Geometric Numerical Methods with Lie Groups

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Abstract. Due to the increasing demands for modeling large-scale and complex systems, designing optimal controls, and conducting optimization tasks, many real-world applications require sophisticated models. Geometric methods are designed to capture the underlying structure of the system at hand and to preserve the global qualitative or geometric properties of the flow, such as symplecticity, volume preservation and symmetry. A survey on three of such structure preserving numerical methods is proposed in the present article. Testing the validity of such simulations is achieved by exhibiting analytically solvable models and comparing the result of simulations with their exact behavior.

Keywords: Geometric integrators · Structure preserving numerical methods · Variational methods

1 Introduction

It is today well established that geometrical methods connected to powerful numerical tools (i.e. Runge-Kutta, Butcher series) can be applied to equations on the Lie algebra to design high order methods and determine their numerical convergence. Beyond these structure preservations, approaches for integration algorithms based on variational principles give a unified treatment of many symplectic numerical schemes. In this context, the Noether theorem [10] allows for a numerical formulation that preserves symmetries and conservation laws.

In the case of homogeneous spaces (smooth manifold on which a Lie group acts transitively), the so-called Lie group integrators, comprising Runge-Kutta-Munthe-Kaas [12] methods is presented shortly in Sect. 2. The main preoccupation is to ensure that discrete solutions are guaranteed to stay on the given manifold. However in this case no particular preservation of symmetries is obtained without further constraints. This is why variational methods are revisited in Sect. 3 to be compared to Lie-Poisson Hamilton-Jacobi algorithm based on generating function for which higher order designs are also available (Sect. 4).

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2 Lie Group Integrators, Runge-Kutta-Munthe-Kaas Methods

The Runge-Kutta-Munhe-Kaas methods (RKMK) developed in a serie of articles [12], are an example of Lie group methods. Let $Y(t)$ be a curve in a matrix Lie group G verifying

$$\dot{Y} = A(t, Y)\, Y, \qquad Y(0) = Y_0 \tag{1}$$

where $A(t, Y) \in \mathfrak{g}$ for all $t, Y \in \mathbb{R} \times G$. The starting point is to describe the solution of (1) as $Y(t) = \exp(\Omega(t))Y_0$ and to deduce an ODE on Ω. Computing the derivative of Y we get

$$\dot{Y}(t) = \frac{d}{dt}\exp(\Omega(t))Y_0 = \mathrm{dexp}_{\Omega(t)}(\dot{\Omega}(t))\, Y_0 = \mathrm{d}^{\mathrm{R}}\mathrm{exp}_{\Omega(t)}(\dot{\Omega}(t))\, Y(t),$$

where the right trivialized derivative $\mathrm{d}^{\mathrm{R}}\mathrm{exp}_{\Omega} := \mathrm{d}R_{\exp(\Omega)^{-1}} \circ \mathrm{dexp}_{\Omega}$ is introduced. Using this expression in (1) and inverting[1] $\mathrm{d}^{\mathrm{R}}\mathrm{exp}_{\Omega}$ a differential equation is obtained for Ω lying on the Lie algebra $\mathfrak{g}$

$$\dot{\Omega}(t) = \mathrm{d}^{\mathrm{R}}\mathrm{exp}^{-1}_{\Omega(t)}\left(A(t, Y(t))\right), \qquad \Omega(0) = 0. \tag{2}$$

The advantage is that the non linear invariants defining the Lie group become linear invariants on the Lie algebra, and will be preserved by any numerical method [7]. This ensures that the solution stays on the Lie group.

The idea behind RKMK methods is to approximate the solution Y of Eq. (1) with a discrete solution (Y_n) by approximately solving Eq. (2) with a general Runge-Kutta method $\dot{\Omega} = f(\Omega)$ with $f = \mathrm{d}^{\mathrm{R}}\mathrm{exp}^{-1}_{\Omega}$ and updating the position via the exponential map. Knowing that $\mathrm{d}^{\mathrm{R}}\mathrm{exp}^{-1}_{\Omega}(\Theta) = \Theta + \sum_{k=1}^{\infty} \frac{B_k}{k!}\,\mathrm{ad}^k_{\Omega}(\Theta)$ where B_k are Bernouilli numbers, a truncated sum up to order q is used in Eq. (2). If the Runke-Kutta method is order p and the truncature order is such that $q \geq p - 2$, then the associated RKMK method is order p [12].

