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Two Algebraic Byways from Differential Equations: Gröbner Bases and Quivers

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Preface

This volume is a collection of several works focusing on differential equations from viewpoints of formal calculus and geometry through applications of quiver theory. This book consists of two parts. The first one introduces the theory of Gröbner bases in their commutative and noncommutative contexts. In particular, the lectures will focus on algorithmic aspects and applications of Gröbner bases to analysis on systems of partial differential equations, effective analysis on rings of differential operators, and homological algebra. The second part constitutes an introduction to representations of quivers, quiver varieties, and their applications to the moduli spaces of meromorphic connections on the complex projective line $\mathbb{P}^1$. All the contributions are presented without assuming any particular background, and the authors have done their best to make the chapters suitable for graduate students.

Gröbner bases and quivers in algebra and geometry. Gröbner bases and more generally linear rewriting systems constitute models for computation in algebras of various types (associative, commutative, Lie…). One of the applications of the theory is to compute normal forms, bases, and more generally Hilbert or Poincaré series. Another important application is a generalization of Gaussian elimination to polynomial systems in various types of algebras (commutative, Weyl algebra…). The theory of Gröbner bases was developed in the twentieth century. Several works had led to the development of computational methods in algebra well before the introduction of algebraic structures such as ideals and algebras and the modern algebraic language. Chapter 1 explains the long and rich developments from the work of M. Janet in 1920 on partial differential equations, elimination theory with seminal works of E. Noether in 1921, and the computational methods in algebraic geometry with the theory of Gröbner bases for commutative algebras developed by B. Buchberger in 1965. In recent years, new algorithms of the theory of Gröbner bases were developed in rings of differentials operators by Oaku–Takayama. In the meanwhile, decision problems in semigroups and groups by A. Thue in 1914 and M. Dehn in 1910 motivate a new combinatorial theory of equivalence relations, the rewriting theory. This theory was expended throughout the twentieth century, in particular with seminal results on confluence by M. Newman in 1942, on completion by Knuth–Bendix in 1970. Rewriting theory had been applied to algebra with works of A. I. Shirshof in 1962 for computing bases in Lie algebras and L. A. Bokut and G. Bergman independently in 1976–1978 for associative algebras. More recently at the end of 1980s, rewriting methods were applied in homological algebra by several authors such as D. J. Anick, C. Squier, K. Brown, and Y. Kobayashi.

Graphical methods in representation theory are rather new in comparison to the theory of Gröbner bases. Nevertheless, many applications are developed in the last decade. In 1934, H. Coxeter classified the finite real reflection groups and represented their fundamental relations in terms of graphs which was applied by E. Witt in 1941 to study the structure of semisimple Lie algebras. H. Weyl, in 1925–1926, and B. L. van der Waerden in 1933 simplified the classification of simple Lie algebras after W. Killing in 1888–1890, but it was E. B. Dynkin in 1946 who used the graphical expression to classify simple Lie algebras, where the name (Coxeter-) Dynkin diagram came from. In 1972, the Dynkin diagrams of type ADE have re-appeared by the work of P. Gabriel in view of the classification of the algebras with finitely many isomorphic classes of simple modules, see Chap. 6. It was only in the 1990s when the so-called quiver varieties were introduced by G. Lusztig for his study on quantum groups and H. Nakajima for his study on gauge theory, see Chap. 7. Their geometric approaches have big impacts not only on representation theory but also on algebraic geometry, for example, the moduli spaces of meromorphic connections on compact Riemann surfaces.

Gröbner bases and applications. The aim of the first part of the volume is to focus on various aspects of the theory of Gröbner bases and of the mathematical problems at the origin of the theory. Chapter 1 briefly reviews the seminal works on constructive methods for computing in ideals by M. Janet in 1920 motivated by integration of partial equation differential systems by C. Riquier and É. Cartan. The main tool introduced by M. Janet is the notion of involutive bases which are particular cases of Gröbner bases. Another domain in application that will be treated is the effective analysis on rings of differential operators. In particular, integral transformations and restriction functors on D-modules will be presented using noncommutative Gröbner bases. Chapters 2 and 3 present algorithmic aspects on D-modules. In particular, Chap. 2 deals with the notion of Gröbner bases in D-modules and their applications to Bernstein–Sato polynomials. An introduction to algorithms for D-modules with Quiver D-modules is also given in Chap. 3. Another aspect of Gröbner bases theory for noncommutative associative algebras is given in Chap. 4. A generalization of noncommutative Gröbner bases without a monomial order and a link between the theory of Gröbner bases and rewriting theory will be also explained. Finally, an application of Gröbner bases to the computation of free resolutions for associative algebras will be given. Chapter 5 will conclude this part with applications of the theory Gröbner bases to computational algebraic statistics.

Quivers and applications. The lectures of this part will be devoted to a geometric application of quivers. In particular, the geometry of the moduli spaces of meromorphic connections on $\mathbb{P}^1$ with irregular singularities is one of the subjects which has been developed recently, and this is the main theme of this part. Chapters 6 and 7 will provide introduction to representations of quivers and quiver varieties. There are some results known by the experts but never explained in the literature. In Chap. 8, the so-called additive Deligne–Simpson problem will be presented including some background materials. Some known results due to Crawley-Boevey and the author himself will also be explained. Geometric aspects of this problem with some recent development will be given in the final chapter (Chap. 9), where the author will recall necessary backgrounds from quiver varieties and symplectic geometry.

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Part I
First Algebraic Byway: Gröbner Bases

Chapter 1
From Analytical Mechanics Problems to Rewriting Theory Through M. Janet's Work

Kenji Iohara and Philippe Malbos

1 Introduction

This chapter is devoted to a survey of the historical background of Gröbner bases for D-modules and linear rewriting theory largely developed in algebra throughout the twentieth century and to present deep relationships between them. Completion methods are the main streams for these computational theories. In the theory of Gröbner bases, they were motivated by algorithmic problems in elimination theory such as computations in quotient polynomial rings modulo an ideal, manipulating algebraic equations, and computing Hilbert series. In rewriting theory, they were motivated by computation of normal forms and linear bases for algebras and computational problems in homological algebra.

In this chapter, we present the seminal ideas of the French mathematician M. Janet on the algebraic formulation of completion methods for polynomial systems. Indeed, the problem of completion already appears in Janet's 1920 thesis [47], which proposed an original approach by formal methods in the study of systems of linear partial differential equations, PDE systems for short. The corresponding constructions were formulated in terms of polynomial systems, but without the notions of ideal and Noetherian induction. These two notions were introduced by Noether in 1921 [68] for commutative rings.

The work of M. Janet was forgotten for about half of a century. It was rediscovered by Schwarz in 1992 in [81]. Our exposition in this chapter does not follow the historical order. The first section deals with the problems that motivate the PDE study undertaken by M. Janet. In Sect. 3, we present completion for monomial PDE

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systems as introduced by Janet in his monograph [51]. This completion used an original division procedure on monomials. In Sect. 4, we present an axiomatization of this Janet notion of division, called *involutive division*, due to V. P. Gerdt. The last two sections concern the case of polynomial PDE systems, with M. Janet's completion method used to reduce a linear PDE system to a canonical form and the axiomatization of the reductions involved in terms of rewriting theory.

1.1 From Analytical Mechanics Problems to Involutive Division

1.1.1 From Lagrange to Janet. The analysis of linear PDE systems was mainly motivated in eighteenth century by the desire to solve problems of analytical mechanics. The seminal work of J.-L. Lagrange gave the first systematic study of PDE systems launched by such problems. The case of PDE of one unknown function of several variables has been treated by J. F. Pfaff. The Pfaff problem will be recalled in Sect. 2.1. This theory was developed in two different directions: toward the general theory of differential invariants and the existence of solutions under given initial conditions. The differential invariants approach will be discussed in Sects. 2.1 and 2.1.4. The question of the existence of solution satisfying some initial conditions was formulated in the Cauchy–Kowalevsky theorem recalled in Sect. 2.1.3.

1.1.2 Exterior Differential Systems. Following the work of H. Grassmann in 1844 which did set up the rules of exterior algebra computations, É. Cartan introduced exterior differential calculus in 1899. This algebraic calculus allowed him to describe a PDE system by an exterior differential system that is independent of the choice of coordinates. This did lead to the so-called Cartan–Kähler theory, reviewed in Sect. 2.2. We will present a geometrical property of involutivity on exterior differential systems in Sect. 2.2.6, which motivates the formal methods introduced by M. Janet for the analysis of linear PDE systems.

1.1.3 Generalizations of the Cauchy–Kowalevsky Theorem. Another origin of the work of M. Janet is the Cauchy–Kowalevsky theorem that gives the initial conditions of solvability of a family of PDE systems that we describe in Sect. 2.1.3. É. Delassus, C. Riquier, and M. Janet attempted to generalize this result to a wider class of linear PDE systems which in turn led them to introduce the computation of a notion of normal form for such systems.

1.1.4 The Janet Monograph. Section 3 presents the historical work that motivated M. Janet to introduce an algebraic algorithm in order to compute normal form of linear PDE systems. In particular, we recall the problem of computation of *inversion of differentiation* introduced by M. Janet in his monograph ≪ *Leçons sur les systèmes d'équations aux dérivées partielles* ≫ on the analysis of linear PDE systems, published in 1929 [51]. Therein, M. Janet introduced formal methods based on polynomial computations for analysis of linear PDE systems. He developed an algorithmic approach for analyzing ideals in the polynomial ring $\mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ of differential operators with constant coefficients. Having the ring isomorphism between this ring and the ring $\mathbb{K}[x_1, \ldots, x_n]$ of polynomials with n variables in mind, M. Janet gave its algorithmic construction in this latter ring. He began by introducing some remarkable properties of monomial ideals. In particular, he recovered Dickson's Lemma [17], assertion that monomial ideals are finitely generated. This property is essential for the Noetherian properties on the set of monomials. Note that M. Janet was not familiar with the axiomatization of the algebraic structure of ideals and the property of Noetherianity already introduced by Noether in [68] and [69]. Note also that the Dickson Lemma was published in 1913 in a paper on number theory in an American journal. Due to the First World War, it took a long time before these works became accessible to the French mathematical community. Janet's algebraic constructions given in his monograph will be recalled in Sect. 3 for monomial systems and in Sect. 5 for polynomial systems.

1.1.5 Janet's Multiplicative Variables. The computations on monomial and polynomial ideals carried out by M. Janet are based on the notion of *multiplicative variable* that he introduced in his thesis [47]. Given an ideal generated by a set of monomials, he distinguished the monomials contained in the ideal and those contained in the complement of the ideal. The notions of multiplicative and non-multiplicative variables appear in order to stratify these two families of monomials. We will recall this notion of multiplicativity of variables in Sect. 3.1.9. This leads to a refinement of the classical division on monomials, nowadays called *Janet's division*.

1.1.6 Involutive Division and Janet's Completion Procedure. The notion of multiplicative variable is local, in the sense that it is defined with respect to a subset $\mathcal{U}$ of the set of all monomials. A monomial u in $\mathcal{U}$ is said to be a Janet divisor of a monomial w with respect to $\mathcal{U}$, if $w = uv$ and all variables occurring in v are multiplicative with respect to $\mathcal{U}$. In this way, we distinguish the set $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$ of monomials having a Janet divisor in $\mathcal{U}$, called *multiplicative* or *involutive cone* of $\mathcal{U}$, and the set $\mathrm{cone}(\mathcal{U})$ of multiple of monomials in $\mathcal{U}$ for the classical division. The Janet division being a refinement of the classical division, the set $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$ is a subset of $\mathrm{cone}(\mathcal{U})$. M. Janet called a set of monomials $\mathcal{U}$ *complete* when this inclusion is an equality.

To a monomial PDE system (Σ) of the form

$$\frac{\partial^{\alpha_1+\ldots+\alpha_n}\varphi}{\partial x_1^{\alpha_1}\ldots\partial x_n^{\alpha_n}} = f_\alpha(x_1, x_2, \ldots, x_n),$$

where $(\alpha_1, \ldots, \alpha_n)$ belongs to a subset I of $\mathbb{N}^n$, M. Janet associated the set of monomials

$$\mathrm{lm}(\Sigma) = \{x_1^{\alpha_1}\ldots x_n^{\alpha_n} \;\mid\; (\alpha_1, \ldots, \alpha_n) \in I\}.$$

The compatibility conditions of the system (Σ) correspond to the factorizations of the monomials ux in $\mathrm{cone}_{\mathcal{J}}(\mathrm{lm}(\Sigma))$, where u is in $\mathrm{lm}(\Sigma)$ and x is a non-multiplicative variable of u with respect to $\mathrm{lm}(\Sigma)$, as explained in Sect. 3.3.1. By definition, for any monomial u in $\mathrm{lm}(\Sigma)$ and x non-multiplicative variable of u with respect to $\mathrm{lm}(\Sigma)$, the monomial ux admits such a factorization if and only if $\mathrm{lm}(\Sigma)$ is complete, see Proposition 3.2.5.

The main procedure presented in Janet's monograph [51] completes in a finite number of operations a finite set of monomials $\mathcal{U}$ to a complete set of monomials $\widetilde{\mathcal{U}}$ that contains $\mathcal{U}$. This procedure consists in analyzing all the local defects of completeness, by adding all the monomials ux where u in $\mathcal{U}$ and x is a non-multiplicative variable for u with respect to $\mathcal{U}$. This procedure will be recalled in Sect. 3.2.9. A generalization of this procedure to any involutive division was given by Gerdt in [25], and is recalled in Sect. 4.2.12.

Extending this procedure to a set of polynomials, M. Janet applied it to linear PDE systems, giving a procedure that transforms a linear PDE system into a complete PDE system with the same set of solutions. This construction is given in Sect. 5.6. In Sect. 6, we present such a procedure for an arbitrary involutive division given by V. P. Gerdt and Blinkov in [27] and its relation to the Buchberger completion procedure in commutative polynomial rings, [7].

1.1.7 The Space of Initial Conditions. In order to stratify the complement of the involutive cone $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$, M. Janet introduced the notion of *complementary monomial*, see Sect. 3.1.13. With this notion, the monomials that generate this complement in a such a way that the involutive cone of $\mathcal{U}$ and the involutive cone of the set $\mathcal{U}^{\complement}$ of complementary monomials form a partition of the set of all monomials, see Proposition 3.2.2.

For each complementary monomial v in $\mathrm{lm}(\Sigma)^{\complement}$, each analytic function in the multiplicative variables of v with respect to $\mathrm{lm}(\Sigma)^{\complement}$ provides an initial condition of the PDE system (Σ) as stated by Theorem 3.3.3.

1.1.8 Polynomial Partial Differential Equations Systems. In Sect. 5, we present the analysis of polynomial PDE systems as Janet [51]. To deal with polynomials, he defined some total orders on the set of derivatives, corresponding to total orders on the set of monomials. We recall them in Sect. 5.1. The definitions on monomial orders given by M. Janet clarified the same notion introduced previously by Riquier in [74]. In particular, he made more explicit the notion of parametric and principal derivatives in order to distinguish the leading derivative in a polynomial PDE. In this way, he extended the algorithms for monomial PDE systems to the case of polynomial PDE systems. In particular, using these notions, he defined the property of completeness for polynomial PDE systems. Namely, a polynomial PDE system is complete if the associated set of monomials corresponding to leading derivatives of the system is complete. Moreover, M. Janet extended the notion of complementary monomials to define the notion of *initial condition* for a polynomial PDE system as in the monomial case.

1.1.9 Initial Conditions. In this way, the notion of completeness provides a suitable framework to discuss the existence and the uniqueness of the initial conditions for a linear PDE system. M. Janet proved that if a linear polynomial PDE system of the form

$$D_i\varphi = \sum_j a_{i,j} D_{i,j}\varphi, \quad i \in I,$$

with one unknown function φ is such that all the functions $a_{i,j}$ are analytic in a neighborhood of a point P in $\mathbb{C}^n$ and if it is complete with respect to some total order, then it admits at most one analytic solution satisfying the initial condition formulated in terms of complementary monomials, see Theorems 5.3.4 and 5.3.6.

1.1.10 Integrability Conditions. A linear polynomial PDE system of the above form is said to be *completely integrable* if it admits an analytic solution for any given initial condition. M. Janet gave an algebraic characterization of complete integrability by introducing integrability conditions formulated in terms of factorization of leading derivatives of the PDE by non-multiplicative variables. These integrability conditions are stated explicitly in Sect. 5.4.4 as generalization to the polynomial situation of the integrability conditions formulated above for monomial PDE systems in Sect. 3.3. M. Janet proved that a linear polynomial PDE system is completely integrable if and only if every integrability condition is trivially satisfied, as stated in Theorem 5.4.7.

1.1.11 Janet's Procedure of Reduction of Linear PDE Systems to a Canonical Form. In order to extend algorithmically the Cauchy–Kowalevsky theorem on the existence and uniqueness of solutions of initial value problems as presented in Sect. 2.1.3, M. Janet considered normal forms of linear PDE systems with respect to a suitable total order on derivatives, satisfying some analytic conditions on coefficients and a complete integrability condition on the system, as defined in Sect. 5.5.2. Such normal forms of PDE systems are called *canonical* by M. Janet.

Procedure 7 is *Janet's method* for deciding if a linear PDE system can be transformed into a completely integrable system. If the system cannot be reduced to a canonical form, the procedure returns the obstructions to such a reduction. Janet's procedure depends on a total order on derivatives of unknown functions of the PDE system. For this purpose, M. Janet introduced a general method to define a total order on derivatives using a parametrization of a weight order on variables and unknown functions, as explained in Sect. 5.1.5. The Janet procedure uses a specific weight order called canonical and defined in Sect. 5.6.2.

The first step of Janet's method consists in applying *autoreduction procedure*, defined in Sect. 5.6.4, in order to reduce any PDE of the system with respect to the total order on derivatives. Namely, two PDE of the system cannot have the same leading derivative, and any PDE of the system is reduced with respect to the leading derivatives of the others PDE, as specified in Procedure 5.

The second step is the *completion procedure*, Procedure 6. In it, the set of leading derivatives of the system defines a complete set of monomials in the sense given in Sect. 5.3.2.

Having transformed the PDE system to an autoreduced and complete system, one can look at its integrability conditions. M. Janet showed that this set of integrability conditions is a finite set of relations that do not contain principal derivatives, as explained in Sect. 5.4.4. Hence, these integrability conditions are $\mathcal{J}$-normal forms and uniquely defined. By Theorem 5.4.7, if all of these normal forms are trivial, then the system is completely integrable. Otherwise, any nontrivial condition in the set of integrability conditions that contains only unknown functions and variables imposes a relation on the initial conditions of the system. If there is no such relation, the procedure is applied again on the PDE system completed by all the integrability conditions. Note that this procedure depends on the Janet division and on a total order on the set of derivatives.

By this algorithmic method, M. Janet did generalize in certain cases the Cauchy–Kowalevsky theorem at the time where the algebraic structures have not been introduced to perform computations with polynomial ideals. This is pioneering work in the field of formal approaches to analysis of PDE systems. Algorithmic methods for dealing with polynomial ideals were developed throughout the twentieth century and extended to a wide range of algebraic structures. In the next subsection, we present some milestones on these formal themes in mathematics.

1.2 Constructive Methods and Rewriting in Algebra Through the Twentieth Century

The constructions developed by M. Janet in his formal theory of linear partial differential equation systems are based on the structure of ideals that he called *module of forms*. This notion corresponds to those introduced previously by Hilbert in [43] with the terminology of *algebraic form*. Notice that Gunther studied such a structure in [39]. The axiomatization of the notion of ideal in an arbitrary ring is due to Noether [68]. As we will explain in this chapter, M. Janet introduced algorithmic methods to compute a family of generators of an ideal having the involutive property and called an *involutive basis*. This property is used to obtain a normal form of linear partial differential equation systems.

Janet's computation of involutive bases is based on a refinement of classical polynomial division, called *involutive division*. He defined a division that is suitable for reduction of linear partial differential equation systems. Thereafter, other involutive divisions were studied, in particular, by Thomas [86] and by Pommaret [72]; we refer to Sect. 4.3 for a discussion on these divisions.

The main purpose is to complete a generating family of an ideal to an involutive basis with respect to a given involutive division. This completion process is quite similar to those introduced by means of the classical division in the theory of Gröbner bases. In fact, involutive bases appear to be particular cases of Gröbner bases. The principle of completion has been developed independently in rewriting theory, which proposes a combinatorial approach to equivalence relations motivated by several computational and decision problems in algebra, computer science, and logic.

1.2.1 Some Milestones in Algebraic Rewriting and Constructive Algebra. The main results in the work of M. Janet rely on constructive methods in linear algebra using the principle of computing normal forms by rewriting and the principle of completion of a generating set of an ideal. These two principles have been developed through the twentieth century in many algebraic contexts with different formulations and in several instances. We review below some important milestones in this long and rich history from T. Seki to the more recent developments.

1683. Seki introduced the notion of resultant and developed the notion of determinant to express the resultant. He also made progress in elimination theory in the Kai-fukudai-no-hō, see, e.g., [94].

1840. Sylvester studied the resultant of two polynomials in [85] and gave an example for two quadratic polynomials.

1882. Kronecker [54] gave the first result in elimination theory using this notion.

1886. Weierstrass proved a fundamental result called *preparation theorem* on the factorization of analytic functions by polynomials. As an application, he obtained a division theorem for rings of convergent series [93].

1890. Hilbert proved that any ideal in a ring of commutative polynomials in a finite set of variables over a field or over the ring of integers is finitely generated [43]. This is the first formulation of the Hilbert basis theorem, which states that every polynomial ring over a Noetherian ring is Noetherian.

1913. In a paper on number theory, L. E. Dickson proved a monomial version of the Hilbert basis theorem by a combinatorial method [17, Lemma A].

1913. In a series of forgotten papers, N. Günther developed algorithmic approaches for polynomials rings [38–40]. A review of Günther's theory can be found in [41].

1914. Dehn described the word problem for finitely presented groups [16]. Using systems of transformations rules, A. Thue studied the problem for finitely presented semigroups [87]. It was only much later, in 1947, that the problem for finitely presented monoids was shown to be undecidable, independently by Post [73] and Markov [64, 65].

1916. Macaulay was one of the pioneers in commutative algebra. In his book *The algebraic theory of modular systems* [59], following the fundamental Hilbert basis theorem, he initiated an algorithmic approach to treat generators of polynomial ideals. In particular, he introduced the notion of H*-basis* corresponding to a monomial version of Gröbner bases.

1920. Janet defended his doctoral thesis [47], which presents a formal study of systems of partial differential equations following works of Ch. Riquier and É. Delassus. In particular, he analyzed completely integrable systems and Hilbert functions of polynomial ideals.

1921. In her seminal paper, *Idealtheorie in Ringbereichen* [68], Noether laid the foundation of general commutative ring theory, and gave one of the first general definitions of a commutative ring. She also formulated the Finite Chain Theorem [68, Satz I, *Satz von der endlichen Kette*].

1923. Noether formulated in [69, 70] concepts of elimination theory in the language of ideals that she had introduced in [68].

1926. Hermann, a student of Noether [42], initiated purely algorithmic approaches to ideals, such as the ideal membership problem and primary decomposition ideals. This work is a fundamental contribution to the emergence of computer algebra.

1927. Macaulay showed in [60] that the Hilbert function of a polynomial ideal I is equal to the Hilbert function of the monomial ideal generated by the set of leading monomials of the elements in I with respect a monomial order. As a consequence, the coefficients of the Hilbert function of a polynomial ideal are polynomial for sufficiently big degree.

1937. Based on early works by Ch. Riquier and Janet, in [86] J. M. Thomas reformulated in the algebraic language of B. L. van der Waerden, *Moderne Algebra* [89, 90], the theory of normal forms of systems of partial differential equations.

1937. In [32], W. Gröbner exhibited the isomorphism between the ring of polynomials with coefficients in an arbitrary field and the ring of differential operators with constant coefficients, see Proposition 3.1.2. The identification of these two rings was used before in the algebraic study of systems of partial differential equations, but without being explicit.

1942. In a seminal paper on rewriting theory, M. Newman presented rewriting as a combinatorial approach to study equivalence relations [66]. He proved a fundamental rewriting result stating that under a termination hypothesis, the confluence property is equivalent to local confluence.

1949. In his monograph *Moderne algebraische Geometrie. Die idealtheoretischen Grundlagen* [33], W. Gröbner surveyed algebraic computation on ideal theory with applications to algebraic geometry.

1962. Shirshov introduced in [83] an algorithmic method to compute normal forms in a free Lie algebra with respect to a family of elements of the Lie algebra satisfying a confluence property. The method is based on a completion procedure. He also proved a version of Newman's lemma for Lie algebras, called *composition lemma*, and deduced a constructive proof of the Poincaré–Birkhoff–Witt theorem.

1964. Hironaka introduced in [44] a division algorithm and proposed the notion of *standard basis*, analogous to the notion of Gröbner basis, for rings of power series in order to solve problems of resolution of singularities in algebraic geometry.

1965. Under the supervision of W. Gröbner, B. Buchberger developed in his Ph.D. thesis an algorithmic theory of Gröbner bases for commutative polynomial algebras [7, 8, 10]. Buchberger gave a characterization of Gröbner bases in terms of *S-polynomials* as well as an algorithm to compute such bases, with a complete implementation in the assembler language of the computer ZUSE Z 23 V.

1967. Knuth and Bendix defined in [53] a completion procedure that completes with respect to a termination a set of equations in an algebraic theory into a confluent term rewriting system. The procedure is similar to Buchberger's completion procedure. We refer the reader to [9] for a historical account of critical pair/completion procedures.

1972. Grauert introduced in [30] a generalization of Weierstrass's preparation division theorem in the language of Banach algebras.

1973. Nivat formulated a critical pair lemma for string rewriting systems and proved that for a terminating rewriting system, the local confluence is decidable [67].

1976, 1978. Bokut in [5] and Bergman in [4] extended the Gröbner bases and Buchberger's algorithm to associative algebras. They obtained the confluence Newman Lemma for rewriting systems in free associative algebras compatible with a monomial order, called, respectively, Diamond Lemma for ring theory and Composition Lemma.

1978. Pommaret introduced in [72] a global involutive division simpler than those introduced by M. Janet.

1980. Schreyer in his Ph.D. thesis [80] gave a method that computes syzygies in commutative multivariate polynomial rings using the division algorithm, see [18, Theorem 15.10].

1980. Huet [45] gave a proof of Newman's lemma using a Noetherian well-founded induction method.

1985. Gröbner basis theory was extended to Weyl algebras by A. Galligo in [24], see also [79].

1997. Gerdt and Blinkov [25, 27] introduced the notion of involutive monomial division and its axiomatization.

1999, 2002. Faugère developed efficient algorithms for computing Gröbner bases, algorithm F4 [20], then an algorithm F5 [21].

2005. Gerdt [26] presented and analyzed an efficient involutive algorithm for computing Gröbner bases.

2012. Bächler, Gerdt, Lange-Hegermann, and Robertz algorithmized in [2] the Thomas decomposition of algebraic and differential systems.

1.3 Conventions and Notations

The set of nonnegative integers is denoted by $\mathbb{N}$. In this chapter, $\mathbb{K}[x_1, \ldots, x_n]$ denotes the polynomial ring on the variables $x_1, \ldots, x_n$ over a field $\mathbb{K}$ of characteristic zero. For a subset G of $\mathbb{K}[x_1, \ldots, x_n]$, we will denote by $\mathrm{Id}(G)$ the ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by G. A polynomial is either zero or it can be written as a finite sum of nonzero *terms*, each term being the product of a scalar in $\mathbb{K}$ by a *monomial*.

1.3.1 Monomials. We denote by $\mathcal{M}(x_1, \ldots, x_n)$ the set of monomials in the ring $\mathbb{K}[x_1, \ldots, x_n]$. For a subset I of $\{x_1, \ldots, x_n\}$ we will denote by $\mathcal{M}(I)$ the set of monomials in $\mathcal{M}(x_1, \ldots, x_n)$ whose variables lie in I. A monomial u in $\mathcal{M}(x_1, \ldots, x_n)$ is written as $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, were the α_i are nonnegative integers. The integer α_i is called the *degree* of the variable x_i in u, it will be also denoted by $\deg_i(u)$. For $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$, we denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

For a finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ and $1 \leqslant i \leqslant n$, we denote by $\deg_i(\mathcal{U})$ the largest degree in the variable x_i of the monomials in $\mathcal{U}$, that is

$$\deg_i(\mathcal{U}) = \max\big(\deg_i(u) \mid u \in \mathcal{U}\big).$$

We call the *cone* of a subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ the set of all multiples of monomials in $\mathcal{U}$, defined by

$$\mathrm{cone}(\mathcal{U}) = \bigcup_{u \in \mathcal{U}} u\mathcal{M}(x_1, \ldots, x_n) = \{\, uv \mid u \in \mathcal{U},\ v \in \mathcal{M}(x_1, \ldots, x_n)\,\}.$$

1.3.2 Homogeneous Polynomials. A *homogenous polynomial* in $\mathbb{K}[x_1, \ldots, x_n]$ is a polynomial for which all nonzero terms have the same degree. A homogenous polynomial is of *degree* p if all its nonzero terms have degree p. We denote by $\mathbb{K}[x_1, \ldots, x_n]_p$ the space of homogeneous polynomials of degree p. The dimension of this space is given by the formula:

$$\Gamma_n^p := \dim\big(\mathbb{K}[x_1, \ldots, x_n]_p\big) = \frac{(p+1)(p+2)\cdots(p+n-1)}{1 \cdot 2 \cdot \cdots \cdot (n-1)}.$$

1.3.3 Monomial Order. Recall that a *monomial order* on $\mathcal{M}(x_1, \ldots, x_n)$ is a relation $\preccurlyeq$ on $\mathcal{M}(x_1, \ldots, x_n)$ satisfying the following three conditions:

(i) $\preccurlyeq$ is a total order on $\mathcal{M}(x_1, \ldots, x_n)$,
(ii) $\preccurlyeq$ is compatible with multiplication, that is, if $u \preccurlyeq u'$, then $uw \preccurlyeq u'w$ for any monomial w in $\mathcal{M}(x_1, \ldots, x_n)$,
(iii) $\preccurlyeq$ is a well-order on $\mathcal{M}(x_1, \ldots, x_n)$, that is, every non-empty subset of $\mathcal{M}(x_1, \ldots, x_n)$ has a smallest element with respect to $\preccurlyeq$.

The *leading term*, *leading monomial*, and *leading coefficient* of a polynomial f of $\mathbb{K}[x_1, \ldots, x_n]$, with respect to a monomial order $\preccurlyeq$, will be denoted by $\mathrm{lt}_\preccurlyeq(f)$, $\mathrm{lm}_\preccurlyeq(f)$, and $\mathrm{lc}_\preccurlyeq(f)$, respectively. For a set F of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$, we will denote by $\mathrm{lm}_{\preccurlyeq}(F)$ the set of leading monomials of the polynomials in F. For simplicity, we will use notations $\mathrm{lt}(f)$, $\mathrm{lm}(f)$, $\mathrm{lc}(f)$, and $\mathrm{lm}(F)$ if there is no danger of confusion.

2 Exterior Differential Systems

Motivated by problems in analytical mechanics, Euler (1707–1783) and Lagrange (1736–1813) initiated the so-called *variational calculus*, cf. [57], which led to the problem of solving partial differential equations, PDEs for short. In this section, we briefly present the evolution of this theory to serve as a guide to M. Janet's contributions. We follow the history to introduce material on exterior differential systems and various PDE problems. For a deeper discussion of the theory of differential equations and the Pfaff problem, we refer the reader to [22, 92] or [11].

