B. Y. Chen Inequalities for Statistical Submanifolds in Sasakian Statistical Manifolds

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Abstract. In this paper, we derive a statistical version of B. Y. Chen inequality for statistical submanifolds in the Sasakian statistical manifolds with constant curvature and discuss the equality case of the inequality. We also give some applications of the inequalities obtained.

Keywords: Chen's inequality · Statistical manifolds · Sasakian statistical manifolds

1 Introduction

In 1989, the notion of statistical submanifolds was introduced and studied by Vos [10]. Though, till the date it has made very little progress due to the hardness to find classical differential geometric approaches for study of statistical submanifolds. Furuhata [6], studied statistical hypersurfaces in the space of Hessian curvature zero and provided some examples as well. In 2017, Furuhata et al. [5] studied Sasakian statistical manifolds and obtained some results. Geometry of statistical submanifolds is still young and efforts are on, so it is growing [1–3, 6–9].

In 1993 Chen [4] has obtained a sharp inequality for the sectional curvature of a submanifold in a real space forms in term of the scalar curvature (intrinsic invariant) and squared mean curvature (extrinsic invariant). Afterward, several geometers obtained similar inequality for various submanifolds in various ambient spaces due to its rich geometric importance.

In the present article, we obtain B. Y. Chen inequality for statistical submanifolds in Sasakian statistical manifold with constant curvature and obtain the equality case of the inequality. We also give some applications of the inequalities we derived.

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2 Preliminaries

Let $(\overline{\mathtt{N}}, \mathtt{g})$ be a Riemannian manifold and $\overline{\nabla}$ and $\overline{\nabla}^*$ be torsion-free affine connections on $\overline{\mathtt{N}}$ such that

$$\mathtt{G}\mathtt{g}(\mathtt{E}, \mathtt{F}) = \mathtt{g}(\overline{\nabla}_{\mathtt{G}}\mathtt{E}, \mathtt{F}) + \mathtt{g}(\mathtt{E}, \overline{\nabla}^*_{\mathtt{G}}\mathtt{F}), \tag{1}$$

for $\mathtt{E}, \mathtt{F}, \mathtt{G} \in \Gamma(T\overline{\mathtt{N}})$. Then Riemannian manifold $(\overline{\mathtt{N}}, \mathtt{g})$ is called a statistical manifold. It is denoted by $(\overline{\mathtt{N}}, \mathtt{g}, \overline{\nabla}, \overline{\nabla}^*)$. The connections $\overline{\nabla}$ and $\overline{\nabla}^*$ are called dual connections. The pair $(\overline{\nabla}, \mathtt{g})$ is said to be a statistical structure.

If $(\overline{\nabla}, \mathtt{g})$ is a statistical structure on $\overline{\mathtt{N}}$, then $(\overline{\nabla}^*, \mathtt{g})$ is also statistical structure on $\overline{\mathtt{N}}$.

For the dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$ we have

$$2\overline{\nabla}^\circ = \overline{\nabla} + \overline{\nabla}^*, \tag{2}$$

where $\overline{\nabla}^\circ$ is Levi-Civita connection for $\mathtt{g}$.

Let $\overline{\mathtt{N}}$ be a $(2m + 1)$-dimensional manifold and let $\mathtt{N}$ be an n-dimensional submanifolds of $\overline{\mathtt{N}}$. Then, the Gauss formulae are [10]

$$\begin{cases} \overline{\nabla}_{\mathtt{E}}\mathtt{F} = \nabla_{\mathtt{E}}\mathtt{F} + \zeta(\mathtt{E}, \mathtt{F}), \\ \overline{\nabla}^*_{\mathtt{E}}\mathtt{F} = \nabla^*_{\mathtt{E}}\mathtt{F} + \zeta^*(\mathtt{E}, \mathtt{F}), \end{cases} \tag{3}$$

where ζ and ζ^* are symmetric, bilinear, imbedding curvature tensors of $\mathtt{N}$ in $\overline{\mathtt{N}}$ for $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively.