Application to the Rigid Body Problem. We consider here the free rigid body problem. Let $\pi \in \mathfrak{so}(3)^* \approx \mathbb{R}^3$ be the angular momentum in the body frame and $\mathbb{J} = \mathrm{diag}(J_1, J_2, J_3)$ the inertia tensor, it verifies the Euler-Poincaré equation $\dot{\pi} = \pi \wedge \xi$, $\pi(0) = \pi_0$ where $\xi = \mathbb{J}^{-1}\pi \in \mathfrak{so}(3)$ and π_0 is the initial angular momentum. In terms of matrix product, this yields

$$\dot{\pi} = \begin{bmatrix} 0 & \frac{\pi_3}{J_3} & -\frac{\pi_2}{J_2} \\ -\frac{\pi_3}{J_3} & 0 & \frac{\pi_1}{J_1} \\ \frac{\pi_2}{J_2} & -\frac{\pi_1}{J_1} & 0 \end{bmatrix} \pi, \qquad \pi(0) = \pi_0. \tag{3}$$

This is in the form of Eq. (1), hence π can be approximately solved using a RKMK method where $SO(3)$ is the acting group. The Lie group $SO(3)$ leaves

[1] Here we made the assumption that $\mathrm{d}^{\mathrm{R}}\mathrm{exp}_{\Omega} : \mathfrak{g} \to \mathfrak{g}$ is invertible, which is the case for SO(3) whenever $\|\Omega\| < \pi$.

the vector space $\mathfrak{so}(3)^*$ invariant, reflecting the conservation of $\|\pi(t)\|$ in time. Applying a Lie group method guarantees the preservation of that constraint, ensuring that the angular momentum $\pi(t)$ lies on the sphere of radius $\|\pi_0\|$ for all t.

Defining Ω with $\pi(t) = \exp(\Omega(t))\pi_0$, the following expression is obtained for Eq. (2)

$$\dot{\Omega} = \sum_{k=0}^{\infty} \frac{B_k}{k!} \operatorname{ad}_{\Omega}^{k}(\mathbb{J}^{-1} \exp(\Omega)\pi_0), \quad \Omega(0) = 0. \tag{4}$$

We build an order 4 RKMK method by truncating the sum (4) up to order 2 and a applying a classical order 4 RK method. The results shown in Fig. 1, outputting the expected behaviour, have been computed for the following parameters:

$$\mathbb{J} = \operatorname{diag}\left(2/3, 1, 2\right), \qquad \pi_0 = \left[\cos\left(\tfrac{\pi}{3}\right)\ 0\ \sin\left(\tfrac{\pi}{3}\right)\right]^T, \qquad h = 0.5s, \qquad N = 200.$$

3 Covariant Variational Methods

Here we build a covariant variational method based on the Hamilton principle associated to a discrete Lagrangian following a similar approach to [5]. We take the case where the configuration space of the system is a Lie group G together with a reduced Lagrangian $\ell : \mathfrak{g} \to \mathbb{R}$.

Let a time step h divide equally the time interval, the set of discrete paths is defined by $\mathcal{C}_d(G) = \left\{ g_d : \{t_k\}_{0\leq k\leq N} \to G \right\}$ where $\forall k,\ t_k = kh$. To determine an approximate trajectory $g_d \in \mathcal{C}_d(G)$ such that $g_k := g_d(k) \approx g(t_k)$, we define a discrete reduced Lagrangian ℓ_d approximating the action

$$\ell_d(\xi_0) \approx \int_{t_0}^{t_1} \ell(\xi)\, \mathrm{d}t$$

where $\xi(0) = \xi_0$ and $\xi = g^{-1}\dot{g}$ such that g is an action extremum on $[t_0, t_1]$. To discretize the relation $\xi = g^{-1}\dot{g}$ we introduce a local diffeomorphism $\tau : \mathfrak{g} \to G$ defined on an open set containing the identity and such that $\tau(0) = e_G$ (the exponential map is an example of such a diffeomorphism). Starting from the reconstruction formula

