Set-Based Extended Functions

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Abstract. In this paper, inspired by the Zadeh approach to the fuzzy connectives in fuzzy set theory and by some applications, we introduce and study set-based extended functions on different universes. After presenting some results for set-based extended functions on a general universe, we focus our investigation on set-based extended functions on some particular universes, including lattices and (bounded) chains. A special attention is devoted to characterization of set-based extended aggregation functions on the unit interval $[0, 1]$.

Keywords: Aggregation function · Extended aggregation function · Extended function · Set-based extended aggregation function · Set-based extended function

1 Introduction

Lotfi Zadeh proposed in his seminal paper [13] to use the minimum and maximum operators for modeling fuzzy intersection and fuzzy union, respectively. This paper focuses on such kinds of fusion procedures that share with Zadeh's proposal a particular property, namely, that these fuzzy connectives can be seen as functions which, for any $n, m \in \mathbb{N}$ and any input vectors $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ and $\mathbf{z} = (z_1, \ldots, z_m) \in [0,1]^m$ such that the sets $\{x_1, \ldots, x_n\}$ and $\{z_1, \ldots, z_m\}$ coincide, provide for input vectors $\mathbf{x}$ and $\mathbf{z}$ the same output values, i.e.,

$$Min(\mathbf{x}) = Min(\mathbf{z}) \quad \text{and} \quad Max(\mathbf{x}) = Max(\mathbf{z}).$$

V. Torra et al. (Eds.): MDAI 2019, LNAI 11676, pp. 41–51, 2019.
https://doi.org/10.1007/978-3-030-26773-5_4

In statistics, for a sample $(x_1, \ldots, x_n)$ several kinds of mean values have been introduced. For example, the arithmetic mean $AM(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} x_i$ is the minimizer of the sum of squares $\sum_{i=1}^{n}(x_i - a)^2$ (Least Squares Method). Minimizing the maximal deviation, i.e., looking for the minimizer of $\max\{|x_i - a| \mid i = 1, \ldots, n\}$ leads to the resulting mean M given by

$$M(\mathbf{x}) = \frac{\min\{x_1, \ldots, x_n\} + \max\{x_1, \ldots, x_n\}}{2}.$$

Observe that repeating or rearrangement of observations does not have any influence on the output of M, i.e., for example, taking a sample

$$\mathbf{z} = (x_1, x_1, x_1, x_2, x_2, x_3, \ldots, x_n),$$

we obtain $M(\mathbf{z}) = M(\mathbf{x})$.

Inspired by the mentioned observations, and taking into account that in most fusion problems the number of values to be fused cannot be fixed a priori, in this paper we will work with extended functions $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, $X \neq \emptyset$, satisfying, in addition, the above discussed property. They will be called set-based extended functions on X (for the definition see below). Evidently, each such set-based extended function depends on the set $\{y_1, \ldots, y_k\}$ of values related to the input vector $(x_1, \ldots, x_n)$, where $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_k\}$ and $\operatorname{card}(\{y_1, \ldots, y_k\}) = k$. Hence, neither the repetition of arguments to be fused nor their rearrangement have any influence on the output result.

We will proceed as follows. First, we propose the concept of set-based extended functions defined for arbitrary but finitely many inputs from some non-empty universe X, with outputs also from X. In the beginning, we examine properties of set-based extended functions acting on a general universe X. The obtained results are contained in Sect. 2. The next section is devoted to the investigation of set-based extended functions on a (bounded) lattice X. In Sect. 4, X is considered to be a (bounded) chain. This section also contains a characterization of set-based extended aggregation functions on $X = [0, 1]$. Finally, some concluding remarks are added.

