Intrinsic Incremental Mechanics

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Abstract. We produce a coordinate free presentation of some concepts usually involved in incremental mechanics (tangent linear stiffness matrix, stability, loading paths for example) but not always well founded. Thanks to the geometric language of vector bundles, a well defined geometrical object may be associated to each of these tools that allows us to understand some latent difficulties linked with these tools due to the absence of a natural connection and also to extend some of our recent results of linear stability to a non linear framework.

Keywords: Vector bundles · Transversality · Vertical derivative

1 Motivations

1.1 Kinematic Structural Stability

For the last ten years, we developed tools to tackle an old question regarding the conflict between two criteria of stability involved in rate-independent mechanical systems. We call these two criteria the divergence Lyapounov criterion (it is the usual one) and the Hill criterion also called the second order work criterion. These two criteria are identical for elastic conservative or for piece-wise rate-independent mechanical systems but they give different critical divergence stability values for elastic non conservative systems or for non linear rate-independent mechanical systems like non associate plastic materials. The usual language in mechanics characterizes these last systems by a non symmetric tangent stiffness matrix $K(p)$ whereas for the first class of system $K(p)$ is symmetric.

Thanks to the new concept of Kinematic Structural Stability (KISS) and an original variational formulation on all the possible kinematic constraints, we proved, in the discrete linear elastic nonconservative framework, that the two criteria become again equivalent ([4] for example). The most elegant proof of this result involves the geometric concept of compression of operator which can be extended to Hilbert spaces and which allowed us to extend the result to continuous linear elastic systems [6]: all the compressions of the operator are one-to-one if and only if the symmetric part of the operator is definite.

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1.2 Geometric Degree of Nonconservativity

In parallel to these stability issues, we also investigated the dual problem which questions the minimal number of kinematic constraints necessary to make conservative the elastic mechanical system. In a linear discrete framework, this number called the geometric degree of nonconservativity (GDNC) is the half of the rank of the skew symmetric part $K_a(p)$ of $K(p)$ [3]. The extension of the GDNC to infinite dimension Hilbert space involved for continuous systems is not obvious whereas the extension to the differentiable non linear framework is possible. Indeed, the skew-symmetric part $K_a(p)$ must be replaced by the exterior derivative $\mathbf{d}\omega_{\mathcal{F}}$ of the 1-form $\omega_{\mathcal{F}}$ defining the corresponding force system on the mechanical system. $\omega_{\mathcal{F}}$ is a section of the cotangent bundle $T^*\mathbb{M}$ of the configuration manifold $\mathbb{M}$ and the GDNC is then the half of the class of the 2-form $\mathbf{d}\omega_{\mathcal{F}}$ [5].

1.3 Main Issue

The problem investigated in this paper is to provide such a non linear extension but for the original KISS issue. Whereas the exterior derivative $\mathbf{d}\omega_{\mathcal{F}}$ provides a "natural" non linear extension of the skew symmetric part $K_a(p)$ of $K(p)$, it appears that there is no such natural extension for $K(p)$ nor for its symmetric part $K_s(p)$.

Indeed the incremental point of view necessitates to make a derivative of $\omega_{\mathcal{F}}$. However there is no natural connection on $\mathbb{M}$ to do it. To solve this problem, we will use the fact that the incremental quasi-static evolution of the mechanical system lies on the nil section of $T^*\mathbb{M}$ (which represents the equilibrium manifold) and we will use this canonical and global section of $T^*\mathbb{M}$ as a horizontal space for the derivative of $\omega_{\mathcal{F}}$. It allows for example to provide an intrinsic meaning of the common concept of tangent stiffness matrix of a system. We have to stress that we stay here within the differentiable framework which means that only hypoelasticity and not plasticity is investigated even if it is the long-term goal of these investigations. We also have to stress that the tools used in these investigations are usual (see [2] or [1] for example) in classical mechanics for so-called Lagrangian or Hamiltonian mechanics or even multisymplectic mechanics. However, here, by principle we do not suppose an Hamiltonian or Lagrangian functions to describe the evolution of the mechanical systems.

2 Some Results

We now present three intrinsic objects or results that geometrically extend more or less usual concepts of the linear framework to the not linear case. A large part of these developments are in [7]. In all these developments, the mechanical system is called Σ and is described by a finite number n of parameters which means that the configuration space is a n dimension manifold $\mathbb{M}$. With this language, any system of forces is represented by a section of the cotangent bundle

$T^*\mathbb{M}$. As usual, we sometimes identify the nil section of $T^*\mathbb{M}$ with $\mathbb{M}$ itself ($0_{T^*\mathbb{M}}(\mathbb{M}) \simeq \mathbb{M}$). We also note $\pi : T^*\mathbb{M} \to \mathbb{M}$ the natural projection so that $T_m^*\mathbb{M} = \pi^{-1}\{m\} \ \forall m \in \mathbb{M}$.