$$g_{i+1} = g_i \tau(h\,\xi_i), \tag{5}$$

we define $\xi_i := \frac{1}{h}\tau^{-1}(g_i^{-1} g_{i+1})$

The discrete action is approximated from the classical action by the sum $S_d(g_d) = \sum_{i=0}^{N-1} \ell_d(\xi_i)$. Applying the Hamilton principle on S_d evaluated on a discrete path g_d yields $\delta S_d(g_d) = \sum_{i=0}^{N-1} \left\langle \frac{\partial \ell_d}{\partial \xi}(\xi_i), \delta\xi_i \right\rangle$. The variation $\delta\xi_i$ is expressed using (5) as

$$\begin{aligned}
\delta\xi_i &= \frac{1}{h}\, \mathrm{d}\tau^{-1}_{g_i^{-1} g_{i+1}} \left(-g_i^{-1}\delta g_i g_i^{-1} g_{i+1} + g_i^{-1}\delta g_{i+1} \right) \\
&= \frac{1}{h}\, \mathrm{d}\tau^{-1}_{\tau(h\xi_i)} \left(\left(-\zeta_i + \operatorname{Ad}_{\tau(h\xi_i)} \zeta_{i+1} \right) \tau(h\xi_i) \right)
\end{aligned}$$

where $\zeta_i = g_i^{-1}\delta g_i$. Here the right trivialized differential $\mathrm{d}^{\mathrm{R}}\tau^{-1} : \mathfrak{g} \to \mathfrak{g}$ defined by $\mathrm{d}^{\mathrm{R}}\tau_\xi^{-1} := T_{\tau(\xi)}\tau^{-1} \circ TR_{\tau(\xi)}$ is introduced, allowing us to write

$$\delta\xi_i = \frac{1}{h}\,\mathrm{d}^{\mathrm{R}}\tau_{h\xi_i}^{-1}\left(-\zeta_i + \mathrm{Ad}_{\tau(h\xi_i)}\,\zeta_{i+1}\right)$$

Using the definition of the adjoint $\langle \pi, A\xi\rangle = \langle A^*\pi, \xi\rangle$ where $\pi \in \mathfrak{g}^*$ and $\xi \in \mathfrak{g}$, the variation of the action functional now reads

$$\delta S_d(g_d) = \sum_{i=0}^{N-1}\left\langle \frac{1}{h}\left(\mathrm{d}^{\mathrm{R}}\tau_{h\xi_i}^{-1}\right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_i), \mathrm{Ad}_{\tau(h\xi_i)}\,\zeta_{i+1} - \zeta_i\right\rangle.$$

Introducting the momentum μ_i associated to ξ_i via the formula

$$\mu_i := \left(\mathrm{d}^{\mathrm{R}}\tau_{h\xi_i}^{-1}\right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_i) \tag{6}$$

and changing the indexes in the sum (discrete integration by part), we finally get, by the independence of ζ_i for all $i \in \{1, \ldots, N-1\}$, the discrete Euler-Poincaré equations

$$\mu_i - \mathrm{Ad}^*_{\tau(h\xi_{i-1})}\,\mu_{i-1} = 0. \tag{7}$$

This allows us to define the general formulation of a covariant method in Algorithm 1 for given boundary conditions g_0 et ξ_0. The momentum μ_i is computed from (7), and the associated $\xi_i \in \mathfrak{g}$ is then deduced from (6). This equation being implicit, it is typically solved using a numerical solver such as a Newton method. Finally, the position is updated via the reconstruction formula (5).

Algorithm 1. General implementation of the covariant variational method.

Data: g_0, ξ_0
$g_1 = g_0\tau\left(h\xi_0\right), \quad \mu_0 = h\left(\mathrm{d}^{\mathrm{R}}\tau_{h\xi_0}^{-1}\right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_0)$
for $i = 1$ **to** $N-1$ **do**
  Compute $\mu_i = \mathrm{Ad}^*_{\tau(h\xi_{i-1})}\,\mu_{i-1}$ (equation (7))
  Find ξ_i **solution of** $\left(\mathrm{d}^{\mathrm{R}}\tau_{h\xi_i}^{-1}\right)^* \frac{\partial \ell_d}{\partial \xi}(\xi_i) - h\mu_i = 0$ (equation (6))
  Update $g_{i+1} = g_i\tau\left(h\xi_i\right)$ (equation (5))
end