2 Set-Based Extended Functions on a General Universe

Suppose that we classify some products and their samples as *good* or *bad* only, i.e., we deal with the universe $X = \{g, b\}$. A function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ assigns to a sample $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ either the value *good*—if all the inputs $x_1, \ldots, x_n$ are *good*, or the value *bad*—in all other cases. The output value $F(\mathbf{x})$ depends on the set $\{x_1, \ldots, x_n\}$ only, namely,

$$F(x_1, \ldots, x_n) = \begin{cases} b & \text{if } b \in \{x_1, \ldots, x_n\}, \\ g & \text{otherwise.} \end{cases}$$

Moreover, if we add any other inputs $y_1, \ldots, y_k$, but such that each of them has already appeared in the original sample, i.e., $y_1, \ldots, y_k \in \{x_1, \ldots, x_n\}$, then

$$F(x_1, \ldots, x_n, y_1, \ldots, y_k) = F(x_1, \ldots, x_n).$$

In what follows, we formalize the above described situation, and define the notion of set-based extended function on a general universe X. We start by recalling the notion of extended function on X.

Definition 2.1. *Let $X \neq \emptyset$. Any function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ will be called an extended function on X.*

Extended functions have open arity, i.e., they can work for any finite number of arguments.

Definition 2.2. *Let $X \neq \emptyset$. A function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ is called a set-based extended function on X if $F(\mathbf{y}) = F(\mathbf{x})$ for any $n, k \in \mathbb{N}$ and all $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$, $\mathbf{y} = (y_1, \ldots, y_k) \in X^k$, such that $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_k\}$.*

Example 2.1. Consider a set X with cardinality $\mathrm{card}(X) > 2$. Let E be a proper subset of X, and $a, b \in X$, $a \neq b$. Define $F_{E,a,b}\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ by

$$F_{E,a,b}(x_1, \ldots, x_n) = \begin{cases} a & \text{if } E \cap \{x_1, \ldots, x_n\} \neq \emptyset, \\ b & \text{otherwise.} \end{cases}$$

Then $F_{E,a,b}$ is a set-based extended function on X. Note that $F_{E,a,b}$ is associative if and only if $a \in E$, where the associativity of a function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ means that

$$F(\mathbf{x}, \mathbf{y}) = F(F(\mathbf{x}), F(\mathbf{y}))$$

for all $\mathbf{x}, \mathbf{y} \in \bigcup_{n\in\mathbb{N}} X^n$.

Example 2.1 is an example of a particular case of the construction of set-based extended functions described in the following proposition.

Proposition 2.1. *Let $X \neq \emptyset$. Let $\mathcal{P} = \{E_1, \ldots, E_k\}$ be a partition of X and $a_1, \ldots, a_k \in X$. Define $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ by*

$$F(\mathbf{x}) = a_i, \quad \text{where } i = \min\{j \in \{1, \ldots, k\} \mid \{x_1, \ldots, x_n\} \cap E_j \neq \emptyset\}. \tag{1}$$

Then F is a set-based extended function on X.

Example 2.2. Let $p \in \mathbb{N}$ and $X = \{1, \ldots, p\}$. Then

- if we consider the partition $\mathcal{P} = \{E_i\}_{i=1}^{p}$, where $E_i = \{i\}$, and $a_i = i$, then (1) defines the function $Min\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ given by $Min(x_1, \ldots, x_n) = \min\{x_1, \ldots, x_n\}$;

– if $\mathcal{P} = \{E_i\}_{i=1}^{p}$, where $E_i = \{p-i+1\}$ and $a_i = p-i+1$, then (1) yields the function Max, $Max(x_1, \ldots, x_n) = \max\{x_1, \ldots, x_n\}$.

Lemma 2.1. *Let $X \neq \emptyset$ and $\mathcal{H}(X) = \{\emptyset \neq E \subseteq X \mid E \text{ is finite}\}$. Then each set-based extended function F on X corresponds in a one-to-one correspondence to a set function $G\colon \mathcal{H}(X) \to X$ given, for each $E = \{x_1, \ldots, x_n\}$ in $\mathcal{H}(X)$, by*

$$G(E) = F(x_1, \ldots, x_n).$$

Clearly, $\mathcal{H}(X)$ is the power set of X except the empty set whenever X is finite.