2.1 Pfaff's Problem

2.1.1 Partial Differential Equations for One Unknown Function. In 1772, Lagrange [56] considered a PDE of the form

$$F(x, y, \varphi, p, q) = 0, \quad \text{with} \quad p = \frac{\partial \varphi}{\partial x} \quad \text{and} \quad q = \frac{\partial \varphi}{\partial y}, \tag{2.1}$$

i.e., a PDE for one unknown function φ of two variables x and y. Lagrange's method to solve this PDE can be summarized as follows.

(i) Express the PDE (2.1) in the form

$$q = F_1(x, y, \varphi, p), \quad \text{with} \quad p = \frac{\partial \varphi}{\partial x} \quad \text{and} \quad q = \frac{\partial \varphi}{\partial y}. \tag{2.2}$$

(ii) Ignore for the moment that $p = \frac{\partial \varphi}{\partial x}$ and consider the 1-form

$$\Omega = d\varphi - pdx - qdy = d\varphi - pdx - F_1(x, y, \varphi, p)dy,$$

where p is regarded as some (not yet fixed) function of x, y, and φ.

(iii) If there exist functions M and Φ of x, y, and φ satisfying $M\Omega = d\Phi$, then $\Phi(x, y, \varphi) = C$ for some constant C. Solving this new equation, we obtain a solution $\varphi = \psi(x, y, C)$ to Eq. (2.2).

2.1.2 Pfaffian Systems. In 1814–15, Pfaff (1765–1825) [71] studied a PDE for one unknown function of n variables; this work was then continued by Jacobi (1804–1851) (cf. [46]). Recall that a PDE of any order is equivalent to a system of first-order PDEs. Thus, we may only think of systems of first-order PDEs with m unknown functions

$$F_k\big(x_1,\ldots,x_n,\varphi^1,\ldots,\varphi^m,\frac{\partial\varphi^a}{\partial x_i}\ (1\leqslant a\leqslant m, 1\leqslant i\leqslant n)\big)=0,\quad \text{for}\quad 1\leqslant k\leqslant r.$$

Introducing new variables p_i^a, the system lives on the space with coordinates (x_i,φ^a,p_i^a) and is given by

$$\begin{cases} F_k(x_i,\varphi^a,p_i^a)=0,\\ d\varphi^a-\displaystyle\sum_{i=1}^n p_i^a dx_i=0,\\ dx_1\wedge\cdots\wedge dx_n\neq 0.\end{cases}$$

Note that the last condition means that the variables $x_1,\ldots,x_n$ are independent. Such a system is called a *Pfaffian system*. One is interested in the questions whether this system admits a solution or not, and if there exists a solution, if it is unique under some conditions. We will refer to these as *Pfaff's problems*.

2.1.3 Cauchy–Kowalevsky's Theorem. A naive approach to Pfaff's problems, with applications to mechanics in mind, is the question of the initial conditions. In series of articles published in 1842, A. Cauchy (1789–1857) studied the system of first-order PDEs:

$$\frac{\partial\varphi^a}{\partial t}=f_a(t,x_1,\ldots,x_n)+\sum_{b=1}^m\sum_{i=1}^n f_{a,b}^i(t,x_1,\ldots,x_n)\frac{\partial\varphi^b}{\partial x_i},\quad \text{for}\quad 1\leqslant a\leqslant m,$$

where f_a, $f_{a,b}^i$ and $\varphi^1,\ldots,\varphi^m$ are functions of $n+1$ variables $t,x_1,\ldots,x_n$. Kowalevsky (1850–1891) [91] in 1875 considered systems of PDEs of the following form: for some $r_a\in\mathbb{Z}_{>0}$ $(1\leqslant a\leqslant m)$,

$$\frac{\partial^{r_a}\varphi^a}{\partial t^{r_a}}=\sum_{b=1}^m\sum_{\substack{j=0\\ j+|\alpha|\leqslant r_a}}^{r_a-1} f_{a,b}^{j,\alpha}(t,x_1,\ldots,x_n)\frac{\partial^{j+|\alpha|}\varphi^b}{\partial t^j\partial x^\alpha},$$

where $f_{a,b}^{j,\alpha}$ and $\varphi^1,\ldots,\varphi^m$ are functions of $n+1$ variables $t,x_1,\ldots,x_n$, and where for a multi-index $\alpha=(\alpha_1,\cdots,\alpha_n)$ in $(\mathbb{Z}_{\geqslant 0})^n$, we set $|\alpha|=\sum_{i=1}^n\alpha_i$ and $\partial x^\alpha=\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}$. They showed that under the hypothesis of analyticity of the coefficients, such a system admits a unique analytic local solution satisfying a given initial condition. This statement is now called the *Cauchy–Kowalevsky theorem*.

2.1.4 Completely Integrable Systems. A first geometric approach to the above problem was undertaken over by Frobenius (1849–1917) [23] and independently by Darboux (1842–1917) [15]. Let X be a differentiable manifold of dimension n. We consider the Pfaffian system

$$\omega_i=0\qquad 1\leqslant i\leqslant r,$$

where ω_i are 1-forms defined on a neighborhood V of a point x in X. Suppose that the family

$$\{(\omega_i)_y\}_{1\leqslant i\leqslant r} \subset T_y^* X$$

is linearly independent for all y in V. For $0 \leqslant p \leqslant n$, let us denote by $\Omega_X^p(V)$ the space of differentiable p-forms on V. A *p-dimensional distribution* $\mathcal{D}$ on X is a subbundle of TX with fiber of dimension p. A distribution $\mathcal{D}$ is *involutive* if, for any vector fields ξ and η taking values in $\mathcal{D}$, the Lie bracket

$$[\xi, , \eta] := \xi\eta - \eta\xi$$

takes values in $\mathcal{D}$ as well. Such a Pfaffian system is said to be *completely integrable*.

G. Frobenius and G. Darboux showed that the ideal I of $\bigoplus_{p=0}^n \Omega_X^p(V)$, generated by the 1-forms $\omega_1, \ldots, \omega_r$, is a differential ideal, i.e., $dI \subset I$, if and only if the distribution $\mathcal{D}$ on V defined as the annihilator of $\omega_1, \ldots, \omega_r$ is involutive.

2.2 The Cartan–Kähler Theory

Here, we give a brief historically oriented exposition of the so-called Cartan–Kähler theory. In particular, we will present the notion of system in involution. For the original treatment by the founders of the theory, we refer the reader to [14, 52], modern introductions are provided in [6, 62], and a quick survey can be found in [95, Appendix].

2.2.1 Differential Forms. Grassmann (1809–1877) [29] introduced in 1844 the first equation-based formulation of the structure of exterior algebra with the anti-commutativity rule

$$x \wedge y = -y \wedge x.$$

Using this setting, Cartan (1869–1951) [11] defined in 1899 the *exterior differential* and *differential p-forms*. He showed that these notions are invariant under arbitrary coordinate transformation. Thanks to these differential structures, several results obtained in the nineteenth century were reformulated in a clear manner.

2.2.2 Exterior Differential Systems. An *exterior differential system* Σ is a finite set of homogeneous differential forms, i.e., $\Sigma \subset \bigcup_p \Omega_X^p$. Cartan [12], in 1901, studied exterior differential systems generated by 1-forms, i.e., Pfaffian systems. Later, Kähler (1906–2000) [52] generalized Cartan's theory to any differential ideal I generated by an exterior differential system. For this reason, the general theory on exterior differential systems is nowadays called the *Cartan–Kähler theory*.

In the rest of this subsection, we discuss briefly the existence theorem for such a system. Since the argument developed here is *local* and we need the Cauchy–

Kowalevsky theorem, we assume that all functions are *analytic* in $x_1, \dots, x_n$ unless otherwise stipulated.

2.2.3 Integral Elements. Let Σ be an exterior differential system on a real analytic manifold X of dimension n such that the ideal generated by Σ is a differential ideal. For $0 \leqslant p \leqslant n$, set $\Sigma^p = \Sigma \cap \Omega^p_X$. We fix x in X. For $p > 0$, a pair (E_p, x), with a p-dimensional vector subspace $E_p \subset T_x X$, is called an *integral p-element* if $\omega|_{E_p} = 0$ for any ω in $\Sigma^p_x := \Sigma^p \cap \Omega^p_{X,x}$, where $\Omega^p_{X,x}$ denotes the space of differential p-forms defined on a neighborhood of x in X. We denote the set of integral elements of dimension p by $I\Sigma^p_x$.

An *integral manifold* Y is a submanifold of X whose tangent space $T_y Y$ at each point y in Y is an integral element. Since the exterior differential system defined by Σ is completely integrable, there exists independent r-functions $\varphi_1(x), \dots, \varphi_r(x)$, called *integrals of motion* or *first integrals*, defined on a neighborhood V of a point $x \in X$ such that their restrictions on $V \cap Y$ are constants.

The *polar space* $H(E_p)$ of an integral element E_p of Σ at the point x is the vector subspace of $T_x X$ generated by the vectors $\xi \in T_x X$ such that $E_p + \mathbb{R}\xi$ is an integral element of Σ.

2.2.4 Regular Integral Elements. Let E_0 be the real analytic subvariety of X defined as the zeros of Σ^0 and let $\mathcal{U}$ be the subset of smooth points. A point in E_0 is called *integral point*. A tangent vector ξ in $T_x X$ is called a *linear integral element* if $\omega(\xi) = 0$ for any $\omega \in \Sigma^1_x$ with $x \in \mathcal{U}$. We define inductively the properties called "regular" and "ordinary" as follows:

(i) The zeroth-order *character* is the integer $s_0 = \max_{x \in \mathcal{U}} \{\dim \mathbb{R}\Sigma^1_x\}$. A point $x \in E_0$ is said to be *regular* if $\dim \mathbb{R}\Sigma^1_x = s_0$, and a linear integral element $\xi \in T_x X$ is called *ordinary* if x is regular.

(ii) Let $E_1 = \mathbb{R}\xi$, where ξ is an ordinary linear integral element. The first-order *character* is the integer s_1 satisfying $s_0 + s_1 = \max_{x \in \mathcal{U}} \{\dim H(E_1)\}$. The ordinary integral 1-element (E_1, x) is said to be *regular* if $\dim H(E_1) = s_0 + s_1$. An integral 2-element (E_2, x) is called *ordinary* if it contains at least one regular linear integral element.

(iii) Assume that all these concepts are defined up to $(p-1)$th step and that $s_0 + s_1 + \cdots + s_{p-1} < n - p + 1$.

The pth-order *character* is the integer s_p satisfying

$$\sum_{i=0}^{p} s_i = \max_{x \in \mathcal{U}} \{\dim H(E_p)\}.$$

An integral p-element (E_p, x) is said to be *regular* if

$$\sum_{i=0}^{p} s_i = \dim H(E_p).$$

The integral p-element (E_p, x) is called *ordinary* if it contains at least one regular integral element (E_{p-1}, x).

Let h be the smallest positive integer such that $\sum_{i=0}^{h} s_i = n - h$. Then, there does not exist an integral $(h+1)$-element. The integer h is called the *genus* of the system Σ. For $0 < p \leqslant h$, one has

$$\sum_{i=0}^{p-1} s_i \leqslant n - p.$$

2.2.5 Theorem *Let* $0 < p \leqslant h$ *be an integer.*

(i) *The case* $\sum_{i=0}^{p-1} s_i = n - p$ *: let* (E_p, x) *be an ordinary integral* p*-element and let* Y_{p-1} *be an integral manifold of dimension* $p-1$ *such that* $(T_x Y_{p-1}, x)$ *is a regular integral* $(p-1)$*-element contained in* (E_p, x)*. Then, there exists a unique integral manifold* Y_p *of dimension* p *containing* Y_{p-1} *such that* $T_x Y_p = E_p$.

(ii) *The case* $\sum_{i=0}^{p-1} s_i < n - p$ *: let* (E_p, x) *be an integral* p*-element and let* Y_{p-1} *be an integral manifold of dimension* $p-1$ *such that* $(T_x Y_{p-1}, x)$ *is a regular integral* $(p-1)$*-element contained in* (E_p, x)*. Then, there is a one-to-one correspondence between the set of real analytic functions* $C^{\omega}(\mathbb{R}^p, \mathbb{R}^{n-p-\sum_{i=0}^{p-1} s_i})$ *and the set of* p*-dimensional integral manifolds* Y_p *containing* Y_{p-1} *such that* $T_x Y_p = E_p$.

This theorem states that a given chain of ordinary integral elements

$$(E_0, x) \subset (E_1, x) \subset \cdots \subset (E_h, x), \qquad \dim E_p = p \quad (0 \leqslant p \leqslant h),$$

one can inductively find an integral manifold Y_p of dimension p such that $Y_0 = \{x\}$, $Y_{p-1} \subset Y_p$ and $T_x Y_p = E_p$. Notice that to obtain Y_p from Y_{p-1}, one applies the Cauchy–Kowalevsky theorem to the PDE system defined by Σ^p and the choice of real analytic functions in the above statement provide a datum to define the integral manifold Y_p.

2.2.6 Systems in Involution. In many applications, the exterior differential systems one considers admit p-independent variables $x_1, \ldots, x_p$. In such a case, we are only interested in the p-dimensional integral manifolds among which no additional relation between $x_1, \ldots, x_p$ is imposed. In general, an exterior differential system Σ for $n - p$ unknown functions and p-independent variables $x_1, \ldots, x_p$ is said to be *in involution* if it satisfies the two following conditions:

1. its genus is larger than or equal to p,
2. the defining equations of the generic ordinary integral p-element introduce no linear relation among $dx_1, \ldots, dx_p$.

2.2.7 Reduced Characters. Consider a family $\mathcal{F}$ of integral elements of dimensions $1, 2, \ldots, p-1$ than can be included in an integral p-element at a generic integral point $x \in X$. Take a local chart with origin x. The *reduced polar system* $H^{\mathrm{red}}(E_i)$ of an integral element at x is the polar system of the restriction of the exterior differential system Σ to the submanifold

$$\{x_1 = x_2 = \cdots = x_p = 0\}.$$

The integers $s'_0, \ldots, s'_{p-1}$, called the *reduced characters*, are defined in such a way that $s'_0 + \cdots + s'_i$ is the dimension of the reduced polar system $H^{\mathrm{red}}(E_i)$ at a generic integral element. For convenience, one sets $s'_p = n - p - (s'_0 + \cdots + s'_{p-1})$.

Let Σ be an exterior differential system of $n - p$ unknown functions of p-independent variables such that the ideal generated by Σ is a differential ideal. É. Cartan showed that Σ is a *system in involution* iff the most general integral p-element in $\mathcal{F}$ depends on $s'_1 + 2s'_2 + \cdots + ps'_p$ independent parameters.

2.2.8 Recent Developments. In 1957, Kuranishi (1924 –) [55] considered the problem of the prolongation of a given exterior differential system and treated what É. Cartan called total case. Here, M. Kuranishi as well as É. Cartan worked locally in the analytic category. After an algebraic approach to the integrability was proposed by Guillemin and Sternberg [34], in 1964, Singer and Sternberg, [84], in 1965 studied some classes of infinite-dimensional systems which can be treated even in the C^∞-category. In 1970s, with the aid of jet bundles and the Spencer cohomology, Pommaret (cf. [72]) considered formally integrable involutive differential systems generalizing the work of M. Janet, in the language of sheaf theory. For other geometric aspects not using sheaf theory, see the books by Griffiths (1938-) [31], and Bryant et al. [6].

3 Monomial PDE Systems

In this section, we present the method introduced by M. Janet under the name "*calcul inverse de la dérivation*" in his monograph [51]. In [51, Chap. I], M. Janet considered *monomial PDE*, that is, PDE of the form

$$\frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}\cdots\partial x_n^{\alpha_n}} = f_{\alpha_1\alpha_2\ldots\alpha_n}(x_1, x_2, \ldots, x_n), \tag{3.1}$$

where φ is an unknown function and the $f_{\alpha_1\alpha_2\ldots\alpha_n}$ are analytic functions of several variables. By an algebraic method, he analyzed the solvability of such an equation, namely, the existence and the uniqueness of an analytic solution φ of the system. Notice that the analyticity condition guarantees the commutativity of partial differentials operators. This property is crucial for the constructions that M. Janet carried out in the ring of commutative polynomials. Note that the first example of PDE that does not admit any solution was found by Lewy in the 1950s in [58].

3.1 Ring of Partial Differential Operators and Multiplicative Variables

3.1.1 Historical Context. In the beginning of 1890s, following collaboration with C. Méray (1835–1911), Riquier (1853–1929) initiated his research on finding normal forms of systems of (infinitely many) PDEs for finitely many unknown functions of finitely many independent variables (see [75] and [76] for more details).

In 1894, Tresse [88] showed that such systems can be always reduced to systems of finitely many PDEs. This is the first result on Noeterianity of a module over a ring of differential operators. Based on this result, É. Delassus (1868–19..) formalized and simplified Riquier's theory. In these works, one already finds an algorithmic approach for analyzing ideals of the ring $\mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$.

It was Janet (1888–1983) who already in his thesis [47], published in 1920, had realized that the latter ring is isomorphic to the ring of polynomials with n variables $\mathbb{K}[x_1, \ldots, x_n]$. At that time, several abstract notions on rings were introduced by E. Noether in Germany but by M. Janet in France was not familiar with them. It was only in 1937 that W. Gröbner (1899–1980) proved this isomorphism.

3.1.2 Proposition [32, Sect. 2.] *There exists a ring isomorphism*

$$\Phi : \mathbb{K}[x_1, \ldots, x_n] \longrightarrow \mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}],$$

from the ring of polynomials in n variables $x_1, \ldots, x_n$ with coefficients in an arbitrary field $\mathbb{K}$ to the ring of differential operators with constant coefficients.

3.1.3 Derivations and Monomials. M. Janet considers monomials in the variables $x_1, \ldots, x_n$ and uses implicitly the isomorphism Φ of Proposition 3.1.2. To a monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, he associates the differential operator

$$D^\alpha := \Phi(x^\alpha) = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

In [51, Chap. I], M. Janet considered finite monomial PDE systems. The equations are of the form (3.1) and since the system has a finitely many equations, the set of monomials associated to it is finite. The first result of the monograph is a finiteness result on monomials stating that a sequence of monomials in which none is a multiple of a preceding one is necessarily finite. M. Janet proved this result by induction on the number of variables. We can formulate it as follows.

3.1.4 Lemma ([51, Sect. 7]) *Let $\mathcal{U}$ be a subset of $\mathcal{M}(x_1, \ldots, x_n)$. If, for any monomials u and u' in $\mathcal{U}$, the monomial u does not divide u', then the set $\mathcal{U}$ is finite.*

This result corresponds to Dickson's Lemma [17], which asserts that every monomial ideal of $\mathbb{K}[x_1, \ldots, x_n]$ is finitely generated.

3.1.5 Stability of the Multiplication. M. Janet paid a special attention to families of monomials with the following property. A subset of monomial $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ is called *multiplicatively stable* if for any monomial u in $\mathcal{M}(x_1, \ldots, x_n)$ such that there exists u' in $\mathcal{U}$ that divides u, one has that u is in $\mathcal{U}$. In other words, the set $\mathcal{U}$ is closed under multiplication by monomials in $\mathcal{M}(x_1, \ldots, x_n)$.

As a consequence of Lemma 3.1.4, if $\mathcal{U}$ is a multiplicatively stable subset of $\mathcal{M}(x_1, \ldots, x_n)$, then it contains only finitely many elements that are not multiples of any other elements in $\mathcal{U}$. Hence, there exists a finite subset $\mathcal{U}_f$ of $\mathcal{U}$ such that for any u in $\mathcal{U}$, there exists u_f in $\mathcal{U}_f$ such that u_f divides u.

3.1.6 Ascending Chain Condition. M. Janet observed another consequence of Lemma 3.1.4: the *ascending chain condition* on multiplicatively stable monomial sets, which he formulated as follows. Any ascending sequence of multiplicatively stable subsets of $\mathcal{M}(x_1, \ldots, x_n)$

$$\mathcal{U}_1 \subset \mathcal{U}_2 \subset \cdots \subset \mathcal{U}_k \subset \cdots$$

is finite. This corresponds to the Noetherian property on the set of monomials in finitely many variables.

3.1.7 Inductive Construction. Let us fix a total order on the variables $x_n > x_{n-1} > \cdots > x_1$. Let $\mathcal{U}$ be a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$. Let us define, for every $0 \leqslant \alpha_n \leqslant \deg_n(\mathcal{U})$,

$$[\alpha_n] = \{u \in \mathcal{U} \mid \deg_n(u) = \alpha_n\}.$$

The family $([0], \ldots, [\deg_n(\mathcal{U})])$ forms a partition of $\mathcal{U}$. We define for every $0 \leqslant \alpha_n \leqslant \deg_n(\mathcal{U})$

$$\overline{[\alpha_n]} = \{u \in \mathcal{M}(x_1, \ldots, x_{n-1}) \mid u x_n^{\alpha_n} \in \mathcal{U}\}.$$

We set for every $0 \leqslant i \leqslant \deg_n(\mathcal{U})$

$$\mathcal{U}'_i = \bigcup_{0 \leqslant \alpha_n \leqslant i} \{u \in \mathcal{M}(x_1, \ldots, x_{n-1}) \mid \text{there exists } u' \in \overline{[\alpha_n]} \text{ such that } u'|u\}.$$

Finally, we set

$$\mathcal{U}_k = \begin{cases} \{u x_n^k \mid u \in \mathcal{U}'_k\}, & \text{if } k < \deg_n(\mathcal{U}), \\ \{u x_n^k \mid u \in \mathcal{U}'_{\deg_n(\mathcal{U})}\}, & \text{if } k \geqslant \deg_n(\mathcal{U}), \end{cases}$$

and $M(\mathcal{U}) = \bigcup_{k \geqslant 0} \mathcal{U}_k$. By this inductive construction, M. Janet obtains the monomial ideal generated by $\mathcal{U}$. Indeed, $M(\mathcal{U})$ coincides with the following set of monomial:

$$\{u \in \mathcal{M}(x_1, \ldots, x_n) \mid \text{there exists } u' \text{ in } \mathcal{U} \text{such that } u'|u\}.$$

3.1.8 Example. Consider the subset $\mathcal{U} = \{x_3 x_2^2, x_3^3 x_1^2\}$ of $\mathcal{M}(x_1, x_2, x_3)$. We have

$$[0] = \emptyset, \quad [1] = \{x_3 x_2^2\}, \quad [2] = \emptyset, \quad [3] = \{x_3^3 x_1^2\}.$$

Hence,

$$\overline{[0]} = \emptyset, \quad \overline{[1]} = \{x_2^2\}, \quad \overline{[2]} = \emptyset, \quad \overline{[3]} = \{x_1^2\}.$$

The set $M(\mathcal{U})$ is defined using of the following subsets:

$$\mathcal{U}'_0 = \emptyset, \quad \mathcal{U}'_1 = \{x_1^{\alpha_1} x_2^{\alpha_2} \mid \alpha_2 \geqslant 2\}, \quad \mathcal{U}'_2 = \mathcal{U}'_1, \quad \mathcal{U}'_3 = \{x_1^{\alpha_1} x_2^{\alpha_2} \mid \alpha_1 \geqslant 2 \text{ ou } \alpha_2 \geqslant 2\}.$$

3.1.9 Janet's Multiplicative Variables [47, Sect. 7]. Let us fix a total order $x_n > x_{n-1} > \cdots > x_1$ on variables. Let $\mathcal{U}$ be a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$. For all $1 \leqslant i \leqslant n$, we define the following subset of $\mathcal{U}$:

$$[\alpha_i, \ldots, \alpha_n] = \{u \in \mathcal{U} \mid \deg_j(u) = \alpha_j \text{ for all } i \leqslant j \leqslant n\}.$$

That is, $[\alpha_i, \ldots, \alpha_n]$ contains monomials of $\mathcal{U}$ of the form $vx_i^{\alpha_i} \cdots x_n^{\alpha_n}$, with v in $\mathcal{M}(x_1, \ldots, x_{i-1})$. The sets $[\alpha_i, \ldots, \alpha_n]$ with $\alpha_i, \ldots, \alpha_n$ in $\mathbb{N}$ form a partition of $\mathcal{U}$. Moreover, for all $1 \leqslant i \leqslant n-1$, we have $[\alpha_i, \alpha_{i+1}, \ldots, \alpha_n] \subseteq [\alpha_{i+1}, \ldots, \alpha_n]$ and the sets $[\alpha_i, \ldots, \alpha_n]$, where $\alpha_i \in \mathbb{N}$, form a partition of $[\alpha_{i+1}, \ldots, \alpha_n]$.

Given a monomial u in $\mathcal{U}$, the variable x_n is said to be *multiplicative for u in the sense of Janet* if

$$\deg_n(u) = \deg_n(\mathcal{U}).$$

For $i \leqslant n-1$, the variable x_i is said to be *multiplicative for u in the sense of Janet* if

$$u \in [\alpha_{i+1}, \ldots, \alpha_n] \quad \text{and} \quad \deg_i(u) = \deg_i([\alpha_{i+1}, \ldots, \alpha_n]).$$

We will denote by $\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u)$ the set of multiplicative variables of u in the sense of Janet with respect to the set $\mathcal{U}$, also called *$\mathcal{J}$-multiplicative variables*.

Note that, by definition, for any u and u' in $[\alpha_{i+1}, \ldots, \alpha_n]$, we have

$$\{x_{i+1}, \ldots, x_n\} \cap \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u) = \{x_{i+1}, \ldots, x_n\} \cap \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u').$$

Accordingly, we will denote this set of multiplicative variables by $\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}([\alpha_{i+1}, \ldots, \alpha_n])$.

3.1.10 Example. Consider the subset $\mathcal{U} = \{x_2x_3, x_2^2, x_1\}$ of $\mathcal{M}(x_1, x_2, x_3)$ with the order
$x_3 > x_2 > x_1$. We have $\deg_3(\mathcal{U}) = 1$; hence, the variable x_3 is $\mathcal{J}$-multiplicative for x_3x_2 and not $\mathcal{J}$-multiplicative for x_2^2 and x_1.

For $\alpha \in \mathbb{N}$, we have $[\alpha] = \{u \in \mathcal{U} \mid \deg_3(u) = \alpha\}$, hence

$$[0] = \{x_2^2, x_1\}, \qquad [1] = \{x_2x_3\}.$$

We have $\deg_2(x_2^2) = \deg_2([0])$, $\deg_2(x_1) \neq \deg_2([0])$ and $\deg_2(x_2x_3) = \deg_2([1])$, so the variable x_2 is $\mathcal{J}$-multiplicative for x_2^2 and x_2x_3 and not $\mathcal{J}$-multiplicative for x_1. Further,

$$[0, 0] = \{x_1\}, \quad [0, 2] = \{x_2^2\}, \quad [1, 1] = \{x_2x_3\},$$

and $\deg_1(x_2^2) = \deg_1([0, 2])$, $\deg_1(x_1) = \deg_1([0, 0])$ and $\deg_1(x_3x_2) = \deg_1([1, 1])$, so the variable x_1 is $\mathcal{J}$-multiplicative for x_1, x_2^2 and x_3x_2.

3.1.11 Janet Divisor. Let $\mathcal{U}$ be a subset of $\mathcal{M}(x_1, \ldots, x_n)$. A monomial u in $\mathcal{U}$ is called *Janet divisor* of a monomial w in $\mathcal{M}(x_1, \ldots, x_n)$ with respect to $\mathcal{U}$, if there is a decomposition $w = uv$, where any variable occurring in v is $\mathcal{J}$-multiplicative with respect to $\mathcal{U}$.

3.1.12 Proposition *Let $\mathcal{U}$ be a subset of $\mathcal{M}(x_1, \ldots, x_n)$ and w be a monomial in $\mathcal{M}(x_1, \ldots, x_n)$. Then w admits in $\mathcal{U}$ at most one Janet divisor with respect to $\mathcal{U}$.*

Proof If u is a Janet divisor of w with respect to $\mathcal{U}$, there is a v in $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u))$ such that $w = uv$. We have $\deg_n(v) = \deg_n(w) - \deg_n(u)$. If $\deg_n(w) \geqslant \deg_n(\mathcal{U})$, then the variable x_n is $\mathcal{J}$-multiplicative and $\deg_n(v) = \deg_n(w) - \deg_n(\mathcal{U})$. If $\deg_n(w) < \deg_n(\mathcal{U})$, then x_n cannot be $\mathcal{J}$-multiplicative and $\deg_n(v) = 0$.

As a consequence, for any Janet divisors u and u' of w in $\mathcal{U}$, we have $\deg_n(u) = \deg_n(u')$ and $u, u' \in [\alpha]$ for some $\alpha \in \mathbb{N}$.

Suppose now that u and u' are two distinct Janet divisors of w in $\mathcal{U}$. There exists $1 < k \leqslant n$ such that $u, u' \in [\alpha_k, \ldots, \alpha_n]$ and $\deg_{k-1}(u) \neq \deg_{k-1}(u')$. Suppose that $\deg_{k-1}(u) > \deg_{k-1}(u')$. Then the variable x_{k-1} cannot be $\mathcal{J}$-multiplicative for u' with respect to $\mathcal{U}$. It follows that u' cannot be a Janet divisor of w. This leads to a contradiction, hence $u = u'$. □

3.1.13 Complementary Monomials. Let $\mathcal{U}$ be a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$. The set of *complementary monomials of $\mathcal{U}$* is the set of monomials

$$\mathcal{U}^{\complement} = \bigcup_{1 \leqslant i \leqslant n} \mathcal{U}^{\complement(i)}, \tag{3.2}$$

where

$$\mathcal{U}^{\complement(n)} = \{x_n^{\beta} \mid 0 \leqslant \beta \leqslant \deg_n(\mathcal{U}) \text{ and } [\beta] = \emptyset\},$$

and for every $1 \leqslant i < n$,

$$\mathcal{U}^{\complement(i)} = \Big\{ x_i^{\beta} x_{i+1}^{\alpha_{i+1}} \ldots x_n^{\alpha_n} \;\Big|\; [\alpha_{i+1}, \ldots, \alpha_n] \neq \emptyset, \\ 0 \leqslant \beta < \deg_i([\alpha_{i+1}, \ldots, \alpha_n]),\; [\beta, \alpha_{i+1}, \ldots, \alpha_n] = \emptyset \Big\}.$$

Note that the union in (3.2) is disjoint, since $\mathcal{U}^{\complement(i)} \cap \mathcal{U}^{\complement(j)} = \emptyset$ for $i \neq j$.

