α-power Sums on Symmetric Cones

Keiko Uohashi(✉)

Tohoku Gakuin University, Tagajo, Miyagi 985-8537, Japan
uohashi@mail.tohoku-gakuin.ac.jp

Abstract. In this paper, we define α-power sums of two or more elements on symmetric cones. For two elements, α-power sums, which are generalized parallel sums, are defined on our previous paper. We mention interpolation for α-power sums, which is not defined on our previous paper. It is shown that the synthesized resistances of α-series parallel circuits naturally correspond to α-power sums. We also mention relations with power sums and arithmetic, geometric, harmonic and α-power means, where α is a parameter of dualistic structure on information geometry.

Keywords: Parallel sum · Power sum · Mean · Operator monotone function · Symmetric cone · Series parallel circuit

1 Introduction

Arithmetic, geometric and harmonic mean are well known means on positive operators [1–3]. α-power mean (or power mean) is a generalized geometric mean, and corresponds to arithmetic, geometric and harmonic mean for $\alpha = 1, 0$ and -1, respectively [4–7]. On a symmetric cone, the α-power mean is the midpoint on the α-geodesic connecting two points, where α is a parameter of dualistic structure on information geometry [8,9].

Parallel sum is the half of harmonic mean [10,11]. However, it seems that few literatures treat sums related to geometric and α-mean for reasons of difficulty of convergence. Then, we define α-power means which are continuous for α and are arithmetic sum, parallel sum for $\alpha = 1, -1$, respectively.

First, we recall definitions and properties on symmetric cones. In Sect. 3, means and monotone functions are mentioned. In Sect. 4, we show definitions of α-power sums and the operator monotone function generating α-power sums. In Sect. 5, we define interpolation for α-power sums. Finally, we show a continuous deformation of the series circuit into the parallel circuit in which resistance elements have fixed resistivity and fixed volumes. The circuits realize arithmetic sum and parallel sum for $\alpha = 1, -1$, respectively.

Applications of α-power mean appear in fields of functional analysis, quantum mechanics, nonextensive statistical mechanics, optimization and information geometry. We expect to find applications of α-power sum as α-power mean.

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2 Symmetric Cones

A vector space V is called a Jordan algebra if a product $*$ defined on V satisfies

$$x * y = y * x, \quad x * (x^2 * y) = x^2 * (x * y) \tag{1}$$

for all $x, y \in V$ by setting $x^2 = x * x$. Let V be an n-dimensional Jordan algebra over $\mathbf{R}$ with an identity element e, i.e., $x * e = e * x = x$. An element $x \in V$ is said to be invertible if there exists $y \in \mathbf{R}[x]$ such that $x * y = e$, where $\mathbf{R}[X]$ is polynomials of X over $\mathbf{R}$. Since $\mathbf{R}[x]$ is an associative algebra, y is unique, called the inverse of x and denoted by $x^{-1} = y$ [8,12,13].

For x in V, let $L(x)$ and $P(x)$ be endomorphisms of V defined by

$$L(x)y = x * y, \quad y \in V \tag{2}$$

$$P(x) = 2L(x)^2 - L(x^2). \tag{3}$$

The following results, about P the quadratic representation of V, are known.

Proposition 1. *([12]) (i) An element x is invertible if and only if $P(x)$ is invertible, and*

$$P(x)x^{-1} = x, \quad P(x)^{-1} = P(x^{-1}). \tag{4}$$

(ii) If x and y are invertible, so is $P(x)y$ and

$$(P(x)y)^{-1} = P(x^{-1})y^{-1}. \tag{5}$$

(iii) For all x and y,

$$P(P(y)x) = P(y)P(x)P(y). \tag{6}$$

Let Ω be an open convex cone on a vector space V. We denote by G the identity component of the linear automorphism group of Ω. If G acts on Ω transitively, Ω is said to be homogeneous. The dual cone of Ω is defined by

$$\Omega^* = \{y \in V \mid (x, y) > 0, \forall x \in \bar{\Omega}\backslash\{0\}\}, \tag{7}$$

where $(\ ,\ )$ is an inner product on V, $\bar{\Omega}$ the closure of Ω. If $\Omega = \Omega^*$, a cone Ω is said to be self-dual. A cone Ω is called symmetric if it is homogeneous and self-dual.

