider the following linear approximation:

$$D(\theta) = 1 + \log(\Theta)(\theta - \theta^0), \tag{18}$$

if $\Theta > 0$ and $D(\theta) = 1$, if $\Theta = 0$.
- L is the luminosity function, given by:

$$L(\mathbf{x},t,\theta) = \mu\, C_t^{\theta-\theta^0}\, \frac{I^0(t)}{I_s}\, e^{-\varphi_1 x_3}, \tag{19}$$

with I^0 the incident light intensity ($\mathrm{W\,m}^{-2}$), I_s ($\mathrm{W\,m}^{-2}$) the light saturation, $\log(C_t)$ (K^{-1}) the phytoplankton growth thermic constant for the reference temperature θ^0, φ_1 (m^{-1}) the light attenuation due to depth, and μ (s^{-1}) the maximum phytoplankton growth rate. We will consider the following linear approximation:

$$L(\mathbf{x}, t, \theta) = \mu \left(1 + \log(C_t)(\theta - \theta^0)\right) \frac{I^0(t)}{I_s} e^{-\varphi_1 x_3}, \tag{20}$$

if $C_t > 0$ and $L(\mathbf{x}, t, \theta) = \mu \frac{I^0(t)}{I_s} e^{-\varphi_1 x_3}$, if $C_t = 0$.

All above coefficients are supposed to be nonnegative, except for the half-saturation constants, that are strictly positive.

3 Mathematical Analysis of the Problem

We will assume the following hypotheses for coefficients and data of the problem:

(a) $\theta^0 \in X_2$
(b) $\theta_S \in L^2(0, T; L^2(\Gamma_S))$
(c) $\theta_N \in L^2(0, T; L^2(\Gamma_N))$
(d) $T_r \in L^5(0, T; L^5(\Gamma_S))$
(e) $\mathbf{v}^0 \in \left[H^2_\sigma(\Omega)\right]^3 = \{\mathbf{v} \in [H^2(\Omega)]^3 : \ \nabla \cdot \mathbf{v} = 0, \ \mathbf{v}_{|\partial\Omega} = \mathbf{0}\} \subset \widetilde{\mathbf{X}}_2$
(f) $g^k \in H^1(0, T)$ with $g^k(0) = 0, \forall k = 1, \ldots, N_{CT}$
(h) $\mathbf{u}^0 \in X_3$
(i) $I_0 \in L^\infty(0, T)$

In order to establish the appropriate framework for mathematical analyzing the state system (8), (5) and (13), we consider, for a Banach space V_1 and a locally convex space V_2 such that $V_1 \subset V_2$, the following Sobolev-Bochner space (cf. Chap. 7 of [6]), for $1 \le p, q \le \infty$:

$$W^{1,p,q}(0, T; V_1, V_2) = \left\{u \in L^p(0, T; V_1) : \ \frac{du}{dt} \in L^q(0, T; V_2)\right\}, \tag{21}$$

where $\frac{du}{dt}$ denotes the derivative of u in the sense of distributions. It is well known that, if both V_1 and V_2 are Banach spaces, then $W^{1,p,q}(0, T; V_1, V_2)$ is also a Banach space endowed with the norm $\|u\|_{W^{1,p,q}(0,T;V_1,V_2)} = \|u\|_{L^p(0,T;V_1)} + \left\|\frac{du}{dt}\right\|_{L^q(0,T;V_2)}$.

For the Modified Navier-Stokes system (8) we consider the following spaces

$$\begin{aligned} \mathbf{X}_1 &= \left\{\mathbf{v} \in [W^{1,3}(\Omega)]^3 : \ \nabla \cdot \mathbf{v} = 0 \ and \ \mathbf{v}_{|\Gamma\setminus(\Gamma_C \cup \Gamma_T)} = \mathbf{0}\right\}, \\ \widetilde{\mathbf{X}}_1 &= \left\{\mathbf{v} \in [W^{1,3}(\Omega)]^3 : \ \nabla \cdot \mathbf{v} = 0 \ and \ \mathbf{v}_{|\Gamma} = \mathbf{0}\right\}. \end{aligned} \tag{22}$$

Associated to the previous spaces, we define:

$$\begin{aligned} \mathbf{W}_1 &= W^{1,\infty,2}(0, T; \mathbf{X}_1, [L^2(\Omega)]^3) \cap \mathcal{C}([0, T]; \mathbf{X}_1), \\ \widetilde{\mathbf{W}}_1 &= W^{1,\infty,2}(0, T; \widetilde{\mathbf{X}}_1, [L^2(\Omega)]^3) \cap \mathcal{C}([0, T]; \widetilde{\mathbf{X}}_1). \end{aligned} \tag{23}$$

For the water temperature (5), we consider the following spaces:

$$X_2 = \{\theta \in H^1(\Omega) : \ \theta_{|\Gamma_S} \in L^5(\Gamma_S)\},$$
$$\widetilde{X}_2 = \{\theta \in X_2 : \ \theta_{|\Gamma_T} = 0\}, \tag{24}$$

and we define the following norm associated to above space X_2:

$$\|\theta\|_{X_2} = \|\theta\|_{H^1(\Omega)} + \|\theta\|_{L^5(\Gamma_S)}. \tag{25}$$

We have that X_2 is a reflexive separable Banach space (cf. Lemma 3.1 of [7]) and $\widetilde{X}_2 \subset L^2(\Omega) \subset \widetilde{X}_2'$ is an evolution triple.

Finally, for the Eutrophication system (13), we define:

$$\mathbf{X}_3 = [H^1(\Omega)]^5$$
$$\widetilde{\mathbf{X}}_3 = \{\mathbf{u} \in \mathbf{X}_3 : \ \mathbf{u}_{|\Gamma_T} = \mathbf{0}\}, \tag{26}$$

and we consider the following spaces associated to $\mathbf{X}_3$:

$$\mathbf{W}_3 = W^{1,2,2}(0, T; \mathbf{X}_3, \mathbf{X}_3'),$$
$$\widetilde{\mathbf{W}}_3 = W^{1,2,2}(0, T; \widetilde{\mathbf{X}}_3, \widetilde{\mathbf{X}}_3'). \tag{27}$$

Now, let's state two lemmas whose demonstration can be found in, respectively, [8, 9], which will allow us to reformulate the state system (8), (5) and (13), as homogeneous Dirichlet problems.

Lemma 1 *There exists a linear continuous extension:*

$$R_{\mathbf{v}} : [H^1(0, T)]^{N_{CT}} \to W^{1,2,2}(0, T; [H^2_\sigma(\Omega)]^3, [H^2_\sigma(\Omega)]^3)$$
$$\mathbf{g} \to R_{\mathbf{v}}(\mathbf{g}) = \boldsymbol{\zeta}_{\mathbf{g}}, \tag{28}$$

such that $\boldsymbol{\zeta}_{\mathbf{g}|_{\partial\Omega}} = \boldsymbol{\phi}_{\mathbf{g}}$, *where* $\boldsymbol{\phi}_{\mathbf{g}}$ *is defined by (2), and* $H^2_\sigma(\Omega) = \{\mathbf{u} \in [H^2(\Omega)]^3 : \nabla \cdot \mathbf{u} = 0\}$.

Remark 1 It is worthwhile emphasizing here that, thanks to the construction done in the proof of Lemma 1, we have

$$\nu \int_\Omega \epsilon(\boldsymbol{\zeta}_{\mathbf{g}}) : \epsilon(\boldsymbol{\eta})\, d\mathbf{x} = 0, \quad \forall \boldsymbol{\eta} \in \widetilde{\mathbf{X}}_1, \tag{29}$$

so, the previous term will disappear in the variational formulation.

Lemma 2 *We have that the following operator is compact*

$$\begin{array}{rl} R_{\mathbf{h}} : [L^2(0,T)]^{N_{CT}} \to & W^{1,2,2}(0,T; H^2(\Omega), H^2(\Omega)) \\ \mathbf{h} \to & R_{\mathbf{h}}(\mathbf{h}) = \zeta_{\mathbf{h}}, \end{array} \tag{30}$$

where:

$$\zeta_{\mathbf{h}}(\mathbf{x},t) = \sum_{k=1}^{N_{CT}} \beta_0(\varphi^k(\mathbf{x})) \int_{-T}^{T} \rho_\epsilon(t-\epsilon-s)\gamma_{\mathbf{h}}^k(s)\,ds, \tag{31}$$

with $\gamma_{\mathbf{h}}^k(s) \in L^2(-T,T)$*, for* $k = 1, 2, \ldots, N_{CT}$*, defined by:*

$$\gamma_{\mathbf{h}}^k(s) = \begin{cases} \frac{1}{\mu(C^k)} \int_{C^k} \theta^0\,d\gamma & \text{if } s \le 0, \\ h^k(s) & \text{if } s > 0, \end{cases} \tag{32}$$

and $\beta_0 : u \in H^{3/2}(\partial\Omega) \to \beta_0(u) \in H^2(\Omega)$ *the right inverse of the classical trace operator* γ_0*, that is,* $(\gamma_0 \circ \beta_0)(u) = u$ *(cf. Theorem 8.3 of [10]). We also have that there exists a constant C, that depends continuously on the space-time configuration of our computational domain and* θ^0*, such that:*

$$\|\zeta_{\mathbf{h}}\|_{W^{1,2,2}(0,T;H^2(\Omega),H^2(\Omega))} \le C(\theta^0)(1 + \|\mathbf{h}\|_{[L^2(0,T)]^{N_{CT}}}). \tag{33}$$

Using the previous lemmas, if we denote by $\mathbf{v} = \mathbf{z} + \boldsymbol{\zeta}_{\mathbf{g}}$, with

$$\boldsymbol{\zeta}_{\mathbf{g}} \in W^{1,2,2}(0,T; [H_\sigma^2(\Omega)]^3, [H_\sigma^2(\Omega)]^3) \tag{34}$$

the extension given by Lemma 1. $\theta = \zeta_{\mathbf{h}_\theta} + \xi$, with $\zeta_{\mathbf{h}_\theta} \in W^{1,2,2}(0,T; H^2(\Omega), H^2(\Omega))$ the extension obtained in Lemma 2, where:

$$h_\theta^k(s) = \frac{1}{\mu(C^k)} \int_{C^k} \theta(s)\,d\gamma, \quad k = 1, 2, \ldots, N_{CT}. \tag{35}$$

Finally, for the eutrophication system, $u^i = w^i + \zeta_{\mathbf{h}_\mathbf{u}^i}$, with

$$\zeta_{\mathbf{h}_\mathbf{u}^i} \in W^{1,2,2}(0,T; H^2(\Omega), H^2(\Omega)) \tag{36}$$

the extension obtained in Lemma 2, where:

$$h_\mathbf{u}^{i,k}(s) = \frac{1}{\mu(C^k)} \int_{C^k} u^i(s)\,d\gamma, \quad k = 1, 2, \ldots, N_{CT}, \quad i = 1, \ldots, 5. \tag{37}$$

Using the previous notations, we can reformulate the state system (8), (5) and (13) in the following way:

$$\begin{cases} \frac{\partial \mathbf{z}}{\partial t} + \nabla(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})\mathbf{z} + \nabla \mathbf{z}\boldsymbol{\zeta}_{\mathbf{g}} \\ -\nabla \cdot \Big(2\nu\epsilon(\mathbf{z}) + 2\nu_{tur} \int_\Omega \big[\epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z}) : \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})\big]^{1/2} \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})\Big) \\ +\nabla p = \alpha_0(\theta - \theta^0)\mathbf{a}_g - \frac{\partial \boldsymbol{\zeta}_{\mathbf{g}}}{\partial t} - \nabla\boldsymbol{\zeta}_{\mathbf{g}}\boldsymbol{\zeta}_{\mathbf{g}} + 2\nu\nabla \cdot \epsilon(\boldsymbol{\zeta}_{\mathbf{g}}), \text{ in } \Omega \times (0, T), \\ \mathbf{z} = \mathbf{0}, \text{ on } \Gamma \times (0, T), \\ \mathbf{z}(0) = \mathbf{v}^0 - \boldsymbol{\zeta}_{\mathbf{g}}(0), \text{ in } \Omega. \end{cases} \tag{38}$$

$$\begin{cases} \frac{\partial \xi}{\partial t} + \mathbf{v} \cdot \nabla\xi - \nabla \cdot (K\nabla\xi) \\ = -\frac{\partial \zeta_{\mathbf{h}_\theta}}{\partial t} - \mathbf{v} \cdot \nabla\zeta_{\mathbf{h}_\theta} + \nabla \cdot (K\nabla\zeta_{\mathbf{h}_\theta}), \text{ in } \Omega \times (0, T), \\ \xi = 0, \text{ on } T^k \times (0, T), \text{ for } k = 1, \dots, N_{CT}, \\ K\frac{\partial \xi}{\partial \mathbf{n}} = -K\frac{\partial \zeta_{\mathbf{h}_\theta}}{\partial \mathbf{n}}, \text{ on } C^k \times (0, T), \text{ for } k = 1, \dots, N_{CT}, \\ K\frac{\partial \xi}{\partial \mathbf{n}} = b_1^N\big(\theta_N - \zeta_{\mathbf{h}_\theta} - \frac{K}{b_1^N}\frac{\partial \zeta_{\mathbf{h}_\theta}}{\partial \mathbf{n}} - \xi\big), \text{ on } \Gamma_N \times (0, T), \\ K\frac{\partial \xi}{\partial \mathbf{n}} = b_1^S\big(\theta_S - \zeta_{\mathbf{h}_\theta} - \frac{K}{b_1^S}\frac{\partial \zeta_{\mathbf{h}_\theta}}{\partial \mathbf{n}} - \xi\big) \\ \quad +b_2^S\big(T_r^4 - |\xi + \zeta_{\mathbf{h}_\theta}|^3(\xi + \zeta_{\mathbf{h}_\theta})\big), \text{ on } \Gamma_S \times (0, T), \\ \xi(0) = \theta^0 - \zeta_{\mathbf{h}_\theta}(0), \text{ in } \Omega. \end{cases} \tag{39}$$

$$\begin{cases} \frac{\partial w^i}{\partial t} + \mathbf{v} \cdot \nabla w^i - \nabla \cdot (\mu^i \nabla w^i) = A^i(\mathbf{x}, t, \theta, \boldsymbol{\zeta}_{\mathbf{h}_\mathbf{u}} + \mathbf{w}) \\ \quad -\frac{\partial \zeta_{\mathbf{h}^i_\mathbf{u}}}{\partial t} - \mathbf{v} \cdot \nabla\zeta_{\mathbf{h}^i_\mathbf{u}} + \nabla \cdot (\mu^i\nabla\zeta_{\mathbf{h}^i_\mathbf{u}}), \text{ in } \Omega \times (0, T), \\ \frac{\partial w^i}{\partial \mathbf{n}} = -\mu^i\frac{\partial \zeta_{\mathbf{h}^i_\mathbf{u}}}{\partial \mathbf{n}}, \text{ on } (\Gamma_S \cup \Gamma_N \cup \Gamma_C) \times (0, T), \\ w^i = 0, \text{ on } T^k \times (0, T), \text{ for } k = 1, \dots, N_{CT}, \\ w^i(0) = u^{0,i} - \zeta_{\mathbf{h}^i_\mathbf{u}}(0), \text{ in } \Omega, \ i = 1, \dots, 5. \end{cases} \tag{40}$$

It is remarkable that the previous systems have homogeneous Dirichlet boundary conditions and then, we can define the concept of solution of the original state system (8), (5) and (13), in terms of the modified state system (38)–(40).

Definition 1 (*The concept of solution for the state system*) An element $(\mathbf{v}, \theta, \mathbf{u}) \in \mathbf{W}_1 \times W_2 \times \mathbf{W}_3$ is a solution for the state system (8), (5) and (13), if there exists an element $(\mathbf{z}, \xi, \mathbf{w}) \in \widetilde{\mathbf{W}}_1 \times \widetilde{W}_2 \times \widetilde{\mathbf{W}}_3$ such that:

– $\mathbf{v} = \mathbf{z} + \boldsymbol{\zeta}_{\mathbf{g}}$, with $\boldsymbol{\zeta}_{\mathbf{g}} \in W^{1,2,2}(0, T; [H^2_\sigma(\Omega)]^3, [H^2_\sigma(\Omega)]^3)$ the extension given by Lemma 1, $\mathbf{z}(0) = \mathbf{v}^0$, a.e. $\mathbf{x} \in \Omega$, and, for each $t \in (0, T)$, $\mathbf{z} \in \widetilde{\mathbf{W}}_1$ is the solution of the following variational formulation:

$$\begin{aligned} \int_\Omega \frac{\partial \mathbf{z}}{\partial t} \cdot \boldsymbol{\eta}\, d\mathbf{x} + \int_\Omega \nabla(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})\mathbf{z} \cdot \boldsymbol{\eta}\, d\mathbf{x} + \int_\Omega \nabla\mathbf{z}\boldsymbol{\zeta}_{\mathbf{g}} \cdot \boldsymbol{\eta}\, d\mathbf{x} + 2\nu \int_\Omega \epsilon(\mathbf{z}) : \epsilon(\boldsymbol{\eta})\, d\mathbf{x} \\ +2\nu_{tur} \int_\Omega \big[\epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z}) : \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z})\big]^{1/2} \epsilon(\boldsymbol{\zeta}_{\mathbf{g}} + \mathbf{z}) : \epsilon(\boldsymbol{\eta})\, d\mathbf{x} \\ = \int_\Omega \mathbf{H}_{\mathbf{g}} \cdot \boldsymbol{\eta}\, d\mathbf{x}, \quad \text{a.e. } t \in (0, T), \quad \forall \boldsymbol{\eta} \in \widetilde{\mathbf{X}}_1. \end{aligned} \tag{41}$$

where:

$$\mathbf{H}_{\mathbf{g}} = \alpha_0(\theta - \theta^0)\mathbf{a}_g - \frac{\partial \boldsymbol{\zeta}_{\mathbf{g}}}{\partial t} - \nabla\boldsymbol{\zeta}_{\mathbf{g}}\boldsymbol{\zeta}_{\mathbf{g}} \in L^2(0, T; [L^2(\Omega)]^3). \tag{42}$$

- $\theta = \xi + \zeta_{\mathbf{h}_\theta}$, with $\zeta_{\mathbf{h}_\theta} \in W^{1,2,2}(0,T; H^2(\Omega), H^2(\Omega))$ the extension obtained in Lemma 2, where $\mathbf{h}_\theta \in [L^2(0,T)]^{N_{CT}}$ is defined by (35), $\xi(0) = \theta^0 - \zeta_{\mathbf{h}}(0)$, a.e. $\mathbf{x} \in \Omega$, and, for each $t \in (0,T)$, $\xi \in \widetilde{W}_2$ is the solution of the following variational formulation:

$$\begin{aligned} \int_\Omega \frac{\partial \xi}{\partial t}\eta \, d\mathbf{x} + \int_\Omega \mathbf{v}\cdot\nabla\xi\eta\, d\mathbf{x} + K\int_\Omega \nabla\xi\cdot\nabla\eta\, d\mathbf{x} + b_1^N \int_{\Gamma_N}\xi\eta\, d\gamma + b_1^S\int_{\Gamma_S}\xi\eta\, d\gamma \\ + b_2^S\int_{\Gamma_S}|\xi+\zeta_{\mathbf{h}}|^3(\xi+\zeta_{\mathbf{h}})\eta\, d\gamma = \int_\Omega H_{\mathbf{h}}\eta\, d\mathbf{x} + \int_{\Gamma_C} g_{\mathbf{h}}^C\eta\, d\gamma + b_1^N\int_{\Gamma_N} g_{\mathbf{h}}^N\eta\, d\gamma \\ + b_1^S\int_{\Gamma_S} g_{\mathbf{h}}^S\eta\, d\gamma + b_2^S\int_{\Gamma_S} T_r^4\eta\, d\gamma, \quad \text{a.e. } t\in(0,T), \quad \forall \eta \in \widetilde{X}_2, \end{aligned} \tag{43}$$

where:

$$\begin{aligned} H_{\mathbf{h}} &= \frac{\partial\zeta_{\mathbf{h}}}{\partial t} - \mathbf{v}\cdot\nabla\zeta_{\mathbf{h}} + \nabla\cdot(K\nabla\zeta_{\mathbf{h}}) \in L^2(0,T;L^2(\Omega)), \\ g_{\mathbf{h}}^C &= -K\frac{\partial\zeta_{\mathbf{h}}}{\partial\mathbf{n}} \in L^2(0,T;L^2(\Gamma_C)), \\ g_{\mathbf{h}}^N &= \theta_N - \zeta_{\mathbf{h}} - \frac{K}{b_1^N}\frac{\partial\zeta_{\mathbf{h}}}{\partial\mathbf{n}} \in L^2(0,T;L^2(\Gamma_N)), \\ g_{\mathbf{h}}^S &= \theta_S - \zeta_{\mathbf{h}} - \frac{K}{b_1^S}\frac{\partial\zeta_{\mathbf{h}}}{\partial\mathbf{n}} \in L^2(0,T;L^2(\Gamma_S)). \end{aligned} \tag{44}$$

- $u^i = w^i + \zeta_{\mathbf{h}_\mathbf{u}^i}$, with $\zeta_{\mathbf{h}_\mathbf{u}^i} \in W^{1,2,2}(0,T;H^2(\Omega),H^2(\Omega))$ the extension obtained in Lemma 2, where, for each $i = 1,\ldots,5$, $\mathbf{h}_\mathbf{u}^i \in [L^2(0,T)]^{N_{CT}}$ is defined by (37), $\mathbf{w}(0) = \mathbf{u}_0 - \boldsymbol{\zeta}_{\mathbf{h}_\mathbf{u}^i}(0)$, a.e. $\mathbf{x}\in\Omega$, and, for each $t\in(0,T)$, $\mathbf{w}\in\widetilde{\mathbf{W}}_3$ is the solution of the following variational formulation:

$$\begin{aligned} \int_\Omega \frac{\partial\mathbf{w}}{\partial t}\cdot\boldsymbol{\eta}\, d\mathbf{x} + \int_\Omega \nabla\mathbf{w}\mathbf{v}\cdot\boldsymbol{\eta}\, d\mathbf{x} + \Lambda_\mu\int_\Omega \nabla\mathbf{w}:\nabla\boldsymbol{\eta}\, d\mathbf{x} = \int_\Omega \mathbf{A}(\theta, \boldsymbol{\zeta}_{\mathbf{h}_\mathbf{u}} + \mathbf{w})\cdot\boldsymbol{\eta}\, d\mathbf{x} \\ + \int_\Omega \mathbf{H}_\mathbf{u}\cdot\boldsymbol{\eta}\, d\mathbf{x} + \int_{\Gamma_S\cup\Gamma_N\cup\Gamma_C}\mathbf{g}_\mathbf{u}\cdot\boldsymbol{\eta}\, d\gamma, \quad \text{a.e. } t\in(0,T), \quad \forall\boldsymbol{\eta}\in\widetilde{\mathbf{X}}_3. \end{aligned} \tag{45}$$

where $\Lambda_\mu = diag(\mu^1,\ldots,\mu^5) \in M_{5\times5}(\mathbb{R})$ is a diagonal matrix with the diffusion coefficients and:

$$\begin{aligned} H_\mathbf{u}^i &= -\frac{\partial\zeta_{\mathbf{h}_\mathbf{u}^i}}{\partial t} - \mathbf{v}\cdot\nabla\zeta_{\mathbf{h}_\mathbf{u}^i} + \nabla\cdot(\mu^i\nabla\zeta_{\mathbf{h}_\mathbf{u}^i}) \in L^2(0,T;L^2(\Omega)),\ i=1,\ldots,5, \\ g_\mathbf{u}^i &= -\mu^i\frac{\partial\zeta_{\mathbf{h}_\mathbf{u}^i}}{\partial\mathbf{n}} \in L^2(0,T;L^2(\Gamma_S\cup\Gamma_N\cup\Gamma_C)),\ i=1,\ldots,5. \end{aligned} \tag{46}$$

Remark 2 We have the following dependence scheme between the elements of state system:

$$\begin{array}{ccccc} \mathbf{g}\rightarrow\mathbf{v} & & \longleftrightarrow & & \theta \\ & \searrow & & \swarrow & \\ & & \mathbf{u} & & \end{array} \tag{47}$$

Thus, we can mathematical analyze the system $\mathbf{g}\rightarrow\mathbf{v}\leftrightarrow\theta$, (8)–(5), first and then, we can study the system $\mathbf{u}$, (13). The coupled system (8)–(5) have been previously treated by authors in [8, 9]. So, in this work, we suppose that, for each $\mathbf{g}$, there exists a solution $(\mathbf{v},\theta)\in\widetilde{\mathbf{W}}_1\times\widetilde{W}_2$ of system (8)–(5) and we will focus our attention in the second row of the previous scheme.

We consider the following operator

$$\begin{aligned} \mathbf{M}_{\mathbf{u}} : (\mathbf{u}^*, \mathbf{h}^*_{\mathbf{u}}) \in [L^2(0,T;L^2(\Omega))]^5 \times [L^2(0,T)]^{5\times N_{CT}} \longrightarrow \\ \mathbf{M}_{\mathbf{u}}(\mathbf{u}^*, \mathbf{h}^*_{\mathbf{u}}) = (\mathbf{u}, \mathbf{h}_{\mathbf{u}}) \in [L^2(0,T;L^2(\Omega))]^5 \times [L^2(0,T)]^{5\times N_{CT}}, \end{aligned} \tag{48}$$

where $\mathbf{u}^* = (u^{1*}, \dots, u^{5*})$, with $u^i \in L^3(0,T;L^3(\Omega))$, $i = 1, \dots, 5$, $\mathbf{h}^*_{\mathbf{u}} = (\mathbf{h}^{1*}_{\mathbf{u}}, \dots, \mathbf{h}^{5*}_{\mathbf{u}})$, with $\mathbf{h}^{i*}_{\mathbf{u}} \in [L^2(0,T)]^{N_{CT}}$, $i = 1, \dots, 5$, $\mathbf{u} = (u^1, \dots, u^5) \in \mathbf{W}_3$, $\mathbf{h}_{\mathbf{u}} = (\mathbf{h}^1_{\mathbf{u}}, \dots, \mathbf{h}^5_{\mathbf{u}}) \in [L^2(0,T)]^{5\times N_{CT}}$, such that:

- For each $i = 1, \dots, 5$, $\zeta_{\mathbf{h}^{i*}_{\mathbf{u}}} \in W^{1,2,2}(0,T; H^2(\Omega), H^2(\Omega))$ is defined by Lemma 2.
- $\mathbf{u} \in \widetilde{\mathbf{W}}_3$ is the solution, in the sense of Definition 1 with obvious modifications, of the following decoupled problem with resolution order $3 \to 2 \to 4 \to 1 \to 5$:

$$\begin{cases} \frac{\partial u^i}{\partial t} + \mathbf{v}\cdot\nabla u^i - \nabla\cdot(\mu^i \nabla u^i) = \widehat{A}^i(\mathbf{x}, t, \theta, \mathbf{u}^*, \mathbf{u}), \text{ in } \Omega \times (0,T), \\ u^i = \zeta_{\mathbf{h}^{i*}_{\mathbf{u}}}, \text{ on } \Gamma_T \times (0,T), \\ \mu^i \frac{\partial u^i}{\partial \mathbf{n}} = 0, \text{ on } (\Gamma_S \cup \Gamma_N \cup \Gamma_C) \times (0,T), \\ u^i(0) = u^i_0, \text{ in } \Omega, \ i = 1, \dots, 5, \end{cases} \tag{49}$$

  where the function $\widehat{\mathbf{A}} = (\widehat{A}^i) : \Omega \times (0,T) \times [\mathbb{R}_+]^6 \times [\mathbb{R}_+]^6 \to \mathbb{R}^5$ is defined by the following expression:

$$\widehat{\mathbf{A}}(\mathbf{x}, t, \theta, \mathbf{u}^*, \mathbf{u}) = \begin{bmatrix} -\frac{C_{nc}L(\mathbf{x},t,\theta)}{K_N+|u^{1*}|}u^{1*}u^2 + C_{nc}K_r u^2 + C_{nc}K_{rd}D(\theta)u^4 \\ \frac{L(\mathbf{x},t,\theta)}{K_N+|u^{1*}|}u^{1*}u^2 - K_r u^2 - K_{mf}u^2 - \frac{K_z}{K_F+|u^{2*}|}u^{2*}u^3 \\ \frac{C_{fz}K_z}{K_F+|u^{2*}|}u^{2*}u^3 - K_{mz}u^3 \\ K_{mf}u^2 + K_{mz}u^3 - K_{rd}D(\theta)u^4 \\ \frac{C_{oc}L(\mathbf{x},t,\theta)}{K_N+|u^{1*}|}u^{1*}u^2 - C_{oc}K_r u^2 - C_{oc}K_{rd}D(\theta)u^4 \end{bmatrix}. \tag{50}$$

- For each $i = 1, \dots, 5$, $\mathbf{h}^i_{\mathbf{u}} \in [L^2(0,T)]^{N_{CT}}$ is such that:

$$h^{k,i}_{\mathbf{u}}(s) = \frac{1}{\mu(C^k)} \int_{C^k} u^i(s)\, d\gamma, \quad k = 1, 2, \dots, N_{CT}. \tag{51}$$

Remark 3 All the equations of the decoupled system (45) can be expressed as follows:

$$\begin{cases} \frac{\partial w}{\partial t} + \mathbf{v}\cdot\nabla w - \nabla\cdot(\mu\nabla w) = k_1 w + k_2, \text{ in } \Omega \times (0,T), \\ \frac{\partial w}{\partial \mathbf{n}} = k_3, \text{ on } \Gamma_1 \times (0,T), \\ w = 0, \text{ on } \Gamma_2 \times (0,T), \\ w(0) = w_0, \text{ in } \Omega, \end{cases} \tag{52}$$

where, $\Gamma_1 = \Gamma_S \cup \Gamma_N \cup \Gamma_C$, $\Gamma_2 = \Gamma_T$, $w_0 \in H^1(\Omega)$, $k_1 \in L^4(0,T;L^3(\Omega))$, $k_2 \in L^2(0,T;L^{3/2}(\Omega))$ and $k_3 \in L^2(0,T;L^2(\Gamma_1))$. It is clear that, $k_3 \in L^2(0,T;L^2(\Gamma_1))$ thanks to Lemma 2 and Theorem 3.1 of [11] for the trace operator $\gamma_1 = \partial^1/\partial\mathbf{n}^1$, as well $\|\gamma_1(\zeta_{\mathbf{h}_\mathbf{u}^{i*}})\|_{L^2(0,T;L^2(\Gamma))} \leq C_3\|\mathbf{h}_\mathbf{u}^{i*}\|_{[L^2(0,T)]^{N_{CT}}}$, $\forall i = 1, \ldots, 5$. In the other hand, it is straightforward to check that for each specie the coefficients k_1 and k_2 satisfy the above regularity.

We have the following existence result for the system (52). The proof of this result can be done using similar techniques as described in [4].

Theorem 1 *Given elements $w_0 \in H^1(\Omega)$, $k_1 \in L^4(0,T;L^3(\Omega))$, $k_2 \in L^2(0,T;L^{3/2}(\Omega))$ and $k_3 \in L^2(0,T;L^2(\Gamma_1))$. There exists and unique element*

$$w \in W^{1,2,2}(0,T;H^1_{0,\Gamma_2}(\Omega), H^1_{0,\Gamma_2}(\Omega)') \cap L^\infty(0,T;L^2(\Omega)), \tag{53}$$

with $w(0) = w_0$ a.e. $x \in \Omega$, that satisfies the following variational formulation:

$$\begin{aligned} \int_\Omega \frac{\partial w}{\partial t}\eta\, d\mathbf{x} + \int_\Omega \mu\nabla w\cdot\nabla\eta\, d\mathbf{x} = \int_\Omega k_1 w\eta\, d\mathbf{x} + \int_\Omega k_2\eta\, d\mathbf{x} \\ + \int_{\Gamma_1} k_3\eta\, d\gamma,\ a.e.\ t \in]0,T[,\ \forall\eta \in H^1_{0,\Gamma_2}(\Omega), \end{aligned} \tag{54}$$

which also verifies the following estimate:

$$\begin{aligned} \|w\|_{W^{1,2,2}(0,T;H^1_{0,\Gamma_0}(\Omega),H^1_{0,\Gamma_0}(\Omega)')} \leq \\ C(\|k_1\|_{L^4(0,T;L^3(\Omega))}, \|w_0\|_{H^1(\Omega)}, \|\mathbf{v}\|_{\mathbf{W}_1})\Big[1 + \|k_2\|_{L^2(0,T;L^{3/2}(\Omega))} \\ + \|k_3\|_{L^2(0,T;L^2(\Gamma_1))}\Big]. \end{aligned} \tag{55}$$

where C is a positive constant that depends on $\|k_1\|_{L^4(0,T;L^3(\Omega))}$, $\|w_0\|_{H^1(\Omega)}$ and $\|\mathbf{v}\|_{\mathbf{W}_1}$.

Remark 4 Thanks to Remark 3 and Theorem 1, we have the following estimates:

$$\|u^3\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} \leq C_3(\|\mathbf{v}\|_{\mathbf{W}_1})\Big[1 + \|\mathbf{h}_\mathbf{u}^{3*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{56}$$

$$\|\mathbf{h}_\mathbf{u}^3\|_{[L^2(0,T)]^{N_{CT}}} \leq C_8(\|\mathbf{v}\|_{\mathbf{W}_1})\Big[1 + \|\mathbf{h}_\mathbf{u}^{3*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{57}$$

$$\begin{aligned} \|u^2\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} \leq C_2(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_\mathbf{u}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} \\ + \|\mathbf{h}_\mathbf{u}^{3*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \end{aligned} \tag{58}$$

$$\begin{aligned} \|\mathbf{h}_\mathbf{u}^2\|_{[L^2(0,T)]^{N_{CT}}} \leq C_7(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_\mathbf{u}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} \\ + \|\mathbf{h}_\mathbf{u}^{3*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \end{aligned} \tag{59}$$

$$\|u^4\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} \leq C_4(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_{\mathbf{u}}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{3*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{4*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{60}$$

$$\|\mathbf{h}_{\mathbf{u}}^{4}\|_{[L^2(0,T)]^{N_{CT}}} \leq C_9(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_{\mathbf{u}}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{3*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{4*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{61}$$

$$\|u^1\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} \leq C_1(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_{\mathbf{u}}^{1*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{3*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{4*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{62}$$

$$\|\mathbf{h}_{\mathbf{u}}^{1}\|_{[L^2(0,T)]^{N_{CT}}} \leq C_6(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_{\mathbf{u}}^{1*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{3*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{4*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{63}$$

$$\|u^5\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} \leq C_5(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_{\mathbf{u}}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{3*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{4*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{5*}\|_{[L^2(0,T)]^{N_{CT}}}\Big], \tag{64}$$

$$\|\mathbf{h}_{\mathbf{u}}^{5}\|_{[L^2(0,T)]^{N_{CT}}} \leq C_{10}(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2})\Big[1 + \|\mathbf{h}_{\mathbf{u}}^{2*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{3*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{4*}\|_{[L^2(0,T)]^{N_{CT}}} + \|\mathbf{h}_{\mathbf{u}}^{5*}\|_{[L^2(0,T)]^{N_{CT}}}\Big]. \tag{65}$$

We must to remark that the previous estimates do not depend on the term $\mathbf{u}^*$.

Now, we will prove the main result of this paper.

Theorem 2 (Existence result for the system (13)) *If there exists coefficient and data such that:*

$$\begin{aligned} C_6(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2}) &\leq 1, \\ C_7(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2}) &\leq 1, \\ C_8(\|\mathbf{v}\|_{\mathbf{W}_1}) &\leq 1, \\ C_9(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2}) &\leq 1, \\ C_{10}(\|\mathbf{v}\|_{\mathbf{W}_1}, \|\theta\|_{W_2}) &\leq 1, \end{aligned} \tag{66}$$

for all $\mathbf{g} \in \mathcal{U}_{ad}$*, then, there will exist positive constants* $\widetilde{C}_i$*,* $i = 1, \ldots, 10$*, such that the operator* $M_{\mathbf{u}} : \mathbf{B_u} \to \mathbf{B_u}$ *defined in (48) has a fixed point solution of the state system (8), (5) and (13) in the sense of Definition 1, where:*

$$\mathbf{B_u} = \Big\{ (\mathbf{u}, \mathbf{h_u}) \in [L^2(0,T; L^2(\Omega))]^5 \times [L^2(0,T)]^{5\times N_{CT}} : \\ \|u^i\|_{L^2(0,T;L^2(\Omega))} \leq \widetilde{C}_i, \ \forall i = 1, \ldots, 5, \\ \|\mathbf{h}^i_{\mathbf{u}}\|_{[L^2(0,T)]^{N_{CT}}} \leq \widetilde{C}_{5+i}, \ \forall i = 1, \ldots, 5 \Big\}. \tag{67}$$

Proof We will apply the Schauder fixed point theorem (cf. Theorem 9.5 of [12]), so will prove that the operator $M_{\mathbf{u}}$ is compact and there exists positive constants $\{\widetilde{C}_k\}_{k=1}^{10}$ such that the operator $M_{\mathbf{u}}$ applies elements from the set $\mathbf{B_u}$ (which is closed and convex) into itself.

– The operator $M_{\mathbf{u}}$ is compact in the sense that it is continuous and that $\overline{M_{\mathbf{u}}(A)}$ is compact whenever A is a bounded subset of $[L^2(0,T; L^2(\Omega)]^5 \times [L^2(0,T)]^{5\times N_{CT}}$. Thus, given a convergent sequence $\{\mathbf{u}^*_n, \mathbf{h}^*_n\}_{n\in\mathbb{N}} \subset [L^2(0,T;\ L^2(\Omega)]^5 \times [L^2(0,T)]^{5\times N_{CT}}$ such that $\mathbf{u}^*_n \to \mathbf{u}^*$ in $[L^2(0,T; L^2(\Omega))]^5$ and $\mathbf{h}^*_n \to \mathbf{h}^*_{\mathbf{u}}$ in $[L^2(0,T)]^{5\times N_{CT}}$, we have that $M_{\mathbf{u}}(\mathbf{u}^*_n, \mathbf{h}^*_n) = (\mathbf{u}_n, \mathbf{h}_n) \in \mathbf{W}_3 \times [H^1(0,T)]^{5\times N_{CT}}$ is such that $\mathbf{u}_n = \mathbf{w}_n + \boldsymbol{\zeta}_{\mathbf{h}^*_n} \in \mathbf{W}_3$, with $\mathbf{w}_n \in \widetilde{\mathbf{W}}_3$ the solution of the following variational formulation:

$$\int_\Omega \frac{\partial \mathbf{w}_n}{\partial t} \cdot \boldsymbol{\eta}\, d\mathbf{x} + \int_\Omega \nabla \mathbf{w}_n \mathbf{v} \cdot \boldsymbol{\eta}\, d\mathbf{x} + \Lambda_\mu \int_\Omega \nabla \mathbf{w}_n : \nabla \boldsymbol{\eta}\, d\mathbf{x} \\ = \int_\Omega \widehat{\mathbf{A}}(\theta, \mathbf{u}^*_n, \boldsymbol{\zeta}_{\mathbf{h}^*_n} + \mathbf{w}_n) \cdot \boldsymbol{\eta}\, d\mathbf{x} + \int_\Omega \mathbf{H}_n \cdot \boldsymbol{\eta}\, d\mathbf{x} + \int_{\Gamma_S \cup \Gamma_N \cup \Gamma_C} \mathbf{g}_n \cdot \boldsymbol{\eta}\, d\gamma, \\ \text{a.e. } t \in (0,T),\ \forall \boldsymbol{\eta} \in \widetilde{\mathbf{X}}_3, \tag{68}$$

where:

$$\begin{aligned} H^i_n &= -\frac{\partial \zeta_{\mathbf{h}^{*i}_n}}{\partial t} - \mathbf{v} \cdot \nabla \zeta_{\mathbf{h}^{*i}_n} + \nabla \cdot (\mu^i \nabla \zeta_{\mathbf{h}^{*i}_n}) \in L^2(0,T; L^2(\Omega)),\ i = 1, \ldots, 5, \\ g^i_n &= -\mu^i \frac{\partial \zeta_{\mathbf{h}^{*i}_n}}{\partial \mathbf{n}} \in L^2(0,T; L^2(\Gamma_S \cup \Gamma_N \cup \Gamma_C)),\ i = 1, \ldots, 5, \\ h^{i,k}_n &= \frac{1}{\mu(C^k)} \int_{C^k} u^i_n\, d\gamma \in H^1(0,T),\ i = 1, \ldots, 5,\ k = 1, \ldots, N_{CT}. \end{aligned} \tag{69}$$

Thanks to Lemma 2, we have that $\boldsymbol{\zeta}_{\mathbf{h}^*_n} \to \boldsymbol{\zeta}_{\mathbf{h}^*}$ in $W^{1,2,2}(0,T; [H^2(\Omega)]^5, [H^2(\Omega)]^5)$, $\mathbf{H}_n \to \mathbf{H_u}$ in $[L^2(0,T; [L^2(\Omega))]^5$ and $\mathbf{g}_n \to \mathbf{g_u}$ in $[L^2(0,T)]^{5\times N_{CT}}$. Taking subsequences if necessary, we have that $\mathbf{u}^*_n \to \mathbf{u}^*$ a.e. $(x,t) \in \Omega \times (0,T)$, $d\mathbf{w}_n/dt \rightharpoonup d\mathbf{w}/dt$ weakly in $L^2(0,T; \widetilde{\mathbf{X}}'_3)$, $\mathbf{w}_n \rightharpoonup \mathbf{w}$ weakly in $L^2(0,T; \widetilde{\mathbf{X}}_3)$ and $\mathbf{w}_n \to \mathbf{w}$ in $[L^{10/3-\epsilon}(0,T, L^{10/3-\epsilon}(\Omega))]^5$, for all $\epsilon > 0$. It is straightforward to prove, using the previous convergences, that we can pass to the limit in the variational formulation (68) and proving that $M_{\mathbf{u}}(\mathbf{u}^*_n, \mathbf{h}^*_n) \to M_{\mathbf{u}}(\mathbf{u}^*, \mathbf{h}^*)$ in $[L^2(0,T; L^2(\Omega))]^5 \times [L^2(0,T)]^{5\times N_{CT}}$, with $(\mathbf{u}, \mathbf{h_u}) = M_{\mathbf{u}}(\mathbf{u}^*, \mathbf{h}^*)$ is such that $\mathbf{u} = \mathbf{w} + \boldsymbol{\zeta}_{\mathbf{h}^*_{\mathbf{u}}} \in \mathbf{W}_3$ and $\mathbf{w} \in \widetilde{\mathbf{W}}_3$ is the solution of the following variational formulation:

$$\begin{aligned}&\int_\Omega \frac{\partial \mathbf{w}}{\partial t}\cdot\boldsymbol{\eta}\,d\mathbf{x}+\int_\Omega \nabla\mathbf{w}\mathbf{v}\cdot\boldsymbol{\eta}\,d\mathbf{x}+\Lambda_\mu\int_\Omega\nabla\mathbf{w}:\nabla\boldsymbol{\eta}\,d\mathbf{x}\\ =&\int_\Omega \widehat{\mathbf{A}}(\theta,\mathbf{u}^*,\boldsymbol{\zeta}_{\mathbf{h}^*_\mathbf{u}}+\mathbf{w})\cdot\boldsymbol{\eta}\,d\mathbf{x}+\int_\Omega\mathbf{H}\cdot\boldsymbol{\eta}\,d\mathbf{x}+\int_{\Gamma_S\cup\Gamma_N\cup\Gamma_C}\mathbf{g}\cdot\boldsymbol{\eta}\,d\gamma,\\ &\text{a.e. } t\in(0,T),\ \forall\boldsymbol{\eta}\in\widetilde{\mathbf{X}}_3.\end{aligned} \tag{70}$$

where:

$$\begin{aligned}H^i&=-\frac{\partial\zeta_{\mathbf{h}^{*i}_\mathbf{u}}}{\partial t}-\mathbf{v}\cdot\nabla\zeta_{\mathbf{h}^{*i}_\mathbf{u}}+\nabla\cdot(\mu^i\nabla\zeta_{\mathbf{h}^{*i}_\mathbf{u}})\in L^2(0,T;L^2(\Omega)),\ i=1,\dots,5,\\ g^i_n&=-\mu^i\frac{\partial\zeta_{\mathbf{h}^{*i}_\mathbf{u}}}{\partial\mathbf{n}}\in L^2(0,T;L^2(\Gamma_S\cup\Gamma_N\cup\Gamma_C)),\ i=1,\dots,5.\end{aligned} \tag{71}$$

Finally, the compactness of $M_\mathbf{u}$ is a direct consequence of the compact embedding of the space $\mathbf{W}_3\times[H^1(0,T)]^{5\times N_{CT}}$ in the space $[L^2(0,T;L^2(\Omega))]^5\times[L^2(0,T)]^5$.