3.1.14 Multiplicative Variables of Complementary Monomials. For any monomial u in $\mathcal{U}^{\complement}$, we define the set ${}^{\complement}\mathrm{Mult}^{\mathcal{U}^{\complement}}$ of *multiplicative variables for u with respect to complementary monomials* in $\mathcal{U}^{\complement}$ as follows. If the monomial u is in $\mathcal{U}^{\complement(n)}$, we set

$${}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}^{\complement(n)}}(u) = \{x_1, \ldots, x_{n-1}\}.$$

For $1 \leqslant i \leqslant n-1$, for any monomial u in $\mathcal{U}^{\complement(i)}$, there exist $\alpha_{i+1}, \ldots, \alpha_n$ such that $u \in [\alpha_{i+1}, \ldots, \alpha_n]$. Then

$$^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}^{\complement(i)}}(u) = \{x_1, \ldots, x_{i-1}\} \cup \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}([\alpha_{i+1}, \ldots, \alpha_n]).$$

Finally, for u in $\mathcal{U}^{\complement}$, there exists a unique $1 \leqslant i_u \leqslant n$ such that $u \in \mathcal{U}^{\complement(i_u)}$. Then we set

$$^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}^{\complement}}(u) = {}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}^{\complement(i_u)}}(u).$$

3.1.15 Example [51, p. 17]. Consider the subset $\mathcal{U} = \{x_3^3x_2^2x_1^2, x_3^3x_1^3, x_3x_2x_1^3, x_3x_2\}$ of $\mathcal{M}(x_1, x_2, x_3)$ with the order $x_3 > x_2 > x_1$. The following table gives the multiplicative variables for each monomial:

$x_3^3x_2^2x_1^2$	x_3	x_2	x_1
$x_3^3x_1^3$	x_3		x_1
$x_3x_2x_1^3$		x_2	x_1
x_3x_2		x_2	

The sets of complementary monomials are

$$\mathcal{U}^{\complement(3)} = \{1, x_3^2\}, \quad \mathcal{U}^{\complement(2)} = \{x_3^3x_2, x_3\},$$
$$\mathcal{U}^{\complement(1)} = \{x_3^3x_2^2x_1, x_3^3x_2^2, x_3^3x_1^2, x_3^3x_1, x_3^3, x_3x_2x_1^2, x_3x_2x_1\}.$$

The following table gives the multiplicative variables for each monomial:

$1, x_3^2$		x_2	x_1
$x_3^3x_2$	x_3		x_1
x_3			x_1
$x_3^3x_2^2x_1, x_3^3x_2^2$	x_3	x_2	
$x_3^3x_1^2, x_3x_1, x_3^3$	x_3		
$x_3x_2x_1^2, x_3x_2x_1$		x_2	

3.2 Completion Procedure

In this subsection, we present the notion of complete system introduced by Janet in [51]. In particular, we recall the completion procedure that he gave in order to complete a finite set of monomials.

3.2.1 Complete Systems. Let $\mathcal{U}$ be a subset of $\mathcal{M}(x_1, \ldots, x_n)$. For a monomial u in $\mathcal{U}$ (resp. in $\mathcal{U}^{\complement}$), M. Janet defined the *involutive cone of u with respect to* $\mathcal{U}$ (resp. *to* $\mathcal{U}^{\complement}$) as the set of monomials

$$\mathrm{cone}_{\mathcal{J}}(u,\mathcal{U}) = \{\, uv \mid v \in \mathcal{M}(\mathrm{Mult}^{\mathcal{U}}_{\mathcal{J}}(u))\,\}$$
$$(\text{resp.}\quad \mathrm{cone}^{\complement}_{\mathcal{J}}(u,\mathcal{U}) = \{\, uv \mid v \in \mathcal{M}({}^{\complement}\mathrm{Mult}^{\mathcal{U}^{\complement}}_{\mathcal{J}}(u))\,\}).$$

The *involutive cone of the set* $\mathcal{U}$ is defined by

$$\mathrm{cone}_{\mathcal{J}}(\mathcal{U}) = \bigcup_{u\in\mathcal{U}} \mathrm{cone}_{\mathcal{J}}(u,\mathcal{U}) \qquad (\text{resp.}\quad \mathrm{cone}^{\complement}_{\mathcal{J}}(\mathcal{U}) = \bigcup_{u\in\mathcal{U}^{\complement}} \mathrm{cone}^{\complement}_{\mathcal{J}}(u,\mathcal{U})).$$

M. Janet called *complete* a set of monomials $\mathcal{U}$ when $\mathrm{cone}(\mathcal{U}) = \mathrm{cone}_{\mathcal{J}}(\mathcal{U})$. An involutive cone is called class in Janet's monograph [51]. The terminology "*involutive*" first appeared in the paper [25] by Gerdt and is standard now. We refer the reader to [63] for a discussion of the relation between this notion and the notion of involutivity in the work of É. Cartan.

3.2.2 Proposition [51, p. 18] *For any finite subset* $\mathcal{U}$ *of* $\mathcal{M}(x_1,\ldots,x_n)$*, we have the partition*

$$\mathcal{M}(x_1,\ldots,x_n) = \mathrm{cone}_{\mathcal{J}}(\mathcal{U}) \amalg \mathrm{cone}^{\complement}_{\mathcal{J}}(\mathcal{U}).$$

3.2.3 A Proof of Completeness by Induction. Let $\mathcal{U}$ be a finite subset of $\mathcal{M}(x_1,\ldots,x_n)$. Consider the partition $[0],\ldots,[\deg_n(\mathcal{U})]$ of monomials in $\mathcal{U}$ by their degrees in x_n. Let $\alpha_1 < \alpha_2 < \cdots < \alpha_k$ be positive integers such that $[\alpha_i]$ is non-empty. Recall that $\overline{[\alpha_i]}$ is the set of monomials u in $\mathcal{M}(x_1,\ldots,x_{n-1})$ such that $ux_n^{\alpha_i}$ is in $\mathcal{U}$. With these notations, the following result gives an inductive method to prove that a finite set of monomials is complete.

3.2.4 Proposition [51, p. 19] *A finite set* $\mathcal{U}$ *is complete if and only if the two following conditions are satisfied:*

- **(i)** *the sets* $\overline{[\alpha_1]},\ldots,\overline{[\alpha_k]}$ *are complete,*
- **(ii)** *for any* $1 \leqslant i < k$*, the set* $\overline{[\alpha_i]}$ *is contained in* $\mathrm{cone}_{\mathcal{J}}(\overline{[\alpha_i+1]})$.

As an immediate consequence of this proposition, M. Janet obtained the following characterization.

3.2.5 Proposition [51, p. 20] *A finite subset* $\mathcal{U}$ *of* $\mathcal{M}(x_1,\ldots,x_n)$ *is complete if and only if, for any* u *in* $\mathcal{U}$ *and any* x *non-multiplicative variable of* u *with respect to* $\mathcal{U}$*,* ux *lies in* $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$.

3.2.6 Example [51, p. 21]. Consider the subset $\mathcal{U} = \{x_5x_4, x_5x_3, x_5x_2, x_4^2, x_4x_3, x_3^2\}$ of $\mathcal{M}(x_1,\ldots,x_5)$. The multiplicative variables are given by the following table:

x_5x_4	x_5	x_4	x_3	x_2	x_1
x_5x_3	x_5		x_3	x_2	x_1
x_5x_2	x_5			x_2	x_1
x_4^2		x_4	x_3	x_2	x_1
x_3x_4			x_3	x_2	x_1
x_3^2			x_3	x_2	x_1

To prove that this set of monomials is complete, we apply Proposition 3.2.5. The completeness follows from the identities

$$
\begin{gathered}
x_5x_3 \cdot x_4 = x_5x_4 \cdot x_3,\\
x_5x_2 \cdot x_4 = x_5x_4 \cdot x_2,\ \ x_5x_2 \cdot x_3 = x_5x_3 \cdot x_2,\\
x_4^2 \cdot x_5 = x_5x_4 \cdot x_4,\\
x_4x_3 \cdot x_5 = x_5x_4 \cdot x_3,\ \ x_4x_3 \cdot x_4 = x_4^2 \cdot x_3,\\
x_3^2 \cdot x_5 = x_5x_3 \cdot x_3,\ \ x_3^2 \cdot x_4 = x_4x_3 \cdot x_3.
\end{gathered}
$$

3.2.7 Examples. For every $1 \leqslant p \leqslant n$, the set of monomials of degree p is complete. Any finite set of monomials of degree 1 is complete.

3.2.8 Theorem (Janet's Completion Lemma, *[51, p. 21]*) *For any finite subset* $\mathcal{U}$ *of* $\mathcal{M}(x_1, \ldots, x_n)$ *there exists a finite set* $J(\mathcal{U})$ *satisfying the following three conditions:*

(i) $J(\mathcal{U})$ *is complete,*
(ii) $\mathcal{U} \subseteq J(\mathcal{U})$,
(iii) $\mathrm{cone}(\mathcal{U}) = \mathrm{cone}(J(\mathcal{U}))$.

3.2.9 Completion Procedure. From Proposition 3.2.5, M. Janet deduced the completion procedure **Complete**$(\mathcal{U})$, Procedure 1, which computes a completion of a finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ [51, p. 21]. M. Janet did not give a proof of the fact that this procedure terminates. We will present a proof of the correctness and termination of this procedure in Sect. 4.2.

Input: $\mathcal{U}$ a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$

Output: A finite set $J(\mathcal{U})$ satisfying the conditions of Theorem 3.2.8.

begin
  $\widetilde{\mathcal{U}} \leftarrow \mathcal{U}$
  while *exist* $u \in \widetilde{\mathcal{U}}$ *and* $x \in \mathrm{NMult}^{\widetilde{\mathcal{U}}}_{\mathcal{J}}(u)$ *such that* ux *is not in* $\mathrm{cone}_{\mathcal{J}}(\widetilde{\mathcal{U}})$ **do**
    Choose such u and x,
    $\widetilde{\mathcal{U}} \leftarrow \widetilde{\mathcal{U}} \cup \{ux\}$.
  end
end

Procedure 1: Complete$(\mathcal{U})$

3.2.10 Example [51, p. 28]. Consider the subset $\mathcal{U} = \{x_3x_2^2, x_3^3x_1^2\}$ of $\mathcal{M}(x_1, x_2, x_3)$ with the order $x_3 > x_2 > x_1$. The following table gives the multiplicative variables for each monomial:

$x_3^3x_1^2$	x_3	x_2	x_1
$x_3x_2^2$		x_2	x_1

We complete the set $\mathcal{U}$ as follows. The monomial $x_3x_2^2 \cdot x_3$ is not in $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$; we set $\widetilde{\mathcal{U}} \leftarrow \mathcal{U} \cup \{x_3^2x_2^2\}$ and we compute multiplicative variables with respect to $\widetilde{\mathcal{U}}$:

$$\begin{array}{l|lll} x_3^3x_1^2 & x_3 & x_2 & x_1 \\ x_3^2x_2^2 & & x_2 & x_1 \\ x_3x_2^2 & & x_2 & x_1 \end{array}$$

The monomial $x_3x_2^2 \cdot x_3$ is in $\text{cone}_{\mathcal{J}}(\widetilde{\mathcal{U}})$, but $x_3^2x_2^2 \cdot x_3$ is not in $\text{cone}_{\mathcal{J}}(\widetilde{\mathcal{U}})$; we set $\widetilde{\mathcal{U}} \leftarrow \widetilde{\mathcal{U}} \cup \{x_3^3x_2^2\}$. The multiplicative variables of this new set of monomials are

$$\begin{array}{l|lll} x_3^2x_2^2 & x_3 & x_2 & x_1 \\ x_3^3x_1^2 & x_3 & & x_1 \\ x_3^2x_2^2 & & x_2 & x_1 \\ x_3x_2^2 & & x_2 & x_1 \end{array}$$

The monomial $x_3x_1^2 \cdot x_2$ is not in $\text{cone}_{\mathcal{J}}(\widetilde{\mathcal{U}})$, the other products are in $\text{cone}_{\mathcal{J}}(\widetilde{\mathcal{U}})$, and we prove that the system

$$\widetilde{\mathcal{U}} = \{ x_3x_2^2, x_3^3x_1^2, x_3^3x_2^2, x_3^3x_2x_1^2, x_3^2x_2^2 \}$$

is complete.

3.3 Inversion of Differentiation

In this subsection, we recall the results of Janet from [51] on the solvability of monomial PDE systems of the form

$$(\Sigma) \qquad D^\alpha \varphi = f_\alpha(x_1, x_2, \ldots, x_n), \qquad \alpha \in \mathbb{N}^n, \tag{3.3}$$

where φ is an unknown function and f_α are analytic functions of several variables. As recalled in Sect. 3.1.1, an infinite set of partial differential equations can be always reduced to a finite set of such equations. This is a consequence of Dickson's Lemma, whose formulation due to M. Janet is given in Lemma 3.1.4. Accordingly, without loss of generality, we can assume that the system (Σ) is finite. Using Proposition 3.1.2, M. Janet associated to each differential operator D^α a monomial x^α in $\mathcal{M}(x_1, \ldots, x_n)$. In this way, to a PDE system (Σ) in the variables $x_1, \ldots, x_n$ he associated a finite set $\text{lm}(\Sigma)$ of monomials. By Theorem 3.2.8, any such set $\text{lm}(\Sigma)$ of monomials can be completed to a finite complete set $J(\text{lm}(\Sigma))$ having the same cone as $\text{lm}(\Sigma)$.

3.3.1 Computation of Inversion of Differentiation. Let us now assume that the set of monomials $\text{lm}(\Sigma)$ is finite and complete. Since the cone of $\text{lm}(\Sigma)$ is equal to the involutive cone of $\text{lm}(\Sigma)$, each monomial u in $\text{lm}(\Sigma)$ and non-multiplicative variable x_i in $\text{NMult}_{\mathcal{J}}^{\text{lm}(\Sigma)}(u)$, admits a decomposition

$$ux_i = vw,$$

where v is in $\mathrm{lm}(\Sigma)$ and w belongs to $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathrm{lm}(\Sigma)}(v))$. To each such a decomposition, it corresponds a compatibility condition of the PDE system (Σ), that is, for $u = x^{\alpha}$, $v = x^{\beta}$ and $w = x^{\gamma}$ with α, β and γ in $\mathbb{N}^n$,

$$\frac{\partial f_{\alpha}}{\partial x_i} = D^{\gamma} f_{\beta}.$$

Let us denote by (C_{Σ}) the set of all such compatibility conditions. M. Janet showed that under the completeness hypothesis this set of compatibility conditions is sufficient for the PDE system (Σ) to be formally integrable in the sense of [72].

3.3.2 The Space of Initial Conditions. Let us consider the set $\mathrm{lm}(\Sigma)^{\complement}$ of complementary monomials of the finite complete set $\mathrm{lm}(\Sigma)$. Suppose that the PDE system (Σ) satisfies the set (C_{Σ}) of compatibility conditions. M. Janet associated to each monomial $v = x^{\beta}$ in $\mathrm{lm}(\Sigma)^{\complement}$ with $\beta \in \mathbb{N}^n$ an analytic function

$$\varphi_{\beta}(x_{i_1}, \ldots, x_{i_{k_v}}),$$

where $\{x_{i_1}, \ldots, x_{i_{k_v}}\} = {}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathrm{lm}(\Sigma)^{\complement}}(v)$. By Proposition 3.2.2, the set of such analytic functions provides a compatible initial condition. Under these assumptions, M. Janet proved the following result.

3.3.3 Theorem [51, p. 25] *Let (Σ) be a finite monomial PDE system such that $\mathrm{lm}(\Sigma)$ is complete. If (Σ) satisfies the compatibility conditions (C_{Σ}), then it admits a unique solution with initial conditions given for any $v = x^{\beta}$ in $\mathrm{lm}(\Sigma)^{\complement}$ with $\beta \in \mathbb{N}^n$ by*

$$D^{\beta}\varphi\Big|_{x_j=0\ \forall x_j \in {}^{\complement}\mathrm{NMult}_{\mathcal{J}}^{\mathrm{lm}(\Sigma)^{\complement}}(v)} = \varphi_{\beta}(x_{i_1}, \ldots, x_{i_{k_v}}),$$

where $\{x_{i_1}, \ldots, x_{i_{k_v}}\} = {}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathrm{lm}(\Sigma)^{\complement}}(v)$.

These initial conditions were called by M. Janet *initial conditions*. A method to obtain these initial conditions is illustrated by the two following examples.

3.3.4 Example [51, p. 26]. Consider the following monomial PDE system (Σ) for the unknown function φ of the variables $x_1, \ldots, x_5$:

$$\frac{\partial^2\varphi}{\partial x_5 \partial x_4} = f_1(x_1, \ldots, x_5), \quad \frac{\partial^2\varphi}{\partial x_5 \partial x_3} = f_2(x_1, \ldots, x_5), \quad \frac{\partial^2\varphi}{\partial x_5 \partial x_2} = f_3(x_1, \ldots, x_5),$$
$$\frac{\partial^2\varphi}{\partial x_4^2} = f_4(x_1, \ldots, x_5), \quad \frac{\partial^2\varphi}{\partial x_4 \partial x_3} = f_5(x_1, \ldots, x_5), \quad \frac{\partial^2\varphi}{\partial x_3^2} = f_6(x_1, \ldots, x_5).$$

The set (C_{Σ}) of compatibility relations of the PDE system (Σ) is a consequence of the identities used in Example 3.2.6 to prove the completeness of the system:

$$
\begin{array}{c|c}
x_5x_3 \cdot x_4 = x_5x_4 \cdot x_3, & \frac{\partial f_2}{\partial x_2} = \frac{\partial f_1}{\partial x_3}, \\
x_5x_2 \cdot x_4 = x_5x_4 \cdot x_2,\ x_5x_2 \cdot x_3 = x_5x_3 \cdot x_2, & \frac{\partial f_3}{\partial x_4} = \frac{\partial f_1}{\partial x_2},\ \frac{\partial f_3}{\partial x_3} = \frac{\partial f_2}{\partial x_2}, \\
x_4^2 \cdot x_5 = x_5x_4 \cdot x_4, & \frac{\partial f_4}{\partial x_5} = \frac{\partial f_1}{\partial x_4}, \\
x_4x_3 \cdot x_5 = x_5x_4 \cdot x_3,\ x_4x_3 \cdot x_4 = x_4^2 \cdot x_3, & \frac{\partial f_5}{\partial x_5} = \frac{\partial f_1}{\partial x_3},\ \frac{\partial f_5}{\partial x_4} = \frac{\partial f_4}{\partial x_3}, \\
x_3^2 \cdot x_5 = x_5x_3 \cdot x_3,\ x_3^2 \cdot x_4 = x_4x_3 \cdot x_3, & \frac{\partial f_6}{\partial x_5} = \frac{\partial f_2}{\partial x_3},\ \frac{\partial f_6}{\partial x_4} = \frac{\partial f_5}{\partial x_3}.
\end{array}
$$

The initial conditions are obtained using the multiplicative variables of the set $\mathrm{lm}(\Sigma)^{\complement}$ of complementary monomials of $\mathrm{lm}(\Sigma)$. We have

$$
\mathrm{lm}(\Sigma)^{\complement(5)} = \mathrm{lm}(\Sigma)^{\complement(4)} = \mathrm{lm}(\Sigma)^{\complement(1)} = \emptyset, \quad \mathrm{lm}(\Sigma)^{\complement(3)} = \{1, x_3, x_4\}, \quad \mathrm{lm}(\Sigma)^{\complement(2)} = \{x_5\}.
$$

The multiplicative variables of these monomials are given in the table

$$
\begin{array}{c|l}
1, x_3, x_4 & x_1, x_2, \\
x_5 & x_1, x_5.
\end{array}
$$

By Theorem 3.3.3, the PDE system (Σ) always admits a unique solution with any given initial conditions of the type

$$
\begin{aligned}
\left.\frac{\partial \varphi}{\partial x_4}\right|_{x_3=x_4=x_5=0} &= \varphi_{0,0,0,1,0}(x_1, x_2), \\
\left.\frac{\partial \varphi}{\partial x_3}\right|_{x_3=x_4=x_5=0} &= \varphi_{0,0,1,0,0}(x_1, x_2), \\
\varphi|_{x_3=x_4=x_5=0} &= \varphi_{0,0,0,0,0}(x_1, x_2), \\
\left.\frac{\partial \varphi}{\partial x_5}\right|_{x_2=x_3=x_4=0} &= \varphi_{0,0,0,0,1}(x_1, x_5).
\end{aligned}
$$

3.3.5 Example. In a last example, M. Janet considered a monomial PDE system where the partial derivatives of the left-hand side do not form a complete set of monomials, namely, the PDE system (Σ) for one unknown function φ of the variables x_1, x_2, x_3, given by

$$
\frac{\partial^3 \varphi}{\partial x_2^2 \partial x_3} = f_1(x_1, x_2, x_3), \qquad \frac{\partial^5 \varphi}{\partial x_1^2 \partial x_3^3} = f_2(x_1, x_2, x_3).
$$

We consider the set of monomials $\mathrm{lm}(\Sigma) = \{x_3x_2^2, x_3^3x_1^2\}$. In Example 3.2.10, we complete $\mathrm{lm}(\Sigma)$ to the complete set of monomials

$$
J(\mathrm{lm}(\Sigma)) = \{\, x_3x_2^2, x_3^3x_1^2, x_3^3x_2^2, x_3^3x_2x_1^2, x_3^2x_2^2 \,\}.
$$

The complementary sets of monomials are

$$J(\mathrm{lm}(\Sigma))^{\complement(3)} = \{1\},\ \ J(\mathrm{lm}(\Sigma))^{\complement(2)} = \{x_3^2x_2, x_3^2, x_3x_2, x_3\},$$
$$J(\mathrm{lm}(\Sigma))^{\complement(1)} = \{x_3^3x_2x_1, x_3^3x_2, x_3^3x_1, x_3^3\}.$$

The multiplicative variables of these monomials are given in the table

$J(\mathrm{lm}(\Sigma))^{\complement(3)}$	$x_1, x_2,$
$J(\mathrm{lm}(\Sigma))^{\complement(2)}$	$x_1.$
$J(\mathrm{lm}(\Sigma))^{\complement(1)}$	$x_3.$

By Theorem 3.3.3, the PDE system (Σ) admits always a unique solution for any given initial conditions of the type

$$\varphi|_{x_3=0} = \varphi_{0,0,0}(x_1,x_2),\quad \left.\frac{\partial\varphi}{\partial x_3}\right|_{x_2=x_3=0} = \varphi_{0,0,1}(x_1),\quad \left.\frac{\partial^2\varphi}{\partial x_3\partial x_2}\right|_{x_2=x_3=0} = \varphi_{0,1,1}(x_1),$$

$$\left.\frac{\partial^2\varphi}{\partial x_3^2}\right|_{x_2=x_3=0} = \varphi_{0,0,2}(x_1),\quad \left.\frac{\partial^3\varphi}{\partial x_3^2\partial x_2}\right|_{x_2=x_3=0} = \varphi_{0,1,2}(x_1),\quad \left.\frac{\partial^3\varphi}{\partial x_3^3}\right|_{x_1=x_2=0} = \varphi_{0,0,3}(x_3),$$

$$\left.\frac{\partial^4\varphi}{\partial x_3^3\partial x_1}\right|_{x_1=x_2=0} = \varphi_{1,0,3}(x_3),\quad \left.\frac{\partial^4\varphi}{\partial x_3^3\partial x_2}\right|_{x_1=x_2=0} = \varphi_{0,1,3}(x_3),\quad \left.\frac{\partial^5\varphi}{\partial x_3^3\partial x_2\partial x_1}\right|_{x_1=x_2=0} = \varphi_{1,1,3}(x_3).$$

4 Monomial Involutive Bases

In this section, we recall a general approach of involutive monomial divisions introduced by Gerdt in [25], see also [27, 28]. In particular, we give the axiomatic properties of an involutive division. The partition of variables into multiplicative and non-multiplicative can be deduced from this axiomatics. In this way, we explain how the notion of multiplicative variable in the sense of Janet can be deduced from a particular involutive division.

4.1 Involutive Division

4.1.1 Involutive Division. An *involutive division* $\mathcal{I}$ on the set of monomials $\mathcal{M}(x_1,\ldots,x_n)$ is defined by a relation $|_{\mathcal{I}}^{\mathcal{U}}$ in $\mathcal{U}\times\mathcal{M}(x_1,\ldots,x_n)$, for every subset $\mathcal{U}$ of $\mathcal{M}(x_1,\ldots,x_n)$, satisfying, for all monomials u, u' in $\mathcal{U}$ and v, w in $\mathcal{M}(x_1,\ldots,x_n)$, the following six conditions:

(i) $u|_{\mathcal{I}}^{\mathcal{U}}w$ implies $u|w$,
(ii) $u|_{\mathcal{I}}^{\mathcal{U}}u$, for all u in $\mathcal{U}$,
(iii) $u|_{\mathcal{I}}^{\mathcal{U}}uv$ and $u|_{\mathcal{I}}^{\mathcal{U}}uw$ if and only if $u|_{\mathcal{I}}^{\mathcal{U}}uvw$,

(iv) if $u|_{\mathcal{I}}^{\mathcal{U}}w$ and $u'|_{\mathcal{I}}^{\mathcal{U}}w$, then $u|_{\mathcal{I}}^{\mathcal{U}}u'$ or $u'|_{\mathcal{I}}^{\mathcal{U}}u$,
(v) if $u|_{\mathcal{I}}^{\mathcal{U}}u'$ and $u'|_{\mathcal{I}}^{\mathcal{U}}w$, then $u|_{\mathcal{I}}^{\mathcal{U}}w$,
(vi) if $\mathcal{U}' \subseteq \mathcal{U}$ and $u \in \mathcal{U}'$, then $u|_{\mathcal{I}}^{\mathcal{U}}w$ implies $u|_{\mathcal{I}}^{\mathcal{U}'}w$.

When there is no danger of confusion, the relation $|_{\mathcal{I}}^{\mathcal{U}}$ will be also denoted by $|_{\mathcal{I}}$.

4.1.2 Multiplicative Monomial. If $u|_{\mathcal{I}}^{\mathcal{U}}w$, by **(i)** there exists a monomial v such that $w = uv$. We say that u is an *$\mathcal{I}$-involutive divisor* of w, that w is an *$\mathcal{I}$-involutive multiple* of u, and that v is *$\mathcal{I}$-multiplicative* for u with respect to $\mathcal{U}$. When the monomial uv is not an involutive multiple of u with respect to $\mathcal{U}$, we say that v is *$\mathcal{I}$-non-multiplicative* for u with respect to $\mathcal{U}$.

We define in the same way the notion of multiplicative (resp. non-multiplicative) variable. We denote by $\mathrm{Mult}_{\mathcal{I}}^{\mathcal{U}}(u)$ (resp. $\mathrm{NMult}_{\mathcal{I}}^{\mathcal{U}}(u)$) the set of multiplicative (resp. non-multiplicative) variables for the division $\mathcal{I}$ of a monomial u with respect to $\mathcal{U}$. We have

$$\mathrm{Mult}_{\mathcal{I}}^{\mathcal{U}}(u) = \{\ x \in \{x_1, \ldots, x_n\} \ \big|\ u|_{\mathcal{I}}^{\mathcal{U}}ux\ \}$$

and thus obtain a partition of the set of variables $\{x_1, \ldots, x_n\}$ into sets of multiplicative and non-multiplicative variables. An involutive division $\mathcal{I}$ is thus entirely defined by a partition

$$\{x_1, \ldots, x_n\} = \mathrm{Mult}_{\mathcal{I}}^{\mathcal{U}}(u) \sqcup \mathrm{NMult}_{\mathcal{I}}^{\mathcal{U}}(u),$$

for any finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ and any u in $\mathcal{U}$, satisfying conditions (**iv**), (**v**) and (**vi**) of Definition 4.1.1. The involutive division $\mathcal{I}$ is then defined by setting $u\ |_{\mathcal{I}}^{\mathcal{U}}\ w$ if $w = uv$ and the monomial v belongs to $\mathcal{M}(\mathrm{Mult}_{\mathcal{I}}^{\mathcal{U}}(u))$. Conditions (**i**), (**ii**), and (**iii**) of Definition 4.1.1 are consequences of this definition.

4.1.3 Example. Consider $\mathcal{U} = \{x_1, x_2\}$ in $\mathcal{M}(x_1, x_2)$ and suppose that $\mathcal{I}$ is an involutive division such that $\mathrm{Mult}_{\mathcal{I}}^{\mathcal{U}}(x_1) = \{x_1\}$ and $\mathrm{Mult}_{\mathcal{I}}^{\mathcal{U}}(x_2) = \{x_2\}$. Then we have

$$x_1 \nmid_{\mathcal{I}} x_1x_2, \quad \text{and} \quad x_2 \nmid_{\mathcal{I}} x_1x_2.$$

4.1.4 Autoreduction. A subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ is said to be *autoreduced* with respect to an involutive division $\mathcal{I}$, or *$\mathcal{I}$-autoreduced*, if it does not contain a monomial $\mathcal{I}$-divisible by another monomial of $\mathcal{U}$.

In particular, by the definition of the involutive division, for any monomials u, u' in $\mathcal{U}$ and any monomial w in $\mathcal{M}(x_1, \ldots, x_n)$, we have $u|_{\mathcal{I}}w$ and $u'|_{\mathcal{I}}w$ implies $u|_{\mathcal{I}}u'$ or $u'|_{\mathcal{I}}u$. As a consequence, if a set of monomials $\mathcal{U}$ is $\mathcal{I}$-autoreduced, then any monomial in $\mathcal{M}(x_1, \ldots, x_n)$ admits at most one $\mathcal{I}$-involutive divisor in $\mathcal{U}$.

4.1.5 Janet Division. We call *Janet division* the division on $\mathcal{M}(x_1, \ldots, x_n)$ given by the multiplicative variables in the sense of M. Janet defined in Sect. 3.1.9. Explicitly, for a subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ and monomials u in $\mathcal{U}$ and w in $\mathcal{M}(x_1, \ldots, x_n)$, we define $u|_{\mathcal{J}}^{\mathcal{U}}w$ if u is a Janet divisor of w as defined in Sect. 3.1.11, that is $w = uv$, where $v \in \mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u))$ and $\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u)$ is the set of Janet's multiplicative variables defined in Sect. 3.1.9.

By Proposition 3.1.12, for a fixed subset of $\mathcal{U}$, any monomial of $\mathcal{M}(x_1, \ldots, x_n)$ has a unique Janet divisor in $\mathcal{U}$ with respect to $\mathcal{U}$. As a consequence, the conditions **(iv)** and **(v)** of Definition 4.1.1 hold trivially for Janet division. Now suppose that $\mathcal{U}' \subseteq \mathcal{U}$ and u is a monomial in $\mathcal{U}'$. If $u|_{\mathcal{J}}^{\mathcal{U}} w$, then there is a decomposition $w = uv$ with $v \in \mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u))$. As $\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u) \subseteq \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}'}(u)$, this implies that $u|_{\mathcal{J}}^{\mathcal{U}'} w$. Hence, the conditions **(vi)** of Definition 4.1.1 holds for Janet division. We have thus proved.

4.1.6 Proposition [27, Proposition 3.6] *Janet division is involutive.*

4.2 Involutive Completion Procedure

4.2.1 Involutive Set. Let $\mathcal{I}$ be an involutive division on $\mathcal{M}(x_1, \ldots, x_n)$ and let $\mathcal{U}$ be a set of monomials. The *involutive cone* of a monomial u in $\mathcal{U}$ with respect to the involutive division $\mathcal{I}$ is defined by

$$\mathrm{cone}_{\mathcal{I}}(u, \mathcal{U}) = \{\, uv \;\big|\; v \in \mathcal{M}(x_1, \ldots, x_n) \text{ and } u|_{\mathcal{I}}^{\mathcal{U}} uv \,\}.$$

The *involutive cone* of $\mathcal{U}$ with respect to the involutive division $\mathcal{I}$ is the following subset of monomials:

$$\mathrm{cone}_{\mathcal{I}}(\mathcal{U}) = \bigcup_{u \in \mathcal{U}} \mathrm{cone}_{\mathcal{I}}(u, \mathcal{U}).$$

Note that the inclusion $\mathrm{cone}_{\mathcal{I}}(\mathcal{U}) \subseteq \mathrm{cone}(\mathcal{U})$ holds for any set $\mathcal{U}$. When the set $\mathcal{U}$ is $\mathcal{I}$-autoreduced, this union is disjoint, thanks to involutivity.

A subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ is $\mathcal{I}$-*involutive* if the following equality holds:

$$\mathrm{cone}(\mathcal{U}) = \mathrm{cone}_{\mathcal{I}}(\mathcal{U}).$$

In other words, a set $\mathcal{U}$ is $\mathcal{I}$-involutive if any multiple of an element u in $\mathcal{U}$ is also the $\mathcal{I}$-involutive multiple of an element v of $\mathcal{U}$. Note that the monomial v can be different from the monomial u, as we have seen in Example 3.2.6.

4.2.2 Involutive Completion. A *completion* of a subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ with respect to an involutive division $\mathcal{I}$, or $\mathcal{I}$-*completion* for short, is a set of monomials $\widetilde{\mathcal{U}}$ satisfying the following three conditions:

(i) $\widetilde{\mathcal{U}}$ is involutive,
(ii) $\mathcal{U} \subseteq \widetilde{\mathcal{U}}$,
(iii) $\mathrm{cone}(\widetilde{\mathcal{U}}) = \mathrm{cone}(\mathcal{U})$.

4.2.3 Noetherianity. An involutive division $\mathcal{I}$ is said to be *Noetherian* if all finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ admits a finite $\mathcal{I}$-completion $\widetilde{\mathcal{U}}$.

4.2.4 Proposition [27, Proposition 4.5] *Janet division is Noetherian.*

4.2.5 Prolongation. Let $\mathcal{U}$ be a subset of $\mathcal{M}(x_1, \ldots, x_n)$. We call *prolongation* of an element u of $\mathcal{U}$ a multiplication of u by a variable x. Given an involutive division $\mathcal{I}$, a prolongation ux is *multiplicative* (resp. *non-multiplicative*) if x is a multiplicative (resp. non-multiplicative) variable.

4.2.6 Local Involutivity. A subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ is *locally involutive with respect to an involutive division* $\mathcal{I}$ if any non-multiplicative prolongation of an element of $\mathcal{U}$ admit an involutive divisor in $\mathcal{U}$. That is

$$\forall u \in \mathcal{U} \quad \forall x_i \in \mathrm{NMult}_{\mathcal{I}}^{\mathcal{U}}(u) \quad \exists v \in \mathcal{U} \quad \text{such that} \quad v|_{\mathcal{I}} u x_i.$$

4.2.7 Example [27, Example 4.8]. By definition, if $\mathcal{U}$ is $\mathcal{I}$-involutive, then it is locally $\mathcal{I}$-involutive. The converse is false in general. Indeed, consider the involutive division $\mathcal{I}$ on $\mathcal{M} = \mathcal{M}(x_1, x_2, x_3)$ defined by

$$\mathrm{Mult}_{\mathcal{I}}^{\mathcal{M}}(x_1) = \{x_1, x_3\}, \quad \mathrm{Mult}_{\mathcal{I}}^{\mathcal{M}}(x_2) = \{x_1, x_2\}, \quad \mathrm{Mult}_{\mathcal{I}}^{\mathcal{M}}(x_3) = \{x_2, x_3\},$$

with $\mathrm{Mult}_{\mathcal{I}}^{\mathcal{M}}(1) = \{x_1, x_2, x_3\}$ and $\mathrm{Mult}_{\mathcal{I}}^{\mathcal{M}}(u)$ is empty for $\deg(u) \geqslant 2$. Then the set $\{x_1, x_2, x_3\}$ is locally $\mathcal{I}$-involutive, but not $\mathcal{I}$-involutive.

4.2.8 Continuity. An involutive division $\mathcal{I}$ is *continuous* if for any finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ and any finite sequence $(u_1, \ldots, u_k)$ of elements in $\mathcal{U}$ for which there exists x_{i_j} in $\mathrm{NMult}_{\mathcal{I}}^{\mathcal{U}}(u_j)$ such that

$$u_k|_{\mathcal{I}} u_{k-1} x_{i_{k-1}}, \ \ldots \ , u_3|_{\mathcal{I}} u_2 x_{i_2}, \ u_2|_{\mathcal{I}} u_1 x_{i_1},$$

it holds that $u_i \neq u_j$, for any $i \neq j$.

For instance, the involutive division in Example 4.2.7 is not continuous. Indeed, there exists the following cycle of divisions:

$$x_2|_{\mathcal{I}} x_1 x_2, \quad x_1|_{\mathcal{I}} x_3 x_1, \quad x_3|_{\mathcal{I}} x_2 x_3, \quad x_2|_{\mathcal{I}} x_1 x_2.$$

4.2.9 From Local to Global Involutivity. Any $\mathcal{I}$-involutive subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ is locally $\mathcal{I}$-involutive. When the division $\mathcal{I}$ is continuous, the converse is also true. Indeed, suppose that $\mathcal{U}$ is locally $\mathcal{I}$-involutive and $\mathcal{I}$ is continuous. Let us show that $\mathcal{U}$ is $\mathcal{I}$-involutive.

Given a monomial u in $\mathcal{U}$ and a monomial w in $\mathcal{M}(x_1, \ldots, x_n)$, we claim that the monomial uw admits an $\mathcal{I}$-involutive divisor in $\mathcal{U}$. If $u|_{\mathcal{I}} uw$, the claim is proved. Otherwise, there exists a non-multiplicative variable x_{k_1} in $\mathrm{NMult}_{\mathcal{I}}^{\mathcal{U}}(u)$ such that $x_{k_1}|w$. By local involutivity, the monomial ux_{k_1} admits an $\mathcal{I}$-involutive divisor v_1 in $\mathcal{U}$. If $v_1|_{\mathcal{I}} uw$, the claim is proved. Otherwise, there exists a non-multiplicative variable x_{k_2} in $\mathrm{NMult}_{\mathcal{I}}^{\mathcal{U}}(v_1)$ such that x_{k_2} divides $\frac{uw}{v_1}$. By local involutivity, the monomial $v_1 x_{k_2}$ admits an $\mathcal{I}$-involutive divisor v_2 in $\mathcal{U}$.

In this way, we construct a sequence $(u, v_1, v_2, \ldots)$ of monomials in $\mathcal{U}$ such that

$$v_1|_{\mathcal{I}} u x_{k_1}, \quad v_2|_{\mathcal{I}} v_1 x_{k_2}, \quad v_3|_{\mathcal{I}} v_2 x_{k_3}, \quad \ldots$$

By the continuity hypothesis, all monomials $v_1, v_2, \ldots$ are distinct. Moreover, all these monomials are divisors of uw, which admits a finite set of distinct divisors. As a consequence, the above sequence is finite. It follows that its last term v_k is an $\mathcal{I}$-involutive monomial of uw. We have thus proved the following result.

4.2.10 Theorem [27, Theorem 4.10] *Let $\mathcal{I}$ be a continuous involutive division. A subset of $\mathcal{M}(x_1, \ldots, x_n)$ is locally $\mathcal{I}$-involutive if and only if it is $\mathcal{I}$-involutive.*

4.2.11 Proposition [27, Corollary 4.11] *Janet division is continuous.*

Input: $\mathcal{U}$ a finite subset of $\mathcal{M}(x_1, \ldots, x_n)$
begin
  $\widetilde{\mathcal{U}} \leftarrow \mathcal{U}$
  while *exist* $u \in \widetilde{\mathcal{U}}$ *and* $x \in \mathrm{NMult}_{\mathcal{I}}^{\widetilde{\mathcal{U}}}(u)$ *such that* ux *does not have an* $\mathcal{I}$*-involutive divisor in* $\widetilde{\mathcal{U}}$ **do**
    Choose such a u and x corresponding to the smallest monomial ux with respect to the monomial order $\preccurlyeq$
    $\widetilde{\mathcal{U}} \leftarrow \widetilde{\mathcal{U}} \cup \{ux\}$
  end
end
Output: $\widetilde{\mathcal{U}}$ the minimal involutive completion of the set $\mathcal{U}$.

Procedure 2: Involutive completion procedure.

4.2.12 Involutive Completion Procedure. Procedure 2 generalizes Janet's completion procedure given in Sect. 3.2.9 to any involutive division. Let us fix a monomial order $\preccurlyeq$ on $\mathcal{M}(x_1, \ldots, x_n)$. Given a set of monomials $\mathcal{U}$, the procedure completes the set $\mathcal{U}$ by all possible non-involutives prolongations of monomials in $\mathcal{U}$.

By introducing the notion of constructive involutive division, Gerdt and Blinkov gave in [27] some conditions on the involutive division $\mathcal{I}$ in order to establish the correctness and the termination of this procedure. A continuous involutive division $\mathcal{I}$ is *constructive* if for any subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ and for any non-multiplicative prolongation ux of a monomial u in $\mathcal{U}$ satisfying the two conditions

(i) ux does not have an $\mathcal{I}$-involutive divisor in $\mathcal{U}$,
(ii) any non-multiplicative prolongation $vy \neq ux$ of a monomial v in $\mathcal{U}$ that divides ux has an $\mathcal{I}$-involutive divisor in $\mathcal{U}$,

the monomial ux cannot be $\mathcal{I}$-involutively divided by a monomial w in $\mathrm{cone}_{\mathcal{I}}(\mathcal{U})$ with respect to $\mathcal{U} \cup \{w\}$.

If $\mathcal{I}$ is a constructive division, then the completion procedure completes the set $\mathcal{U}$ to an involutive set. We refer the reader to [27, Theorem 4.14] for a proof of the correctness and termination of the completion procedure under these hypotheses.

4.2.13 Example. An application of this procedure to the set of monomials $\mathcal{U} = \{x_3x_2^2, x_3^3x_1^2\}$ given by Janet in [51] is presented in Sect. 3.2.10.

4.3 Others Involutive Approaches

For the analysis of differential systems, several other notions of multiplicative variables were studied by J. M. Thomas 1937 and J.-F. Pommarret in 1978. Other examples of involutive divisions can be found in [28].

4.3.1 Thomas Division. In [86], Thomas introduced an involutive division that differs from that of M. Janet, also used in the analysis of differential systems. The multiplicative variables in the sense of Thomas's division for a monomial u with of a finite subset $\mathcal{U}$ of $\mathcal{M}(x_1, \ldots, x_n)$ are defined by the rule

$$x_i \in \mathrm{Mult}_{\mathcal{T}}^{\mathcal{U}}(u) \quad \text{if} \quad \deg_i(u) = \deg_i(\mathcal{U}).$$

In particular, we have $u|_{\mathcal{T}}^{\mathcal{U}} w$ if $w = uv$ and for all variables x_i in v, we have $\deg_i(u) = \deg_i(\mathcal{U})$. The Thomas division is a Noetherian and continuous involutive division. We refer the reader to [27] for detailed proofs of these results. Note also that the Janet division is a refinement of the Thomas division, in the sense that for any finite set of monomials $\mathcal{U}$ and any monomial u in $\mathcal{U}$, the following inclusions hold:

$$\mathrm{Mult}_{\mathcal{T}}^{\mathcal{U}}(u) \subseteq \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u) \quad \text{and} \quad \mathrm{NMult}_{\mathcal{J}}^{\mathcal{U}}(u) \subseteq \mathrm{NMult}_{\mathcal{T}}^{\mathcal{U}}(u).$$

4.3.2 Pommaret Division. In [72], Pommaret studied an involutive division that is defined globally, that is, the multiplicative variables for the Pommaret division do not depend on a given subset of monomials. In this way, Pommaret's division can be defined on an infinite set of monomials.

Fix an order on the variables $x_1 > x_2 > \cdots > x_n$. Given a monomial $u = x_1^{\alpha_1} \cdots x_k^{\alpha_k}$, with $\alpha_k > 0$, the Pommaret multiplicative variables for u are defined by the rule

$$x_j \in \mathrm{Mult}_{\mathcal{P}}^{\mathcal{M}(x_1,\ldots,x_n)}(u), \quad \text{if } j \geqslant k, \qquad \text{and} \qquad x_j \in \mathrm{NMult}_{\mathcal{P}}^{\mathcal{M}(x_1,\ldots,x_n)}(u), \quad \text{if } j < k.$$

Set $\mathrm{Mult}_{\mathcal{P}}^{\mathcal{M}(x_1,\ldots,x_n)}(1) = \{x_1, \ldots, x_n\}$. The Pommaret division is a continuous involutive division that is not Noetherian [27]. The Janet division is a refinement of the Pommaret division, that is, for an autoreduced finite set of monomials $\mathcal{U}$, the following inclusions hold for any monomial u in $\mathcal{U}$:

$$\mathrm{Mult}_{\mathcal{P}}^{\mathcal{U}}(u) \subseteq \mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}}(u) \quad \text{and} \quad \mathrm{NMult}_{\mathcal{J}}^{\mathcal{U}}(u) \subseteq \mathrm{NMult}_{\mathcal{P}}^{\mathcal{U}}(u).$$

Finally, let us remark that the separation of variables into multiplicative and nonmultiplicative ones in the Pommaret division was used first by Janet in [51, Sect. 20]. For this reason, the terminology *Pommaret division* does not reflect correctly the history of the theory. We refer the reader to the monograph by Seiler [82, Section 3.5] for a historical account.

5 Polynomial Partial Differential Equations Systems

In this section, we extend the results on monomial systems presented in Sect. 3 to linear (polynomial) systems. All PDE systems are considered in analytic categories, meaning that all unknown functions, coefficients, and initial conditions are assumed to be analytic. In the first part, we recall the notion of principal derivative with respect to an order on derivatives introduced by M. Janet. This notion is used to give an algebraic characterization of complete integrability conditions of a PDE system. Then we present a procedure that decides whether a given finite linear PDE system can be transformed into a completely integrable linear PDE system. Finally, we recall the algebraic formulation of involutivity introduced by Janet in [51].

5.1 Parametric and Principal Derivatives

5.1.1 Motivations. In [51, Chapter 2], M. Janet first considered the following PDE for one unknown function on $\mathbb{C}^n$:

$$\frac{\partial^2\varphi}{\partial x_n^2} = \sum_{1\leqslant i,\,j<n} a_{i,j}(x)\frac{\partial^2\varphi}{\partial x_i\partial x_j} + \sum_{1\leqslant i<n} a_i(x)\frac{\partial^2\varphi}{\partial x_i\partial x_n} + \sum_{r=1}^{n} b_r(x)\frac{\partial\varphi}{\partial x_r} + c(x)\varphi + f(x), \tag{5.1}$$

where the functions $a_{i,j}(x)$, $a_i(x)$, $b_r(x)$, $c(x)$ and $f(x)$ are analytic functions in a neighborhood of a point $P = (x_1^0, \dots, x_n^0)$ in $\mathbb{C}^n$. Given two analytic functions φ_1 and φ_2 in a neighborhood U_Q of a point $Q = (x_1^0, \dots, x_{n-1}^0)$ in $\mathbb{C}^{n-1}$, M. Janet studied the problem of the existence of solutions of equation (5.1) with the initial condition

$$\varphi|_{x_n=x_n^0} = \varphi_1, \qquad \left.\frac{\partial\varphi}{\partial x_n}\right|_{x_n=x_n^0} = \varphi_2, \tag{5.2}$$

in a neighborhood of the point Q. In Sect. 5.4.2, we will formulate such condition for higher order linear PDE systems with several unknown functions, called initial condition.

5.1.2 Principal and Parametric Derivatives. In order to treat the problems of the existence and uniqueness of a solution of Eq. (5.1) under the initial condition (5.2), M. Janet introduced the notions of parametric and principal derivatives defined as follows. The partial derivatives $D^\alpha\varphi$, with $\alpha = (\alpha_1, \dots, \alpha_n)$, of an analytic function φ are determined by

- **(i)** φ_1 and its derivatives for $\alpha_n = 0$,
- **(ii)** φ_2 and its derivatives for $\alpha_n = 1$,

in the neighborhood U_Q. These derivatives for $\alpha_n = 0$ and $\alpha_n = 1$ are called *parametric*, while the derivatives for $\alpha_n \geqslant 2$, i.e., the derivatives of $\frac{\partial^2\varphi}{\partial x_n^2}$, are called *principal*. Note that the values of the principal derivatives at the point P are entirely given by φ_1

and φ_2 and by their derivatives thanks to Eq. (5.1). Note that the notion of parametric derivative corresponds to a parametrization of the initial conditions of the system.

5.1.3 Janet's Orders on Derivatives. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ be in $\mathbb{N}^n$. Let φ be an analytic function. The derivative $D^\alpha\varphi$ is said to be *posterior* (resp. *anterior*) to $D^\beta\varphi$ if

$$|\alpha| > |\beta| \quad (\text{resp. } |\alpha| < |\beta|) \qquad \text{or} \qquad |\alpha| = |\beta| \text{ and } \alpha_n > \beta_n \quad (\text{resp.} \alpha_n < \beta_n).$$

Obviously, any derivative of φ admits only finitely many anterior derivatives of φ. Using this notion of posteriority, M. Janet showed the existence and uniqueness of the solution to Eq. (5.1) under the initial conditions (5.2).

In his monograph, M. Janet gave several generalizations of the above notion of posteriority. The first one corresponds to the degree lexicographic order [51, Sect. 22], formulated as follows:

(i) for $|\alpha| \neq |\beta|$, the derivative $D^\alpha\varphi$ is called *posterior* (resp. *anterior*) to $D^\beta\varphi$, if $|\alpha| > |\beta|$ (resp. $|\alpha| < |\beta|$),
(ii) for $|\alpha| = |\beta|$, the derivative $D^\alpha\varphi$ is called *posterior* (resp. *anterior*) to $D^\beta\varphi$ if the first nonzero difference

$$\alpha_n - \beta_n\ , \quad \alpha_{n-1} - \beta_{n-1}\ , \quad \dots \quad , \ \alpha_1 - \beta_1,$$

is positive (resp. negative).

5.1.4 Generalization. Let us consider the following generalization of equation (5.1):

$$D\varphi = \sum_{i \in I} a_i D_i \varphi + f, \tag{5.3}$$

where D and the D_i are differential operators such that $D_i\varphi$ is anterior to $D\varphi$ for all i in I. The derivative $D\varphi$ and all its derivatives are called *principal derivatives of the Eq.* (5.3). All the other derivatives of u are called *parametric derivatives of the Eq.* (5.3).

5.1.5 Weight Order. Further generalization of these order relations was given by M. Janet by introducing the notion of *cote*, which corresponds to a parametrization of a weight order defined as follows. Let us fix a positive integer s. We define a *weight matrix*

$$C = \begin{bmatrix} C_{1,1} & \dots & C_{n,1} \\ \vdots & & \vdots \\ C_{1,s} & \dots & C_{n,s} \end{bmatrix}$$

that associates to each variable x_i nonnegative integers $C_{i,1}, \dots, C_{i,s}$, called the s*-weights* of x_i. This notion was called *cote* by Janet in [51, Sect. 22] following the terminology introduced by Riquier [75]. Ritt used the term *mark* in [77]. For each derivative $D^\alpha\varphi$, with $\alpha = (\alpha_1, \dots, \alpha_n)$, of an analytic function φ, we associate the *s-weight* $\Gamma(C) = (\Gamma_1, \ldots, \Gamma_s)$, where the Γ_k are defined by

$$\Gamma_k = \sum_{i=1}^{n} \alpha_i C_{i,k}.$$

Given two monomial partial differential operators D^α and D^β as in Sect. 5.1.3, we say that $D^\alpha\varphi$ is *posterior* (resp. *anterior*) to $D^\beta\varphi$ with respect to the weight matrix C if

(i) $|\alpha| \neq |\beta|$ and $|\alpha| > |\beta|$ (resp. $|\alpha| < |\beta|$), or
(ii) $|\alpha| = |\beta|$ and the first nonzero difference

$$\Gamma_1 - \Gamma'_1, \quad \Gamma_2 - \Gamma'_2, \quad \ldots \quad , \Gamma_s - \Gamma'_s,$$

is positive (resp. negative).

In this way, we define an order on the set of monomial partial derivatives, called *weight order*. Note that, by setting $C_{i,k} = \delta_{i+k,n+1}$, we recover the Janet order defined in Sect. 5.1.3.

5.2 First-Order PDE Systems

We consider first the resolution of first-order PDE systems.

5.2.1 Complete Integrability. In [51, Sect. 36], M. Janet considered a first-order PDE system of the form

$$(\Sigma) \qquad \frac{\partial\varphi}{\partial y_\lambda} = f_\lambda(y_1, \ldots, y_h, z_1, \ldots, z_k, \varphi, q_1, \ldots, q_k) \qquad (1 \leqslant \lambda \leqslant h), \quad (5.4)$$

where φ is an unknown function of the independent variables $y_1, \ldots, y_h, z_1, \ldots, z_k$, with $h + k = n$ and $q_i = \frac{\partial\varphi}{\partial z_i}$. It is assumed that the functions f_λ are analytic in a neighborhood of a point P. M. Janet wrote down explicitly the integrability condition of the PDE systems (Σ) as the equality

$$\frac{\partial}{\partial y_\lambda}\left(\frac{\partial\varphi}{\partial y_\mu}\right) = \frac{\partial}{\partial y_\mu}\left(\frac{\partial\varphi}{\partial y_\lambda}\right),$$

for any $1 \leqslant \lambda, \mu \leqslant h$. Differentiating (5.4), we deduce that

$$\begin{aligned}\frac{\partial}{\partial y_\lambda}\left(\frac{\partial\varphi}{\partial y_\mu}\right) &= \frac{\partial f_\mu}{\partial y_\lambda} + \frac{\partial\varphi}{\partial y_\lambda}\frac{\partial f_\mu}{\partial\varphi} + \sum_{i=1}^{k}\frac{\partial f_\mu}{\partial q_i}\frac{\partial^2\varphi}{\partial y_\lambda\partial z_i},\\ &= \frac{\partial f_\mu}{\partial y_\lambda} + f_\lambda\frac{\partial f_\mu}{\partial\varphi} + \sum_{i=1}^{k}\frac{\partial f_\mu}{\partial q_i}\left(\frac{\partial f_\lambda}{\partial z_i} + q_i\frac{\partial f_\lambda}{\partial\varphi}\right) + \sum_{i,j=1}^{k}\frac{\partial f_\lambda}{\partial q_i}\frac{\partial f_\mu}{\partial q_j}\frac{\partial^2\varphi}{\partial z_i\partial z_j}.\end{aligned}$$

Hence, the integrability condition reads

$$\begin{aligned}&\frac{\partial}{\partial y_\lambda}\left(\frac{\partial\varphi}{\partial y_\mu}\right)-\frac{\partial}{\partial y_\mu}\left(\frac{\partial\varphi}{\partial y_\lambda}\right)\\&=\frac{\partial f_\mu}{\partial y_\lambda}+f_\lambda\frac{\partial f_\mu}{\partial\varphi}+\sum_{i=1}^{k}\frac{\partial f_\mu}{\partial q_i}\left(\frac{\partial f_\lambda}{\partial z_i}+q_i\frac{\partial f_\lambda}{\partial\varphi}\right)-\frac{\partial f_\lambda}{\partial y_\mu}-f_\mu\frac{\partial f_\lambda}{\partial\varphi}-\sum_{i=1}^{k}\frac{\partial f_\lambda}{\partial q_i}\left(\frac{\partial f_\mu}{\partial z_i}+q_i\frac{\partial f_\mu}{\partial\varphi}\right)\\&=0,\end{aligned} \tag{5.5}$$

for any $1 \leqslant \lambda \neq \mu \leqslant h$. When the PDE system (Σ) in (5.4) satisfies relation (5.5), it is said to be *completely integrable*.

5.2.2 Theorem *Suppose that the PDE system (Σ) in (5.4) is completely integrable. Let P be a point in $\mathbb{C}^n$ and $\varphi(z_1, \ldots, z_k)$ be an analytic function in the neighborhood of the point $\pi(P)$, where $\pi : \mathbb{C}^n \to \mathbb{C}^k$ denotes the canonical projection $(y_1, \ldots, y_h, z_1, \ldots z_k) \mapsto (z_1, \ldots, z_k)$. Then, the system (Σ) admits only one analytic solution satisfying $u = \varphi \circ \pi$ in a neighborhood of the point P.*

5.3 Higher Order Finite Linear PDE Systems

In [51, Sect. 39], M. Janet discussed the existence of solutions of a finite linear PDE system for one unknown function φ in which each equation is of the form

$$(\Sigma) \qquad D_i\varphi = \sum_j a_{i,j} D_{i,j}\varphi, \quad i \in I. \tag{5.6}$$

All the functions $a_{i,j}$ are assumed to be analytic in a neighborhood of a point P in $\mathbb{C}^n$.

5.3.1 Principal and Parametric Derivatives. Consider Janet's order $\preccurlyeq_J$ on derivatives as the generalization defined in Sect. 5.1.3. We assume that each equation of the system (Σ) defined by (5.6) satisfies the following two conditions:

(i) $D_{i,j}\varphi$ is anterior to $D_i\varphi$, for any i in I,
(ii) all the D_i's for i in I are distinct.

We extend the notion of principal derivative introduced in Sect. 5.1.4 for one PDE equation to a system of the form (5.6) as follows. The derivative $D_i\varphi$, for i in I, and all its derivatives are called *principal derivatives* of the PDE system (Σ) in (5.6) with respect to Janet's order. Any other derivative of φ is called *parametric derivative*.

5.3.2 Completeness with Respect to Janet's Order. Fix an order $x_n > x_{n-1} > \cdots > x_1$ on variables. By the isomorphism of Proposition 3.1.2, which identifies monomial partial differential operators with monomials in $\mathcal{M}(x_1, \ldots, x_n)$, we associate to the set of operators D_i's, i in I, defined in Sect. 5.3.1, a set $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)$ of monomials. By definition, the set $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)$ contains the monomials associated to leading derivatives of the PDE system (Σ) with respect to Janet's order.

The PDE system (Σ) is said to be *complete* with respect to Janet's order $\preccurlyeq_J$ if the set of monomials $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)$ is complete in the sense of Sect. 3.2.1. Procedure 6 is a completion procedure that transforms a finite linear PDE system into an equivalent complete linear PDE system.

By definition, the set of principal derivatives corresponds, via the isomorphism of Proposition 3.1.2, to the multiplicative cone of the monomial set $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)$. Hence, when (Σ) is complete, the set of principal derivatives corresponds to the involutive cone of $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)$. By Proposition 3.2.2, there is a partition

$$\mathcal{M}(x_1, \ldots, x_n) = \mathrm{cone}_{\mathcal{J}}(\mathrm{lm}_{\preccurlyeq_J}(\Sigma)) \amalg cone^{\complement}_{\mathcal{J}}(\mathrm{lm}_{\preccurlyeq_J}(\Sigma)).$$

It follows that the set of parametric derivatives of a complete PDE system (Σ) corresponds to the involutive cone of the set of monomials $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)^{\complement}$.

5.3.3 Initial Conditions. Consider the set $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)^{\complement}$ of complementary monomials of $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)$, as defined in Sect. 3.1.13. To a monomial x^β in $\mathrm{lm}_{\preccurlyeq_J}(\Sigma)^{\complement}$, with $\beta = (\beta_1, \ldots, \beta_n)$ in $\mathbb{N}^n$ and

$${}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathrm{lm}_{\preccurlyeq_J}(\Sigma)^{\complement}}(x^\beta) = \{x_{i_1}, \ldots, x_{i_{k_\beta}}\},$$

we associate an arbitrary analytic function

$$\varphi_\beta(x_{i_1}, \ldots, x_{i_{k_\beta}}).$$

Using these functions, M. Janet defined an *initial condition*:

$$(C_\beta) \qquad D^\beta \varphi\Big|_{x_j = 0\ \forall x_j \in {}^{\complement}\mathrm{NMult}_{\mathcal{J}}^{\mathrm{lm}_{\preccurlyeq_J}(\Sigma)^{\complement}}(x^\beta)} = \varphi_\beta(x_{i_1}, \ldots, x_{i_{k_\beta}}).$$

Then he introduced an *initial condition for the Eq.* (5.6) with respect to the Janet order as the set

$$\{\, C_\beta \mid x^\beta \in \mathrm{lm}_{\preccurlyeq_J}(\Sigma)^{\complement} \,\}. \tag{5.7}$$

5.3.4 Theorem [51, Sect. 39] *If the PDE system* (Σ) *in (5.6) is complete with respect to Janet's order* $\preccurlyeq_J$*, then it admits at most one analytic solution satisfying the initial condition (5.7).*

5.3.5 PDE Systems with Several Unknown Functions. The construction of initial conditions given in Sect. 5.3.3 for one unknown function can be extended to linear PDE systems on $\mathbb{C}^n$ with several unknown functions using a weight order. Consider a linear PDE system with m unknown analytic functions $\varphi^1, \ldots, \varphi^m$ of the form

$$(\Sigma) \qquad D^\alpha \varphi^r = \sum_{(\beta,s) \in \mathbb{N}^n \times \{1,2,\ldots,m\}} a_{\alpha,\beta}^{r,s} D^\beta \varphi^s, \qquad \alpha \in I^r, \tag{5.8}$$

for $1 \leqslant r \leqslant m$, where I^r is a finite subset of $\mathbb{N}^n$ and the $a^{r,s}_{\alpha,\beta}$ are analytic functions.

For such a system, we define a weight order as follows. Fix a positive integer s. To any variable x_i we associate $s+1$ weights $C_{i,0}, C_{i,1}, \ldots, C_{i,s}$ by setting $C_{i,0} = 1$ and taking $C_{i,1}, \ldots, C_{i,s}$ as defined in Sect. 5.1.5. To each unknown function φ^j, we associate $s+1$ weights $T_0^{(j)}, T_1^{(j)} \ldots, T_s^{(j)}$. With these data, we define the $s+1$ weights $\Gamma_0^{(j)}, \Gamma_1^{(j)}, \ldots, \Gamma_s^{(j)}$ of the partial derivative $D^\alpha \varphi^j$ with $\alpha = (\alpha_1, \ldots, \alpha_n)$ in $\mathbb{N}^n$ by setting

$$\Gamma_k^{(j)} = \sum_{i=1}^{n} \alpha_i C_{i,k} + T_k^{(j)} \qquad (0 \leqslant k \leqslant s).$$

We define the notions of anteriority and posteriority on derivatives with respect to this weight order, denoted by $\preccurlyeq_{wo}$, as it is done in Sect. 5.3.1 for systems with one unknown function. In particular, we define the notions of principal and parametric derivatives in a similar way to the case of systems with one unknown function.

Now suppose that the system (5.8) is written in the form

$$(\Sigma) \qquad D^\alpha \varphi^r = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\} \\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} a^{r,s}_{\alpha,\beta} D^\beta \varphi^s, \qquad \alpha \in I^r. \tag{5.9}$$

We can formulate the notion of completeness with respect to the weight order $\preccurlyeq_{wo}$ as in Sect. 5.3.2. Let $\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma, \varphi^r)$ be the set of monomials associated to leading derivatives D^α of all PDE in (Σ) such that α belongs to I^r. The PDE system (Σ) is *complete* with respect to $\preccurlyeq_{wo}$, if for any $1 \leqslant r \leqslant m$, the set of monomials $\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma, \varphi^r)$ is complete in the sense of Sect. 3.2.1. Finally, we can formulate, as in (5.7), an initial condition for the linear PDE system (5.9) with respect to such a weight order:

$$\{\, C_{\beta,r} \ \mid\ x^\beta \in \mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma, \varphi^r)^{\complement}, \ \text{ for } 1 \leqslant r \leqslant m \,\}. \tag{5.10}$$

5.3.6 Theorem [51, Sect. 40] *If the PDE system (Σ) in (5.9) is complete with respect to a weight order $\preccurlyeq_{wo}$, then it admits at most one analytic solution satisfying the initial condition (5.10).*

M. Janet asserted that this result could be proved in a way similar to the proof of Theorem 5.3.4.

5.4 Completely Integrable Higher Order Linear PDE Systems

In this subsection, we will introduce integrability conditions for higher order linear PDE systems with several unknown functions. The main result, Theorem 5.4.7, characterizes algebraically the complete integrability property for complete PDE systems. It states that, under the completeness property, the complete integrability condition is equivalent to all integrability conditions being trivially satisfied. In this subsection, we will assume that the linear PDE systems are complete. In Sect. 5.6 we will provide Procedure 6 that transforms a linear PDE system of the form (5.9) into a complete linear PDE system with respect to a weight order.