3 Means and Operator Monotone Functions

We consider a symmetric cone Ω a set of positive operators.

Let $x = \sum_{i=1}^{r} \lambda_i p_i$ be a spectral decomposition of $x \in V$, where r and $\{p_1, \ldots, p_r\}$ are the rank and a Jordan frame of V, respectively, and $\lambda_1, \ldots, \lambda_r$ are eigenvalues of x [12]. For a function $f(t)$ on an interval $\mathbf{I} \subseteq \mathbf{R}$, $f(x)$ is defined by

$$f(x) = \sum_{i=1}^{r} f(\lambda_i) p_i \tag{8}$$

if $\lambda_1, \ldots, \lambda_r \in \mathbf{I}$. A function $f(t)$ on an interval $\mathbf{I} \subseteq \mathbf{R}$ satisfying Inequation () is called an operator monotone function on $\mathbf{I}$.

$$a \leq b \Rightarrow f(a) \leq f(b), \tag{9}$$

where a and $b \in \Omega$ have eigenvalues on $\mathbf{I}$, respectively.

A binary operation $\sigma : (a, b) \in \bar{\Omega} \times \bar{\Omega} \mapsto a\sigma b \in \bar{\Omega}$ is called an operator connection if the following requirements are fulfilled.

(i) Monotonicity; $a \leq c$ and $b \leq d$ imply $a\sigma b \leq c\sigma d$,
(ii) Transformer inequality; $P(c)(a\sigma b) \leq (P(c)(a))\sigma(P(c)(b))$,
(iii) Semi-continuity; $a_n \downarrow a$ and $b_n \downarrow b$ imply $(a_n \sigma b_n) \downarrow a\sigma b$,

where $a \leq b$ (resp. $a < b$) is $b - a \in \bar{\Omega}$ (resp. in Ω) [1,8].

On transformer inequality, it holds that $P(c)(a\sigma b) = (P(c)(a))\sigma(P(c)(b))$ for Ω. If satisfying normalization $e\sigma e = e$, an operator connection σ is called an operator mean (or a mean).

It is known that α-power mean on Ω is generated by

$$a\sigma^{(\alpha)}b = P(a^{\frac{1}{2}})f^{(\alpha)}(P(a^{-\frac{1}{2}})b), \quad -1 \leq \alpha \leq 1, \tag{10}$$

where f is an operator monotone function defined by

$$f^{(\alpha)}(t) = \left(\frac{1 + t^\alpha}{2}\right)^{\frac{1}{\alpha}} \quad (\alpha \neq 0), \quad f^{(0)}(t) = \sqrt{t} \tag{11}$$

[6,8]. Arithmetic, geometric and harmonic mean are described by $a\sigma^{(1)}b$, $a\sigma^{(0)}b$ and $a\sigma^{(-1)}b$, respectively. In particular, for positive definite matrices A and B, they are

(i) arithmetic mean; $A\sigma^{(1)}B = (A + B)/2$,
(ii) geometric mean; $A\sigma^{(0)}B = A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$,
(iii) harmonic mean; $A\sigma^{(-1)}B = ((A^{-1} + B^{-1})/2)^{-1}$,
(iv) α-power mean; $A\sigma^{(\alpha)}B = ((A^\alpha + B^\alpha)/2)^{1/\alpha}$.

For scalar A and B, the geometric mean is $A\#B = \sqrt{AB}$.

4 α-power Sums and Operator Monotone Functions

In our previous paper, we defined α-power sum via an operator monotone function, which interpolates generalized sum between arithmetic sum and parallel sum [14].

For $-1 \leq \alpha \leq 1$, a function

$$f^{(\alpha)}(t) = \frac{(1 + t)^{1+\alpha}}{1 + t^\alpha}, \quad t > 0 \tag{12}$$

is an operator monotone function on $\{t | t^\alpha - \alpha t^{\alpha-1} + \alpha + 1 > 0\}$. Function (12) is obviously monotone increasing and operator monotone with α.