The $\overline{\mathtt{R}}$ and $\overline{\mathtt{R}}^*$ be Riemannian curvature tensor fields of $\overline{\nabla}$ and $\overline{\nabla}^*$, respectively. Then [10]

$$\begin{aligned} \mathtt{g}(\overline{\mathtt{R}}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) &= \mathtt{g}(\mathtt{R}(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) + \mathtt{g}(\zeta(\mathtt{E}, \mathtt{G}), \zeta^*(\mathtt{F}, \mathtt{W})) \\ &\quad - \mathtt{g}(\zeta^*(\mathtt{E}, \mathtt{W}), \zeta(\mathtt{F}, \mathtt{G})), \end{aligned} \tag{4}$$

and

$$\begin{aligned} \mathtt{g}(\overline{\mathtt{R}}^*(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) &= \mathtt{g}(\mathtt{R}^*(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) + \mathtt{g}(\zeta^*(\mathtt{E}, \mathtt{G}), \zeta(\mathtt{F}, \mathtt{W})) \\ &\quad - \mathtt{g}(\zeta(\mathtt{E}, \mathtt{W}), \zeta^*(\mathtt{F}, \mathtt{G})), \end{aligned} \tag{5}$$

where

$$\mathtt{g}(\overline{\mathtt{R}}^*(\mathtt{E}, \mathtt{F})\mathtt{G}, \mathtt{W}) = -\mathtt{g}(\mathtt{G}, \overline{\mathtt{R}}(\mathtt{E}, \mathtt{F})\mathtt{W}). \tag{6}$$

Let us denote the normal bundle of $\mathtt{N}$ by $T\mathtt{N}^\perp$. The linear transformations A_N and A^*_N are defined by

$$\begin{cases} \mathtt{g}(A_N\mathtt{E}, \mathtt{F}) = \mathtt{g}(\zeta(\mathtt{E}, \mathtt{F}), N), \\ \mathtt{g}(A^*_N\mathtt{E}, \mathtt{F}) = \mathtt{g}(\zeta^*(\mathtt{E}, \mathtt{F}), N), \end{cases} \tag{7}$$

for any $N \in \Gamma(T\mathtt{N}^{\perp})$ and $\mathtt{E}, \mathtt{F} \in \Gamma(T\mathtt{N})$. The corresponding Weingarten formulas are [10]

$$\begin{cases} \overline{\nabla}_{\mathtt{E}} N = -A^{*}_{N}\mathtt{E} + \nabla^{\perp}_{\mathtt{E}} N, \\ \overline{\nabla}^{*}_{\mathtt{E}} N = -A_{N}\mathtt{E} + \nabla^{*\perp}_{\mathtt{E}} N, \end{cases} \tag{8}$$

where $N \in \Gamma(T\mathtt{N}^{\perp})$, $\mathtt{E} \in \Gamma(T\mathtt{N})$ and $\nabla^{\perp}_{\mathtt{E}}$ and $\nabla^{*\perp}_{\mathtt{E}}$ are Riemannian dual connections with respect to the induced metric on $\Gamma(T\mathtt{N}^{\perp})$.

Let $\overline{\mathtt{N}}$ be an odd dimensional manifold and ϕ be a tensor of type $(1,1)$, ξ a vector field, and a 1-form η on $\overline{\mathtt{N}}$ satisfying the conditions

$$\eta(\xi) = 1,$$
$$\phi^2\mathtt{E} = -\mathtt{E} + \eta(\mathtt{E})\xi,$$

for any vector field $\mathtt{E}$ on $\overline{\mathtt{N}}$, then $\overline{\mathtt{N}}$ is said to have an almost contact structure (ϕ, ξ, η).

Definition 1. *An almost contact structure* $(\phi, \xi, \mathtt{g})$ *on* $\overline{\mathtt{N}}$ *is said to be a Sasakian structure if*

$$(\overline{\nabla}^{\circ}_{\mathtt{E}}\phi)\mathtt{F} = \mathtt{g}(\mathtt{F}, \xi)\mathtt{E} - \mathtt{g}(\mathtt{F}, \mathtt{E})\xi,$$

holds for any $\mathtt{E}, \mathtt{F} \in T\overline{\mathtt{N}}$.