2.1 Tangent Stiffness Operator

Let $\mathcal{F}$ be a force system described by a section $\omega_{\mathcal{F}}$ of $T^*\mathbb{M}$ and $m_e \in \mathbb{M}$ an equilibrium configuration of Σ subjected to $\mathcal{F}$. Then, considering the derivative $d\omega_{\mathcal{F}}$ of $\omega_{\mathcal{F}}$ (and NOT the exterior derivative $\mathbf{d}\omega_{\mathcal{F}}$ as above), we have

$$\begin{aligned} d\omega_{\mathcal{F}}(m_e) : T_{m_e}\mathbb{M} \to T_{(m_e,0)}T^*\mathbb{M} = T_{x_e}\mathbb{M} \oplus \pi_{m_e}^{-1} \\ u \mapsto u + d\omega_{\mathcal{F}}(m_e)^{ver}(m_e)(u) \end{aligned} \tag{1}$$

where $d\omega_{\mathcal{F}}(m_e)^{ver}(m_e)$ is a linear map and then belongs to $\mathcal{L}(T_{m_e}\mathbb{M}, T_{m_e}^*\mathbb{M})$. Then, we are led to put the

Definition 1. *The above linear map $d\omega_{\mathcal{F}}^{ver}(m_e) \in \mathcal{L}(T_{m_e}\mathbb{M}, T_{m_e}^*\mathbb{M})$ is called the tangent stiffness operator or the tangent stiffness tensor of Σ at m_e. Because of the involved spaces, it is a covariant 2-tensor on the vector space $T_{m_e}\mathbb{M}$. It obviously depends on m_e and on the force system $\mathcal{F}$.*

In local coordinates on the manifold $\mathbb{M}$, this tensor is represented by a square matrix of $\mathcal{M}_n(\mathbb{R})$: it is the "usual" tangent stiffness matrix K at the equilibrium m_e and for the force system $\mathcal{F}$.

2.2 T-Stability

We adopt the following

Definition 2. *Let $\mathcal{F}$ be a force system described by a section $\omega_{\mathcal{F}}$ of $T^*\mathbb{M}$ and $m_e \in \mathbb{M}$ an equilibrium configuration of Σ subjected to $\mathcal{F}$. Thus $\omega_{\mathcal{F}}(m_e) = 0 = 0_{T^*\mathbb{M}}(m_e)$. m_e is then called Transervality-stable or T-stable if $\omega_{\mathcal{F}}$ intersects or cuts transversally the nil section $0_{T^*\mathbb{M}}$.*

This definition is then purely geometric and does not involves the tangent stiffness. The infinitesimal characterization of the transversality of the intersection of manifolds leads to the following property:

Proposition 1. *m_e is T-stable if and only if $d\omega_{\mathcal{F}}^{ver}(m_e)$ is an invertible map.*

The T-transversality leads then to the usual characterization of the divergence Lyapounov stability.

2.3 KISS Issue

The KISS issue necessitates to consider all the submanifolds $\mathbb{V}$ of (embedded in) $\mathbb{M}$ and also the definition of loading paths $\mathcal{L} = (\mathbb{M}, \omega_{\mathcal{L}})$ on $\mathbb{M}$. The above

T-stability can be extended to loading paths which are then called regular loading paths (see [7] for more precisions). Then, for any embedded submanifold $j : \mathbb{V} \to \mathbb{M}$ of $\mathbb{M}$ we may define by a pullback j^* the induced loading path $\mathcal{L}_{\mathbb{V}}$ on $\mathbb{V}$ and is called the subloading path $\mathcal{L}_{\mathbb{V}} = (\mathbb{V}, \omega_{\mathcal{L}_{\mathbb{V}}})$ of $\mathcal{L}$. We then have the following extension of the KISS result to the non linear framework:

Theorem 1. *Let $\mathcal{L}$ be a regular loading path and $m_{e,\sigma} = \pi(\omega_{\mathcal{L}}(\sigma)) \in \mathbb{M}$. If the symmetric part $d\omega_{\mathcal{L}}(\sigma)^{ver,s}(m_{e,\sigma})$ of $d\omega_{\mathcal{L}}(\sigma)^{ver}(m_{e,\sigma})$ is a degenerated $(0,2)$ symmetric tensor then there is a submanifold $\mathbb{V} \ni m_{e,\sigma}$ of $\mathbb{M}$ such that $\mathcal{L}_{\mathbb{V}} = (\mathbb{V}, \omega_{\mathcal{L}_{\mathbb{V}}})$ is singular at σ.*

3 Some Open Questions

For future works, the fundamental open problem is to establish such a geometric framework to tackle similar questions for plastic evolutions. Two main issues are then to describe the internal irreversibility and to take into account the non differentiability along the incremental evolutions.

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