



# Expectation Variables on a Para-Contact Metric Manifold Exactly Derived from Master Equations

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**Abstract.** Based on information and para-contact metric geometries, in this paper a class of dynamical systems is formulated for describing time-development of expectation variables. Here such systems for expectation variables are exactly derived from continuous-time master equations describing nonequilibrium processes.

**Keywords:** Master equations · Para-contact manifolds · Nonequilibrium statistical mechanics · Information geometry

## 1 Introduction

Information geometry is a geometrization of mathematical statistics [1, 2], and its differential geometric aspects and applications in statistics have been investigated. Examples of applications of information geometry include thermodynamics, and some links between equilibrium thermodynamics and information geometry have been clarified. In addition, links between information geometry and contact geometry have been argued [3, 4]. In this context, it was found that para-Sasakian geometry is suitable for describing thermodynamics [5, 6], where para-Sasakian manifolds are para-contact metric manifolds satisfying some additional condition. We then ask how para-contact metric manifolds describe thermodynamics.

In this paper a class of nonequilibrium thermodynamic processes are formulated on a para-contact metric manifold. Most of discussions in this paper have been in [7], and those involving an almost para-contact structure are given in this contribution.

## 2 Preliminaries

In this paper manifolds are assumed smooth and connected. In addition tensor fields are assumed smooth and real. The set of vector fields on a manifold  $\mathcal{M}$  is denoted by  $\mathcal{STM}$ , and the Lie derivative along  $X \in \mathcal{STM}$  by  $\mathcal{L}_X$ .

In this section definitions and some existing statements are summarized. (see [5, 8]).

Let  $\mathcal{M}$  be a  $(2n + 1)$ -dimensional manifold ( $n \geq 1$ ). An *almost para-contact structure* on  $\mathcal{M}$  is a triplet  $(\phi, \xi, \lambda)$ , where  $\xi$  is a vector field,  $\lambda$  a one-form,  $\phi : \mathcal{STM} \rightarrow \mathcal{STM}$  a  $(1, 1)$ -tensor field such that

$$(i) : \phi^2 = \text{Id} - \lambda \otimes \xi, \quad (ii) : \lambda(\xi) = 1, \quad \text{and} \quad (iii) : \ker(\lambda) = \text{Im}(\phi) = \mathcal{D}^+ + \mathcal{D}^-,$$

where  $\ker(\lambda) := \{X \in \mathcal{STM} \mid \lambda(X) = 0\}$ ,  $\text{Im}(\phi) := \{\phi(X, -) \in \mathcal{STM} \mid X \in \mathcal{STM}\}$ ,  $\mathcal{D}^\pm$  are eigen-spaces whose eigenvalues are  $\pm 1$ , and  $\text{Id}$  is an identity operator. A pseudo Riemannian metric tensor field  $g$  satisfying

$$g(\phi X, \phi Y) = -g(X, Y) + \lambda(X)\lambda(Y), \quad \forall X, Y \in \mathcal{STM}$$

is referred to as a *metric tensor compatible with an almost para-contact structure*. It is verified for non-compact manifolds that any almost para-contact structure admits a metric tensor field compatible with an almost para-contact structure. Then  $(\mathcal{M}, \phi, \xi, \lambda, g)$  is referred to as an *almost para-contact metric manifold*.

On almost para-contact metric manifolds, one can show that

$$\lambda\phi = 0, \quad \phi\xi = 0, \quad \lambda(X) = g(X, \xi), \quad g(\xi, \xi) = 1, \quad \text{and} \quad g(\phi X, Y) + g(X, \phi Y) = 0,$$

for  $\forall X, Y \in \mathcal{STM}$ . If  $g$  of an almost para-contact metric manifold satisfies

$$g(X, \phi Y) = \frac{1}{2}d\lambda(X, Y), \quad \forall X, Y \in \mathcal{STM} \tag{1}$$

then,  $(\mathcal{M}, \phi, \xi, \lambda, g)$  is referred to as a *para-contact metric manifold*, where the convention of the numerical factor  $d\lambda(X, Y) = X\lambda(Y) - Y\lambda(X) - \lambda([X, Y])$ , ( $[X, Y] := XY - YX$ ) has been adopted. Para-Sasakian manifolds are para-contact metric manifolds satisfying the so-called normality condition.