Definition 2 ([5]). *A quadruple* $(\overline{\nabla}, \mathtt{g}, \phi, \xi)$ *is called a Sasakian statistical structure on* $\overline{\mathtt{N}}$ *if* $(\overline{\nabla}, \mathtt{g})$ *is a statistical structure,* $(\mathtt{g}, \phi, \xi)$ *is a Sasakian structure on* $\overline{\mathtt{N}}$ *and the formula*

$$K_{\mathtt{E}}\phi\mathtt{F} + \phi K_{\mathtt{E}}\mathtt{F} = 0$$

holds for any $\mathtt{E}, \mathtt{F} \in T\overline{\mathtt{N}}$*, where* $K_{\mathtt{E}}\mathtt{F} = \overline{\nabla}_{\mathtt{E}}\mathtt{F} - \overline{\nabla}^{\circ}_{\mathtt{E}}\mathtt{F}$.

Definition 3 ([5]). *Let* $(\overline{\mathtt{N}}, \overline{\nabla}, \mathtt{g}, \phi, \xi)$ *be a Sasakian statistical manifold and* $c \in \mathtt{R}$*. The Sasakian statistical structure is said to be of constant* ϕ*-sectional curvature* c *if the curvature tensor* $\overline{S}$ *is given by*

$$\begin{aligned} \overline{S}(\mathtt{E}, \mathtt{F})\mathtt{G} = {} & \frac{c+3}{4}\{\mathtt{g}(\mathtt{F}, \mathtt{G})\mathtt{E} - \mathtt{g}(\mathtt{E}, \mathtt{G})\mathtt{F}\} + \frac{c-1}{4}\{\mathtt{g}(\phi\mathtt{F}, \mathtt{G})\phi\mathtt{E} - \mathtt{g}(\phi\mathtt{E}, \mathtt{G})\phi\mathtt{F} \\ & - 2\mathtt{g}(\phi\mathtt{E}, \mathtt{F})\phi\mathtt{G} - \mathtt{g}(\mathtt{F}, \xi)\mathtt{g}(\mathtt{G}, \xi)\mathtt{E} + \mathtt{g}(\mathtt{E}, \xi)\mathtt{g}(\mathtt{G}, \xi)\mathtt{F} + \mathtt{g}(\mathtt{F}, \xi)\mathtt{g}(\mathtt{G}, \mathtt{E})\xi \\ & - \mathtt{g}(\mathtt{E}, \xi)\mathtt{g}(\mathtt{G}, \mathtt{F})\xi\}, \quad \text{where} \quad \mathtt{E}, \mathtt{F}, \mathtt{G} \in T\overline{\mathtt{N}} \end{aligned} \tag{9}$$

and

$$2\overline{S}(\mathtt{E}, \mathtt{F})\mathtt{G} = \overline{\mathtt{R}}(\mathtt{E}, \mathtt{F})\mathtt{G} + \overline{\mathtt{R}}^{*}(\mathtt{E}, \mathtt{F})\mathtt{G}. \tag{10}$$

We denote a Sasakian statistical manifold with constant ϕ*-sectional curvature* c *by* $\overline{\mathtt{N}}(c)$.

Let ξ be tangent to the submanifolds $\mathtt{N}$ and let $\{e_1, \ldots, e_n = \xi\}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be tangent orthonormal frame and normal orthonormal frame, respectively, on $\mathtt{N}$. Then, the mean curvature vector fields $\mathtt{H}, \mathtt{H}^*, \mathtt{H}^\circ$ are given by

$$\mathtt{H} = \frac{1}{n}\sum_{i=1}^{n} \zeta(e_i, e_i), \tag{11}$$

$$\mathtt{H}^* = \frac{1}{n}\sum_{i=1}^{n} \zeta^*(e_i, e_i), \tag{12}$$

and

$$\mathtt{H}^\circ = \frac{1}{n}\sum_{i=1}^{n} \zeta^\circ(e_i, e_i). \tag{13}$$

We also set

$$\|\zeta\|^2 = \sum_{i,j=1}^{n} \mathtt{g}(\zeta(e_i, e_j), \zeta(e_i, e_j)), \tag{14}$$

$$\|\zeta^*\|^2 = \sum_{i,j=1}^{n} \mathtt{g}(\zeta^*(e_i, e_j), \zeta^*(e_i, e_j)), \tag{15}$$

and

$$\|\zeta^\circ\|^2 = \sum_{i,j=1}^{n} \mathtt{g}(\zeta^\circ(e_i, e_j), \zeta^\circ(e_i, e_j)). \tag{16}$$

The second fundamental form ζ° (resp. ζ, or ζ^*) has several geometric properties due to which we got following different classes of the submanifolds.