– There exists positive constants $\{\widetilde{C}_k\}_{k=1}^{10}$ such that the operator $M_\mathbf{u}$ applies elements from the set $\mathbf{B}_\mathbf{u}$ into itself. If we define the following constants:

$$\begin{aligned}\widetilde{C}_1&=\frac{C_1}{(1-C_6)(1-C_7)(1-C_8)(1-C_9)}, & \widetilde{C}_6&=\frac{C_6}{(1-C_6)(1-C_7)(1-C_8)(1-C_9)},\\ \widetilde{C}_2&=\frac{C_2}{(1-C_7)(1-C_8)}, & \widetilde{C}_7&=\frac{C_7}{(1-C_7)(1-C_8)},\\ \widetilde{C}_3&=\frac{C_3}{(1-C_8)}, & \widetilde{C}_8&=\frac{C_8}{(1-C_8)},\\ \widetilde{C}_4&=\frac{C_4}{(1-C_7)(1-C_8)(1-C_9)}, & \widetilde{C}_9&=\frac{C_9}{(1-C_7)(1-C_8)(1-C_9)},\\ \widetilde{C}_5&=\frac{C_5}{(1-C_7)(1-C_8)(1-C_9)(1-C_{10})}, & \widetilde{C}_{10}&=\frac{C_{10}}{(1-C_7)(1-C_8)(1-C_9)(1-C_{10})},\end{aligned} \tag{72}$$

we have that $(\mathbf{u},\mathbf{u}_\mathbf{u})=M_\mathbf{u}(\mathbf{u}^*,\mathbf{h}^*_\mathbf{u})\in\mathbf{B}_\mathbf{u}$, for all $(\mathbf{u}^*,\mathbf{h}^*_\mathbf{u})\in\mathbf{B}_\mathbf{u}$. Indeed, given an element $(\mathbf{u}^*,\mathbf{h}^*_\mathbf{u})\in\mathbf{B}_\mathbf{u}$, we have, thanks to the estimates (62)–(63), (58)–(59), (56)–(57), (60)–(61) and (64)–(65), that $(\mathbf{u},\mathbf{u}_\mathbf{u})=M_\mathbf{u}(\mathbf{u}^*,\mathbf{h}^*_\mathbf{u})$ satisfies the following estimates:

$$\begin{aligned}\|u^1\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')}&\le C_1\Bigg[1+\frac{C_6}{(1-C_6)(1-C_7)(1-C_8)(1-C_9)}\\ &+\frac{C_7}{(1-C_7)(1-C_8)}+\frac{C_8}{(1-C_8)}+\frac{C_9}{(1-C_7)(1-C_8)(1-C_9)}\Bigg]=\widetilde{C}_1,\\ \|\mathbf{h}^1_\mathbf{u}\|_{[L^5(0,T)]^{N_{CT}}}&\le C_6\Bigg[1+\frac{C_6}{(1-C_6)(1-C_7)(1-C_8)(1-C_9)}\\ &+\frac{C_7}{(1-C_7)(1-C_8)}+\frac{C_8}{(1-C_8)}+\frac{C_9}{(1-C_7)(1-C_8)(1-C_9)}\Bigg]=\widetilde{C}_6,\end{aligned} \tag{73}$$

$$\begin{aligned}\|u^2\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')}&\le C_2\left[1+\frac{C_7}{(1-C_7)(1-C_8)}+\frac{C_8}{(1-C_8)}\right]=\widetilde{C}_2,\\ \|\mathbf{h}^2_\mathbf{u}\|_{[L^2(0,T)]^{N_{CT}}}&\le C_7\left[1+\frac{C_7}{(1-C_7)(1-C_8)}+\frac{C_8}{(1-C_8)}\right]=\widetilde{C}_7,\end{aligned} \tag{74}$$

$$
\begin{aligned}
\|u^3\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} &\leq C_3\left[1+\tfrac{C_8}{(1-C_8)}\right]=\widetilde{C}_3,\\
\|\mathbf{h}_{\mathbf{u}}^3\|_{[L^2(0,T)]^{N_{CT}}} &\leq C_8\left[1+\tfrac{C_8}{(1-C_8)}\right]=\widetilde{C}_8,
\end{aligned}
\tag{75}
$$

$$
\begin{aligned}
\|u^4\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} &\leq C_4\Big[1+\tfrac{C_7}{(1-C_7)(1-C_8)}+\tfrac{C_8}{(1-C_8)}\\
&\quad+\tfrac{C_9}{(1-C_7)(1-C_8)(1-C_9)}\Big]=\widetilde{C}_4,\\
\|\mathbf{h}_{\mathbf{u}}^4\|_{[L^2(0,T)]^{N_{CT}}} &\leq C_9\Big[1+\tfrac{C_7}{(1-C_7)(1-C_8)}+\tfrac{C_8}{(1-C_8)}\\
&\quad+\tfrac{C_9}{(1-C_7)(1-C_8)(1-C_9)}\Big]=\widetilde{C}_9,
\end{aligned}
\tag{76}
$$

$$
\begin{aligned}
\|u^5\|_{W^{1,2,2}(0,T;H^1(\Omega),H^1(\Omega)')} &\leq C_5\Big[\tfrac{C_7}{(1-C_7)(1-C_8)}+\tfrac{C_8}{(1-C_8)}\\
&\quad+\tfrac{C_9}{(1-C_7)(1-C_8)(1-C_9)}+\tfrac{C_{10}}{(1-C_7)(1-C_8)(1-C_9)(1-C_{10})}\Big]=\widetilde{C}_5,\\
\|\mathbf{h}_{\mathbf{u}}^5\|_{[L^2(0,T)]^{N_{CT}}} &\leq C_{10}\Big[\tfrac{C_7}{(1-C_7)(1-C_8)}+\tfrac{C_8}{(1-C_8)}\\
&\quad+\tfrac{C_9}{(1-C_7)(1-C_8)(1-C_9)}+\tfrac{C_{10}}{(1-C_7)(1-C_8)(1-C_9)(1-C_{10})}\Big]=\widetilde{C}_{10}.
\end{aligned}
\tag{77}
$$

Thus $(\mathbf{u}, \mathbf{h_u}) \in \mathbf{B_u}$.

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The Fučík Spectrum as Two Regular Curves

Jiří Kadlec and Petr Nečesal

Abstract In this paper, we investigate the structure of the Fučík spectrum for the second order boundary value problem with one non-local boundary condition. We provide a new compact form of the implicit description of the Fučík spectrum in the first quadrant. Presented compact form of the implicit description can be easily implemented in numerical computing packages or computer algebra systems and also leads to a suitable parametrization of the Fučík spectrum. We prove that the Fučík spectrum consists of two regular curves of C^1 class and parametrizations for both regular curves are provided. Presented approach can be adopted for problems with other non-local boundary conditions.

Keywords Asymmetric nonlinearities · Fučík spectrum · Non-local boundary condition · Continuous curve · Reparametrization · Regular curve

1 Introduction

The Fučík spectrum was introduced by Dancer [4] and Fučík [9], who studied the solvability of the Dirichlet problem

$$u''(x) + g(u(x)) = f(x), \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0, \tag{1}$$

where the nonlinearity g is jumping, i.e.,

$$\lim_{s \to -\infty} \frac{g(s)}{s} = \alpha \neq \beta = \lim_{s \to +\infty} \frac{g(s)}{s}. \tag{2}$$

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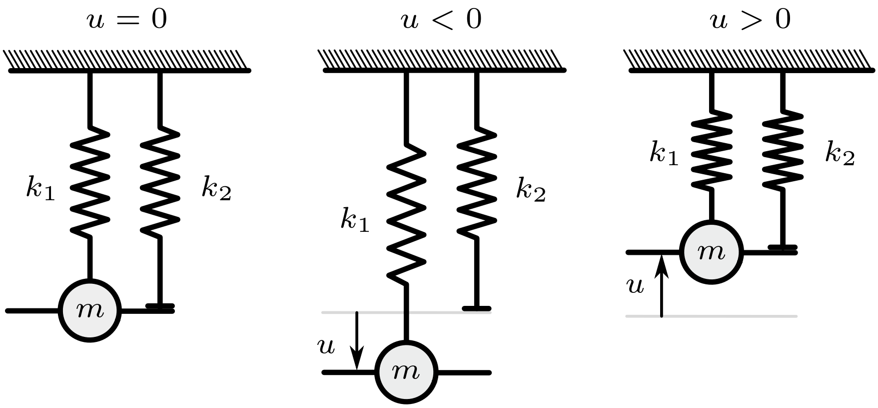

Fig. 1 A spring-mass system with two springs (linear one with stiffness k_1 and one-sided one with stiffness k_2)

They recognized that the solvability of (1) depends crucially on the fact whether or not the problem (u^+ and u^- are the positive and negative parts of u, respectively)

$$u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, \quad x \in (0, \pi), \quad u(0) = u(\pi) = 0, \qquad (3)$$

possesses a non-trivial solution. The set of all pairs (α, β) for which the problem (3) has a non-trivial solution is called the Fučík spectrum and consists of countably many unbounded curves given by ($n \in \mathbb{N}$)

$$(\alpha - 1)(\beta - 1) = 0, \quad \tfrac{n}{\sqrt{\alpha}} + \tfrac{n}{\sqrt{\beta}} = 1, \quad \tfrac{n+1}{\sqrt{\alpha}} + \tfrac{n}{\sqrt{\beta}} = 1, \quad \tfrac{n}{\sqrt{\alpha}} + \tfrac{n+1}{\sqrt{\beta}} = 1.$$

In [12, 16], Lazer and McKenna called the jumping nonlinearity g satisfying (2) as *asymmetric* nonlinearity because it can be considered as a restoring term in asymmetric oscillators. One of the simplest mechanical systems which exhibits asymmetric oscillations is a spring-mass system with one linear spring (with stiffness k_1) and one one-sided spring (with stiffness k_2), which affects the system only if the displacement u is positive with respect to the equilibrium position (see Fig. 1). The corresponding equation of motion takes the form

$$mu'' + (k_1 + k_2)u^+ - k_1 u^- = f, \qquad (4)$$

where f represents external forces, which are usually periodic. Despite its simplicity, this model can serve as a good example of the behaviour of much more complex asymmetric systems like suspension bridges (see [5] for details). The questions of solvability of the problem (4) (i.e., the existence, uniqueness and bifurcation of solutions) depend substantially on the position of the point $(\alpha, \beta) = (k_1 + k_2, k_1)$ with respect to the corresponding Fučík spectrum (see [6, 18]).

There are currently many papers studying the structure of the Fučík spectrum for various particular differential operators, let us mention here only some of them [1–3, 8, 10, 11, 17, 20–24]. Unfortunately, the Fučík problems that can be solved directly and provide new non-trivial structures of the Fučík spectrum are quite rare. On the other hand, due to recent results in [23, 24], it seems that the differential operators with integral type boundary conditions can provide full analytical description of the Fučík spectrum with such non-trivial structures. Let us note that boundary value problems with integral type boundary conditions naturally occur in hydrodynamic problems, as well as in semiconductor, thermostat, and thermal conduction problems (see [27] and references therein).

In the case of discrete variants of particular differential operators, there are several results regarding the Fučík spectrum that are available in [7, 13–15, 19, 25]. Finally, according to our best knowledge, R. Švarc was the first one who investigated the Fučík spectrum for matrices. In [26], he shows that the Fučík spectrum can exhibit strange behaviour even for small 4-by-4 matrices.

In this paper, we deal with the following boundary value problem with one non-local boundary condition in the integral form

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0, \pi), \\ u(0) = 0, \quad \int_0^\pi u(x)\,\mathrm{d}x = 0, \end{cases} \tag{5}$$

where $\alpha, \beta \in \mathbb{R}$ and $u^\pm(x) := \max\{\pm u(x), 0\}$. The aim of this paper is to study the set of all pairs $(\alpha, \beta) \in \mathbb{R}^2$ such that the problem (5) has a non-trivial solution $u \in C^2[0, \pi]$. The set of such pairs (α, β) is known as the Fučík spectrum for the problem (5) and let us denote it as

$$\Sigma := \left\{(\alpha, \beta) \in \mathbb{R}^2 : \text{ the problem (5) has a non-trivial solution } u\right\}.$$

The structure of the Fučík spectrum Σ has already been investigated by Sergejeva in [21] and thus, let us briefly recall the well-known description of this set Σ. For this purpose, let us consider the following initial value problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in \mathbb{R}, \\ u(0) = 0, \quad u'(0) = 1, \end{cases} \tag{6}$$

where $\alpha, \beta \in \mathbb{R}$, $u \in C^2(\mathbb{R})$. The Fučík spectrum Σ is symmetric with respect to the diagonal $\alpha = \beta$ (see Fig. 2) and thus, we have the following decomposition

$$\Sigma = \tilde{\Sigma} \cup \{(\alpha, \beta) : (\beta, \alpha) \in \tilde{\Sigma}\}, \tag{7}$$

where

$$\tilde{\Sigma} := \Big\{(\alpha, \beta) \in \mathbb{R}^2 : \text{the solution } u \text{ of the initial value problem (6) satisfies } \int_0^\pi u(x)\,\mathrm{d}x = 0\Big\}. \tag{8}$$

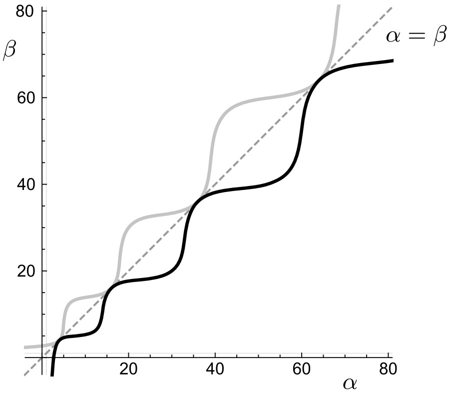

Fig. 2 The Fučík spectrum Σ (black and gray curves) and its part $\tilde{\Sigma}$ (black curves for $\beta \leq \alpha$) in $\alpha\beta$-plane (left) and in ab-plane (right), where $\alpha = \operatorname{sign}(a)a^2$ and $\beta = \operatorname{sign}(b)b^2$

At first, a pair $(\alpha, \beta) \in \tilde{\Sigma}$ with $\beta < 0$ if and only if $\alpha = a^2$, $\beta = -b^2$ and

$$\cosh\left(b\pi - \tfrac{b}{a}\pi\right) = 1 + 2\tfrac{b^2}{a^2} \quad \text{for } a > 1,\ b < 0.$$

At second, a pair $(\alpha, 0) \in \tilde{\Sigma}$ if and only if $\alpha = \left(1 + \frac{2}{\pi}\right)^2$. And finally, a pair $(\alpha, \beta) \in \tilde{\Sigma}$ with $\beta > 0$ if and only if $\alpha = a^2$, $\beta = b^2$ and

$$\cos\left(b\pi - n\left(\tfrac{b}{a} + 1\right)\pi\right) = 1 + 2n\left(\tfrac{b^2}{a^2} - 1\right) \tag{9}$$

for

$$n\tfrac{1}{a} + (n-1)\tfrac{1}{b} < 1 \leq n\tfrac{1}{a} + n\tfrac{1}{b}, \quad n \in \mathbb{N}, \tag{10}$$

and

$$\cos\left(a\pi - n\left(\tfrac{a}{b} + 1\right)\pi\right) = 1 + 2n\left(1 - \tfrac{a^2}{b^2}\right) \tag{11}$$

for

$$n\tfrac{1}{a} + n\tfrac{1}{b} < 1 \leq (n+1)\tfrac{1}{a} + n\tfrac{1}{b}, \quad n \in \mathbb{N}. \tag{12}$$

Thus, the Fučík spectrum Σ is described implicitly by parts using levels of transcendental functions of two variables a and b. Let us note that the points $(\lambda, \lambda) \in \Sigma$ on the diagonal $\alpha = \beta$ are determined by the eigenvalues λ of the following linear problem corresponding to (5) for $\alpha = \beta = \lambda$

$$\begin{cases} u''(x) + \lambda u(x) = 0, & x \in (0, \pi), \\ u(0) = 0, & \int_0^\pi u(x)\,\mathrm{d}x = 0. \end{cases} \tag{13}$$

Moreover, all eigenvalues of (13) are of the form $\lambda_n = 4n^2$, $n \in \mathbb{N}$, and coincide with all positive solutions of the following equation

$$\cos\left(\pi\sqrt{\lambda}\right) = 1. \tag{14}$$

In this paper, we prove the following main results.

1. For $\alpha, \beta > 0$, we prove that $(\alpha, \beta) \in \tilde{\Sigma}$ if and only if

$$\mathcal{G}\left(\sqrt{\frac{\beta}{\alpha}}, \frac{2\pi\sqrt{\alpha\beta}}{\sqrt{\alpha}+\sqrt{\beta}}\right) = 1 + 2\left(\sqrt{\beta} - \sqrt{\alpha}\right), \tag{15}$$

   where the function $\mathcal{G} = \mathcal{G}(k, t)$ is 2π-periodic in the second variable t and it is defined by parts on the Cartesian product $(0, +\infty) \times [0, 2\pi]$ (see Fig. 3, Definition 3 and Theorem 1 for $a = \sqrt{\alpha}$ and $b = \sqrt{\beta}$). Let us point out that $\mathcal{G}(1, t) \equiv \cos t$ and thus, for $\alpha = \beta = \lambda > 0$, the Eq. (15) simplifies to (14).
2. In Theorem 2, we provide the parametrization of the set $\tilde{\Sigma}$ by the continuous curve γ. Moreover, in the proof of Theorem 3, we show that this curve γ can be reparametrized to obtain a regular curve. Thus, we prove that the Fučík spectrum Σ consists of two regular curves of C^1 class.

Remark 1 Let us note that the condition (15) in a compact form can be expanded into the form of known Eqs. (9) and (11) with inequality conditions in (10) and in (12) (see Remark 4 for details). The advantage of the implicit condition (15) is that

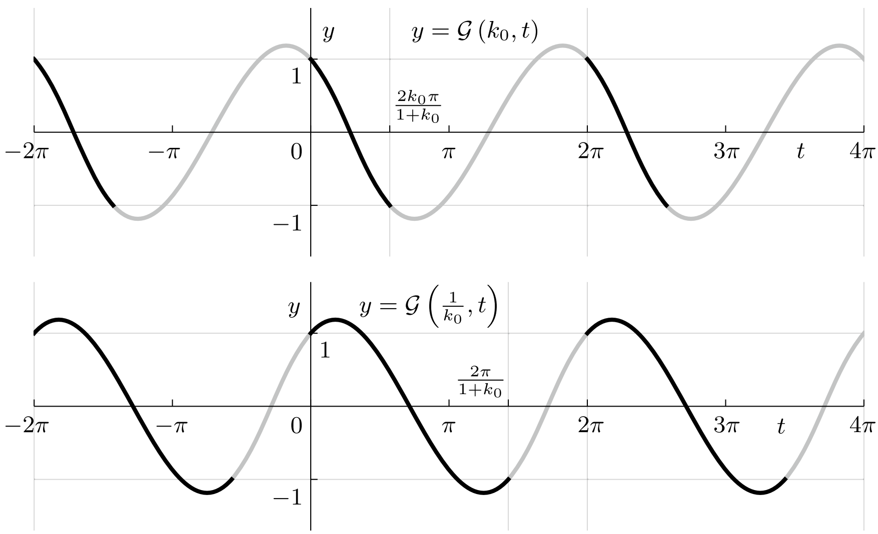

Fig. 3 Graphs of 2π-periodic functions $\mathcal{G}(k_0, \cdot)$ and $\mathcal{G}\left(\frac{1}{k_0}, \cdot\right)$ for $k_0 = \frac{2}{5}$

it can be directly used to visualize the set $\tilde{\Sigma}$ in the first quadrant of the $\alpha\beta$-plane by a standard numerical procedure that generates a contour plot for a function of two variables α and β. Indeed, in the condition (15), the function $\mathcal{G} = \mathcal{G}(k, t)$ can be easily implemented by the modulo operation $t \pmod{2\pi}$ since $\mathcal{G}$ is 2π-periodic in the second variable t according to Definition 3.

2 Preliminaries

In this part, let us recall some basic facts concerning both problems (5) and (6). If u is the solution of (5) for $\alpha = \alpha_0 \in \mathbb{R}$ and $\beta = \beta_0 \in \mathbb{R}$ then $v(x) = -u(x)$ is the solution of (5) for $\alpha = \beta_0$ and $\beta = \alpha_0$. Thus, the Fučík spectrum Σ is symmetric with respect to the diagonal $\alpha = \beta$ (see the decomposition of Σ in (7) and Fig. 2). Moreover, if u is a solution of the boundary value problem (5) then $v(x) = c \cdot u(x)$ with $c > 0$ is also a solution of (5). Thus, to describe all pairs (α, β) of the Fučík spectrum Σ, it is enough to restrict to pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which the boundary value problem (5) has a non-trivial solution u with $u'(0) = 1$ or to pairs $(\alpha, \beta) \in \mathbb{R}^2$ for which the solution u of the initial value problem (6) satisfies

$$\int_0^\pi u(x)\,\mathrm{d}x = 0, \tag{16}$$

which justifies the definition of $\tilde{\Sigma}$ in (8).

If u is the solution of the initial value problem (6) then it is straightforward to verify that (see Fig. 4)

$$u(x) = \begin{cases} \frac{1}{\sqrt{-\alpha}} \sinh\left(\sqrt{-\alpha}x\right) & \text{for } 0 \le x \quad\quad \text{and } \alpha < 0, \\ x & \text{for } 0 \le x \quad\quad \text{and } \alpha = 0, \\ \frac{1}{\sqrt{\alpha}} \sin\left(\sqrt{\alpha}x\right) & \text{for } 0 \le x \le x_1 \text{ and } \alpha > 0, \end{cases}$$

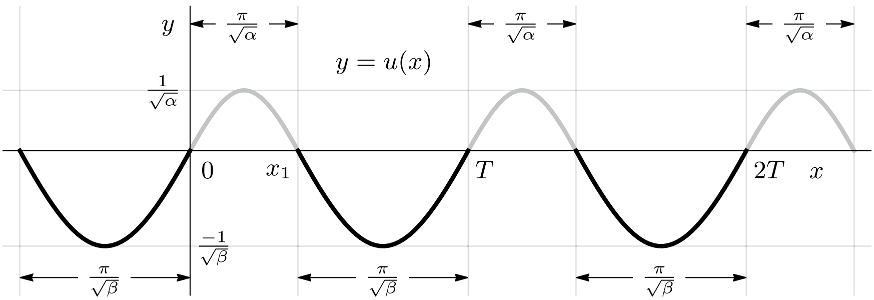

Fig. 4 The graph of T-periodic solution u of the initial value problem (6) for $0 < \beta < \alpha$

where $x_1 := \frac{\pi}{\sqrt{\alpha}}$ is the first positive zero of u for $\alpha > 0$, and that

$$u(x) = \begin{cases} -\frac{1}{\sqrt{-\beta}} \sinh\left(\sqrt{-\beta}(x - x_1)\right) & \text{for } x_1 \le x \quad\quad \text{and } \alpha > 0,\ \beta < 0, \\ -(x - x_1) & \text{for } x_1 \le x \quad\quad \text{and } \alpha > 0,\ \beta = 0, \\ -\frac{1}{\sqrt{\beta}} \sin\left(\sqrt{\beta}(x - x_1)\right) & \text{for } x_1 \le x \le T \text{ and } \alpha > 0,\ \beta > 0, \end{cases}$$

where $T := \frac{\pi}{\sqrt{\alpha}} + \frac{\pi}{\sqrt{\beta}}$ for $\alpha, \beta > 0$. Moreover, u is T-periodic for $\alpha, \beta > 0$. Thus, the necessary condition for the solution u of (6) to satisfy the integral condition (16) is $\alpha > 1$. Indeed, for $\alpha \le 1$, the solution u is only positive on $(0, \pi)$.

3 Implicit Description of the Fučík Spectrum for $\alpha, \beta > 0$

In this section, we investigate the Fučík spectrum Σ in the first quadrant of the $\alpha\beta$-plane. Thus, for $a, b > 0$, let us consider the following initial value problem

$$\begin{cases} u''(x) + a^2 u^+(x) - b^2 u^-(x) = 0, \quad x \in \mathbb{R}, \\ u(0) = 0, \quad u'(0) = a \cdot b > 0, \end{cases} \tag{17}$$

where $u \in C^2(\mathbb{R})$. Let us recall that the solution u of this problem (17) is T-periodic with $T = \frac{\pi}{a} + \frac{\pi}{b}$ and

$$u(x) = \begin{cases} b \sin(ax) & \text{for } x \in \left[0, \frac{\pi}{a}\right), \\ -a \sin\left(b\left(x - \frac{\pi}{a}\right)\right) & \text{for } x \in \left[\frac{\pi}{a}, T\right]. \end{cases} \tag{18}$$

Now, let us define the set

$$\mathcal{M} := \Big\{(a, b) \in \mathbb{R}^+ \times \mathbb{R}^+ : \text{the solution } u \text{ of the initial value problem (17) satisfies } \int_0^\pi u(x)\,\mathrm{d}x = 0\Big\}.$$

Remark 2 If $(a, b) \in \mathcal{M}$ then $(a^2, b^2) \in \tilde{\Sigma}$, where $\tilde{\Sigma}$ determines the Fučík spectrum Σ (see the decomposition of Σ in (7)). And vice versa, if $(\alpha, \beta) \in \tilde{\Sigma}$ with $\alpha, \beta > 0$ then $(\sqrt{\alpha}, \sqrt{\beta}) \in \mathcal{M}$.

In the following part, we rewrite the integral condition (16) into another form such that π as the right end point of the integration interval will be the solution of an equation containing a periodic function.

Definition 1 For $a, b > 0$, let us define

$$F(x) := \int_0^x u(t)\,\mathrm{d}t, \quad x \in \mathbb{R},$$

where u is the solution of the initial value problem (17).

Using the function F, the integral condition (16) reads $F(\pi) = 0$. Unfortunately, according to the following lemma, the function F is periodic if and only if $a = b$.

Lemma 1 *For the function F, we have*

$$\forall x \in \mathbb{R}: \ F(x+T) = F(x) + F(T), \tag{19}$$

where $T = \frac{\pi}{a} + \frac{\pi}{b}$. Moreover, the function F is periodic if and only if $a = b$.

Proof Since u is the T-periodic function, we get

$$F(x+T) = \int_0^x u(t)\,\mathrm{d}t + \int_x^{x+T} u(t)\,\mathrm{d}t = F(x) + \int_0^T u(t)\,\mathrm{d}t = F(x) + F(T)$$

for all $x \in \mathbb{R}$. Moreover, using (18), we obtain

$$F(T) = \int_0^{\frac{\pi}{a}} u(t)\,\mathrm{d}t + \int_{\frac{\pi}{a}}^{\frac{\pi}{a}+\frac{\pi}{b}} u(t)\,\mathrm{d}t = \frac{2b}{a} - \frac{2a}{b} = \frac{2T}{\pi}(b-a), \tag{20}$$

which finishes the proof.

Definition 2 For $a, b > 0$, let us define

$$G(x) := 1 - \int_0^x (u(t) - \overline{u})\ \mathrm{d}t, \quad x \in \mathbb{R}, \qquad \overline{u} := \frac{1}{T}\int_0^T u(t)\,\mathrm{d}t,$$

where $T = \frac{\pi}{a} + \frac{\pi}{b}$ and u is the solution of the initial value problem (17).

Let us note that in Definition 2, $\overline{u}$ is the mean value of u over the interval $[0, T]$ and it depends only on a and b. Moreover, for all $x \in \mathbb{R}$, we have

$$G(x) = 1 - F(x) + \tfrac{F(T)}{T}x, \tag{21}$$

and thus, the integral condition (16) can be written in the following way

$$G(\pi) = 1 + \tfrac{F(T)}{T}\pi. \tag{22}$$

According to the following lemma, the left hand side of the equality (22) is given by the value of the periodic function G at π.

Lemma 2 *The function G is T-periodic with $T = \frac{\pi}{a} + \frac{\pi}{b}$.*

Proof For all $x \in \mathbb{R}$, we obtain using (21) and (19) that

$$\begin{aligned} G(x+T) &= 1 - F(x+T) + \tfrac{F(T)}{T}(x+T) \\ &= 1 - F(x) - F(T) + \tfrac{F(T)}{T}x + F(T) \\ &= G(x). \end{aligned}$$

In the following definition, we introduce the function $\mathcal{G} = \mathcal{G}(k, t)$ which is 2π-periodic in the second variable (see Figs. 3 and 5) and we use it to evaluate the values of the T-periodic function G (see (25) in the proof of Theorem 1). Let us note that $\mathcal{G}(1, t) \equiv \cos t$.

Definition 3 Let us define the function $P : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ as

$$P(k,t) := \left(k - \frac{1}{k}\right)\frac{t}{\pi} + 1, \quad k > 0,\ t \in \mathbb{R}, \tag{23}$$

and the function $\mathcal{G} : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ which is 2π-periodic in the second variable

$$\forall k > 0\ \forall t \in \mathbb{R} :\ \mathcal{G}(k, t + 2\pi) = \mathcal{G}(k, t)$$

and is given for $k > 0$ and $t \in [0, 2\pi]$ by

$$\mathcal{G}(k,t) := \begin{cases} k\cos\left(\frac{1+k}{2k}t\right) + P(k,t) - k & \text{for } 0 \le t < \frac{k}{1+k}2\pi, \\ \frac{1}{k}\cos\left(\frac{1+k}{2}(2\pi - t)\right) + P(k, t-\pi) - k & \text{for } \frac{k}{1+k}2\pi \le t \le 2\pi. \end{cases}$$

The following theorem provides the necessary and sufficient condition for a pair (a, b) to be in the set $\mathcal{M}$. This condition contains the function $\mathcal{G}$ and has straightforward implementation in numerical computing packages or computer algebra systems since the 2π-periodicity of the function $\mathcal{G}$ in its second variable t can be easily implemented by the modulo operation $t \pmod{2\pi}$.

Theorem 1 *We have that* $(a, b) \in \mathcal{M}$ *if and only if* $a, b > 0$ *and*

$$\mathcal{G}\left(\frac{b}{a}, \frac{2ab\pi}{a+b}\right) = 1 + 2(b-a). \tag{24}$$

Proof Let u be the solution of the initial value problem (17) for $a, b > 0$. Then the integral condition (16) can be equivalently written as (22), where the function $G = G(x)$ is given in Definition 2. We claim that

$$\forall x \in \mathbb{R} :\ G(x) = \mathcal{G}\left(\frac{b}{a}, \frac{2ab}{a+b}x\right), \tag{25}$$

which implies that the equality (22) is exactly the equality (24) since we have $\frac{F(T)}{T}\pi = 2\,(b-a)$ due to (20).

It remains to prove (25). If we denote $k := \frac{b}{a}$ then the equality in (25) can be equivalently written as $G(x) = \mathcal{G}\left(k, \frac{2\pi}{T}x\right)$, where $T = \frac{a+b}{ab}\pi$. Thus, in order to justify (25), it is enough to show that

$$\forall x \in [0, T] :\ G(x) = \mathcal{G}\left(k, \tfrac{2\pi}{T}x\right) \tag{26}$$

since G is T-periodic function and $\mathcal{G}$ is 2π-periodic function in the second variable. At first, using (21) and (20), we get

$$G(x) = 1 - F(x) + \tfrac{2}{\pi}(b-a)x \tag{27}$$

and thus, we obtain using (18) that

$$G(x) = \begin{cases} \frac{b}{a}\cos(ax) - \frac{b}{a} + \frac{2}{\pi}(b-a)x + 1 & \text{for } 0 \le x < \frac{\pi}{a}, \\ -\frac{a}{b}\cos\left(b\left(x - \frac{\pi}{a}\right)\right) + \frac{a}{b} - \frac{2b}{a} + \frac{2}{\pi}(b-a)x + 1 & \text{for } \frac{\pi}{a} \le x \le T. \end{cases}$$

At second, in the case of $x \in \left[0, \frac{\pi}{a}\right)$, we have $0 \le t := \frac{2\pi}{T}x = \frac{2ab}{a+b}x < \frac{2b\pi}{a+b} = \frac{2k\pi}{1+k}$ and

$$\begin{aligned} G(x) &= G\left(\tfrac{a+b}{2ab}t\right) \\ &= \tfrac{b}{a}\cos\left(\tfrac{1}{2}\left(1 + \tfrac{a}{b}\right)t\right) - \tfrac{b}{a} + \left(\tfrac{b}{a} - \tfrac{a}{b}\right)\tfrac{t}{\pi} + 1 \\ &= k\cos\left(\tfrac{1}{2}\left(1 + \tfrac{1}{k}\right)t\right) - k + \left(k - \tfrac{1}{k}\right)\tfrac{t}{\pi} + 1 \\ &= k\cos\left(\tfrac{k+1}{2k}t\right) - k + P(k,t) \\ &= \mathcal{G}(k,t). \end{aligned}$$

At third, in the case of $x \in \left[\frac{\pi}{a}, T\right]$, we have $\frac{2k\pi}{1+k} \le t := \frac{2\pi}{T}x \le 2\pi$ and

$$\begin{aligned} G(x) &= G\left(\tfrac{a+b}{2ab}t\right) \\ &= \tfrac{a}{b}\cos\left(\tfrac{1}{2}\left(1 + \tfrac{b}{a}\right)(t-2\pi)\right) - \tfrac{b}{a} + \left(\tfrac{b}{a} - \tfrac{a}{b}\right)\tfrac{t-\pi}{\pi} + 1 \\ &= \tfrac{1}{k}\cos\left(\tfrac{1}{2}(1+k)(t-2\pi)\right) - k + \left(k - \tfrac{1}{k}\right)\tfrac{t-\pi}{\pi} + 1 \\ &= \tfrac{1}{k}\cos\left(\tfrac{1+k}{2}(t-2\pi)\right) - k + P(k, t-\pi) \\ &= \mathcal{G}(k,t). \end{aligned}$$

Thus, (26) is justified, which finishes the proof.

The next statement follows directly from the previous Theorem 1 and says that finding all pairs $(a,b) \in \mathcal{M}$ is equivalent to finding all pairs (k,t) with $k, t > 0$ such that $\mathcal{G}(k,t) = P(k,t)$. Moreover, for fixed $k = k_0 > 0$, each positive solution t_i of the equation $\mathcal{G}(k_0,t) = P(k_0,t)$ (see Fig. 5) is in one to one correspondence with a point $(a_i, b_i) \in \mathcal{M}$ located on the line $b = k_0 a$ (see Fig. 6).

Corollary 1 *We have that* $(a,b) \in \mathcal{M}$ *if and only if*

$$a = \left(1 + \tfrac{1}{k}\right)\tfrac{t}{2\pi}, \quad b = (1+k)\tfrac{t}{2\pi}, \quad k, t > 0, \tag{28}$$

and

$$\mathcal{G}(k,t) = P(k,t). \tag{29}$$

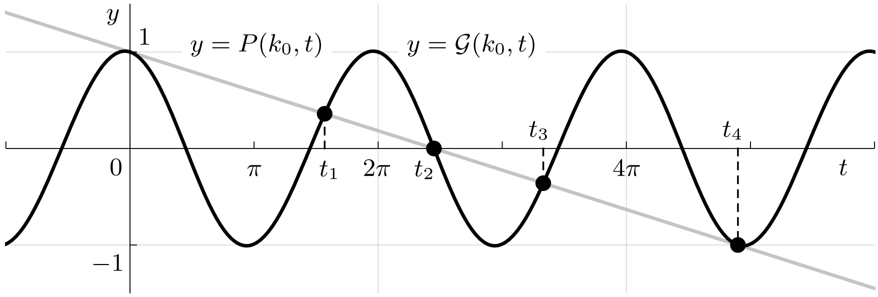

Fig. 5 The graph of 2π-periodic function $\mathcal{G}(k_0, \cdot)$ (the black curve) and the graph of the linear function $P(k_0, \cdot)$ (the gray line) for $k_0 = \sqrt{\frac{2}{3}}$ and all positive solutions t_1, t_2, t_3 and t_4 of the equation $\mathcal{G}(k_0, t) = P(k_0, t)$

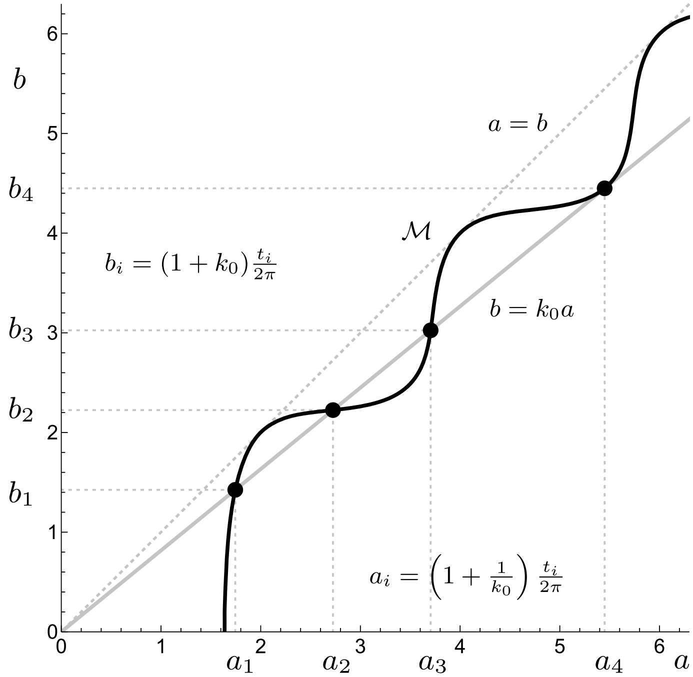

Fig. 6 The intersection points (a_i, b_i), $i = 1, \ldots, 4$, of the set $\mathcal{M}$ (the black curve) and the line $b = k_0 a$ (the gray line) for $k_0 = \sqrt{\frac{2}{3}}$, which are uniquely determined by positive solutions t_1, t_2, t_3 and t_4 of the equation $\mathcal{G}(k_0, t) = P(k_0, t)$

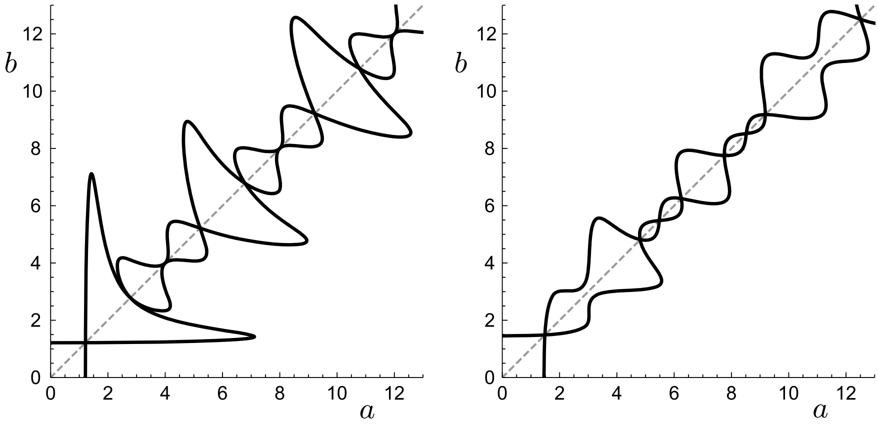

Fig. 7 The Fučík spectrum Σ_{pc} (black curves) for $p = \frac{\pi}{2}$, $c = -\frac{1}{3}$ (left) and for $p = \frac{2\pi}{7}$, $c = -\frac{1}{3}$ (right) in ab-plane, where $a = \sqrt{\alpha} > 0$ and $b = \sqrt{\beta} > 0$

Proof The condition (24) in Theorem 1 can be also written as $\mathcal{G}(k,t) = P(k,t)$, where

$$k = \tfrac{b}{a}, \quad t = \tfrac{2ab}{a+b}\pi, \quad a, b > 0. \tag{30}$$

Indeed, $P(k,t) = \left(k - \frac{1}{k}\right)\frac{t}{\pi} + 1 = \left(\frac{b}{a} - \frac{a}{b}\right)\frac{2ab}{a+b} + 1 = 1 + 2(b-a)$. The inverse transformation in change of variables (30) has the form of (28). Indeed, we have $\frac{t}{2\pi} = \frac{ab}{a+b} = \frac{ak}{1+k}$ since $b = ka$.