Coordinate expressions were given for a para-contact metric manifold (and a para-Sasakian manifold)  $(\mathcal{M}, \phi, \xi, \lambda, g)$  in [5]. They are summarized here. Let  $(x, y, z)$  be coordinates for  $\mathcal{M}$  with  $x = \{x^1, \dots, x^n\}$  and  $y = \{y_1, \dots, y_n\}$  such that  $\lambda = dz - y_a dx^a$  where the Einstein convention has been used. Introduce the pseudo-Riemannian metric tensor field, referred to as the *Mruagala metric tensor field* [11],

$$g^M = \frac{1}{2}dx^a \otimes dy_a + \frac{1}{2}dy_a \otimes dx^a + \lambda \otimes \lambda, \tag{2}$$

which is shown to induce a para-contact metric manifold. In what follows we consider the case where  $y_a > 0$  for all  $a \in \{1, \dots, n\}$ . Introduce the co-frame  $\{\hat{\theta}^0, \hat{\theta}^1_-, \hat{\theta}^1_+, \dots, \hat{\theta}^n_-, \hat{\theta}^n_+\}$  and frame  $\{e_0, e_1^-, e_1^+, \dots, e_n^-, e_n^+\}$  with

$$\begin{aligned} \hat{\theta}^0 &:= \lambda, & \hat{\theta}^\pm_a &:= \frac{1}{2\sqrt{y_a}} [y_a dx^a \pm dy_a], & (\text{no sum over } a), \\ e_0 &:= \xi, & e^\pm_a &:= \sqrt{y_a} \left[ \frac{1}{y_a} \left( \frac{\partial}{\partial x^a} + y_a \frac{\partial}{\partial z} \right) \pm \frac{\partial}{\partial y_a} \right], & (\text{no sum over } a), \end{aligned}$$

so that

$$\widehat{\theta}^0(e_0) = 1, \quad \widehat{\theta}_+^a(e_b^+) = \widehat{\theta}_-^a(e_b^-) = \delta_b^a, \quad \text{others vanish,}$$

where  $\delta_b^a$  is the Kronecker delta, giving unity for  $a = b$  and zero otherwise. One can then show that

$$g^M = \widehat{\theta}^0 \otimes \widehat{\theta}^0 + \sum_{a=1}^n \widehat{\theta}_+^a \otimes \widehat{\theta}_+^a - \sum_{a=1}^n \widehat{\theta}_-^a \otimes \widehat{\theta}_-^a, \quad \xi = \frac{\partial}{\partial z}, \quad \phi = -\widehat{\theta}_-^a \otimes e_a^+ - \widehat{\theta}_+^a \otimes e_a^-.$$

Then, introducing the abbreviation  $\phi(X) := \phi(X, -) \in STM$  for  $X \in STM$ , one has  $\phi(e_a^+) = -e_a^-$ ,  $\phi(e_a^-) = -e_a^+$ , and  $\phi(e_0) = 0$ .

In the context of geometry of thermodynamics, contact manifold is identified with the so-called thermodynamic phase space [12]. This manifold is defined as follows (see [9] for details). Let  $\mathcal{C}$  be a  $(2n + 1)$ -dimensional manifold ( $n = 1, 2, \dots$ ), and  $\lambda$  a one-form. If  $\lambda$  satisfies

$$\lambda \wedge \underbrace{d\lambda \wedge \dots \wedge d\lambda}_n \neq 0,$$

then the pair  $(\mathcal{C}, \lambda)$  is referred to as a *contact manifold*, and  $\lambda$  a *contact one-form*. It has been known as the Darboux theorem that there exists a special set of coordinates  $(x, y, z)$  with  $x = \{x^1, \dots, x^n\}$  and  $y = \{y_1, \dots, y_n\}$  such that  $\lambda = dz - y_a dx^a$ . It follows from (1) that para-contact metric manifolds are contact manifolds.

The *Legendre submanifold*  $\mathcal{A} \subset \mathcal{C}$  is an  $n$ -dimensional submanifold where  $\lambda|_{\mathcal{A}} = 0$  holds. One can verify that

$$\mathcal{A}_\varpi = \left\{ (x, y, z) \mid y_a = \frac{\partial \varpi}{\partial x^a}, \quad \text{and} \quad z = \varpi(x) \right\}, \quad (3)$$

is a Legendre submanifold, where  $\varpi : \mathcal{C} \rightarrow \mathbb{R}$  is a function of  $x$  on  $\mathcal{C}$ . The submanifold  $\mathcal{A}_\varpi$  is referred to as the *Legendre submanifold generated by  $\varpi$* , and is used for describing equilibrium thermodynamic systems [12].

