Deformed Exponential and the Behavior of the Normalizing Function

Francisca Leidmar Josue Vieira[1(✉)], Rui Facundo Vigelis[2], Luiza Helena Felix de Andrade[3], and Charles Casimiro Cavalcante[4]

[1] Department of Mathematics, Regional University of Cariri, Crato-CE, Brazil
leidmar.vieira@urca.br
[2] Computer Engineering, Campus Sobral, Federal University of Ceará, Sobral-CE, Brazil
rfvigelis@ufc.br
[3] Center of Exact and Natural Sciences, Federal Rural University of Semi-Arid Region, Mossoró-RN, Brazil
luizafelix@ufersa.edu.br
[4] Department of Teleinformatics Engineering, Federal University of Ceará, Fortaleza-CE, Brazil
charles@ufc.br

Abstract. In this paper we consider the statistical manifold defined in terms of a deformed exponential φ. For the non-atomic case we establish a relation between the behavior of the deformed exponential function and the Δ_2-condition and analyze the comportment of the normalizing function near to the boundary of its domain. In the purely atomic case we find an equivalent condition to the behavior that characterizes the deformed exponential discussed in this work. Moreover, we prove a consequence from the fact the Musielak-Orlicz function does not satisfy the δ_2-condition.

Keywords: Deformed exponential · φ-Families of probability distribuitions · Musielak-Orlicz space · Normalizing function

1 Introduction

Consider (T, Σ, μ), a σ-finite, non-atomic measure space and $\mathcal{P}_\mu$ the set of μ-equivalent strictly positive probability densities. Let $\varphi : \mathbb{R} \to (0, \infty)$ be a deformed exponential, which is a convex function such that $\lim_{u\to-\infty} \varphi(u) = 0$ and $\lim_{u\to\infty} \varphi(u) = \infty$. The φ-families were constructed based on the replacement of the classical exponential function by a deformed exponential function [6,8], which satisfies the property that there exists a measurable function $u_0 : T \to (0, \infty)$ such that [11]

$$\int_T \varphi(c + \lambda u_0) d\mu < \infty, \qquad \text{for all } \lambda > 0, \tag{1}$$

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for each measurable function $c\colon T \to \mathbb{R}$ satisfying $\int_T \varphi(c)d\mu < \infty$. The construction of a φ-families is based on Musielak-Orlicz spaces [5].

Another statement with respect to deformed exponential that satisfies condition (1), is that two probability densities in $\mathcal{P}_\mu$ can be connected by an open arc [2,3,9,13]. This allows us to define the generalization of Rényi divergence and consequently a family of α-connections [4,7].

In the non-atomic case, it was proved in [10] that if the Musielak-Orlicz function does not satisfy the Δ_2-condition, then the boundary of the parametrization domain is non-empty. Moreover, the authors considered that condition (1) occurs and studied the behavior of the normalizing function near to the boundary of their domain. In [1], it was considered that condition (1) does not occur and analyzed the behavior of the normalizing function near to the boundary of its domain in points that are not in the Musielak-Orlicz class.

In this work, we establish a relationship between condition (1) and the Δ_2-condition, which allows us to study points on the boundary of the parametrization domain. Furthermore, continuing the discussion initiated in [1], we analyse the behavior of the normalizing function in the points that are in the Musielak-Orlicz class. More precisely, we observe that regardless of the condition (1) occurs, given a function in the Musielak-Orlicz class, we have that the normalizing function converges near the boundary of its domain. In the purely atomic case, where μ is a counting measure on the set $T = \mathbb{N}$, as in [5], the analogous of the Δ_2-condition will be denoted by δ_2-condition. In this case, we find an equivalent condition to (1) and a result related to a Musielak-Orlicz function Φ_c which does not satisfy the δ_2-condition.

This paper is organized as follows. In Sect. 2 we recall some important results about φ-families of probability distributions. In Sects. 3 and 4 we develop our main results regarding the condition described in Eq. (1) and for the purely atomic case. Finally, in Sect. 5 we state our conclusions and future perspectives for later works.