Example 1 Let us consider the following boundary value problem

$$\begin{cases} u''(x) + \alpha u^+(x) - \beta u^-(x) = 0, & x \in (0,\pi), \\ u(0) = 0, \quad c \cdot \int_0^p u(x)\,\mathrm{d}x + \int_p^\pi u(x)\,\mathrm{d}x = 0, \end{cases} \tag{31}$$

where $0 < p < \pi$, $c \in \mathbb{R}$, and let us denote its corresponding Fučík spectrum as

$$\Sigma_{pc} := \left\{(\alpha,\beta) \in \mathbb{R}^2 : \text{ the problem (31) has a non-trivial solution } u\right\}.$$

Let us note that for $c = 1$, the Fučík spectrum Σ_{pc} coincides with the Fučík spectrum Σ for the problem (5). A pair (α,β) with $\alpha, \beta > 0$ belongs to Σ_{pc} if and only if $(\alpha,\beta) = (a^2, b^2)$ or $(\alpha,\beta) = (b^2, a^2)$ and $a, b > 0$ satisfy

$$\mathcal{G}\left(\tfrac{b}{a}, \tfrac{2ab}{a+b}\pi\right) + (c-1)\cdot\mathcal{G}\left(\tfrac{b}{a}, \tfrac{2ab}{a+b}p\right) = c + 2(b-a)\left(1 + \tfrac{p}{\pi}(c-1)\right). \tag{32}$$

Indeed, using (27), we have

$$F(x) = \int_0^x u(t)\,\mathrm{d}t = 1 - G(x) + \tfrac{2}{\pi}(b-a)x$$

and thus, the integral condition in (31) reads $(c-1)\cdot F(p)+F(\pi)=0$ and can be written as

$$(c-1)\cdot\left(1-G(p)+\tfrac{2}{\pi}(b-a)p\right)+1-G(\pi)+2(b-a)=0,$$

which justifies (32) if we take into account (25). See Fig. 7 for the Fučík spectrum Σ_{pc} for $c=-\frac{1}{3}$ and two different settings of p. Let us note that both pictures in Fig. 7 were obtained using the implicit condition (32) and a standard numerical procedure that generates a contour plot for a function of two variables a and b.

4 The Fučík Spectrum as Parametrized Curves

In this section, we show how to parametrize the set $\tilde{\Sigma}$ as a continuous curve (see Theorem 2) and as a regular curve (see Theorem 3). Let us begin with the parametrization in the fourth quadrant of the $\alpha\beta$-plane. Thus, let us consider the initial value problem (6) for $\alpha=a^2>0$ and $\beta=-b^2\le 0$, i.e.

$$\begin{cases} u''(x)+a^2u^+(x)+b^2u^-(x)=0, & x\in\mathbb{R},\\ u(0)=0, \quad u'(0)=1, \end{cases} \tag{33}$$

where $a>0$, $b\le 0$, $u\in C^2(\mathbb{R})$, and define the set (see Fig. 8)

$$\mathcal{N}:=\Big\{(a,b)\in\mathbb{R}^+\times\mathbb{R}_0^- : \text{the solution } u \text{ of the initial value problem (33) satisfies } \int_0^\pi u(x)\,\mathrm{d}x=0\Big\}.$$

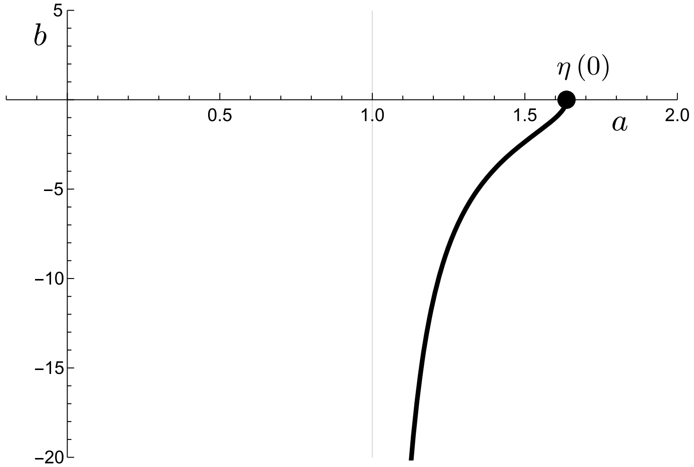

Fig. 8 The set $\mathcal{N}$ as the continuous curve η (the black curve) in ab-plane

Remark 3 If $(a, b) \in \mathcal{N}$ then $(a^2, -b^2) \in \tilde{\Sigma}$, where $\tilde{\Sigma}$ determines the Fučík spectrum Σ (see the decomposition of Σ in (7)). And vice versa, if $(\alpha, \beta) \in \tilde{\Sigma}$ with $\alpha > 0$ and $\beta \leq 0$ then $(\sqrt{\alpha}, -\sqrt{-\beta}) \in \mathcal{N}$.

Lemma 3 *The set* $\mathcal{N}$ *is a continuous curve* $\eta : (-\infty, 0] \to \mathbb{R}^2$ *with the parametrization* $\eta(s) := (\eta_1(s), \eta_2(s))$*, where functions* $\eta_1, \eta_2 : (-\infty, 0] \to \mathbb{R}$ *are defined as*

$$\eta_1(s) := \begin{cases} 1 - \frac{s}{\pi}\sqrt{\frac{2}{\cosh s - 1}} & \textit{for } s < 0, \\ 1 + \frac{2}{\pi} & \textit{for } s = 0, \end{cases} \tag{34}$$

and

$$\eta_2(s) := \begin{cases} \frac{s}{\pi} - \sqrt{\frac{\cosh s - 1}{2}} & \textit{for } s < 0, \\ 0 & \textit{for } s = 0. \end{cases} \tag{35}$$

Proof Let u be the solution of the initial value problem (33) for $a > 0$ and $b < 0$. Then we have

$$u(x) = \begin{cases} \frac{1}{a}\sin(ax) & \text{for } 0 \leq x \leq x_1, \\ -\frac{1}{b}\sinh(b(x - x_1)) & \text{for } x_1 < x, \end{cases}$$

where $x_1 = \frac{\pi}{a} > 0$ is the first positive zero of u. For $0 < a \leq 1$, we have $x_1 \geq \pi$, the solution u is only positive on the interval $(0, \pi)$ and thus, the integral condition (16) cannot be satisfied. On the other hand, for $a > 1$, we have $x_1 < \pi$ and the integral condition (16) reads

$$\begin{aligned} &\tfrac{2}{a^2} + \tfrac{1}{b^2}\left(1 - \cosh\left(\left(b - \tfrac{b}{a}\right)\pi\right)\right) = 0, \\ &\qquad\qquad \cosh\left(\left(b - \tfrac{b}{a}\right)\pi\right) = 1 + 2\tfrac{b^2}{a^2}, \quad a > 1,\ b < 0. \end{aligned} \tag{36}$$

Now, using the following change of variables

$$s = \left(b - \tfrac{b}{a}\right)\pi, \quad k = \tfrac{b}{a}, \quad a > 1,\ b < 0, \tag{37}$$

conditions in (36) can be equivalently written as

$$\cosh s = 1 + 2k^2, \quad s < 0,\ k < 0,$$

or as

$$k = -\sqrt{\frac{\cosh s - 1}{2}}, \quad s < 0. \tag{38}$$

The inverse transformation in change of variables (37) has the form

$$a = 1 + \tfrac{s}{k\pi}, \quad b = k + \tfrac{s}{\pi}, \quad s < 0,\ k < 0,$$

and thus, using (38), we get

$$a = 1 - \frac{s}{\pi}\sqrt{\frac{2}{\cosh s - 1}} = \eta_1(s), \quad b = \frac{s}{\pi} - \sqrt{\frac{\cosh s - 1}{2}} = \eta_2(s).$$

If u is the solution of the initial value problem (33) for $a > 1$ and $b = 0$ then the integral condition (16) reads $\frac{2}{a^2} - \frac{\pi^2}{2}\left(1 - \frac{1}{a}\right)^2 = 0$, which implies $a = 1 + \frac{2}{\pi}$. Finally, it is straightforward to verify the continuity of functions η_1 and η_2, which finishes the proof.

Lemma 4 *The set $\mathcal{M}$ is a continuous curve $\mu : (0, +\infty) \to \mathbb{R}^2$ with the parametrization $\mu(s) := (\mu_1(s), \mu_2(s))$, where functions $\mu_1, \mu_2 : (0, +\infty) \to \mathbb{R}$ are defined as*

$$\mu_1(s) := \begin{cases} n + \dfrac{(s - n\pi + \pi)\sqrt{2n}}{\pi\sqrt{2n - 1 - \cos s}} & \text{for } s \in (2n\pi - 2\pi, 2n\pi - \pi],\ n \in \mathbb{N}, \\[2ex] \dfrac{s - n\pi + \pi}{\pi} + \sqrt{\dfrac{n}{2}}\sqrt{2n + 1 + \cos s} & \text{for } s \in (2n\pi - \pi, 2n\pi],\ n \in \mathbb{N}, \end{cases} \tag{39}$$

and

$$\mu_2(s) := \begin{cases} \dfrac{s - n\pi + \pi}{\pi} + \sqrt{\dfrac{n}{2}}\sqrt{2n - 1 - \cos s} & \text{for } s \in (2n\pi - 2\pi, 2n\pi - \pi],\ n \in \mathbb{N}, \\[2ex] n + \dfrac{(s - n\pi + \pi)\sqrt{2n}}{\pi\sqrt{2n + 1 + \cos s}} & \text{for } s \in (2n\pi - \pi, 2n\pi],\ n \in \mathbb{N}. \end{cases} \tag{40}$$

Proof Let u be the solution of the initial value problem (17) for $a, b > 0$. Using Corollary 1, the integral condition (16) can be written as (29), where k and t are given by (30) as $k = \frac{b}{a}$ and $t = \frac{2ab}{a+b}\pi$. Now, let us split the proof according to the value of t.

1. For $0 < t \leq \frac{k}{1+k}2\pi$, the condition (29) is not satisfied. Indeed, the condition (29) can be written as
$$k \cos\left(\tfrac{1+k}{2k}t\right) + P(k, t) - k = P(k, t)$$

or as $\cos\left(\frac{1+k}{2k}t\right) = 1$, which cannot be satisfied since $0 < \frac{1+k}{2k}t \leq \pi$.

2. In this case, let us consider

$$2(n-1)\pi + \tfrac{k}{1+k}2\pi < t \leq 2n\pi, \quad n \in \mathbb{N}. \tag{41}$$

Using 2π-periodicity of $\mathcal{G}$ in the second variable, the condition (29) reads

$$\begin{aligned} \mathcal{G}(k, t-2(n-1)\pi) &= P(k,t), \\ \tfrac{1}{k}\cos\left(\tfrac{1+k}{2}(2\pi - t + 2(n-1)\pi)\right) + P(k, t-(2n-1)\pi) - k &= P(k,t), \\ \tfrac{1}{k}\cos\left(\tfrac{1+k}{2}(t-2n\pi)\right) + P(k,t) - (k - \tfrac{1}{k})(2n-1) - k &= P(k,t), \\ \cos\left(\tfrac{1+k}{2}(t-2n\pi)\right) - 2n\left(k^2-1\right) - 1 &= 0. \end{aligned} \tag{42}$$

Now, if we denote

$$s = \tfrac{1+k}{2}(t-2n\pi) + 2n\pi - \pi \tag{43}$$

then according to (41), we have $2n\pi - 2\pi < s \leq 2n\pi - \pi$ and the condition (42) reads $\cos s + 2n(k^2-1) + 1 = 0$ or (recall that $k > 0$)

$$k = \frac{\sqrt{2n-1-\cos s}}{\sqrt{2n}}. \tag{44}$$

Using (43), we get $t = 2n\pi + \frac{2}{1+k}(s - 2n\pi + \pi)$, and thus, using (44) and the inverse transformation (28) in change of variables (30), we obtain

$$\begin{aligned} b &= (1+k)\,n + \frac{s-2n\pi+\pi}{\pi} \\ &= \left(1 + \frac{\sqrt{2n-1-\cos s}}{\sqrt{2n}}\right) n + \frac{s-2n\pi+\pi}{\pi} \\ &= \frac{s-n\pi+\pi}{\pi} + \sqrt{\frac{n}{2}}\sqrt{2n-1-\cos s} \\ &= \mu_2(s). \end{aligned} \tag{45}$$

Finally, since $a = \frac{1}{k}b$, we obtain using (44) and (45) that

$$a = \frac{(s-n\pi+\pi)\sqrt{2n}}{\pi\sqrt{2n-1-\cos s}} + n = \mu_1(s).$$

3. Now, let us consider

$$2n\pi < t \leq 2n\pi + \tfrac{k}{1+k}2\pi, \quad n \in \mathbb{N}. \tag{46}$$

If we take into account 2π-periodicity of $\mathcal{G}$ in the second variable, the condition (29) reads

$$
\begin{aligned}
\mathcal{G}(k, t - 2n\pi) &= P(k, t), \\
k \cos\left(\tfrac{1+k}{2k}\,(t - 2n\pi)\right) + P(k, t - 2n\pi) - k &= P(k, t), \\
\cos\left(\tfrac{1+k}{2k}\,(t - 2n\pi)\right) - 2n\left(1 - \tfrac{1}{k^2}\right) - 1 &= 0.
\end{aligned} \tag{47}
$$

If we denote

$$
s = \tfrac{1+k}{2k}\,(t - 2n\pi) + 2n\pi - \pi \tag{48}
$$

then according to (46), we have $2n\pi - \pi < s \leq 2n\pi$ and the condition (47) reads $\cos s + 1 + 2n\left(1 - \frac{1}{k^2}\right) = 0$ or (recall that $k > 0$)

$$
k = \frac{\sqrt{2n}}{\sqrt{1 + 2n + \cos s}}. \tag{49}
$$

Using (48), we get $t = 2n\pi + \frac{2k}{1+k}(s - 2n\pi + \pi)$ and thus, using the inverse transformation (28) and (49), we obtain

$$
\begin{aligned}
a &= \left(1 + \frac{1}{k}\right) n + \frac{s - 2n\pi + \pi}{\pi} \\
&= \left(1 + \frac{\sqrt{1 + 2n + \cos s}}{\sqrt{2n}}\right) n + \frac{s - 2n\pi + \pi}{\pi} \\
&= \frac{s - n\pi + \pi}{\pi} + \sqrt{\frac{n}{2}}\sqrt{2n + 1 + \cos s} \\
&= \mu_1(s).
\end{aligned} \tag{50}
$$

Since $b = ka$, we conclude using (49) and (50) that

$$
b = \frac{(s - n\pi + \pi)\sqrt{2n}}{\pi\sqrt{2n + 1 + \cos s}} + n = \mu_2(s).
$$

Finally, it is straightforward to verify the continuity of functions μ_1 and μ_2, which finishes the proof.

Remark 4 Let us note that using the transformation (30), the Eq. (42) with conditions in (41) can be identified as (9) and (10), as well as the Eq. (47) with conditions in (46) coincide with (11) and (12).

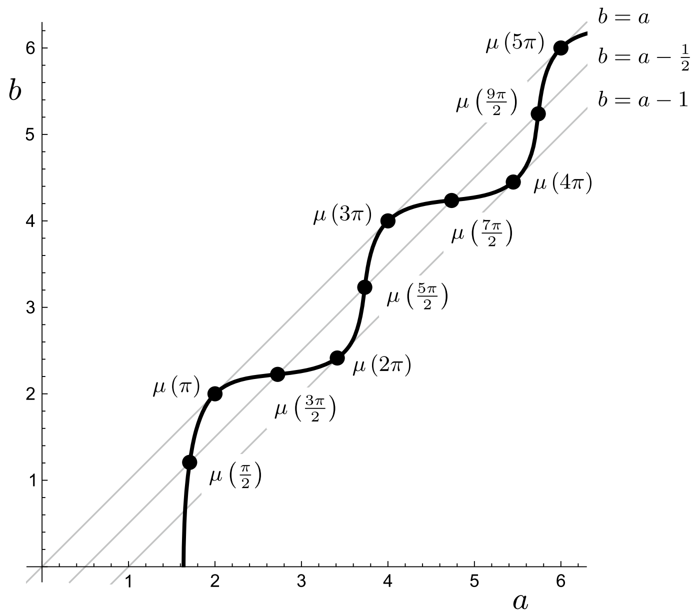

Fig. 9 The set $\mathcal{M}$ as the continuous curve μ (the black curve) in ab-plane

Remark 5 Let us note that for $n \in \mathbb{N}$, the following points belong to the set $\mathcal{M}$ (see Fig. 9):

$$\mu(2n\pi - \pi) = (2n, 2n), \tag{51}$$

$$\mu(2n\pi) = \left(n + 1 + \sqrt{n(n+1)}, n + \sqrt{n(n+1)}\right), \tag{52}$$

$$\mu\left(2n\pi - \tfrac{3\pi}{2}\right) = \tfrac{1}{2}\left(2n + \sqrt{(2n-1)2n}, 2n - 1 + \sqrt{(2n-1)2n}\right), \tag{53}$$

$$\mu\left(2n\pi - \tfrac{\pi}{2}\right) = \tfrac{1}{2}\left(2n + 1 + \sqrt{2n(2n+1)}, 2n + \sqrt{2n(2n+1)}\right). \tag{54}$$

Moreover, let us point out that points in (51) and in (52) lie on lines $b = a$ and $b = a - 1$, in particular. The points in (53) and in (54) lie on the line $b = a - \frac{1}{2}$.

Theorem 2 *The set $\tilde{\Sigma}$ is a continuous curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ with the parametrization $\gamma(s) := (\gamma_1(s), \gamma_2(s))$, where continuous functions $\gamma_1, \gamma_2 : \mathbb{R} \to \mathbb{R}$ are defined using* (34)*,* (35)*,* (39) *and* (40) *as*

$$\gamma_1(s) := \begin{cases} \eta_1^2(s) & \text{for } s \le 0, \\ \mu_1^2(s) & \text{for } s > 0, \end{cases} \qquad \gamma_2(s) := \begin{cases} -\eta_2^2(s) & \text{for } s \le 0, \\ \mu_2^2(s) & \text{for } s > 0. \end{cases} \tag{55}$$

The Fučík spectrum Σ *has the following parametrization*

$$\Sigma = \{(\gamma_1(s), \gamma_2(s)) : s \in \mathbb{R}\} \cup \{(\gamma_2(s), \gamma_1(s)) : s \in \mathbb{R}\} .$$

Proof The first statement follows directly from Lemmas 3 and 4 if we take into account Remarks 2 and 3. It is straightforward to verify the continuity of the curve γ at $s = 0$. The second statement follows from the decomposition of Σ in (7).

Theorem 3 *The Fučík spectrum* Σ *consists of two regular curves of* C^1 *class, which are symmetric with respect to the diagonal* $\alpha = \beta$ *and connect all the points* (λ_n, λ_n), $n \in \mathbb{N}$, *on that diagonal given by the eigenvalues* $\lambda_n = 4n^2$ *of the problem* (13).

Proof According to Theorem 2, the Fučík spectrum Σ consists of two continuous curves, which are symmetric with respect to the diagonal $\alpha = \beta$ and one of them is the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$. The curve γ is located below the diagonal $\alpha = \beta$ and has the parametrization $\gamma(s) = (\gamma_1(s), \gamma_2(s))$ given by (55). Moreover, this curve γ connects all the points $(\lambda_n, \lambda_n) = (4n^2, 4n^2)$, $n \in \mathbb{N}$, on the diagonal $\alpha = \beta$ since $\gamma(2n\pi - \pi) = (4n^2, 4n^2)$ due to (51). Unfortunately, the curve γ is not a regular curve since the derivative γ' does not exists for $s = 2n\pi$, $n \in \mathbb{N}$. Indeed, the one-sided derivatives of γ at $s = 2n\pi$, $n \in \mathbb{N}$, have the following form

$$\gamma'_-(2n\pi) = \left(2\mu_1(2n\pi)\mu'_{1-}(2n\pi), 2\mu_2(2n\pi)\mu'_{2-}(2n\pi)\right),$$
$$\gamma'_+(2n\pi) = \left(2\mu_1(2n\pi)\mu'_{1+}(2n\pi), 2\mu_2(2n\pi)\mu'_{2+}(2n\pi)\right),$$

where

$$\mu'_{1-}(2n\pi) = \tfrac{1}{\pi}, \qquad \mu'_{2-}(2n\pi) = \tfrac{1}{\pi}\sqrt{\tfrac{n}{n+1}},$$
$$\mu'_{1+}(2n\pi) = \tfrac{1}{\pi}\sqrt{\tfrac{n+1}{n}}, \quad \mu'_{2+}(2n\pi) = \tfrac{1}{\pi}.$$

On the other hand, it is possible to reparametrize the curve γ in the following way to obtain a differentiable curve γ_{d} of class C^1. Thus, let us define

$$\gamma_{\mathrm{d}}(t) := \gamma(\varphi(t)), \quad t \in \mathbb{R},$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is the continuous surjective function defined as

$$\varphi(t) := \begin{cases} t & \text{for } t \le 2\pi, \\ 2\pi n + \dfrac{t - 2\pi H_n}{\sqrt{n+1}} & \text{for } t \in (2\pi H_n, 2\pi H_{n+1}], \ n \in \mathbb{N}, \end{cases}$$

and H_n is the generalized harmonic number of order $-\frac{1}{2}$ of n given by $H_n = \sum_{k=1}^{n} \sqrt{k}$. Let us note φ is the piecewise linear and strictly increasing function and we have $\varphi'(t) = 1$ for $t < 2\pi$ and $\varphi'(t) = \frac{1}{\sqrt{n+1}}$ for $t \in (2\pi H_n, 2\pi H_{n+1})$, $n \in \mathbb{N}$. The original curve γ is defined by parts with the connection points $\gamma(s)$ for $s = (n-1)\pi$, $n \in \mathbb{N}$. The reparametrized curve γ_{d} has the connection points

$\gamma_{\mathrm{d}}(t) = \gamma(\varphi(t))$ for $t = 0, t = \pi, t = 2\pi H_n$ and $t = \pi H_n + \pi H_{n+1}, n \in \mathbb{N}$. Indeed, we have $\varphi(2\pi H_n) = 2\pi n$ and $\varphi(\pi H_n + \pi H_{n+1}) = 2\pi n + \pi$ for $n \in \mathbb{N}$. Thus, for $n \in \mathbb{N}$, we get

$$\begin{aligned}
\gamma_{\mathrm{d}}'(\pi) &= \gamma'(\pi) = \left(\tfrac{4}{\pi}, \tfrac{4}{\pi}\right), \\
\gamma_{\mathrm{d}}'(\pi H_n + \pi H_{n+1}) &= \gamma'(2\pi n + \pi)\varphi'(\pi H_n + \pi H_{n+1}) \\
&= \left(\tfrac{4}{\pi}\sqrt{n+1}, \tfrac{4}{\pi}\sqrt{n+1}\right), \\
\gamma_{\mathrm{d}}'(2\pi H_n) &= \gamma_-'(2\pi n)\,\varphi_-'(2\pi H_n) \\
&= \gamma_+'(2\pi n)\,\varphi_+'(2\pi H_n) \\
&= \left(\tfrac{2}{\pi}\left(\sqrt{n} + \sqrt{n+1} + \tfrac{1}{\sqrt{n}}\right), \tfrac{2}{\pi}\left(\sqrt{n} + \sqrt{n+1} - \tfrac{1}{\sqrt{n+1}}\right)\right),
\end{aligned}$$

where we used (51), (52) and the following one-sided derivatives of $\gamma_{\mathrm{d}}(t) = (\gamma_1(\varphi(t)), \gamma_2(\varphi(t)))$ for $t > 0$

$$\begin{aligned}
\gamma_{\mathrm{d}-}'(t) &= \left(\gamma_{1-}'(\varphi(t))\varphi_-'(t), \gamma_{2-}'(\varphi(t))\varphi_-'(t)\right) \\
&= \left(2\mu_1(\varphi(t))\mu_{1-}'(\varphi(t))\varphi_-'(t), 2\mu_2(\varphi(t))\mu_{2-}'(\varphi(t))\varphi_-'(t)\right), \\
\gamma_{\mathrm{d}+}'(t) &= \left(\gamma_{1+}'(\varphi(t))\varphi_+'(t), \gamma_{2+}'(\varphi(t))\varphi_+'(t)\right) \\
&= \left(2\mu_1(\varphi(t))\mu_{1+}'(\varphi(t))\varphi_+'(t), 2\mu_2(\varphi(t))\mu_{2+}'(\varphi(t))\varphi_+'(t)\right).
\end{aligned}$$

Moreover, we have that $\gamma_{\mathrm{d}}'(0) = \gamma'(0) = (0, 0)$ since

$$\begin{aligned}
\gamma_{1-}'(0) &= \lim_{h\to 0-} 2\eta_1(h)\eta_1'(h) = 0, & \gamma_{1+}'(0) &= \lim_{h\to 0+} 2\mu_1(h)\mu_1'(h) = 0, \\
\gamma_{2-}'(0) &= \lim_{h\to 0-} -2\eta_2(h)\eta_2'(h) = 0, & \gamma_{2+}'(0) &= \lim_{h\to 0+} 2\mu_2(h)\mu_2'(h) = 0,
\end{aligned}$$

where we used that

$$\lim_{h\to 0-} \eta_1'(h) = \lim_{h\to 0+} \mu_1'(h) = 0, \quad \lim_{h\to 0-} \eta_2'(h) = \lim_{h\to 0+} \mu_2'(h) = \tfrac{1}{2} + \tfrac{1}{\pi}.$$

To conclude, the first derivative γ_{d}' exists for all $t \in \mathbb{R}$ and it is straightforward to verify its continuity.

The norm of the first derivative $\left\|\gamma_{\mathrm{d}}'\right\|$ is zero for $t = 0$ and never vanishes for $t \neq 0$, which can be justified by the following estimates from below

$$\begin{aligned}
\eta_2'(s) &= \tfrac{1}{\pi} - \tfrac{\sinh s}{2\sqrt{2}\sqrt{\cosh s - 1}} &&> \tfrac{1}{2} + \tfrac{1}{\pi} && \text{for } s < 0, \\
\mu_2'(s) &= \tfrac{1}{\pi} + \tfrac{\sin s}{2\sqrt{2n-1-\cos s}}\sqrt{\tfrac{n}{2}} &&> \tfrac{1}{\pi} && \text{for } s \in (2n\pi - 2\pi, 2n\pi - \pi), n \in \mathbb{N}, \\
\mu_1'(s) &= \tfrac{1}{\pi} - \tfrac{\sin s}{2\sqrt{2n+1+\cos s}}\sqrt{\tfrac{n}{2}} &&> \tfrac{1}{\pi} && \text{for } s \in (2n\pi - \pi, 2n\pi), n \in \mathbb{N}.
\end{aligned}$$

Finally, to remove the singularity of the curve γ_{d} at $t = 0$, let us define

$$\gamma_{\mathrm{r}}(t) := \gamma_{\mathrm{d}}(\varrho(t)) = \gamma(\varphi(\varrho(t))), \quad t \in \mathbb{R}, \tag{56}$$

where $\varrho : \mathbb{R} \to \mathbb{R}$ is the continuous surjective function defined as

$$\varrho(t) := \operatorname{sign}(t)\sqrt{|t|}.$$

Let us note ϱ is the strictly increasing function and $\varrho'(0) = +\infty$. Moreover, we have that $\gamma_{\mathtt{r}}'(0) = \left(\frac{1}{3\pi^2} + \frac{1}{6\pi}, \frac{1}{\pi^2} + \frac{1}{\pi} + \frac{1}{4}\right)$ since

$$(\gamma_1 \circ \varrho)'(0) = \lim_{h\to 0} \gamma_1'(\varrho(h))\varrho'(h) = \tfrac{1}{3\pi^2} + \tfrac{1}{6\pi},$$
$$(\gamma_2 \circ \varrho)'(0) = \lim_{h\to 0} \gamma_2'(\varrho(h))\varrho'(h) = \tfrac{1}{\pi^2} + \tfrac{1}{\pi} + \tfrac{1}{4}.$$

For $t \neq 0$, we have $\gamma_{\mathtt{r}}'(t) = \gamma_{\mathtt{d}}'(\varrho(t))\varrho'(t)$ and thus, we get that $\left\|\gamma_{\mathtt{r}}'(t)\right\| > 0$. To conclude, $\gamma_{\mathtt{r}}$ is the differentiable curve of class C^1 such that $\left\|\gamma_{\mathtt{r}}'\right\|$ never vanishes, i.e. $\gamma_{\mathtt{r}}$ is a regular curve.

Remark 6 Let us note that the regular curve $\gamma_{\mathtt{r}}$ defined in (56) is not an analytic curve. Indeed, it is straightforward to verify that $\gamma_{\mathtt{r}}'''$ does not exist at $t = \pi^2$.

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Duffing Equation with Nonlinearities Between Eigenvalues

Petr Tomiczek

Abstract In this article, we investigate the periodic nonlinear second order ordinary differential equation with damping

$$u''(x) + r(x)\,u'(x) + g(x, u(x)) = f(x)\,, \quad x \in [0, 2\pi]\,,$$
$$u(0) = u(2\pi)\,,\ u'(0) = u'(2\pi)\,,$$

where g is a L^1-Caratheodory function, $r \in C([0, 2\pi])$, $r', f \in L^1(0, 2\pi)$. We obtain a solution to this problem if a quotient $\frac{g(x,s)}{s}$ lies between $0, \frac{1}{4} + \tilde{r}(x)$ and $\frac{1}{4} + \tilde{r}(x), 1 + \tilde{r}(x)$ or in interval $(n^2 + \tilde{r}(x), (n + 1)^2 + \tilde{r}(x)), n \in \mathbb{N}$, where $\tilde{r}(x) = \frac{r(x)^2}{4} + \frac{r(x)'}{2}$. We use variational method and suppose that for functions $u = u(x, a)$ satisfying $\lim_{a \to \pm\infty} u(x, a) = \pm\infty$ the function $F(s) = \int_0^{2\pi} \int_0^s \big[-r'(x)u(x, a) + g(x, u(x, a)) - f(x) \,\big]\, da\, dx$ has a critical point.

Keywords Second order ODE · Periodic problem · Variational method · Critical point

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1 Introduction

In this article, we study the nonlinear periodic boundary value problem

$$\begin{aligned} u''(x) + r(x)\,u'(x) + g(x, u(x)) &= f(x)\,, \quad x \in [0, 2\pi]\,, \\ u(0) = u(2\pi)\,,\ u'(0) &= u'(2\pi)\,, \end{aligned} \tag{1}$$

where L^1-Caratheodory's function $g\colon [0, 2\pi] \times \mathbb{R} \to \mathbb{R}$, $r \in C([0, 2\pi])$ and r', $f \in L^1(0, 2\pi)$ are 2π-periodic.

In papers [1] authors investigated Lineard equation and supposed that $\gamma(x) \leq \liminf_{|s|\to\infty} \frac{g(x,s)}{s} \leq \limsup_{|s|\to\infty} \frac{g(x,s)}{s} \leq \Gamma(x)\,, \Gamma(x) \leq 1$ with the strict inequality on a subset of $[0, 2\pi]$ of positive measure (i.e. $\Gamma(x) \ll 1$) and $\gamma(x)$ satisfies $\int_0^{2\pi} \gamma(x)\,dx \geq 0$, $\int_0^{2\pi} \gamma^+(x)\,dt > 0$ where $\gamma^+(x) = \max_{x\in[0,2\pi]} \{\gamma(x), 0\}$. In paper [2] $r(x) = c, c \in \mathbb{R}$ (see also [3]) and authors assume that $\beta(x) \ll \frac{g(x,s)}{s} \ll \frac{c^2}{4} + 1$ for all $s \in \mathbb{R}$, where $\beta(x) \in C([0, 2\pi])$, $\beta(0) = \beta(2\pi)$ and $\frac{1}{2\pi} \int_0^{2\pi} \beta(x)\,dt > 0$. For positive solutions of periodic problem see [4–6]. Others have studied problem (1) with jumping nonlinearities [7, 8] using topological method.

In this article, we choose another strategy of proof which rely essentially on a variational method (see also [9]). We denote

$$R(x) = e^{\int_0^x \frac{1}{2} r(\xi)\,d\xi}\,, \quad \tilde{r}(x) = \frac{r(x)^2}{4} + \frac{r(x)'}{2}\,, \quad w(x) = R(x)u(x)\,.$$

We multiply equation in (1) by R and we obtain for w an equivalent equation

$$w''(x) - \tilde{r}(x)\,w(x) + R(x)g(x, \tfrac{w(x)}{R(x)}) = R(x)f(x)\,.$$

We take $a \in \mathbb{R}$ and modify the previous equation to

$$w''(x) - \tilde{r}(x)\,w(x) + R(x)g(x, \tfrac{w(x)}{R(x)} + a) = R(x)f(x) \tag{2}$$

Then we will investigate the corresponding functional

$$J_a(w) = \frac{1}{2} \int_0^{2\pi} \left[(w')^2 + \tilde{r}w^2\right] dx - \int_0^{2\pi} \left[R^2 G(x, \tfrac{w}{R} + a) - Rfw\right] dx\,, \tag{3}$$

where $G(x, s) = \int_0^s g(x, \xi)\,d\xi$. We use that a critical point $w_a \in H$ of J_a, which satisfies

$$\langle J_a'(w_a), z\rangle = \int_0^{2\pi} \left[w'z' + \frac{c^2}{4}\,wz\right] dx - \int_0^{2\pi} \left[e^{\frac{c}{2}x} g\left(x, \frac{w}{e^{\frac{c}{2}x}} + a\right) z - e^{\frac{c}{2}x} fz\right] dx = 0$$

for all $z \in H = W_0^{1,2}(0, 2\pi)$, is also a weak solution to the Dirichlet problem ($w_a(0) = w_a(2\pi) = 0$) and vice versa. The usual regularity argument for ODE proves immediately (see Fučík [10]) that any weak solution to (2) is also a classical solution to (2).

We assume that the nonlinearity g satisfies

$$\begin{aligned}
&\text{(a)} \quad \frac{g(x,s)}{s} - \tilde{r}(x) \le \frac{1}{4} - \varepsilon \\
&\text{or} \\
&\text{(b)} \quad \frac{1}{4} + \varepsilon \le \frac{g(x,s)}{s} - \tilde{r}(x) \le 1 - \varepsilon \\
&\text{or} \\
&\text{(c)} \quad n^2 + \varepsilon \le \frac{g(x,s)}{s} - \tilde{r}(x) \le (n+1)^2 - \varepsilon \quad n \in \mathbb{N}
\end{aligned} \tag{4}$$

for a.e. $x \in (0, \pi)$, for all $s \in \mathbb{R}$, with some $\varepsilon > 0$.

There exist functions $a_+(x), a_-(x) \in L^1(0, T)$ and a constant $s_1 \in \mathbb{R}^+$ such that for a.e. $x \in (0, 2\pi)$

$$g(x,s) \le a_-(x) \quad \text{for} \quad s \le -s_1\,, \qquad g(x,s) \ge a_+(x) \quad \text{for} \quad s \ge s_1\,. \tag{5}$$

Furthermore there exists a critical point of the following function F,

$$F(s) = \int_0^{2\pi} \int_0^s \big[-r'(x)u(x,a) + g(x, u(x,a)) - f(x) \big] \, da \, dx\,.$$

Precisely, let $u = u(x,a)$, $u : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}$, $u(\cdot, a) \in C(\mathbb{R})$ for each $a \in \mathbb{R}$, $u(x, \cdot) \in C[0, 2\pi]$ for each $x \in [0, 2\pi]$ such that $\lim_{a \to \pm\infty} u(x,a) = \pm\infty$ on $[0, 2\pi]$. We suppose that for such $u = u(a,x)$ there exists $s_0 \in \mathbb{R}$ such that

$$F'(s_0) = 0\,. \tag{6}$$

We note this assumption is fulfilled if the right hand side f satisfies the orthogonal condition $\int_0^{2\pi} f(x)\, dx = 0$, $r' = 0$ and g satisfies the sign condition $g(x,s)s \ge 0$ (Fredholm alternative for a nonlinear equation).

Similarly to [11] we firstly investigate the Dirichlet problem. Then, we apply this result for finding periodic solutions. For the seek of simplicity we suppose in (4) $\varepsilon > 0$, but we can investigate also resonant case $\varepsilon = 0$ and suppose Landesman–Lazer type conditions (see [8, 12]).

2 Preliminaries

Notation: We shall use the classical space $C^k(0, 2\pi)$ (with a norm $\| \cdot \|_{C^k(0,2\pi)}$) of functions whose k-th derivative is continuous and the space $L^p(0, 2\pi)$ (with a norm $\| \cdot \|_p$) of measurable real-valued functions whose p-th power of the absolute value is Lebesgue integrable.

We denote $H = W_0^{1,2}(0, 2\pi)$ with the norm $\|u\| = (\int_0^{2\pi} [\,(u')^2\,]\,dt)^{\frac{1}{2}}$.

By a solution to (1) we mean a function $u \in C^1(0, 2\pi)$ such that u' is absolutely continuous, u satisfies the boundary conditions and the Eq. (1) is satisfied a.e. on $(0, 2\pi)$.

We study (1) by using variational methods. We investigate the functional J_a, see (3).

We say that $w \in H$ is a critical point of J_a, if

$$\langle J_a'(w), v\rangle = \int_0^{2\pi} \big[w'v' + \tilde{r}wv\big]\,dx - \int_0^{2\pi} \big[R(g(x, \tfrac{w}{R} + a) - f)v\big]\,dx = 0 \quad (7)$$

for all $v \in H$.

Let E be a Banach space. We say that $J_a : E \to \mathbb{R}$ satisfies the Palais–Smale condition (PS) if every sequence $(u_n) \subset E$ for which $J_a(u_n)$ is bounded and $J_a'(u_n) \to 0$ (as $n \to \infty$) possesses a convergent subsequence.

To find a point w such that $J_a'(w) = 0$ we prove that the functional J_a has a minimum (see [13]) or saddle point (see [14]) using the following theorems.

Theorem 1 *Let E be a Banach space and $J_a : E \to \mathbb{R}$ be Gâteaux differentiable, lower semicontinuous and bounded from below. Let J_a satisfies the Palais–Smale condition then J_a reaches its minimum.*

Theorem 2 (Saddle Point Theorem) *Let $H = \overline{H} \oplus \widehat{H}$, $\dim \overline{H} < \infty$ and $\dim \widehat{H} = \infty$. Let $J : H \to \mathbb{R}$ be a functional such that $J \in C^1(H, \mathbb{R})$ and*

(a) there exists a bounded neighbourhood D of 0 in $\overline{H}$ and a constant α such that $J/\partial D \le \alpha$,
(b) there is a constant $\beta > \alpha$ such that $J/\widehat{H} \ge \beta$,
(c) J satisfies the Palais–Smale condition (PS).

Then the functional J has a critical point in H.

In this section we introduce Lemma which will be used in the proof of the main result.

Lemma 1 (continuity) *Let $(a_n) \subset \mathbb{R}$ be a sequence such that $\lim_{n\to\infty} a_n = a_0$. We put $a = a_n$ in the definition of the functional J_a (7) and to each a_n find critical point $w_{a_n} \in H$ of the functional J_{a_n}. Then the sequence $(w_{a_n}) \subset H$ contains subsequence $(w_{a_{n_k}}) \subset H$ such that $w_{a_{n_k}} \to w_0$, $w_0 \in H$ and w_0 is a critical point of J_{a_0}.*

Proof The critical point w_{a_n} of J_a satisfies

$$\int_0^{2\pi}\big[w_{a_n}' z' + \tilde r\, w_{a_n} z\big]\,dx - \int_0^{2\pi}\big[R\,g(x, \tfrac{w_{a_n}}{R} + a_n)z - Rfz\big]\,dx = 0\,, \tag{8}$$

for all $z \in H$.