Application to the Rigid Body Problem. A rigid body is represented by an element of the rotation group $SO(3)$. The reduced Lagrangian for this system is defined for $\xi \in \mathfrak{so}(3)$ as the rotation kinetic energy $\ell(\xi) := 1/2\,\langle \mathbb{J}\xi, \xi\rangle$. We chose to approximate this Lagrangian with ℓ_d defined by $\ell_d(\xi_0) := h\ell(\xi_0) = \frac{h}{2}\langle \mathbb{J}\xi_0, \xi_0\rangle$ and choose the local diffeomorphism τ to be defined as the Cayley map $\tau := \mathrm{cay} : \mathfrak{so}(3) \to SO(3)$ (details can be found in [4]).

The results of the application of Algorithm 1 for the parameters given in Sect. 2 are also plotted on Fig. 1 for comparison.

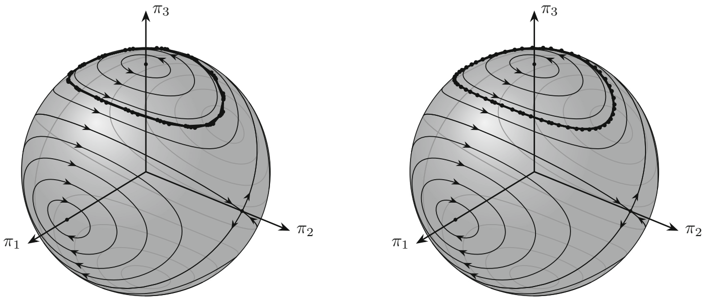

Fig. 1. Numerical angular momentum of the rigid body problem computed for RKMK4 (left) and covariant method (right); both exactly lie one the sphere. Exact solutions have been plotted for comparison.

4 Methods Based on Generating Functions, Hamilton-Jacobi Equation

4.1 Classical Case

The Hamilton Jacobi equation plays an important role in the development of numerical integrators that preserve the symplectic structure. In this section Hamilton-Jacobi theory is approached from the point of view of extended phase space as it is presented by Marsden [11] (p 206) and Arnold [1] (chapter 9). A link between Hamilton-Jacobi integrators and variational integrators could also be find in ([10]).

By definition, canonical transformations preserve the (pre)-symplectic 2-form, which can be deduced from the differential of the Poincaré-Cartan form. Let us consider a canonical transformation in the extended phase space $(t, q, p) \mapsto (T, Q, P)$ depicted in Fig. 2. Let (t, q, p) be coordinate functions in some chart of extended phase space considered as a manifold M. The Poincaré-Cartan form $\theta = p\,\mathrm{d}q - H\,\mathrm{d}t$ is a differential 1-form on M for which $H(t, q, p)$ is a Hamiltonian function. The coordinates (t, Q, P) can be considered as giving another chart on M associated to the 1-form $\Theta = P\,\mathrm{d}Q - K\,\mathrm{d}t$ with a corresponding Hamiltonian function $K(T, Q, P)$.

As it is well-know, it is possible to find four[2] generating functions depending of all mixes of old and new variables: (q, Q), (q, P), (p, Q), or (p, P). It appears that the second kind (q, P) of generating function is easily used to generate an infinitesimal transformation closed to the identity. And in turn, defines, by construction, a structure preserving numerical method. The mixed coordinates system (t, q, P) may be related to the previous ones through two mappings h and f: such that

$$h : (t, q, P) \mapsto p(t, q, P) \quad \text{and} \quad f : (t, q, P) \mapsto Q(t, q, P)$$

[2] At least four since many generating functions can be constructed.

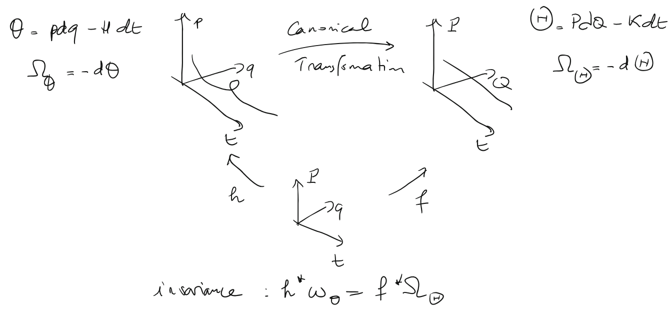

Fig. 2. Canonical transformation $(t, q, p) \mapsto (T, Q, P)$. Independent variables (q, P) are used to construct the second kind of generating function $G(t, q, P)$.