2 φ-Families of Probability Distribution: Revisiting Some Results

In this section we recall some results and we fix notations that will be important for the understanding of the text. The Musielak-Orlicz space L^{Φ_c} and Musielak-Orlicz class $\tilde{L}^{\Phi_c}$, when the Musielak-Orlicz function is $\Phi_c(t,u) = \varphi(t, c(t)+u) - \varphi(t, c(t))$ are denoted by L^φ_c and $\tilde{L}^\varphi_c$, respectively, and defined as [11]

$$L^\varphi_c = \left\{ u \in L^0 : \int_T \varphi(c + \lambda u) d\mu < \infty \text{ for each } \lambda \in (-\varepsilon, \varepsilon), \exists \;\; \varepsilon > 0 \right\},$$

$$\tilde{L}^\varphi_c = \left\{ u \in L^0 : \int_T \varphi(c+u) d\mu < \infty \right\},$$

are used to build the sets $B^\varphi_c = \left\{ u \in L^\varphi_c ; \int_T u \varphi'_+(c) d\mu = 0 \right\}$,

$$\mathcal{K}^\varphi_c = \left\{ u \in L^\varphi_c ; \; \int_T \varphi(c + \lambda u) < \infty \text{ for each } \lambda \in (-\varepsilon, 1+\varepsilon), \;\; \exists \;\; \varepsilon > 0 \right\}$$

and the parametrization $\varphi_c : \mathcal{B}_c^\varphi \to \mathcal{F}_c^\varphi$, where $\varphi_c(u) = \varphi(c + u - \psi(u)u_0)$, for each $u \in \mathcal{B}_c^\varphi = B_c^\varphi \cap \mathcal{K}_c^\varphi$. The mapping $\psi : \mathcal{B}_c^\varphi \to [0, \infty)$ is called *normalizing function* and is defined in such way that $\varphi_c(u) = \varphi(c + u - \psi(u)u_0) \in \mathcal{P}_\mu$.

We say that the function Φ_c satisfies the Δ_2-condition ($\Phi_c \in \Delta_2$) if it can be found a constant $K > 0$ and a non-negative function f belonging to Musielak-Orlicz class $\tilde{L}_c^\varphi$, such that $\Phi_c(t, u) \le K\Phi_c(t, u)$, for all $u > f(t)$ and μ-a.e. $t \in T$. In [10] was proved that if Φ_c does not satisfy the Δ_2-condition then the boundary of $\mathcal{B}_c^\varphi$ is non-empty. A function $u \in B_c^\varphi$ belongs to the boundary of $\mathcal{B}_c^\varphi$(denoted by $\partial\mathcal{B}_c^\varphi$) if, and only if, $\int_T \varphi(c + \lambda u)d\mu < \infty$ for all $\lambda \in (0, 1)$ and $\int_T \varphi(c + \lambda u)d\mu = \infty$, for each $\lambda > 1$.

In [10], supposing that (1) is satisfied and given u, $w \in \partial\mathcal{B}_c^\varphi$ such that $\int_T \varphi(c + w)d\mu < \infty$ and $\int_T \varphi(c + u)d\mu = \infty$, then $\psi(\alpha w) \to \beta$, with $\beta \in (0, \infty)$ as $\alpha \uparrow 1$ and $\psi'_+(\alpha u) \to \infty$ as $\alpha \uparrow 1$. This last result was complemented by [1] when it was proved that given $u \in \partial\mathcal{B}_c^\varphi$ such that $\int_T \varphi(c + u)d\mu = \infty$ we have that $\lim_{\alpha\uparrow 1} \psi(\alpha u) = \infty$, as $\alpha \uparrow 1$.

Now, supposing that (1) does not occur, in [1] was shown that there exists $u \in \partial\mathcal{B}_c^\varphi$ such that $\int_T \varphi(c+u)d\mu = \infty$ but $\psi(\alpha u) \to \beta$, with $\beta \in (0, \infty)$ as $\alpha \uparrow 1$. In the next section we analyse the behavior of the normalizing function in the points of the boundary of $\mathcal{B}_c^\varphi$ such that $\int_T \varphi(c + u)d\mu < \infty$.