We suppose that the sequence (w_{a_n}) is unbounded and we put $v_n = \frac{w_{a_n}}{\|w_{a_n}\|}$. Then there exists $v_0 \in H$ such that $v_n \rightharpoonup v_0$ in H and due to compact embedding H into $C([0, 2\pi])$ $v_n \to v_0$ in $C([0, 2\pi])$ (taking a subsequence if it is necessary). We divide (8) by $\|w_{a_n}\|$ and put $z = v_n$ then

$$\int_0^{2\pi}\Big[(v_n')^2 + \tilde r\, v_n^2\Big]\,dx - \int_0^{2\pi}\Big[\frac{R\,g(x, \frac{w_{a_n}}{R} + a_n)v_n}{\|w_{a_n}\|} - \frac{Rf v_n}{\|w_{a_n}\|}\Big]\,dx = 0\,. \tag{9}$$

We note that

$$\frac{R\,g(x, \frac{w_{a_n}}{R} + a_n)v_n}{\|w_{a_n}\|} = \frac{g(x, \frac{w_{a_n}}{R} + a_n)v_n}{\frac{w_{a_n}}{R} + a_n}\,\frac{w_{a_n} + Ra_n}{\|w_{a_n}\|}$$

and wee pass to the limit in (9). We use $\int_0^{2\pi}(v_0')^2\,dx \le \liminf_{n\to\infty}\int_0^{2\pi}(v_n')^2\,dx = 1$ (the weak sequential lower semi-continuity of the Hilbert norm) and according to (4) case (a) we obtain

$$\int_0^{2\pi}(v_0')^2\,dx - \Big(\frac{1}{4} - \varepsilon\Big)\int_0^{2\pi}(v_0)^2\,dx \le 1 - \Big(\frac{1}{4} - \varepsilon\Big)\int_0^{2\pi}(v_0)^2\,dx \le 0 \tag{10}$$

a contradiction, since for $w \in H$ it holds

$$\int_0^{2\pi}(w')^2\,dx \ge \frac{1}{4}\int_0^{2\pi} w^2\,dx\,. \tag{11}$$

For g satisfying (4) case (b) we split $H = \overline{H} \oplus \widehat{H}$, where $\overline{H} = \text{span}\{\sin\frac{x}{2}\}$, $\widehat{H} = \text{span}\{\sin x, \sin 2x, \ldots\}$. We denote $w = \overline{w} + \widehat{w}$, where $\overline{w} \in \overline{H}$, $\widehat{w} \in \widehat{H}$. We divide (8) by $\|w_{a_n}\|$ and we put $z = \overline{v}_n - \widehat{v}_n$ then

$$\begin{aligned}&\int_0^{2\pi}\Big[(\overline{v}_n')^2 - (\widehat{v}_n')^2 + \tilde r\,((\overline{v}_n)^2 - (\widehat{v}_n)^2)\Big]\,dx - \\ &\int_0^{2\pi}\Big[\frac{g(x, \frac{w_{a_n}}{R} + a_n)}{\frac{w_{a_n}}{R} + a_n}\Big((\overline{v}_n)^2 - (\widehat{v}_n)^2 + \frac{Ra_n}{\|w_{a_n}\|}\Big) - \frac{Rf v_n}{\|w_{a_n}\|}\Big]\,dx = 0\,.\end{aligned} \tag{12}$$

Passing to the limit in (12) we get

$$\int_0^{2\pi} (\overline{v}_0')^2 - \Big(\frac{1}{4}+\varepsilon\Big)(\overline{v}_0)^2\,dx - \int_0^{2\pi} (\widehat{v}_0')^2 - (1-\varepsilon)(\widehat{v}_0)^2\,dx \geq 0 \tag{13}$$

a contradiction, since for $\overline{w} \in \overline{H}$, $\widehat{w} \in \widehat{H}$ it holds

$$\int_0^{2\pi} (\overline{w}')^2\,dx = \frac{1}{4}\int_0^{2\pi} \overline{w}^2\,dx, \quad \int_0^{2\pi} (\widehat{w}')^2\,dx \geq \int_0^{2\pi} \widehat{w}^2\,dx\,. \tag{14}$$

For g satisfying (4) case (c) we split $H = \overline{H} \oplus \widehat{H}$, where $\overline{H} = \operatorname{span}\{\sin\frac{x}{2}, \sin x, \dots, \sin nx\}$, $\widehat{H} = \operatorname{span}\{\sin(n+1)x, \sin(n+2)x, \dots\}$ and repeat previous steps (12)–(14) to obtain a contradiction.

Therefore the sequence (w_{a_n}) is bounded and there exists $w_0 \in H$ such that $w_{a_n} \rightharpoonup w_0$ in H, $w_{a_n} \to w_0$ in $L^2(0, 2\pi)$, $C([0, 2\pi])$ (taking a subsequence if it is necessary).

We put $n = m$ in (8) and subtract this equality from (8) (with n) we obtain

$$\begin{aligned}\lim_{\substack{n\to\infty\\ m\to\infty}} \Big\{ &\int_0^{2\pi} \Big[(w_{a_n} - w_{a_m})'z' + \tilde{r}\,(w_{a_n} - w_{a_m})\,z\Big]\,dx \\ &- \int_0^{2\pi} \Big[R\,\Big(g(x, \frac{w_{a_n}}{R} + a_n) - g(x, \frac{w_{a_m}}{R} + a_m)\Big)\,z\Big]\,dx\Big\} = 0\,.\end{aligned} \tag{15}$$

The convergence $w_{a_n} \to w_0$ in $C([0, 2\pi])$, (15) and $a_n \to a_0$ yield

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \int_0^{2\pi} \Big[\,R\Big(g(x, \frac{w_{a_n}}{R} + a_n) - g(x, \frac{w_{a_m}}{R} + a_m)\Big)\,\big(w_{a_n} - w_{a_m}\big)\,\Big]\,dx = 0\,. \tag{16}$$

We set $z = w_{a_n} - w_{a_m}$ in (8), then using (16) we get

$$\lim_{\substack{n\to\infty\\ m\to\infty}} \int_0^{2\pi} \Big[(w_{a_n}' - w_{a_m}')^2 + \tilde{r}\,(w_{a_n} - w_{a_m})^2\Big]\,dx = 0\,. \tag{17}$$

Hence the strong convergence $w_{a_n} \to w_0$ in $L^2(0, 2\pi)$ and (17) imply the strong convergence $w_{a_n} \to w_0$ in H and we can pass to the limit in (8). We obtain

$$\int_0^{2\pi} \Big[w_0'z' + \tilde{r}\,w_0 z\Big]\,dx - \int_0^{2\pi} \Big[R\,g(x, \frac{w_0}{R} + a_0)z - Rfz\Big]\,dx = 0\,, \tag{18}$$

for all $z \in H$. Hence w_0 is a critical point of J_{a_0} with $a = a_0$.

Remark 1 We have proved that to each $a \in \mathbb{R}$ there exist function $u_a = \frac{w_a}{R} + a$ such that the $A : \mathbb{R} \to H$, $A(a) = u_a$ is a continuous operator.

3 Existence Theorem

Theorem 3 *We assume that the nonlinearity g satisfies the assumption (4) case (a) or (4) case (b) or (4) case (c) and the assumption (5). Furthermore let $u = u(x, a)$, $u : \mathbb{R} \times [0, 2\pi] \to \mathbb{R}$, $u(\cdot, a) \in C(\mathbb{R})$ for each $a \in \mathbb{R}$, $u(x, \cdot) \in C[0, 2\pi]$ for each $x \in [0, 2\pi]$ such that $\lim_{a\to\pm\infty} u(x, a) = \pm\infty$ on $[0, 2\pi]$. We suppose that for such $u = u(a, x)$ the function F, where*

$$F(s) = \int_0^{2\pi} \int_0^{s} \left[-r'(x)u(x, a) + g(x, u(x, a)) - f(x) \right] da\, dx,$$

satisfies the assumption (6). Then Problem (1) has at least one solution.

Proof Firstly we suppose (4) case (a), hence

$$\frac{2G(x, s)}{s^2} - \tilde{r}(x) \leq \frac{1}{4} - \varepsilon \tag{19}$$

for a.e. $x \in (0, \pi)$, for all $s \in \mathbb{R}$, with some $\varepsilon > 0$. We use theorem 1 with a space $E = H$. We prove that J_a is a continuous (consequently lower semicontinuous). Let $w_n \to w_0$ in H then due to the compact imbedding $w_n \to w_0$ in $C([0, 2\pi])$ we get

$$\begin{aligned} \lim_{n\to\infty} J_a(w_n) &= \lim_{n\to\infty} \frac{1}{2} \int_0^{2\pi} \left[(w_n')^2 + \tilde{r}\, w_n^2 \right] dx \\ &\quad - \int_0^{2\pi} \left[R^2\, G(x, \frac{w_n}{R} + a) - R\, f\, w_n \right] dx \\ &= \frac{1}{2} \int_0^{2\pi} \left[(w_0')^2 + \tilde{r}\, w_0^2 \right] dx \\ &\quad - \int_0^{2\pi} \left[R^2\, G(x, \frac{w_0}{R} + a) - R\, f\, w_0 \right] dx \; = J_a(w_0)\,. \end{aligned} \tag{20}$$

The second equality in (20) follows from equicontinuity $w_n \rightrightarrows w_0$ and continuity $G(x, s)$ in the variable s. Hence J_a is continuous.

Now we prove that J_a is bounded from below. Due to the compact imbedding of H into $C([0, 2\pi])$, $L^2(0, 2\pi)$, ($\|w\|_{C([0,2\pi])} \leq k\|w\|$), ($\|w\|_2^2 \leq 4\|w\|^2$) and (19) we get

$$
\begin{aligned}
J_a(w) &= \frac{1}{2}\int_0^{2\pi}\left[(w')^2+\tilde{r}\,w^2\right]dx-\int_0^{2\pi}\left[R^2G(x,\tfrac{w}{R}+a)-Rfw\right]dx\\
&\geq \frac{1}{2}\int_0^{2\pi}\left[(w')^2+\tilde{r}\,w^2\right]dx\\
&\quad -\int_0^{2\pi}\left[R^2\left(\tfrac{1}{2}(\tfrac{1}{4}-\varepsilon+\tilde{r})(\tfrac{w}{R}+a)^2\right)-Rfw\right]dx\\
&\geq \frac{1}{2}\|w\|^2-\frac{1}{2}(\tfrac{1}{4}-\varepsilon)\|w\|_2^2-\|R(\tfrac{1}{4}-\varepsilon+\tilde{r})a+Rf\|_1k\|w\|\\
&\quad -\|\tfrac{1}{2}(\tfrac{1}{4}-\varepsilon+\tilde{r})a^2R^2\|_1 \qquad (21)\\
&\geq \frac{1-(1-4\varepsilon)}{2}\|w\|^2-\|R((\tfrac{1}{4}-\varepsilon+\tilde{r})a+f)\|_1k\|w\|-k_2\\
&= 2\varepsilon\|w\|^2-k_1\,\|w\|-k_2\,,\quad k_1,k_2\in\mathbb{R}\,.
\end{aligned}
$$

Hence the functional J_a is bounded from below.

Now we show that J_a satisfies the Palais–Smale condition. We suppose for the sequence $(w_n)\subset H$, there exists a constant c_1 such that

$$|J_a(w_n)|\leq c_1 \tag{22}$$

and

$$\lim_{n\to\infty}\|J_a'(w_n)\|=0\,. \tag{23}$$

Using (19), (21), (22) we obtain

$$2\varepsilon\,\|w_n\|^2-k_1\,\|w_n\|-k_2\leq c_1\,. \tag{24}$$

This implies the sequence (w_n) is bounded. Then there exists $w_0\in H$ such that $w_n\rightharpoonup w_0$ in H, $w_n\to w_0$ in $L^2(0,2\pi)$, $C([0,2\pi])$ (taking a subsequence if it is necessary).

Let (z_k) be an arbitrary sequence bounded in H. It follows from (23) and the Schwarz inequality that

$$
\begin{aligned}
&\left|\lim_{\substack{n\to\infty\\k\to\infty}}\int_0^{2\pi}\left[w_n'z_k'+\tilde{r}w_nz_k\right]dx-\int_0^{2\pi}\left[R\left(g(\frac{w_n}{R}+a)z_k-fz_k\right)\right]dx\right|\\
&=|\lim_{\substack{n\to\infty\\k\to\infty}}J_a'(w_n)z_k|\leq\lim_{\substack{n\to\infty\\k\to\infty}}\|J_a'(w_n)\|\cdot\|z_k\|=0\,.
\end{aligned} \tag{25}
$$

We compare (25) with (8) and repeat steps (15)–(17) to obtain the strong convergence $w_n\to w_0$ in H. Therefore J_a satisfies the Palais–Smale condition.

For g satisfying (4) case (b) we again split $H=\overline{H}\oplus\widehat{H}$ and we shall prove that the functional J_a satisfies the assumptions in Theorem 2 (Saddle Point Theorem).

(a) For $w\in\overline{H}$ it holds $4\|w\|^2\,dx=\|w\|_2^2$ and we use inequality $\frac{2G(x,s)}{s^2}-\tilde{r}(x)\geq\frac{1}{4}+\varepsilon$. Hence (see also (21))

$$\begin{aligned} J_a(w) &= \frac{1}{2}\int_0^{2\pi}\big[(w')^2+\tilde r w^2\big]\,dx - \int_0^{2\pi}\big[R^2G(x,\tfrac{w}{R}+a) - Rfw\big]\,dx \\ &\le \frac{1}{2}\|w\|^2 - \frac{1}{2}(\tfrac14+\varepsilon)\|w\|_2^2 + \|R(\tfrac14+\varepsilon+\tilde r)a + Rf\|_1 k\|w\| \\ &\quad +\|\tfrac12(\tfrac14+\varepsilon+\tilde r)a^2R^2\|_1 \\ &\le -2\varepsilon\|w\|^2 + k_1\,\|w\| + k_2\,, \quad k_1,k_2\in\mathbb{R}\,, \end{aligned} \tag{26}$$

We have proved that $\lim_{\|w\|\to\infty} J_a(w) = -\infty$ and assumption (a) of Theorem 2 is satisfied.

(b) Similarly for $w \in \widehat{H}$ it holds $\|w\|^2\,dx \ge \|w\|_2^2$ and $\frac{2G(x,s)}{s^2} - \tilde r(x) \le 1-\varepsilon$. Hence

$$\begin{aligned} J_a(w) &= \frac{1}{2}\int_0^{2\pi}\big[(w')^2+\tilde r w^2\big]\,dx - \int_0^{2\pi}\big[R^2G(x,\tfrac{w}{R}+a) - Rfw\big]\,dx \\ &\ge \frac{1}{2}\|w\|^2 - \frac{1}{2}(1-\varepsilon)\|w\|_2^2 - \|R(1-\varepsilon+\tilde r)a + Rf\|_1 k\|w\| \\ &\quad -\|\tfrac12(1-\varepsilon+\tilde r)a^2R^2\|_1 \\ &\ge \frac{\varepsilon}{2}\|w\|^2 - k_3\,\|w\| - k_4\,, \quad k_3,k_4\in\mathbb{R}\,, \end{aligned} \tag{27}$$

We have proved assumption (b) of Theorem 2.

(c) To prove the Palais–Smale condition we again compare (25) with (8) and repeat steps (12)–(17) to obtain the strong convergence $w_n \to w_0$ in H.

We have proved that to each a there exists critical point $w_a \in H$ of the functional J_a, see (2). The usual regularity argument for ODE proves immediately (see Fučík [10]) that any weak solution to (2) is also a solution in the sense mentioned above.

Let (a_n) be sequence such that $\lim\limits_{n\to\infty} a_n = \infty$ and (w_{a_n}) be a corresponding sequence of the critical points of the functional J_a with $a = a_n$.

We will prove that $\lim\limits_{n\to\infty}(\frac{w_n}{R}(x)+a_n) = \infty$ a.e. on $[0,2\pi]$. We suppose for contradiction there are $\varepsilon > 0$, $k\in\mathbb{R}$, $(\alpha_n,\beta_n)\subset[0,2\pi]$, meas $(\alpha_n,\beta_n)\ge\varepsilon>0$ such that $\frac{w_n}{R}(x)+a_n \le k$ on (α_n,β_n).

We choose in (7) test function $v_n \in H$, $\|v_n\|_2 \le k_1$, $k_1\in\mathbb{R}$. We use assumption (5) if $\lim\limits_{n\to\infty}(\frac{w_n}{R}+a_n) = -\infty$ and obtain there exists $k_2\in\mathbb{R}$ for all $n\in\mathbb{N}$ such that

$$\int_0^{2\pi}\big[R(g(x,\tfrac{w_n}{R}+a) - f)v_n\big]\,dx < k_2. \tag{28}$$

We set test function $v_n(x) = 0$ on $[0,2\pi]\setminus(\alpha_n,\beta_n)$, $v_n'(\alpha_n) = v_n'(\beta_n) = 0$, then

$$\int_0^{2\pi}\big[w_n'v_n' + \tilde r w_n v_n\big]\,dx = \int_0^{2\pi}\big[w_n(-v_n''+\tilde r v_n)\big]\,dx$$

and we put $-v_n'' + \tilde{r} v_n = h(x) < 0$ on (α_n, β_n). We note that $\frac{w_n}{R}(x) \le k - a_n \to -\infty$ and $R > 0$. Hence

$$\lim_{n\to\infty} \int_0^{2\pi} \left[w_n' v_n' + \tilde{r} w_n v_n \right] dx = +\infty \tag{29}$$

a contradiction with (7) and (28). We prove $\lim\limits_{a_n\to-\infty} (\frac{w_n}{R} + a_n) = -\infty$ similarly. Since (a_n) was an arbitrary sequence, we get $\lim\limits_{a\to\pm\infty} (\frac{w_a}{R} + a) = \pm\infty$.

We denote $u(x,a) = \frac{w_a}{R}(x) + a$, note that $u(x,a)$ is a solution to (1) with boundary condition $u(0,a) = u(2\pi, a) = a$ and investigate function

$$F(s) = \int_0^{2\pi} \int_0^{s} \left[-r'(x) u(x,a) + g(x, u(x,a)) - f(x) \right] da\, dx\,.$$

Using Lemma 1 we conclude $F'(s) = \int_0^{2\pi} \left[-r'(x) u(x,s) + g(x, u(x,s)) - f(x) \right] dx$. We get by (6) that there exists a constant $s_0 \in \mathbb{R}$ such that

$$F'(s_0) = \int_0^{2\pi} \left[-r'(x) u(x, s_0) + g(x, u(x, s_0)) - f(x) \right] dx = 0\,. \tag{30}$$

Integrating (1) over $[0, 2\pi]$ with $u_0 = u(x, s_0)$ we obtain

$$\int_0^{2\pi} \left[u_0'' + r\, u_0' + r' u_0 \right] dx + \int_0^{2\pi} \left[-r' u_0 + g(x, u_0) - f \right] dx = 0.$$

Since $\int_0^{2\pi} \left[r\, u_0' + r' u_0 \right] dx = [r\, u_0]_0^{2\pi} = 0$ and (30), we obtain $u_0'(0) = u_0'(2\pi)$ and the function u_0 is a solution to the periodic problem (1).

The proof of Theorem 3 is completed.

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Comparison and Uniqueness Results for the Periodic Boundary Value Problem for Linear First-Order Differential Equations Subject to a Functional Perturbation

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Abstract We improve some comparison results for the periodic boundary value problem related to a first-order differential equation perturbed by a functional term. The comparison results presented cover many cases as differential equations with delay, differential equations with maxima and integro-differential equations. The interesting case of functional perturbation with piecewise constant arguments is also analyzed.

Keywords Boundary value problems · Comparison results · Uniqueness of solution

1 Introduction

The comparison principles are important tools for the study of the properties of the solution to differential and integral equations. In this sense, we can find many monographs devoted to the development of estimates for functions satisfying a certain differential inequality. For instance, see [1–6]. With the help of these estimates, different techniques are applied to deduce the positivity of the solutions to differential, difference or integral equations [7] or iterative techniques in order to approximate the solutions to nonlinear differential equations [8]. Some other papers on this topic are, for instance, [9–11, 21–26]

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The study of comparison results for functional differential equations with piecewise constant arguments has received special attention. See [12] for first-order problems and [13–16] for the second-order case. Some results on boundary value problems with causal operators can be found in [17].

Section 2 is devoted to present a general formulation of the problem and a key result for this work, which gives conditions to assure the existence of a nonpositive solution. In Sect. 3, we analyse some particular cases of the general equation, such as retarded functional differential equations, equations with minima and integro-differential equations. In Sect. 4, we provide an extension of the above-mentioned result of Sect. 2, where the main conditions are imposed in subintervals induced by a partition of the interval where the problem is formulated. In Sect. 5, by bearing in mind few remarks based on the previous results, we obtain analogous results to obtain nonnegative solutions. Finally, in Sect. 6 we join all the conditions for nonnegativity and nonpositivity to deduce several uniqueness results.

2 General Comparison Result

Let $I = [0, T]$, $p : L^1(I) \to L^1(I)$ and consider the problem

$$\begin{cases} v'(t) + Mv(t) + [p(v)](t) = \sigma(t), \quad a.e.\ t \in I, \\ \qquad v(0) = v(T) + \lambda. \end{cases} \tag{1}$$

We introduce the following conditions

$$p(w) \geq 0 \text{ a.e. on } I, \text{ if } w \in C(I),\ w \geq 0 \text{ on } I, \tag{2}$$

and

$$\begin{cases} \text{for all } a < b \in I \text{ and } w \in C(I) \text{ with } \min\limits_{[0,b]} w \leq 0, \\ \text{we have } \int_a^b [p(w)](s)\, e^{Ms}\, ds \geq \min\limits_{s\in[0,b]} (w(s)\, e^{Ms}), \end{cases} \tag{3}$$

Theorem 1 *If $v \in W^{1,1}(I)$ is a solution of problem (1), $M > 0$, $\lambda \leq 0$, $\sigma \leq 0$ a.e. on I and p satisfies (2) and (3), then $v \leq 0$ a.e. on I.*

Proof If $v \geq 0$ on I, then, using (2), we get

$$v'(t) = \sigma(t) - Mv(t) - [p(v)](t) \leq 0$$

for a.e. $t \in I$ and v is monotonic nonincreasing. Then v is a constant function since $v(0) \leq v(T)$. Let $v(t) = k$ with $k \geq 0$. Then, by (2),

$$0 \leq Mk = \sigma(t) - [p(v)](t) \leq 0, \ a.e.\ t \in I,$$

and $v(t) = k = 0$ for all $t \in I$.

This shows that either $v \equiv 0$ or there exists at least one point $t_* \in I$ with $v(t_*) < 0$. If $v \leq 0$ on I is not true, then there will exist $t_0 \in I$ such that $v(t_0) > 0$. Consider the function

$$z(t) = v(t)e^{Mt},\ t \in I.$$

The signs of v and z are the same.

Then

$$v'(t)e^{Mt} + Mv(t)e^{Mt} \leq -[p(v)](t)\, e^{Mt},\ a.e.\ t \in I,$$

that is,

$$z'(t) \leq -[p(v)](t)\, e^{Mt},\ a.e.\ t \in I. \tag{4}$$

For this function z, it is true that $z(0) = v(0)$, $z(t_*) < 0$ and $z(t_0) > 0$. We will distinguish two cases:

Case 1: $v(0) \leq 0$.

Let $t_1 \in [0, t_0)$ such that

$$z(t_1) = \min_{[0,t_0]} z \leq 0.$$

Integrating (4) from t_1 to t_0 and taking into account the inequality (3), we obtain

$$\begin{aligned} -z(t_1) < z(t_0) - z(t_1) &\leq -\int_{t_1}^{t_0} [p(v)](s)\, e^{Ms}\, ds \\ &\leq -\min_{s\in[0,t_0]} (v(s)\, e^{Ms}) = -\min_{s\in[0,t_0]} z(s) = -z(t_1), \end{aligned}$$

that is a contradiction.

Case 2: $v(0) > 0$.

Here, $z(0) > 0$ and $v(T) \geq v(0) > 0$, so that $z(T) > 0$. Let $t_2 \in (0, T)$ with

$$z(t_2) = \min_{s\in[0,T]} z(s) < 0.$$

Integrating (4) on $[t_2, T]$ and using (3) again, we get

$$\begin{aligned} -z(t_2) < z(T) - z(t_2) &\leq -\int_{t_2}^{T} [p(v)](s)\, e^{Ms}\, ds \\ &\leq -\min_{s\in[0,T]} (v(s)\, e^{Ms}) = -\min_{s\in[0,T]} z(s) = -z(t_2), \end{aligned}$$

which again is a contradiction.

This proves that $v \leq 0$ on I. ■

Note that condition (3) can be expressed in the following equivalent terms.

Let $\widehat{w}(t) = w(t)e^{-Mt}$

$$\begin{cases} \int_a^b [p(\widehat{w})](s)\, e^{Ms}\, ds \geq \min_{[0,b]} w, \\ \text{for all}\, a < b \in I \,\text{and}\, w \in C(I) \text{with} \min_{[0,b]} w \leq 0. \end{cases}$$

3 Particular Cases

3.1 Retarded Functional Differential Equations

If

$$[p(w)](t) = Nw(\theta(t)),$$

with $N > 0$, $\theta : I \longrightarrow I$ such that $p : L^1(I) \longrightarrow L^1(I)$ and

$$\theta(t) \leq t,\ \ a.e.\ t \in I,$$

then p satisfies the hypothesis (2). Indeed, if $w \geq 0$ on I, then

$$[p(w)](t) = Nw(\theta(t)) \geq 0 \text{ for } t \in I.$$

If the following condition holds

$$N \int_0^T e^{M(s-\theta(s))}\, ds \leq 1,$$

then (3) is satisfied. Indeed,

$$\begin{aligned} &\int_a^b [p(w)](s)\, e^{Ms}\, ds = \int_a^b Nw(\theta(s))\, e^{Ms}\, ds \\ &= \int_a^b Nw(\theta(s))\, e^{M\theta(s)}\, e^{M(s-\theta(s))}\, ds \geq \min_{s\in[0,b]} (w(s)e^{Ms})\, N \int_a^b e^{M(s-\theta(s))}\, ds \\ &\geq \min_{s\in[0,b]} (w(s)e^{Ms})\, N \int_0^T e^{M(s-\theta(s))}\, ds \geq \min_{s\in[0,b]} (w(s)e^{Ms}), \end{aligned}$$

for all $a < b \in I$ and $w \in C(I)$ with $\min_{[0,b]} w \leq 0$. Here we have used that

$$w(\theta(t))\, e^{M\theta(t)} \geq \min_{s\in[0,b]} (w(s)e^{Ms}),\ \text{ a.e. } t \in [a,b],$$

that is true since $\theta(t) \leq t$, a.e. $t \in I$.

Corollary 1 *If* $M > 0$, $N \geq 0$, $\theta : I \to I$, $\theta(t) \leq t$, *a.e. on* I *and* $v \in W^{1,1}(I)$ *are such that* $p : L^1(I) \to L^1(I)$ *and*

$$\begin{cases} v'(t) + Mv(t) + Nv(\theta(t)) \leq 0, \;\; a.e.\ t \in I, \\ v(0) \leq v(T), \\ N\int_0^T e^{M(s-\theta(s))}\, ds \leq 1, \end{cases}$$

then $v \leq 0$ *a.e. on* I.

This result improves Corollary 2 in [18].

An important case is $\theta(t) = [t]$, where $[\cdot]$ is the floor function. In this case, $p(v) \in L^1(I)$, for any $v \in L^1(I)$, and we obtain the estimate

$$N \int_0^T e^{M(s-[s])}\, ds \leq 1. \tag{5}$$

If $T \leq 1$, then $[t] = 0$, at least for $t \in [0, T)$ and (5) becomes

$$N \int_0^T e^{Ms}\, ds = \frac{N}{M}(e^{MT} - 1) \leq 1.$$

Note that this is not a trivial case, since we can find a function v with

$$v'(t) + Mv(t) + Nv(0) \leq 0,\ t \in I,$$

$$v(0) < v(T)$$

and $v(T) > 0$. Set, for instance, $M = 1$, $N = 5$, $T = \frac{1}{2}$ and $v(t) = t - \frac{1}{4}$, for $t \in [0, \frac{1}{2}]$. In this case,

$$\frac{N}{M}(e^{MT} - 1) = 5(\sqrt{e} - 1) > 1.$$

If $T > 1$, let $k \in \mathbb{N}$, such that $k < T \leq k + 1$. Then

$$\begin{aligned} N \int_0^T e^{M(s-[s])}\, ds &= N\left[\sum_{i=1}^{k} \int_{i-1}^{i} e^{M(s-i+1)}\, ds + \int_k^T e^{M(s-k)}\, ds\right] \\ &= N\left[\sum_{i=1}^{k} \frac{1}{M}(e^M - 1) + \frac{1}{M}(e^{M(T-k)} - 1)\right] \\ &= \frac{N}{M}[k(e^M - 1) + (e^{M(T-k)} - 1)]. \end{aligned}$$

This leads to the condition

$$\frac{N}{M}[k(e^{M}-1)+e^{M(T-k)}-1]\le 1.$$

If $k=0$, it coincides with the case $T\le 1$. However, this estimate can be improved, as we will show below.

3.2 Minimum Case

If p satisfies the three following conditions considered in Theorem 5 of [18],

$$p(w)\in L^{\infty}(I),\ \text{ for every } w\in C(I), \tag{6}$$

$$\operatorname{ess}\inf_{t\in[0,\tau]}[p(w)](t)\ge N\min_{[0,\tau]} w,\ \text{ for } \tau\in I \text{ and } w\in C(I), \tag{7}$$

$$\frac{N}{M}(e^{MT}-1)<1,$$

for a certain $N\ge 0$, then p satisfies (2) and (3).

Indeed, let $w\in C(I)$, $w\ge 0$ on I, then

$$[p(w)](t)\ge \operatorname{ess}\inf_{s\in[0,t]}[p(w)](s)\ge N\min_{[0,t]} w\ge 0,\ \ a.e.\ t\in I,$$

and (2) holds.

Now, let $a<b\in I$ and $w\in C(I)$ with $\min_{[0,b]} w\le 0$, then

$$[p(w)](t)\ge \operatorname{ess}\inf_{s\in[0,t]}[p(w)](s)\ge N\min_{[0,t]} w,\ \text{ for a.e. } t\in[a,b].$$

We have

$$\begin{aligned}
&\int_a^b [p(w)](s)\,e^{Ms}\,ds\ge \int_a^b N\left(\min_{[0,s]} w\right)e^{Ms}\,ds\\
&=\int_a^b N\left(\min_{t\in[0,s]} w(t)\,e^{Mt}e^{-Mt}\right)e^{Ms}\,ds\\
&\ge \int_a^b N\left(\min_{t\in[0,s]}\left(\min_{t\in[0,b]} w(t)\,e^{Mt}\right)e^{-Mt}\right)e^{Ms}\,ds\\
&=\int_a^b N\left(\min_{t\in[0,b]} w(t)\,e^{Mt}\right)\left(\max_{t\in[0,s]} e^{-Mt}\right)e^{Ms}\,ds
\end{aligned}$$

$$= N \min_{t\in[0,b]} (w(t)\, e^{Mt}) \int_a^b e^{Ms}\, ds$$
$$= N \min_{t\in[0,b]} (w(t)\, e^{Mt}) \, \frac{1}{M}(e^{Mb} - e^{Ma}).$$

Also, $e^{Mb} - e^{Ma} \le e^{MT} - 1$ and taking into account that $\min_{[0,b]} w \le 0$, then

$$\int_a^b [p(w)](s)\, e^{Ms}\, ds \ge \frac{N}{M}(e^{MT} - 1) \min_{t\in[0,b]} w(t)\, e^{Mt} \ge \min_{t\in[0,b]} w(t)\, e^{Mt}$$

and (3) is valid, even when $\frac{N}{M}(e^{MT} - 1) = 1$.
Thus, we have proved the following

Corollary 2 *If $M > 0$, $N \ge 0$ and $v \in W^{1,1}(I)$ are such that*

$$\begin{cases} v'(t) + Mv(t) + N[p(v)](t) \le 0, \;\; a.e.\ t \in I, \\ v(0) \le v(T), \\ \frac{N}{M}(e^{MT} - 1) \le 1, \end{cases}$$

where $p : L^1(I) \to L^1(I)$ satisfies (6) and (7), then $v \le 0$ a.e. on I.

This result improves Theorem 5 of [18].

Note that the new result applies to functions $p : L^1(I) \longrightarrow L^1(I)$, such that

$$[p(w)](t) \ge N \min_{[0,t]} w, \;\; a.e.\, t \in I,$$

for $w \in C(I)$ and

$$\frac{N}{M}(e^{MT} - 1) \le 1.$$

3.3 Integral Case

Corollary 3 *Let $M > 0$, $N \ge 0$ and suppose that $p : L^1(I) \longrightarrow L^1(I)$ satisfies that*

$$[p(w)](t) \ge N \int_0^t w(s)\, ds, \;\; a.e.\, t \in I, \; for\, w \in C(I). \tag{8}$$

If $v \in W^{1,1}(I)$ is such that

$$\begin{cases} v'(t) + Mv(t) + N[p(v)](t) \le 0, \;\; a.e.\ t \in I, \\ v(0) \le v(T), \\ \frac{N}{M^2}(e^{MT} - MT - 1) \le 1, \end{cases}$$

then $v \le 0$ a.e. on I.

Proof If $w \in C(I)$, $w \geq 0$, then $[p(w)] \geq 0$ a.e. on I, so that (2) is valid. We will prove that (3) holds under the estimate

$$\frac{N}{M^2}(e^{MT} - MT - 1) \leq 1.$$

Indeed, for $a < b \in I$ and $w \in C(I)$ with $\min_{[0,b]} w \leq 0$, it can be proved that

$$\int_0^s w(r)\,dr \geq \int_0^s \left(\min_{r\in[0,b]} w(r)e^{Mr}\right) e^{-Mr}\,dr$$
$$= \left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \int_0^s e^{-Mr}\,dr = \left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \frac{1 - e^{-Ms}}{M},$$

for $s \in [a, b]$ and, therefore,

$$\int_a^b [p(w)](s)\, e^{Ms}\,ds \geq \int_a^b N \int_0^s w(r)\,dr\, e^{Ms}\,ds$$
$$\geq \int_a^b N \left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \frac{1 - e^{-Ms}}{M} e^{Ms}\,ds$$
$$= \frac{N}{M}\left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \int_a^b (e^{Ms} - 1)\,ds$$
$$\geq \frac{N}{M}\left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \int_0^T (e^{Ms} - 1)\,ds$$
$$= \frac{N}{M}\left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \frac{e^{MT} - MT - 1}{M}$$
$$= \left(\min_{r\in[0,b]} w(r)e^{Mr}\right) \frac{N}{M^2}(e^{MT} - MT - 1) \geq \min_{s\in[0,b]} (w(s)e^{Ms}).$$

This means that (3) is valid. ■

The estimate

$$\frac{N}{M^2}(e^{MT} - MT - 1) \leq 1$$

is better than the following (that can be obtained analogously to Proposition 2 [19])

$$\frac{NT}{M}(e^{MT} - 1) \leq 1,$$

since

$$\frac{N}{M^2}(e^{MT} - MT - 1) < \frac{NT}{M}(e^{MT} - 1).$$

Indeed, this is equivalent to

$$e^{MT} - MT - 1 < MT(e^{MT} - 1),$$

or

$$e^{MT} - 1 - MTe^{MT} < 0,$$

but the function

$$\varphi(x) = e^x - 1 - xe^x$$

satisfies that $\varphi(0) = 0$ and $\varphi'(x) = -xe^x < 0$, for $x > 0$, so that φ is nonincreasing and negative for $x > 0$. Therefore, the assertion is true.

4 Generalization of Theorem 1

Occasionally, the equation

$$v'(t) + Mv(t) + [p(v)](t) = \sigma(t) \ \ a.e. \ t \in I,$$

can be split into several equations

$$v'(t) + Mv(t) + [p(v)](t) = \sigma(t), \ \ a.e. \ t \in [\alpha_i, \alpha_{i+1}],$$

where $i = 0, 1, \ldots, k$ and $\{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1} = T\}$ is a partition of $[0, T]$. This is possible, for instance, when $[p(v)](t)$ only takes into account the values of v in $[\alpha_i, t]$, for $t \in [\alpha_i, \alpha_{i+1}]$ and $i = 0, 1, \ldots, k$. Precisely, this was the case for the delayed equation

$$v'(t) + Mv(t) + Nv([t]) \leq 0, \ t \in I,$$

studied in last section. Here, if $T > 1$ and $k < T \leq k + 1$, we can take $\alpha_i = i$, for $i = 0, 1, \ldots, k$, $\alpha_{k+1} = T$ and $[p(v)](t) = Nv([t]) = Nv(i)$, for $t \in [i, i + 1)$, $i = 0, 1, \ldots, k - 1$, $[p(v)](t) = Nv(k)$, for $t \in [k, T]$.

In this formulation we include also the case of delayed equations with delay function θ satisfying

$$\theta(t) \in [\alpha_i, t], \quad \text{for a.e. } t \in [\alpha_i, \alpha_{i+1}], \ i = 0, 1, \ldots, k.$$

Now, we will prove a result in the spirit of Theorem 1 but adapted to the property of p cited above.

Theorem 2 *Let $v \in W^{1,1}(I)$ be a solution of (1), $M > 0$, $\lambda \leq 0$, $\sigma \leq 0$ a.e. on I and $p : L^1(I) \to L^1(I)$ such that (2) holds. Suppose that there exists $\{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1} = T\}$ a partition of $[0, T]$ such that $[p(v)](t)$ only depends on the values of v in $[\alpha_i, t]$, for a.e. $t \in [\alpha_i, \alpha_{i+1}]$, $i = 0, 1, \ldots, k$ and that*

$$\begin{cases} \int_a^b [p(w)](s)\, e^{Ms}\, ds \geq \min_{s\in[\alpha_i,b]} (w(s)\, e^{Ms}),\ \forall i = 0, 1, \ldots, k, \\ \alpha_i \leq a < b \leq \alpha_{i+1}\ and\, w \in C([\alpha_i, \alpha_{i+1}])with\ \min_{[\alpha_i,b]} w \leq 0. \end{cases} \tag{9}$$

Then $v \leq 0$ *a.e. on* I.

Proof We have that

$$v'(t) + Mv(t) + [p(v)](t) \leq 0,\ \ a.e.\ t \in [\alpha_i, \alpha_{i+1}),\ \ i = 0, 1, \ldots, k.$$

If $v \geq 0$ on I, then, by (2),

$$v'(t) \leq -Mv(t) - [p(v)](t) \leq 0$$

for a.e. $t \in [\alpha_i, \alpha_{i+1})$, $i = 0, 1, \ldots, k$ and v is nonincreasing on $[\alpha_i, \alpha_{i+1})$, for $i = 0, 1, \ldots, k$. But v is continuous, so that v is nonincreasing on I and v is a constant function since $v(0) \leq v(T)$. Then $v(t) = k$ with $k \geq 0$ and

$$0 \leq Mk \leq -[p(v)](t) \leq 0,\ a.e.\ t \in I,$$

therefore, $v(t) = k = 0$, for all $t \in I$.

Now, suppose that $v(t_\star) < 0$ for $t_\star \in [\alpha_p, \alpha_{p+1})$ and some $p \in \{0, 1, \ldots, k\}$, or $v(T) < 0$.

Consider the function

$$z(t) = v(t)e^{Mt},\ t \in I.$$

Then

$$z'(t) \leq -[p(v)](t)\, e^{Mt},\ a.e.\ t \in [\alpha_i, \alpha_{i+1}),\ \ i = 0, 1, \ldots, k. \tag{10}$$

If $v(t_\star) < 0$, with $t_\star \in [\alpha_p, \alpha_{p+1})$ and $p \in \{0, 1, \ldots, k\}$, then we will prove that $v(T) \leq 0$. If $z(\alpha_{p+1}) > 0$ then there exists $t_0 \in [\alpha_p, \alpha_{p+1})$ such that $z(t_0) = \min_{[\alpha_p,\alpha_{p+1}]} z < 0$ and integrating (10) between t_0 and α_{p+1} we obtain that

$$\begin{aligned} -z(t_0) &< z(\alpha_{p+1}) - z(t_0) \\ &\leq -\int_{t_0}^{\alpha_{p+1}} [p(v)](s)\, e^{Ms}\, ds \leq -\min_{s\in[\alpha_p,\alpha_{p+1}]} (v(s)\, e^{Ms}) = -z(t_0), \end{aligned}$$

that is absurd. Therefore, $z(\alpha_{p+1}) \leq 0$.

Now, if $p = k$, then $v(T) \leq 0$. If $p < k$, we will prove that $z(\alpha_{p+2}) \leq 0$. If $z(\alpha_{p+2}) > 0$, then $t_1 \in [\alpha_{p+1}, \alpha_{p+2}]$ is such that $z(t_1) = \min_{[\alpha_{p+1},\alpha_{p+2}]} z \leq 0$ and integrating (10) between t_1 and α_{p+2} we get another contradiction. If $p + 1 = k$, we achieve $v(T) \leq 0$ and if $p + 1 < k$ we repeat this process until we have that $p + j = k + 1$, $z(\alpha_{p+j}) = z(T) \leq 0$ and also $v(T) \leq 0$.

In both cases, we have that $v(T) \leq 0$. Then $z(0) = v(0) \leq v(T) \leq 0$. If there exists $t_2 \in (0, \alpha_1)$ such that $z(t_2) > 0$, then $\min_{[0,t_2]} z = z(t_3) \leq 0$. Integrating (10) for $i = 0$ in $[t_3, t_2]$, we get that

$$-z(t_3) < z(t_2) - z(t_3) \leq -\int_{t_3}^{t_2} [p(v)](s)\, e^{Ms}\, ds \leq -\min_{s\in[0,t_2]} (v(s)\, e^{Ms}) = -z(t_3),$$

that is a contradiction. This implies that $z \leq 0$ on $[0, \alpha_1)$ and $z(\alpha_1) \leq 0$, since z is continuous. Following an analogous procedure in the interval $[\alpha_1, \alpha_2]$, we get that $z \leq 0$ on that interval, and so on, until the interval $[\alpha_k, T]$, we will prove that $z \leq 0$ on I and, therefore, $v \leq 0$ on I. ■

Thus, Theorem 1 is a particular case of Theorem 2, where the partition of $[0, T]$ is trivial

$$0 = \alpha_0 < \alpha_1 = T.$$

Let us see how Theorem 2 improves the result obtained for the delayed differential inequality with delay function $\theta(t) = [t]$, $t \in I$, that is

$$\begin{cases} v'(t) + Mv(t) + Nv([t]) \leq 0, \quad a.e.\ t \in I, \\ v(0) \leq v(T). \end{cases}$$

As we have pointed out, if $k \in \mathbb{N}$, $k < T \leq k + 1$, we consider the partition given by $\alpha_i = i$, $i = 0, \ldots, k$, $\alpha_{k+1} = T$. Let $i \in \{0, \ldots, k\}$, $\alpha_i = i \leq a < b \leq \alpha_{i+1}$ and $w \in C([\alpha_i, \alpha_{i+1}])$ with $\min_{[\alpha_i,b]} w \leq 0$. Then

$$\begin{aligned}
&\int_a^b Nw([s])\, e^{Ms}\, ds = \\
&= N \int_a^b w([s]) e^{M[s]}\, e^{-M[s]}\, e^{Ms}\, ds \geq N \min_{s\in[i,b]} (w(s)e^{Ms}) \int_a^b e^{-M[s]}\, e^{Ms}\, ds \\
&= N \min_{s\in[i,b]} (w(s)e^{Ms}) \int_a^b e^{M(s-i)}\, ds \geq N \min_{s\in[i,b]} (w(s)e^{Ms}) \int_i^{i+1} e^{M(s-i)}\, ds \\
&= N \min_{s\in[i,b]} (w(s)e^{Ms}) \frac{1}{M}(e^M - 1) = \frac{N}{M}(e^M - 1) \min_{s\in[i,b]} (w(s)e^{Ms}).
\end{aligned}$$

If the following condition holds

$$N \int_i^{i+1} e^{M(s-\theta(s))}\, ds \leq 1$$

or, equivalently,

$$\frac{N}{M}(e^M - 1) \leq 1,$$

then

$$\int_a^b Nw([s])\, e^{Ms}\, ds \geq \min_{s\in[i,b]} (w(s)e^{Ms}),$$

for $\alpha_i \leq a < b < \alpha_{i+1}$, $i \in \{0, 1, \ldots, k\}$ and (9) is valid.