As shown in [3] and [6], a class of relaxation processes, initial states approach to the equilibrium state as time develops, can be formulated as contact Hamiltonian vector fields on contact manifolds. This statement on a class of contact Hamiltonian vector fields can be summarized as follows.

**Proposition 1** (*Legendre submanifold as an attractor, [3]*). *Let  $(\mathcal{C}, \lambda)$  be a  $(2n + 1)$ -dimensional contact manifold with  $\lambda$  being a contact form,  $(x, y, z)$  its coordinates so that  $\lambda = dz - y_a dx^a$ , and  $\varpi$  a function depending only on  $x$ . Then, one has*

1. *The contact Hamiltonian vector field associated with the contact Hamiltonian  $h : \mathcal{C} \rightarrow \mathbb{R}$  such that  $h(x, y, z) = \varpi(x) - z$ , gives*

$$\frac{d}{dt} x^a = 0, \quad \frac{d}{dt} y_a = \frac{\partial \varpi}{\partial x^a} - y_a, \quad \frac{d}{dt} z = \varpi(x) - z. \quad (4)$$

2. The Legendre submanifold generated by  $\varpi$ , given by (3), is an invariant manifold for the contact Hamiltonian vector field.
3. Every point on  $\mathcal{C} \setminus \mathcal{A}_\varpi$  approaches to  $\mathcal{A}_\varpi$  along an integral curve as time develops. Equivalently  $\mathcal{A}_\varpi$  is an attractor in  $\mathcal{C}$ .
4. Let  $\{x(0), y(0), z(0)\}$  be a point on  $\mathcal{C} \setminus \mathcal{A}_\varpi$ . Then for any  $t \in \mathbb{R}$ ,

$$h(x(t), y(t), z(t)) = \exp(-t) h(x(0), y(0), z(0)).$$

### 3 Solvable Master Equations

In this section a set of master equations with particular Markov kernels is introduced, and then its solvability is shown.

Let  $\Gamma$  be a set of finite discrete states,  $t \in \mathbb{R}$  time, and  $p(j, t) dt$  a probability that a state  $j \in \Gamma$  is found in between  $t$  and  $t + dt$ . The first objective is to realize a given distribution function  $p_\theta^{\text{eq}}$  that can be written as

$$p_\theta^{\text{eq}}(j) = \frac{\pi_\theta(j)}{Z(\theta)}, \quad Z(\theta) := \sum_{j \in \Gamma} \pi_\theta(j)$$

where  $\theta \in \Theta \subset \mathbb{R}^n$  is a parameter set with  $\theta = \{\theta^1, \dots, \theta^n\}$ , and  $Z : \Theta \rightarrow \mathbb{R}$  the so-called partition function so that  $p_\theta^{\text{eq}}$  is normalized:  $\sum_{j \in \Gamma} p_\theta^{\text{eq}}(j) = 1$ .

In what follows, attention is focused on a class of master equations. Let  $p : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  be a time-dependent probability function. Then, consider the set of master equations

$$\frac{\partial}{\partial t} p(j, t) = \sum_{j' (\neq j)} [w(j|j') p(j', t) - w(j'|j) p(j, t)], \tag{5}$$

where  $w : \Gamma \times \Gamma \rightarrow I$ , ( $I := [0, 1] \subset \mathbb{R}$ ) is such that  $w(j|j')$  denotes a probability that a state jumps from  $j'$  to  $j$ . With (5) and the assumptions

$$w_\theta(j|j') = p_\theta^{\text{eq}}(j), \quad \text{and} \quad p_\theta^{\text{eq}}(j) \neq 0, \quad \forall j \in \Gamma,$$

one derives the *solvable master equations*:

$$\frac{\partial}{\partial t} p(j, t) = p_\theta^{\text{eq}}(j) - p(j, t). \tag{6}$$

An explicit form of  $p(j, t)$  is obtained by solving (6). Then the following proposition can easily be shown.

**Proposition 2** (*Solutions of the master equations, [7]*). *The solution of (6) is*

$$p(j, t) = e^{-t} p(j, 0) + (1 - e^{-t}) p_\theta^{\text{eq}}(j), \quad \text{from which} \quad \lim_{t \rightarrow \infty} p(j, t) = p_\theta^{\text{eq}}(j).$$

With this proposition, one notices that every solution  $p$  depends on  $\theta$ . Taking into account this,  $p(j, t)$  is denoted  $p(j, t; \theta)$ . Also notice that the equilibrium state is realized with (6) as the time-asymptotic limit.