If each the (pre-)symplectic forms $\omega_\theta = -\,\mathrm{d}\theta$ and $\Omega_\Theta = -\,\mathrm{d}\Theta$ are invariantly associated to one another, their pull-back should agree: $h^*\omega_\theta = f^*\Omega_\Theta$. Since the operator (d) and (*) commute, that means $\mathrm{d}(h^*\theta) = \mathrm{d}(f^*\Theta)$. Consequently, $h^*\theta$ and $f^*\Theta$ differ from a closed form $dS = h^*\theta - f^*\Theta$ which is

$$dS(t, q, P) = h^* \left(p\,\mathrm{d}q - H\,\mathrm{d}t\right) - f^* \left(P\,\mathrm{d}Q - K\,\mathrm{d}T\right).$$

Replacing $P\,\mathrm{d}Q = \mathrm{d}(QP) - Q\,\mathrm{d}P$ and introducing $G = (f^*Q)P + S$, one computes

$$\frac{\partial G}{\partial t}\,\mathrm{d}t + \frac{\partial G}{\partial q}\,\mathrm{d}q + \frac{\partial G}{\partial P}\,\mathrm{d}P = h^* \left(p\,\mathrm{d}q - H\,\mathrm{d}t\right) - f^* \left(Q\,\mathrm{d}P - K\,\mathrm{d}T\right)$$

and obtains

$$\begin{cases} f^*K = h^*H + \frac{\partial G}{\partial t} \\ f^*Q = \frac{\partial G}{\partial P} \\ h^*p = \frac{\partial G}{\partial q} \end{cases} \mapsto \begin{cases} K(t, Q(t, q, P), P) = H(t, q, p(t, q, P)) + \frac{\partial G}{\partial t} \\ Q(t, q, P) = \frac{\partial G}{\partial P} \\ p(t, q, P) = \frac{\partial G}{\partial q} \end{cases} \tag{8}$$

Now suppose that $G(t, q, P)$ satisfies the so-called Hamilton-Jacobi equation,

$$H(t, q, \frac{\partial G}{\partial q}), + \frac{\partial G}{\partial t} = 0 \tag{9}$$

for a given time dependent Hamiltonian H. This equation is obtained by taking $K \equiv 0$ in (8-a). The generating function G generates a time dependent canonical transformation ψ that transforms the Hamiltonian vector fields X_H to equilibrium: $\psi_* X_H = X_{K=0}$. That means that the integral curves of X_K are represented by straight lines in the image space. The vector field has been "integrated" by the transformation (see Fig. 2).

The choice of the second kind of generating function is convenient to easily generate the identity transformation. Choosing $G = qP$ in (8b) and (8c) reads $Q = \frac{\partial G}{\partial P} = q$ and $p = \frac{\partial G}{\partial q} = P$. So, a canonical (infinitesimal) transformation is obtained by plugging the ansatz

$$G(t,q,P) = qP + \sum_{m=1}^{\infty} \frac{t^m}{m!} G_m(q,P) = qP + tG_1(q,P) + \frac{t^2}{2} G_2(q,P) + \dots \quad (10)$$

into the Hamilton-Jacobi Eq. (9). Equating coefficients of equal powers of t gives

$$G_1 = -H(t,q,P), \quad G_2 = -\frac{\partial H}{\partial p}\frac{\partial G_1}{\partial q}, \quad G_3 = -\frac{\partial H}{\partial p}\frac{\partial G_2}{\partial q} - \frac{\partial^2 H}{\partial p^2}\frac{\partial G_1}{\partial q} \quad G_4 = \dots$$

A numerical method of the order k is obtained by truncating the serie (10) to a certain order k (see also [2]). The remaining variables (p,Q) are computed using the generating function G in (8b) and (8c): $Q = \frac{\partial G}{\partial P}$ and $p = \frac{\partial G}{\partial q}$. Putting (q,p) in the left-hand size, the numerical algorithm is finally

$$\begin{cases} q = Q - \sum_{m=1}^{k} \frac{t^m}{m!} \frac{\partial G_m}{\partial P}(q,P) \\ p = P + \sum_{m=1}^{k} \frac{t^m}{m!} \frac{\partial G_m}{\partial q}(q,P) \end{cases}$$