Note that properties of the set function $G\colon \mathcal{H}(X) \to X$ can be transformed into new kinds of properties of the related set-based extended function F on X, as is shown in the following example.

Example 2.3. Consider $X = \mathbb{N}$ and define $G\colon \mathcal{H}(\mathbb{N}) \to \mathbb{N}$ by $G(E) = \sum_{i \in E} i$.

Obviously, G is monotone non-decreasing, because for all E_1, E_2 in $\mathcal{H}(\mathbb{N})$, $G(E_1) \leq G(E_2)$ whenever $E_1 \subseteq E_2$. G is also additive, i.e.,

$$G(E_1 \cup E_2) = G(E_1) + G(E_2) \quad \text{whenever} \quad E_1 \cap E_2 = \emptyset.$$

The set-based extended function $F\colon \bigcup_{n \in \mathbb{N}} \mathbb{N}^n \to \mathbb{N}$ corresponding to G, is given by

$$F(x_1, \ldots, x_n) = \sum_{i \in \mathbb{N}} i \cdot \min\left\{1, \sum_{j=1}^{n} \mathbf{1}_{\{i\}}(x_j)\right\},$$

and is neither monotone non-decreasing nor additive in the standard case, because, given any $n \in \mathbb{N}$, the relation $\mathbf{x} \leq \mathbf{y}$ does not imply $F(\mathbf{x}) \leq F(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{N}^n$, and similarly, the additivity property F($\mathbf{x} + \mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$ does not hold for all $\mathbf{x}, \mathbf{y} \in \mathbb{N}^n$.

However, F is monotone non-decreasing with respect to the partial order $\preceq$ on $\bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, defined as follows: for any $n, k \in \mathbb{N}$ and all $\mathbf{x} \in \mathbb{N}^n$, $\mathbf{y} \in \mathbb{N}^k$,

$$\mathbf{x} \preceq \mathbf{y} \text{ whenever } n \leq k \text{ and } x_i = y_i \text{ for all } i \leq n.$$

Indeed, then for all $\mathbf{x}, \mathbf{y} \in \bigcup_{n \in \mathbb{N}} \mathbb{N}^n$, if $\mathbf{x} \preceq \mathbf{y}$ then $F(\mathbf{x}) \leq F(\mathbf{y})$.

Similarly, F is concatenation additive, i.e., if $\{x_1, \ldots, x_n\} \cap \{y_1, \ldots, y_k\} = \emptyset$, then $F(\mathbf{x}, \mathbf{y}) = F(\mathbf{x}) + F(\mathbf{y})$.

We still give another example illustrating Lemma 2.1.

Example 2.4. Consider $X = \{0, 1\}$. Then a function $F\colon \bigcup_{n \in \mathbb{N}} \{0, 1\}^n \to \{0, 1\}$ is an extended Boolean function. The cardinality of X is $\text{card}(X) = 2$, $\mathcal{H}(X) = \{\{0\}, \{1\}, \{0, 1\}\}$, i.e., $\text{card}(\mathcal{H}(X)) = 3$, thus there are exactly $2^3 = 8$ set functions $G_i\colon \mathcal{H}(X) \to \{0, 1\}$, $i = 1, \ldots, 8$. Consequently, there are 8 set-based extended Boolean functions F_i, where F_i corresponds to G_i by Lemma 2.1. The results are summarized in Table 1.