5.4.1 Formal Solutions. Consider a linear PDE system (Σ) of the form (5.9) with unknown functions $\varphi^1, \ldots, \varphi^m$ and independent variables $x_1, \ldots, x_n$. Assume that (Σ) is complete; hence, the set of monomials $\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma, \varphi^r) = \{x^\alpha \mid \alpha \in I^r\}$ is complete for all $1 \leqslant r \leqslant m$. For the remaining part of this subsection, we will denote $\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma, \varphi^r)$ by $\mathcal{U}_r$. Let $(\mathrm{cone}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma))$ denote the PDE system

$$\Phi(u)(D^\alpha \varphi^r) = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} \Phi(u)\left(a^{r,s}_{\alpha,\beta} D^\beta \varphi^s\right), \qquad 1 \leqslant r \leqslant m,$$

for $\alpha \in I^r$ and $u \in \mathcal{M}(\mathrm{Mult}(x^\alpha, \mathcal{U}_r))$.

We use the PDE system $(\mathrm{cone}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma))$ to compute the values of the principal derivative at a point $P^0 = (x_1^0, \ldots, x_n^0)$ of $\mathbb{C}^n$. We call *formal solutions* of the PDE system (Σ) at the point P^0 the elements $\varphi^1, \ldots, \varphi^m$ in $\mathbb{C}[[x_1 - x_1^0, \ldots, x_n - x_n^0]]$ which are solutions of (Σ). If the system (Σ) admits an analytic solution, then these formal solutions are convergent series and give analytic solutions of (Σ) on a neighborhood of the point P^0.

5.4.2 Initial Conditions. We are interested in condition under which the system (Σ) admits a solution for any given initial condition. The initial conditions are parametrized by the set $\mathcal{U}_r^{\complement}$ of complementary monomials of the set of monomials $\mathcal{U}_r$ as in Sect. 5.3.3. Explicitly, for $1 \leqslant r \leqslant m$, to a monomial x^β in $\mathcal{U}_r^{\complement}$, with β in $\mathbb{N}^n$ and ${}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}_r^{\complement}}(x^\beta) = \{x_{i_1}, \ldots, x_{i_{k_r}}\}$, we associate an arbitrary analytic function

$$\varphi_{\beta,r}(x_{i_1}, \ldots, x_{i_{k_r}}).$$

Then by *initial condition* one means the following data:

$$(C_{\beta,r}) \qquad D^\beta \varphi^r \Big|_{x_j = x_j^0\ \forall x_j \in {}^{\complement}\mathrm{NMult}_{\mathcal{J}}^{\mathcal{U}_r^{\complement}}(x^{\beta_r})} = \varphi_{\beta,r}(x_{i_1}, \ldots, x_{i_{k_r}}).$$

Then, as the *initial condition* for the system (Σ) in (5.8), one takes the set

$$\bigcup_{1\leqslant r\leqslant m} \{\, C_{\beta,r} \mid x^{\beta_r} \in \mathcal{U}_r^{\complement} \,\}. \tag{5.11}$$

Note that M. Janet calls *degree of generality* of the solution of the PDE system (Σ) the dimension of the initial conditions of the system, that is

$$\underset{u\in\mathcal{U}_r^{\complement}}{\mathrm{Max}} \left| {}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}_r^{\complement}}(u) \right|.$$

5.4.3 $\mathcal{J}$-Normal Form. Suppose that the PDE system (Σ) is complete. Given a linear equation E among the unknown functions $\varphi^1, \ldots, \varphi^m$ and variables $x_1, \ldots, x_n$, a *$\mathcal{J}$-normal form of E with respect to the system* (Σ) is an equation obtained from E by the reduction process that replaces principal derivatives by parametric derivatives by means of a procedure similar to **RightReduce** given in Procedure 5.

5.4.4 Integrability Conditions. Given $1 \leqslant r \leqslant m$ and $\alpha \in I^r$, let x_i in $\mathrm{NMult}_{\mathcal{J}}^{\mathcal{U}_r}(x^\alpha)$ be a non-multiplicative variable. Apply the partial derivative $\Phi(x_i) = \frac{\partial}{\partial x_i}$ to the equation

$$D^\alpha \varphi^r = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} a_{\alpha,\beta}^{r,s} D^\beta \varphi^s.$$

This yields the PDE

$$\Phi(x_i)(D^\alpha\varphi^r) = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} \left(\frac{\partial a_{\alpha,\beta}^{r,s}}{\partial x_i} D^\beta\varphi^s + a_{\alpha,\beta}^{r,s}\Phi(x_i)(D^\beta\varphi^s) \right). \quad (5.12)$$

Using the system $(\mathrm{cone}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma))$, we can rewrite the PDE (5.12) as a PDE formulated in terms of parametric derivatives and independent variables. The set of monomials $\mathcal{U}_r$ being complete, there exists α' in $\mathbb{N}^n$ with $x^{\alpha'}$ in $\mathcal{U}_r$ and u in $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}_r}(x^{\alpha'}))$ such that $x_i x^\alpha = u x^{\alpha'}$. Then we have $\Phi(x_i)D^\alpha = \Phi(u)D^{\alpha'}$ and we obtain the equation

$$\sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} \left(\frac{\partial a_{\alpha,\beta}^{r,s}}{\partial x_i} D^\beta\varphi^s + a_{\alpha,\beta}^{r,s}\Phi(x_i)(D^\beta\varphi^s) \right) = \sum_{\substack{(\beta',s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^{\beta'}\varphi^s \prec_{wo} D^{\alpha'}\varphi^r}} \Phi(u)(a_{\alpha',\beta'}^{r,s} D^{\beta'}\varphi^s). \quad (5.13)$$

Using the equations of the system $(\mathrm{cone}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma))$, we replace all principal derivatives in the equation (5.13) by parametric derivatives and independent variables. The order $\preccurlyeq_{wo}$ being well-founded this process is terminating. Moreover, when the PDE system (Σ) is complete this reduction process is confluent in the sense that any transformations of an Eq. (5.13) end with a unique $\mathcal{J}$-normal form. The set of resulting $\mathcal{J}$-normal forms is denoted by $\mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma)$.

5.4.5 Remarks. Since the system (Σ) is complete, any Eq. (5.13) is reduced to a unique normal form. Such a normal form allows us to judge whether a given integrability condition is trivial or not.

Recall that the parametric derivatives correspond to the initial conditions. Hence, a nontrivial relation in $\mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$ provides a nontrivial relation among the initial conditions. In this way, we can decide whether the system (Σ) is completely integrable or not.

5.4.6 Completely Integrable Systems. A complete linear PDE system (Σ) of the form (5.9) is said to be *completely integrable* if it admits an analytic solution for any given initial condition (5.11). For the geometrical interpretation of this condition, we refer the reader to Sect. 2.1.4.

5.4.7 Theorem [51, Sect. 42] *Let* (Σ) *be a complete finite linear PDE system of the form (5.9). Then the system* (Σ) *is completely integrable if and only if every relation in* $\mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma)$ *is a trivial identity.*

A proof of this result is given in [51, Sect. 43]. Note that the condition in this theorem is equivalent to asserting that any relation (5.13) is an algebraic consequence of a PDE equation of the system $(\mathrm{cone}_{\mathcal{J},\preccurlyeq_{wo}}(\Sigma))$.

5.5 Canonical Forms of Linear PDE Systems

In this subsection, we recall from [51] the notion of canonical linear PDE system. A canonical system is a normal form with respect to a weight order on derivatives, and such that it satisfies some analytic conditions, allowing to extend the Cauchy–Kowalevsky theorem given in Sect. 2.1.3. Note that this terminology refers to a notion of normal form, but it does not correspond to the well-known notion for a rewriting system meaning both terminating and confluence. In this chapter, we present canonical systems with respect to a weight order as it has done in Janet's monograph [51], but we point out here that this notion can be defined with any total order on derivatives.

5.5.1 Autoreduced PDE Systems. Let (Σ) be a finite linear PDE system. Suppose that a weight order $\preccurlyeq_{wo}$ is fixed on the set of unknown functions $\varphi^1, \ldots, \varphi^m$ of (Σ) and their derivatives, as defined in Sect. 5.3.5. Suppose also that each equation of the system (Σ) can be expressed in the form

$$(\Sigma^{(\alpha,r)}) \qquad D^\alpha \varphi^r = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} a^{(\alpha,r)}_{(\beta,s)} D^\beta \varphi^s,$$

so that

$$(\Sigma) = \bigcup_{(\alpha,r)\in I} \Sigma^{(\alpha,r)}, \tag{5.14}$$

the union being indexed by a multiset I. The support of the equation $(\Sigma^{(\alpha,r)})$ is defined by

$$\mathrm{Supp}(\Sigma^{(\alpha,r)}) = \{\, (\beta, s) \mid a^{(\alpha,r)}_{(\beta,s)} \neq 0 \,\}.$$

For $1 \leqslant r \leqslant m$, consider the set of monomials $\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma, \varphi^r)$ corresponding to leading derivatives, that is, monomial x^α such (α, r) belongs to I. The system (Σ) is said to be

(i) *$\mathcal{J}$-left-reduced with respect to $\preccurlyeq_{wo}$* if for any (α, r) in I there exist no (α', r) in I and nontrivial monomial x^γ in $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma,\varphi^r)}(x^{\alpha'}))$ such that $x^\alpha = x^\gamma x^{\alpha'}$;

(ii) *$\mathcal{J}$-right-reduced with respect to $\preccurlyeq_{wo}$* if, for any (α, r) in I and any (β, s) in $\mathrm{Supp}(\Sigma^{(\alpha,r)})$, there exist no (α', s) in I and nontrivial monomial x^γ in $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathrm{lm}_{\preccurlyeq_{wo}}(\Sigma,\varphi^r)}(x^{\alpha'}))$ such that $x^\beta = x^\gamma x^{\alpha'}$;

(iii) *$\mathcal{J}$-autoreduced with respect to $\preccurlyeq_{wo}$* if it is both $\mathcal{J}$-left-reduced and $\mathcal{J}$-right-reduced with respect to $\preccurlyeq_{wo}$.

5.5.2 Canonical PDE Systems. A PDE system (Σ) is said to be *$\mathcal{J}$-canonical with respect a weight order $\preccurlyeq_{wo}$* if it satisfies the following five conditions

(i) it consists of finitely many equations, and each equation can be expressed in the form

$$D^\alpha \varphi^r = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\} \\ D^\beta\varphi^s \prec_{wo} D^\alpha\varphi^r}} a_{(\beta,s)}^{(\alpha,r)} D^\beta \varphi^s,$$

(ii) the system (Σ) is $\mathcal{J}$-autoreduced with respect to $\preccurlyeq_{wo}$;

(iii) the system (Σ) is complete;

(iv) the system (Σ) is completely integrable;

(v) the coefficients $a_{(\beta,s)}^{(\alpha,r)}$ of the equations in **(i)** and the initial conditions of (Σ) are analytic.

Under these assumptions, the system (Σ) admits a unique analytic solution satisfying appropriate initial conditions parametrized by complementary monomials as in Sect. 5.3.3.

5.5.3 Remark. We note that the notion of canonicity proposed by Janet in [51] does not impose the condition of being $\mathcal{J}$-autoreduced, even if M. Janet did mentioned this autoreduced property for some simple cases. The autoreduced property implies the minimality of the system. This fact was formulated by Gerdt and Blinkov in [28] with the notion of minimal involutive basis.

5.5.4 Example. In [51, Sect. 44], M. Janet studied the following linear PDE system with one unknown function φ:

$$(\Sigma) \qquad \begin{cases} p_{54} = p_{11}, \\ p_{53} = p_{41}, \\ p_{52} = p_{31}, \\ p_{44} = p_{52}, \\ p_{43} = p_{21}, \\ p_{33} = p_{42}, \end{cases}$$

where $p_{i,j}$ denotes $\dfrac{\partial^2 \varphi}{\partial x_i \partial x_j}$. In Example 3.2.6, we have shown that the left-hand sides of the equations of this system form a complete set of monomials. Let us define the following weights for the variables:

x_1	x_2	x_3	x_4	x_5
1	0	1	1	2
0	0	0	1	1

We deduce the following weights for the second derivatives:

p_{22}	p_{21}, p_{32}	p_{42}	p_{11}, p_{31}, p_{33}	p_{52}, p_{41}, p_{43}	p_{44}	p_{51}, p_{53}	p_{54}	p_{55}
0	1	1	2	2	2	3	3	4
0	0	1	0	1	2	1	2	2

As seen in Example 3.3.4, given any four analytic functions

$$\varphi_0(x_1, x_2), \quad \varphi_3(x_1, x_2), \quad \varphi_4(x_1, x_2), \quad \varphi_5(x_1, x_5),$$

there exists a unique solution of the PDE system (Σ). Note that the initial condition is given by

$$\begin{aligned}
\varphi|_{x_3=x_3^0, x_4=x_4^0, x_5=x_5^0} &= \varphi_{0,0,0,0,0}(x_1, x_2), \\
\left.\frac{\partial \varphi}{\partial x_3}\right|_{x_3=x_3^0, x_4=x_4^0, x_5=x_5^0} &= \varphi_{0,0,1,0,0}(x_1, x_2), \\
\left.\frac{\partial \varphi}{\partial x_4}\right|_{x_3=x_3^0, x_4=x_4^0, x_5=x_5^0} &= \varphi_{0,0,0,1,0}(x_1, x_2), \\
\left.\frac{\partial \varphi}{\partial x_5}\right|_{x_2=x_2^0, x_3=x_3^0, x_4=x_4^0} &= \varphi_{0,0,0,0,1}(x_1, x_5).
\end{aligned}$$

We set

$$\begin{array}{rcl|ccccc}
A &=& p_{54} - p_{11} & x_5 & x_4 & x_3 & x_2 & x_1 \\
B &=& p_{53} - p_{41} & x_5 & & x_3 & x_2 & x_1 \\
C &=& p_{52} - p_{31} & x_5 & & & x_2 & x_1 \\
D &=& p_{44} - p_{52} & & x_4 & x_3 & x_2 & x_1 \\
E &=& p_{43} - p_{21} & & & x_3 & x_2 & x_1 \\
F &=& p_{33} - p_{42} & & & x_3 & x_2 & x_1
\end{array}$$

where the variables on the right correspond to the multiplicative variables of the first term. In order to decide if the system (Σ) is completely integrable it suffices to check if the terms

$$B_4, C_4, C_3, D_5, E_5, E_4, F_5, F_4$$

are linear combinations of derivatives of the terms A, B, C, D, E, F with respect to their multiplicative variables. Here Y_i denotes the derivative $\frac{\partial}{\partial x_i} Y$ of a term Y. Finally, we observe that

$$
\begin{aligned}
&B_4 = A_3 - D_1 - C_1, \\
&C_4 = A_2 - E_1, \qquad C_3 = B_2 - F_1, \\
&D_5 = A_4 - B_1 - C_5, \\
&E_5 = A_3 - C_1, \qquad E_4 = D_3 + B_2, \\
&F_5 = B_3 - A_2 + E_1, \qquad F_4 = E_3 - D_2 - C_2.
\end{aligned}
$$

As a consequence, the system (Σ) is completely integrable; hence, it is $\mathcal{J}$-canonical.

5.6 Reduction of a PDE System to a Canonical Form

In his monograph [51], M. Janet did not talk about the correctness of the procedures that he introduced in order to reduce a finite linear PDE system to a canonical form. In this section, we explain how to transform a finite linear PDE system with several unknown functions by derivation, elimination, and autoreduction, into an equivalent linear PDE system that is either in canonical form, or an incompatible system. For linear PDE systems with constant coefficients, the correctness of the procedure can be verified easily.

5.6.1 Equivalence of PDE System. Janet's procedure transforms by reduction and completion a finite linear PDE system into a new PDE system, which is equivalent to the original one. In his work, M. Janet dit not explain this notion of equivalence which can be described as follows. Consider two finite linear PDE systems with m unknown functions and n independent variables,

$$
(\Sigma^l) \qquad \sum_{j=1}^{m} p_{i,j}^l \varphi^j = 0, \qquad i \in I^l,
$$

for $l = 1, 2$, where $p_{i,j}^l$ are linear differential operators. We say that the PDE systems (Σ^1) and (Σ^2) are *equivalent* if the sets of solutions of the two systems coincide. This notion can be also formulated by saying that the D-modules generated by the families of differentials operators $(p_{i,1}^1, \ldots, p_{i,m}^1)$ for $i \in I^1$ and $(p_{i,1}^2, \ldots, p_{i,m}^2)$ for $i \in I^2$ coincide.

5.6.2 A Canonical Weight Order. Consider a finite linear PDE system (Σ) of m unknown functions $\varphi^1, \ldots, \varphi^m$ of the independent variables $x_1, \ldots, x_n$. To these variables and functions, we associate the following weights:

x_1	x_2	…	x_{n-1}	x_n	φ^1	φ^2	…	φ^m
1	1	…	1	1	0	0	…	0
0	0	…	0	0	1	2	…	m
0	0	…	0	1	0	0	…	0
0	0	…	1	0	0	0	…	0
⋮	⋮		⋮	⋮	⋮	⋮		⋮
0	1	…	0	0	0	0	…	0
1	0	…	0	0	0	0	…	0

The weight order on monomial partial derivatives defined in Sect. 5.1.5 induced by this weight system is total. Following M. Janet, this order is called *canonical weight order* and is denoted by $\preccurlyeq_{cwo}$.

5.6.3 Combination of Equations. Consider the PDE system (Σ) with the canonical weight order $\preccurlyeq_{cwo}$ defined in Sect. 5.6.2. We assume that the system (Σ) is given in the same form as (5.14) and that each equation of the system is written in the form

$$(E_i^{(\alpha,r)}) \qquad D^\alpha \varphi^r = \sum_{\substack{(\beta,s)\in\mathbb{N}^n\times\{1,2,\ldots,m\}\\ D^\beta\varphi^s \prec_{cwo} D^\alpha\varphi^r}} a_{(\alpha,r),i}^{(\beta,s)} D^\beta \varphi^s, \qquad i \in I^{(\alpha,r)}.$$

The *leading pair* (α, r) of the equation $E_i^{(\alpha,r)}$ will be denoted by $\mathrm{ldeg}_{\preccurlyeq_{cwo}}(E_i^{\alpha,r})$. We will denote by $\mathrm{Ldeg}_{\preccurlyeq_{cwo}}(\Sigma)$ the subset of $\mathbb{N}^n \times \{1, \ldots, m\}$ consisting of leading pairs of the equations forming the system (Σ):

$$\mathrm{Ldeg}_{\preccurlyeq_{cwo}}(\Sigma) = \{\, \mathrm{ldeg}_{\preccurlyeq_{cwo}}(E) \mid E \text{ is an equation of } \Sigma \,\}.$$

The canonical weight order $\preccurlyeq_{cwo}$ induces a total order on $\mathbb{N}^n \times \{1, \ldots, m\}$ denoted by $\prec_{lp}$. We will denote by $K(\alpha, r, i)$ the set of pairs (β, s) of running indices in the sum of the equation $E_i^{(\alpha,r)}$. Given i and j in $I^{(\alpha,r)}$, we set

$$(\alpha_{i,j}, r_{i,j}) = \mathrm{Max}\big((\beta, s) \in K(\alpha, r, i) \cup K(\alpha, r, j) \mid a_{(\alpha,r),i}^{(\beta,s)} \neq a_{(\alpha,r),j}^{(\beta,s)}\big).$$

We define

$$b_{(\alpha,r)}^{(\alpha_{i,j},r_{i,j})} = \begin{cases} a_{(\alpha,r),i}^{(\alpha_{i,j},r_{i,j})}, & \text{if } (\alpha_{i,j}, r_{i,j}) \in K(\alpha, r, i) \setminus K(\alpha, r, j),\\ -a_{(\alpha,r),i}^{(\alpha_{i,j},r_{i,j})}, & \text{if } (\alpha_{i,j}, r_{i,j}) \in K(\alpha, r, j) \setminus K(\alpha, r, i),\\ a_{(\alpha,r),i}^{(\alpha_{i,j},r_{i,j})} - a_{(\alpha,r),i}^{(\alpha_{i,j},r_{i,j})}, & \text{if } (\alpha_{i,j}, r_{i,j}) \in K(\alpha, r, i) \cap K(\alpha, r, j), \end{cases} \tag{5.15}$$

and we denote by $E_{i,j}^{(\alpha,r)}$ the equation

$$D^{\alpha_{i,j}} \varphi^{r_{i,j}} = \sum_{\substack{(\beta,s)\in K(\alpha,r,j)\\ (\beta,s)\prec_{lp}(\alpha_{i,j},r_{i,j})}} c_{(\alpha_{i,j},r_{i,j}),j}^{(\beta,s)} D^\beta \varphi^s - \sum_{\substack{(\beta,s)\in K(\alpha,r,i)\\ (\beta,s)\prec_{lp}(\alpha_{i,j},r_{i,j})}} c_{(\alpha_{i,j},r_{i,j}),i}^{(\beta,s)} D^\beta \varphi^s, \tag{5.16}$$

where, for any $k = i, j$,

$$c^{(\beta,s)}_{(\alpha_{i,j},r_{i,j}),k} = a^{(\beta,s)}_{(\alpha,r),k} / b^{(\alpha_{i,j},r_{i,j})}_{(\alpha,r)}.$$

Equation (5.16) corresponds to a combination of the two equations $E_i^{(\alpha,r)}$ and $E_j^{(\alpha,r)}$ and accordingly it will be denoted by $\mathbf{Combine}_{\preccurlyeq_{cwo}}(E_i^{(\alpha,r)}, E_j^{(\alpha,r)})$. Procedure 3 adds to a set of PDE equations (Σ) an equation E by combination.

Input:
- A canonical weight order $\preccurlyeq_{cwo}$ for $\varphi^1, \dots, \varphi^m$ and $x_1, \dots, x_n$.
- (Σ) a finite linear PDE system with unknown functions $\varphi^1, \dots, \varphi^m$ of independent variables $x_1, \dots, x_n$ given in the same form as (5.14) such that the leading derivatives are different.
- E be a linear PDE in the same form as (5.14).

begin
    $\Gamma \leftarrow \Sigma$
    $(\beta, s) \leftarrow \mathrm{ldeg}_{\preccurlyeq_{cwo}}(E)$
    if $(\beta, s) \notin \mathrm{Ldeg}_{\preccurlyeq_{cwo}}(\Gamma)$ **then**
        $\Gamma \leftarrow \Gamma \cup \{E\}$
    end
    else
        let $E^{(\beta,s)}$ be the equation of the system (Σ) whose leading pair is (β, s).
        $C \leftarrow \mathbf{Combine}_{\preccurlyeq_{cwo}}(E^{(\beta,s)}, E)$
        $\mathbf{Add}_{\preccurlyeq_{cwo}}(\Gamma, C)$
    end
end

Output: Γ a PDE system equivalent to the system obtained from (Σ) by adding equation E.

Procedure 3: $\mathbf{Add}_{\preccurlyeq_{cwo}}(\Sigma, E)$

Note that at each step of the procedure $\mathbf{RightReduce}_{\mathcal{J},\preccurlyeq_{cwo}}$ the running system Γ remains $\mathcal{J}$-left-reduced. Combining this procedure with the procedure $\mathbf{LeftReduce}_{\mathcal{J},\preccurlyeq_{cwo}}$ we obtain the following autoreduce procedure that transform a PDE system into a autoreduced PDE system.

5.6.4 Procedure $\mathbf{Autoreduce}_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$. Let us fix a canonical weight order $\preccurlyeq_{cwo}$ for $\varphi^1, \dots, \varphi^m$ and $x_1, \dots, x_n$. Let (Σ) be a finite linear PDE system given in the same form as (5.14), with unknown functions $\varphi^1, \dots, \varphi^m$ of the independent variables $x_1, \dots, x_n$. We assume that the leading derivatives of (Σ) are all different. The procedure $\mathbf{Autoreduce}_{\mathcal{J},\preccurlyeq_{cwo}}$ transforms the PDE system (Σ) into an $\mathcal{J}$-autoreduced PDE system equivalent to (Σ), by applying successively the procedures $\mathbf{LeftReduce}_{\mathcal{J},\preccurlyeq_{cwo}}$ and $\mathbf{RightReduce}_{\mathcal{J},\preccurlyeq_{cwo}}$. An algebraic version of this procedure is given in Procedure 9. Let us remark that the autoreduction procedure given in Janet's monographs corresponds to the $\mathbf{LeftReduce}_{\mathcal{J},\preccurlyeq_{cwo}}$, so does not deal with right reduction of equations.

Input:
- A canonical weight order $\preccurlyeq_{cwo}$ for $\varphi^1, \ldots, \varphi^m$ and $x_1, \ldots, x_n$.
- (Σ) a finite linear PDE system with unknown functions $\varphi^1, \ldots, \varphi^m$ of independent variables $x_1, \ldots, x_n$ given in the same form as (5.14) such that the leading derivatives are different.

begin

$\Gamma \leftarrow \Sigma$

$I \leftarrow \mathrm{Ldeg}_{\preccurlyeq_{cwo}}(\Gamma)$

$\mathcal{U}_r \leftarrow \{x^\alpha \mid (\alpha, r) \in I\}$

while $\big($*exist* $(\alpha, r), (\alpha', r)$ *in* I *and a non-trivial monomial* x^γ *in* $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}_r}(x^{\alpha'}))$ *such that* $x^\alpha = x^\gamma x^{\alpha'}\big)$ **do**

$\Gamma \leftarrow \Gamma \setminus \{E^{(\alpha,r)}\}$

Let $D^\gamma E^{(\alpha',r)}$ be the equation obtained from the equation $E^{(\alpha',r)}$ by applying the operator D^γ to the two sides.

$C \leftarrow \mathbf{Combine}_{\preccurlyeq_{cwo}}(E^{(\alpha,r)}, D^\gamma E^{(\alpha',r)})$

$\mathbf{Add}_{\preccurlyeq_{cwo}}(\Gamma, C)$

end

end

Output: Γ a $\mathcal{J}$-left-reduced PDE system with respect to $\preccurlyeq_{cwo}$ that is equivalent to (Σ).

Procedure 4: LeftReduce$_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$

Note that the procedure **Autoreduce**$_{\mathcal{J},\preccurlyeq_{cwo}}$ fails if and only if the procedure **Combine**$_{\preccurlyeq_{cwo}}$ fails. This occurs when the procedure **Combine**$_{\preccurlyeq_{cwo}}$ is applied to equations $E_i^{(\alpha,r)}$ and $E_j^{(\alpha,r)}$ and some coefficients $b_{(\alpha,r)}^{(\alpha_{i,j},r_{i,j})}$, as defined in (5.15), vanish at some point of $\mathbb{C}^n$. In particular, the procedure **Autoreduce**$_{\mathcal{J},\preccurlyeq_{cwo}}$ does not fail when all the coefficients are constant. This constraint on the coefficients of the system concerns only the left reduction and was not discussed in Janet's monograph. As a consequence, we have the following result.

5.6.5 Theorem *If* (Σ) *is a finite linear PDE system with constant coefficients, the procedure* **Autoreduce**$_{\mathcal{J},\preccurlyeq_{cwo}}$ *terminates and produces a finite autoreduced PDE system that is equivalent to* (Σ).

5.6.6 Completion Procedure of a PDE System. Consider a finite linear PDE system (Σ) with the canonical weight order $\preccurlyeq_{cwo}$ given in Sect. 5.6.2. If the system (Σ) is $\mathcal{J}$-autoreduced, then the following procedure **Complete**$_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$ transforms the system (Σ) into a finite complete $\mathcal{J}$-autoreduced linear PDE system. This completion procedure appears in Janet's monograph [51] but not in an explicit way.

5.6.7 Completion and Integrability Conditions. In Procedure 6, the set $\mathcal{P}r_r$ contains all the obstructions to the completeness of a system. The procedure **Complete**$_{\mathcal{J},\preccurlyeq_{cwo}}$ adds to the system the necessary equations in order to eliminate all these obstructions. The equations added to the system have the form

$$D^\beta \varphi^r = \mathrm{Rhs}(E^{(\beta,r)}) - a_{(\beta,r)}^{(\delta,r)} D^\delta \varphi^r + a_{(\beta,r)}^{(\delta,r)} D^\gamma(\mathrm{Rhs}(E^{(\alpha,r)}))$$

Input:
- A canonical weight order $\preccurlyeq_{cwo}$ for $\varphi^1, \ldots, \varphi^m$ and $x_1, \ldots, x_n$.
- (Σ) a finite linear PDE system with unknown functions $\varphi^1, \ldots, \varphi^m$ of independent variables $x_1, \ldots, x_n$ that is given in the same form as (5.14) and that is $\mathcal{J}$-left reduced with respect to $\preccurlyeq_{cwo}$.

```
begin
  Γ ← Σ
  Γ' ← Γ
  I ← Ldeg_{≼cwo}(Γ)
  // The canonical weight order ≼cwo induces a total
  // order on the set I of leading pairs denoted by ≼lp
  (δ, t) ← max(I) with respect to ≼lp
  while Γ' ≠ ∅ do
    Γ' ← Γ' \ {E^(δ,t)}
    I ← I \ {(δ, t)}
    S ← Supp(E^(δ,t))
    U_r ← {x^α | (α, r) ∈ I}
    while (exist (β, r) in S, (α, r) in I and a non-trivial monomial x^γ in M(Mult_J^{U_r}(x^α))
    such that x^β = x^γ x^α) do
      Γ ← Γ \ {E^(δ,t)}
      C ← E^(δ,t) − a^(β,r)_(δ,t) D^β φ^r + a^(β,r)_(δ,t) D^γ(Rhs(E^(α,r)))
      Add_{≼cwo}(Γ, C)
    end
  end
end
```

Output: Γ a $\mathcal{J}$-right-reduced PDE system with respect to $\preccurlyeq_{cwo}$ that is equivalent to (Σ).

Procedure 5: RightReduce$_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$

with $\delta \neq \beta$ and lead to the definition of a new integrability condition of the form (5.13) by using the construction given in Sect. 5.4.4.

5.6.8 Janet's Procedure. Given a finite linear PDE system (Σ) with the canonical weight order $\preccurlyeq_{cwo}$ defined in Sect. 5.6.2, *Janet's procedure* **Janet**$_{\mathcal{J},\preccurlyeq_{cwo}}$ either transforms the system (Σ) into a PDE system (Γ) that is $\mathcal{J}$-canonical with respect to $\preccurlyeq_{cwo}$, or computes an obstruction to the feasibility of such a transformation. In the first case, the solutions of the $\mathcal{J}$-canonical system (Γ) are solutions of the initial system (Σ). In the second case, the obstruction corresponds to a nontrivial relation on the initial conditions. We refer the reader to [81] or [78] for a deeper discussion on this procedure and its implementations.

Applying the procedures **Autoreduce**$_{\mathcal{J}}$ and **Complete**$_{\mathcal{J}}$ successively, the first step of the procedure consists in reducing the given PDE system (Σ) to a PDE system (Γ) that is $\mathcal{J}$-autoreduced and complete with respect to $\preccurlyeq_{cwo}$.