### 4 Time-Development of Observables

In this section differential equations describing time-development of observables are derived with the solvable master equations under some assumptions. Then, the time-asymptotic limit of such observables is stated. Here *observable* in this paper is defined as a function that does not depend on a random variable or a state. Thus expectation values with respect to a probability distribution function are observables.

Let  $\mathcal{O}_a : \Gamma \rightarrow \mathbb{R}$  be a function with  $a \in \{1, \dots, n\}$ , and  $p : \Gamma \times \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$  a distribution function that follows (6). Then

$$\langle \mathcal{O}_a \rangle_\theta(t) := \sum_{j \in \Gamma} \mathcal{O}_a(j) p(j, t; \theta), \quad \text{and} \quad \langle \mathcal{O}_a \rangle_\theta^{\text{eq}} := \sum_{j \in \Gamma} \mathcal{O}_a(j) p_\theta^{\text{eq}}(j),$$

are referred to as the *expectation variable* of  $\mathcal{O}_a$  with respect to  $p$ , and that with respect to  $p_\theta^{\text{eq}}$ , respectively.

If an equilibrium distribution function belongs to the exponential family, then the function  $\Psi^{\text{eq}} : \Theta \rightarrow \mathbb{R}$  with

$$\Psi^{\text{eq}}(\theta) := \ln \left( \sum_{j \in \Gamma} e^{\theta^b \mathcal{O}_b(j)} \right), \tag{7}$$

plays various roles. Here and in what follows, (7) is assumed to exist. In the context of information geometry, this function is referred to as a  $\theta$ -potential. Discrete distribution functions are considered in this paper and it has been known that such distribution functions belong to the exponential family, then  $\Psi^{\text{eq}}$  in (7) also plays a role throughout this paper. The value  $\Psi^{\text{eq}}(\theta)$  can be interpreted as the negative dimension-less free-energy. It follows from (7) that

$$\langle \mathcal{O}_a \rangle_\theta^{\text{eq}} = \frac{\partial \Psi^{\text{eq}}}{\partial \theta^a}.$$

One then can generalize  $\Psi^{\text{eq}}$  defined at equilibrium state to a function defined in nonequilibrium states as  $\Psi : \Theta \times \mathbb{R} \rightarrow \mathbb{R}$ ,

$$\Psi(\theta, t) := \left( \frac{1}{J^0} \sum_{j \in \Gamma} \frac{p(j, t; \theta)}{p_\theta^{\text{eq}}(j)} \right) \Psi^{\text{eq}}(\theta), \quad \text{where} \quad J^0 := \sum_{j' \in \Gamma} 1.$$

Since  $p_\theta^{\text{eq}}(j) \neq 0$  and  $\Psi^{\text{eq}}(\theta) < \infty$  by assumptions, the function  $\Psi$  exists. Generalizing the idea for the equilibrium case, the function  $\Psi$  may be interpreted as a nonequilibrium negative dimension-less free-energy.

A set of differential equations for  $\{\langle \mathcal{O}_a \rangle_\theta\}$  and  $\Psi$  can be derived as follows.

**Proposition 3** (*Dynamical system obtained from the master equations, [7]*). *Let  $\theta$  be a time-independent parameter set characterizing a discrete distribution function  $p_\theta^{\text{eq}}$ . Then  $\{\langle \mathcal{O}_a \rangle_\theta\}$  and  $\Psi$  are solutions to the differential equations on  $\mathbb{R}^{2n+1}$*

$$\frac{d}{dt} \theta^a = 0, \quad \frac{d}{dt} \langle \mathcal{O}_a \rangle_\theta = - \langle \mathcal{O}_a \rangle_\theta + \frac{\partial \Psi^{\text{eq}}}{\partial \theta^a}, \quad \text{and} \quad \frac{d}{dt} \Psi = - \Psi + \Psi^{\text{eq}}.$$

*Remark 1.* The explicit time-dependence for this system is obtained as  $\theta^a(t) = \theta^a(0)$ , and  $\Psi(\theta, t) = e^{-t} [\Psi(0) - \Psi^{\text{eq}}(\theta)] + \Psi^{\text{eq}}(\theta)$ , and

$$\langle \mathcal{O}_a \rangle_\theta(t) = e^{-t} \left[ \langle \mathcal{O}_a \rangle_\theta(0) - \frac{\partial \Psi^{\text{eq}}}{\partial \theta^a} \right] + \frac{\partial \Psi^{\text{eq}}}{\partial \theta^a}.$$

From these, one can verify that the time-asymptotic limit of these variables are those defined at equilibrium. In this paper this dynamical system is referred to as the *moment dynamical system*.