Table 1. Set-based extended Boolean functions

$G_i \backslash E$	{0}	{1}	{0,1}	$F_i(\mathbf{x})$
G_1	0	0	0	0
G_2	0	0	1	$\bigvee_{j,k} \lvert x_j - x_k \rvert$
G_3	0	1	0	$\bigwedge_j x_j$
G_4	0	1	1	$\bigvee_j x_j$
G_5	1	0	0	$1 - F_4(\mathbf{x})$
G_6	1	0	1	$1 - F_3(\mathbf{x})$
G_7	1	1	0	$1 - F_2(\mathbf{x})$
G_8	1	1	1	$1 - F_1(\mathbf{x})$

Proposition 2.2. *Fix $X = \{1, 2, \ldots, k\}$. Consider a permutation $\sigma\colon X \to X$ and a total order $\preceq_\sigma$ on X determined by σ, given by*

$$x \preceq_\sigma y \text{ if and only if } \sigma^{-1}(x) \leq \sigma^{-1}(y).$$

Let $G_\sigma\colon \mathcal{H}(X) \to X$, $G_\sigma(E) = \min_{\preceq_\sigma}\{x \mid x \in E\}$. Then the set-based extended function $F_\sigma\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, $F_\sigma(\mathbf{x}) = G_\sigma(\{x_1, \ldots, x_n\})$, is symmetric, associative, and with neutral element $e = \sigma(n)$, but in general, F_σ need not be monotone.

Recall that $e \in X$ is a neutral element of an extended function F on X, if for all $n \in \mathbb{N}$, and all $\mathbf{x} \in X^n$, with $e = x_i$ for some $i \in \{1, \ldots, n\}$, we have

$$F(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n) = F(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

Obviously, in Proposition 2.2, there are $k!$ set-based extended functions F_σ.

Remark 2.1. In Proposition 2.2, if for each $x, y \in X$,

$$x < y < e \Rightarrow \sigma^{-1}(x) < \sigma^{-1}(y) \text{ and } x > y > e \Rightarrow \sigma^{-1}(x) < \sigma^{-1}(y),$$

then F_σ is an idempotent uninorm (and only in that case). There are 2^{k-1} idempotent uninorms on X.

Note that the previous result for idempotent uninorms was also proved by Zemánková in [12].

We now summarize some properties related to general set-based functions.

Proposition 2.3. *Let $X \neq \emptyset$. Set-based extended functions on X have the following properties.*

(i) Each set-based extended function on X is symmetric.

(ii) For any function $V\colon X^k \to X$ and any set-based extended functions $F_1,\dots,F_k$ on X, also the composite $F = V(F_1,\dots,F_k)\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ is a set-based extended function on X.

(iii) For any function $V\colon X \to X$ and a any set-based extended function F on X, also the composites $V(F), F(V)\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, given by

$$V(F)(\mathbf{x}) = V(F(\mathbf{x})) \quad\text{and}\quad F(V)(\mathbf{x}) = F(V(x_1),\dots,V(x_n)),$$

respectively, are set-based extended functions.

Proposition 2.4. *Let $X_i \neq \emptyset$, $i = 1,\dots,k$, and let X be the Cartesian product of X_i, $X = X_1 \times \cdots \times X_k$. For any set-based extended functions F_i on X_i, $i = 1,\dots,k$, the function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, defined by*

$$F((x_1^{(1)},\dots,x_k^{(1)}),\dots,(x_1^{(n)},\dots,x_k^{(n)})) = (F_1(x_1^{(1)},\dots,x_1^{(n)}),\dots,F_k(x_k^{(1)},\dots,x_k^{(n)})),$$

is a set-based extended function on X.

The following theorem shows that some algebraic properties of a function $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ already ensure that F is a set-based extended function on X.