As it can be seen, the first step may be implicit for the variable q. But when it is solved, the second step is explicit for p. The symplectic Euler method is an example of such methods of order 1 with $G_1 = -H(q,P)$ given by

$$\begin{cases} q = Q + t\frac{\partial H}{\partial P}(q,P) \\ p = P - t\frac{\partial H}{\partial q}(q,P) \end{cases}$$

for which the "discrete Hamiltonian structure" is easily recognizable.

4.2 Lie-Poisson Hamilton-Jacobi Integrators

Following the same approach as the preceding section, the Hamilton-Jacobi theory is reduced from T^*G to $\mathfrak{g}^*$, the dual Lie algebra. Let (t, q_0, π_0) be coordinate functions in some chart of extended phase space considered as a manifold $M = \mathbb{R} \times G \times \mathfrak{g}^*$ (see Fig. 3). The 1-form

$$\theta = \pi_0 \lambda_{q_0} - H\,\mathrm{d}t$$

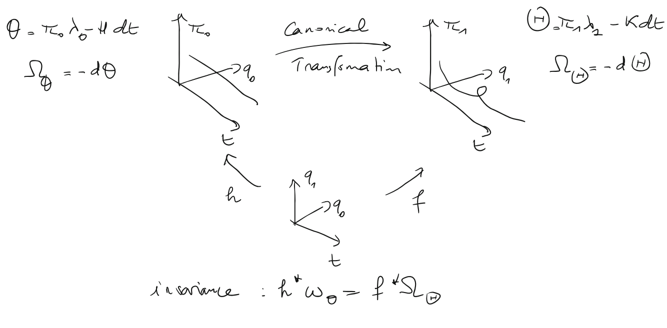

Fig. 3. Canonical transformation using the dual Lie algebra $(t, q_0, \pi_0) \mapsto (t, q_1, \pi_1)$. Independent variables (q_1, π_1) are used to construct the first kind of generating function $S(t, q_1, \pi_1)$.

is the reduced Poincaré-Cartan for which the Maurer-Cartan form is defined by $\lambda_{q_0}(v) = (L_{q_0^{-1}})_*(v)$. The coordinates (t, q_1, π_1) can be considered as giving another chart in M associated to the 1-form $\Theta = \pi_1 \lambda_{q_1} - K\,\mathrm{d}t$ with $\lambda_{q_1}(v) = (L_{q_1^{-1}})_*(v)$. The mixed coordinates system (t, q_0, q_1) may be related to the previous ones through two mappings $h : (t, q_0, q_1) \mapsto \pi_0(t, q_0, q_1)$ and $f(t, q_0, q_1) \mapsto \pi_1(t, q_0, q_1)$.

For the left invariant system, the Hamiltonian function is left invariant. It is then natural to seek for left invariant generating functions satisfying $S_t(q_0, q_1) = S_t(hq_0, hq_1), \forall h \in G$. Choosing $h = q_0^{-1}$ we can construct a left invariant function $\bar{S}_t$ given by

$$S_t(q_0, q) = S_t(e, q_0^{-1} q) = S_t(e, g) = \bar{S}_t(g), \quad g = q_0^{-1} q_1.$$

The invariance of the (pre-)symplectic forms $\omega_\theta = -\,\mathrm{d}\theta$ and $\Omega_\Theta = -\,\mathrm{d}\Theta$ gives now rise to a function $\bar{S}_t(g)$ such that

$$\mathrm{d}\bar{S}_t = f^*\Theta - h^*\theta = f^*\left(\pi\lambda_{q_1} - K\,\mathrm{d}t\right) - h^*\left(\pi_0\lambda_{q_0} - H\,\mathrm{d}t\right) \tag{11}$$