- A submanifold is said to be totally geodesic submanifold with respect to $\overline{\nabla}^\circ$ (resp. $\overline{\nabla}$, or $\overline{\nabla}^*$), if the second fundamental form ζ° (resp. ζ, or ζ^*) vanishes identically, that is $\zeta^\circ = 0$ (resp. $\zeta = 0$, or $\zeta^* = 0$).
- A submanifold is said to be minimal submanifold with respect to $\overline{\nabla}^\circ$ (resp.$\overline{\nabla}$, or $\overline{\nabla}^*$), if the mean curvature vector $\mathtt{H}^\circ$ (resp.$\mathtt{H}$, or $\mathtt{H}^*$) vanishes identically, that is $\mathtt{H}^\circ = 0$ (resp. $\mathtt{H} = 0$, or $\mathtt{H}^* = 0$).

Let $K(\pi)$ denotes the sectional curvature of a Riemannian manifold $\mathtt{N}$ of the plane section $\pi \subset T_p\mathtt{N}$ at a point $p \in \mathtt{N}$. If $\{e_1, \ldots, e_n\}$ be the orthonormal basis of $T_p\mathtt{N}$ and $\{e_{n+1}, \ldots, e_{2m+1}\}$ be the orthonormal basis of $T_p^\perp\mathtt{N}$ at any $p \in \mathtt{N}$, then

$$\tau(p) = \sum_{1\le i<j\le n} K(e_i \wedge e_j), \tag{17}$$

where τ is the scalar curvature. The normalized scalar curvature ρ is defined as

$$2\tau = n(n-1)\rho. \tag{18}$$

We also put

$$\zeta_{ij}^{\gamma} = \mathtt{g}(\zeta(e_i, e_j), e_\gamma), \qquad \zeta_{ij}^{*\gamma} = \mathtt{g}(\zeta^*(e_i, e_j), e_\gamma),$$

$i, j \in 1, \ldots, n, \gamma \in \{n+1, \ldots, 2m+1\}$.

3 B. Y. Chen Inequalities

In this section, we obtain statistical version of well known B. Y. Chen inequality for statistical submanifolds of Sasakian statistical manifolds with constant ϕ-sectional curvature.

Theorem 1. *Let* $\mathtt{N}$ *be a statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{N}}(c)$ *with* $\sum_\alpha \left[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha}\right] = 2\sum_\alpha \zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}$ *such that the structure vector field* ξ *of* $\overline{\mathtt{N}}(c)$ *is tangent to* $\mathtt{N}$*. Then*

$$\begin{aligned} K(\pi) \leq \tau &+ \frac{c+3}{4}(1+n-n^2) + \frac{c-1}{4}\big\{3(\Theta(\pi) - \|P\|^2) - \Phi(\pi) - 2(1-n)\big\} \\ &+ \frac{n^2}{2}(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) - 2n^2\|\mathtt{H}^\circ\|^2 + \|\zeta\|\|\zeta^*\|, \end{aligned} \tag{19}$$

where $\Theta(\pi) = \mathtt{g}^2(\phi e_1, e_2)$, $\Phi(\pi) = \eta^2(e_1) + \eta^2(e_2)$, $\pi = e_1 \wedge e_2$ *and* $||P||^2 = \mathtt{g}^2(\phi e_i, e_j)\big\}$*. Moreover, the equality holds if* ζ *and* ζ^* *are parallel. That is*