Then one computes the set **IntCond**$_{\mathcal{J},\preccurlyeq_{cwo}}(\Gamma)$ of integrability conditions of the system (Γ). Recall from Sect. 5.4.4 that this set is a finite set of relations that do

Input:
- A canonical weight order $\preccurlyeq_{cwo}$ for $\varphi^1, \ldots, \varphi^m$ and $x_1, \ldots, x_n$.
- (Σ) a finite $\mathcal{J}$-autoreduced linear PDE system with unknown functions $\varphi^1, \ldots, \varphi^m$ of independent variables $x_1, \ldots, x_n$ given in the same form as (5.14) and whose leading derivatives are different.

begin
$\Gamma \leftarrow \Sigma$
$\Xi \leftarrow \emptyset$
for $r = 1, \ldots, m$ **do**
while $\Xi = \emptyset$ **do**
$I \leftarrow \mathrm{Ldeg}_{\preccurlyeq_{cwo}}(\Gamma)$
$\mathcal{U}_r \leftarrow \{x^\alpha \mid (\alpha, r) \in I\}$
$\mathcal{P}r_r \leftarrow \big\{ \frac{\partial E}{\partial x} \mid E \in \Gamma,\ x \in \mathrm{NMult}_{\mathcal{J}}^{\mathcal{U}_r}(x^\delta)$ with $(\delta, r) = \mathrm{ldeg}(E)$ and $xx^\delta \notin \mathrm{cone}_{\mathcal{J}}(\mathcal{U}_r)\big\}$
$C \leftarrow 0$
while $\mathcal{P}r_r \neq \emptyset$ *and* $C = 0$ **do**
choose $E^{(\beta,r)}$ in $\mathcal{P}r_r$, whose leading pair (β, r) is minimal with respect to $\preccurlyeq_{cwo}$.
$\mathcal{P}r_r \leftarrow \mathcal{P}r_r \setminus \{E^{(\beta,r)}\}$
$C \leftarrow E^{(\beta,r)}$
$S_C \leftarrow \mathrm{Supp}(C)$
while *exist* (δ, r) *in* S_C, (α, r) *in* I *and* x^γ *in* $\mathcal{M}(\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}_r}(x^\alpha))$ *such that* $x^\delta = x^\gamma x^\alpha$ **do**
$C \leftarrow C - a_{(\beta,r)}^{(\delta,r)} D^\delta \varphi^r + a_{(\beta,r)}^{(\delta,r)} D^\gamma(\mathrm{Rhs}(E^{(\alpha,r)}))$
$S_C \leftarrow \mathrm{Supp}(C)$
end
end
if $C \neq 0$ **then**
$\Gamma \leftarrow$ **Autoreduce**$_{\mathcal{J}, \preccurlyeq_{cwo}}(\Gamma \cup \{C\})$
end
else
$\Xi \leftarrow \Gamma$
end
end
end
end

Output: (Ξ) a linear $\mathcal{J}$-autoreduced PDE system equivalent to (Σ) and that is complete with respect to $\preccurlyeq_{cwo}$.

Procedure 6: Complete$_{\mathcal{J}, \preccurlyeq_{cwo}}(\Sigma)$

not contain principal derivatives. Hence, these integrability conditions are $\mathcal{J}$-normal forms with respect to (Γ). Since the system (Γ) is complete, these normal forms are unique, and by Theorem 5.4.7, if all of these normal forms are trivial, then the system (Γ) is completely integrable. Otherwise, the procedure takes a nontrivial condition

$\mathcal{R}$ in the set $\mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{cwo}}(\Gamma)$ and distinguishes two cases. If the relation $\mathcal{R}$ is among functions $\varphi^1,\ldots,\varphi^m$ and variables $x_1,\ldots,x_n$, then it imposes a relation on the initial conditions of the system (Γ). In the other case, the set $\mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{cwo}}(\Gamma)$ contains at least one PDE involving a derivative of one of the functions $\varphi^1,\ldots,\varphi^m$ and the procedure $\mathbf{Janet}_{\mathcal{J},\preccurlyeq_{cwo}}$ is applied again to the PDE system (Σ) completed by all the PDE equations in $\mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{cwo}}(\Gamma)$.

5.6.9 Remarks. If the procedure stops at the first loop, that is, if C consists only of trivial identities, then the system (Σ) is reducible to the $\mathcal{J}$-canonical form (Γ) equivalent to (Σ).

When the set C contains an integrability condition involving at least one derivative of the unknown functions, the procedure is applied again to the system $(\Sigma)\cup C$. Notice that it could be also possible to recall the procedure on $(\Gamma)\cup C$, but as done in Janet's monograph [51], we choose to restart the procedure on $(\Sigma)\cup C$ in order to have a PDE system where each equation has a clear meaning, namely, it comes either from the initial problem or from the integrability condition.

Input:
- A canonical weight order $\preccurlyeq_{cwo}$ for $\varphi^1,\ldots,\varphi^m$ and $x_1,\ldots,x_n$.
- (Σ) a finite linear PDE system with unknown functions $\varphi^1,\ldots,\varphi^m$ of independent variables $x_1,\ldots,x_n$ given in the same form as (5.14) and whose leading derivatives are different.

begin
- $\Gamma \leftarrow \mathbf{Autoreduce}_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$
- $\Gamma \leftarrow \mathbf{Complete}_{\mathcal{J},\preccurlyeq_{cwo}}(\Gamma)$
- $C \leftarrow \mathbf{IntCond}_{\mathcal{J},\preccurlyeq_{cwo}}(\Gamma)$
- **if** *C consists only of trivial identities* **then**
  - **return** The PDE system (Σ) is transformable to a $\mathcal{J}$-canonical system (Γ).
- **end**
- **if** *C contains a non-trivial relation $\mathcal{R}$ among functions $\varphi^1,\ldots,\varphi^m$ and variables $x_1,\ldots,x_n$* **then**
  - **return** The PDE system (Σ) is not reducible to a $\mathcal{J}$-canonical system and the relation $\mathcal{R}$ imposes a non-trivial relation on the initial conditions of the system (Γ).
- **end**
- **else**
  - // *C contains a non-trivial relation among the functions $\varphi^1,\ldots,\varphi^m$, the variables $x_1,\ldots,x_n$,*
  - // *and at least one derivative of one of the functions $\varphi^1,\ldots,\varphi^m$.*
  - $\Sigma \leftarrow \Sigma \cup \{C\}$
  - $\mathbf{Janet}_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$.
- **end**

end

Output: Complete integrability of the system (Σ) and its obstructions to be reduced to a $\mathcal{J}$-canonical form with respect to $\preccurlyeq_{cwo}$.

Procedure 7: $\mathbf{Janet}_{\mathcal{J},\preccurlyeq_{cwo}}(\Sigma)$

Finally, note that the procedure **Janet**$_{\mathcal{J},\preccurlyeq_{cwo}}$ fails on a PDE system (Σ) if and only if the procedure **Autoreduce**$_{\mathcal{J},\preccurlyeq_{cwo}}$ fails on $(\Sigma) \cup C$, where C consists of the potential nontrivial relations among the unknown functions and the variables added during the process, as explained in Sect. 5.6.4. In particular, by Theorem 5.6.5, if (Σ) is a finite linear PDE system with constant coefficients, the procedure **Autoreduce**$_{\mathcal{J},\preccurlyeq_{cwo}}$ terminates and produces a finite autoreduced PDE system equivalent to (Σ).

5.6.10 Example. In [51, Sect. 47], M. Janet studied the PDE system

$$(\Sigma) \qquad \begin{cases} p_{33} = & x_2 p_{11}, \\ p_{22} = & 0, \end{cases}$$

where $p_{i_1 \ldots i_k}$ denotes the derivative $\dfrac{\partial^k \varphi}{\partial x_{i_1} \ldots \partial x_{i_k}}$ of an unknown function φ of the independent variables x_1, x_2, x_3. The set of monomials of the left-hand side of the system (Σ) is $\mathcal{U} = \{x_3^2, x_2^2\}$. The set $\mathcal{U}$ is not complete. Indeed, for instance the monomial $x_3 x_2^2$ is not in the involutive cone $\mathrm{cone}_{\mathcal{J}}(\mathcal{U})$. If we complete the set $\mathcal{U}$ by the monomial $x_3 x_2^2$ we obtain a complete set $\widetilde{\mathcal{U}} := \mathcal{U} \cup \{x_3 x_2^2\}$. The PDE system (Σ) is then equivalent to the PDE system

$$(\Gamma) \qquad \begin{cases} p_{33} = & x_2 p_{11}, \\ p_{322} = & 0, \\ p_{22} = & 0. \end{cases}$$

Note that $p_{322} = \partial_{x_3} p_{22} = 0$. The table of multiplicative variables with respect to the set $\widetilde{\mathcal{U}}$ is given by

x_3^2	x_3	x_2	x_1
$x_3 x_2^2$		x_2	x_1
x_2^2		x_2	x_1

We deduce that there exists only one nontrivial compatibility condition, which reads

$$\begin{aligned} p_{3322} &= \partial_{x_3} p_{322} = \partial_{x_2}^2 p_{33}, \qquad (x_3 \cdot x_3 x_2^2 = (x_2)^2 \cdot x_3^2) \\ &= \partial_{x_2}^2 (x_2 p_{11}) = 2 p_{211} + x_2 p_{2211} = 2 p_{211} = 0, \qquad (p_{2211} = \partial_{x_1}^2 p_{22} = 0). \end{aligned}$$

Hence, $p_{211} = 0$ is a nontrivial relation of the system (Γ). Hence, the PDE system (Σ) is not completely integrable. Then, we consider the new PDE system given by

$$(\Sigma') \qquad \begin{cases} p_{33} = & x_2 p_{11}, \\ p_{22} = & 0, \\ p_{211} = & 0. \end{cases}$$

The associated set of monomials $\mathcal{U}' = \{x_3^2, x_2^2, x_2 x_1^2\}$ is not complete. It can be completed to the complete set $\widetilde{\mathcal{U}'} := \mathcal{U}' \cup \{x_3 x_2^2, x_3 x_2 x_1^2\}$. The PDE system (Σ') is then equivalent to the following PDE system:

$$(\Gamma') \quad \begin{cases} p_{33} = & x_2 p_{11}, \\ p_{322} = & 0, \\ p_{3211} = & 0, \\ p_{22} = & 0, \\ p_{221} = & 0. \end{cases}$$

Note that $p_{322} = \partial_{x_3} p_{22}$ and $p_{3211} = \partial_{x_3} p_{211}$. The multiplicative variables with respect to the set of monomials $\mathcal{U}'$ are given by the following table:

x_3^2	x_3 x_2 x_1
$x_3x_2^2$	x_2 x_1
$x_3x_2x_1^2$	x_1
x_2^2	x_2 x_1
$x_2x_1^2$	x_1

We deduce that the only nontrivial compatibility relation is

$$\begin{aligned} p_{33211} &= \partial_{x_3}(p_{3211}) = 0, \\ &= \partial_{x_1}^2 \partial_{x_2}(p_{33}) = \partial_{x_1}^2 \partial_{x_2}(x_2 p_{11}), \\ &= \partial_{x_1}^2 (p_{11} + x_2 p_{211}) = p_{1111}, \qquad \text{since } p_{211} = 0. \end{aligned}$$

We see that $p_{1111} = 0$ is a nontrivial relation of the system (Γ'). Hence, the system (Σ') is not completely integrable. Now consider the new PDE system given by

$$(\Sigma'') \quad \begin{cases} p_{33} = & x_2 p_{11}, \\ p_{22} = & 0, \\ p_{211} = & 0, \\ p_{1111} = & 0. \end{cases}$$

The associated set of monomials $\mathcal{U}'' = \{x_3^2, x_2^2, x_2x_1^2, x_1^4\}$ is not complete. It can be completed to the set of monomials $\widetilde{\mathcal{U}''} := \mathcal{U}'' \cup \{x_3x_2^2, x_3x_2x_1^2, x_3x_1^4\}$. The PDE system (Σ'') is seen to be equivalent to the system

$$(\Gamma'') \quad \begin{cases} p_{33} = & x_2 p_{11}, \\ p_{322} = & 0, \\ p_{31111} = & 0, \\ p_{22} = & 0, \\ p_{211} = & 0, \\ p_{1111} = & 0. \end{cases}$$

Note that $p_{322} = \partial_{x_2} p_{22}$ and $p_{31111} = \partial_{x_3} p_{1111}$. All the compatibility conditions are trivial identities, and by Theorem 5.4.7 we deduce that the PDE (Σ'') obtained from the initial PDE system (Σ) by adding compatibility conditions is completely integrable.

5.6.11 Remark. Let us mention that using a procedure similar to the one presented in this section, Janet in [51, Sect. 48] gave a constructive proof of a result obtained previously by Tresse [88] asserting that an infinite linear PDE system can be reduced to a finite linear PDE system.

5.7 Algebra, Geometry, and PDEs

The notion of ideal first appeared in the work of R. Dedekind. It appeared also in a seminal paper [43] of Hilbert, where he developed the theory of ideals in polynomial rings. In particular, he proved Noetherianity results, such as the Noetherianity of the ring of polynomials over a field, a result known now as Hilbert's basis theorem. In his works on PDE systems [48–50], M. Janet used the notion of ideal generated by homogeneous polynomials under the terminology of *module of forms*, which he defined as follows. He called *form* a homogeneous polynomial with several variables and he defined a *module* of forms as an algebraic system satisfying the two following conditions:

- **(i)** if a form f belongs to the system, then the form hf belongs to the system for every form h,
- **(ii)** if f and g are two forms of the same order in the system, then the form $f + g$ belongs to the system.

Finally, in [51, Sect. 51], M. Janet recalls Hilbert's basis theorem.

5.7.1 Characteristic Functions of Homogeneous Ideals. In [51, Sect. 51], M. Janet recalled the Hilbert description of the problem of finding the number of independent conditions so that a homogenous polynomial of order p belongs to a given homogeneous ideal. These independent conditions correspond to the independent linear forms that annihilate all homogeneous polynomials of degree p in the ideal. Janet recalled from [43] that this number of independent conditions is expressed as a polynomial in p for sufficiently large p.

Let I be a homogenous ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by polynomials $f_1, \ldots, f_k$. Given a monomial order on $\mathcal{M}(x_1, \ldots, x_n)$, we can assume that all the leading coefficients are equal to 1. For any $p \geqslant 0$, consider the homogeneous component of degree p so that $I = \bigoplus_p I_p$, with

$$I_p := I \cap \mathbb{K}[x_1, \ldots x_n]_p.$$

Recall that

$$\dim I_p \leqslant \dim \left(\mathbb{K}[x_1, \ldots, x_n]_p\right) = \Gamma_n^p.$$

The number of independent conditions such that a homogeneous polynomial of order p belongs to the ideal I is given by the difference

$$\chi(p) := \Gamma_n^p - \dim I_p.$$

This is the number of monomials of degree p that cannot be divided by the monomials $\mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_k)$. The function $\chi(p)$ corresponds to a coefficient of the Hilbert series of the ideal I and is called the *characteristic function* of the ideal I, or *postulation* by Janet in [51, Sect. 52]. We refer the reader to [18] for the definition of Hilbert series of polynomial rings and their applications. In Sect. 5.8, we will show that the function $\chi(p)$ is polynomial for sufficiently large p. Finally, note that the set of monomials that cannot be divided by the monomials $\mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_k)$ consists of a finite number of classes of complementary monomials.

5.7.2 Geometric Remark. M. Janet made the following geometric observation about the characteristic function. Suppose that p is sufficiently large so that the function $\chi(p)$ is polynomial. Let $\lambda - 1$ be the degree of the leading term of the polynomial $\chi(p)$. Consider the projective variety $V(I)$ defined by

$$V(I) = \{a \in \mathbb{P}^{n-1} \mid f(a) = 0 \text{ for all } f \text{ in } I\}.$$

The integer $\mu = \mathrm{lc}(\chi(p))(\lambda - 1)!$ corresponds to the degree of the variety $V(I)$ [43]. If $\chi(p) = 0$ then the variety $V(I)$ is empty, in the other cases $V(I)$ is a subvariety of $\mathbb{P}^{n-1}$ of dimension $\lambda - 1$.

5.7.3 Example [51, Sect. 53]. Consider the monomial ideal I of $\mathbb{K}[x_1, x_2, x_3]$ generated by x_1^2, x_1x_2, and x_2^2. The characteristic function $\chi(p)$ of the ideal I is constant and equal to 3. The unique point that annihilates the ideal I is (0, 0, 1), with multiplicity 3. This result is compatible with the fact that the zeros of the ideal J generated by the polynomials

$$(x_1 - ax_3)(x_1 - bx_3), \qquad (x_1 - ax_3)(x_2 - cx_3), \qquad (x_2 - cx_3)(x_2 - dx_3),$$

consists of the three points

$$(a, c, 1), \qquad (a, d, 1), \qquad (b, c, 1).$$

5.7.4 The Ideal – PDE Dictionary. Let I be a homogeneous ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by a set $F = \{f_1, \ldots, f_k\}$ of polynomials. For a fixed monomial order on $\mathcal{M}(x_1, \ldots, x_n)$, we set $\mathcal{U} = \mathrm{lm}(F)$. Consider the ring isomorphism Φ from $\mathbb{K}[x_1, \ldots, x_n]$ to $\mathbb{K}[\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}]$ given in Proposition 3.1.2. To each polynomial f in I, we associate a PDE $\Phi(f)\varphi = 0$. In this way, the ideal I defines a PDE system $(\Sigma(I))$. Let λ and μ be the integers associated to the characteristic function

$\chi(p)$ as defined in 5.7.2. The maximal number of arguments of the arbitrary analytic functions used to define the initial conditions

$$\{ C_\beta \mid x^\beta \in \mathcal{U}^{\complement} \}$$

of the PDE system $(\Sigma(I))$, as defined in (5.7), corresponds to λ, explicitly,

$$\lambda = \max_{v \in \mathcal{U}^{\complement}} |{}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}^{\complement}}(v)|,$$

where $\mathcal{U}^{\complement}$ denotes the set of complementary monomials of $\mathcal{U}$. Moreover, the number of arbitrary analytic functions with λ arguments in the initial conditions $\{ C_\beta \mid x^\beta \in \mathcal{U}^{\complement} \}$ is equal to μ, that is

$$\mu = \Big| \{ v \in \mathcal{U}^{\complement} \text{ such that } |{}^{\complement}\mathrm{Mult}_{\mathcal{J}}^{\mathcal{U}^{\complement}}(v)| = \lambda \} \Big|.$$

Conversely, let (Σ) be a PDE system with one unknown function φ of the independent variables $x_1, \ldots, x_n$. Denote by $\mathrm{ldo}(\Sigma)$ the set of differential operators associated to the principal derivatives of PDE in (Σ), with respect to Janet's order on derivatives defined in Sect. 5.1.3. The isomorphism Φ associates to any monomial differential operator $\frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$ in $\mathrm{ldo}(\Sigma)$ a monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in $\mathcal{M}(x_1, \ldots, x_n)$.

Denote by $I(\Sigma)$ the ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by $\Phi^{-1}(\mathrm{ldo}(\Sigma))$. Note that, by construction, the ideal $I(\Sigma)$ is monomial and for any monomial u in $I(\Sigma)$ the derivative $\Phi(u)\varphi$ is a principal derivative of the PDE system (Σ) as defined in Sect. 5.3.1. In [51, Sect. 54], M. Janet called *characteristic form* any element of the ideal $I(\Sigma)$.

In this way, M. Janet concluded that the degree of generality of the solutions of a linear PDE system with one unknown function is described by the leading term of the characteristic function of the ideal of characteristic forms defined in Sect. 5.7.1.

5.7.5 The Particular Case of First-Order Systems. Consider a completely integrable first-order linear PDE system (Σ). The number λ, defined in Sect. 5.7.4, which is equal to the maximal number of arguments of the arbitrary functions used to define the initial conditions of the system (Σ), is also equal in this case to the cardinality of the set $\mathcal{U}^{\complement}$ of complementary monomials of the set of monomials $\mathcal{U} = \Phi^{-1}(\mathrm{ldo}(\Sigma))$.

5.8 Involutive Systems

In this subsection, we recall the algebraic formulation of involutive systems as introduced by M. Janet. This formulation first appeared in its work in [48] and [49]. But notice that this notion comes from the work of Cartan in [13].

5.8.1 Characters and Derived Systems. Let I be a proper ideal of $\mathbb{K}[x_1, \ldots, x_n]$ generated by homogeneous polynomials. M. Janet introduced the *characters* of the homogeneous component I_p as the nonnegative integers $\sigma_1, \sigma_2, \ldots, \sigma_n$ defined inductively by the formula

$$\dim\left(I_p + \left(\sum_{i=1}^{h} \mathbb{K}[x_1, \ldots, x_n]_{p-1} x_i\right)\right) = \dim(I_p) + \sigma_1 + \cdots + \sigma_h, \qquad 1 \leqslant h \leqslant n.$$

Note that the sum $\sigma_1 + \sigma_2 + \cdots + \sigma_n$ corresponds to the codimension of I_p in $\mathbb{K}[x_1, \ldots, x_n]_p$.

Given a positive integer λ, we set

$$J_{p+\lambda} = \mathbb{K}[x_1, \ldots, x_n]_\lambda I_p.$$

We define the nonnegative integers $\sigma_1^{(\lambda)}, \sigma_2^{(\lambda)}, \ldots, \sigma_n^{(\lambda)}$ by the relations

$$\dim\left(J_{p+\lambda} + \left(\sum_{i=1}^{h} \mathbb{K}[x_1, \ldots, x_n]_{p+\lambda-1} x_i\right)\right)$$
$$= \dim(J_{p+\lambda}) + \sigma_1^{(\lambda)} + \cdots + \sigma_h^{(\lambda)}, \qquad 1 \leqslant h \leqslant n.$$

For $\lambda = 1$, M. Janet called J_{p+1} the *derived system* of I_p. Let us mention some properties of these numbers proved by M. Janet.

5.8.2 Lemma *We set $\sigma'_h = \sigma_h^{(1)}$ and $\sigma''_h = \sigma_h^{(2)}$ for $1 \leqslant h \leqslant n$. Then,*

(i) $\sigma'_1 + \sigma'_2 + \cdots + \sigma'_n \leqslant \sigma_1 + 2\sigma_2 + \cdots + n\sigma_n$.
(ii) *If $\sigma'_1 + \sigma'_2 + \cdots + \sigma'_n = \sigma_1 + 2\sigma_2 + \cdots + n\sigma_n$, the two following relations hold:*

  (a) $\sigma''_1 + \sigma''_2 + \cdots + \sigma''_n = \sigma'_1 + 2\sigma'_2 + \cdots + n\sigma'_n$.
  (b) $\sigma'_h = \sigma_h + \sigma_{h+1} + \cdots + \sigma_n$.

We refer the reader to [51] for a proof of the relations of Lemma 5.8.2.

5.8.3 Involutive Systems. The homogenous component I_p is said to be in *involution* when

$$\sigma'_1 + \sigma'_2 + \cdots + \sigma'_n = \sigma_1 + 2\sigma_2 + \cdots + n\sigma_n.$$

Following properties **(ii)–(a)** of Lemma 5.8.2, if the component I_p is in involution, then the component I_{p+k} is in involution for all $k \geqslant 0$.

5.8.4 Proposition [51, Sect. 56 & Sect. 57] *The characters of a homogeneous component I_p satisfy the two following properties:*

(i) $\sigma_1 \geqslant \sigma_2 \geqslant \cdots \geqslant \sigma_n$.
(ii) *if $I_p \neq \{0\}$, then $\sigma_n = 0$.*

5.8.5 Polynomiality of Characteristic Functions. Suppose that the homogeneous component I_p is in involution. We claim that the characteristic function $\chi(P)$ defined in Sect. 5.7.1 is polynomial for $P \geqslant p$. Indeed, using Lemma 5.8.2, we show by induction that for any $1 \leqslant h < n$ and any positive integer λ, it holds that

$$\sigma_h^{(\lambda)} = \sum_{k=0}^{n-h-1} \binom{\lambda + k - 1}{k} \sigma_{h+k}.$$

The codimension of $I_{p+\lambda}$ in $\mathbb{K}[x_1, \ldots, x_n]_{p+\lambda}$ is given by

$$\begin{aligned}\sum_{h=1}^{n-1} \sigma_h^{(\lambda)} &= \sum_{h=1}^{n-1} \sum_{k=0}^{n-h-1} \binom{\lambda + k - 1}{k} \sigma_{h+k} = \sum_{i=1}^{n-1} \left(\sum_{k=0}^{i-1} \binom{\lambda + k - 1}{k} \right) \sigma_i \\ &= \sum_{i=1}^{n-1} \left(\sum_{k=0}^{i-1} \binom{P - p + k - 1}{k} \right) \sigma_i = \sum_{i=1}^{n-1} \binom{P - p + i - 1}{i - 1} \sigma_i .\end{aligned}$$

This proves the polynomiality of the characteristic function of the ideal I for sufficiently large p.

5.9 Concluding Remarks

Recall that the so-called Cartan–Kähler theory is concerned with the Pfaffian systems on a differentiable (or analytic) manifold and its aim is to determine whether a given system is prolongeable to a completely integrable system or an incompatible system. The Cartan–Kähler method relies on a geometrical argument, which is to construct integral submanifolds of the system inductively. Here, a step of the induction is to find an integral submanifold of dimension $i + 1$ containing the integral submanifold of dimension i, and their theory does not allow to deduce whether such step can be achieved or not.

Janet's method is, even if it works only locally, completely algebraic and algorithmic so that it partially completes the parts where the Cartan–Kähler theory does not work.

According to these works, there are two seemingly different notions of involutivity, the one by G. Frobenius, G. Darboux, and É. Cartan and the other by M. Janet. The fact is that at each step of the induction in the Cartan–Kähler theory, one has to study a system of PDE. The system is called in *involution* (compare with those in Sects. 2.2.6 with Sect. 5.8) if it can be written in a canonical form, as defined in Sect. 5.5.2, perhaps after a change of coordinates, if necessary. Following Janet's algebraic definition of involutivity, several involutive methods were developed for polynomial and differential systems, [72, 86]. In these approaches, a differential system is involutive when its non-multiplicative derivatives are consequences of multiplicative derivatives. In [25, 27], Gerdt gave an algebraic characterization of involutivity for polynomial systems. Gerdt's approach is presented in the next section.

6 Polynomial Involutive Bases

In this section, we present the algebraic definition of involutivity for polynomial systems given by Gerdt in [25, 27]. In particular, we relate the notion of involutive basis for a polynomial ideal to the notion of Gröbner basis.

6.1 Involutive Reduction on Polynomials

6.1.1 Involutive Basis. Recall that a *monomial ideal* I of $\mathbb{K}[x_1, \ldots, x_n]$ is an ideal generated by monomials. An *involutive basis* of the ideal I with respect to an involutive division $\mathcal{I}$ is an involutive set of monomials $\mathcal{U}$ that generates I. By the Dickson Lemma [17], every monomial ideal I admits a finite set of generators. When the involutive division $\mathcal{I}$ is Noetherian as defined in Sect. 4.2.3, this generating set admits a finite $\mathcal{I}$-completion that forms an involutive basis of the ideal I. As a consequence, we deduce the following result.

6.1.2 Proposition *Let $\mathcal{I}$ be a Noetherian involutive division on $\mathcal{M}(x_1, \ldots, x_n)$. Every monomial ideal of $\mathbb{K}[x_1, \ldots, x_n]$ admits an $\mathcal{I}$-involutive basis.*

The objective of this section is to show how to extend this result to polynomial ideals with respect to a monomial order. In the remainder of this subsection, we assume that a monomial order $\preccurlyeq$ is fixed on $\mathcal{M}(x_1, \ldots, x_n)$.

6.1.3 Multiplicative Variables for a Polynomial. Let $\mathcal{I}$ be an involutive division on $\mathcal{M}(x_1, \ldots, x_n)$. Let F be a set of polynomials from $\mathbb{K}[x_1, \ldots, x_n]$, and let f be a polynomial in F. We define the set of $\mathcal{I}$*-multiplicative* (resp. $\mathcal{I}$*-non-multiplicative*) *variables* of the polynomial f with respect to F and the monomial order $\preccurlyeq$ by setting

$$\mathrm{Mult}_{\mathcal{I},\preccurlyeq}^{F}(f) = \mathrm{Mult}_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(F)}(\mathrm{lm}_{\preccurlyeq}(f)) \quad (\text{resp. } \mathrm{NMult}_{\mathcal{I},\preccurlyeq}^{F}(f) = \mathrm{NMult}_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(F)}(\mathrm{lm}_{\preccurlyeq}(f))).$$

Note that the $\mathcal{I}$-multiplicative variables depend on the monomial order $\preccurlyeq$ used to determine the leading monomials of the polynomials of F.

6.1.4 Polynomial Reduction. Polynomial division can be described as a rewriting operation as follows. Given polynomials f and g in $\mathbb{K}[x_1, \ldots, x_n]$, we say that f is *reducible modulo g with respect to $\preccurlyeq$*, if there is a term λu in f whose monomial u is divisible by $\mathrm{lm}_{\preccurlyeq}(g)$ for the usual monomial division. In this case, we denote such a reduction by $f \overset{g_{\preccurlyeq}}{\longrightarrow} h$, where

$$h = f - \frac{\lambda u}{\mathrm{lt}_{\preccurlyeq}(g)} g.$$

For a set G of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$, we define a rewriting system corresponding to the division modulo G by considering the relation reduction $\overset{G_{\preccurlyeq}}{\longrightarrow}$ defined by

$$\overset{G_{\preccurlyeq}}{\longrightarrow} = \bigcup_{g \in G} \overset{g_{\preccurlyeq}}{\longrightarrow} .$$

We will denote by $\overset{G_{\preccurlyeq}}{\longrightarrow}^*$ the reflexive and transitive closure of the relation $\overset{G_{\preccurlyeq}}{\longrightarrow}$.

6.1.5 Involutive Reduction. In a same way, we define a notion of reduction with respect to an involutive division $\mathcal{I}$ on $\mathcal{M}(x_1, \ldots, x_n)$. Let g be a polynomial in $\mathbb{K}[x_1, \ldots, x_n]$. A polynomial f in $\mathbb{K}[x_1, \ldots, x_n]$ is said to be *$\mathcal{I}$-reducible modulo g* with respect to the monomial order $\preccurlyeq$, if there is a term λu of f, with $\lambda \in \mathbb{K} - \{0\}$ and $u \in \mathcal{M}(x_1, \ldots, x_n)$, such that

$$u = \mathrm{lm}_{\preccurlyeq}(g)v \quad \text{and} \quad v \in \mathcal{M}(\mathrm{Mult}_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(G)}(g)).$$

Such an $\mathcal{I}$-reduction is denoted by $f \underset{\mathcal{I}}{\overset{g_{\preccurlyeq}}{\longrightarrow}} h$, where

$$h = f - \frac{\lambda}{\mathrm{lc}_{\preccurlyeq}(g)} g v = f - \frac{\lambda u}{\mathrm{lt}_{\preccurlyeq}(g)} g.$$

6.1.6 Involutive Normal Forms. Let G be a set of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$. A polynomial f is said to be *$\mathcal{I}$-reducible modulo G* with respect to the monomial order $\preccurlyeq$, if there exists a polynomial g in G such that f is $\mathcal{I}$-reducible modulo g. We will denote by $\underset{\mathcal{I}}{\overset{G_{\preccurlyeq}}{\longrightarrow}}$ this reduction relation defined by

$$\underset{\mathcal{I}}{\overset{G_{\preccurlyeq}}{\longrightarrow}} = \bigcup_{g \in G} \underset{\mathcal{I}}{\overset{g_{\preccurlyeq}}{\longrightarrow}} .$$

The polynomial f is said to be in *$\mathcal{I}$-irreducible modulo G* if it is not $\mathcal{I}$-reducible modulo G. A *$\mathcal{I}$-normal form of a polynomial f* is an $\mathcal{I}$-irreducible polynomial h such that there is a sequence of reductions from f to h:

$$f \underset{\mathcal{I}}{\overset{G_{\preccurlyeq}}{\longrightarrow}} f_1 \underset{\mathcal{I}}{\overset{G_{\preccurlyeq}}{\longrightarrow}} f_2 \underset{\mathcal{I}}{\overset{G_{\preccurlyeq}}{\longrightarrow}} \cdots \underset{\mathcal{I}}{\overset{G_{\preccurlyeq}}{\longrightarrow}} h,$$

The procedure $\textbf{InvReduction}_{\mathcal{I},\preccurlyeq}(f, G)$ computes a normal form of f modulo G with respect to the division $\mathcal{I}$. The proofs of its correctness and termination can be carried out as in the case of the division procedure for the classical polynomial division, see for instance [3, Proposition 5.22].