So computing $\mathrm{d}\bar{S}_t = \frac{\partial \bar{S}_t}{\partial t}\,\mathrm{d}t + \frac{\partial \bar{S}_t}{\partial g}\,\mathrm{d}g$, it appears that $\mathrm{d}g$ must also be computed in term of λ_{q_0} and λ_q,

$$\begin{aligned} \mathrm{d}g = \mathrm{d}(q_0^{-1} q_1) &= \mathrm{d}q_0^{-1} q_1 + q_0^{-1}\,\mathrm{d}q_1 = -q_0^{-1}\,\mathrm{d}q_0 q_0^{-1} q_1 + q_0^{-1} q_1 q_1^{-1}\,\mathrm{d}q_1 \\ &= -\lambda_{q_0} g + g\lambda_{q_1} = -(R_g)_*\lambda_{q_0} + (L_g)_*\lambda_{q_1}. \end{aligned}$$

So, comparing the expression $\mathrm{d}\bar{S}_t = \frac{\partial \bar{S}_t}{\partial t}\,\mathrm{d}t - \frac{\partial \bar{S}_t}{\partial g}(R_g)_*\lambda_{q_0} + \frac{\partial \bar{S}_t}{\partial g}(L_g)_*\lambda_{q_1}$ with (11), one obtains

$$\begin{cases} h^*H = f^*K + \frac{\partial \bar{S}_t}{\partial t} \\ f^*\pi_1 = (L_g)^*\frac{\partial \bar{S}_t}{\partial g} \\ h^*\pi_0 = (R_g)^*\frac{\partial \bar{S}_t}{\partial g} \end{cases} \mapsto \begin{cases} H(t, \pi_0(t, g)) = K(t, \pi_1(t, g)) + \frac{\partial \bar{S}_t}{\partial t} \\ \pi_1(t, g) = (L_g)^*\frac{\partial \bar{S}_t}{\partial g} \\ \pi_0(t, g) = (R_g)^*\frac{\partial \bar{S}_t}{\partial g} \end{cases} \tag{12}$$

For $H \equiv 0$, this yields the Lie-Poisson Hamilton-Jacobi equation

$$K\left(t, (L_g)^*\frac{\partial \bar{S}_t}{\partial g}\right) + \frac{\partial \bar{S}_t}{\partial t} = 0, \quad g = q_0^{-1} q_1 \tag{13}$$

So Eq. (12c)

$$\pi_0(t, g) = (R_g)^*\frac{\partial \bar{S}_t}{\partial g} \tag{14}$$

plugged into Eq. (12b) gives

$$\pi_1(t, g) = Ad_g^*\pi_0(t, g) \tag{15}$$

Marsden [6,9], Li [8] and de Degio [3] obtained a slightly different result using the convention $g = q_1^{-1} q_0$. Nevertheless, one can obtain a Lie-Poisson integrator by approximately solving the Lie-Poisson Hamilton-Jacobi Eq. (13) and then using (14) and (15) to generate the algorithm. This last Eq. (15) manifestly preserves the co-adjoint orbit $\mathcal{O}_{\pi_0} = \{\pi \in \mathfrak{g}^* | \pi = Ad_g^* \pi_0, \forall g \in G\}$. As in the classical case, one can generate algorithms of arbitrary accuracy by approximating the generative function by an ansatz such as the one given by (10), i.e $\bar{S}_t(g) = S_0(g) + \sum_{m=1}^{\infty} \frac{t^m}{m!} S_m(g)$ The main difficulty is to determine S_0 that can generate the identity map. Marsden propose to use in [6] the function $S_0 = trace(Ad_g^*)$ and astoundingly, de Diego [3], approximating the solution by taking the Taylor series in t of S up to order k, mention $S_0 = 0$.

Li [8] propose to reformulate the above theory of a generating function on $TG*$ by the exponential mapping in terms of algebra variable. For $g \in G$, choose $\xi \in \mathfrak{g}$ so that $g = \exp \xi$. He use Channel and Scovel's [2] results for which $S_0 = (\xi, \xi)/2)$.

4.3 Conclusions and Future Research

In our case, our perspective is to relate the Lie-Poisson Hamilton-Jacobi algorithm to the Euler-Poincaré algorithm developed in Sect. 3 based on the Cayley map. In particular, since Eqs. (15) and (7) are the same in both algorithm, it will be instructive to compare the approximation of the Lie-Poisson Hamilton-Jacobi Eq. (13) to the relationship between μ and ξ given by Eq. (6).

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