Theorem 2.1. *Let $X \neq \emptyset$. Let $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ be symmetric, idempotent and associative. Then F is a set-based extended function on X.*

Proof: Let F satisfy the given assumptions. For any $n \in \mathbb{N}$ and each $\mathbf{x} = (x_1,\dots,x_n) \in X^n$ with $\operatorname{card}(\{x_1,\dots,x_n\}) = k$, let $\{x_1,\dots,x_n\} = \{y_1,\dots,y_k\}$. Then there is a partition $\{I_1,\dots,I_k\}$ of $\{x_1,\dots,x_n\}$ given by

$$I_i = \{j \in \{1,\dots,n\} \mid x_j = y_i\}.$$

Then, writing $I_i = \{j_{i1},\dots,j_{im_i}\}$, where $m_i = \operatorname{card}(I_i)$, we have

$$\begin{aligned} F(\mathbf{x}) &= F(x_{j_{11}},\dots,x_{j_{1m_1}},x_{j_{21}},\dots,x_{j_{2m_2}},\dots,x_{j_{k1}},\dots,x_{j_{km_k}}) \\ &= F(F(x_{j_{11}},\dots,x_{j_{1m_1}}),F(x_{j_{21}},\dots,x_{j_{2m_2}}),\dots,F(x_{j_{k1}},\dots,x_{j_{km_k}})) \\ &= F(y_1,\dots,y_k), \end{aligned}$$

where the first equality follows from the symmetry of F, the second one from its associativity, and the third one follows from the idempotency of F. Obviously, for all $\mathbf{x},\mathbf{z} \in \bigcup_{n\in\mathbb{N}} X^n$, such that $\{x_1,\dots,x_n\} = \{y_1,\dots,y_k\} = \{z_1,\dots,z_m\}$, we have $F(\mathbf{x}) = F(\mathbf{z})$, and hence F is a set-based extended function on X. □

Note that neither idempotency nor associativity are necessary properties for being F a set-based extended function, see Example 2.1 and Proposition 2.1.

3 Set-Based Extended Functions on Lattices

In this section we consider X to be a carrier of a lattice $(X, \leq)$. For any fixed $a \in X$, we define a function $F_a\colon \bigcup\limits_{n\in\mathbb{N}} X^n \to X$ by

$$F_a(\mathbf{x}) = \begin{cases} \bigvee\limits_i x_i & \text{if } \bigvee\limits_i x_i < a, \\ \bigwedge\limits_i x_i & \text{if } \bigwedge\limits_i x_i > a, \\ a & \text{otherwise.} \end{cases}$$

Obviously, F_a is symmetric and idempotent, and its associativity can also be verified. By Theorem 2.1, F_a is a set-based extended function on X. Moreover, F_a is monotone non-decreasing, and thus it is an extended aggregation function on X, see [7] (because of the idempotency of F_a we need not consider X to be a bounded lattice). Observe that if X is bounded, with top and bottom elements $\mathbf{1}_X$ and $\mathbf{0}_X$, respectively, then $F_{\mathbf{1}_X} = \vee$ is the standard join on X, and $F_{\mathbf{0}_X} = \wedge$ is the standard meet on X. By Theorem 2.1, any idempotent uninorm F on a bounded (distributive) lattice X [8], is a set-based extended function on X. Similarly, idempotent nullnorms on bounded lattices, see [9], are set-based extended functions.

Proposition 3.1. *Let $(X, \leq)$ be an ordinal sum of lattices $(X_i, \leq_i)_{i\in I}$, and let for any $i \in I$, $F_i\colon \bigcup\limits_{n\in\mathbb{N}} X_i^n \to X_i$ be a set-based extended function on X_i. Define $F\colon \bigcup\limits_{n\in\mathbb{N}} X^n \to X$ by*

$$F(x_1, \ldots, x_n) = F_i(y_1, \ldots, y_k),$$

where

$$\begin{aligned} i &= \min\{j \in I \mid \{x_1, \ldots, x_n\} \cap X_j \neq \emptyset\}, \\ k &= \operatorname{card}(\{j \in \{1, \ldots, n\} \mid x_j \in X_i\}), \\ \{y_1, \ldots, y_k\} &= \{x_j \mid x_j \in X_i\}. \end{aligned}$$

Then F is a set-based extended function on X. Moreover, F is monotone non-decreasing if and only if all F_i, $i \in I$, are of that property, and it is idempotent if and only if all F_i, $i \in I$, are idempotent.