We have proved the following result:

Corollary 4 *If $M > 0$, $N \geq 0$, $T > 1$ and $v \in W^{1,1}(I)$ are such that*

$$\begin{cases} v'(t) + Mv(t) + Nv([t]) \leq 0, \quad a.e.\ t \in I, \\ v(0) \leq v(T), \\ \frac{N}{M}(e^M - 1) \leq 1, \end{cases}$$

then $v \leq 0$ a.e. on I.

Compare this result with Theorem 1 [20].
In the case $T \leq 1$, the estimate $\frac{N}{M}(e^{MT} - 1) \leq 1$ was obtained in Sect. 2.

Corollary 5 *Let $\{\alpha_0 = 0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k < \alpha_{k+1} = T\}$ a partition of $[0, T]$ and*

$$\theta_i(t) : [\alpha_i, \alpha_{i+1}) \to [\alpha_i, \alpha_{i+1}), \quad \theta_i(t) \leq t, \ \textit{for a.e. } t \in [\alpha_i, \alpha_{i+1}).$$

If $p : L^1(I) \to L^1(I)$ is such that

$$[p(v)](t) = Nv(\theta_i(t)), \quad \textit{for } t \in [\alpha_i, \alpha_{i+1}), i = 0, 1, \ldots, k-1,$$

$$[p(v)](t) = Nv(\theta_k(t)), \quad \textit{for } t \in [\alpha_k, \alpha_{k+1}]$$

and

$$N \int_{\alpha_i}^{\alpha_{i+1}} e^{M(s-\theta_i(s))}\, ds \leq 1,$$

for a certain $N \geq 0$, then (9) is valid. As a consequence, if $v \in W^{1,1}(I)$ is a solution of problem (1) with $M > 0$, $\lambda \leq 0$ and $\sigma \leq 0$ a.e. on I, then $v \leq 0$ a.e. on I.

Proof Let $i \in \{0, \ldots, k\}$ $\alpha_i \leq a < b \leq \alpha_{i+1}$ and $w \in C([\alpha_i, \alpha_{i+1}])$ with $\min_{[\alpha_i,b]} w \leq 0$. Then, using the properties of θ_i, we get

$$\int_a^b Nw(\theta_i(s))\, e^{Ms}\, ds = N \int_a^b w(\theta_i(s))e^{M\theta_i(s)}\, e^{-M\theta_i(s)}\, e^{Ms}\, ds$$

$$\geq N \min_{s\in[\alpha_i,b]} (w(s)e^{Ms}) \int_a^b e^{M(s-\theta_i(s))}\, ds$$

$$\geq N \min_{s\in[\alpha_i,b]} (w(s)e^{Ms}) \int_{\alpha_i}^{\alpha_{i+1}} e^{M(s-\theta_i(s))}\, ds \geq \min_{s\in[\alpha_i,b]} (w(s)e^{Ms}).$$

■

If $\alpha_i = i, i = 0, \ldots, k, \alpha_{k+1} = T$ and $\theta_i(t) = [t] = i$, for $t \in [\alpha_i, \alpha_{i+1})$, then we obtain the result given in Corollary 4.

If the delay is a piecewise constant function, $\theta_i(t) = \alpha_i$, for a.e. $t \in [\alpha_i, \alpha_{i+1})$, where

$$\{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1} = T\},$$

then

$$N \int_{\alpha_i}^{\alpha_{i+1}} e^{M(t-\alpha_i)}\, dt = \frac{N}{M}(e^{M(\alpha_{i+1}-\alpha_i)} - 1) \leq \frac{N}{M}(e^{M\tau} - 1)$$

and we obtain the following estimate on the constants in order to guarantee the validity of (9):

$$\frac{N}{M}(e^{M\tau} - 1) \leq 1,$$

where $\tau = \max\{\alpha_{i+1} - \alpha_i \;:\; i = 0, \ldots, k\}$.

Corollary 6 *If* $p : L^1(I) \to L^1(I)$ *and* $\{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1} = T\}$ *are such that*

$$[p(w)](t) \geq N \min_{[\alpha_i, t]} w, \quad a.e.\, t \in [\alpha_i, \alpha_{i+1}),$$

for $i = 0, 1, \ldots, k$*,* $w \in C([\alpha_i, \alpha_{i+1}])$ *and the estimate*

$$\frac{N}{M}(e^{M\tau} - 1) \leq 1$$

where $\tau = \max\{\alpha_{i+1} - \alpha_i \;:\; i = 0, \ldots, k\}$ *holds, then any solution* $v \in W^{1,1}(I)$ *of (1) with* $M > 0$*,* $\lambda \leq 0$*,* $\sigma \leq 0$ *a.e. on* I*, satisfies that* $v \leq 0$ *a.e. on* I*.*

Proof It is easy to check that (9) is true if $\frac{N}{M}(e^{M\tau} - 1) \leq 1$ holds. ■

Corollary 7 *Suppose that* $p : L^1(I) \longrightarrow L^1(I)$ *and*

$$\{0 = \alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1} = T\}$$

are such that

$$[p(w)](t) \geq N \int_{\alpha_i}^{t} w(s)\, ds, \quad a.e.\, t \in [\alpha_i, \alpha_{i+1}),$$

for $i = 0, 1, \ldots, k$*,* $w \in C([\alpha_i, \alpha_{i+1}])$ *and*

$$\frac{N}{M^2}(e^{M\tau} - M\tau - 1) \leq 1,$$

where $\tau = \max\{\alpha_{i+1} - \alpha_i \;:\; i = 0, \ldots, k\}$*. If* $v \in W^{1,1}(I)$ *is a solution of (1), where* $M > 0$*,* $\lambda \leq 0$*,* $\sigma \leq 0$ *a.e. on* I*, then* $v \leq 0$ *a.e. on* I*.*

Proof It can be proved that (9) holds if

$$\frac{N}{M^2}[e^{M(\alpha_{i+1}-\alpha_i)} - M(\alpha_{i+1}-\alpha_i) - 1] \leq 1,$$

for all $i = 0, 1, \ldots, k$.

However, the function $\phi(y) = e^y - y - 1$ is nondecreasing for $y > 0$, so that

$$\phi(M(\alpha_{i+1}-\alpha_i)) \leq \phi(M\tau),$$

for $i = 0, 1, \ldots, k$ and

$$\frac{N}{M^2}[e^{M(\alpha_{i+1}-\alpha_i)} - M(\alpha_{i+1}-\alpha_i) - 1] \leq \frac{N}{M^2}(e^{M\tau} - M\tau - 1),$$

for all $i = 0, 1, \ldots, k$ where the equality is valid for at least one index i. ■

5 Nonnegativity of Solutions

Let $I = [0, T]$, $p : L^1(I) \to L^1(I)$ and consider the problem (1) again, that is,

$$\begin{cases} v'(t) + Mv(t) + [p(v)](t) = \sigma(t), \quad a.e.\ t \in I, \\ v(0) = v(T) + \lambda. \end{cases}$$

Consider the following conditions

$$p(w) \leq 0 \text{ a.e. on } I, \text{ if } w \in C(I),\ w \leq 0 \text{ on } I \tag{11}$$

and

$$\begin{cases} \text{for all } a < b \in I \text{ and } w \in C(I) \text{ with } \max_{[0,b]} w \geq 0, \\ \text{we have } \int_a^b [p(w)](s)\, e^{Ms}\, ds \leq \max_{s\in[0,b]} (w(s)\, e^{Ms}), \end{cases} \tag{12}$$

Remark 1 The former conditions can be included in the framework of Sect. 2 by considering $q(v) = -p(-v)$. Then, the results of this section are a direct consequence of the ones in the past sections.

Theorem 3 *If $v \in W^{1,1}(I)$ is a solution of problem (1), $M > 0$, $\lambda \geq 0$, $\sigma \geq 0$ a.e. on I and p satisfies (11) and (12), then $v \geq 0$ a.e. on I.*

Note that condition (12) can be expressed in the following way, as well.

Let $\widehat{w}(t) = w(t)e^{-Mt}$

$$\begin{cases} \int_a^b [p(\widehat{w})](s)\, e^{Ms}\, ds \le \max_{[0,b]} w, \\ \text{for all}\, a < b \in I \,\text{and}\, w \in C(I) \text{ with } \max_{[0,b]} w \ge 0. \end{cases}$$

Next, we see some particular cases of the general framework.

5.1 Retarded Functional Differential Equations

Corollary 8 *Let $M > 0$, $N \ge 0$, $\theta : I \to I$, $\theta(t) \le t$, a.e. on I,*

$$[p(w)](t) = Nw(\theta(t)),$$

such that $p : L^1(I) \longrightarrow L^1(I)$. If $v \in W^{1,1}(I)$ is a solution of (1), where $\lambda \ge 0$, $\sigma \ge 0$ a.e. on I, then $v \ge 0$ a.e. on I.

What we have proved is that if $M > 0$, $N \ge 0$, $\theta : I \to I$, $\theta(t) \le t$ a.e. on I and $v \in W^{1,1}(I)$ are such that $p : L^1 \to L^1$ and

$$\begin{cases} v'(t) + Mv(t) + Nv(\theta(t)) \ge 0, \quad a.e.\ t \in I, \\ v(0) \ge v(T), \\ N \int_0^T e^{M(s-\theta(s))}\, ds \le 1, \end{cases}$$

then $v \ge 0$ a.e. on I.

In the case where $\theta(t) = [t]$, $t \in [0, T]$, where $[\cdot]$ is the greatest integer function, we obtain the estimate

$$\frac{N}{M}(e^{MT} - 1) \le 1, \text{ if } T \le 1$$

and

$$\frac{N}{M}[k(e^{M} - 1) + e^{M(T-k)} - 1] \le 1, \text{ if}\, T > 1 \,\text{and}\, k \in \mathbb{N} \,\text{such that}\, k < T \le k + 1.$$

In the last case, this is not the best estimate we can obtain.

5.2 Maximum Case

Corollary 9 *Let $M > 0$, $\lambda \ge 0$, $\sigma \ge 0$ a.e. on I and $p : L^1(I) \longrightarrow L^1(I)$ such that*

$$p(w) \in L^{\infty}(I), \textit{ for every}\, w \in C(I), \tag{13}$$

$$\begin{gathered} \textit{There exists } N \ge 0 \textit{such that} \\ \operatorname{ess\,sup}_{t\in[0,\tau]}[p(w)](t) \le N \max_{[0,\tau]} w, \textit{ for}\, \tau \in I \textit{and}\, w \in C(I) \end{gathered} \tag{14}$$

and

$$\frac{N}{M}(e^{MT}-1)\leq 1.$$

Then, if $v \in W^{1,1}(I)$ *is a solution of (1),* $v \geq 0$ *a.e. on* I.

5.3 Integral Case

Corollary 10 *Let* $M > 0$, $N \geq 0$ *and suppose that* $p : L^1(I) \longrightarrow L^1(I)$ *satisfies that*

$$[p(w)](t) \leq N \int_0^t w(s)\, ds, \quad a.e.\, t \in I, \ for\, w \in C(I). \tag{15}$$

If $v \in W^{1,1}(I)$ *is a solution of (1), where* $\lambda \geq 0$, $\sigma \geq 0$ *a.e. on* I *and the estimate*

$$\frac{N}{M^2}(e^{MT}-MT-1)\leq 1$$

holds, then $v \geq 0$ *a.e. on* I.

6 Uniqueness of Solution

The above results provide several uniqueness results for periodic boundary problems.

Corollary 11 *Let* $M > 0$ *and* $v \in W^{1,1}(I)$ *a solution to the problem*

$$\begin{cases} v'(t) + Mv(t) + [p(v)](t) = 0, \quad a.e.\ t \in I, \\ \qquad\qquad v(0) = v(T), \end{cases} \tag{16}$$

where $p : L^1(I) \to L^1(I)$ *is such that*

$$p(w) \geq 0\, a.e.\ on\, I, \ if\ w \in C(I),\ w \geq 0\ on\ I, \tag{17}$$

$$p(w) \leq 0\, a.e.\ on\, I, \ if\ w \in C(I),\ w \leq 0\ on\ I, \tag{18}$$

$$\begin{cases} for\ all\, a < b \in I and\, w \in C(I) with \min\limits_{[0,b]} w \leq 0, \\ we\ have\ \int_a^b [p(w)](s)\, e^{Ms}\, ds \geq \min\limits_{s\in[0,b]} (w(s)\, e^{Ms}), \end{cases} \tag{19}$$

and

$$\begin{cases} \textit{for all } a < b \in I \textit{ and } w \in C(I) \textit{ with } \max\limits_{[0,b]} w \geq 0, \\ \textit{we have } \int_a^b [p(w)](s)\, e^{Ms}\, ds \leq \max\limits_{s\in[0,b]} (w(s)\, e^{Ms}), \end{cases} \tag{20}$$

(that is, the conditions (2), (11), (3) and (12) are valid).
Then, $v \equiv 0$ *a.e. on* I.

Now, consider some particular cases.

Corollary 12 *If* $M > 0$, $N \geq 0$, $\theta : I \to I$, $\theta(t) \leq t$, *a.e. on* I *and* $v \in W^{1,1}(I)$ *are such that*

$$\begin{cases} v'(t) + Mv(t) + Nv(\theta(t)) = 0, \quad a.e.\ t \in I, \\ v(0) = v(T), \\ N \int_0^T e^{M(s-\theta(s))}\, ds \leq 1, \end{cases}$$

where the operator $[p(w)](t) = Nw(\theta(t))$ *satisfies that* $p : L^1(I) \longrightarrow L^1(I)$, *then* $v \equiv 0$ *a.e. on* I.

Corollary 13 *Let* $M > 0$ *and* $v \in W^{1,1}(I)$ *a solution to the problem*

$$\begin{cases} v'(t) + Mv(t) + [p(v)](t) = 0, \quad a.e.\ t \in I, \\ v(0) = v(T), \end{cases} \tag{21}$$

where $p : L^1(I) \to L^1(I)$ *satisfies the three following conditions:*

(a) $p(w) \in L^\infty(I)$, *for* $w \in C(I)$,
(b) there exists $N \geq 0$, *such that, for* $\tau \in I$ *and* $w \in C(I)$

$$\begin{cases} \operatorname{ess} \inf\limits_{t\in[0,\tau]} [p(w)](t) \geq N \min\limits_{[0,\tau]} w, \\ \operatorname{ess} \sup\limits_{t\in[0,\tau]} [p(w)](t) \leq N \max\limits_{[0,\tau]} w, \end{cases}$$

(c) $\frac{N}{M}(e^{MT} - 1) \leq 1$.

Then, $v \equiv 0$ *a.e. on* I.

Corollary 14 *Let* $M > 0$, $N \geq 0$ *and* $v \in W^{1,1}(I)$ *such that*

$$\begin{cases} v'(t) + Mv(t) + N \int_0^t w(s)\, ds = 0, \quad a.e.\ t \in I, \\ v(0) = v(T), \end{cases} \tag{22}$$

and

$$\frac{N}{M^2}(e^{MT} - MT - 1) \leq 1,$$

then $v \equiv 0$ *a.e. on* I.

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Time-Fractional Optimal Control of Initial Value Problems on Time Scales

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Abstract We investigate Optimal Control Problems (OCP) for fractional systems involving fractional-time derivatives on time scales. The fractional-time derivatives and integrals are considered, on time scales, in the Riemann–Liouville sense. By using the Banach fixed point theorem, sufficient conditions for existence and uniqueness of solution to initial value problems described by fractional order differential equations on time scales are known. Here we consider a fractional OCP with a performance index given as a delta-integral function of both state and control variables, with time evolving on an arbitrarily given time scale. Interpreting the Euler–Lagrange first order optimality condition with an adjoint problem, defined by means of right Riemann–Liouville fractional delta derivatives, we obtain an optimality system for the considered fractional OCP. For that, we first prove new fractional integration by parts formulas on time scales.

Keywords Fractional derivatives and integrals on time scales · Initial value problems · Optimal control

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1 Introduction

Let $\mathbb{T}$ be a time scale, that is, a nonempty closed subset of $\mathbb{R}$. We consider the following initial value problem:

$$\begin{aligned} {}_{t_0}^{\mathbb{T}}D_t^{\alpha} y(t) = f(t, y(t)), \quad t \in [t_0, t_0 + a] = \mathcal{J} \subseteq \mathbb{T}, \quad 0 < \alpha < 1, \\ {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} y(t_0) = 0, \end{aligned} \tag{1}$$

where ${}_{t_0}^{\mathbb{T}}D_t^{\alpha}$ is the (left) Riemann–Liouville fractional derivative operator or order α defined on $\mathbb{T}$ and ${}_{t_0}^{\mathbb{T}}I_t^{1-\alpha}$ is the (left) Riemann–Liouville fractional integral operator or order $1-\alpha$ defined on $\mathbb{T}$, as introduced in [24] (see also [45, 46]), and function $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ is a right-dense continuous function. Necessary and sufficient conditions for the existence and uniqueness of solution to problem (1) are already discussed in [24]. Here, our goal is to prove optimality conditions for such systems.

Fractional Calculus (FC) is a generalization of classical calculus. It has been reported in the literature that systems described using fractional derivatives give a more realistic behavior. There exists many definitions of a fractional derivative. Commonly used fractional derivatives are the classical Riemann–Liouville and Caputo derivatives on continuous time scales. Fractional derivatives and integrals of Riemann–Liouville and Caputo types have a vast number of applications, across many fields of science and engineering. For example, they can be used to model controllability, viscoelastic flows, chaotic systems, Stokes problems, thermo-elasticity, several vibration and diffusion processes, bioengineering problems, and many other complex phenomena: see, e.g., [6, 11] and references therein.

Fractional optimal control problems on a continuous time scale have attracted several authors in the last two decades, and many techniques have been developed for solving such problems, involving classical fractional derivatives. Agrawal [6, 7] presented a general formulation and proposed a numerical method to solve such problems. In those papers, the fractional derivative was defined in the Riemann–Liouville sense and the formulation was obtained by means of a fractional variational principle and the Lagrange multiplier technique. Using new techniques, Frederico and Torres [32, 33] obtained Noether-like theorems for fractional optimal control problems in both Riemann–Liouville and Caputo senses. In [39, 40], Mophou and N'Guérékata studied the fractional optimal control of diffusion equations involving the classical Riemann–Liouville derivatives. In [43], Ozdemir investigated the fractional optimal control problem of a distributed system in cylindrical coordinates whose dynamics are defined in the classical Riemann–Liouville sense. For the state of the art and many generalizations, see the recent books [9, 37].

The theory of fractional differential equations, specifically the question of existence and uniqueness of solutions, is a research topic of great importance [1, 12, 34]. Another important area of study is dynamic equations on time scales, which goes back to 1988 and the work of Aulbach and Hilger, and has been used with success to unify differential and difference equations [5, 10, 26]. Starting with a linear dynamic equation, Bastos et al. have introduced the notion of fractional-order derivative on time scales, involving time-scale analogues of Riemann–Liouville operators [15–17]. Another approach originate from the inverse Laplace transform on time scales [18]. After such pioneer work, the study of fractional calculus on time scales developed in a popular research subject: see [19, 22, 23, 25, 42] and the more recent references [2, 20, 21, 38, 41].

To the best of our knowledge, the study of fractional optimal control problems for dynamical systems on time scales is under-developed, at least when compared to the continuous and discrete cases [31, 36]. Motivated by this fact, in this paper an Optimal Control Problem (OCP) for fractional initial value systems involving fractional-time derivatives on time scales is considered. The fractional-time derivative and integral are considered in the Riemann–Liouville sense on time scales, as introduced in [24]. We prove necessary optimality conditions for such OCPs. The performance index of the Fractional Optimal Control Problem (FOCP) is considered as a non-autonomous delta integral of a function depending on state and control variables, and where the dynamic control system is expressed by a delta-differential system. Interpreting the Euler–Lagrange first order optimality condition with an adjoint problem, defined by means of the time-scale right fractional derivative in the sense of Riemann–Liouville, we obtain an optimality system for the FOCP on time scales.

2 Preliminaries

In this section, we collect notations, definitions, and results, which are needed in the sequel. We use $C(\mathcal{J}, \mathbb{R})$ for the Banach space of continuous functions y with the norm $\|y\|_\infty = \sup\{|y(t)| : t \in \mathcal{J}\}$, where $\mathcal{J}$ is a time-scale interval.

2.1 Time-Scale Essentials

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of $\mathbb{R}$. The reader interested on the calculus on time scales is referred to the books [26, 27]. For a survey, see [5]. Any time scale $\mathbb{T}$ is a complete metric space with the distance $d(t, s) = |t - s|$, $t, s \in \mathbb{T}$. Consequently, according to the well-known theory of general metric spaces, we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets, etc. In particular, for a given number $\delta > 0$, the δ-neighborhood $\mathrm{U}_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t, s) < \delta$. We also have, for functions $f : \mathbb{T} \to \mathbb{R}$, the concepts of limit, continuity, and the properties of continuous functions on a general complete metric space. Roughly speaking, the calculus on time scales begins by introducing and investigating the concept of derivative for functions $f : \mathbb{T} \to \mathbb{R}$. In the definition of derivative, an important role is played by the so-called jump operators.

Definition 1 (*See* [27]) Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$, and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$.

Remark 1 In Definition 1, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(M) = M$ if $\mathbb{T}$ has a maximum M) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(m) = m$ if $\mathbb{T}$ has a minimum m).

If $\sigma(t) > t$, then we say that t is right-scattered; if $\rho(t) < t$, then t is said to be left-scattered. Points that are simultaneously right-scattered and left-scattered are called isolated. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense; if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) := \sigma(t) - t$. The derivative makes use of the set $\mathbb{T}^\kappa$, which is obtained from the time scale $\mathbb{T}$ as follows: if $\mathbb{T}$ has a left-scattered maximum M, then $\mathbb{T}^\kappa := \mathbb{T} \setminus \{M\}$; otherwise, $\mathbb{T}^\kappa := \mathbb{T}$.

Definition 2 (*Delta derivative* [4]) Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^\kappa$. We define

$$f^\Delta(t) := \lim_{s \to t} \frac{f(\sigma(s)) - f(t)}{\sigma(s) - t}, \quad t \neq \sigma(s),$$

provided the limit exists. We call $f^\Delta(t)$ the delta derivative (or Hilger derivative) of f at t. Moreover, we say that f is delta differentiable on $\mathbb{T}^\kappa$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$. The function $f^\Delta : \mathbb{T}^\kappa \to \mathbb{R}$ is then called the (delta) derivative of f on $\mathbb{T}^\kappa$.

Definition 3 (*See* [27]) A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by C_{rd}. Similarly, a function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right-sided limits exist (finite) at right-dense points in $\mathbb{T}$. The set of ld-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by C_{ld}.

Definition 4 (*See* [27]) Let $[a, b]$ denote a closed bounded interval in $\mathbb{T}$. A function $F : [a, b] \to \mathbb{R}$ is called a delta antiderivative of function $f : [a, b) \to \mathbb{R}$ provided F is continuous on $[a, b]$, delta differentiable on $[a, b)$, and $F^\Delta(t) = f(t)$ for all $t \in [a, b)$. Then, we define the Δ-integral of f from a to b by

$$\int_a^b f(t)\Delta t := F(b) - F(a).$$

Proposition 1 (See [8]) *Suppose* $\mathbb{T}$ *is a time scale and* f *is an increasing continuous function on the time-scale interval* $[a, b]$. *If* F *is the extension of* f *to the real interval* $[a, b]$ *given by*

$$F(s) := \begin{cases} f(s) & \text{if } s \in \mathbb{T}, \\ f(t) & \text{if } s \in (t, \sigma(t)) \notin \mathbb{T}, \end{cases}$$

then

$$\int_a^b f(t)\Delta t \le \int_a^b F(t)dt.$$

2.2 Fractional Derivative and Integral on Time Scales

We adopt a recent notion of fractional derivative on time scales introduced in [24], which is based on the notion of fractional integral on time scales $\mathbb{T}$. This is in contrast with [22, 23, 25], where first a notion of fractional differentiation on time scales is introduced and only after that, with the help of such concept, the fraction integral is defined. The classical gamma and beta functions are used.

Definition 5 (*Gamma function*) For complex numbers with a positive real part, the gamma function $\Gamma(t)$ is defined by the following convergent improper integral:

$$\Gamma(t) := \int_0^\infty x^{t-1}e^{-x}dx.$$

Definition 6 (*Beta function*) The beta function, also called the Euler integral of first kind, is the special function $\mathrm{B}(x, y)$ defined by

$$\mathrm{B}(x, y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt, \quad x > 0, \quad y > 0.$$

Remark 2 The gamma function satisfies the following property: $\Gamma(t+1) = t\Gamma(t)$. The beta function can be expressed through the gamma function by

$$\mathrm{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

Definition 7 (*Fractional integral on time scales* [24]) Suppose $\mathbb{T}$ is a time scale, $[a, b]$ is an interval of $\mathbb{T}$, and h is an integrable function on $[a, b]$. Let $0 < \alpha < 1$. Then the left fractional integral of order α of h is defined by

$${}_a^{\mathbb{T}}I_t^\alpha h(t) := \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s.$$

The right fractional integral of order α of h is defined by

$${}_t^{\mathbb{T}}I_b^\alpha h(t) := \int_t^b \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} h(s)\Delta s,$$

where Γ is the gamma function.

Definition 8 (*Riemann–Liouville fractional derivative on time scales* [24]) Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The left Riemann–Liouville fractional derivative of order α of h is defined by

$$ {}_a^{\mathbb{T}}D_t^{\alpha} h(t) := \left({}_a^{\mathbb{T}}I_t^{1-\alpha} h(t)\right)^{\Delta} = \frac{1}{\Gamma(1-\alpha)} \left(\int_a^t (t-s)^{-\alpha} h(s)\Delta s\right)^{\Delta}. \tag{2} $$

The right Riemann–Liouville fractional derivative of order α of h is defined by

$$ {}_t^{\mathbb{T}}D_b^{\alpha} h(t) := -\left({}_t^{\mathbb{T}}I_b^{1-\alpha} h(t)\right)^{\Delta} = \frac{-1}{\Gamma(1-\alpha)} \left(\int_t^b (s-t)^{-\alpha} h(s)\Delta s\right)^{\Delta}. $$

Definition 9 (*Caputo fractional derivative on time scales* [8]) Let $\mathbb{T}$ be a time scale, $t \in \mathbb{T}$, $0 < \alpha < 1$, and $h : \mathbb{T} \to \mathbb{R}$. The left Caputo fractional derivative of order α of h is defined by

$$ {}_a^{\mathbb{T}C}D_t^{\alpha} h(t) := {}_a^{\mathbb{T}}I_t^{1-\alpha}(h^{\Delta}(t)) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} h^{\Delta}(s)\Delta s. $$

The right Caputo fractional derivative of order α of h is defined by

$$ {}_t^{\mathbb{T}C}D_b^{\alpha} h(t) := -{}_t^{\mathbb{T}}I_b^{1-\alpha}(h^{\Delta}(t)) = \frac{-1}{\Gamma(1-\alpha)} \int_t^b (s-t)^{-\alpha} h^{\Delta}(s)\Delta s. $$

The relation between the left/right RLFD and the left/right CFD is as follows:

$$ {}_a^{\mathbb{T}C}D_t^{\alpha} x(t) = {}_a^{\mathbb{T}}D_t^{\alpha} x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{(k-\alpha)}, $$

$$ {}_t^{\mathbb{T}C}D_b^{\alpha} x(t) = {}_t^{\mathbb{T}}D_b^{\alpha} x(t) - \sum_{k=0}^{n-1} \frac{x^{(k)}(b)}{\Gamma(k-\alpha+1)} (b-t)^{(k-\alpha)}. $$

If x and $x^{(i)}$, $i = 1, \ldots, n-1$, vanish at $t = a$, then ${}_a^{\mathbb{T}}D_t^{\beta} x(t) = {}_a^{\mathbb{T}C}D_t^{\beta} x(t)$, and if they vanish at $t = b$, then ${}_t^{\mathbb{T}}D_b^{\beta} x(t) = {}_t^{\mathbb{T}C}D_b^{\beta} x(t)$. Furthermore, ${}_a^{\mathbb{T}C}D_t^{\alpha}\, c = 0$, where c is a constant, and

$$ {}_a^{\mathbb{T}C}D_t^{\alpha}\, t^n = \begin{cases} 0, \text{ for } n \in N_0 \text{ and } n < [\alpha], \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)} t^{n-\alpha}, \text{ for } n \in \mathbb{N}_0 \text{ and } n \geq [\beta], \end{cases} $$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Remark 3 If $\mathbb{T} = \mathbb{R}$, then Definition 8 gives the classical left and right Riemann–Liouville fractional derivatives [44]. Similar comment for Definition 9. For different extensions of the fractional derivative to time scales using the Caputo approach, see [18]. For local approaches to fractional calculus on time scales, we refer the reader to [22, 23, 25]. Here we restrict ourselves to the delta approach to time scales. Analogous definitions are, however, trivially obtained for the nabla approach to time scales by using the duality theory of [3, 30].

2.3 Properties of the Time-Scale Fractional Operators

We recall some fundamental properties of the fractional operators on time scales.

Proposition 2 (See Proposition 15 of [24]) *Let* $\mathbb{T}$ *be a time scale with derivative* Δ, *and* $0 < \alpha < 1$. *Then,* ${}_a^{\mathbb{T}}D_t^{\alpha} = \Delta \circ {}_a^{\mathbb{T}}I_t^{1-\alpha}$.

Proposition 3 (See Proposition 16 of [24]) *For any function* h *integrable on* $[a, b]$, *the Riemann–Liouville* Δ*-fractional integral satisfies* ${}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\beta} = {}_a^{\mathbb{T}}I_t^{\alpha+\beta}$ *for* $\alpha > 0$ *and* $\beta > 0$.

Proposition 4 (See Proposition 17 of [24]) *For any function* h *integrable on* $[a, b]$ *one has* ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} h = h$.

Corollary 1 (See Corollary 18 of [24]) *For* $0 < \alpha < 1$, *we have* ${}_a^{\mathbb{T}}D_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{-\alpha} = Id$ *and* ${}_a^{\mathbb{T}}I_t^{-\alpha} \circ {}_a^{\mathbb{T}}I_t^{\alpha} = Id$, *where* Id *denotes the identity operator.*

Definition 10 (*See* [24]) For $\alpha > 0$, we denote by ${}_a^{\mathbb{T}}I_t^{\alpha}([a, b])$ the space of functions that can be represented by the Riemann–Liouville Δ integral of order α of some $C([a, b])$-function.

Theorem 1 (See Theorem 20 of [24]) *Let* $f \in C([a, b])$ *and* $\alpha > 0$. *Function* $f \in {}_a^{\mathbb{T}}I_t^{\alpha}([a, b])$ *if and only if* ${}_a^{\mathbb{T}}I_t^{1-\alpha} f \in C^1([a, b])$ *and* $\left({}_a^{\mathbb{T}}I_t^{1-\alpha} f(t)\right)\big|_{t=a} = 0$.

Theorem 2 (See Theorem 21 of [24]) *Let* $\alpha > 0$ *and* $f \in C([a, b])$ *satisfy the conditions in Theorem 1. Then,* $\left({}_a^{\mathbb{T}}I_t^{\alpha} \circ {}_a^{\mathbb{T}}D_t^{\alpha}\right)(f) = f$.

2.4 Existence of Solutions to Fractional IVPs on Time Scales

Let $\mathbb{T}$ be a time scale and $\mathcal{J} = [t_0, t_0 + a] \subset \mathbb{T}$. Consider the fractional order initial value problem (1) defined on $\mathbb{T}$. Then the function $y \in C(\mathcal{J}, \mathbb{R})$ is a solution of problem (1) if ${}_{t_0}^{\mathbb{T}}D_t^{\alpha} y(t) = f(t, y)$ on $\mathcal{J}$ and ${}_{t_0}^{\mathbb{T}}I_t^{\alpha} y(t_0) = 0$.

Theorem 3 (See Theorem 24 of [24]) *If* $f : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ *is a rd-continuous bounded function for which there exists* $M > 0$ *such that* $|f(t, y)| \le M$ *for all* $t \in \mathcal{J}$ *and* $y \in \mathbb{R}$, *then problem* (1) *has a solution on* $\mathcal{J}$.

3 Main Results

We begin by proving formulas of integration by parts in Sect. 3.1, which are then used in Sect. 3.2 to prove necessary optimality conditions for nonlinear Riemann–Liouville fractional optimal control problems (FOCPs) on time scales.

3.1 Fractional Integration by Parts on Time Scales

Our first result gives integration by parts formulas for fractional integrals and derivatives on time scales. For the relation between integration on time scales and Lebesgue integration we refer the reader to [29].

Theorem 4 *Let $\alpha > 0$, $p, q \geq 1$, and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$, where $p \neq 1$ and $q \neq 1$ in the case when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$. Moreover, let*

$$ {}_a^{\mathbb{T}}I_t^{\alpha}(L_p) := \left\{ f : f = {}_a^{\mathbb{T}}I_t^{\alpha} g,\ g \in L_p(a, b) \right\} $$

and

$$ {}_t^{\mathbb{T}}I_b^{\alpha}(L_p) := \left\{ f : f = {}_t^{\mathbb{T}}I_b^{\alpha} g,\ g \in L_p(a, b) \right\}. $$

The following integration by parts formulas hold.

(a) If $\varphi \in L_p(a, b)$ and $\psi \in L_q(a, b)$, then

$$ \int_a^b \varphi(t) \left({}_a^{\mathbb{T}}I_t^{\alpha} \psi \right)(t)\Delta t = \int_a^b \psi(t) \left({}_t^{\mathbb{T}}I_b^{\alpha} \varphi \right)(t)\Delta t. \tag{3} $$

(b) If $g \in {}_t^{\mathbb{T}}I_b^{\alpha}(L_p)$ and $f \in {}_a^{\mathbb{T}}I_t^{\alpha}(L_q)$, then

$$ \int_a^b g(t) \left({}_a^{\mathbb{T}}D_t^{\alpha} f \right)(t)\Delta t = \int_a^b f(t) \left({}_t^{\mathbb{T}}D_b^{\alpha} g \right)(t)\Delta t. \tag{4} $$

(c) For Caputo fractional derivatives, if $g \in {}_t^{\mathbb{T}}I_b^{\alpha}(L_p)$ and $f \in {}_a^{\mathbb{T}}I_t^{\alpha}(L_q)$, then

$$ \int_a^b g(t) \left({}_a^{\mathbb{T}C}D_t^{\alpha} f \right)(t)\Delta t = \left[{}_t^{\mathbb{T}}I_b^{1-\alpha} g(t) \cdot f(t) \right]_a^b + \int_a^b f(\sigma(t)) \left({}_t^{\mathbb{T}}D_b^{\alpha} g \right)(t)\Delta t $$

and

$$ \int_a^b g(t) \left({}_t^{\mathbb{T}C}D_b^{\alpha} f \right)(t)\Delta t = -\left[{}_a^{\mathbb{T}}I_t^{1-\alpha} g(t) \cdot f(t) \right]_a^b + \int_a^b f(\sigma(t)) \left({}_a^{\mathbb{T}}D_t^{\alpha} g \right)(t)\Delta t. $$

Proof (a) If $\varphi \in L_p(a,b)$ and $\psi \in L_q(a,b)$, then, from Definition 7, we get

$$\int_a^b \varphi(t)\left({}_a^{\mathbb{T}}I_t^\alpha \psi\right)(t)\Delta t = \int_a^b \varphi(t)\left(\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}\psi(s)\Delta s\right)\Delta t.$$

Interchanging the order of integrals (see [24]), we reach at

$$\int_a^b \varphi(t)\left({}_a^{\mathbb{T}}I_t^\alpha \psi\right)(t)\Delta t = \int_a^b \psi(t)\left({}_t^{\mathbb{T}}I_b^\alpha \varphi\right)(t)\Delta t.$$

(b) If $g \in {}_t^{\mathbb{T}}I_b^\alpha(L_p)$ and $f \in {}_a^{\mathbb{T}}I_t^\alpha(L_q)$, then, from Definition 8, we get

$$\int_a^b g(t)\left({}_a^{\mathbb{T}}D_t^\alpha f\right)(t)\Delta t = \int_a^b g(t)\left(\frac{1}{\Gamma(1-\alpha)}\left(\int_a^t (t-s)^{-\alpha} f(s)\Delta s\right)^\Delta\right)\Delta t.$$

Interchanging the order of integrals, we obtain that

$$\int_a^b g(t)\left({}_a^{\mathbb{T}}D_t^\alpha f\right)(t)\Delta t = \int_a^b f(t)\left({}_t^{\mathbb{T}}D_b^\alpha g\right)(t)\Delta t.$$

(c) If $g \in {}_t^{\mathbb{T}}I_b^\alpha(L_p)$ and $f \in {}_a^{\mathbb{T}}I_t^\alpha(L_q)$, then, from Definition 9, we get

$$\int_a^b g(t)\left({}_a^{\mathbb{T}C}D_t^\alpha f\right)(t)\Delta t = \int_a^b g(t)\left(\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha} f^\Delta(s)\Delta s\right)\Delta t.$$

Interchanging the order of the integrals, and by using integration by parts on time scales, we conclude that

$$\int_a^b g(t)\left({}_a^{\mathbb{T}C}D_t^\alpha f\right)(t)\Delta t = \int_a^b f(\sigma(t))\left({}_t^{\mathbb{T}}D_b^\alpha g\right)(t)\Delta t + \left[{}_t^{\mathbb{T}}I_b^{1-\alpha} g(t)\cdot f(t)\right]_a^b.$$

The second relation is obtained in a similar way. □

3.2 Nonlinear Riemann–Liouville FOCPs on Time Scales

Let $\mathbb{T}$ be a given time scale with $t_0, t_f \in \mathbb{T}$ and let us consider a control system given by the fractional differential equation

$$ {}_{t_0}^{\mathbb{T}}D_t^\alpha x(t) = f(x(t), u(t), t), \quad t \in \mathbb{T}, \tag{5}$$

subject to

$$ {}_{t_0}^{\mathbb{T}}I_t^{1-\alpha} x(t_0) = x_0, \tag{6}$$

where $x(t) \in \mathbb{R}^n$ and $u(t) \in \mathbb{R}^m$ are the state and control vectors, respectively, function $f : \mathbb{R}^{n\times m\times 1} \to \mathbb{R}^n$ is a nonlinear vector function, and $x_0 \in \mathbb{R}^n$ is the specified initial state vector. A similar problem is studied in [14] for problems involving AB derivatives in Caputo sense on continuous time scales. Here we study it within Riemann–Liouville derivatives on arbitrary time scales. In order to achieve a desired behavior in terms of performance requirements, we select a cost index for the dynamical system (5)–(6). In selecting the performance index, the designer attempts to define a mathematical expression that, when minimized, indicates that the system is performing in the most desirable manner. Thus, choosing a performance cost index is a translation of system's physical requirements into mathematical terms [13]. For the fractional dynamic system (5)–(6), we choose the following performance index:

$$J[x,u] = \int_{t_0}^{t_f} L(x(t), u(t), t)\Delta t \longrightarrow \min, \tag{7}$$

where $L : \mathbb{R}^{n\times m\times 1} \to \mathbb{R}$ is a scalar function. In the following, we derive a necessary optimality condition corresponding to the considered fractional optimal control problem (5)–(7). Under given considerations, the following theorem holds true.