Definition 1. *([14]) Let $f^{(\alpha)}(t)$ be a function defined by Eq. (12). For $-1 \leq \alpha \leq 1$, we define the α-power sum $:^{(\alpha)}$ of a and $b \in \Omega$ by*

$$a :^{(\alpha)} b = P(a^{\frac{1}{2}}) f^{(\alpha)}(P(a^{-\frac{1}{2}})b). \tag{13}$$

Theorem 1. *([14]) The α-power sum $:^{(\alpha)}$ of a and $b \in \Omega$ corresponds to arithmetic sum $a+b$ for $\alpha = 1$, and to parallel sum $a : b = (a^{-1}+b^{-1})^{-1}$ for $\alpha = -1$. The 0-power sum $a :^{(0)} b$ is arithmetic mean $(a+b)/2$.*

Corollary 1. *([14]) For positive definite matrices A and B, α-power sums are*

(i) arithmetic sum; $A :^{(1)} B = A + B$,
(ii) 0-power sum; $A :^{(0)} B = (A+B)/2$ (arithmetic mean),
(iii) parallel sum; $A :^{(-1)} B = (A^{-1} + B^{-1})^{-1}$ (the half of harmonic mean),
(iv) α-power sum; $A :^{(\alpha)} B = (A^\alpha + B^\alpha)^{-1/2}(A + B)^{1+\alpha}(A^\alpha + B^\alpha)^{-1/2}$ $= (A+B)^{(1+\alpha)/2}(A^\alpha + B^\alpha)^{-1}(A+B)^{(1+\alpha)/2}$.

For scalar A and B, the α-power sum is

$$A :^{(\alpha)} B = \frac{(A+B)^{1+\alpha}}{A^\alpha + B^\alpha}. \tag{14}$$

If defined by $(A^\alpha + B^\alpha)^{1/\alpha}$ which is α-power mean without normalization property, generalized sum diverges to $\pm\infty$ as $\alpha = 0$. The α-power sum by Eqs. (12), (13) possesses continuity at $\alpha = 0$. It satisfies (i) Monotonicity for elements with eigenvalues on an interval $\{t | t^\alpha - \alpha t^{\alpha-1} + \alpha + 1 > 0\}$. It satisfies (ii) Transformer inequality and (iii) Semi-continuity on $\bar{\Omega}$ (resp. Ω).

The α-power sum of $a_1, \ldots, a_n \in \Omega$ for $n \geq 2$ is defined as follows.

Definition 2. *For $-1 \leq \alpha \leq 1$, we define the α-power sum of $a_1, \ldots, a_n \in \Omega$ for $n \geq 2$ by*

$$a_1 :^{(\alpha)} \cdots :^{(\alpha)} a_n = P(a_1^{\frac{1}{2}}) P((e + \sum_{i=2}^{n} P(a_1^{-\frac{1}{2}})a_i)^{\frac{1+\alpha}{2}})(e + \sum_{i=2}^{n}(P(a_1^{-\frac{1}{2}})a_i)^\alpha)^{-1}. \tag{15}$$

If $n = 2$, the α-power sum $a_1 :^{(\alpha)} a_2$ defined by Definition 2 coincides with $a_1 :^{(\alpha)} a_2$ defined by Definition 1 for a_1 and $a_2 \in \Omega$. In general, it holds that $(a_1 :^{(\alpha)} a_2) :^{(\alpha)} a_3 \neq a_1 :^{(\alpha)} a_2 :^{(\alpha)} a_3$ for a_1, a_2 and $a_3 \in \Omega$.

We obtain the next theorem similar to Corollary 1.

Theorem 2. *For positive definite matrices $A_1, \ldots, A_n$, $n \geq 2$, α-power sums are*

(i) arithmetic sum; $A_1 :^{(1)} \cdots :^{(1)} A_n = A_1 + \cdots + A_n$,
(ii) 0-power sum; $A_1 :^{(0)} \cdots :^{(0)} A_n = (A_1 + \cdots + A_n)/n$,
(iii) parallel sum; $A_1 :^{(-1)} \cdots :^{(-1)} A_n = (A_1^{-1} + \cdots + A_n^{-1})^{-1}$,
(iv) α-power sum; $A_1 :^{(\alpha)} \cdots :^{(\alpha)} A_n$
$= (A_1^\alpha + \cdots + A_n^\alpha)^{-1/2}(A_1 + \cdots + A_n)^{1+\alpha}(A_1^\alpha + \cdots + A_n^\alpha)^{-1/2}$
$= (A_1 + \cdots + A_n)^{(1+\alpha)/2}(A_1^\alpha + \cdots + A_n^\alpha)^{-1}(A_1 + \cdots + A_n)^{(1+\alpha)/2}$.