3 The Condition (1) on the deformed exponential and Its consequences

In this section, we notice that regardless of the occurence condition (1) if $u \in \partial\mathcal{B}_c^\varphi$ is such that $\int_T \varphi(c+u)d\mu < \infty$, then the *normalizing function* converges near to the boundary of its domain. Moreover, we relate condition (1) and condition Δ_2.

Remark 1. *Consider the deformed exponential function φ. Given $u \in \partial\mathcal{B}_c^\varphi$ such that $\int_T \varphi(c + u)d\mu < \infty$, we have that $\psi(\alpha u) \to \beta$, with $\beta \in (0, \infty)$ as $\alpha \uparrow 1$. In fact, since $\int_T \varphi(c + u)d\mu < \infty$, we have $\int_T \varphi(c + u - \lambda u_0)d\mu < \infty$ for all $\lambda > 0$. Suppose that $\psi(\alpha u) \uparrow \infty$, as $\alpha \uparrow 1$. Then, for all $A > 0$, there exists $\delta > 0$, such that $0 < 1 - \alpha < \delta \Rightarrow \psi(\alpha u) > A$. Since $\int_T \varphi(c + u - \lambda u_0)d\mu < \infty$ for all $\lambda > 0$, we have that there exists $\gamma > \lambda$, such that $\int_T \varphi(c+u-\gamma u_0)d\mu < 1$. In particular, take $A = \gamma$. Then, from of the Dominated Convergence Theorem it follows that*

$$\begin{aligned} 1 &= \lim_{\alpha\uparrow 1} \int_T \varphi(c + \alpha u - \psi(\alpha u)u_0)d\mu \\ &\le \lim_{\alpha\uparrow 1} \int_T \varphi(c + \alpha u - \gamma u_0)d\mu \\ &= \int_T \varphi(c + u - \gamma u_0)d\mu \\ &< 1, \end{aligned}$$

which is an absurd. Therefore, we obtain the desired result.

From the previous remark, supposing that the deformed exponential function φ does not satisfy the condition (1), we can find $u \in \partial\mathcal{B}_c^{\varphi}$, such that $\int_T \varphi(c+u)d\mu < \infty$ and $\psi(\alpha u)$ converges as $\alpha \uparrow 1$.

We have the following condition that is equivalent the condition (1).

Proposition 1. *[1, Proposition 2] We say that a deformed exponential function φ and a measurable function $u_0 : T \to (0,\infty)$ satisfy the condition (1) if and only if, for some measurable function $c : T \to \mathbb{R}$ such that $\varphi(c)$ is μ-integrable, we can find constants $\overline{\lambda}, \alpha > 0$ and a non-negative function $f \in \widetilde{L}^{\Phi_c}$ such that*

$$\alpha\Phi_c(t,u) \leq \Phi_{c-\overline{\lambda}u_0}(t,u), \text{ for all } u > f(t), \tag{2}$$

where $\Phi_c(t,u) = \varphi(t, c(t)+u(t)) - \varphi(c(t))$ is a Musielak-Orlicz function.

For what follows we will define $I_{\Phi_c}(u(t)) = \int_T \Phi_c(t, |u(t)|)d\mu$ for any $u \in L^0$.

Lemma 1. *[1, Lemma 3] Consider $c : T \to [0,\infty)$ a measurable function such that $\int_T \varphi(c)d\mu < \infty$. Suppose that, for each $\lambda > 0$, we cannot find $\alpha > 0$ and $f \in \widetilde{L}^{\Phi_c}$ such that*

$$\alpha\Phi_c(t,u) \leq \Phi_{c-\lambda u_0}(t,u), \text{ for all } u > f(t). \tag{3}$$

Then we can find a strictly decreasing sequence $0 < \lambda_n \downarrow 0$, a sequence $\{u_n\}$ of measurable functions finite-value and a sequence $\{A_n\}$ of measurables sets pairwise disjoint such that

$$I_{\Phi_c}(u_n\chi_{A_n}) = 1 \text{ and } I_{\Phi_{c-\lambda_n u_0}}(u_n\chi_{A_n}) \leq 2^{-n}, \text{ for all } n \geq 1. \tag{4}$$

In the next corollary, we prove that there exists a relationship between the condition (1) and the Δ_2-condition. For this, we use the fact of that if $\Phi_c \in \Delta_2$ then $L_c^{\varphi} = \widetilde{L}_c^{\varphi}$, and consequently $I_{\Phi_c}(u) < \infty$, for every $u \in L_c^{\varphi}$.