More information on ordinal sum of lattices can be found, e.g., in [3].

4 Set-Based Extended Aggregation Functions on Chains

In this section we consider X to be a (bounded) chain. A total order on X has an important impact on characterization of monotone set-based extended functions on X.

Proposition 4.1. *Let X be a chain. Then $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$ is a monotone non-decreasing (non-increasing) set-based extended function if and only if for each $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$ we have*

$$F(\mathbf{x}) = D(Min(\mathbf{x}), Max(\mathbf{x})), \tag{2}$$

for some monotone non-decreasing (non-increasing) function $D\colon X^2 \to X$.

Proof: It is not difficult to see that representation of F in the form (2) is sufficient for being F a monotone non-decreasing (non-increasing) set-based extended function on X. We only prove a necessary condition.

Let F be a monotone non-decreasing set-based extended function on a chain X. As F is symmetric, with no loss of generality, we can only consider elements $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} X^n$ such that $x_1 \leq \cdots \leq x_n$. Then $x_1 = Min(\mathbf{x})$, $x_n = Max(\mathbf{x})$ and we can write

$$\begin{aligned} F(x_1, x_n) &= F(x_1, \ldots, x_1, x_n) \leq F(x_1, x_2, \ldots, x_{n-1}, x_n) \leq F(x_1, x_n, \ldots, x_n) \\ &= F(x_1, x_n), \end{aligned} \tag{3}$$

which yields $F(\mathbf{x}) = F(Min(\mathbf{x}), Max(\mathbf{x}))$. Putting $D = F|_{X^2}$, we obtain the required representation in the form (2). The monotonicity of D follows from the monotonicity of F. To get the result for a monotone non-increasing F, it is enough to reverse the inequalities in (3). □

Now we provide a characterization of set-based extended aggregation functions acting on a bounded chain X, in particular on $X = [0,1]$. In what follows, we only recall the notion of extended aggregation function on $[0,1]$, for more details on (extended) aggregation functions and their properties we recommend, e.g., [4,7,10], see also [1,2].

Definition 4.1. *A function $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ is an extended aggregation function on $[0,1]$ if A is monotone non-decreasing and satisfies the boundary conditions, i.e.,*

(i) for all elements $\mathbf{0} = (0, \ldots, 0), \mathbf{1} = (1, \ldots, 1) \in \bigcup_{n\in\mathbb{N}} [0,1]^n$, $A(\mathbf{0}) = 0$ and $A(\mathbf{1}) = 1$;
(ii) for all $\mathbf{x}, \mathbf{y} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$ we have $A(\mathbf{x}) \leq A(\mathbf{y})$ whenever $\mathbf{x} \leq \mathbf{y}$.

Note that for $\mathbf{x}, \mathbf{y} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$ we have $\mathbf{x} \leq \mathbf{y}$ if and only if $\mathbf{x}$ and $\mathbf{y}$ are n-tuples of the same arity n satisfying $x_i \leq y_i$ for each $i = 1, \ldots, n$.

We will also work with n-ary aggregation functions on $[0,1]$, i.e., functions

$$A_{(n)}\colon [0,1]^n \to [0,1]$$

which satisfy boundary conditions (i) and monotonicity conditions (ii) from Definition 4.1 for a considered fixed $n \in \mathbb{N}$. Clearly, given an extended aggregation function A on $[0,1]$, the function $A_{(n)} = A_{|_{[0,1]^n}}$ is an n-ary aggregation function.

Definition 4.2. *A function* $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *is a set-based extended aggregation function if* A *is an extended aggregation function on* $[0,1]$ *satisfying the set-based property, i.e., for all* $n, k \in \mathbb{N}$*, and all* $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$ *and* $\mathbf{y} = (y_1, \ldots, y_k) \in [0,1]^k$*,* $A(\mathbf{y}) = A(\mathbf{x})$ *whenever* $\{x_1, \ldots, x_n\} = \{y_1, \ldots, y_k\}$*.*

It can be shown that set-based extended aggregation functions on $[0,1]$ can be completely characterized as follows.