For scalar $A_1, \ldots, A_n$, $n \geq 2$, the α-power sum is

$$A_1 :^{(\alpha)} \cdots :^{(\alpha)} A_n = \frac{(A_1 + \cdots + A_n)^{1+\alpha}}{A_1^\alpha + \cdots + A_n^\alpha}. \tag{16}$$

Proof. The theorem is proved by calculations similar to techniques on the proof of Corollary 1.

Remark 1. For scalar $A_1, \ldots, A_n$, $n \geq 2$, the α-power sum (16) is the arithmetic sum $A_1 + \cdots + A_n$ multiplied by the ratio of the α-coordinate for $A_1 + \cdots + A_n$ and the arithmetic sum for the α-coordinates A_i^α, $i = 1, \ldots, n$.

5 Interpolation for α-power Sums

Uhlmann's interpolation for an α-power mean $\sigma^{(\alpha)}$ $(-1 \leq \alpha \leq 1)$ is defined by an operator monotone function

$$f_s^{(\alpha)}(t) = (1 - s + st^\alpha)^{\frac{1}{\alpha}} \ \ (\alpha \neq 0), \quad f_s^{(0)}(t) = t^s, \quad 0 \leq s \leq 1 \tag{17}$$

[4,5]. We define interpolation for α-power sums as follows.

Definition 3. *For $-1 \leq \alpha \leq 1$, we define interpolation $:_s^{(\alpha)}$ for an α-power sum $:^{(\alpha)}$ on a symmetric cone Ω by $a :_s^{(\alpha)} b = P(a^{\frac{1}{2}}) f^{(\alpha)}(P(a^{-\frac{1}{2}})b)$, $a, b \in \Omega$, where*

$$f_s^{(\alpha)}(t) = \frac{(2(1-s) + 2st)^{1+\alpha}}{2(1-s) + 2st^\alpha} = \frac{2^\alpha (1 - s + st)^{1+\alpha}}{1 - s + st^\alpha} \tag{18}$$

We have the next theorem via simple calculations.

Theorem 3. *For $-1 \leq \alpha \leq 1$ and $a, b \in \Omega$, they hold that*

$$a :_0^{(\alpha)} b = 2^\alpha a, \quad a :_{\frac{1}{2}}^{(\alpha)} b = a :^{(\alpha)} b, \quad a :_1^{(\alpha)} b = 2^\alpha b. \tag{19}$$

Proof. For a function (18), they hold that

$$f_0^{(\alpha)}(t) = 2^\alpha, \quad f_{\frac{1}{2}}^{(\alpha)}(t) = \frac{(1+t)^{1+\alpha}}{1+t^\alpha}, \quad f_1^{(\alpha)}(t) = 2^\alpha t. \tag{20}$$

Thus, we obtain Eq. (19).

Corollary 2. *For $\alpha = 1, 0$ and -1, interpolation $:_s^{(\alpha)}$ between a and $b \in \Omega$ is described as follows, respectively.*

(i) $a :_s^{(1)} b = 2((1-s)a + sb)$ *(interpolation for arithmetic sum)*

$$a :_0^{(1)} = 2a, \quad a :_{\frac{1}{2}}^{(1)} = a + b, \quad a :_1^{(1)} = 2b \tag{21}$$

(ii) $a :_s^{(0)} b = (1-s)a + sb$ *(interpolation for arithmetic mean)*

$$a :_0^{(0)} = a, \quad a :_{\frac{1}{2}}^{(0)} = \frac{1}{2}(a+b), \quad a :_1^{(0)} = b \tag{22}$$

(iii) $a :_s^{(-1)} b = (2((1-s)a^{-1} + sb^{-1}))^{-1}$ *(interpolation for the half of harmonic mean)*

$$a :_0^{(-1)} = \frac{1}{2}a, \quad a :_{\frac{1}{2}}^{(-1)} = (a^{-1} + b^{-1})^{-1}, \quad a :_1^{(-1)} = \frac{1}{2}b \tag{23}$$

Corollary 3. *For $-1 \leq \alpha \leq 1$ and for scalar A and B, it holds that*

$$A :_s^{(\alpha)} B = \frac{2^\alpha((1-s)A + sB)^{1+\alpha}}{(1-s)A^\alpha + sB^\alpha}. \tag{24}$$

6 Series Parallel Circuits Realizing α-power Sums

In our previous paper, we show series parallel circuits realizing α-power sums of two positive numbers [14]. In this section, we show series parallel circuits realizing α-power sums of two or more positive numbers.