Proposition 2. *If the deformed exponential function φ does not satisfiy the condition (1), then $\Phi_c \notin \Delta_2$.*

Proof. Suppose that (1) does not occur. Take $\lambda > 0$. Then, there exists a $n_0 \in \mathbb{N}$, such that $\lambda > \lambda_n$, for all $n \geq n_0$. By [1, Proposition 2] and Lemma 1, we can take $u = \sum_{n=n_0}^{\infty} u_n\chi_{A_n}$. Since $u \in L_c^{\varphi}$ and $I_{\Phi_c}(u) = \infty$ it follows the result.

The reciprocal of the Proposition 2 is not valid, since the exponential function satisfies the condition (1) but does not satisfy the Δ_2-condition.

Proposition 3 *Suposing that the deformed exponential φ does not satisfy (1) and that $u_0 \in E_c^{\varphi}$, then there exists $w \in \partial\mathcal{B}_c^{\varphi}$ such that $\int_T \varphi(c+w)d\mu < \infty$, $\int_T \varphi(c+w+\overline{\lambda}u_0)d\mu = \infty$, for all $\overline{\lambda} > 0$ and $\psi(\alpha w) \to \beta$, with $\beta \in (0,\infty)$, as $\alpha \uparrow 1$.*

Proof. Let $\{\lambda_n\}$, $\{u_n\}$ and $\{A_n\}$ be defined as in Lemma 1. Given $\alpha > 1$, there exists $n_1 \in \mathbb{N}$, such that $\alpha(u_n - \lambda_n u_0) > u_n$ for all $n \geq n_1$. Given any $\overline{\lambda} > 0$, there exists $n_2 \in \mathbb{N}$ such that $\overline{\lambda} > \lambda_n$ for all $n \geq n_2$. Take $n_0 = \max\{n_1, n_2\}$. Consider $B = \cup_{n=n_0}^{\infty} A_n$ and $u = \sum_{n=n_0}^{\infty}(u_n - \lambda_n u_0)\chi_{A_n}$. Then,

$$\begin{aligned}\int_T \varphi(c+u)d\mu &= \int_T \varphi\left(c + \sum_{n=n_0}^{\infty}(u_n - \lambda_n u_0)\chi_{A_n}\right)d\mu \\ &= \int_{T\smallsetminus B}\varphi(c)d\mu + \sum_{n=n_0}^{\infty}\left\{\int_{A_n}\varphi(c)d\mu + I_{\Phi_c}(u_n - \lambda_n u_0)\chi_{A_n}\right\} \\ &\leq \int_T \varphi(c)d\mu + \sum_{n=n_0}^{\infty}\left\{I_{\Phi_{c-\lambda_n u_0}}(u_n\chi_{A_n})\right\} \\ &< \infty.\end{aligned} \tag{5}$$

For $\alpha \in (0,1)$ we have

$$\int_T \varphi(c+\alpha u)d\mu \leq \underbrace{\alpha\int_T \varphi(c+u-\lambda u_0)d\mu + (1-\alpha)\textstyle\int_T \varphi\left(c + \dfrac{\alpha\lambda}{1-\alpha}u_0\right)d\mu}_{<\infty}$$

and considering $\alpha > 1$ we have

$$\begin{aligned}\int_T \varphi(c+\alpha u)d\mu &= \int_{T\smallsetminus B}\varphi(c)d\mu + \sum_{n=n_0}^{\infty}\int_{A_n}\varphi(c+\alpha(u_n - \lambda_n u_0))d\mu \\ &\geq \int_T \varphi(c)d\mu + \sum_{n=n_0}^{\infty} I_{\Phi_c}(u_n\chi_{A_n})d\mu \\ &= \infty.\end{aligned}$$