## 5 Geometric Description of Dynamical Systems

Several geometrization of nonequilibrium states for some models and methods have been proposed. Yet, suffice to say that there remains no general consensus on how best to extend a geometry of equilibrium states to a geometry of nonequilibrium states. In this section, a geometrization of nonequilibrium states is proposed for the moment dynamical system.

### 5.1 Geometry of Equilibrium States

Equilibrium states are identified with the Legendre submanifolds generated by functions in the context of geometric thermodynamics [10, 12]. Besides, in the context of information geometry, equilibrium states are identified with dually flat spaces [1]. Combining these identifications, one has the following.

**Proposition 4** *(A contact manifold and a strictly convex function induce a dually flat space, [3]). Let  $(\mathcal{C}, \lambda)$  be a contact manifold,  $(x, y, z)$  a set of coordinates such that  $\lambda = dz - y_a dx^a$  with  $x = \{x^1, \dots, x^n\}$  and  $y = \{y_1, \dots, y_n\}$ , and  $\varpi$  a strictly convex function depending only on  $x$ . Then,  $((\mathcal{C}, \lambda), \varpi)$  induces an  $n$ -dimensional dually flat space*

To apply the proposition above to physical systems, the coordinate sets  $x$  and  $y$  are chosen such that  $x^a$  and  $y_a$  form a thermodynamic conjugate pair for each  $a$ . Here it is assumed that such thermodynamic variables can be defined even for nonequilibrium states, and that they are consistent with those variables defined at equilibrium. In addition to this, the physical dimension of  $\varpi$  should be equal to that of  $y_a dx^a$ . Also  $\Psi$  and its Legendre transform are chosen as  $\varpi$ .

### 5.2 Geometry of Nonequilibrium States

So far geometry of equilibrium states have been discussed. One remaining issue is how to give the physical meaning of the set outside  $\mathcal{A}_\varpi$ ,  $\mathcal{C} \setminus \mathcal{A}_\varpi$ . A natural interpretation of  $\mathcal{C} \setminus \mathcal{A}_\varpi$  would be some set of nonequilibrium states. We make this interpretation in this paper.

As shown in Proposition 2, initial states approach to the equilibrium state as time develops. This can be reformulated on contact manifolds and para-contact

metric manifolds. In the contact geometric framework of nonequilibrium thermodynamics, the equilibrium state is identified with a Legendre submanifold. Then, as found in [3] and [6], some dynamical systems expressing nonequilibrium process can be identified with a class of contact Hamiltonian vector fields on a contact manifold. The above claim also holds on para-contact metric manifolds.

**Geometry of Moment Dynamical System.** Proposition 3 is written in a contact geometric language here. In what follows phase space is identified with a  $(2n + 1)$ -dimensional para-contact metric manifold  $(\mathcal{C}, \phi, \xi, \lambda, g^M)$ .

As shown below, the moment dynamical system is a contact Hamiltonian system.

**Proposition 5** (*Moment dynamical system as a contact Hamiltonian system, [7]*). *The dynamical system in Proposition 3 can be written as a contact Hamiltonian system.*

One is interested in how a  $(1, 1)$ -tensor field  $\phi$  plays a role for geometric nonequilibrium thermodynamics. To give an answer, one needs the following.

**Lemma 1.** *Let  $\{\dot{x}_a\}, \{\dot{y}_a\}, \dot{z}$  be some functions, and  $X_0$  the vector field*

$$X_0 = \dot{x}^a \frac{\partial}{\partial x^a} + \dot{y}_a \frac{\partial}{\partial y_a} + \dot{z} \frac{\partial}{\partial z}.$$

*Then,  $\phi(X_0)$  and  $\phi^2(X_0)$  are calculated as*

$$\phi^\mu(X_0) = (-1)^\mu \dot{x}^a \left( \frac{\partial}{\partial x^a} + y_a \frac{\partial}{\partial z} \right) + \dot{y}_a \frac{\partial}{\partial y_a}, \quad \mu = 1, 2.$$

*Proof.* Throughout this proof, the Einstein convention is not used. With the local expressions shown in Sect. 2, one has

$$\hat{\theta}_\pm^a(X_0) = \frac{\sqrt{y_a}}{2} \dot{x}^a \pm \frac{\dot{y}_a}{2\sqrt{y_a}},$$

$$e_a^+ + e_a^- = \frac{2}{\sqrt{y_a}} \left( \frac{\partial}{\partial x^a} + y_a \frac{\partial}{\partial z} \right), \quad \text{and} \quad e_a^+ - e_a^- = 2\sqrt{y_a} \frac{\partial}{\partial y_a}.$$