Input: a polynomial f in $\mathbb{K}[x_1, \ldots, x_n]$ and a finite subset G of $\mathbb{K}[x_1, \ldots, x_n]$.

begin
  $h \leftarrow f$
  while *exist* g *in* G *and a term* t *of* h *such that* $\mathrm{lm}_{\preccurlyeq}(g)|_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(G)} \frac{t}{\mathrm{lc}_{\preccurlyeq}(t)}$ **do**
    choose such a g
    $h \leftarrow h - \frac{t}{<_{\preccurlyeq}(g)} g$
  end
end

Output: h a $\mathcal{I}$-normal form of the polynomial f with respect to the monomial order $\preccurlyeq$

Procedure 8: InvReduction$_{\mathcal{I},\preccurlyeq}(f, G)$

6.1.7 Remarks. Note that the involutive normal form of a polynomial f is not unique; in general, it depends on the order in which the reductions are applied. Suppose that for each polynomial f we have a $\mathcal{I}$-normal form with respect to the monomial order $\preccurlyeq$, denoted by $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f)$. Denote by $\mathrm{nf}^G_{\preccurlyeq}(f)$ a normal form of a polynomial f obtained by the classical division procedure. In general, the equality $\mathrm{nf}^G_{\preccurlyeq}(f) = \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f)$ does not hold. For example, let $G = \{x_1, x_2\}$ and consider the Thomas division $\mathcal{T}$ defined in Sect. 4.3.1. Then $\mathrm{nf}^G_{\preccurlyeq}(x_1x_2) = 0$, while $\mathrm{nf}^G_{\mathcal{T},\preccurlyeq}(x_1x_2) = x_1x_2$ because the monomial x_1x_2 is a $\mathcal{T}$-irreducible modulo G.

6.1.8 Autoreduction. Recall from Sect. 4.1.4 that a set of monomials $\mathcal{U}$ is $\mathcal{I}$-autoreduced with respect to an involutive division $\mathcal{I}$ if it does not contain a monomial $\mathcal{I}$-divisible by another monomial of $\mathcal{U}$. In that case, every monomial in $\mathcal{M}(x_1, \ldots, x_n)$ admits at most one $\mathcal{I}$-involutive divisor in $\mathcal{U}$.

A set G of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ is said to be $\mathcal{I}$*-autoreduced* with respect to the monomial order $\preccurlyeq$, if it satisfies the two following conditions:

(i) (*left* $\mathcal{I}$*-autoreducibility*) the set of leading monomials $\mathrm{lm}_{\preccurlyeq}(G)$ is $\mathcal{I}$-autoreduced,
(ii) (*right* $\mathcal{I}$*-autoreducibility*) for any g in G, there is no term $\lambda u \neq \mathrm{lt}_{\preccurlyeq}(g)$ of g, with $\lambda \neq 0$ and $u \in \mathrm{cone}_{\mathcal{I}}(\mathrm{lm}_{\preccurlyeq}(G))$.

Note that the condition **(i)**, (resp. **(ii)**) corresponds to the left-reducibility (resp. right-reducibility) property given in Sect. 5.5.2. Any finite set G of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ can be transformed by Procedure 9 into a finite $\mathcal{I}$-autoreduced set that generates the same ideal. The proofs of the correctness and termination are immediate consequences of the property of involutive division.

```
Input: G a finite subset of K[x_1, ..., x_n].
begin
    H ← G
    H' ← ∅
    while exist h ∈ H and g ∈ H \ {h} such that h is I-reducible modulo g with respect to ≼
    do
        choose such a h
        H' ← H \ {h}
        h' ← nf^{H'}_{I,≼}(h)
        if h' = 0 then
            H ← H'
        end
        else
            H ← H' ∪ {h'}
        end
    end
end
Output: H an I-autoreduced set generating the same ideal as G does.
```

Procedure 9: Autoreduce$_{\mathcal{I},\preccurlyeq}(G)$

6.1.9 Proposition **[27, Theorem 5.4]** *Let G be an $\mathcal{I}$-autoreduced set of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ and f be a polynomial in $\mathbb{K}[x_1, \ldots, x_n]$. Then $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) = 0$ if and only if the polynomial f can be written in the form*

$$f = \sum_{i,j} \beta_{i,j} g_i v_{i,j},$$

where $g_i \in G$, $\beta_{i,j} \in \mathbb{K}$ and $v_{i,j} \in \mathcal{M}(\mathrm{Mult}_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(G)}(\mathrm{lm}_{\preccurlyeq}(g_i)))$, with $\mathrm{lm}_{\preccurlyeq}(v_{i,j}) \neq \mathrm{lm}_{\preccurlyeq}(v_{i,k})$ if $j \neq k$.

Proof Suppose that $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) = 0$. Then there exists a sequence of involutive reductions modulo G,

$$f = f_0 \xrightarrow[\mathcal{I}]{g_1} f_1 \xrightarrow[\mathcal{I}]{g_2} f_2 \xrightarrow[\mathcal{I}]{g_3} \ldots \xrightarrow[\mathcal{I}]{g_{k-1}} f_k = 0,$$

terminating on 0. For any $1 \leqslant i \leqslant k$, we have

$$f_i = f_{i-1} - \frac{\lambda_{i,j}}{\mathrm{lc}_{\preccurlyeq}(g_i)} g_i v_{i,j},$$

with $v_{i,j}$ in $\mathcal{M}(\mathrm{Mult}_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(G)}(\mathrm{lm}_{\preccurlyeq}(g_i)))$. This shows the equality.

Conversely, suppose that f can be written in the indicated form. Then the leading monomial $\mathrm{lm}_{\preccurlyeq}(f)$ admits an involutive $\mathcal{I}$-divisor in $\mathrm{lm}_{\preccurlyeq}(G)$. Indeed, the leading monomial of the decomposition of f has the form

$$\mathrm{lm}_{\preccurlyeq}\left(\sum_{i,j} g_i v_{i,j}\right) = \mathrm{lm}_{\preccurlyeq}(g_{i_0}) v_{i_0,j_0}.$$

The monomial $\mathrm{lm}_{\preccurlyeq}(g_{i_0})$ is an involutive divisor of $\mathrm{lm}_{\preccurlyeq}(f)$, and by the autoreduction hypothesis, such a divisor is unique. Hence, the monomial $\mathrm{lm}_{\preccurlyeq}(g_{i_0})v_{i_0,j_0}$ does not divide other monomials of the form $\mathrm{lm}_{\preccurlyeq}(g_i)v_{i,j}$. We apply the reduction $g_{i_0}v_{i_0,j_0} \overset{g_{i_0}\preccurlyeq}{\underset{\mathcal{I}}{\longrightarrow}} 0$ to the decomposition. In this way, we define a sequence of reductions ending on 0. This proves that $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) = 0$. □

6.1.10 Uniqueness and Additivity of Involutive Normal Forms. From decomposition Proposition 6.1.9, we deduce two important properties of involutive normal forms. Let G be an $\mathcal{I}$-autoreduced set of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ and f be a polynomial. Suppose that $h_1 = \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f)$ and $h_2 = \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f)$ are two involutive normal forms of f. From the involutive reduction procedure that computes this two normal forms, we deduce two decompositions

$$h_1 = f - \sum_{i,j} \beta_{i,j} g_i v_{i,j}, \qquad h_2 = f - \sum_{i,j} \beta'_{i,j} g_i v'_{i,j}.$$

As a consequence, $h_1 - h_2$ admits a decomposition as in Proposition 6.1.9, hence $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(h_1 - h_2) = 0$. The polynomial $h_1 - h_2$ being in normal form, we deduce that $h_1 = h_2$. This shows the uniqueness of the involutive normal form modulo an autoreduced set of polynomials.

In a same manner, we prove the following additivity formula for any polynomial f and f':

$$\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f + f') = \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) + \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f').$$

6.2 Involutive Bases

Fix a monomial order $\preccurlyeq$ on $\mathcal{M}(x_1, \ldots, x_n)$.

6.2.1 Involutive Bases. Let I be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$. A subset G of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ is an $\mathcal{I}$-*involutive basis* of the ideal I with respect the monomial order $\preccurlyeq$, if G is $\mathcal{I}$-autoreduced and satisfies the following property:

$$\forall g \in G,\ \forall u \in \mathcal{M}(x_1, \ldots, x_n), \quad \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(gu) = 0.$$

In other words, for any polynomial g in G and any monomial u in $\mathcal{M}(x_1, \ldots, x_n)$, there is a sequence of involutive reductions:

$$gu \overset{g_1 \preccurlyeq}{\underset{\mathcal{I}}{\longrightarrow}} f_1 \overset{g_2 \preccurlyeq}{\underset{\mathcal{I}}{\longrightarrow}} f_2 \overset{g_3 \preccurlyeq}{\underset{\mathcal{I}}{\longrightarrow}} \cdots \overset{g_{k-1} \preccurlyeq}{\underset{\mathcal{I}}{\longrightarrow}} 0,$$

with g_i in G. In particular, we recover the notion of involutive sets of monomials given in Sect. 4.2.1. Indeed, if G is an $\mathcal{I}$-involutive basis, then $\mathrm{lm}_{\preccurlyeq}(G)$ is an $\mathcal{I}$-involutive set of monomials of $\mathcal{M}(x_1, \ldots, x_n)$.

6.2.2 Proposition *Let $\mathcal{I}$ be an involutive division on $\mathbb{K}[x_1, \ldots, x_n]$ and G be a $\mathcal{J}$-involutive subset of $\mathbb{K}[x_1, \ldots, x_n]$. A polynomial of $\mathbb{K}[x_1, \ldots, x_n]$ is reducible with respect to G if and only if it is $\mathcal{I}$-reducible modulo G.*

Proof Let f be a polynomial in $\mathbb{K}[x_1, \ldots, x_n]$. By the definition of the involutive reduction, if f is $\mathcal{I}$-reducible modulo G, then it is reducible for the relation $\overset{G \preccurlyeq}{\longrightarrow}$. Conversely, suppose that f is reducible by a polynomial g in G. That is, there exists a term λu in f, where λ is a nonzero scalar and u is a monomial in $\mathcal{M}(x_1, \ldots, x_n)$ such that $u = \mathrm{lm}_{\preccurlyeq}(g)v$, where $v \in \mathcal{M}(x_1, \ldots, x_n)$. The set G being involutive, we have $\mathrm{nf}^{G}_{\mathcal{I},\preccurlyeq}(gv) = 0$. By Proposition 6.1.9, the polynomial gv can written in the form

$$gv = \sum_{i,j} \beta_{i,j} g_i v_{i,j},$$

where $g_i \in G$, $\beta_{i,j} \in \mathbb{K}$, and $v_{i,j} \in \mathcal{M}(\mathrm{Mult}_{\mathcal{I}}^{\mathrm{lm}_{\preccurlyeq}(G)}(\mathrm{lm}_{\preccurlyeq}(g_i)))$. In particular, this shows that the monomial u admits an involutive divisor in G. □

6.2.3 Uniqueness of Normal Forms. Let us mention an important consequence of Proposition 6.2.2 given in [27, Theorem 7.1]. Let G be a $\mathcal{J}$-involutive subset of $\mathbb{K}[x_1, \ldots, x_n]$, for any reduction procedure that computes a normal form $\mathrm{nf}^{G}_{\preccurlyeq}(f)$ of a polynomial f in $\mathbb{K}[x_1, \ldots, x_n]$ and any involutive reduction procedure that computes an involutive normal form $\mathrm{nf}^{G}_{\mathcal{I},\preccurlyeq}(f)$, as a consequence of the uniqueness of the involutive normal form and Proposition 6.2.2, we have

$$\mathrm{nf}^{G}_{\preccurlyeq}(f) = \mathrm{nf}^{G}_{\mathcal{I},\preccurlyeq}(f).$$

6.2.4 Example. We set $\mathcal{U} = \{x_1, x_2\}$. We consider the deglex order induced by $x_2 > x_1$ and the Thomas division $\mathcal{T}$. The monomial x_1x_2 is $\mathcal{T}$-irreducible modulo $\mathcal{U}$. Hence, it does not admit zero as $\mathcal{T}$-normal form and the set $\mathcal{U}$ cannot be an $\mathcal{T}$-involutive basis of the ideal generated by $\mathcal{U}$. In turn, the set $\{x_1, x_2, x_1x_2\}$ is a $\mathcal{T}$-involutive basis of the ideal generated by $\mathcal{U}$.

We now consider the Janet division $\mathcal{J}$. We have $\deg_2(\mathcal{U}) = 1$, $[0] = \{x_1\}$ and $[1] = \{x_2\}$. The $\mathcal{J}$-multiplicative variables are given by the table

u	$\mathrm{Mult}^{\mathcal{U}}_{\mathcal{J}}(u)$	
x_1	x_1	
x_2	x_1	x_2

It follows that the monomial x_1x_2 is not $\mathcal{J}$-reducible by x_1 modulo $\mathcal{U}$. However, it is $\mathcal{J}$-reducible by x_2. We conclude that the set $\mathcal{U}$ forms a $\mathcal{J}$-involutive basis.

As an immediate consequence of involutive bases, the involutive reduction procedure provides a decision method of the ideal membership problem, as stated by the following result.

6.2.5 Proposition [27, Corollary 6.4] *Let I be an ideal of $\mathbb{K}[x_1, \ldots, x_n]$, and G be an $\mathcal{I}$-involutive basis of I with respect to a monomial order $\preccurlyeq$. For any polynomial f of $\mathbb{K}[x_1, \ldots, x_n]$, we have $f \in I$ if and only if $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) = 0$.*

Proof If $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) = 0$, then the polynomial f can be written in the form Proposition 6.1.9. This shows that f belongs to the ideal I. Conversely, suppose that f belongs to I. Then it can be decomposed in the form

$$f = \sum_i h_i g_i,$$

where $h_i = \sum_j \lambda_{i,j} u_{i,j} \in \mathbb{K}[x_1, \ldots, x_n]$. Since the set G is $\mathcal{I}$-involutive, we have $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(u_{i,j} g_i) = 0$, for any monomials $u_{i,j}$ and g_i in G. By the linearity of the operator $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(-)$, we see that $\mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(f) = 0$. □

6.2.6 Local Involutivity. Gerdt and Blinkov introduced in [27] the notion of local involutivity for a set of polynomials. A set G of polynomials in $\mathbb{K}[x_1, \ldots, x_n]$ is said to be *locally involutive* if the following condition holds:

$$\forall g \in G,\ \forall x \in \mathrm{NMult}_{\mathcal{I}}^{\mathrm{lm}_\preccurlyeq(G)}(\mathrm{lm}_\preccurlyeq(g)), \quad \mathrm{nf}^G_{\mathcal{I},\preccurlyeq}(gx) = 0.$$

For a continuous involutive division $\mathcal{I}$, they prove that an $\mathcal{I}$-autoreduced set of polynomials is involutive if and only if it is locally involutive [27, Theorem 6.5]. This local involutivity criterion is essential for computing the completion of a set of polynomials into an involutive basis. Note that this result is analogous to the critical pair lemma in rewriting theory stating that a rewriting system is locally confluent if and only if all its critical pairs are confluent, see, e.g., [36, 37]. Together with the Newman Lemma stating that for terminating rewriting, local confluence and confluence are equivalent properties, this gives a constructive method to prove confluence in a terminating rewriting system by analyzing the confluence of critical pairs.

6.2.7 Completion Procedure. For a given monomial order $\preccurlyeq$ on $\mathcal{M}(x_1, \ldots, x_n)$ and a continuous and constructive involutive division $\mathcal{I}$, as defined in [27, Definition 4.12], Procedure 10 computes an $\mathcal{I}$-involutive basis of an ideal from a set of generators of the ideal. We refer the reader to [27, Sect. 8] or [19, Sect. 4.4] for the correctness of this procedure and conditions for its termination. This procedure is in the same vein as the completion procedure for rewriting systems by Knuth and Bendix [53], and completion procedure for commutative polynomials by Buchberger [7].

```
Input: F a finite set of polynomials in 𝕂[x_1, ..., x_n].
begin
    F' ← Autoreduce_{𝓘,≼}(F)
    G ← ∅
    while G = ∅ do
        𝒫r ← {fx | f ∈ F', x ∈ NMult^{F'}_{𝓘,≼}(f)}
        p' ← 0
        while 𝒫r ≠ ∅ and p' = 0 do
            choose p in 𝒫r such that lm_≼(p) is minimal with respect to ≼.
            𝒫r ← 𝒫r \ {p}
            p' ← InvReduction_{𝓘,≼}(p, F')
        end
        if p' ≠ 0 then
            F' ← Autoreduce_{𝓘,≼}(F' ∪ {p'})
        end
        else
            G ← F'
        end
    end
end
Output: G an 𝓘-involutive basis of the ideal generated by F with respect to the monomial
        order ≼.
```

Procedure 10: InvolutiveCompletionBasis$_{\mathcal{I},\preccurlyeq}(F)$

6.2.8 Example: Computation of an Involutive Basis. Let I be the ideal of $\mathbb{Q}[x_1, x_2]$ generated by the set $F = \{f_1, f_2\}$, where the polynomial f_1 and f_2 are defined by

$$f_1 = x_2^2 - 2x_1x_2 + 1,$$
$$f_2 = x_1x_2 - 3x_1^2 - 1.$$

We compute an involutive basis of the ideal I with respect to the Janet division $\mathcal{J}$ and the deglex order induced by $x_2 > x_1$. We have $\mathrm{lm}(f_1) = x_2^2$ and $\mathrm{lm}(f_2) = x_1x_2$, hence the following $\mathcal{J}$-reductions

$$x_2^2 \xrightarrow[\mathcal{J}]{f_1} 2x_1x_2 - 1, \qquad x_1x_2 \xrightarrow[\mathcal{J}]{f_2} 3x_1^2 + 1.$$

The polynomial f_1 is $\mathcal{J}$-reducible by f_2, and we have

$$f_1 \xrightarrow[\mathcal{J}]{f_2} x_2^2 - 2(3x_1^2 + 1) + 1 = x_2^2 - 6x_1^2 - 1.$$

Thus, we set $f_3 = x_2^2 - 6x_1^2 - 1$ and we consider the reduction

$$x_2^2 \xrightarrow[\mathcal{J}]{f_3} 6x_1^2 + 1.$$

The set $F' = \{f_2, f_3\}$ is $\mathcal{J}$-autoreduced and generates the ideal I. Let us compute the multiplicative variables of the polynomials f_2 and f_3. We have $\deg_2(F') = \deg_2(\{x_2^2, x_1x_2\}) = 2, [1] = \{x_1x_2\}$ and $[2] = \{x_2^2\}$. Hence, the $\mathcal{J}$-multiplicative variables are given by the table

f	$\text{lm}(f)$	$\text{Mult}_{\mathcal{J}}^{F'}(f)$	
f_2	x_1x_2	x_1	
f_3	x_2^2	x_1	x_2

The polynomial $f_2x_2 = x_1x_2^2 - 3x_1^2x_2 - x_2$ is the only non-multiplicative prolongation to consider. This prolongation can be reduced as follows:

$$f_2x_2 \xrightarrow[\mathcal{J}]{f_3} 6x_1^3 + x_1 - 3x_1^2x_2 - x_2 \xrightarrow[\mathcal{J}]{f_2} -3x_1^3 - 2x_1 - x_2.$$

We set $f_4 = -3x_1^3 - 2x_1 - x_2$; the associated reduction of f_4 is

$$x_1^3 \xrightarrow[\mathcal{J}]{f_4} -\frac{2}{3}x_1 - \frac{1}{3}x_2,$$

and we set $F' = \{f_2, f_3, f_4\}$. We have $\deg_2(F') = 2$, $[0] = \{x_1^3\}$, $[1] = \{x_1x_2\}$ and $[2] = \{x_2^2\}$. Hence, the $\mathcal{J}$-multiplicative variables are given by the table

f	$\text{lm}(f)$	$\text{Mult}_{\mathcal{J}}^{F'}(f)$	
f_2	x_1x_2	x_1	
f_3	x_2^2	x_1	x_2
f_4	x_1^3	x_1	

There are two non-multiplicative prolongations to consider:

$$f_2x_2 = x_1x_2^2 - 3x_1^2x_2 - x_2, \qquad f_4x_2 = -3x_1^3x_2 - 2x_1x_2 - x_2^2.$$

We have $\text{lm}(f_2x_2) = x_1x_2^2 < \text{lm}(f_4x_2) = x_1^3x_2$. Hence, the prolongation f_2x_2 must be examined first. We have the following reductions:

$$f_2x_2 \xrightarrow[\mathcal{J}]{f_3} 6x_1^3 + x_1 - 3x_1^2x_2 - x_2 \xrightarrow[\mathcal{J}]{f_2} -3x_1^3 - 2x_1 - x_2 \xrightarrow[\mathcal{J}]{f_4} 0.$$

Hence, there is no polynomial to add. The other non-multiplicative prolongation is f_4x_2, which can be reduced to an $\mathcal{J}$-irreducible polynomial as follows:

$$f_4x_2 \xrightarrow[\mathcal{J}]{f_2} -3x_1^3x_2 - 6x_1^2 - x_2^2 - 2 \xrightarrow[\mathcal{J}]{f_3} -3x_1^3x_2 - 12x_1^2 - 3$$

$$\xrightarrow[\mathcal{J}]{f_2} -9x_1^4 - 15x_1^2 - 3 \xrightarrow[\mathcal{J}]{f_4} 3x_1x_2 - 9x_1^2 - 3 \xrightarrow[\mathcal{J}]{f_2} 0.$$

All the non-multiplicative prolongations are $\mathcal{J}$-reducible to 0; consequently, the set F' is a Janet basis of the ideal I.

6.3 Involutive Bases and Gröbner Bases

In this subsection, we recall the notion of Gröbner basis and we show that any involutive basis is a Gröbner basis. We fix a monomial order $\preccurlyeq$ on $\mathcal{M}(x_1, \dots, x_n)$.

6.3.1 Gröbner Bases. A subset G of $\mathbb{K}[x_1, \dots, x_n]$ is a *Gröbner basis* with respect to the monomial order $\preccurlyeq$ if it is finite and satisfies one of the following equivalent conditions:

(i) $\overset{G_\preccurlyeq}{\longrightarrow}$ is Church-Rosser,

(ii) $\overset{G_\preccurlyeq}{\longrightarrow}$ is confluent,

(iii) $\overset{G_\preccurlyeq}{\longrightarrow}$ is locally confluent,

(iv) $\overset{G_\preccurlyeq}{\longrightarrow}$ has unique normal forms,

(v) $f \overset{G_\preccurlyeq}{\longrightarrow}{}^* 0$, for all polynomial f in $\mathrm{Id}(G)$,

(vi) every polynomial f in $\mathrm{Id}(G) \setminus \{0\}$ is reducible modulo G,

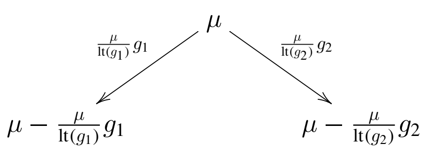
(vii) for any term t in $\mathrm{lt}_\preccurlyeq(\mathrm{Id}(G))$, there is g in G such that $\mathrm{lt}_\preccurlyeq(g)$ divides t,

(viii) $S_\preccurlyeq(g_1, g_2) \overset{G_\preccurlyeq}{\longrightarrow}{}^* 0$ for all g_1, g_2 in G, where

$$S_\preccurlyeq(g_1, g_2) = \frac{\mu}{\mathrm{lt}_\preccurlyeq(g_1)} g_1 - \frac{\mu}{\mathrm{lt}_\preccurlyeq(g_2)} g_2,$$

with $\mu = \mathrm{ppcm}(\mathrm{lm}_\preccurlyeq(g_1), \mathrm{lm}_\preccurlyeq(g_2))$, is the *S-polynomial* of g_1 and g_2 with respect to the monomial order $\preccurlyeq$,

(xi) any critical pair



with $\mu = \text{ppcm}(\text{lm}(g_1), \text{lm}(g_2))$, of the relation $\overset{G_{\preccurlyeq}}{\longrightarrow}$ is confluent.

We refer the reader to [3, Theorem 5.35] for proofs of these equivalences, see also [35, Section 3] [61]. The proofs of the equivalence of conditions **(i)–(iv)** are classical results for terminating rewriting systems. Note that condition **(viii)** corresponds to the Buchberger criterion [7] and condition **(ix)** is a formulation of this criterion in rewriting terms. We refer to [1, Chapter 8] for the rewriting interpretation of the Buchberger algorithm.

A *Gröbner basis of an ideal* I of $\mathbb{K}[x_1, \dots, x_n]$ with respect to a monomial order $\preccurlyeq$ is a Gröbner basis with respect to $\preccurlyeq$ that generates the ideal I. This can be also be formulated saying that G is a generating set for I such that $\text{Id}(\text{lt}(G)) = \text{Id}(\text{lt}(I))$.

6.3.2 Involutive Bases and Gröbner Bases. Let I be an ideal of $\mathbb{K}[x_1, \dots, x_n]$. Suppose that G is an involutive basis of the ideal I with respect to an involutive division $\mathcal{I}$ and the monomial order $\preccurlyeq$. In particular, the set G generates the ideal I. For every g_1 and g_2 in G, we consider the S-polynomial $S_{\preccurlyeq}(g_1, g_2)$ with respect to $\preccurlyeq$. By definition, the polynomial $S_{\preccurlyeq}(g_1, g_2)$ belongs to the ideal I. By the involutivity of the set G, it follows from Sect. 6.2.3 and Proposition 6.2.5 that we have

$$\text{nf}^G(S_{\preccurlyeq}(g_1, g_2)) = \text{nf}^G_{\mathcal{I}}(S_{\preccurlyeq}(g_1, g_2)) = 0.$$

In this way, G is a Gröbner basis of the ideal I by the Buchberger criterion **(viii)**. We have thus proved the following result due to V. P. Gerdt and Y. A. Blinkov.

6.3.3 Theorem [27, Corollary 7.2] *Let $\preccurlyeq$ be a monomial order on $\mathcal{M}(x_1, \dots, x_n)$ and $\mathcal{I}$ be an involutive division on $\mathbb{K}[x_1, \dots, x_n]$. Any $\mathcal{I}$-involutive basis of an ideal I of $\mathbb{K}[x_1, \dots, x_n]$ is a Gröbner basis of I.*

Since the involutive division used to define involutive bases is a refinement of the classical division with respect to which the Gröbner bases are defined, the converse of this result is false in general.

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Chapter 2
Gröbner Bases in D-Modules: Application to Bernstein-Sato Polynomials

Rouchdi Bahloul

In this chapter, we introduce Gröbner bases in a particular non-commutative ring and we show how they can be applied in a geometric context. In Sect. 2, we introduce the Weyl algebra and we present Gröbner bases in this ring. In Sect. 3, we define Bernstein-Sato polynomials and ideals, which are objects related to functional analysis and singularities. In Sect. 4, we present how Gröbner bases can be used for the computation of Bernstein-Sato ideals. In the final section, we present Bernstein-Sato polynomials from the point of view adopted in the next chapter.

1 Introduction

Let $\mathbb{K}$ be a field of characteristic 0 and $f, g \in \mathbb{K}[x]$ be two polynomials of one variable x. One can consider the euclidean division (or division by the degree) of f by g. This can be done as follows. Let us write $f = ax^d + f'$ and $g = bx^e + g'$ with $a, b \in \mathbb{K}^*$, $f', g' \in \mathbb{K}[x]$ and $\deg(f') < d$ and $\deg(g') < e$. Assume that $d \geq e$. Write

$$\begin{aligned} f &= \frac{a}{b}x^{d-e} \cdot bx^e + f' \\ &= \frac{a}{b}x^{d-e} \cdot (g - g') + f' \\ &= \frac{a}{b}x^{d-e} \cdot g + \underbrace{(f' - \frac{a}{b}x^{d-e} \cdot g')}_{f_1}. \end{aligned}$$

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Now, if $\deg f_1 \geq \deg g$ then we can restart the process with f_1. What we obtain is the existence of $Q, R \in \mathbb{K}[x]$ such that $f = Q \cdot g + R$ and $\deg(R) < \deg(g)$ or $R = 0$.

If we are given two polynomials $f, g \in \mathbb{K}[x, y]$ of two variables, we may have many ways to reduce f by g. For example if $f = x^4 + xy^3$ and $g = x^3 + xy$ then we may decide that the "leading" monomial of g is x^3 or xy.

$$f = x(g - xy) + xy^3 = xg - x^2y + xy^3 \quad \text{if the leading monomial is } x^3$$

or

$$f = x^4 + y^2(g - x^3) = y^2g + x^4 - x^3y^2 \quad \text{if the leading monomial is } xy.$$

Using the first reduction, the x-degree decreased but the y-degree and the total degree did not. With the second reduction, the y-degree decreased but the total degree increased. This shows that we can't find a canonical way to reduce and we loose uniqueness in the reduction process.

Moreover, $\mathbb{K}[x]$ is a principal domain but $\mathbb{K}[x, y]$ is not. Thus, given generators of an ideal I in $\mathbb{K}[x, y]$ and a polynomial $f \in \mathbb{K}[x, y]$, it is not trivial to decide whether f belongs to I or not. Gröbner bases theory is a nice tool for solving this problem. Given an ideal I in $\mathbb{K}[x_1, \ldots, x_n]$, a Gröbner basis is a special set of generators of I. Gröbner bases was initially defined in polynomial rings but the theory can be extended to some noncommutative algebras.

Now, let us introduce a noncommutative ring by defining the n-th Weyl algebra $\mathbf{A}_n(\mathbb{K})$ ($n \geq 1$ being a fixed integer). Let us consider $\mathrm{End}(\mathbb{K}[x])$ the set of $\mathbb{K}$-linear endomorphims of $\mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$. The Weyl algebra is the subalgebra of $\mathrm{End}(\mathbb{K}[x])$ generated by

$$S = \{\widehat{x_i} \mid i = 1, \ldots, n\} \cup \{\frac{\partial}{\partial x_i} \mid i = 1, \ldots, n\}$$

where $\widehat{x_i} : \mathbb{K}[x] \to \mathbb{K}[x],\ f(x) \mapsto x_i f(x)$ and $\frac{\partial}{\partial x_i}$ denotes the partial derivative with respect to x_i. One can easily see that $\mathbf{A}_n(\mathbb{K})$ is not commutative, indeed :

$$(\frac{\partial}{\partial x_i}\widehat{x_i})(f(x)) = \frac{\partial}{\partial x_i}(x_i f) = 1 \times f + x_i \times \frac{\partial f}{\partial x_i} = (\widehat{1} + \widehat{x_i}\frac{\partial}{\partial x_i})(f(x)).$$

Thus, in $\mathbf{A}_n(\mathbb{K})$: $\frac{\partial}{\partial x_i}\widehat{x_i} - \widehat{x_i}\frac{\partial}{\partial x_i} = \widehat{1}$. In this chapter, we shall deal with Gröbner bases in $\mathbf{A}_n(\mathbb{K})$ and show how we can use them to study Bernstein-Sato polynomials. Let us introduce them.