Then, since (5) occurs, it follows from Remark 1 that $\psi(\alpha u) \to \beta$, with $\beta \in (0,\infty)$, as $\alpha \uparrow 1$. Take $\lambda > 0$, such that $w = \sum_{n=n_0}^{\infty}\alpha(u_n - \lambda_n u_0) - \lambda u_0\chi_{T\smallsetminus B} \in B_c^{\varphi}$. Clearly, $\int_T \varphi(c+w)d\mu = \infty$, $\int_T \varphi(c+\alpha w)d\mu < \infty$ for $\alpha \in (0,1)$ and $\int_T \varphi(c+\alpha w)d\mu = \infty$ for $\alpha > 1$. Therefore, $w \in \partial\mathcal{B}_c^{\varphi}$ and we obtain

$$\begin{aligned}\int_T \varphi(c+w+\overline{\lambda}u_0)d\mu &> \int_{T\smallsetminus B}\varphi(c-\lambda u_0)d\mu + \sum_{n=n_0}^{\infty}\int_{A_n}\varphi(c+u_n-\lambda_n u_0+\overline{\lambda}u_0)d\mu \\ &> \int_{T\smallsetminus B}\varphi(c-\lambda u_0)d\mu + \sum_{n=n_0}^{\infty}\left(I_{\Phi_c}(u_n) - \int_{A_n}\varphi(c)d\mu\right) \\ &= \infty\end{aligned}$$

4 Purely Atomic Case

This section is devoted to present new results involving the condition (1) and the δ_2-condition in the purely atomic case. In this case, the integrals present in the definition of condition (1) are replaced by sums and the functions be replaced by sequences of functions.

Given $\Phi = \{\Phi_i\}$ a Musielak-Orlicz function, we define the Musielak-Orlicz class in the purely atomic case as being $\widetilde{\ell^{\Phi}} = \{u \in L^0; \sum_{i=1}^{\infty} \Phi_i(c) < \infty\}$.

In the next proposition, we find a condition equivalent to the sequence of functions $u_0 = \{u_{0,i}\}$ and a deformed exponential function φ satisfying the condition (1). In order to provide such result we will need of the following lemma.

Lemma 2. *Let Ψ and Φ be finite-valued Musielak-Orlicz functions. Then, the inclusion $\widetilde{\ell^{\Phi}} \subseteq \widetilde{\ell^{\Psi}}$ holds if, and only if, there exist ε, $\alpha > 0$ and a sequence of non-negative real numbers $f = \{f_i\} \in \widetilde{\ell^{\Phi}}$ such that*

$$\alpha \Psi_i(u) \leq \Phi_i(u), \text{ for all } u > f_i \text{ with } \Phi_i(u) < \varepsilon. \tag{6}$$

The proof of the Lemma 2 is analogous to the proof provided in [5, Theorem 8.4].

Proposition 4. *A sequence $u_0 = \{u_{0,i}\}$ and a deformed exponential function φ satisfy the condition (1) if, and only if, for some sequence $c = \{c_i\}$ of numbers such that $\sum_{i=1}^{\infty} \varphi(c_i) = 1$, we can find constants ε, λ, $\alpha > 0$ and a sequence $f = \{f_i\} \in \widetilde{\ell^{\Phi_c}}$ of non-negative real numbers such that*

$$\alpha \Phi_{c,i}(u) \leq \Phi_{c-\lambda u_{0,i}}(u), \text{ for all } u > f_i \text{ with } \Phi_{c-\lambda u_{0,i}}(u) < \varepsilon. \tag{7}$$

The proof of the Proposition 4 is analogous to the proof provided in [1, Proposition 2].

The authors in [12] have proven that if a Musielak-Orlicz function Φ does not satisfy the Δ_2-condition then we can find sequences satisfying some conditions and, from this, we obtain functions admitting the properties in (10).