Theorem 5 (Necessary optimality conditions) *Let* $(x(\cdot), u(\cdot))$ *be a minimizer of problem* (5)–(7)*. Then, there exists a function* $\lambda(\cdot)$ *for which the triplet* $(x(\cdot), \lambda(\cdot), u(\cdot))$ *satisfies:*

(i) the Hamiltonian system

$$\begin{cases} {}^{\mathbb{T}}_{t_0}D^{\alpha}_{t} x(t) = \dfrac{\partial \mathcal{H}}{\partial \lambda}(x(t), \lambda(t), u(t), t), & t \in \mathbb{T}, \\ {}^{\mathbb{T}}_{t}D^{\alpha}_{t_f} \lambda(t) = \dfrac{\partial \mathcal{H}}{\partial x}(x(t), \lambda(t), u(t), t), & t \in \mathbb{T}; \end{cases} \tag{8}$$

(ii) the stationary condition

$$\frac{\partial \mathcal{H}}{\partial u}(x(t), \lambda(t), u(t), t) = 0, \quad t \in \mathbb{T}, \tag{9}$$

where $\mathcal{H}$ *is a scalar function, called the* Hamiltonian*, defined by*

$$\mathcal{H}(x, \lambda, u, t) = L(x, u, t) + \lambda^T f(x, u, t). \tag{10}$$

Proof To deduce the necessary optimality conditions that the optimal pair $(x(\cdot), u(\cdot))$ must satisfy, we use the Lagrange multiplier technique to adjoin the dynamic constraint (5) to the performance index (7). Thus, we form the augmented functional

$$J_a[x, \lambda, u] = \int_{t_0}^{t_f} \left[\mathcal{H}(x(t), \lambda(t), u(t), t) - \lambda^T(t)\, {}^{\mathbb{T}}_{t_0}D^{\alpha}_{t} x(t)\right] \Delta t, \tag{11}$$

where $\lambda(t) \in \mathbb{R}^n$ is the Lagrange multiplier, also known as the costate or adjoint variable. Taking the first variation of the augmented performance index $J_a[x, \lambda, u]$ given by (11), we obtain that

$$\delta J_a[x, \lambda, u] = \int_{t_0}^{t_f} \Bigg\{ \left[\frac{\partial \mathcal{H}}{\partial x}\right]^T \delta x(t) + \left[\frac{\partial \mathcal{H}}{\partial \lambda} - {}_{t_0}^{\mathbb{T}}D_t^{\alpha} x(t)\right]^T \delta\lambda(t) + \left[\frac{\partial \mathcal{H}}{\partial u}\right]^T \delta u(t) - \lambda^T(t)\, {}_{t_0}^{\mathbb{T}}D_t^{\alpha} \delta x(t) \Bigg\} \Delta t. \tag{12}$$

Using the fractional integration by parts formula (4), the last integral in (12) can be written as

$$\int_{t_0}^{t_f} \lambda^T(t)\, {}_{t_0}^{\mathbb{T}}D_t^{\alpha} \delta x(t) \Delta t = \int_{t_0}^{t_f} \left({}_{t}^{\mathbb{T}}D_{t_f}^{\alpha} \lambda(t)\right)^T \delta x(t) \Delta t. \tag{13}$$

Using (13) in (12), we deduce that

$$\delta J_a[x, \lambda, u] = \int_{t_0}^{t_f} \Bigg\{ \left[\frac{\partial \mathcal{H}}{\partial x} - {}_{t}^{\mathbb{T}}D_{t_f}^{\alpha} \lambda(t)\right]^T \delta x(t) + \left[\frac{\partial \mathcal{H}}{\partial \lambda} - {}_{t_0}^{\mathbb{T}}D_t^{\alpha} x(t)\right]^T \delta\lambda(t) + \left[\frac{\partial \mathcal{H}}{\partial u}\right]^T \delta u(t) \Bigg\} \Delta t. \tag{14}$$

The necessary condition for an extremal asserts that the first variation of $J_a[x, \lambda, u]$ must vanish along the extremal for all independent variations $\delta x(t)$, $\delta\lambda(t)$, and $\delta u(t)$. Because of this, all factors multiplying a variation in Eq. (14) must vanish. We obtain conditions (8)–(9). □

Equations (8)–(9) represent the Euler–Lagrange equations of the FOCP (5)–(7). Note that Theorem 5 covers fractional optimal control problems defined on isolated time scales with a non-constant graininess, as well as variational problems on time scales that are partially continuous and partially discrete, i.e., on hybrid time scales.

3.3 An Illustrative Example

Let $\mathbb{T}$ be a time scale with $0, T \in \mathbb{T}$. Consider the control system

$${}_{0}^{\mathbb{T}}D_t^{\alpha} x(t) = u(t), \quad t \in [0, T]_{\mathbb{T}}, \tag{15}$$

subject to the initial condition

$${}_{0}^{\mathbb{T}}I_t^{1-\alpha} x(0) = x_0, \tag{16}$$

where the control u belongs to L^2. Consider the problem of minimizing

$$J[x,u] = \frac{1}{2}\left(||x - z||_{L^2}^2 + N||u||_{L^2}^2\right)$$

subject to (15)–(16), where $z \in L^2$ and $N > 0$ are fixed/given. In agreement with Theorem 5, the optimal control u is characterized by (15)–(16) with the adjoint system

$${}_t^{\mathbb{T}}D_T^{\alpha}\lambda(t) = x(t) - z(t), \quad t \in [0, T]_{\mathbb{T}},$$

and with the optimality condition

$$u(t) = -\frac{\lambda(t)}{N}.$$

4 Conclusion

We studied optimal control problems for fractional initial values systems involving fractional-time derivatives on time scales. As a main result, a necessary optimality condition is proved. In the formulation of the optimal control problem, the control u takes values in $\mathbb{R}^m$. As future work, it would be interesting to consider the case where the control takes values on a closed subset of $\mathbb{R}^m$. This question is far from being trivial [28, 35] and needs further developments.

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Relationship Between Green's Functions for Even Order Linear Boundary Value Problems

Alberto Cabada and Lucía López-Somoza

Abstract In this paper we will show several properties of the Green's functions related to various boundary value problems of arbitrary even order. In particular, we will write the expression of the Green's functions related to the general differential operator of order $2n$ coupled to Neumann, Dirichlet and mixed boundary conditions, as a linear combination of the Green's functions corresponding to periodic conditions on a different interval. This will allow us to ensure the constant sign of various Green's functions and to deduce spectral results.

Keywords Green's functions · Comparison principles · Boundary value problems

1 Introduction

Boundary value problems have been widely studied. This is due to the fact that these problems arise in many areas to model from most of physical problems to biological or economical ones.

It is very well-known that the solutions of a given boundary value problem coincide with fixed points of related integral operators which have as kernel the associated Green's function in each case. So, the Green's functions play a very important role in the study of boundary value problems.

In particular, some of the main techniques applied to prove the existence of solutions of nonlinear boundary value problems are, among others, monotone iterative techniques (see [6, 8, 10]), lower and upper solutions method (see [1, 5]) or fixed

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points theorems (see [6, 9]). In all these cases, the constant sign of the associated Green's functions is usually fundamental to prove such results.

Traditionally, the most studied boundary value problems have been the periodic and the two-point ones. In this paper we will take advantage of such studies by finding some connections between Green's functions of various separated two point boundary conditions and Green's functions of periodic problem. The key idea is that the expression of the Green's function related to each two-points case can be obtained as a linear combination of the Green's function of periodic problems.

From these expressions relating the different Green's functions, we will be able to compare their constant sign. These results will allow us to obtain some comparison principles which assure that, under certain hypotheses, the solution of a boundary value problem under some suitable conditions is bigger in every point than the solution of the same equation under another type of boundary conditions.

We will also obtain a decomposition of the spectrum of some problems as a combination of the other ones and some relations of order between the first eigenvalues of the considered problems.

The paper is organized as follows: Sect. 2 includes some preliminary results and proves a symmetry property which will be satisfied by some Green's functions. In Sect. 3, the aforementioned decomposition of Green's functions is fully detailed. Section 4 proves some results relating the constant sign of Green's function and includes also some counterexamples, showing that some properties which hold for second order boundary value problems are not true for higher order ones. In Sect. 5, both the spectra and the first eigenvalues of the considered problems are related. Section 6 proves some point-by-point relations between different Green's functions and also between solutions of some linear problem under several boundary conditions.

It must be pointed out that the study developed in Sects. 3–6 has been done in [3] for the particular case of Hill's equation. This is generalized here for any $2n$-th order boundary value problem. However, some arguments which worked for Hill's equation (mainly the ones related with oscillation theory) do not hold for $n > 1$. This implies that some of the results proved in this paper will be weaker than the corresponding ones for Hill's equation (in particular, as we have mentioned, some counterexamples will be shown in Sect. 4).

2 Preliminary Results

Consider the $2n$-order general linear operator

$$L\,u(t) \equiv u^{(2n)}(t) + a_{2n-1}(t)\,u^{(2n-1)}(t) + \cdots + a_1(t)\,u'(t) + a_0(t)\,u(t), \quad t \in I, \tag{1}$$

with $I \equiv [0, T]$, $a_k : I \to \mathbb{R}$, $a_k \in L^{\alpha}(I)$, $\alpha \geq 1, k = 0, \ldots, 2n-1$.

We will work with the space

$$W^{2n,1}(I) = \left\{u \in C^{2n-1}(I) :\ u^{(2n-1)} \in \mathcal{AC}(I)\right\},$$

where $\mathcal{AC}(I)$ denotes the set of absolutely continuous functions on I. In particular, we will consider $X \subset W^{2n,1}(I)$ a Banach space such that the following definition is satisfied.

Definition 1 Given a Banach space X, operator L is said to be nonresonant in X if and only if the homogeneous equation

$$L\,u(t) = 0 \quad \text{a. e. } t \in I, \qquad u \in X,$$

has only the trivial solution.

It is very well known that if $\sigma \in L^1(I)$ and operator L is nonresonant in X, then the nonhomogeneous problem

$$L\,u(t) = \sigma(t) \quad \text{a. e. } t \in I, \quad u \in X,$$

has a unique solution given by

$$u(t) = \int_0^T G[T](t,s)\,\sigma(s)\,\mathrm{d}s, \qquad \forall\, t \in I,$$

where $G[T]$ denotes the Green's function related to operator L on X and it is uniquely determined. See [2] for details.

We will introduce now an auxiliary linear operator, whose coefficients will be defined from the ones of operator L as follows:

$$\begin{aligned}\widetilde{L}\,u(t) \equiv u^{(2n)}(t) + \hat{a}_{2n-1}(t)\,u^{(2n-1)}(t) + \tilde{a}_{2n-2}(t)\,u^{(2n-2)}(t) + \cdots \\ + \hat{a}_1(t)\,u'(t) + \tilde{a}_0(t)\,u(t), \quad t \in J \equiv [0, 2T],\end{aligned}$$

where $\tilde{a}_{2k}$ and $\hat{a}_{2k+1}$, $k = 0, \ldots, n-1$, are the even and odd extensions of a_{2k} and a_{2k+1} to J.

We obtain the following symmetric property for Green's functions related to operator $\widetilde{L}$.

Lemma 1 *Let $X \subset W^{2n,1}(J)$ be a Banach space such that operator $\widetilde{L}$ is nonresonant in X and let $G[2T]$ denote the corresponding Green's function. Moreover, suppose that if $v \in X$ and $w(t) := v(2T - t)$, $t \in J$, then $w \in X$. Then the following equality holds:*

$$G[2T](t,s) = G[2T](2T - t, 2T - s) \qquad \forall\,(t,s) \in J \times J. \tag{2}$$

Proof Let $\tilde{\sigma} \in L^1(J)$ be arbitrarily chosen and consider the problem

$$\widetilde{L}v(t) = \widetilde{\sigma}(t), \ \text{a.e. } t \in J, \ v \in X.$$

Since operator $\widetilde{L}$ is nonresonant in X, this problem has a unique solution v which is given by

$$v(t) = \int_0^{2T} G[2T](t,s)\,\tilde{\sigma}(s)\,\mathrm{d}s.$$

On the other hand, due to the fact that $\tilde{a}_{2k}(t) = \tilde{a}_{2k}(2T-t)$ and $\hat{a}_{2k+1}(t) = -\hat{a}_{2k+1}(2T-t)$, it is easy to verify that $w(t) = v(2T-t)$ is the unique solution of the problem

$$\widetilde{L}w(t) = \widetilde{\sigma}(2T-t), \text{ a.e. } t \in J, \ w \in X.$$

Therefore,

$$w(t) = \int_0^{2T} G[2T](t,s)\,\tilde{\sigma}(2T-s)\,\mathrm{d}s = \int_0^{2T} G[2T](t,2T-s)\,\tilde{\sigma}(s)\,\mathrm{d}s.$$

Now, since

$$w(t) = v(2T-t) = \int_0^{2T} G[2T](2T-t,s)\,\tilde{\sigma}(s)\,\mathrm{d}s,$$

and $\tilde{\sigma} \in L^1(J)$ is arbitrary, we arrive at the following equality

$$G[2T](2T-t,s) = G[2T](t,2T-s) \qquad \forall\,(t,s) \in J \times J$$

or, which is the same,

$$G[2T](t,s) = G[2T](2T-t,2T-s) \qquad \forall\,(t,s) \in J \times J.$$

In addition, we will consider another auxiliary operator $\widetilde{\widetilde{L}}$ which will be constructed from $\widetilde{L}$ in the same way than $\widetilde{L}$ has been constructed from L, that is:

$$\widetilde{\widetilde{L}}\,u(t) \equiv u^{(2n)}(t) + \hat{\hat{a}}_{2n-1}(t)\,u^{(2n-1)}(t) + \tilde{\tilde{a}}_{2n-2}(t)\,u^{(2n-2)}(t) + \cdots + \hat{\hat{a}}_1(t)\,u'(t) + \tilde{\tilde{a}}_0(t)\,u(t),$$

$t \in [0,4T]$, where $\tilde{\tilde{a}}_{2k}$ and $\hat{\hat{a}}_{2k+1}$, $k = 0,\ldots,n-1$, are the even and odd extensions to the interval $[0,4T]$ of $\tilde{a}_{2k}$ and $\hat{a}_{2k+1}$, respectively.

To finish with this preliminary section, we will show two particular cases of some more general spectral results given in [2, Lemmas 1.8.25 and 1.8.33]. For these results we need to introduce a new differential operator.

For any $\lambda \in \mathbb{R}$, consider operator $L[\lambda]$ defined from operator L given by

$$L[\lambda]\,u(t) \equiv u^{(2n)}(t) + a_{2n-1}(t)\,u^{(2n-1)}(t) + \cdots + a_1(t)\,u'(t) + (a_0(t)+\lambda)\,u(t), \quad t \in I.$$

In particular, note that $L \equiv L[0]$. When working with this operator, to stress the dependence of the Green's function on the parameter λ, we will denote by $G[\lambda, T]$

the Green's function related to $L[\lambda]$. Again, note that $G[T] \equiv G[0, T]$. Analogous notation can we used for $\widetilde{L}[\lambda]$ and $\widetilde{\widetilde{L}}[\lambda]$, whose related Green's functions will be denoted by $G[\lambda, 2\,T]$ and $G[\lambda, 4\,T]$, respectively.

Lemma 2 *Suppose that operator L is nonresonant in a Banach space X, its related Green's function $G[T]$ is nonpositive on $I \times I$, and satisfies condition*

(N_g) *There is a continuous function $\phi(t) > 0$ for all $t \in (0, T)$ and $k_1, k_2 \in L^1(I)$, such that $k_1(s) < k_2(s) < 0$ for a. e. $s \in I$, satisfying*

$$\phi(t)\,k_1(s) \leq G[T](t, s) \leq \phi(t)\,k_2(s), \quad \textit{for a. e. } (t, s) \in I \times I.$$

Then $G[\lambda, T]$ is nonpositive on $I \times I$ if and only if $\lambda \in (-\infty, \lambda_1(T))$ or $\lambda \in [-\bar{\mu}(T), \lambda_1(T))$, with $\lambda_1(T) > 0$ the first eigenvalue of operator L in X and $\bar{\mu}(T) \geq 0$ such that $L[-\bar{\mu}(T)]$ is nonresonant in X and the related nonpositive Green's function $G[-\bar{\mu}(T), T]$ vanishes at some point of the square $I \times I$.

Lemma 3 *Suppose that operator L is nonresonant in a Banach space X, its related Green's function $G[T]$ is nonnegative on $I \times I$, and satisfies condition*

(P_g) *There is a continuous function $\phi(t) > 0$ for all $t \in (0, T)$ and $k_1, k_2 \in L^1(I)$, such that $0 < k_1(s) < k_2(s)$ for a. e. $s \in I$, satisfying*

$$\phi(t)\,k_1(s) \leq G[T](t, s) \leq \phi(t)\,k_2(s), \quad \textit{for a. e. } (t, s) \in I \times I.$$

Then $G[\lambda, T]$ is nonnegative on $I \times I$ if and only if $\lambda \in (\lambda_1(T), \infty)$ or $\lambda \in (\lambda_1(T), \bar{\mu}(T)]$, with $\lambda_1(T) < 0$ the first eigenvalue of operator L in X and $\bar{\mu}(T) \geq 0$ such that $L[\bar{\mu}(T)]$ is nonresonant in X and the related nonnegative Green's function $G[\bar{\mu}(T), T]$ vanishes at some point of the square $I \times I$.

3 Decomposing Green's Functions

In this section we will obtain the expression of the Green's function of different two-point boundary value problems (Neumann, Dirichlet and Mixed problems) as a sum of Green's functions of other related problems.

A similar decomposition has been made in [3] for the particular case of $n = 1$ and $a_1 \equiv 0$, which will be generalised here. There, the authors worked with Hill's operator, namely

$$\mathcal{L}\,u(t) \equiv u''(t) + a(t)\,u(t),$$

with $a \in L^\alpha(I)$, $\alpha \geq 1$.

In this case, we will deal with some problems related to operators L and $\widetilde{L}$ and the periodic problem related to $\widetilde{\widetilde{L}}$, which we describe in the sequel:

- Neumann problem on the interval $[0, T]$:

$$\begin{cases} L\,u(t) = \sigma(t), \quad \text{a. e. } t \in I, \\ u^{(2k+1)}(0) = u^{(2k+1)}(T) = 0, \; k = 0, \dots, n-1. \end{cases} \tag{N, T}$$

- Dirichlet problem on the interval $[0, T]$:

$$\begin{cases} L\,u(t) = \sigma(t), \quad \text{a. e. } t \in I, \\ u^{(2k)}(0) = u^{(2k)}(T) = 0, \; k = 0, \dots, n-1. \end{cases} \tag{D, T}$$

- Mixed problem 1 on the interval $[0, T]$:

$$\begin{cases} L\,u(t) = \sigma(t), \quad \text{a. e. } t \in I, \\ u^{(2k+1)}(0) = u^{(2k)}(T) = 0, \; k = 0, \dots, n-1. \end{cases} \tag{M_1, T}$$

- Mixed problem 2 on the interval $[0, T]$:

$$\begin{cases} L\,u(t) = \sigma(t), \quad \text{a. e. } t \in I, \\ u^{(2k)}(0) = u^{(2k+1)}(T) = 0, \; k = 0, \dots, n-1. \end{cases} \tag{M_2, T}$$

- Periodic problem on the interval $[0, 2\,T]$:

$$\begin{cases} \widetilde{L}\,u(t) = \tilde{\sigma}(t), \quad \text{a. e. } t \in J, \\ u^{(k)}(0) = u^{(k)}(2\,T), \; k = 0, \dots, 2n-1. \end{cases} \tag{$P, 2T$}$$

- Antiperiodic problem on the interval $[0, 2\,T]$:

$$\begin{cases} \widetilde{L}\,u(t) = \tilde{\sigma}(t), \quad \text{a. e. } t \in J, \\ u^{(k)}(0) = -u^{(k)}(2\,T), \; k = 0, \dots, 2n-1. \end{cases} \tag{$A, 2T$}$$

- Neumann problem on the interval $[0, 2\,T]$:

$$\begin{cases} \widetilde{L}\,u(t) = \tilde{\sigma}(t), \quad \text{a. e. } t \in J, \\ u^{(2\,k+1)}(0) = u^{(2\,k+1)}(2\,T) = 0, \; k = 0, \dots, n-1. \end{cases} \tag{$N, 2T$}$$

- Dirichlet problem on the interval $[0, 2\,T]$:

$$\begin{cases} \widetilde{L}\,u(t) = \tilde{\sigma}(t), \quad \text{a. e. } t \in J, \\ u^{(2\,k)}(0) = u^{(2\,k)}(2\,T) = 0, \; k = 0, \dots, n-1. \end{cases} \tag{$D, 2T$}$$

- Periodic problem on the interval $[0, 4\,T]$:

$$\begin{cases} \widetilde{\widetilde{L}}\,u(t) = \tilde{\tilde{\sigma}}(t), \quad \text{a. e. } t \in [0, 4\,T], \\ u^{(k)}(0) = u^{(k)}(4\,T), \; k = 0, \dots, 2n-1. \end{cases} \tag{$P, 4T$}$$

Now, we will show how to relate the expressions of different Green's functions. The following argument is analogous to the one made in [3] for the case $n = 1$.

3.1 Neumann Problem

To begin with, we will decompose the Green's function related to problem (N, T) as sum of the Green's function related to $(P, 2T)$ evaluated in the same point and of the same function evaluated in another point which satisfies a symmetric relation.

First, suppose that operator L is nonresonant in the space

$$X_{N,T} = \left\{u \in W^{2n,1}(I) : \; u^{(2k+1)}(0) = u^{(2k+1)}(T) = 0, \; k = 0, \ldots, n-1\right\}.$$

Moreover, assume that $\widetilde{L}$ is nonresonant in

$$X_{P,2T} = \left\{u \in W^{2n,1}(J) : \; u^{(k)}(0) = u^{(k)}(2T), \; k = 0, \ldots, 2n-1\right\}.$$

Then, if we denote by $G_N[T]$ and $G_P[2T]$ the Green's functions related to problems (N, T) and $(P, 2T)$, respectively, reasoning as in [3], we can deduce that

$$G_N[T](t, s) = G_P[2T](t, s) + G_P[2T](2T - t, s) \qquad \forall\, (t, s) \in I \times I. \tag{3}$$

Previous expression lets us obtain the exact value at every point of the Green's function for the Neumann problem by means of the values of the periodic one, as long as both Green's functions exist.

Analogously, assuming that $\widetilde{L}$ is nonresonant in

$$X_{N,2T} = \left\{u \in W^{2n,1}(J) : \; u^{(2k+1)}(0) = u^{(2k+1)}(2T) = 0, \; k = 0, \ldots, n-1\right\},$$

and denoting by $G_N[2T]$ the Green's function related to $(N, 2T)$, we deduce that

$$G_N[T](t, s) = G_N[2T](t, s) + G_N[2T](2T - t, s) \qquad \forall\, (t, s) \in I \times I, \tag{4}$$

or, using (3),

$$\begin{aligned} G_N[T](t, s) = {} & G_P[4T](t, s) + G_P[4T](4T - t, s) + G_P[4T](2T - t, s) \\ & + G_P[4T](2T + t, s). \end{aligned} \tag{5}$$

3.2 Dirichlet Problem

Now, we will do an analogous decomposition for the Green's function related to problem (D, T).

To this end, we will assume that operator L is nonresonant in

$$X_{D,T} = \left\{u \in W^{2n,1}(I) : \; u^{(2k)}(0) = u^{(2k)}(T) = 0, \; k = 0, \ldots, n-1\right\}.$$

Again, we will also assume that $\widetilde{L}$ is nonresonant in $X_{P,2T}$.

Now, denoting by $G_D[T]$ the Green's function related to (D, T), we obtain:

$$G_D[T](t,s) = G_P[2\,T](t,s) - G_P[2\,T](2\,T - t, s) \qquad \forall\,(t,s) \in I \times I. \tag{6}$$

On the other hand, assuming that $\widetilde{L}$ is nonresonant in

$$X_{D,2T} = \left\{u \in W^{2n,1}(J) : \; u^{(2k)}(0) = u^{(2k)}(2T) = 0, \; k = 0, \ldots, n-1\right\},$$

and denoting by $G_D[2\,T]$ the Green's function related to $(D, 2\,T)$, we obtain that

$$G_D[T](t,s) = G_D[2\,T](t,s) - G_D[2\,T](2\,T - t, s) \qquad \forall\,(t,s) \in I \times I \tag{7}$$

or, using (6),

$$\begin{aligned} G_D[T](t,s) = {} & G_P[4\,T](t,s) - G_P[4\,T](4\,T - t, s) - G_P[4\,T](2\,T - t, s) \\ & + G_P[4\,T](2\,T + t, s). \end{aligned} \tag{8}$$

3.3 Mixed Problems

The same arguments of two previous sections are applicable to problems (M_1, T) and (M_2, T), by assuming the nonresonant character of operator L in

$$X_{M_1,T} = \left\{u \in W^{2n,1}(I) : \; u^{(2k+1)}(0) = u^{(2k)}(T) = 0, \; k = 0, \ldots, n-1\right\}$$

or

$$X_{M_2,T} = \left\{u \in W^{2n,1}(I) : \; u^{(2k)}(0) = u^{(2k+1)}(T) = 0, \; k = 0, \ldots, n-1\right\},$$

respectively. However, these problems will not be related to periodic ones but to $(A, 2\,T)$. Therefore, we will assume for both cases that operator $\widetilde{L}$ is nonresonant in

$$X_{A,2T} = \left\{u \in W^{2n,1}(J) : \; u^{(k)}(0) = -u^{(k)}(2T), \; k = 0, \ldots, 2\,n-1\right\}.$$

Then we arrive at the following decompositions:

$$G_{M_1}[T](t,s) = G_A[2\,T](t,s) - G_A[2\,T](2\,T-t,s) \qquad \forall\,(t,s)\in I\times I, \tag{9}$$

$$G_{M_2}[T](t,s) = G_A[2\,T](t,s) + G_A[2\,T](2\,T-t,s) \qquad \forall\,(t,s)\in I\times I. \tag{10}$$

Again, $G_{M_1}[T]$ and $G_{M_2}[T]$ can also be related to $G_N[2T]$ and $G_D[2T]$, respectively:

$$G_{M_1}[T](t,s) = G_N[2\,T](t,s) - G_N[2\,T](2\,T-t,s) \qquad \forall\,(t,s)\in I\times I, \tag{11}$$

$$G_{M_2}[T](t,s) = G_D[2\,T](t,s) + G_D[2\,T](2\,T-t,s) \qquad \forall\,(t,s)\in I\times I, \tag{12}$$

or, using (3) and (6),

$$\begin{aligned} G_{M_1}[T](t,s) = {} & G_P[4\,T](t,s) + G_P[4\,T](4\,T-t,s) - G_P[4\,T](2\,T-t,s) \\ & - G_P[4\,T](2\,T+t,s), \end{aligned} \tag{13}$$

$$\begin{aligned} G_{M_2}[T](t,s) = {} & G_P[4\,T](t,s) - G_P[4\,T](4\,T-t,s) + G_P[4\,T](2\,T-t,s) \\ & - G_P[4\,T](2\,T+t,s). \end{aligned} \tag{14}$$

On the other hand, it is also possible to obtain a direct relation between the Green's functions of the two mixed problems.

Consider the following operator defined from L by taking the reflection of the coefficients

$$\check{L}\,u(t) = u^{(2n)}(t) + \sum_{k=0}^{2n-1} (-1)^k\, a_k(T-t)\, u^{(k)}(t),$$

for all $t\in I$, and let $\check{G}_{M_2}[T]$ be the Green's function related to the Mixed problem 2 associated with $\check{L}$, namely,

$$\begin{cases} \check{L}\,u(t) = \check{\sigma}(t), & t\in I, \\ u^{(2k)}(0) = u^{(2k+1)}(T) = 0, & k=0,\dots,n-1. \end{cases} \tag{15}$$

Now, let u be the unique solution of problem (M_1,T). If we define $v(t) = u(T-t)$, it is easy to check that v is a solution of problem (15) for the particular case of taking $\check{\sigma}(t) = \sigma(T-t)$. Therefore,

$$v(t) = \int_0^T \check{G}_{M_2}[T](t,s)\,\sigma(T-s)\;\mathrm{d}s = \int_0^T \check{G}_{M_2}[T](t,T-s)\,\sigma(s)\;\mathrm{d}s.$$

On the other hand,

$$v(t) = u(T-t) = \int_0^T G_{M_1}[T](T-t,s)\,\sigma(s)\;\mathrm{d}s.$$

Since previous equalities are valid for all $\sigma \in L^1(I)$, we deduce that

$$G_{M_1}[T](T-t,s) = \check{G}_{M_2}[T](t,T-s)$$

or, which is the same,

$$G_{M_1}[T](T-t,T-s) = \check{G}_{M_2}[T](t,s). \tag{16}$$

Analogously, if we denote by $\check{G}_{M_1}[T]$ the Green's function related to the Mixed problem 1 associated with $\check{L}$, namely,

$$\begin{cases} \check{L}\,u(t) = \check{\sigma}(t), \quad t \in I, \\ u^{(2k+1)}(0) = u^{(2k)}(T) = 0, \quad k = 0,\dots,n-1, \end{cases} \tag{17}$$

repeating the previous reasoning, we reach to the following connecting expression

$$G_{M_2}[T](T-t,T-s) = \check{G}_{M_1}[T](t,s). \tag{18}$$

3.4 Connecting Relations Between Different Problems

On the other hand, assuming again the nonresonant character of all the considered operators in the corresponding spaces, if we sum different combinations of the previous equalities, we obtain more connecting expressions between the considered Green's functions. The results are the following:

- From (3) to (6), we deduce:

$$\begin{aligned} G_P[2T](t,s) &= 1/2\,(G_N[T](t,s) + G_D[T](t,s)) \qquad \forall\,(t,s) \in I \times I, \\ G_P[2T](2T-t,s) &= 1/2\,(G_N[T](t,s) - G_D[T](t,s)) \qquad \forall\,(t,s) \in I \times I. \end{aligned} \tag{19}$$

- From (9) to (10):

$$\begin{aligned} G_A[2T](t,s) &= 1/2\,\big(G_{M_2}[T](t,s) + G_{M_1}[T](t,s)\big) \qquad \forall\,(t,s) \in I \times I, \\ G_A[2T](2T-t,s) &= 1/2\,\big(G_{M_2}[T](t,s) - G_{M_1}[T](t,s)\big) \qquad \forall\,(t,s) \in I \times I. \end{aligned} \tag{20}$$

- From (4) to (11):

$$\begin{aligned} G_N[2T](t,s) &= 1/2\,\big(G_N[T](t,s) + G_{M_1}[T](t,s)\big) \qquad \forall\,(t,s) \in I \times I, \\ G_N[2T](2T-t,s) &= 1/2\,\big(G_N[T](t,s) - G_{M_1}[T](t,s)\big) \qquad \forall\,(t,s) \in I \times I. \end{aligned} \tag{21}$$

- From (7) to (12):

$$G_D[2T](t,s) = 1/2\left(G_{M_2}[T](t,s) + G_D[T](t,s)\right) \qquad \forall\,(t,s) \in I \times I,$$
$$G_D[2T](2T-t,s) = 1/2\left(G_{M_2}[T](t,s) - G_D[T](t,s)\right) \qquad \forall\,(t,s) \in I \times I. \tag{22}$$

- From (5), (8), (13) to (14):

$$G_P[4T](t,s) = 1/4\left(G_N[T](t,s) + G_D[T](t,s) + G_{M_1}[T](t,s) + G_{M_2}[T](t,s)\right).$$

4 Decomposition of the Spectra

In this section we will show first how the spectra of the considered problems can be connected. The results here generalise those proved in [3].

We will denote by $\Lambda_N[T]$, $\Lambda_D[T]$, $\Lambda_{M_1}[T]$, $\Lambda_{M_2}[T]$, $\Lambda_P[2T]$, $\Lambda_A[2T]$, $\Lambda_N[2T]$, $\Lambda_D[2T]$ and $\Lambda_P[4T]$ the set of eigenvalues of problems (N,T), (D,T), (M_1,T), (M_2,T), $(P,2T)$, $(A,2T)$, $(N,2T)$, $(D,2T)$ and $(P,4T)$, respectively. Then, arguing as in [3], we obtain the following equalities:

$$\Lambda_N[T] \cup \Lambda_D[T] = \Lambda_P[2T],$$

$$\Lambda_N[T] \cup \Lambda_{M_1}[T] = \Lambda_N[2T],$$

$$\Lambda_D[T] \cup \Lambda_{M_2}[T] = \Lambda_D[2T],$$

$$\Lambda_{M_1}[T] \cup \Lambda_{M_2}[T] = \Lambda_A[2T],$$

$$\Lambda_N[T] \cup \Lambda_D[T] \cup \Lambda_{M_1}[T] \cup \Lambda_{M_2}[T] = \Lambda_P[4T].$$

Finally, if we denote by $\check{\Lambda}_{M_2}[T]$ and $\check{\Lambda}_{M_1}[T]$ the set of eigenvalues of problems (15) and (17), respectively, from (16) to (18) we deduce that

$$\Lambda_{M_1}[T] = \check{\Lambda}_{M_2}[T] \quad \text{and} \quad \Lambda_{M_2}[T] = \check{\Lambda}_{M_1}[T].$$

As an immediate consequence we have the following result.

Corollary 1 *If $a_k(t) = (-1)^k a_k(T-t)$ for all $k = 0, \ldots, 2n-1$, then the spectra of the two mixed problems coincides, that is,*

$$\Lambda_{M_1}[T] = \Lambda_{M_2}[T].$$

Moreover, if we denote by $\lambda_0^N[T]$, $\lambda_0^D[T]$, $\lambda_0^{M_1}[T]$, $\lambda_0^{M_2}[T]$, $\lambda_0^P[2T]$, $\lambda_0^A[2T]$, $\lambda_0^N[2T]$, $\lambda_0^D[2T]$ and $\lambda_0^P[4T]$ the first eigenvalue of problems (N,T), (D,T),

(M_1, T), (M_2, T), $(P, 2T)$, $(A, 2T)$, $(N, 2T)$, $(D, 2T)$ and $(P, 4T)$, respectively, from the connecting expressions proved in Sect. 3, we deduce the relations below.

Theorem 1 *Assume that all the previously considered spectra are not empty, the first eigenvalue of each problem is simple and its related eigenfunction has constant sign. Then, the following equalities are fulfilled for any* $a_0, \ldots, a_{2n-1} \in L^1(I)$*:*

1. $\lambda_0^N[T] = \lambda_0^P[2T] < \lambda_0^D[T]$.
2. $\lambda_0^N[T] = \lambda_0^N[2T] < \lambda_0^{M_1}[T]$.
3. $\lambda_0^N[T] = \lambda_0^P[4T]$.
4. $\lambda_0^{M_2}[T] = \lambda_0^D[2T] < \lambda_0^D[T]$.
5. $\lambda_0^N[T] < \lambda_0^{M_2}[T]$.
6. $\lambda_0^A[2T] = \min\left\{\lambda_0^{M_1}[T],\ \lambda_0^{M_2}[T]\right\}$.

Proof Assertion 1 is proved in the following way: as we have seen above, the spectrum of $(P, 2T)$ is decomposed as $\Lambda_P[2T] = \Lambda_N[T] \cup \Lambda_D[T]$, which implies that

$$\lambda_0^P[2T] = \min\left\{\lambda_0^N[T],\ \lambda_0^D[T]\right\}.$$

Consider now the even extension to J of the eigenfunction associated to $\lambda_0^N[T]$. This extension has constant sign on J and, moreover, it satisfies periodic boundary conditions, so it is a constant sign eigenfunction of $(P, 2T)$. On the contrary, the odd extension to J of the eigenfunction associated to $\lambda_0^D[T]$ is a sign changing eigenfunction of $(P, 2T)$. Thus, since the eigenfunction related to the first eigenvalue of each problem has constant sign, we deduce that $\lambda_0^N[T] = \lambda_0^P[2T] < \lambda_0^D[T]$.

An analogous argument is valid to prove Assertion 2, by taking into account that $\Lambda_N[2T] = \Lambda_N[T] \cup \Lambda_{M_1}[T]$.

Assertion 3 is deduced from the two previous one. Indeed, Assertion 1 implies that $\lambda_0^N[2T] = \lambda_0^P[4T]$ and, from Assertion 2, we deduce the equality.

Assertion 4 is proved analogously to Assertions 1 and 2, taking into account the decomposition $\Lambda_D[2T] = \Lambda_D[T] \cup \Lambda_{M_2}[T]$.

Now Assertion 5 can be deduced from 1, 2 to 4. Indeed, Assertion 1 implies that $\lambda_0^N[2T] < \lambda_0^D[2T]$ and, using Assertions 2 and 4, $\lambda_0^N[T] = \lambda_0^N[2T] < \lambda_0^D[2T] = \lambda_0^{M_2}[T]$.

Finally, Assertion 6 is immediate from $\Lambda_A[2T] = \Lambda_{M_1}[T] \cup \Lambda_{M_2}[T]$. □

Remark 1 With respect to the hypothesis that all the considered spectra are not empty note that, as a consequence of the relations proved at the beginning of this section, if one of those spectra is not empty, we could ensure that some others are not empty too.

On the other hand, there are several results which ensure that, under some suitable conditions, the first eigenvalue of a boundary value problem is simple and its related eigenfunction has constant sign, for instance, Krein–Rutman Theorem.

Sufficient conditions to ensure that all the hypotheses required in previous theorem are fulfilled can be found in [7]. First, we can deduce from Theorem 1 in such reference that if there exists some λ for which the Green's function $G[\lambda, T]$ has constant sign and the spectrum of such problem is not empty, then the eigenfunction related to the first eigenvalue has constant sign.

Moreover, from Theorem 2 in [7] it is deduced that if there exists some λ for which the Green's function $G[\lambda, T]$ has strict constant sign on $[0, T] \times (0, T)$ then the spectrum of such problem is not empty, the first eigenvalue is simple and its related eigenfunction has strict constant sign.

Finally, from Theorem 2' in [7] we can ensure that if there exists some λ for which $G[\lambda, T]$ has strict constant sign on $(0, T) \times (0, T)$ and there exists a continuous function ϕ, positive on $(0, T)$, such that

$$\frac{G[\lambda, T](t, s)}{\phi(t)}$$

is continuous on $[0, T] \times [0, T]$ and positive on $[0, T] \times (0, T)$, then the spectrum of such problem is not empty, the first eigenvalue is simple and its related eigenfunction has strict constant sign.

Analogously, if conditions given in Lemmas 2 or 3 hold for some λ, then we are also able to deduce that the spectrum of such problem is not empty, the first eigenvalue is simple and its related eigenfunction has constant sign. Details of this can be seen in [2], where it is proved that Lemmas 2 or 3 imply that Krein–Rutman's Theorem holds.

Finally, we must note that, since the eigenfunctions of the considered problems are related, the constant sign of the eigenfunction associated with the first eigenvalue of a problem implies (in some cases) the constant sign of the eigenfunction of other problems.

5 Constant Sign of Green's Functions

In [3] and [4, Sect. 3.4], for $n = 1$, some results relating the constant sign of various Green's functions have been proved. The result is the following:

Theorem 2 ([3, Corollary 4.8]) *For $n = 1$ and $a_1 \equiv 0$, the following properties hold:*

1. $G_P[2T] < 0$ *on* $J \times J$ *if and only if* $G_N[T] < 0$ *on* $I \times I$. *This is equivalent to* $G_N[2T] < 0$ *on* $J \times J$.
2. $G_P[2T] > 0$ *on* $(0, 2T) \times (0, 2T)$ *if and only if* $G_N[T] > 0$ *on* $(0, T) \times (0, T)$.
3. *If* $G_N[2T] > 0$ *on* $(0, 2T) \times (0, 2T)$ *then* $G_N[T] > 0$ *on* $(0, T) \times (0, T)$.
4. *If* $G_P[2T] < 0$ *on* $J \times J$ *then* $G_D[2T] < 0$ *on* $(0, 2T) \times (0, 2T)$.
5. *If* $G_P[2T] > 0$ *on* $(0, 2T) \times (0, 2T)$ *then* $G_D[2T] < 0$ *on* $(0, 2T) \times (0, 2T)$.
6. *If* $G_N[T]$ *(or* $G_P[2T]$*) has constant sign on* $I \times I$, *then* $G_D[T] < 0$ *on* $(0, T) \times (0, T)$, $G_{M_1}[T] < 0$ *on* $[0, T) \times [0, T)$ *and* $G_{M_2}[T] < 0$ *on* $(0, T] \times (0, T]$.

7. $G_D[2\,T] < 0$ *on* $(0, 2\,T) \times (0, 2\,T)$ *if and only if* $G_{M_2}[T] < 0$ *on* $(0, T] \times (0, T]$.
8. *If* $G_{M_2}[T] < 0$ *on* $(0, T] \times (0, T]$ *or* $G_{M_1}[T] < 0$ *on* $[0, T) \times [0, T)$ *then* $G_D[T] < 0$ *on* $(0, T) \times (0, T)$.

Some of the previous results can be extended to the more general case considered in this paper, although some of them are no longer true. We will show now these results and some counterexamples for the cases which do not hold anymore.

From (3), (4) to (12) we deduce that

Corollary 2 *The following properties are fulfilled for any* $a_0, \ldots, a_{2n-1} \in L^1(I)$*:*

1. *If* $G_P[2T] \le 0$ *on* $J \times J$*, then* $G_N[T] \le 0$ *on* $I \times I$.
2. *If* $G_P[2T] \ge 0$ *on* $J \times J$*, then* $G_N[T] \ge 0$ *on* $I \times I$.
3. *If* $G_N[2T] \le 0$ *on* $J \times J$*, then* $G_N[T] \le 0$ *on* $I \times I$.
4. *If* $G_N[2T] \ge 0$ *on* $J \times J$*, then* $G_N[T] \ge 0$ *on* $I \times I$.
5. *If* $G_D[2T] \le 0$ *on* $J \times J$*, then* $G_{M_2}[T] \le 0$ *on* $I \times I$.
6. *If* $G_D[2T] \ge 0$ *on* $J \times J$*, then* $G_{M_2}[T] \ge 0$ *on* $I \times I$.

The reciprocal of Assertions 1 and 2 in Corollary 2 holds for constant coefficients. This occurs as a consequence of the following property:

Lemma 4 ([2, Sect. 1.4]) *Let* $L_n\, u(t) \equiv u^{(n)}(t) + a_{n-1}(t)\, u_{n-1}(t) + \cdots + a_1(t)\, u'(t) + a_0(t)\, u(t)$*,* $t \in I$ *be a* n*-th order linear operator and let* $G_P[T]$ *denote the Green's function related to the periodic problem*

$$\begin{cases} L_n\, u(t) = 0, & t \in I, \\ u^{(k)}(0) = u^{(k)}(T), & k = 0, \ldots, n-1. \end{cases}$$

If the coefficients a_k*,* $k = 0, \ldots, n-1$*, involved in the definition of operator* L_n *are constant, then the Green's function is constant over the straight lines of slope one, that is, it satisfies the following property*

$$G_P[T](t,s) = \begin{cases} G_P[T](t-s, 0), & 0 \le s \le t \le T, \\ G_P[T](T+t-s, 0), & \textit{otherwise.} \end{cases}$$

As a consequence, we arrive at the following result.