Combining these, one has

$$\begin{aligned} \phi(X_0) &= \sum_a \left[ -\hat{\theta}_-^a(X_0) e_a^+ + \hat{\theta}_+^a(X_0) e_a^- \right] \\ &= \sum_a \left[ -\frac{\sqrt{y_a}}{2} \dot{x}^a (e_a^+ + e_a^-) + \frac{\dot{y}_a}{2\sqrt{y_a}} (e_a^+ - e_a^-) \right] \\ &= \sum_a \left[ -\dot{x}^a \left( \frac{\partial}{\partial x^a} + y_a \frac{\partial}{\partial z} \right) + \dot{y}_a \frac{\partial}{\partial y_a} \right]. \end{aligned}$$

For  $\phi^2(X_0)$ , substituting  $\lambda(X_0) = \dot{z} - \sum_a y_a \dot{x}^a$  into  $\phi^2(X_0) = X_0 - \lambda(X_0)\xi$ , one has the desired expression. □

Applying this Lemma, one has the following.

**Theorem 1** (*Roles of  $\phi$  of  $X_h$  for the moment dynamical system*). *Let  $X_h$  be the contact Hamiltonian vector field in Proposition 1. Then*

$$\mathcal{L}_{\phi(X_h)}h = \mathcal{L}_{\phi^2(X_h)}h = 0.$$

*Proof.* Substituting  $\dot{x}^a = 0$  into  $\phi^\mu(X_0)$  in Lemma 1, one has

$$\phi^\mu(X_h) = \dot{y}_a \frac{\partial}{\partial y_a}, \quad \text{where} \quad \dot{y}_a = \frac{\partial \varpi}{\partial x^a} - y_a, \quad \mu = 1, 2.$$

Then, with  $\partial h / \partial y_a = 0$ , one has

$$\mathcal{L}_{\phi^\mu(X_h)}h = [\phi^\mu(X_h)]h = 0, \quad \mu = 1, 2.$$

□

This states that the  $h$  is preserved along  $\phi^\mu(X_h) \in \mathcal{STC}$ , which should be compared with the case of  $\mathcal{L}_{X_h}h$ :

$$\mathcal{L}_{X_h}h = -\dot{z} = -(\varpi(x) - z) = -h.$$

**Curve Length from the Equilibrium State.** In nonequilibrium statistical physics, attention is often concentrated on how far a state is close to the equilibrium state. In general, to define and measure such a distance in terms of geometric language, length of a curve can be used. In Riemannian geometry, length is a measure for expressing how far given two points are away, where these points are connected with an integral curve of a vector field on a manifold.

The following can easily be proven.

**Lemma 2** ([7]). *The length between a state and the equilibrium state for the moment dynamical system calculated with (2) is*

$$l[X_\Psi]_\infty^t := \int_\infty^t \sqrt{g^M(X_\Psi, X_\Psi)} dt = |h(\theta, \langle \mathcal{O} \rangle_\theta, \Psi)|, \tag{8}$$

where  $\langle \mathcal{O} \rangle_\theta = \{\langle \mathcal{O}_1 \rangle_\theta, \dots, \langle \mathcal{O}_n \rangle_\theta\}$ ,  $h$  is such that  $h(\theta, \langle \mathcal{O} \rangle_\theta, \Psi) = \Psi^{\text{eq}}(\theta) - \Psi$  (see Proposition 1), and  $X_\Psi$  its corresponding contact Hamiltonian vector field. Then the convergence rate for (8) is exponential.

Combining Lemma 2 and discussions in the previous sections, one arrives at the main theorem in this paper.

**Theorem 2** (*Geometric description of the expectation variables and its convergence*). *The moment dynamical system derived from solvable master equations are described on a para-contact metric manifold, and its convergence rate associated with the metric tensor field (2) is exponential.*



## 6 Conclusions

This paper has offered a viewpoint that expectation variables of the moment dynamical system derived from master equations can be described on a para-contact metric manifold. To give a geometric description of these variables a contact Hamiltonian vector field has been introduced on a para-contact metric manifold. Also, roles of the  $(1, 1)$ -tensor field  $\phi$  have been clarified in this paper (Theorem 1). Then, with the Mrugala metric tensor field, the convergence rate has been shown to be exponential on this para-contact metric manifold (Theorem 2).

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