$$\zeta = k\zeta^*, k \in \mathbb{R}^+. \tag{20}$$

Proof. From (4), (5), (9) and (10), we have

$$\begin{aligned} \mathtt{g}(\mathtt{R}(\mathtt{E},\mathtt{F})\mathtt{G},\mathtt{W}) + \mathtt{g}(\mathtt{R}^*(\mathtt{E},\mathtt{F})\mathtt{G},\mathtt{W}) = \frac{c+3}{2}\{&\mathtt{g}(\mathtt{F},\mathtt{G})\mathtt{g}(\mathtt{E},\mathtt{W}) - \mathtt{g}(\mathtt{E},\mathtt{G})\mathtt{g}(\mathtt{F},\mathtt{W}\} \\ + \frac{c-1}{2}\{&\mathtt{g}(\phi\mathtt{F},\mathtt{G})\mathtt{g}(\phi\mathtt{E},\mathtt{W}) - \mathtt{g}(\phi\mathtt{E},\mathtt{G})\mathtt{g}(\phi\mathtt{F},\mathtt{W}) - 2\mathtt{g}(\phi\mathtt{E},\mathtt{F})\mathtt{g}(\phi\mathtt{G},\mathtt{W}) \\ - &\mathtt{g}(\mathtt{F},\xi)\mathtt{g}(\mathtt{G},\xi)\mathtt{g}(\mathtt{E},\mathtt{W}) + \mathtt{g}(\mathtt{E},\xi)\mathtt{g}(\mathtt{G},\xi)\mathtt{g}(\mathtt{F},\mathtt{W}) + \mathtt{g}(\mathtt{F},\xi)\mathtt{g}(\mathtt{G},\mathtt{E})\mathtt{g}(\xi,\mathtt{W}) \\ - &\mathtt{g}(\mathtt{E},\xi)\mathtt{g}(\mathtt{G},\mathtt{F})\mathtt{g}(\xi,\mathtt{W})\} - \mathtt{g}(\zeta(\mathtt{E},\mathtt{G}),\zeta^*(\mathtt{F},\mathtt{W})) + \mathtt{g}(\zeta^*(\mathtt{E},\mathtt{W}),\zeta(\mathtt{F},\mathtt{G})) \\ - &\mathtt{g}(\zeta^*(\mathtt{E},\mathtt{G}),\zeta(\mathtt{F},\mathtt{W})) + \mathtt{g}(\zeta(\mathtt{E},\mathtt{W}),\zeta^*(\mathtt{F},\mathtt{G})). \end{aligned} \tag{21}$$

Putting $\mathtt{F} = \mathtt{W} = e_i$ and $\mathtt{E} = \mathtt{G} = e_j$, in (21), we get

$$\begin{aligned} \mathtt{g}(\mathtt{R}(e_i,e_j)e_j,e_i) + \mathtt{g}(\mathtt{R}^*(e_i,e_j)e_j,e_i) = \frac{c+3}{2}\{&\mathtt{g}(e_j,e_j)\mathtt{g}(e_i,e_i) - \mathtt{g}(e_i,e_j)\mathtt{g}(e_j,e_i\} \\ + \frac{c-1}{2}\{&\mathtt{g}(\phi e_j,e_j)\mathtt{g}(\phi e_i,e_i) - \mathtt{g}(\phi e_i,e_j)\mathtt{g}(\phi e_j,e_i) \\ - &2\mathtt{g}(\phi e_i,e_j)\mathtt{g}(\phi e_j,e_i) - \mathtt{g}(e_j,\xi)\mathtt{g}(e_j,\xi)\mathtt{g}(e_i,e_i) \\ + &\mathtt{g}(e_i,\xi)\mathtt{g}(e_j,\xi)\mathtt{g}(e_j,e_i) + \mathtt{g}(e_j,\xi)\mathtt{g}(e_j,e_i)\mathtt{g}(\xi,e_i) \end{aligned}$$

$$- \mathrm{g}(e_i,\xi)\mathrm{g}(e_j,e_j)\mathrm{g}(\xi,e_i)\} - \mathrm{g}(\zeta(e_i,e_j),\zeta^*(e_j,e_i)) + \mathrm{g}(\zeta^*(e_i,e_i),\zeta(e_j,e_j)) - \mathrm{g}(\zeta^*(e_i,e_j),\zeta(e_j,e_i)) + \mathrm{g}(\zeta(e_i,e_i),\zeta^*(e_j,e_j)). \tag{22}$$