Theorem 3 *If all the coefficients* $a_0, \ldots, a_{2n-1}$ *of operator* L *defined in* (1) *are constant, then the following properties hold:*

1. $G_P[2T] \le 0$ *on* $J \times J$ *if and only if* $G_N[T] \le 0$ *on* $I \times I$.
2. $G_P[2T] \ge 0$ *on* $J \times J$ *if and only if* $G_N[T] \ge 0$ *on* $I \times I$.

Proof From Corollary 2, the Assertion is equivalent to prove that if $G_P[2\,T]$ changes sign, then $G_N[T]$ will also change sign. Indeed, assume that there exist two pairs of values (t_1, s_1) and (t_2, s_2) such that $G_P[2\,T](t_1, s_1) < 0$ and $G_P[2\,T](t_2, s_2) > 0$. As $G_P[2\,T](t, s) = G_P[2\,T](s, t)$ for all $(t, s) \in J \times J$, we may assume, without loss of generality, that $s_1 \le t_1$ and $s_2 \le t_2$.

If all the coefficients $a_0, \dots, a_{2n-1}$ are constant then, from Lemma 4, it holds that

$$G_P[2T](t,s) = \begin{cases} G_P[2T](t-s,0), & 0 \le s \le t \le 2T, \\ G_P[2T](2T+t-s,0), & \text{otherwise.} \end{cases}$$

Therefore, it is fulfilled that

$$G_P[2T](t_1,s_1) = G_P[2T](t_1-s_1,0) \quad \text{and} \quad G_P[2T](t_2,s_2) = G_P[2T](t_2-s_2,0).$$

On the other hand, from equality (2) and the fact that the Green's function satisfies the periodic boundary conditions (see [2, Definition 1.4.1]), it holds that

$$G_P[2T](t_1-s_1,0) = G_P[2T](2T-t_1+s_1,2T) = G_P[2T](2T-t_1+s_1,0),$$

$$G_P[2T](t_2-s_2,0) = G_P[2T](2T-t_2+s_2,2T) = G_P[2T](2T-t_2+s_2,0).$$

Now, we will distinguish two possibilities:

- If $t_1 - s_1 \le T$, then

$$\begin{aligned} G_N[T](t_1-s_1,0) &= G_P[2T](t_1-s_1,0) + G_P[2T](2T-t_1+s_1,0) \\ &= 2\,G_P[2T](t_1-s_1,0) < 0. \end{aligned}$$

- When $t_1 - s_1 > T$, we have

$$\begin{aligned} G_N[T](2T-t_1+s_1,0) &= G_P[2T](2T-t_1+s_1,0) + G_P[2T](t_1-s_1,0) \\ &= 2\,G_P[2T](t_1-s_1,0) < 0. \end{aligned}$$

Analogously, if $t_2 - s_2 \le T$, then

$$G_N[T](t_2-s_2,0) = 2\,G_P[2T](t_2-s_2,0) > 0,$$

and, if $t_2 - s_2 > T$, then

$$G_N[T](2T-t_2+s_2,0) = 2\,G_P[2T](t_2-s2,0) > 0.$$

It is clear that, in any of the cases, $G_N[T]$ changes its sign and the result holds. □

The following counterexample shows that the converse of Assertion 2 in Corollary 2 is not true in general for nonconstant coefficients.

Example
Consider the Neumann problem on $[0,T] = [0,2]$ related to operator

$$L\,u(t) = u^{(4)}(t) + ((t-2)^4 + \lambda)\,u(t), \quad t \in [0,2], \tag{23}$$

and the periodic problem on $[0, 2\,T] = [0, 4]$ related to

$$\widetilde{L}\,u(t) \equiv u^{(4)}(t) + ((t-2)^4 + \lambda)\,u(t), \quad t \in [0, 4]\,. \tag{24}$$

By numerical approach, it can be seen that $G_N[T]$ is nonpositive for $\lambda \in \left(\lambda_1, \lambda_0^N[T]\right)$, where $\lambda_1 \approx -2.26$ and $\lambda_0^N[T] = \lambda_0^P[2\,T] \approx -1.746$. Moreover it is nonnegative for $\lambda \in \left(\lambda_0^N[T], \lambda_2\right)$, with $\lambda_2 \approx 4.11$. However, $G_P[2\,T]$ is nonpositive for $\lambda \in \left(\lambda_1, \lambda_0^P[2\,T]\right)$ and nonnegative for $\lambda \in \left(\lambda_0^P[2\,T], \lambda_3\right)$, with $\lambda_3 \approx 5.95$.

Despite this, we remark that the interval of values of λ for which $G_N[T]$ and $G_P[2\,T]$ are nonpositive is exactly the same.

Remark 2 It must be pointed out that the converse of Assertion 2 in Corollary 2 also holds for several examples with non constant coefficients. However we have not been able to prove the existence of any general condition under which this Assertion holds.

Furthermore, up to this moment, we have not been able to find a counterexample for the converse of Assertion 1. So, it remains as an open problem to know if Assertion 1 is or not an equivalence for $n \geq 2$.

The following counterexample shows that the converse of Assertions 3 and 4 in Corollary 2 does not hold in general, not even in the constant case:

Example
Consider the following Neumann problem with constant coefficients on $[0, T] = \left[0, \frac{3}{2}\right]$ related to the following operator

$$L\,u(t) \equiv u^{(4)}(t) + \lambda\,u(t), \quad t \in \left[0, \frac{3}{2}\right],$$

and the Neumann problem on $[0, 2\,T] = [0, 3]$ related to

$$\widetilde{L}\,u(t) \equiv u^{(4)}(t) + \lambda\,u(t), \quad t \in [0, 3]\,,$$

By numerical approach, it can be seen that $G_N[T]$ is nonpositive for $\lambda \in \left(\lambda_4, \lambda_0^N[T]\right)$, with $\lambda_4 \approx -6.1798$ and $\lambda_0^N[T] = \lambda_0^N[2\,T] = 0$, and nonnegative for $\lambda \in \left(\lambda_0^N[T], \lambda_5\right)$, with $\lambda_5 \approx 24.7192$. However, $G_N[2\,T]$ is nonpositive for $\lambda \in \left(\lambda_6, \lambda_0^N[2\,T]\right)$, with $\lambda_6 \approx -0.3862$, and nonnegative for $\lambda \in \left(\lambda_0^N[2\,T], \lambda_7\right)$, with $\lambda_7 \approx 1.5449$.

So, the converse of Assertions 3 and 4 does not hold for these operators.

The following counterexample shows that the converse of Assertions 5 and 6 in Corollary 2 is not true in general, not even in the constant case:

Example
Consider the Mixed problem 2 with constant coefficients on $[0, T] = [0, 1]$ related to operator

$$L\,u(t) \equiv u^{(4)}(t) + \lambda\,u(t), \quad t \in [0, 1]\,,$$

and the Dirichlet problem on $[0, 2\,T] = [0, 2]$ related to

$$\widetilde{L}\, u(t) \equiv u^{(4)}(t) + \lambda\, u(t), \quad t \in [0, 2]\,.$$

It can be seen that $G_{M_2}[T]$ is nonpositive for $\lambda \in \left(\lambda_8, \lambda_0^{M_2}[T]\right)$, with $\lambda_8 \approx -31.2852$ and $\lambda_0^{M_2}[T] = \lambda_0^{D}[2\,T] = -\frac{\pi^4}{16}$, and nonnegative for $\lambda \in \left(\lambda_0^{M_2}[T], \lambda_9\right)$, with $\lambda_9 \approx 389.6365$. However, $G_D[2\,T]$ is nonpositive for $\lambda \in \left(\lambda_{10}, \lambda_0^{D}[2\,T]\right)$, where $\lambda_{10} \approx -14.8576$, and nonnegative for $\lambda \in \left(\lambda_0^{D}[2\,T], \lambda_{11}\right)$, with $\lambda_{11} \approx 59.4303$.

We will see now some more counterexamples which show that Assertions 4, 5, 6 and 8 in Theorem 2 do not hold, in general, for $n > 1$.

Next example shows that Assertions 4 and 5 in Theorem 2 are not true in general.

Example

Consider the periodic and Dirichlet problems on the same interval $[0, 2\,T] = [0, 3]$ related to operator

$$\widetilde{L}\, u(t) \equiv u^{(4)}(t) + (t\,(t-3) + \lambda)\, u(t), \quad t \in [0, 3]. \tag{25}$$

By numerical approach, we have obtained that for $\lambda = -1.5$, $G_P[2\,T]$ is negative while $G_D[2\,T]$ changes its sign on $J \times J$.

Moreover, for $\lambda = 15$, $G_P[2\,T]$ is positive while $G_D[2\,T]$ changes sign again.

We will see in the two following examples that none of the implications given in Assertion 6 in Theorem 2 holds for $n > 1$.

Example

Consider now $[0, T] = [0, 2]$ and operators L and $\widetilde{L}$ given in (23) and (24).

For $\lambda = -2$, one can check that both $G_P[2\,T]$ and $G_N[T]$ are nonpositive, meanwhile $G_D[T]$ and $G_{M_1}[T]$ are nonnegative.

For $\lambda = 2$, it occurs that both $G_P[2\,T]$ and $G_N[T]$ are nonnegative, meanwhile $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ are nonnegative.

Example

Take now $[0, T] = \left[0, \frac{3}{2}\right]$, operator L given by

$$L\, u(t) \equiv u^{(4)}(t) + (t(t-3) + \lambda)\,, \quad t \in \left[0, \frac{3}{2}\right]$$

and operator $\widetilde{L}$ given in (25).

In this case, for $\lambda = 1.5$, it occurs that $G_P[2\,T]$ and $G_N[T]$ are nonpositive, meanwhile $G_{M_2}[T]$ is nonnegative.

Finally, we will show that Assertion 8 in Theorem 2 does not hold either when $n > 1$.

Example
Consider again $[0, T] = [0, 2]$ and operators L and $\widetilde{L}$ given in (23) and (24). In this case, for $\lambda = -6$, $G_{M_1}[T]$ is nonpositive but $G_D[T]$ is nonnegative. Similarly, for $\lambda = -2$, $G_{M_2}[T]$ is nonpositive but $G_D[T]$ is nonnegative.

Finally, from the relations given in Theorem 1, together with the general characterization given in Lemmas 2 and 3, it can be deduced the following corollary.

Corollary 3 *Assume that we are in conditions to apply Lemmas 2 and 3, that is, all the considered Green's functions $G[\lambda, T]$ (or $G[\lambda, 2T]$, $G[\lambda, 4T]$, with the suitable subscript for each case) are:*

- *nonpositive on $I \times I$ if and only if $\lambda \in (-\infty, \lambda_1)$ or $\lambda \in [-\bar{\mu}, \lambda_1)$, with $\lambda_1 > 0$ the first eigenvalue of operator L_n coupled with the corresponding boundary conditions and $\bar{\mu} \geq 0$ such that $L_n[-\bar{\mu}]$ is nonresonant on X and the related nonpositive Green's function $G[-\bar{\mu}]$ vanishes at some point of the square $I \times I$.*
- *nonnegative on $I \times I$ if and only if $\lambda \in (\lambda_1, \infty)$ or $\lambda \in (\lambda_1, \bar{\mu}]$, with $\lambda_1 < 0$ the first eigenvalue of operator L_n coupled with the corresponding boundary conditions and $\bar{\mu} \geq 0$ such that $L_n[\bar{\mu}]$ is nonresonant on X and the related nonnegative Green's function $G[\bar{\mu}]$ vanishes at some point of the square $I \times I$.*

Then the following relations between the constant sign of Green's functions are valid for any $a_0, \ldots, a_{2n-1} \in L^1(I)$:

- *If $G_N[T]$ is nonpositive on $I \times I$, then $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_N[2T]$ is nonpositive on $J \times J$, then $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_P[2T]$ is nonpositive on $J \times J$, then $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_P[4T]$ is nonpositive on $[0, 4T] \times [0, 4T]$, then $G_N[T]$, $G_D[T]$, $G_{M_1}[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*
- *If $G_{M_2}[T]$ is nonpositive on $I \times I$, then $G_D[T]$ either changes sign or is nonpositive on $I \times I$.*
- *If $G_D[2T]$ is nonpositive on $J \times J$, then $G_D[T]$ and $G_{M_2}[T]$ either change sign or are nonpositive on $I \times I$.*

6 Comparison Principles

In this section we will use the connecting expressions for Green's functions obtained in Sect. 3 to compare the values that several Green's functions take point by point. It must be pointed out that, since the relations between the constant sign of the Green's functions are not as strong as for the case $n = 1$, the results in this section will also be weaker (in some cases) than the ones obtained in [3].

However, some results which could not be deduced for $n = 1$ hold for $n > 1$. This will be the case of Item 4 in Corollary 5 or Item 1 in Theorem 6, which do not make sense for the case $n = 1$ because, in such a case, $G_D[2\,T]$ can never be nonnegative on $J \times J$.

First, from (19), we obtain the following result.

Corollary 4 *1. If $G_P[2\,T] \geq 0$ on $J \times J$, then $G_N[T](t,s) \geq |G_D[T](t,s)|$ for all $(t,s) \in I \times I$.*
2. If $G_P[2\,T] \leq 0$ on $J \times J$, then $G_N[T](t,s) \leq -|G_D[T](t,s)|$ for all $(t,s) \in I \times I$.

The difference between this case and the particular one with $n = 1$ and $a_1 \equiv 0$ is that when $n = 1$, the constant sign of $G_P[2\,T]$ ensures not only the constant sign of $G_N[T]$ but also that of $G_D[T]$. Thus, in such case, we would substitute $|G_D[T](t,s)|$ by $-G_D[T](t,s)$ in the inequalities given in previous corollary.

As a consequence of previous corollary, we can compare the solutions of (N,T) and (D,T):

Theorem 4 *Let u_N be the unique solution of problem (N,T) for $\sigma = \sigma_1$ and u_D the unique solution of problem (D,T) for $\sigma = \sigma_2$. Then*

1. *If $G_P[2\,T] \geq 0$ on $J \times J$ and $|\sigma_2(t)| \leq \sigma_1(t)$ a.e. $t \in I$, then $|u_D(t)| \leq u_N(t)$ for all $t \in I$.*
2. *If $G_P[2\,T] \leq 0$ on $J \times J$ and $0 \leq \sigma_2(t) \leq \sigma_1(t)$ a.e. $t \in I$, then $u_N(t) \leq 0$ and $u_N(t) \leq u_D(t)$ for all $t \in I$.*
3. *If $G_P[2\,T] \leq 0$ on $J \times J$ and $\sigma_1(t) \leq \sigma_2(t) \leq 0$ a.e. $t \in I$, then $u_N(t) \geq 0$ and $u_D(t) \leq u_N(t)$ for all $t \in I$.*

Proof 1. Since $G_P[2\,T] \geq 0$ on $J \times J$ then, from Corollary 4, it holds that

$$|u_D(t)| = \left| \int_0^T G_D[T](t,s)\,\sigma_2(s)\;\mathrm{d}s \right| \leq \int_0^T |G_D[T](t,s)|\,|\sigma_2(s)|\;\mathrm{d}s$$
$$\leq \int_0^T G_N[T](t,s)\,\sigma_1(s)\;\mathrm{d}s = u_N(t).$$

2. Since $G_P[2\,T] \leq 0$ on $J \times J$ then, from Corollary 4, since $\sigma_1(s) \geq 0$ a.e. $s \in I$,

$$G_N[T](t,s)\,\sigma_1(s) \leq -|G_D[T](t,s)|\,\sigma_1(s), \quad \forall\,(t,s) \in I \times I.$$

Moreover, from $\sigma_2(s) \leq \sigma_1(s)$ and $\sigma_2(s) \geq 0$ a.e. $s \in I$, we deduce that

$$-|G_D[T](t,s)|\,\sigma_1(s) \leq -|G_D[T](t,s)|\,\sigma_2(s) \leq G_D[T](t,s)\,\sigma_2(s), \quad \forall\,(t,s) \in I \times I.$$

Therefore, for all $t \in I$, we have

$$u_N(t) = \int_0^T G_N[T](t,s)\,\sigma_1(s)\,\mathrm{d}s \le \int_0^T -|G_D[T](t,s)|\,\sigma_1(s)\,\mathrm{d}s$$
$$\le \int_0^T -|G_D[T](t,s)|\,\sigma_2(s)\,\mathrm{d}s \le \int_0^T G_D[T](t,s)\,\sigma_2(s)\,\mathrm{d}s = u_D(t).$$

Finally, the fact that $u_N \le 0$ on I is deduced from $G_N[T] \le 0$ and $\sigma_1 \ge 0$.

3. Since $G_P[2\,T] \le 0$ on $J \times J$ then, from Corollary 4, it can be deduced that

$$G_N[T](t,s) \le G_D[T](t,s) \text{ and } G_N[T](t,s) \le 0, \quad \forall\,(t,s) \in I \times I$$

and so, since $\sigma_2(s) \le 0$ and $\sigma_1(s) \le \sigma_2(s)$ a.e. $s \in I$,

$$G_D[T](t,s)\,\sigma_2(s) \le G_N[T](t,s)\,\sigma_2(s) \le G_N[T](t,s)\,\sigma_1(s), \quad \forall\,(t,s) \in I \times I$$

Therefore,

$$u_D(t) = \int_0^T G_D[T](t,s)\,\sigma_2(s)\,\mathrm{d}s \le \int_0^T G_N[T](t,s)\,\sigma_1(s)\,\mathrm{d}s = u_N(t).$$

Finally, the fact that $u_N \ge 0$ on I is deduced from $G_N[T] \le 0$ and $\sigma_1 \le 0$.

□

So, the main difference with respect to the case $n = 1$ is that when $n = 1$ we are able to ensure the constant sign of function u_D, which does not happen in this case.

Analogously, from (21) and (22), the constant sign of $G_N[2\,T]$ and $G_D[2\,T]$ lets us deduce some point-by-point relation between various Green's functions.

Corollary 5 *1. If $G_N[2\,T] \ge 0$ on $J \times J$, then $G_N[T](t,s) \ge |G_{M_1}[T](t,s)|$ for all $(t,s) \in I \times I$.*
2. If $G_N[2\,T] \le 0$ on $J \times J$, then $G_N[T](t,s) < -|G_{M_1}[T](t,s)|$ for all $(t,s) \in I \times I$.
3. If $G_D[2\,T] \le 0$ on $J \times J$, then $G_{M_2}[T](t,s) < -|G_D[T](t,s)|$ for all $(t,s) \in I \times I$.
4. If $G_D[2\,T] \ge 0$ on $J \times J$, then $G_{M_2}[T](t,s) \ge |G_D[T](t,s)|$ for all $(t,s) \in I \times I$.

As a consequence of the previous corollary, we deduce the following comparison principles among the solutions of the corresponding problems. The arguments are similar to the ones used in the proof of Theorem 4.

Theorem 5 *Let u_N be the unique solution of problem (N,T) for $\sigma = \sigma_1$ and u_{M_1} the unique solution of problem (M_1,T) for $\sigma = \sigma_2$. Then*

1. If $G_N[2\,T] \ge 0$ on $J \times J$ and $|\sigma_2(t)| \le \sigma_1(t)$ a.e. $t \in I$, then $|u_{M_1}(t)| \le u_N(t)$ for all $t \in I$.

2. *If* $G_N[2T] \leq 0$ *on* $J \times J$ *and* $0 \leq \sigma_2(t) \leq \sigma_1(t)$ *a.e.* $t \in I$, *then* $u_N(t) \leq 0$ *and* $u_N(t) \leq u_{M_1}(t)$ *for all* $t \in I$.
3. *If* $G_N[2T] \leq 0$ *on* $J \times J$ *and* $\sigma_1(t) \leq \sigma_2(t) \leq 0$ *a.e.* $t \in I$, *then* $u_N(t) \geq 0$ *and* $u_{M_1}(t) \leq u_N(t)$ *for all* $t \in I$.

Theorem 6 *Let* u_{M_2} *be the unique solution of problem* (M_2, T) *for* $\sigma = \sigma_1$ *and* u_D *the unique solution of problem* (D, T) *for* $\sigma = \sigma_2$. *Then, it holds that:*

1. *If* $G_D[2T] \geq 0$ *on* $J \times J$ *and* $|\sigma_2(t)| \leq \sigma_1(t)$ *a.e.* $t \in I$, *then* $|u_D(t)| \leq u_{M_2}(t)$ *for all* $t \in I$.
2. *If* $G_D[2T] \leq 0$ *on* $J \times J$ *and* $0 \leq \sigma_2(t) \leq \sigma_1(t)$ *a.e.* $t \in I$, *then* $u_{M_2}(t) \leq 0$ *and* $u_{M_2}(t) \leq u_D(t)$ *for all* $t \in I$.
3. *If* $G_D[2T] \leq 0$ *on* $J \times J$ *and* $\sigma_1(t) \leq \sigma_2(t) \leq 0$ *a.e.* $t \in I$, *then* $u_{M_2}(t) \geq 0$ *and* $u_D(t) \leq u_{M_2}(t)$ *for all* $t \in I$.

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Random Evolution Equations with Bounded Fractional Integral-Feedback

Abdelkader Moulay and Abdelghani Ouahab

Abstract We consider abstract evolution with random parameter. We introduce the notion of stabilization with respect to the random parameter and fractional integral-feedback. More precisely study the well-posedness and polynomial stabilization result for random evolution equation with fractional integral-feedback. Finally we give some applications to random heat and wave equations with fractional integral-feedback and bounded damping.

Keywords Cauchy problem · C_0−semigroup · Stabilization · Random evolution · Fractional integral · Caputo's derivative · Heat equation · Wave equation

2010 Mathematics Subject Classification 35R60 · 93D15 · 47D06

1 Introduction

In the last decades mathematical deterministic stabilization and observability theory has been extensively developed to handle various models of ordinary and partial differential equations. Stability of the wave equation have been extensively studied in the literature, we can refer to [16–18, 28] and references therein. For the stability results of Petrovsky, elastodynamic, Mawxell's and Schrödinger system we see [2, 8, 9, 13, 15, 21]. Recently, various authors have obtained stabilization results for Schrödinger, wave equations with delay [1, 3, 4, 6, 14, 20, 22–24].

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Many problems in physics, engineering, biology and social sciences lead to mathematical equations. In

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these equations, the coefficients and the other parameters have their origin in experimental data and represent some kind of average value. Therefore in many instances due to wide variations of the data or even due to our own ignorance, it is appropriate to abandon the deterministic model in favor of a stochastic one. Important contributions to the study of the mathematical aspects of such random equations have been undertaken in [10, 25, 29] among others. In this paper, we analyze the random stabilization property of solutions of abstract first order differential equation. Very recently, the issue of random countability and observability (or averaged controllability) in [31] both in finite or infinite dimensional space. Averaged controllability and observability for semilinear partial differential equations as Schrödinger, heat and wave equations with different boundary conditions has bee initial by and it has attracted considerable attention, as for example [32–34].

This paper is organized as follows. In Sect. 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Sect. 3, we prove that that random abstract dissipative evolution equation with fractional integral feedback is well-posedness, where the damping is bounded. In Sect. 4 we obtain the same polynomial stabilization of random evolutions with fractional integral damped. Finally, in Sect. 5, we illustrate our abstract results by some applications to heat and wave equations.

2 Abstract Setting and Augmented Model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, H be a separable Hilbert space with norm and inner product denoted respectively by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ and U be a separable Hilbert space (which will be identified to its dual space) with norm and inner product respectively denoted by $\|\cdot\|_U$ and $\langle\cdot,\cdot\rangle_U$. Let $\{A(\omega)\}_{\omega\in\Omega}$ be a family of linear operators has the following conditions

$(C1)$ $A(\cdot) : D(A(\cdot)) \subset H \to H$ is a unbounded self-adjoint and strictly positive operator.

$(C2)$ $B(\cdot) \in \mathcal{L}(U, H)$, and there exists $M \geq 0$ such that

$$\|B(\omega)\|^2_{\mathcal{L}(U,H)} \leq M, \quad \forall \omega \in \Omega.$$

We consider the the abstract Cauchy problem

$$\begin{cases} u'(t,\omega) = -A(\omega)u(t,\omega) - B(\omega)B^*(\omega)I^{1-\alpha,\eta}u(t,\omega), \quad t > 0, \\ u(0,\omega) = u_0(\omega), \end{cases} \tag{1}$$

where $I^{\alpha,\eta}$ is the the integral fractional order by the following

$$I^{\alpha,\eta}w(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{1-\alpha}e^{-\eta(t-s)}w(s)ds, \quad 0 < \alpha < 1,\ \eta \geq 0 \tag{2}$$

and $\partial_t^{\alpha,\eta}$ is the generalized Caputo's fractional order derivative defined by

$$\partial^{\alpha,\eta} w(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\eta(t-s)} w'(s) ds, \quad 0 < \alpha < 1, \ \eta \geq 0. \tag{3}$$

From (2) and (3) we have

$$\partial_t^{\alpha,\eta} w(t) = I^{1-\alpha,\eta} w'(t). \tag{4}$$

By the same technique as in the proof of ([19], Theorem 1) we have the following theorem concerned the reformulation of the model (1) into an augmented system.

Theorem 2.1 *For all* $\xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d$ *with* $|\xi|^2 = \xi_1^2 + \cdots + \xi_d^2$ *we define the function*

$$p(\xi) = |\xi|^{\frac{2\alpha-d}{2}}.$$

Then the relation between the 'input' V and the 'output' O of the following system

$$\begin{cases} \partial_t \phi(\xi, t, \omega) = -(|\xi|^2 + \eta)\phi(\xi, t, \omega) + p(\xi) V(t, \omega) \ , \ \xi \in \mathbb{R}^d, t > 0, \\ \phi(\xi, 0, \omega) \quad = 0, \qquad \xi \in \mathbb{R}^d, \quad \omega \in \Omega \\ O(t, \omega) \qquad = \gamma \int_{\mathbb{R}^d} p(\xi)\phi(\xi, t, \omega) d\xi, \end{cases} \tag{5}$$

where $V(\cdot, \omega) \in C([0, +\infty), H), \quad \forall \omega \in \Omega$ *and* $V \in L^1(\Omega, \mathbb{P}, H)$ *is given by*

$$O(t, \omega) = I^{1-\alpha,\eta} V(t, \omega)$$

and

$$\gamma = \frac{2\sin(\alpha\pi)\Gamma(\frac{d}{2}+1)}{d\pi^{\frac{d}{2}+1}}.$$

We finish this preliminary section by giving the following technical result.

Lemma 2.1 *If* $\lambda > 0$, *then*

$$\int_{\mathbb{R}^d} \frac{p(\xi)^2}{\lambda + |\xi|^2 + \eta} d\xi < +\infty.$$

Proof We define the spherical coordinates by

$$\begin{cases} \xi_1 \quad = r\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{d-3})\sin(\theta_{d-2})\sin(\theta_{d-1}) \\ \xi_2 \quad = r\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{d-3})\sin(\theta_{d-2})\cos(\theta_{d-1}) \\ \xi_3 \quad = r\sin(\theta_1)\sin(\theta_2)\ldots\sin(\theta_{d-3})\cos(\theta_{d-2}) \\ \vdots \\ \xi_{d-1} = r\sin(\theta_1)\cos(\theta_2) \\ \xi_d \quad = r\sin(\theta_1), \end{cases} \tag{6}$$

where $r = |\xi|,\ \theta_j \in [0, \pi], \quad 1 \le j \le d-2$ and $\theta_{d-1} \in [0, 2\pi]$. The Jacobian determinant of (6), given by

$$J = r^{d-1}\Pi_{j=1}^{d-2} \sin^{d-1-j}(\theta_j).$$

Then

$$\int_{\mathbb{R}^d} \frac{p(\xi)^2}{\lambda + |\xi|^2 + \eta} d\xi = 2\int_0^{+\infty} \frac{r^{2\alpha-1}}{\lambda + r^2 + \eta} \Pi_{j=1}^{d-2} \left(\int_0^{\pi} \sin^{d-1-j}(\theta_j) d\theta_j \right) \int_0^{2\pi} \sin(\theta_{d-1}) d\theta_{d-1} dr.$$

By induction, we can easily prove that

$$\Pi_{j=1}^{d-2} \left(\int_0^{\pi} \sin^{d-1-j}(\theta_j) d\theta_j \right) \int_0^{2\pi} \sin(\theta_{d-1}) d\theta_{d-1} = \frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)}.$$

This implies that

$$\int_{\mathbb{R}^d} \frac{p(\xi)^2}{\lambda + |\xi|^2 + \eta} d\xi = \frac{2d\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \int_0^{+\infty} \frac{r^{2\alpha-1}}{\lambda + r^2 + \eta} dr.$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^d} \frac{p(\xi)^2}{\lambda + |\xi|^2 + \eta} d\xi &= \frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \int_0^{+\infty} \frac{x^{\alpha-1}}{\lambda + x + \eta} dx \\
&= \frac{d\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \int_{\lambda+\eta}^{+\infty} \frac{(y-\lambda-\eta)^{\alpha-1}}{y} dy \\
&= \frac{d(\lambda+\eta)^{\alpha-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \int_0^1 (1-z)^{\alpha-1} z^{-\alpha} dz \\
&= \frac{d(\lambda+\eta)^{\alpha-1}\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} \mathbf{B}(\alpha, 1-\alpha),
\end{aligned}$$

where **B** is a beta function. □

Lemma 2.2 *Let $\lambda \in \mathbb{R}^*$, then for any $\eta > 0$, we have*

$$\int_{\mathbb{R}^d} \frac{p(\xi)^2}{i\lambda + |\xi|^2 + \eta} d\xi < +\infty.$$

By using Theorem 2.1 and (4), system (1) may be recast into the following random augmented system

$$\begin{cases} u'(t,\omega) + A(\omega)u(t,\omega) + \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\phi(\xi,t,\omega)d\xi &= 0, \quad t>0, \\ \partial_t\phi(\xi,t,\omega) + (|\xi|^2+\eta)\phi(\xi,t,\omega) - p(\xi)B^*(\omega)u(t,\omega) = 0, \ \xi \in \mathbb{R}^d \\ u(0,\omega) = u_0(\omega) \quad \phi(\xi,0,\omega) = 0, \end{cases} \tag{7}$$

where the constant γ and the function $p(\xi)$ are defined in Theorem 2.1.

3 Well-Posedness

In this section, we will prove the existence and uniqueness of solution of (7) using semigroup theory. We begin with the functional spaces.

$$\widetilde{V} = L^2(\mathbb{R}^d, H), \quad \langle u, v\rangle_{\widetilde{V}} = \int_{\mathbb{R}^d} \langle u(y), v(y)\rangle_H dy, \ \|u\|_2^2 = \int_{\mathbb{R}^d} \|u(y)\|_H^2 dy,$$

and the following Hilbert space

$$\mathcal{H} = H \times \widetilde{V}$$

with inner product

$$\left\langle \begin{pmatrix} u_1 \\ \phi_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ \phi_2 \end{pmatrix} \right\rangle_{\mathcal{H}} = \langle u_1, u_2\rangle_H + \gamma \int_{\mathbb{R}^d} \langle \phi_1(\xi), \phi_2(\xi)\rangle_H d\xi.$$

Let $Y(\cdot,\omega) = \begin{pmatrix} u(\cdot,\omega) \\ \phi(\cdot,\xi,\omega) \end{pmatrix}$, $Y_0(\omega) = \begin{pmatrix} u_0(\omega) \\ 0 \end{pmatrix}$, $\omega \in \Omega$ and rewrite (7) as

$$\begin{cases} Y'(t,\omega) = \mathcal{A}(\omega)Y(t,\omega), \quad t>0, \\ Y(0,\omega) = Y_0(\omega), \end{cases} \tag{8}$$

where $\mathcal{A} : D(\mathcal{A}(\omega)) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{A}(\omega)\begin{pmatrix} u(t,\omega) \\ \phi(t,\xi,\omega) \end{pmatrix} = \begin{pmatrix} -A(\omega)u(t,\omega) - \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi \\ \\ -(|\xi|^2+\eta)\phi(t,\xi,\omega) + p(\xi)B^*(\omega)u(t,\omega) \end{pmatrix} \tag{9}$$

with domain

$$D(\mathcal{A}(\omega)) = \left\{ \begin{array}{l} \begin{pmatrix} u \\ \phi \end{pmatrix} \in \mathcal{H} : -A(\omega)u(t,\omega) - \gamma B(\omega) \int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi \in H, \\ |\xi|\phi \in L^2(\mathbb{R}^d, H) - (|\xi|^2+\eta)\phi + p(\xi)B^*(\omega)u \in L^2(\mathbb{R}^d, H) \end{array} \right\}. \tag{10}$$

Theorem 3.1 *For every $\omega \in \Omega$, the operator $\mathcal{A}(\omega)$ defined in (8) and (10), generates C_0-semigroup of contraction $e^{t\mathcal{A}(\omega)}$ on $\mathcal{H}$.*

Proof Let $\omega \in \Omega$. By Lumer Phillips theorem (see [26, Theorem 4.3]) we show that $\mathcal{A}(\omega)$ is $m-$dissipative. We first prove that $\mathcal{A}(\omega)$ is dissipative. Let $(u,\phi) \in D(\mathcal{A}(\omega))$. Thus

$$\begin{aligned} \left\langle \mathcal{A}(\omega)\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u \\ \phi \end{pmatrix} \right\rangle_{\mathcal{H}} &= \langle -A(\omega)u(t,\omega) - \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi, u\rangle_H \\ &+ \gamma \int_{\mathbb{R}^d} \langle -(|\xi|^2+\eta)\phi + p(\xi)B^*(\omega)u, \phi\rangle_H d\xi \\ &= \langle -A(\omega)u, u\rangle_H - \gamma \int_{\mathbb{R}^d} p(\xi)\langle B(\omega)\phi(t,\xi,\omega), u\rangle_H d\xi \\ &- \gamma(|\xi|^2+\eta)\int_{\mathbb{R}^d} \langle \phi, \phi\rangle_H d\xi + \gamma \int_{\mathbb{R}^d} p(\xi)\langle B^*(\omega)u, \phi\rangle_H d\xi. \end{aligned}$$

Since $A(\omega)$ is positive,

$$\Re\left(\left\langle \mathcal{A}(\omega)\begin{pmatrix} u \\ \phi \end{pmatrix}, \begin{pmatrix} u \\ \phi \end{pmatrix} \right\rangle_{\mathcal{H}}\right) = -\gamma(|\xi|^2+\eta)\int_{\mathbb{R}^d} \|\phi(t,\xi,\omega)\|_H^2 d\xi \le 0.$$

Consequently $\mathcal{A}(\omega)$, is dissipative.

Next, we would like to show that $(\lambda I - \mathcal{A}(\omega))$ is surjective for some $\lambda > 0$. For this purpose, let $(f, g) \in \mathcal{H}$, there is an other $Y = (u,\phi) \in D(\mathcal{A}(\omega))$ solution of the following system of equations:

$$\begin{cases} \lambda u + A(\omega)u + \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi = f, \\ \lambda\phi + (|\xi|^2+\eta)\phi - p(\xi)B^*(\omega)u = g. \end{cases} \tag{11}$$

By the second equation of (11) we can find ϕ as

$$\phi = \frac{p(\xi)B^*(\omega)u + g}{\lambda + |\xi|^2 + \eta}. \tag{12}$$

By (11) and (12), we get

$$\lambda u + A(\omega)u + \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\frac{p(\xi)B^*(\omega)u + g(\xi)}{\lambda + |\xi|^2 + \eta}d\xi = f.$$

Therefore

$$\lambda u + A(\omega)u + \gamma \int_{\mathbb{R}^d} \frac{p^2(\xi)}{\lambda + |\xi|^2 + \eta} d\xi B(\omega)B^*(\omega)u = \gamma B(\omega) \int_{\mathbb{R}^d} \frac{p(\xi)g(\xi)}{\lambda + |\xi|^2 + \eta} d\xi + f.$$

Then

$$C(\omega)u = \gamma B(\omega) \int_{\mathbb{R}^d} \frac{p(\xi)g(\xi)}{\lambda + |\xi|^2 + \eta} d\xi + f, \tag{13}$$

where

$$C(\omega) = \lambda I + A(\omega) + \gamma \int_{\mathbb{R}^d} \frac{p^2(\xi)}{\lambda + |\xi|^2 + \eta} d\xi B(\omega)B^*(\omega).$$

Using that $A(\omega)$ is strict positive operator and self-adjoint operator and $B(\omega)$ is linear bounded operator. Thus from proposition 3.3.5 in [30], $-A(\omega)$ is dissipative and

$$\|(\lambda I + A(\omega))^{-1}\|_{\mathcal{L}(H)} \leq \frac{1}{\lambda} \tag{14}$$

and $C^*(\omega)$ is strict positive operators. Hence $N(C^*(\omega)) = \{0\}$. It follows that

$$\overline{R(C(\omega))} = (N(C^*(\omega)))^{\perp} = H.$$

Now, we show that $C(\omega)$ is closed operator. Let $(u_n, C(\omega)u_n) \in R(C(\omega))$ such that

$$u_n \to u, \quad C(\omega)u_n \to v, \quad \text{as } n \to \infty.$$

Then (14) and the continuity of $B(\omega)$ implies

$$(\lambda I + A(\omega))^{-1} C(\omega)u_n \to u + \gamma(\lambda I + A(\omega))^{-1} \int_{\mathbb{R}^d} \frac{p^2(\xi)}{\lambda + |\xi|^2 + \eta} d\xi B(\omega)B^*(\omega)u, \quad n \to \infty.$$

Therefore,

$$v = \lambda u + A(\omega)u + \gamma \int_{\mathbb{R}^d} \frac{p^2(\xi)}{\lambda + |\xi|^2 + \eta} d\xi B(\omega)B^*(\omega)u.$$

As consequence of Theorem 3.1 and Proposition 4.1 of [27], the system (7) is well-posed in the energy space $\mathcal{H}$. □

Theorem 3.2 *For every $Y_0 = (u_0(\cdot), 0) : \Omega \to \mathcal{H}$ random variable, the system* (8) *has unique random mild solution*

$$Y(\cdot, \omega) \in C(\mathbb{R}_+, \mathcal{H}), \quad \forall \omega \in \Omega.$$

If $Y_0(\omega) = (u_0(\omega), 0) \in D(\mathcal{A}(\omega))$, $\omega \in \Omega$, then the system (7) *has unique solution*

$$Y(.,\omega) \in C^1(\mathbb{R}_+, D(\mathcal{A}(\omega))) \cap C(\mathbb{R}_+, \mathcal{H}), \quad \omega \in \Omega.$$

Proof From Theorem 3.1, $\mathcal{A}(\omega)$ generate a C_0-semigroup of contractions. We denote this semigroup by $e^{\mathcal{A}(\omega)t}$, $t \geq 0$; $\omega \in \Omega$. By the Grandall–Liggett formula we have

$$e^{t\mathcal{A}(\omega)}Y = \lim_{n\to\infty}\left(I - \frac{t}{n}\mathcal{A}(\omega)\right)^{-n} Y.$$

From Proposition 4.1 of [27], we know that

$$\omega \mapsto \left[\left(I - \frac{t}{n}\mathcal{A}(\omega)\right)^{-1}\right]^n Y$$

is measurable, which implies that $\omega \mapsto e^{\mathcal{A}(\omega)t}$ is measurable. Using the continuity properties of semigroup $t \to e^{t\mathcal{A}(\omega)}$, we get

$$\omega \mapsto e^{t\mathcal{A}(\omega)}Y_0(\omega),$$

is measurable. Then $(\omega, t) \mapsto e^{\mathcal{A}(\omega)t}Y_0(\omega)$ is a unique random solution of the problem (7). □

4 Random Stabilization

In this section we introduce the following notation of random stabilizations.

Definition 4.1 Let $\{A(\omega)\}_{\omega\in\Omega}$ be a family generator of a strong continuous semigroup of contractions $(S(t,\omega))_{t\geq 0}$ on H. We say that the C_0-semiroup $(S(t,\omega))_{t\geq 0}$ is

(i) Strong stable if

$$\lim_{t\to+\infty}\int_\Omega \|S(t,\omega)x\|_H^2 d\mathbb{P}(\omega) = 0, \ \forall x \in H.$$

(ii) Uniformly stable if

$$\lim_{t\to+\infty}\int_\Omega \|S(t,\omega)\|_{\mathcal{L}(H)}^2 d\mathbb{P}(\omega) = 0.$$

(iii) Random exponentially stable if there exist $M, \delta \geq 0$ such that

$$\int_{\Omega} \|S(t,\omega)x\|_H^2 d\mathbb{P}(\omega) \leq Me^{-\delta t} \int_{\Omega} \|x\|_H^2 d\mathbb{P}(\omega), \quad \forall t > 0,\ \forall x \in H.$$

(iv) Random polynomially stable if there exist $M, l > 0$, such that

$$\int_{\Omega} \|S(t,\omega)x\|_H^2 d\mathbb{P}(\omega) \leq \frac{M}{t^l} \int_{\Omega} \|x\|_H^2 d\mathbb{P}(\omega), \quad \forall t > 0,\ \forall x \in H.$$

Theorem 4.1 ([5]) *Assume that A is the generator of a strongly continuous semigroup of contractions $(S(t))_{t\geq 0}$ on a reflexive Banach space X. If*

- *A has no pure imaginary eigenvalues.*
- *$\sigma(A) \cap i\mathbb{R}$ is countable.*

Then $(S(t))_{t\geq 0}$ is strongly stable.

Theorem 4.2 (Borichev-Tomilov [11]) *Let $S(t) = e^{At}$ be a C_0-semigroup on a Hilbert space. If*

$$i\mathbb{R} \subset \rho(A) \quad \textit{and} \quad \lim_{\lambda\in\mathbb{R},\ \lambda\to+\infty} \sup \|\psi(|\lambda|)^{-1}(i\lambda I - A)^{-1}\|_{\mathcal{L}(H)} < +\infty,$$

then there exists $c > 0$ such that

$$\|e^{At}U_0\|_{\mathcal{L}(H)} \leq \frac{c}{t^l}\|U_0\|_{D(A)}, \quad U_0 \in D(A).$$

By simple calculation we show that (u, ϕ) the regular solution of (7) satisfies the random energy identity

$$E(t,\omega) = \frac{1}{2}\|u(t,\omega)\|_H^2 + \frac{\gamma}{2}\int_{\mathbb{R}^d} \|\phi(t,\xi,\omega)\|_H^2 d\xi, \quad \forall\omega \in \Omega,\ t \geq 0, \tag{15}$$

is the random energy of x at time t.