The sequence $\Phi = \{\Phi_i\}$ does satisfy the δ_2-condition if, and only if, for every $\lambda \in (0,1)$, there exist constants $\varepsilon > 0$, $\alpha \in (0,1)$, and a non-negative sequence $f = \{f_i\}$ with $I_{\Phi}(f) < \infty$ such that

$$\alpha \Phi_i(u) \leq \Phi_i(\lambda u), \text{ for all } u > f_i \text{ with } \Phi_i(u) < \varepsilon. \tag{8}$$

Lemma 3. *Let $\Phi = \{\Phi_i\}$ be a finite-valued Musielak-Orlicz function which does not satisfy the δ_2-condition. Then we can find a strictly increasing sequence $\{\lambda_n\}$ in $(0,1)$ converging upward to 1, and sequences $\{u_n\}$ and $\{A_n\}$ of finite-valued real numbers, and pairwise disjoint sets in $\mathbb{N}$, respectively, such that*

$$1 - 2^{-n} \leq I_{\Phi}(u_n \chi_{A_n}) \leq 1 \text{ and } I_{\Phi}(\lambda_n u_n \chi_{A_n}) \leq 2^{-n}, \tag{9}$$

for all $n \geq 1$.

Proof. Suppose that the Musielak-Orlicz function Φ does not satisfy the δ_2-condition. Let $\{\lambda_n\}$ be a strictly increasing sequence in $(0,1)$ such that $\lambda_n \uparrow 1$. For each $n \geq 1$, we define the non-negative sequence $u_n = \{u_{n,i}\}$ by

$$u_{n,i} = \sup\{u > 0; 2^{-n}\Phi_i(u) \leq \Phi_i(\lambda_n u) \text{ and } \Phi_i(u) < 2^{-n}\},$$

where we adopt the convention $\sup \emptyset = 0$. Since (8) is not satisfied, we have that $I_{\Phi}(u_n) = \infty$ for each $n \geq 1$. Because $\Phi_i(u_{n,i}) \leq 2^{-n}$, we can find an increasing sequence $\{k_n\} \subset \mathbb{N}$ such that

$$1 - 2^{-n} \leq \sum_{i=k_{n-1}}^{k_n} \Phi_i(u_{n,i}) \leq 1.$$

The second inequality above in conjunction with $2^{-n}\Phi_i(u_{n,i}) \geq \Phi_i(\lambda_n u_{n,i})$ implies that

$$\sum_{i=k_{n-1}}^{k_n} \Phi_i(\lambda_n u_{n,i}) \leq 2^{-n}.$$

Thus, expression (9) follows with $A_n = [k_{n-1}, k_n - 1] \cap \mathbb{N}$.

Similar to what was done in [12, Remark 3.12], using Lemma 3 let $\Phi = \{\Phi_i\}$ be a finite-valued Musielak-Orlicz function does not satisfying the δ_2-condition. Then we can find functions $u_* = \sum_{n=1}^{\infty} \lambda_n u_n \chi_{A_n}$ and $u^* = \sum_{n=1}^{\infty} u_n \chi_{A_n}$ in L^{Φ} such that

$$\begin{cases} I_{\Phi}(\lambda u_*) < \infty, \text{ for } \ 0 \leq \lambda \leq 1, \\ I_{\Phi}(\lambda u_*) = \infty, \text{ for } \ 1 < \lambda, \end{cases} \quad \begin{cases} I_{\Phi}(\lambda u^*) < \infty, \text{ for } \ 0 \leq \lambda < 1, \\ I_{\Phi}(\lambda u^*) = \infty, \text{ for } \ 1 \leq \lambda. \end{cases} \tag{10}$$

5 Conclusion

We conclude that in the case non-atomic regardless of the condition (1) occurs, the normalizing function converges to a finite value near the boundary of its domain in the case that the functions u belong to the Musielak-Orlicz class. We prove that if the condition (1) does not occur, then the Musielak-Orlicz function does not satisfy the Δ_2-condition. Another important fact is that in the purely atomic case we find an equivalence for the occurence of condition (1) and given a Musielak-Orlicz function not satisfying the δ_2-condition, we find functions in Musielak-Orlicz space satisfying the equations in (10). The perspective for future works is to study, in the case purely atomic, the possibility of relating the condition (1) with the δ_2-condition and the behavior of the normalizing function near the boundary of $\mathcal{B}_c^{\varphi}$. We also want to investigate the behavior of normalizing function, considering that the deformed exponential function is not injective in all its domain.

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