Theorem 4.1. *Let* $A\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ *be an extended aggregation function on* $[0,1]$*.* A *is a set-based extended aggregation function on* $[0,1]$ *if and only if for all* $\mathbf{x} \in \bigcup_{n\in\mathbb{N}} [0,1]^n$ *we have*

$$A(\mathbf{x}) = A(Min(\mathbf{x}), Max(\mathbf{x})). \quad (4)$$

For more results on set-based extended aggregation functions on $[0,1]$, see [11].

By the previous theorem, set-based extended aggregation functions on $[0,1]$ are generated by binary aggregation functions; there is a one-to-one correspondence between the set of all set-based extended aggregation functions and the set of all symmetric binary aggregation functions. Observe that in the case of an associative symmetric binary aggregation function $A\colon [0,1]^2 \to [0,1]$ there are two possible ways how to extend it into an extended aggregation function. On the one hand, based on formula (2), one can define the function $A_\Box\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ by

$$A_\Box(\mathbf{x}) = A(Min(\mathbf{x}), Max(\mathbf{x})),$$

and on the other hand, using the associativity of A, one can define the function $A_\triangle\colon \bigcup_{n\in\mathbb{N}} [0,1]^n \to [0,1]$ by

$$A_\triangle(x_1) = x_1,\ A_\triangle(x_1, x_2) = A(x_1, x_2),$$

and for all $n \geq 3$,

$$A_\triangle(x_1, \ldots, x_n) = A(A_\triangle(x_1, \ldots, x_{n-1}), x_n).$$

Due to Proposition 2.1, $A_\Box = A_\triangle$ if and only if a binary aggregation function A is idempotent, i.e., $A(x,x) = x$ for all $x \in [0,1]$. Note that this is, e.g., the case of idempotent uninorms [6,12], and also the case of idempotent nullnorms [5] (compare F_a introduced in Sect. 3). As a negative example, consider the standard product $A(x_1, x_2) = x_1 x_2$. Then $A_\triangle(x_1, \ldots, x_n) = \prod_{i=1}^{n} x_i$ is the standard product, which, if $n \neq 2$, differs from $A_\Box(\mathbf{x}) = (Min(\mathbf{x})) \cdot Max(\mathbf{x}))$.

5 Concluding Remarks

In this paper, we have introduced and discussed set-based extended functions, which can be seen as a generalization of extended functions $F\colon \bigcup_{n\in\mathbb{N}} X^n \to X$, which are symmetric, idempotent and associative. In the case when X is a lattice, the introduced set-based extended functions can be viewed as a particular generalization of joins, meets, idempotent uninorms and idempotent nullnorms. In the case of bounded chains, we have shown the existence of a one-to-one correspondence between set-based aggregation functions A and symmetric binary aggregation functions D given by

$$A(\mathbf{x}) = D(Min(\mathbf{x}), Max(\mathbf{x})).$$

Based on the presented approach, in our future research we intend to solve how to relate aggregation of input values $x_1, \ldots, x_n$ to aggregation of inputs $x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}$, where $x_{n+1}, \ldots, x_{n+k}$ are some additionally obtained observations.

Acknowledgement. R. Mesiar and A. Šeliga kindly acknowledge the support of the grant VEGA 1/0006/19, and A. Kolesárová is grateful for the support of the grant VEGA 1/0614/18. All these three authors also acknowledge the support of the project of Science and Technology Assistance Agency under the contract No. APVV–18–0052. D. Gómez and J. Montero kindly acknowledge the support of the projects TIN205-66471-P (Government of Spain), S2013/ICE-2845 (State of Madrid) and Complutense University research group GR3/14-910149. Moreover, the authors thank M. Botur for inspirative personal discussion.

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