Let the symbol of a parallel sum $A_1 : \cdots : A_n$ be also one of the circuit connecting resistances $A_1, \ldots, A_n$ in parallel. We suppose that electric resistances R_j, $j = 1, \ldots, n$ consist of element with fixed resistivity 1 and fixed cross-sectional areas 1, and that lengths of resistances R_j, $j = 1, \ldots, n$ are $R_j > 0$, respectively. Then, the synthetic resistance of the parallel circuit connecting n resistances with resistivity 1 and length R_j and with cross-sectional areas $R_1/(R_1 + \cdots + R_n), \ldots, R_n/(R_1 + \cdots + R_n)$ is R_j for each j. We give a continuous deformation of $R_1 + \cdots + R_n$ into $R_1 : \cdots : R_n$, using resistances R_{ij} with cross-sectional areas $(R_i/(R_1 + \cdots + R_n))^{(1+\alpha)/2}$, lengths $(R_i/(R_1 + \cdots + R_n))^{(1-\alpha)/2} R_j$ and volumes $R_i R_j/(R_1 + \cdots + R_n)$, $i, j = 1, \ldots, n$, respectively (Fig. 1). Note that, for each i, j, the volume $R_i R_j/(R_1 + \cdots + R_n)$ is constant for all $-1 \leq \alpha \leq 1$.

Theorem 4. *Let $R_j > 0$, $j = 1, \ldots, n$ be constant real numbers, and for $-1 \leq \alpha \leq 1$,*

$$R_{ij} = \left(\frac{R_1 + \cdots + R_n}{R_i}\right)^\alpha R_j\ , \quad i, j = 1, \ldots, n \tag{25}$$

be resistances in an electric circuit. Then, the synthetic resistance of the series circuit connecting parallel circuits $R_{1j} : \cdots : R_{nj}$, $j = 1, \ldots, n$, which we call the α-series parallel circuit, is the α-power sum of $R_1, \ldots, R_n$, i. e.,

$$R_1 :^{(\alpha)} \cdots :^{(\alpha)} R_n = \frac{(R_1 + \cdots + R_n)^{1+\alpha}}{R_1^\alpha + \cdots + R_n^\alpha}. \tag{26}$$

Proof. It follows from Eq. (25) that the synthetic resistance of the series circuit connecting $R_{1j} : \cdots : R_{nj}$, $j = 1, \ldots, n$ is

$$\begin{aligned}
&(R_{11} : \cdots : R_{n1}) + \cdots + (R_{1n} : \cdots : R_{nn}) \\
&= (R_{11}^{-1} + \cdots + R_{n1}^{-1})^{-1} + \cdots + (R_{1n}^{-1} + \cdots + R_{nn}^{-1})^{-1} \\
&= ((R_1 + \cdots + R_n)^{-\alpha} R_1^{\alpha} R_1^{-1} + \cdots + (R_1 + \cdots + R_n)^{-\alpha} R_n^{\alpha} R_1^{-1})^{-1} + \ldots \\
&\quad + ((R_1 + \cdots + R_n)^{-\alpha} R_1^{\alpha} R_n^{-1} + \cdots + (R_1 + \cdots + R_n)^{-\alpha} R_n^{\alpha} R_n^{-1})^{-1} \\
&= (R_1 + \cdots + R_n)^{\alpha} (R_1^{\alpha} + \cdots + R_n^{\alpha})^{-1} (R_1 + \cdots + R_n) \\
&= (R_1 + \cdots + R_n)^{1+\alpha} (R_1^{\alpha} + \cdots + R_n^{\alpha})^{-1} = R_1 :^{(\alpha)} \cdots :^{(\alpha)} R_n \,.
\end{aligned}$$

Remark 2. For $\alpha = 1$, it holds that

$$R_{1j} : \cdots : R_{nj} = (R_{1j}^{-1} + \cdots + R_{nj}^{-1})^{-1} = R_j \,, \quad j = 1, \ldots, n \,.$$

Then, the 1-series parallel circuit is equivalent to series circuit $R_1 + \cdots + R_n$ (Fig. 2).