Applying summation $1 \leq i,j \leq n$ and using (11)–(16) in (22), we obtain

$$\begin{aligned}\sum_{1\leq i,j\leq n} [\mathrm{g}(\mathtt{R}(e_i,e_j)e_j,e_i) + \mathrm{g}(\mathtt{R}^*(e_i,e_j)e_j,e_i)] &= \frac{c+3}{2}n(n-1) + 2n^2\mathrm{g}(\mathtt{H},\mathtt{H}^*)\\ &+ \frac{c-1}{2}\{2(1-n) + 3\mathrm{g}^2(\phi e_i,e_j)\} - \mathrm{g}(\zeta(e_i,e_j),\zeta^*(e_j,e_i))\\ &- \mathrm{g}(\zeta^*(e_i,e_j),\zeta(e_j,e_i))\\ &= \frac{c+3}{2}n(n-1) + n^2\big\{\mathrm{g}(\mathtt{H}^*+\mathtt{H},\mathtt{H}^*+\mathtt{H}) - \mathrm{g}(\mathtt{H},\mathtt{H}) - \mathrm{g}(\mathtt{H}^*,\mathtt{H}^*)\big\}\\ &+ \frac{c-1}{2}\big\{2(1-n) + 3\mathrm{g}^2(\phi e_i,e_j)\big\}\\ &- \big\{\mathrm{g}(\zeta(e_i,e_j)+\zeta^*(e_j,e_i),\zeta^*(e_i,e_j)+\zeta(e_j,e_i))\\ &- \mathrm{g}(\zeta(e_i,e_j),\zeta(e_i,e_j)) - \mathrm{g}(\zeta^*(e_j,e_i),\zeta^*(e_j,e_i))\big\}. \end{aligned} \tag{23}$$

Since from Eq. (2) $2\mathtt{H}^\circ = \mathtt{H} + \mathtt{H}^*$, it follows from the above equation that

$$\begin{aligned} 2\tau = &\frac{c+3}{2}n(n-1) + \frac{c-1}{2}\big\{2(1-n) + 3\|P\|^2\big\}\\ &+ 4n^2\|\mathtt{H}^\circ\|^2 - n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) + 4\|\zeta^\circ\|^2 - (\|\zeta\|^2 + \|\zeta^*\|^2). \end{aligned} \tag{24}$$

On the other hand we know that

$$\begin{aligned} K(\pi) &= \frac{1}{2}\big[\mathrm{g}(\mathtt{R}(e_1,e_2)e_2,e_1) + \mathrm{g}(\mathtt{R}^*(e_1,e_2)e_2,e_1)\big]\\ &= \frac{1}{2}\big[\mathrm{g}(\overline{\mathtt{R}}(e_1,e_2)e_2,e_1) + \mathrm{g}(\overline{\mathtt{R}}^*(e_1,e_2)e_2,e_1)\\ &- 2\mathrm{g}(\zeta^*(e_1,e_2),\zeta(e_2,e_1)) + 2\mathrm{g}(\zeta(e_1,e_1),\zeta^*(e_2,e_2))\big]\\ &= \mathrm{g}(\overline{S}(e_1,e_2)e_2,e_1) + \sum_\alpha \Big[\frac{1}{2}\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \frac{1}{2}\zeta_{11}^{\alpha}\zeta_{22}^{*\alpha} - \zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}\Big]. \end{aligned} \tag{25}$$

Taking inner product of (9) with $\mathtt{W}$ and setting $\mathtt{E} = \mathtt{W} = e_1$ and $\mathtt{F} = \mathtt{G} = e_2$, we find

$$\begin{aligned} \mathrm{g}(\overline{S}(e_1,e_2)e_2,e_1) &= \frac{c+3}{4}\big\{\mathrm{g}(e_2,e_2)\mathrm{g}(e_1,e_1) - \mathrm{g}(e_1,e_2)\mathrm{g}(e_2,e_1)\big\}\\ &+ \frac{c-1}{4}\big\{\mathrm{g}(\phi e_2,e_2)\mathrm{g}(\phi e_1,e_1) - \mathrm{g}(\phi e_1,e_2)\mathrm{g}(\phi e_2,e_1)\\ &- 2\mathrm{g}(\phi e_1,e_2)\mathrm{g}(\phi e_2,e_1) - \mathrm{g}(e_2,\xi)\mathrm{g}(e_2,\xi)\mathrm{g}(e_1,e_1)\\ &+ \mathrm{g}(e_1,\xi)\mathrm{g}(e_2,\xi)\mathrm{g}(e_2,e_1) + \mathrm{g}(e_2,\xi)\mathrm{g}(e_2,e_1)\mathrm{g}(\xi,e_1)\\ &- \mathrm{g}(e_1,\xi)\mathrm{g}(e_2,e_2)\mathrm{g}(\xi,e_1)\big\}\\ &= \frac{c+3}{4} + \frac{c-1}{4}\big\{\mathrm{g}(\phi e_2,e_2)\mathrm{g}(\phi e_1,e_1) + 3\mathrm{g}^2(\phi e_1,e_2) \end{aligned}$$