Lemma 4.1 *Let (u, ϕ) be a regular solution of the system (7). Then, the energy functional defined by (15) satisfies*

$$E'(t,\omega) = -\gamma \int_{\mathbb{R}^d} (|\xi|^2 + \eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi, \quad \forall\omega \in \Omega,\ t \geq 0, \tag{16}$$

and

$$E(t,\omega) - E(0,\omega) = -\int_0^t \int_{\mathbb{R}^d} (|\xi|^2 + \eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi dt, \quad \forall\omega \in \Omega,\ t \geq 0. \tag{17}$$

Proof Multiplying the first equation of system (7) by u and the second equation by ϕ, we get

$$\langle u', u\rangle_H = \langle -A(\omega)u(t,\omega) - \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi, u\rangle_H \tag{18}$$

and

$$\begin{aligned}\langle \phi', \phi\rangle_{\widetilde{V}} &= \gamma \textstyle\int_{\mathbb{R}^d} \left\langle -(|\xi|^2+\eta)\phi + p(\xi)B^*(\omega)u, \phi\right\rangle_H d\xi \\ &\quad +\gamma \textstyle\int_{\mathbb{R}^d} p(\xi)\left\langle B^*(\omega)u(t,\omega), \phi(t,\xi,\omega)\right\rangle_H d\xi.\end{aligned} \tag{19}$$

Adding (18) and (19), one has

$$\frac{1}{2}\frac{d}{dt}\|u(t,\omega)\|_H^2 + \frac{1}{2}\int_{\mathbb{R}^d}\|\phi(t,\xi,\omega)\|_H^2 d\xi = -\gamma\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi.$$

Therefore

$$E'(t,\omega) = -\gamma\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi \le 0, \quad \omega\in\Omega,\ t\ge 0.$$

By integration we get

$$E(t,\omega) - E(0,\omega) = -\gamma\int_0^t\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi dt, \quad \forall\omega\in\Omega,\ t\ge 0.$$

□

Proposition 4.3 *For every $u_0 \in L^2(\Omega, H, \mathbb{P})$ the solution of (8) satisfy.*

$$\int_0^T\int_\Omega\int_{\mathbb{R}^d}\|\phi(t,\xi,\omega)\|_H^2 d\mathbb{P}(\omega)d\xi dt \le (2\eta)^{-1}\int_\Omega E(0,\omega)d\mathbb{P}(\omega), \quad \forall T>0. \tag{20}$$

Proof Let $u_0 \in L^2(\Omega, H, \mathbb{P})$, then $Y_0(\omega) = (u_0(\omega), 0) \in D(\mathcal{A}(\omega))$. From (17), we get

$$E(t,\omega) - E(0,\omega) = -\gamma\int_0^t\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi dt.$$

Thus

$$E(t,\omega) - E(0,\omega) \le -\gamma\eta\int_0^t\int_{\mathbb{R}^d}\|\phi(t,\xi,\omega)\|_H^2 d\xi dt.$$

Therefore

$$E(t,\omega) + \gamma\eta\int_0^t\int_{\mathbb{R}^d}\|\phi(s,\xi,\omega)\|_H^2 ds d\xi \le E(0,\omega), \quad t>0.$$

So, for every $T > 0$ we have

$$\int_0^T \int_{\mathbb{R}^d} \|\phi(t,\xi,\omega)\|_H^2 dt d\xi \leq (2\gamma\eta)^{-1} \int_\Omega \|u_0(\omega)\|_H^2 d\mathbb{P}(\omega).$$

Since $D(A(\omega))$ is dense in H, there exist a sequence $Y_0^n(\omega) = (u_0^n(\omega), 0) \in D(\mathcal{A}(\omega))$ such that $(Y_0^n(\omega))_{n\in\mathbb{N}}$ converge to Y_0 in $\mathcal{H}$. □

Proposition 4.4 *For every* $u_0 \in L^2(\Omega, D(\mathcal{A}(\omega)), \mathbb{P})$ *the solution of* (8) *satisfies the inequality:*

$$\int_0^T \int_\Omega \|I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H^2 d\mathbb{P}(\omega)dt \leq A_\eta \int_\Omega \|u_0(\omega)\|_H^2 d\mathbb{P}(\omega), \quad \forall T > 0. \tag{21}$$

Proof From Theorem 2.1, we have

$$\gamma \int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi = I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)$$

Hence

$$\|I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H \leq \gamma \int_{\mathbb{R}^d} p(\xi)\|\phi(t,\xi,\omega)\|_H d\xi.$$

By Cauchy–Schwarz inequality, we get

$$\|I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H \leq \left(\int_{\mathbb{R}^d} (|\xi|^2+\eta)^{-1} p(\xi)^2 d\xi\right)^{\frac{1}{2}} \left(\gamma \int_{\mathbb{R}^d} (|\xi|^2+\eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi\right)^{\frac{1}{2}}$$

and

$$\|B(\omega)I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H \leq M\left(\int_{\mathbb{R}^d} (|\xi|^2+\eta)^{-1} p(\xi)^2 d\xi\right)^{\frac{1}{2}} \left(\gamma \int_{\mathbb{R}^d} (|\xi|^2+\eta)\|\phi(t,\xi,\omega)\|_H^2 d\xi\right)^{\frac{1}{2}}.$$

On the other hand, from (16) we have

$$\|I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H^2 \leq -CE'(t,\omega)$$

and

$$\|B(\omega)I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H^2 \leq -MCE'(t,\omega),$$

where

$$C = \int_{\mathbb{R}^d} \frac{p(\xi)^2}{|\xi|^2+\eta} d\xi.$$

By integrating with respect to time and a random parameter in above inequalities, we get

$$\int_0^T \int_\Omega \|I^{1-\alpha,\eta} B^*(\omega)u(t,\omega)\|_H^2 dt d\xi \leq C \int_\Omega E(0,\omega)d\mathbb{P}(\omega)$$

and

$$\int_0^T \int_\Omega \|B(\omega)I^{1-\alpha,\eta}B^*(\omega)u(t,\omega)\|_H^2 dt d\xi \leq MC \int_\Omega E(0,\omega)d\mathbb{P}(\omega).$$

□

Proposition 4.5 *For every* $u_0 \in L^2(\Omega, D(\mathcal{A}(\omega)), \mathbb{P})$ *the solution of* (8) *satisfies the inequality:*

$$\int_\Omega \int_0^T \left(\gamma \int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi\right)^2 d\mathbb{P}(\omega)dt \leq B_\eta \int_\Omega \|u_0(\omega)\|_H^2 d\mathbb{P}(\omega), \quad \forall T > 0. \tag{22}$$

For polynomial stability we consider the following problem

$$\begin{cases} u'(t,\omega) = -A(\omega)u(t,\omega) - B(\omega)B^*(\omega)u(t,\omega), & t > 0, \\ u(0,\omega) = u_0(\omega) \end{cases} \tag{23}$$

Theorem 4.6 *For every random variable* $u_0 : \Omega \to H$*, there exists a unique random solution* $x : \Omega \to C([0,+\infty), H)$*. Moreover if for every* $\omega \in \Omega$ *we have* $u_0(\omega) \in D(A_d(\cdot))$ *then* $u(\cdot) \in C([0,+\infty), D(A_d(\cdot))) \cap C^1([0,+\infty), H)$*.*

Proof Let $\omega \in \Omega$. By Lumer Phillips Theorem we show that $A_d(\omega)$ is $m-$dissipative. We first prove that $A_d(\omega) = -A(\omega) - B(\omega)B^*(\omega)$ is dissipative. Let $u \in D(A_d(\omega))$. Thus

$$\begin{aligned} \langle A_d(\omega)u, u\rangle_H &= -\langle A(\omega)u + B(\omega)B^*(\omega)u, u\rangle_H \\ &= -\langle A(\omega)u, u\rangle_H - \langle B^*(\omega)u, B^*(\omega)u\rangle_H \\ &= -\langle A(\omega)u, u\rangle_H - \|B^*(\omega)u\|_U^2. \end{aligned}$$

Since $A(\omega)$ is positive,

$$\Re(\langle A_d(\omega)u, u\rangle) = -\|B^*(\omega)u\|_U^2 \leq 0.$$

Consequently $A_d(\omega)$, is dissipative.

Next, we would like to show that $(\lambda I - A_d(\omega))$ is surjective for some $\lambda > 0$. Let $v \in H$, we need $u \in D(A_d(\omega))$ such that

$$\lambda u - A_d(\omega)u = v.$$

Hence

$$\lambda u + A(\omega)u + B(\omega)B^*(\omega)u = v.$$

Let $\bigwedge(u,\xi) = \langle \lambda u + A(\omega)u + B(\omega)B^*(\omega)u, \xi\rangle_H$. For $u \in D(A^{\frac{1}{2}}(\omega))$ we can writing $\bigwedge$ in the following form

$$\begin{aligned}\bigwedge(u,\xi) &= \langle \lambda u + A(\omega)u + B(\omega)B^*(\omega)u, \xi\rangle_H \\ &= \langle \lambda u, \xi\rangle_H + \langle \lambda A(\omega)u, \xi\rangle_H + \langle B(\omega)B^*(\omega)u, \xi\rangle_U \\ &= \lambda\langle u, \xi\rangle_H + \lambda\langle A^{\frac{1}{2}}(\omega)u, A^{\frac{1}{2}}(\omega)\xi\rangle_H + \langle B^*(\omega)u, B^*(\omega)\xi\rangle_U, \quad \forall \xi \in D(A^{\frac{1}{2}}(\omega)).\end{aligned}$$

Now we show that $\bigwedge$ is continuous and coercive on $V = D(A^{\frac{1}{2}}(\omega))$ and $\|u\|_V := \|A^{\frac{1}{2}}(\omega)u\|_H$. Let $(u, \xi) \in V \times V$ then we have

$$\begin{aligned}\left|\bigwedge(u,\xi)\right| &\leq |\lambda\langle u, \xi\rangle_H| + |\lambda||\langle A^{\frac{1}{2}}(\omega)u, A^{\frac{1}{2}}(\omega)\xi\rangle_H| + |\langle B^*(\omega)u, B^*(\omega)\xi\rangle_U| \\ &\leq |\lambda|\|u\|_H\|\xi\|_H + |\lambda|\|A^{\frac{1}{2}}(\omega)u\|_H\|A^{\frac{1}{2}}\xi\|_H + \|B^*(\omega)u\|_U\|B^*(\omega)\xi\|_U.\end{aligned}$$

By continuously embedded of $V \subset H$, there exists $C > 0$ such that

$$\left|\bigwedge(u,\xi)\right| \leq |C\|u\|_V\|\xi\|_V.$$

It is clear that

$$\begin{aligned}\left|\bigwedge(u,u)\right| &= \left|\lambda\langle u, u\rangle + \lambda\langle A^{\frac{1}{2}}(\omega)u, A^{\frac{1}{2}}u\rangle + \langle B^*(\omega)u, B^*(\omega)u\rangle_U\right| \\ &\geq |\lambda|\|u\|_V^2, \quad \forall u \in V.\end{aligned}$$

By the Lax–Milgram lemma, we conclude that there exists unique $u \in V$ such that

$$\lambda u - A_d(\omega)u = v.$$

Then $(\omega, t) \mapsto e^{A_d(\omega)t}u_0(\omega)$ is a unique random solution of the problem (23). □

Let ψ be an increasing function in $\mathbb{R}_+$. We assume that $A_d(\omega)$ satisfies the following conditions

$$i\mathbb{R} \subset \rho(A_d(\omega)) \quad \text{and} \quad \lim_{\lambda\in\mathbb{R},\ \lambda\to+\infty} \sup \|\psi(|\lambda|)^{-1}(i\lambda I - A_d(\omega))^{-1}\| \leq C(\omega) < +\infty, \ \ \omega \in \Omega \tag{24}$$

and

$$\lim_{\lambda\in\mathbb{R},\ \lambda\to+\infty} \int_\Omega \sup \|\psi(|\lambda|)^{-1}(i\lambda I - A_d(\omega))^{-1}\| d\mathbb{P}(\omega) < +\infty. \tag{25}$$

Proposition 4.7 *We assume that the conditions* (24), (25) *hold with* $\psi(|\lambda|) = |\lambda|^{\sqrt{l}}$. *If* $C(\omega)$ *is independence to* ω. *Then there exists* $C > 0$ *such that*

$$\int_\Omega \|e^{A_d(\omega)t}x\|_H d\mathbb{P}(\omega) \leq \frac{C}{t^{l^{-1}}} \int_\Omega \|x\| D(A_d(\omega)) d\mathbb{P}(\omega)$$

and

$$\int_\Omega \|e^{A_d(\omega)t}x\|_H^2 d\mathbb{P}(\omega) \leq \frac{C^2}{t^{l^{-1}}} \int_\Omega \|x\|_{D(A_d(\omega))}^2 d\mathbb{P}(\omega)$$

Proof Form Theorem 2.4 in [11] we have

$$\|e^{A_d(\omega)t}x\|_H \leq \frac{C}{t^{\frac{1}{\sqrt{l}}}}\|x\|_{D(A_d(\omega))}.$$

Consequently

$$\int_\Omega \|e^{A_d(\omega)t}x\|_H^2 d\mathbb{P}(\omega) \leq \frac{C^2}{t^{\frac{1}{l}}}\int_\Omega \|x\|_{D(A_d(\omega))}^2 d\mathbb{P}(\omega).$$

□

Theorem 4.8 *Assume that* $i\mathbb{R} \subset \rho(\mathcal{A}(\omega))$, $\forall\omega \in \Omega$ *and the condition* (24) *holds. Let* $\eta > 0$*, there exists* $C > 0$ *such that*

$$\|(i\lambda I - \mathcal{A}(\omega))^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^{1-\alpha}\psi(|\lambda|), \quad \forall\,\omega \in \Omega. \tag{26}$$

Proof We only need to prove that

$$\sup_{\lambda\in\mathbb{R},\ |\lambda|\to\infty} |\lambda|^{\alpha-1}\psi(|\lambda|)\|(i\lambda I - \mathcal{A}(\omega))^{-1}\|_{\mathcal{L}(\mathcal{H})} < \infty, \quad \forall\,\omega \in \Omega.$$

For this, we using an argument of contradiction. For this purpose, assume that (26) is false, then there exist a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ with $\lim_{n\to+\infty}\lambda_n = +\infty$ and a sequence $(u_n, \phi_n) \in D(\mathcal{A}(\omega))$ such that

$$\|(u_n, \phi_n)\|_{\mathcal{H}} = 1. \tag{27}$$

and

$$|\lambda_n|^{1-\alpha}\psi(|\lambda_n|)(i\lambda_n I - \mathcal{A})(u_n, \phi_n) = (f_n, g_n) \to 0, \quad \text{in } \mathcal{H}. \tag{28}$$

Since

$$\lim_{n\to\infty}\Re\left(\left\langle\begin{pmatrix} f_n \\ g_n\end{pmatrix}, \begin{pmatrix} u_n \\ \phi_n\end{pmatrix}\right\rangle_{\mathcal{H}}\right) = \lim_{n\to\infty}\gamma|\lambda_n|^{1-\alpha}\psi(|\lambda_n|)\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi_n(t,\xi,\omega)\|_H^2 d\xi$$

Hence

$$\lim_{n\to\infty}\Re\left(\left\langle\begin{pmatrix} f_n \\ g_n\end{pmatrix}, \begin{pmatrix} u_n \\ \phi_n\end{pmatrix}\right\rangle_{\mathcal{H}}\right) = 0. \tag{29}$$

Detailing Eq. (28), we obtain

$$f_n = |\lambda_n|^{1-\alpha}\psi(|\lambda_n|)\left(i\lambda_n u_n(t,\omega) + A(\omega)u_n(t,\omega) + \gamma B(\omega)\int_{\mathbb{R}^d} p(\xi)\phi_n(t,\xi,\omega)d\xi\right) \tag{30}$$

and

$$|\lambda_n|^{1-\alpha}\psi(|\lambda_n|)\left(i\lambda_n\phi_n(t,\xi,\omega)+(|\xi|^2+\eta)\phi_n(t,\xi,\omega)-p(\xi)B^*(\omega)u_n(t,\omega)\right)=g_n. \tag{31}$$

Combining Eqs. (30) and (31), we obtain that

$$\phi_n=\frac{p(\xi)B^*(\omega)u_n}{i\lambda_n+|\xi|^2+\eta}+\frac{g_n}{C_n(i\lambda_n+|\xi|^2+\eta)},\quad \text{with } C_n=|\lambda_n|^{1-\alpha}\psi(|\lambda_n|) \tag{32}$$

and

$$\begin{aligned} f_n &= C_n\,(i\lambda_n u_n+A(\omega)u_n)+\gamma\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{i\lambda_n+|\xi|^2+\eta}B(\omega)B^*(\omega)u_n\\ &\quad+\gamma B(\omega)\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+|\xi|^2+\eta}. \end{aligned} \tag{33}$$

We multiply (32) by $|\xi|^{\frac{2-d}{2}}$

$$|\xi|^{\frac{2-d}{2}}\phi_n=\frac{|\xi|^{\frac{2-d}{2}}p(\xi)B^*(\omega)u_n}{i\lambda_n+|\xi|^2+\eta}+\frac{|\xi|^{\frac{2-d}{2}}g_n}{C_n(i\lambda_n+|\xi|^2+\eta)}.$$

□

Then

$$\begin{aligned}\left|\int_{\mathbb{R}^d}\frac{|\xi|^{\alpha+1-d}d\xi}{i\lambda_n+|\xi|^2+\eta}\right|\|B^*(\omega)u_n\|_H &\le \left(\int_{\mathbb{R}^d}\frac{|\xi|^{2-d}}{|\xi|^2+\eta}d\xi\right)^{\frac12}\left(\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi_n(t,\xi,\omega)\|_H^2 d\xi\right)^{\frac12}\\ &\quad+C_n^{-1}\left(\int_{\mathbb{R}^d}\frac{|\xi|^{2-d}}{|\lambda_n|^2+(\xi^2+\eta)^2}d\xi\right)^{\frac12}\|g_n\|_{\widetilde{V}}\end{aligned}$$

hence

$$\begin{aligned}(|\lambda_n|+\eta)^{\frac{\alpha-1}{2}}\|B^*(\omega)u_n\|_H &\le C\left(\int_{\mathbb{R}^d}\frac{|\xi|^{2-d}}{|\xi|^2+\eta}d\xi\right)^{\frac12}\left(\int_{\mathbb{R}^d}(|\xi|^2+\eta)\|\phi_n\|_H^2 d\xi\right)^{\frac12}\\ &\quad+C_n^{-1}(|\lambda_n|+\eta)^{\frac{-1}{2}}\|g_n\|_{\widetilde{V}}.\end{aligned}$$

This imply that

$$|\lambda_n|^{1-\alpha}\psi(|\lambda_n|)\|B^*(\omega)u_n\|_H^2\to 0,\quad \text{as } n\to\infty.$$

It is clear that $A_d(\omega)$ generate C_0-semigroup of contraction and $A_d(\omega)$ is stable in the sense of condition, then there exists unique $v_n\in D(A_d(\omega))$ such that

$$-\lambda_n v_n+iAv_n+iB(\omega)B^*(\omega)v_n=u_n, \tag{34}$$

and we have the following estimate

$$\|v_n\|_H\le C(\omega)\psi(|\lambda_n|)\|u_n\|_H \tag{35}$$

Taking the inner product in H of (34) with v_n, we get

$$-\lambda_n \|v_n\|_H^2 + i\|A^{\frac{1}{2}}(\omega)v_n\|_H + i\|B^*(\omega)v_n\|_H^2 = \langle u_n, v_n\rangle_H. \tag{36}$$

The imaginary part of (36), using Cauchy Schwarz inequality, we obtain

$$\|B^*(\omega)v_n\|_H^2 \leq \|u_n\|_H\|v_n\|_H \leq C(\omega)\psi(|\lambda_n|)\|u_n\|_H. \tag{37}$$

Taking the inner product of with $i\lambda_n v_n$, one has

$$\begin{aligned}\lambda_n\|u_n\|_H^2 &= i\lambda_n\langle B^*u_n, B^*v_n\rangle_H - i\lambda_n C_n^{-1}\langle f_n, v_n\rangle_H + i\lambda_n\gamma C_n^{-1}\left\langle B(\omega)\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+\xi^2+\eta}, v_n\right\rangle_H \\ &\quad + i\lambda_n\gamma C_n^{-1}\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{i\lambda_n+\xi^2+\eta}\left\langle B^*(\omega)u_n, B^*(\omega)v_n\right\rangle.\end{aligned}$$

Then

$$\begin{aligned}\|u_n\|_H^2 &= i\langle B^*u_n, B^*v_n\rangle_H - iC_n^{-1}\langle f_n, v_n\rangle_H + i\gamma C_n^{-1}\left\langle B(\omega)\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+\xi^2+\eta}, v_n\right\rangle_H \\ &\quad + i\gamma C_n^{-1}\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{i\lambda_n+\xi^2+\eta}\left\langle_H B^*(\omega)u_n, B^*(\omega)v_n\right\rangle_H.\end{aligned} \tag{38}$$

Using the estimates (35), (37) and Lemma 2.1, we get

$$\begin{aligned}|i\langle B^*u_n, B^*v_n\rangle_H| &\leq \|B^*u_n\|_H\|B^*v_n\|_H \\ &\leq \sqrt{\|u_n\|_H}\sqrt{C(\omega)\psi(|\lambda_n|)\|B^*(u_n)\|_H^2} \to 0, \quad \text{as } n\to\infty,\end{aligned} \tag{39}$$

$$\begin{aligned}|iC_n^{-1}\langle f_n, v_n\rangle_H| &\leq C_n^{-1}\|f_n\|_H\|v_n\|_H \\ &\leq C(\omega)\frac{\|f_n\|_H\|u_n\|_H}{|\lambda_n|^{1-\alpha}} \to 0, \quad \text{as } n\to\infty,\end{aligned} \tag{40}$$

$$\begin{aligned}\left|i\gamma C_n^{-1}\left\langle B(\omega)\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+\xi^2+\eta}, v_n\right\rangle_H\right| &= \left|i\gamma C_n^{-1}\left\langle\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+\xi^2+\eta}, B^*(\omega)v_n\right\rangle_H\right| \\ &\leq |\gamma|C_n^{-1}\left\|\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+\xi^2+\eta}\right\|_H\|B^*(\omega)v_n\|_H \\ &\leq |\gamma|C_n^{-1}\left(\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{\lambda_n^2+(|\xi|^2+\eta)^2}\right)^{\frac{1}{2}}\|g_n\|_{\widetilde{V}}\|B^*(\omega)v_n\|_H \\ &\leq |\gamma|C_n^{-1}\left(\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{\lambda_n^2+(|\xi|^2+\eta)^2}\right)^{\frac{1}{2}}\|g_n\|_{\widetilde{V}}\sqrt{\|u_n\|_H}\sqrt{C(\omega)\psi(|\lambda_n|)} \\ &\leq \frac{|\gamma|\sqrt{C(\omega)}\sqrt{\|u_n\|_H}}{|\lambda_n|^{1-\alpha}\sqrt{\psi(|\lambda_n|)}}\left(\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{\lambda_n^2+(|\xi|^2+\eta)^2}\right)^{\frac{1}{2}}\|g_n\|_{\widetilde{V}}\end{aligned}$$

so, there exists $C_* > 0$ such that

$$\left|i\gamma C_n^{-1}\left\langle B(\omega)\int_{\mathbb{R}^d}\frac{p(\xi)g_n d\xi}{i\lambda_n+\xi^2+\eta}, v_n\right\rangle_H\right| \le \frac{C_*|\gamma|\sqrt{C(\omega)}}{|\lambda_n|^{1-\alpha}\sqrt{\psi(|\lambda_n|)}(|\lambda_n|^2+\eta^2)^{1-\frac{\alpha}{2}}}\|g_n\|_{\widetilde{V}} \to 0, \quad \text{as } n\to\infty, \tag{41}$$

and there exists $\bar{C}_* > 0$ such that

$$\left|i\gamma C_n^{-1}\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{i\lambda_n+\xi^2+\eta}\langle B^*(\omega)u_n, B^*(\omega)v_n\rangle_H\right| \le \frac{\sqrt{C(\omega)\psi(|\lambda_n|)\|B^*(u_n)\|_H^2}}{|\lambda_n|^{1-\alpha}(|\lambda_n|^2+\eta^2)^{\frac{1-\alpha}{2}}\psi(|\lambda_n|)}$$

then

$$\left|i\gamma C_n^{-1}\int_{\mathbb{R}^d}\frac{p^2(\xi)d\xi}{i\lambda_n+\xi^2+\eta}\langle B^*(\omega)u_n, B^*(\omega)v_n\rangle_H\right| \to 0, \quad \text{as } n\to\infty. \tag{42}$$

From (39)–(42), we get

$$\|u_n\| \to 0, \quad \text{as } n\to\infty.$$

Hence by (39), we have

$$\|\phi_n\|_V \to 0, \quad \text{as } n\to\infty.$$

Therefore

$$(u_n, \phi_n) \to 0, \quad \text{as } n\to\infty,$$

which contradiction with (28).

5 Applications

5.1 Heat Equation with Fractional Integral Damping

In this part we study the well posedness of random heat equation with an internal fractional integral damping in a bounded domain G of $\mathbb{R}^d$ with smooth boundary $\Gamma = \partial G$. Let us consider the following random heat equation with fractional integral feedback

$$\begin{cases} u_t - \Delta u + a(x,\omega)I^{1-\alpha,\eta}u(x,t,\omega) = 0, \; (x,t)\in G\times(0,\infty), \quad \omega\in\Omega \\ u(x,t,\omega) = 0, \; \Gamma\times(0,\infty), \quad \omega\in\Omega \\ u(x,0,\omega) = u_0(\omega), \; \omega\in\Omega, \end{cases} \tag{43}$$

where $a : G \times \Omega \to \mathbb{R}_+$ be a positive random variable and $u_0 : \ \Omega \to L^2(G)$ be a random variable. System (43) may be recast into the augmented model

$$\begin{cases} u_t - \Delta u + \gamma\sqrt{a(x,\omega)}\int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi = 0, & x \in G, t > 0, \\ \partial_t \phi(t,\xi,\omega) + (|\xi|^2+\eta)\phi(\xi,t,\omega) = p(\xi)\sqrt{a(x,\omega)}u(x,t,\omega), & \xi \in \mathbb{R}^d, \\ u(x,t,\cdot) = 0, & x \in \Gamma,\ t > 0, \\ u(x,0,\omega) = u_0(\omega) \quad \phi(\xi,0,\omega) = 0,\ x \in G,\ \omega \in \Omega. \end{cases} \tag{44}$$

The operator $A_d = -\Delta$ is strict positive and auto-adjoint operator in $H = L^2(G)$, $D(A_d) = H_0^1(G)$. We shall use the semigroup method to demonstrate the global existence and uniqueness of solution, for this purpose we rewrite the system (44) as evolution equation for

$$\begin{cases} Y'(t,\omega) = \mathcal{A}(\omega)Y(t,\omega), \quad t > 0, \\ Y(0,\omega) = Y_0(\omega), \end{cases} \tag{45}$$

where $Y(\cdot,\omega) = \begin{pmatrix} u(\cdot,\cdot,\omega) \\ \phi(\xi,\cdot,\omega) \end{pmatrix}$, $\quad Y_0(\omega) = \begin{pmatrix} u_0(\cdot,\omega) \\ 0 \end{pmatrix}$, $\omega \in \Omega$, $\mathcal{A} : D(\mathcal{A}(\omega)) \subset \mathcal{H} \to \mathcal{H}$ defined by

$$\mathcal{A}\begin{pmatrix} u(\cdot,t,\omega) \\ \phi(\xi,t,\omega) \end{pmatrix} = \begin{pmatrix} -A_d u(\cdot,t,\omega) - \gamma\sqrt{a(\cdot,\omega)}\int_{\mathbb{R}^d} p(\xi)\phi(\xi,t,\omega)d\xi \\ -(|\xi|^2+\eta)\phi(t,\xi,\omega) + p(\xi)\sqrt{a(\cdot,\omega)}u(\cdot,t,\omega) \end{pmatrix} \tag{46}$$

with domain in the Hilbert space $\mathcal{H} = H_0^1(G) \times L^2(\mathbb{R}^d, H)$,

$$D(\mathcal{A}(\omega)) = \left\{ \begin{array}{l} \begin{pmatrix} u \\ \phi \end{pmatrix} \in \mathcal{H} : -A_d u(\cdot,t,\omega) - \gamma\sqrt{a(\cdot,\omega)}\int_{\mathbb{R}^d} p(\xi)\phi(\xi,t,\omega)d\xi \in H, \\ |\xi|\phi \in L^2(\mathbb{R}^d, H) - (|\xi|^2+\eta)\phi + p(\xi)\sqrt{a(\cdot,\omega)}u \in L^2(\mathbb{R}^d, H) \end{array} \right\}. \tag{47}$$

Theorem 5.1 *The operator $\mathcal{A}(\omega)$ defined in (45) and (47), generates C_0- semigroup of contraction $e^{t\mathcal{A}(\omega)}$ on $\mathcal{H}$.*

5.2 Integral Fractional Damped Random Wave Equation

For the second application let us consider the wave equation with internal damping. More precisely, let $G \subset \mathbb{R}^d$ with smooth boundary $\Gamma = \partial G$. We consider the following random wave equation with fractional integral damping

$$\begin{cases} \partial^2 u_t - \Delta u + a(x,\omega) I^{1-\alpha,\eta} u(x,t,\omega) = 0, & \omega \in \Omega, \quad (x,t) \in G \times (0,\infty), \\ u(x,t,\omega) = 0, & \Gamma \times [0,+\infty), \ \omega \in \Omega \\ u(\cdot,0,\cdot) = u_0(\cdot,\omega), \quad \partial_t u(x,0,\omega) = u_1(x,\omega) \ , & \omega \in \Omega, \quad x \in G. \end{cases} \tag{48}$$

where $a(x,.)$ is a random nonnegative function satisfying that there exist a nonempty subset $\mathcal{O}_a$ of G and strict positive constat a_0 such that

$$a(\omega,x) \geq a_0, \text{ a.e. } x \in \mathcal{O}_a, \quad \forall \omega \in \Omega \tag{49}$$

and $(u_0(\cdot,\omega); u_1(\cdot,\omega)) \in H_0^1(G) \times L^2(G)$. The system (48) can be written as

$$\begin{cases} \partial^2 u_t - \Delta u + \gamma\sqrt{a(x,\omega)} \int_{\mathbb{R}^d} p(\xi)\phi(t,\xi,\omega)d\xi = 0, & x \in G, \ t > 0, \\ \partial_t \phi(t,\xi,\omega) + (|\xi|^2 + \eta)\phi(t,\xi,\omega) = p(\xi)\sqrt{a(x,\omega)} u(t,x,\omega), & \xi \in \mathbb{R}^d \\ u(x,t,\omega) = 0, & \Gamma \times [0,+\infty) \\ u(\cdot,0,\cdot) = u_0(\cdot,\omega), \quad \partial_t u(x,0,\omega) = u_1(x,\omega) & \omega \in \Omega, \quad x \in G. \end{cases} \tag{50}$$

The operator $\mathcal{A}_*(\omega) : D(\mathcal{A}_*(\omega)) \subset L^2(G) \to L^2(G)$ corresponding to the Cauchy problem of the system (50) given by

$$\mathcal{A}_* \begin{pmatrix} u(\cdot,t,\omega) \\ v \\ \phi(\xi,t,\omega) \end{pmatrix} = \begin{pmatrix} v \\ -A_d u(\cdot,t,\omega) - \gamma\sqrt{a(\cdot,\omega)} \int_{\mathbb{R}^d} p(\xi)\phi(\xi,t,\omega)d\xi \\ -(|\xi|^2 + \eta)\phi(\xi,t,\omega) + p(\xi)\sqrt{a(\cdot,\omega)} u(\cdot,t,\omega) \end{pmatrix}$$

with domain in the Hilbert space $\mathcal{H}_* = H_0^1(G) \times H \times L^2(\mathbb{R}^d, H)$,

$$D(\mathcal{A}_*(\omega)) = \left\{ \begin{array}{l} \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} \in \mathcal{H}_* : v \in H_0^1(G), \ -A_d u(t,\cdot,\omega) - \gamma\sqrt{a(\cdot,\omega)} \int_{\mathbb{R}^d} p(\xi)\phi(\xi,t,\omega)d\xi \in H, \\ |\xi|\phi \in L^2(\mathbb{R}^d, H) - (|\xi|^2 + \eta)\phi + p(\xi)\sqrt{a(\cdot,\omega)} u \in L^2(\mathbb{R}^d, H) \end{array} \right\}.$$

Consequently, by using Theorem 3.2, we obtain the following result.

Theorem 5.2 *(1) If $U_0 \in D(\mathcal{A}_*(\omega))$, then system (50) has a unique strong random solution*

$$U(\omega) \in C^1(\mathbb{R}_+, \mathcal{H}_*) \cap C(\mathbb{R}_+, D(\mathcal{A}_*(\omega))), \quad \omega \in \Omega.$$

(2) If $U_0 \in \mathcal{H}_*$, then system (50) has a unique weak random solution

$$U(\omega) \in C(\mathbb{R}_+, \mathcal{H}_*), \quad \omega \in \Omega.$$

Now we have the following lemma.

Lemma 5.1 *Let $\eta > 0$ and $\lambda \in \mathbb{R}$ then for every $\omega \in \Omega$, the operator $(i\lambda I - \mathcal{A}_*(\omega))$ is bijective.*

Proof Let $(u, v, \phi) \in D(\mathcal{A}_*(\omega))$ such that

$$(i\lambda I - \mathcal{A}_*(\omega)) \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = 0,$$

then

$$\begin{cases} i\lambda u - v = 0 \\ i\lambda v - A_d u(\cdot, t, \omega) - \gamma\sqrt{a(\cdot, \omega)} \int_{\mathbb{R}^d} p(\xi)\phi(\xi, t, \omega)d\xi = 0 \\ i\lambda\phi + (|\xi|^2 + \eta)\phi(\xi, t, \omega) - p(\xi)\sqrt{a(\cdot, \omega)}u(\cdot, t, \omega) = 0. \end{cases}$$

Hence

$$i\lambda v = -\lambda^2 u, \quad \phi(\xi, t, \omega) = \frac{p(\xi)\sqrt{a(\cdot, \omega)}u(\cdot, t, \omega)}{i\lambda + |\xi|^2 + \eta}$$

and

$$-\lambda^2 u + \Delta u(\cdot, t, \omega) - \gamma a(\cdot, \omega)u \int_{\mathbb{R}^d} \frac{p^2(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi = 0 \tag{51}$$

Multiplying (51) by $\overline{u}$ and integrating over G, by Green's formula we obtain

$$-\lambda^2 \int_G u^2 dx - \int_G |\nabla u|^2 dx - \gamma \int_G a(\cdot, \omega)u^2 dx \int_{\mathbb{R}^d} \frac{p^2(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi = 0.$$

Then

$$\lambda^2 \int_G |\nabla u|^2 dx + \gamma \int_G a(\cdot, \omega)u^2 dx \int_{\mathbb{R}^d} \frac{(|\xi|^2 + \eta)p^2(\xi)}{\sqrt{\lambda^2 + (|\xi|^2 + \eta)^2}} d\xi = 0$$

and

$$\lambda \int_G u^2 dx + \lambda \int_G a(\cdot, \omega)u^2 dx \int_{\mathbb{R}^d} \frac{p^2(\xi)}{\sqrt{\lambda^2 + (|\xi|^2 + \eta)^2}} d\xi = 0.$$

This imply that $Ker(i\lambda I - \mathcal{A}_*(\omega)) = \{0\}$. So $i\lambda I - \mathcal{A}_*(\omega)$ is injective. Now we show that $i\lambda I - \mathcal{A}_*(\omega)$ is subjective. Let $(f, g, h) \in \mathcal{H}$, we solve the system of equations

$$\begin{cases} i\lambda u - v = f \\ i\lambda v + \Delta u(\cdot, t, \omega) - \gamma\sqrt{a(\cdot, \omega)} \int_{\mathbb{R}^d} p(\xi)\phi(\xi, t, \omega)d\xi = g \\ i\lambda\phi + (|\xi|^2 + \eta)\phi(\xi, t, \omega) - p(\xi)\sqrt{a(\cdot, \omega)}u(\cdot, t, \omega) = h. \end{cases} \tag{52}$$

$$-i\lambda u + \Delta u - \gamma a(\cdot, \omega)u \int_{\mathbb{R}^d} \frac{p^2(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi = g + f + \gamma\sqrt{a(\cdot, \omega)} \int_{\mathbb{R}^d} \frac{p(\xi)h}{i\lambda + |\xi|^2 + \eta} d\xi$$

Hence

$$i\lambda u - Au = -g - f - \gamma\sqrt{a(\cdot, \omega)} \int_{\mathbb{R}^d} \frac{p(\xi)h}{i\lambda + |\xi|^2 + \eta} d\xi \tag{53}$$

where

$$Au = -\Delta u + \gamma a(\cdot, \omega)u \int_{\mathbb{R}^d} \frac{p^2(\xi)}{i\lambda + |\xi|^2 + \eta} d\xi, \ u \in H_0^1(G).$$

By using Lax–Milgram's lemma we can easy to show that A is isomorphism from $H_0^1(G)$ onto $H^{-1}(G)$. Other hand form the compact embedding $H_0^1(G) \hookrightarrow L^2(G)$ and $L^2(G) \hookrightarrow H^{-1}(G)$. Then A^{-1} is a compact operator in $H_0^1(G)$. For $\lambda \in \mathbb{R}^*$ we thanks to Fredholm's alternative, the operator $(I - i\lambda A^{-1})$ is bijective in $H_0^1(G)$, thus the Eq. (53) has unique solution in $H_0^1(G)$. This implies that the operator $(i\lambda I - \mathcal{A}_*(\omega))$ is surjective. In the case $\lambda = 0$ we us Lax–Milgram's lemma the unique solution of the Eq. (53). □

For polynomial stability, we consider the initial boundary value problem

$$\begin{cases} \partial^2 u_t - \Delta u + a(x, \omega)u = 0, & (x, t) \in G \times (0, \infty), \ \omega \in \Omega, \\ u(x, t, \omega) = 0, & \Gamma \times [0, +\infty), \ \omega \in \Omega, \\ u(\cdot, 0, \cdot) = u_0(\cdot, \omega), \quad \partial_t u(x, 0, \omega) = u_1(x, \omega), \ \omega \in \Omega, \quad x \in G. \end{cases} \tag{54}$$

This problem enters into our previous framework, if we take $H = L^2(G)$ and the operator $\mathcal{A}_0(\omega) : D(\mathcal{A}(\omega)) \subset \mathcal{H} \to \mathcal{H}$ defined dy

$$\mathcal{A}_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -\Delta u - a(\cdot, \omega)u \end{pmatrix},$$

where $\mathcal{H} = H_0^1(G) \times H$ with domain

$$D(\mathcal{A}_0(\omega)) = \{(u, v) \in \mathcal{H} : \Delta u - a(\cdot, \omega)u \in H, \ v \in H_0^1(G)\}.$$

From lemma 5.1 we conclude that

$$i\mathbb{R} \subset \rho(\mathcal{A}_*(\omega)), \quad i\mathbb{R} \subset \rho(\mathcal{A}_0(\omega)) \quad \forall \omega \in \Omega. \tag{55}$$

For polynomial stabilization of the system (54) we assume a geometric constriction condition (GC) on $\mathcal{O}_a$ is satisfies (see [12]) or geometric control condition (GGC) is satisfied (see [7]).

Definition 5.1 The couple $(\mathcal{O}_a, T)$ satisfies (GCC) if every geodesic of G, travelling with speed 1 and issued at $t = 0$ enters the open set $\mathcal{O}_a$ before the time T.

(GC) There exist open sets $G_j \subset G$ with piecewise smooth boundary ∂G_j, and points $x_j^0 \in \mathbb{R}^n$, $j = 1, 2, \ldots, m$, such that

$$G_i \cap G_j = \emptyset \text{ for all } 1 \le i < j \le m,$$

and

$$G \cap \mu_\delta \left[(\bigcup_{j=1}^{m} \Gamma_j) \cup G \setminus \bigcup_{j=1}^{m} G_j \right] \subset \mathcal{O}_a \text{ for some } \delta > 0,$$

where

$$\mathcal{O}_a = \{x \in G : a(x) > 0\}$$

and

$$\mu_\delta(B) = \bigcup_{x \in B} \left\{ y \in \mathbb{R}^n : |x - y| < \delta \right\}, \text{ for } B \subset \mathbb{R}^n$$

$$\Gamma_j = \{x \in \partial G_j : (x - x_j^0)\nu_j > 0\},$$

where ν_j is the unit normal vector pointing into the exterior of G_j.

Under the deterministic case of initial data we have the following result.

Theorem 5.3 *Under the assumptions that* (49), (GC) *or* (GCC) *and for* $\eta > 0$ *the random operator* $\mathcal{A}_*(\cdot)$ *generates a contraction of semigroup satisfying*

$$\int_\Omega \|e^{\mathcal{A}_*(\omega)t} U\|_{\mathcal{H}_*} d\mathbb{P}(\omega) \le \frac{C}{t^{l-1}} \int_\Omega \|U\|_{D(\mathcal{A}_*(\omega))} d\mathbb{P}(\omega), \quad \forall\, U \in L^1(\Omega, D(\mathcal{A}_*(\omega)), \mathbb{P}) \textit{ for some } l \in \mathbb{R}_+.$$

Moreover the system (50) *is polynomial stable.*

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