Remark 3. For $\alpha = 0$, it holds that

$$R_{ij} = R_j, \quad i, j = 1, \ldots, n$$

(Fig. 3).

If $n = 2$, the 0-series parallel circuit is equivalent to the balanced Wheatstone bridge connecting two R_1 in parallel and two R_2 in parallel [15].

Remark 4. For $\alpha = -1$, it holds that

$$R_{i1} + \cdots + R_{in} = (R_1 + \cdots + R_n)^{-1} R_i R_1 + \cdots + (R_1 + \cdots + R_n)^{-1} R_i R_n = R_i \,,$$

$i = 1, \ldots, n$. Then, the (-1)-series parallel circuit is equivalent to parallel circuit $R_1 : \cdots : R_n$ (Fig. 4).

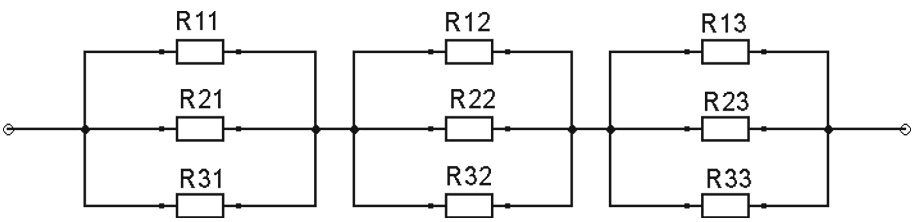

Fig. 1. The α-series parallel circuit ($n = 3$).

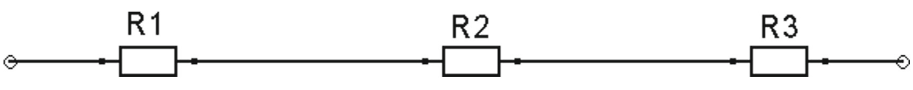

Fig. 2. The series circuit ($\alpha = 1$) ($n = 3$).

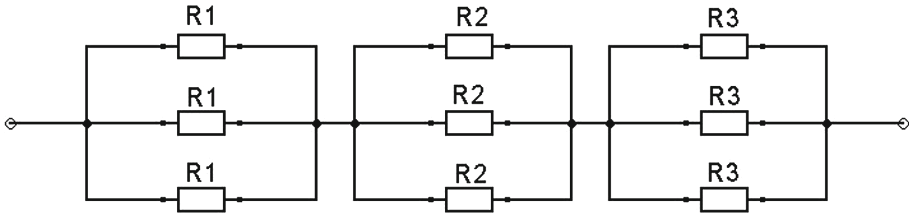

Fig. 3. The 0-series parallel circuit ($n = 3$).

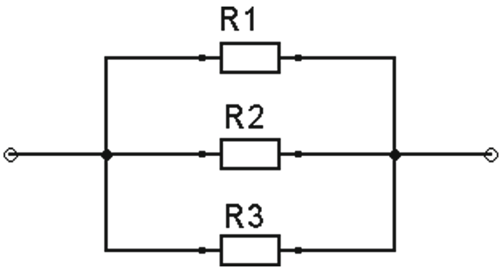

Fig. 4. The parallel circuit ($\alpha = -1$) ($n = 3$).

7 Conclusions

In this paper, we defined α-power sums of two or more elements on symmetric cones. They are generalized sums for arithmetic and parallel sums. We compared monotone functions of α-power sums and means. We also mentioned interpolation describing weighted sums for each α-power sum.

It was shown that the synthesized resistances of α-series parallel circuits naturally correspond to α-power sums. An α-series parallel circuit is the series circuit and the parallel circuit for $\alpha = 1, -1$, respectively.

The assumed medium of the resistances is free to deform. The results may be applicable to the comparison of the electrical properties of metal elements. In addition, characteristics such as fluid and blood flow may be compared with characteristics of the electrical circuit. Applications to fluid in tubes that combine in series and parallel in complexity are also conceivable.

It is a future subject to investigate these through α-power sum and information geometry.

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