$$- \mathtt{g}^2(e_2,\xi) - \mathtt{g}^2(e_1,\xi)\}$$

$$= \frac{c+3}{4} + \frac{c-1}{4}\big\{\mathtt{g}(\phi e_2, e_2)\mathtt{g}(\phi e_1, e_1) + 3\Theta(\pi) - \Phi(\pi)\big\}. \quad (26)$$

From (25) and (26), we get

$$2K(\pi) = \frac{c+3}{2} + \frac{c-1}{2}\big\{3\Theta(\pi) - \Phi(\pi)\big\} + \sum_{\alpha}\big[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha} - 2\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}\big]. \quad (27)$$

Taking into account (24) and (27), we have

$$\begin{aligned} 2K(\pi) - 2\tau = {} & \frac{c+3}{2} + \frac{c-1}{2}\big\{3\Theta(\pi) - \Phi(\pi)\big\} + \sum_{\alpha}\big[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha} - 2\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}\big] \\ & - \frac{c+3}{2}n(n-1) - \frac{c-1}{2}\big\{2(1-n) + 3\|P\|^2\big\} - 4n^2\|\mathtt{H}^\circ\|^2 \\ & + n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) + 4n\mathtt{C}^\circ - n(\mathtt{C} + \mathtt{C}^*) \\ = {} & \frac{c+3}{2}(1 + n - n^2) + \frac{c-1}{2}\big\{3\Theta(\pi) - \Phi(\pi) - 2(1-n) - 3\|P\|^2\big\} \\ & + \sum_{\alpha}\big[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha} - 2\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}\big] - 4n^2\|\mathtt{H}^\circ\|^2 \\ & + n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) + 4\|\zeta^\circ\|^2 - (\|\zeta\|^2 + \|\zeta^*\|^2). \end{aligned} \quad (28)$$

On the other hand,

$$\begin{aligned} \|\zeta + \zeta^*\|^2 &= g(\zeta + \zeta^*, \zeta + \zeta^*) \\ &= \|\zeta\|^2 + g(\zeta, \zeta^*) + g(\zeta^*, \zeta) + \|\zeta^*\|^2 \\ &= \|\zeta\|^2 + 2g(\zeta, \zeta^*) + \|\zeta^*\|^2 \\ &\le \|\zeta\|^2 + 2\|\zeta\|\|\zeta^*\| + \|\zeta^*\|^2, \end{aligned} \quad (29)$$

and the equality holds if

$$\zeta = k\zeta^*, \quad k \in \mathbb{R}^+ \quad (30)$$

Equation (29) implies

$$\|\zeta\|^2 + \|\zeta^*\|^2 \ge \|\zeta + \zeta^*\|^2 - 2\|\zeta\|\|\zeta^*\| \quad (31)$$

Using (31) in (28), we obtain

$$\begin{aligned} 2K(\pi) - 2\tau \le {} & \frac{c+3}{2}(1 + n - n^2) + \frac{c-1}{2}\big\{3\Theta(\pi) - \Phi(\pi) - 2(1-n) - 3\|P\|^2\big\} \\ & + \sum_{\alpha}\big[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha} - 2\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}\big] - 4n^2\|\mathtt{H}^\circ\|^2 + n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) \\ & + 4\|\zeta^\circ\|^2 - \|\zeta + \zeta^*\|^2 + 2\|\zeta\|\|\zeta^*\| \\ = {} & \frac{c+3}{2}(1 + n - n^2) + n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) + 2\|\zeta\|\|\zeta^*\| \end{aligned}$$

$$+ \frac{c-1}{2}\left\{3\Theta(\pi) - \Phi(\pi) - 2(1-n) - 3\|P\|^2\right\}$$
$$+ \sum_{\alpha}\left[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha} - 2\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}\right] - 4n^2\|\mathtt{H}^{\circ}\|^2. \quad (32)$$

Using the hypothesis of the theorem in (32), we have

$$2K(\pi) - 2\tau \leq \frac{c+3}{2}(1+n-n^2) + \frac{c-1}{2}\left\{3\Theta(\pi) - \Phi(\pi) - 2(1-n) - 3\|P\|^2\right\}$$
$$- 4n^2\|\mathtt{H}^{\circ}\|^2 + n^2(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) + 2\|\zeta\|\|\zeta^*\|. \quad (33)$$

Moreover, equality holds if and only if it satisfies (30). Hence we have the required result.

The following result is immediate consequence of Theorem 1.

Corollary 1. *Let* $\mathtt{N}$ *be a statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{N}}(c)$ *with* $\sum_{\alpha}\left[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha}\right] = 2\sum_{\alpha}\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}$ *such that the structure vector field* ξ *of* $\overline{\mathtt{N}}(c)$ *is tangent to* $\mathtt{N}$*. Then*

$$K(\pi) - \tau \leq \frac{c+3}{4}(1+n-n^2) + \frac{c-1}{4}\left\{3(\Theta(\pi) - \|P\|^2) - \Phi(\pi) - 2(1-n)\right\}, \quad (34)$$

if $\mathtt{N}$ *is totally geodesic with respect to* $\overline{\nabla}$ *or* $\mathtt{N}$ *is totally geodesic with respect to* $\overline{\nabla}^*$*.*

Further, we state similar result when the structure vector field ξ of $\overline{\mathtt{N}}(c)$ is normal to $\mathtt{N}$.

Theorem 2. *Let* $\mathtt{N}$ *be a statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{N}}(c)$ *with* $\sum_{\alpha}\left[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha}\right] = 2\sum_{\alpha}\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}$ *such that the structure vector field* ξ *of* $\overline{\mathtt{N}}(c)$ *is Normal to* $\mathtt{N}$*. Then*

$$K(\pi) \leq \tau + \frac{c+3}{4}(1+n-n^2) + \frac{c-1}{4}\left\{3\Theta(\pi) - 3\|P\|^2\right\}$$
$$+ \frac{n^2}{2}(\|\mathtt{H}\|^2 + \|\mathtt{H}^*\|^2) - 2n^2\|\mathtt{H}^{\circ}\|^2 + \|\zeta\|\|\zeta^*\|. \quad (35)$$

From the above result we deduce the following corollary.

Corollary 2. *Let* $\mathtt{N}$ *be a statistical submanifold in a Sasakian statistical manifold* $\overline{\mathtt{N}}(c)$ *with* $\sum_{\alpha}\left[\zeta_{11}^{*\alpha}\zeta_{22}^{\alpha} + \zeta_{11}^{\alpha}\zeta_{22}^{*\alpha}\right] = 2\sum_{\alpha}\zeta_{12}^{*\alpha}\zeta_{12}^{\alpha}$ *such that the structure vector field* ξ *of* $\overline{\mathtt{N}}(c)$ *is Normal to* $\mathtt{N}$*. Then*

$$K(\pi) \leq \tau + \frac{c+3}{4}(1+n-n^2) + \frac{c-1}{4}\left\{3\Theta(\pi) - 3\|P\|^2\right\}, \quad (36)$$

if $\mathtt{N}$ *is totally geodesic with respect to* $\overline{\nabla}$ *or* $\mathtt{N}$ *is totally geodesic with respect to* $\overline{\nabla}^*$*.*

4 Conclusion and Future work

We obtained the B. Y. Chen inequality for the statistical submanifolds in Sasakian statistical manifolds having constant curvature. In fact, this is the first such attempt for any statistical case. Therefore, I hope it will open the door for the researcher to obtain such inequality, which has the great geometric importance, for different ambient such as **Holomorphic statistical manifolds, Kenmotsu Statistical manifolds, Cosymplectic statistical manifolds, Quaternion Kaehler-like statistical manifolds** etc. with constant curvatures. The forthcoming challenge is to improve the result by weakening the condition.

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