Let $f = f(x_1, \ldots, x_n)$ be a polynomial mapping on $\mathbb{R}^n$ and assume f to be positive. For $s \in \mathbb{C}$ such that the real part $\Re(s) > 0$ we can consider the distribution f^s: indeed for a test function φ (that is an infinitely differentiable function on $\mathbb{R}^n$ with a compact support), the integral

$$\langle f^s, \varphi \rangle = \int_{\mathbb{R}^n} (f(x))^s \varphi(x) dx$$

is well defined since $\Re(s) > 0$. Thus we have a map from $\{s \in \mathbb{C} \mid \Re(s) > 0\}$ to the set $\mathcal{D}'(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n$. One can show that this map is holomorphic in the sense that given a test function φ, the map $s \mapsto \langle f^s, \varphi \rangle \in \mathbb{R}$ is holomorphic on the set of the complex numbers s such that $\Re(s) > 0$. During the international congress of Mathematics in 1954, I. M. Gelfand asked if we can extend meromorphically this function to the whole plan $\mathbb{C}$. Two positive answers were given by Bernstein and Gelfand in [5] and by Atiyah in [1]. These two proofs used Hironaka's resolution of singularities. They proved that there is a meromorphic extension and the set of poles is contained in $\{-1/N, -2/N, \dots\}$ where N is some positive integer depending on f.

In 1972, Bernstein [4] gave another proof of this extension by proving the existence of a functional equation. This proof was purely algebraic and did not use the resolution of singularities. He proved that there exists a non-zero polynomial $b(s) \in \mathbb{R}[s]$ and an operator $P(s) \in \mathbf{A}_n(\mathbb{R})[s]$ such that

$$(\star) \qquad b(s) f^s = P(s) f^{s+1}.$$

Using this identity, one can prove that the poles of the extension are contained in the set

$$\{\lambda - k \mid \lambda \in \mathbb{C}, k \in \mathbb{N}, b(\lambda) = 0\}.$$

The set of $b(s)$ satisfying the relation above is an ideal. Its monic generator is called the (global) Bernstein-Sato polynomial of f. Kashiwara proved, using the resolution of the singularities, that the roots of the Bernstein-Sato polynomial are rational. This rationality allows one to recover the results of Bernstein, Gelfand, and Atiyah. There exists a local version of the Bernstein-Sato polynomial and it is directly related to the singularities of f (see [17]).

Now, when we are given several polynomials or functions $f_1, \dots, f_p$ then we can consider a functional equation which generalizes $(\star)$ and we get an ideal in $\mathbb{K}[s_1, \dots, s_p]$ called the Bernstein-Sato ideal. We shall see how Gröbner bases can be used to compute these ideals and how one can produce a stratification of space and obtain strata on which the local Bernstein-Sato ideal is constant.

2 Gröbner Bases and Rings of Differential Operators

In this section, we introduce Gröbner bases in the Weyl algebra. In the introduction we defined the Weyl Algebra as a subring of $\mathrm{End}(\mathbb{K}[x])$. Here we propose two other ways to define $\mathbf{A}_n(\mathbb{K})$.

2.1 Rings of Differential Operators

In this text n denotes a positive integer and $\mathbb{K}$ denotes a field of characteristic 0. In the sequel, let R be one of the following rings:

(i) $R = \mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$,
(ii) $R = \mathbb{K}[[x]] := \mathbb{K}[[x_1, \ldots, x_n]]$ (the formal power series ring),
(iii) $R = \mathbb{C}\{x\} = \mathbb{C}\{x_1, \ldots, x_n\}$ (the complex convergent power series ring).

Then we set

$$D(R) := R\langle \partial_1, \ldots, \partial_n \rangle$$

as the free module over R generated by the symbols ∂_i with the following commuting relation:

$$\partial_i \cdot a - a \cdot \partial_i = \frac{\partial a}{\partial x_i}$$

where $a \in R$ and $i \in \{1, \ldots, n\}$. When $R = \mathbb{K}[x]$, $D(R)$ is denoted $\mathbf{A}_n(\mathbb{K})$ and called the n-th Weyl algebra. Here is another definition for $\mathbf{A}_n(\mathbb{K})$:

$$\mathbf{A}_n(\mathbb{K}) = \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

with the following relations

$$\forall i, \quad \partial_i \cdot x_i - x_i \cdot \partial_i = 1 \quad ; \qquad \forall i \neq j, \quad \partial_i \cdot x_j - x_j \cdot \partial_i = 0.$$

We leave the next proposition as an exercise.

2.1.1 Proposition. *The ring R is a left $D(R)$-module if we set:*

$$\partial_i \bullet g(x) := \frac{\partial g(x)}{\partial x_i} \; ; \qquad a(x) \bullet g(x) := a(x) \cdot g(x)$$

where $a(x) \in R \subset D(R)$ and $g(x) \in R$.

Any non-zero element of $D(R)$ has a unique presentation of the form:

$$P = \sum_{\beta \in \mathbb{N}^n} a_\beta(x) \partial^\beta$$

where $\partial^\beta := \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$, $a_\beta(x) \in R$. The previous sum is finite.

Proof. The existence is easy an left to the reader. For unicity, we have to prove that if $P \in D(R)$ has two presentations: $P = \sum_{\beta \in \mathbb{N}^n} a_\beta(x) \partial^\beta$ and $P = \sum_{\beta \in \mathbb{N}^n} b_\beta(x) \partial^\beta$, then $a_\beta(x) = b_\beta(x)$ for any $\beta \in \mathbb{N}^n$. Thus we are reduced to prove that if P is written $P = \sum_{\beta \in \mathbb{N}^n} a_\beta(x) \partial^\beta$ and P is zero in $D(R)$ then $a_\beta(x) = 0$ for any β. Assume, by contradiction, that not all the $a_\beta(x)$ are zero. Take $\beta = (\beta_1, \ldots, \beta_n)$ such that $a_\beta \neq 0$

and $\beta_1 + \cdots + \beta_n$ is minimal. Then P applied to x^β is equal to $a_\beta(x) \cdot \beta_1! \cdots \beta_n!$ and it is not zero. $\square$

As a consequence, we have a formal presentation of the form:

$$P = \sum_{\alpha,\beta \in N^n} c_{\alpha,\beta} x^\alpha \partial^\beta$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $c_{\alpha,\beta}$ is an element of $\mathbb{K}$ or $\mathbb{C}$. This sum is not always finite if $R = \mathbb{K}[[x]]$ or $R = \mathbb{C}\{x\}$.

2.2 Orders

As we said, any element of $D(R)$ is a (finite or infinite) combination over $\mathbb{K}$ or $\mathbb{C}$ of elements of the form $x^\alpha \partial^\beta$. These elements are called monomials.
A monomial order on the monomials

$$x^\alpha \partial^\beta := x_1^{\alpha_1} \cdots x_n^{\alpha_n} \partial_1^{\beta_1} \cdots \partial_n^{\beta_n}$$

is an order $\preceq$ such that for any $\alpha, \beta, \alpha', \beta', \alpha'', \beta'' \in \mathbb{N}^n$,

$$x^\alpha \partial^\beta \preceq x^{\alpha'} \partial^{\beta'} \Rightarrow x^{\alpha+\alpha''} \partial^{\beta+\beta''} \preceq x^{\alpha'+\alpha''} \partial^{\beta'+\beta''}.$$

The order $\preceq$ is called admissible if:

$$\forall i = 1, \ldots, n, \quad x_i \partial_i \succ 1.$$

In the next result, we introduce new commutative variables: $\xi_1, \ldots, \xi_n$.

2.2.1 Lemma. (Dickson's lemma). *Let M be a family of monomials of $\mathbb{K}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n]$. Then there exists a finite subset F of M such that $M \subset \langle F \rangle$.*

This is by noetherianity of $\mathbb{K}[x, \xi]$.

2.2.2 Lemma. *Let $\preceq$ be a monomial order on the monomials x^α, $\alpha \in \mathbb{N}^n$. The following facts are equivalent.*

1. *For any i, $x_i \succ 1$.*
2. *Any non-empty set $M \subset \mathbb{K}[x]$ of monomials admits a minimum.*
3. *Any decreasing sequence of monomials stops.*

Proof. Assume that assertion 1 holds. Let M be a set of monomial. By Dickson's lemma, there exists a finite set F in M such that $M \subset \langle F \rangle$. The set F being finite, let m_0 be its minimum. For any $m \in M$, there is some $f \in F$ and a monomial m' such that $m = m' f$. Since $m_0 \preceq f$, $m' m_0 \preceq m' f = m$ but $m' \succeq 1$ then $m' m_0 \succeq m_0$.

Finally $m \succeq m_0$. This proves assertion 2.
Assume that assertion 2 holds. Suppose that assertion 3 is not true then we obtain a set with no minimum which contradicts 2. Thus assertion 3 is proven.
Assume that assertion 3 holds. By contradiction, suppose that there is some i for which $\alpha_i \prec 1$. Then multiplying by α_i, we get $\alpha_i^2 \prec \alpha_i \prec 1$. This shows that we can construct a decreasing sequence which never stops and this is a contradiction with 3, thus we proved assertion 1. □

2.2.3 Definition. An admissible order that satisfies one of the equivalent statements of this lemma is called a well-order.

2.3 Gröbner Bases in $\mathbf{A}_n(\mathbb{K})$

Let us fix an admissible monomial order $\preceq$ on the monomials $x^\alpha \partial^\beta$ ($\alpha, \beta \in \mathbb{N}^n$). Let $\xi_1, \ldots, \xi_n$ be commutative indeterminates. Let P be non-zero element of $\mathbf{A}_n(\mathbb{K})$. It has a unique presentation as:

$$P = \sum_{\alpha,\beta \in \mathbb{N}^n} c_{\alpha,\beta} x^\alpha \partial^\beta.$$

Set $x^{\alpha_0}\partial^{\beta_0} = \max_{\preceq}\{x^\alpha \partial^\beta \mid c_{\alpha,\beta} \neq 0\}$.
We introduce the following objects:

(i) The leading monomial of P with respect to $\preceq$: $\mathrm{lm}_{\preceq}(P) := x^{\alpha_0}\partial^{\beta_0}$,
(ii) The leading coefficient of P with respect to $\preceq$: $\mathrm{lc}_{\preceq}(P) := c_{\alpha_0,\beta_0}$,
(iii) The leading term of P with respect to $\preceq$: $\mathrm{lt}_{\preceq}(P) := \mathrm{lc}_{\preceq}(P)\mathrm{lm}_{\preceq}(P)$,
(iv) The initial term of P with respect to $\preceq$: $\mathrm{in}_{\preceq}(P) := c_{\alpha_0,\beta_0} x^{\alpha_0}\xi^{\beta_0}$.

In the sequel, when no confusion is possible, we shall write $\mathrm{lm}(P)$ for $\mathrm{lm}_{\preceq}(P)$. We shall write in a same way $\mathrm{lc}(P)$, $\mathrm{lt}(P)$ and $\mathrm{in}(P)$. Let us remark that since $\preceq$ is an admissible monomial order, we have $\mathrm{in}(PQ) = \mathrm{in}(P)\mathrm{in}(Q)$ for any non-zero $P, Q \in \mathbf{A}_n(\mathbb{K})$.
Let I be a non-zero left ideal of $\mathbf{A}_n(\mathbb{K})$.

2.3.1 Definition. We define the following set in $\mathbb{K}[x, \xi]$

$$\mathrm{in}(I) := \{\mathrm{in}(P) \mid P \in I \setminus \{0\}\} \cup \{0\}.$$

This set is an ideal of $\mathbb{K}[x, \xi]$ called the initial ideal of I (with respect to $\preceq$).

We leave as an exercise the fact that $\mathrm{in}(I)$ is an ideal.

2.3.2 Definition. A Gröbner basis of I w.r.t. $\preceq$ is a finite set $G \subset I$ such that

$$\mathrm{in}(I) = \langle \{\mathrm{in}(g),\ g \in G\} \rangle.$$

By Dickson's lemma, a Gröbner basis always exists.
Now let us introduce the notion of reduction (or division process).

Reduction
Given $P \in \mathbf{A}_n(\mathbb{K})$, $G = \{P_1, \ldots, P_m\} \subset \mathbf{A}_n(\mathbb{K})$, consider

$$m = \mathrm{in}(P) = c_{\alpha,\beta} x^\alpha \xi^\beta \text{ and } m_i = \mathrm{in}(P_i) = c^i_{\alpha(i),\beta(i)} x^{\alpha(i)} \xi^{\beta(i)}, \text{ for } i = 1, \ldots, m.$$

If m is divisible by one of the m_i's then a m-reduction (here "m" stands for monomial) of P by G is

$$P' := P - \frac{c_{\alpha,\beta}}{c^i_{\alpha(i),\beta(i)}} x^{\alpha(i)-\alpha} \partial^{\beta(i)-\beta} \cdot P.$$

Such a reduction is not unique since we may have several possible P_i's for which m_i divides m. The key remark is that since $\preceq$ is admissible, we have:

$$\mathrm{lm}(P') \prec \mathrm{lm}(P).$$

If $\preceq$ is a well-order then any sequence of m-reduction stops. The result of such a sequence is called a reduction (or a normal form) of P by G with respect to $\preceq$.

2.3.3 Proposition. *Let $G = \{P_1, \ldots, P_m\}$ be a Gröbner basis of I. Assume $\preceq$ to be a well-order. For any $P \in \mathbf{A}_n(\mathbb{K})$, we have: $P \in I$ if and only if any reduction of P by G is* 0. *Furthermore, for $P \in I$, we have*

$$P := \sum_i Q_i P_i$$

where for any i, $Q_i \in \mathbf{A}_n(\mathbb{K})$ and $\mathrm{lm}(P) \succeq \mathrm{lm}(Q_i P_i)$ *if $Q_i \neq 0$.*
This is called a standard representation *of P with respect to G (and $\preceq$).*

In the sequel, we shall describe Buchberger's criterion and Buchberger's algorithm. For this purpose, we need to define the S-operator.

2.3.4 Definition. Let $P, P' \in \mathbf{A}_n(\mathbb{K})$. Set $m = \mathrm{in}(P) = c_{\alpha,\beta} x^\alpha \xi^\beta$ and $m' = \mathrm{in}(P')$ $= c'_{\alpha',\beta'} x^{\alpha'} \xi^{\beta'}$. The S-operator of P and P' is

$$S(P, P') = \frac{1}{c_{\alpha,\beta}} x^{A-\alpha} \partial^{B-\beta} \cdot P - \frac{1}{c'_{\alpha',\beta'}} x^{A-\alpha'} \partial^{B-\beta'} \cdot P'$$

where $A = (A_1, \ldots, A_n) \in \mathbb{N}^n$, $B = (B_1, \ldots, B_n) \in \mathbb{N}^n$ and $A_i = \max(\alpha_i, \alpha'_i)$ and $B_i = \max(\beta_i, \beta'_i)$ for any i.

2.3.5 Theorem (Buchberger's criterion). Suppose that $\preceq$ is a well-order. Let $G = \{P_1, \ldots, P_m\}$ be a set of generators of a left ideal I of $\mathbf{A}_n(\mathbb{K})$. Then G is a $\preceq$-Gröbner basis of I if and only if for any $(i, j) \in \{1, \ldots, m\}$, the reduction of $S(P_i, P_j)$ by G is 0.

For the proof and other developments, see [12, 13, 15]. Now, we are able to describe Buchberger's algorithm.

2.3.6 Buchberger's Algorithm. Let $G = \{P_1, \ldots, P_m\}$ be a family of generators of a given left ideal I of $\mathbf{A}_n(\mathbb{K})$.

1. Set $G_0 = G$.
2. Assume $G_0, \ldots, G_k$ are constructed.
   For any $P, P' \in G_k$, consider a reduction $R(P, P')$ of $S(P, P')$ by G_k.
   Define $\mathcal{R}_k = \{R(P, P') \mid (P, P') \in G_k^2,\ R(P, P') \neq 0\}$.
3. If $\mathcal{R}_k$ is empty then the algorithm stops.
   If $\mathcal{R}_k$ is not empty then define $G_{k+1} = G_k \cup \mathcal{R}_k$ and return to step 2.

The algorithm always stops because the order $\preceq$ is a well-order. By Buchberger's criterion, the last constructed G_k is a Gröbner basis of I.

2.3.7 Remark. If $\preceq$ is not a well-order, there exists a homogenization process that enables the construction of a Gröbner basis, see e.g., [12].

2.3.8 Elimination of Variables. In this last paragraph, we give an example of the use of Gröbner bases, called elimination. We shall do an elimination of the variables $\partial_1, \ldots, \partial_n$ but this can be done for any family of variables. One has only to adapt the order. Fix any well-order $\preceq_0$. Define the following order $\preceq$ by:

$$x^\alpha \partial^\beta \prec x^{\alpha'} \partial^{\beta'} \iff \begin{cases} \sum_i \beta_i < \sum_i \beta'_i \\ \text{or } (\sum_i \beta_i = \sum_i \beta'_i \text{ and } x^\alpha \partial^\beta \prec_0 x^{\alpha'} \partial^{\beta'}). \end{cases}$$

2.3.9 Proposition. Let G be a Gröbner basis of a left ideal $I \subset \mathbf{A}_n(\mathbb{K})$ then the ideal $I \cap \mathbb{K}[x] \subset \mathbb{K}[x]$ is generated by $G \cap \mathbb{K}[x]$. In fact $G \cap \mathbb{K}[x]$ is a Gröbner basis of $I \cap \mathbb{K}[x]$ with respect to the restriction of the order $\preceq$ to the monomials of the form x^α.

Proof. Let $f \in I \cap \mathbb{K}[x]$. We have a standard representation

$$f = \sum_i Q_i P_i$$

where $G = \{P_1, \ldots, P_m\}$, $\mathrm{lm}(f) \succeq \mathrm{lm}(Q_i P_i)$ if $Q_i \neq 0$. For $Q_i \neq 0$, the definition of $\preceq$ implies that $Q_i, P_i \in \mathbb{K}[x]$. □

To go further: one can generalize the definition of Gröbner bases in $\mathbb{K}[[x]]$, $\mathbb{C}\{x\}$ and $D(R)$ where $R = \mathbb{K}[[x]]$ or $R = \mathbb{C}\{x\}$. We also have a notion of division (which is not a finite process) and Buchberger's criterion and algorithm still work. For all these generalizations, one can refer to [12]. The first paper where Gröbner bases in the Weyl Algebra appeared is [11]. In [7], the authors defined and used standard bases in $\mathbb{C}\{x\}[\partial_x]$ in the one dimensional case $n = 1$.

3 Bernstein-Sato Polynomials and Ideals

3.1 Introduction of the Bernstein-Sato Polynomial of One Function

First let us consider the polynomial case. Let us fix a non-zero polynomial $f \in \mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$. Introduce a new single variable s. Let us denote by $D = \mathbf{A}_n(\mathbb{K})$. Set $\mathcal{L} = \mathbb{K}[x, \frac{1}{f}, s] \cdot f^s$. This is the free module of rank 1 generated by the symbol f^s over $\mathbb{K}[x, \frac{1}{f}, s]$.
We have an action of $D[s]$ on $\mathcal{L}$ if we set:

- $u \bullet (g \cdot f^s) = (ug) \cdot f^s$,
- $\partial_i \bullet (g \cdot f^s) = (\frac{\partial g}{\partial x_i} + sg\frac{1}{f}) \cdot f^s$

where $u \in \mathbb{K}[x, s]$.

3.1.1 Theorem (see Bernstein [4]). There exists a non-zero polynomial $b(s) \in \mathbb{K}[s]$ and there exists $P(s) \in D[s]$ such that

$$b(s) f^s = P(s) f^{s+1}.$$

Here $P(s) f^{s+1}$ means $(P(s) f) \bullet f^s = P(s) \bullet (f \cdot f^s)$.
Such a polynomial is called a (global) Bernstein-Sato polynomial associated with f. The monic Bernstein-Sato polynomial of least degree is called the (global) Bernstein-Sato polynomial of f, or the (global) b-function of f. We shall denote it by: $b_{f,\mathrm{glob}}(s)$ or $b_f(s)$.

These objects can be generalized to other rings. We can take $f \in R = \mathbb{C}\{x\}$ or $f \in \mathbb{K}[[x]]$. Then $\mathcal{L} = R[\frac{1}{f}, s] \cdot f^s$ is again a $D(R)[s]$-module.
The existence of a Bernstein-Sato polynomial in these last cases is proven by Kashiwara [16] in the case where $f \in \mathbb{C}\{x\}$ and by Björk [6] when $f \in \mathbb{K}[[x]]$.

As we recalled it in the introduction, the existence of a (global) Bernstein-Sato polynomial is related to the question of Gelfand on meromorphic extensions but the case when $f \in \mathbb{C}\{x\}$ has been the most studied from a geometrical point of view. Here is a basic result on the link between $b_f(s)$ and the geometric properties of the germ of the analytic variety defined by f.

3.1.2 Proposition. *Take $f \in \mathbb{C}\{x\}$. The following assertions hold.*

(i) *f is unit of $\mathbb{C}\{x\}$ if and only if $b_f(s) = 1$.*
(ii) *If f is not a unit then: f is smooth if and only if $b_f(s) = s + 1$.*

Proof. **(i)** If f is invertible then $\frac{1}{f} \cdot f^{s+1} = 1 \cdot f^s$. Conversely, assume that $P(s)f^{s+1} = f^s$ for some $P(s) \in D(C\{x\})[s]$. Putting $s = -1$, we obtain: $P(-1) \cdot 1 = f^{-1}$. But $P(-1) \cdot 1 \in \mathbb{C}\{x\}$ then $f^{-1} \in \mathbb{C}\{x\}$.

(ii) We shall prove the left-right implication. For the converse, one can refer to [8]. Without loss of generality, we may assume that $\frac{\partial f}{\partial x_1}(0) \neq 0$; so $\frac{\partial f}{\partial x_1}$ is a unit. We have: $\frac{\partial}{\partial x_1} \cdot f^{s+1} = (s+1)\frac{\partial f}{\partial x_1} f^s$ then $(\frac{\partial f}{\partial x_1})^{-1} \frac{\partial}{\partial x_1} \cdot f^{s+1} = (s+1) f^s$.

In 1975, when $f \in \mathbb{C}\{x\}$ has an isolated singularity, Malgrange (see [17]) showed a strong link between the roots of $b_f(s)$ and the eigenvalues of the monodromy on the cohomology of the Milnor fiber.

3.2 Generalization to Several Functions

Here, $f = (f_1, \ldots, f_p) \in \mathbb{K}[x]^p = \mathbb{K}[x_1, \ldots, x_n]^p$ is given where $f_j \neq 0$ for $j = 1, \ldots, p$. We introduce the new indeterminates $s_1, \ldots, s_p$. Define $\mathcal{L} = \mathbb{K}[x, \frac{1}{f_1 \cdots f_p}, s_1, \ldots, s_p] \cdot f^s$ as the free module of rank 1. Here f^s has to be thought as $f_1^{s_1} \cdots f_p^{s_p}$. Put $D = \mathbf{A}_n(\mathbb{K})$, then $D[s] = D[s_1, \ldots, s_p]$ acts on $\mathcal{L}$ in the following way:

- $u \bullet (g \cdot f^s) := (ug) \cdot f^s$
- $\partial_i \bullet (g \cdot f^s) = (\frac{\partial g}{\partial x_i} + \sum_{j=1}^p s_j \frac{\partial f_j}{\partial x_i} f_j^{-1} g) \cdot f^s$

where $u \in \mathbb{K}[x, s]$.

Remark that $\partial_i(g \cdot f^s)$ is nothing but the partial derivation of the product $g \cdot f^s$. A Bernstein-Sato polynomial associated with $f = (f_1, \ldots, f_p)$ is a polynomial $b(s) \in \mathbb{K}[s]$ such that

$$b(s) f^s \in D[s] \cdot f^{s+1}.$$

Here $f^{s+1} := f_1 \cdots f_p \cdot f^s$.

Bernstein also proved that there exists a non-zero such polynomial. Then we can define the (global) Bernstein-Sato ideal as the ideal of $\mathbb{K}[s]$ of such polynomials $b(s)$. We will denote it by $\mathcal{B}_{\text{glob}}(f)$. This ideal is not principal in general (see [10] and also [3]). Now one can generalize these objects to $f_j \in \mathbb{C}\{x\}$ or $f_j \in \mathbb{K}[[x]]$. For $f_j \in \mathbb{C}\{x\}$, Sabbah (see [22, 23]) and Bahloul [2] proved the existence of non-zero Bernstein-Sato polynomials. The Bernstein-Sato ideal obtained here is denoted by $\mathcal{B}_{\text{an}}(f)$. For $f_j \in \mathbb{K}[[x]]$ the existence of Bernstein-Sato polynomials is still open. Nevertheless, the Bernstein-Sato ideal here is denoted by $\mathcal{B}_{\text{form}}(f)$.

3.2.1 Local Bernstein-Sato Ideal

We still consider $f = (f_1, \dots, f_p) \in \mathbb{K}[x]^p$. Take $a_0 \in \mathbb{K}^n$.
Let us consider the localized ring $R = \mathbb{K}[x]_{a_0} := \{\frac{u}{v} \mid u, v \in \mathbb{K}[x], v(a_0) \neq 0\}$. Set $D_{a_0} = D(R)$. We still can consider the following equation

$$b(s) f^s \in D_{a_0}[s] f^{s+1}.$$

The set of such polynomials $b(s)$ is an ideal called the local Bernstein-Sato ideal at a_0, and denoted by $\mathcal{B}_{a_0}(f)$. It is clear that $\mathcal{B}_{a_0}(f) \neq \{0\}$ because it contains $\mathcal{B}_{\text{glob}}(f)$. Here is a proposition on the link between all the different Bernstein-Sato ideals.

3.2.2 Proposition. *Take $f \in \mathbb{C}\{x\}^p$.*

- *Assume that $f \in \mathbb{C}[x]^p$. Take $a_0 = 0 \in \mathbb{K}^n$ then $\mathcal{B}_{a_0}(f) = \mathcal{B}_{\text{an}}(f)$.*
- *$\mathcal{B}_{\text{form}}(f) \neq \{0\}$ and we have $\mathcal{B}_{\text{form}}(f) = \mathcal{B}_{\text{an}}(f)$.*

For the proof, one can refer to [8].

3.2.3 Proposition. *Take $f \in \mathbb{K}[x]^p$ and assume that $\mathbb{K}$ is algebraically closed. We have*

$$\mathcal{B}_{\text{glob}}(f) = \bigcap_{a \in \mathbb{K}^n} \mathcal{B}_a(f).$$

Proof. The left-right inclusion is trivial, let us prove the converse one. Take $b(s) \in \cap_a \mathcal{B}_a(f)$. Then for any $a \in \mathbb{K}^n$, there exist $P_a \in D_a[s]$ such that $b(s) f^s = P_a f^{s+1}$. We may write $P_a = \frac{1}{h_a(x)} Q_a$ with $Q_a \in D[s]$ and $h(x) \in \mathbb{K}[x]$ such that $h(a) \neq 0$. So we have $h_a(x) b(s) f^s = Q_a f^{s+1}$.

Consider the ideal $H = \sum_{a \in \mathbb{K}^n} \mathbb{K}[x] h_a$. The zero set $V(H)$ is empty. So by Hilbert's Nullstellensatz: $\sqrt{H} = I(V(H)) = I(\emptyset) = \mathbb{K}[x]$ so $1 \in H$. Therefore there exist $a_1, \dots, a_d \in \mathbb{K}^n$ and $u_1, \dots, u_d \in \mathbb{K}[x]$ such that $1 = \sum_{i=1}^d u_i h_{a_i}$. As a consequence

$$b(s) f^s = \sum_1^d u_i h_{a_i} b(s) f^s = (\sum_1^d u_i Q_{a_i}) f^{s+1}$$

and $b(s)$ is a global Bernstein-Sato polynomial. □

For this result, one can refer to [8] and [19].

3.3 Stratification Results

We still have $f \in \mathbb{K}[x]^p$ but $\mathbb{K}$ is not assumed to be algebraically closed.

3.3.1 Lemma. *Take $\emptyset \neq A \subset \mathbb{K}^n$. There exists an open set U of $\mathbb{K}^n$ for the Zariski topology such that $U \cap A \neq \emptyset$ and the map $U \cap A \ni a \mapsto \mathcal{B}_a(f)$ is constant.*

Proof. Take $a_1 \in A$. If $b(s) \in \mathcal{B}_{a_1}(f)$ then we can write

$$h(x)b(s)f^s \in D[s]f^{s+1}$$

where $h(x) \in \mathbb{K}[x]$ satisfies $h(a_1) \neq 0$. Thus if we set $U_1 = \mathbb{K}^n \smallsetminus V(h)$ then for any $a \in U_1$, $\mathcal{B}_{a_1}(f) \subseteq \mathcal{B}_a(f)$. If for any $a \in U_1 \cap A$ we have $\mathcal{B}_{a_1}(f) = \mathcal{B}_a(f)$ then the proof is done. If not, take $a_2 \in U_1 \cap A$ such that $\mathcal{B}_{a_1}(f) \subsetneq \mathcal{B}_{a_2}(f)$. As before, there exists an open set U_2 of $\mathbb{K}^n$ such that for any $a \in U_2$, $\mathcal{B}_{a_2}(f) \subseteq \mathcal{B}_a(f)$. If for any $a \in U_2 \cap A$ we have $\mathcal{B}_{a_2}(f) = \mathcal{B}_a(f)$ then the proof is done. If not there is some $a_3 \in U_2 \cap A$ such that $\mathcal{B}_{a_2}(f) \subsetneq \mathcal{B}_{a_3}(f)$. Thus we have the following strict inclusions: $\mathcal{B}_{a_1}(f) \subsetneq \mathcal{B}_{a_2}(f) \subsetneq \mathcal{B}_{a_3}(f)$. By noetherianity of $\mathbb{K}[s]$ this process stops. □

3.3.2 Proposition. **(i)** *The set* $\{\mathcal{B}_a(f) \mid a \in \mathbb{K}^n\}$ *is finite.*
(ii) *There exists a finite partition of* $\mathbb{K}^n$*:*

$$\mathbb{K}^n = \bigsqcup_j (U_j \smallsetminus V_j)$$

where U_j *and* V_j *are open sets of* $\mathbb{K}^n$*, such that the map* $a \mapsto \mathcal{B}_a(f)$ *is constant on each* $U_j \smallsetminus V_j$*.*

Notice that the constant maps associated with two different sets $U_j \smallsetminus V_j$ may be the same.

Proof. Assume by contradiction that the set in 1 is infinite. By using the previous lemma, let U_1 be an open set such that the map $a \mapsto \mathcal{B}_a(f)$ is constant on U_1. Let $Z_1 := \mathbb{K}^n \smallsetminus U_1$. We again apply the lemma to Z_1 and we obtain an open set U_2 such that the map $a \mapsto \mathcal{B}_a(f)$ is constant on $U_2 \cap Z_1$.

Set $Z_2 = Z_1 \smallsetminus (U_2 \cap Z_1)$. We have $Z_2 \subsetneq Z_1$. (Indeed, if $Z_2 = Z_1$ then we have a finite number of $\mathcal{B}_a(f)$'s). We continue the construction and we get a strictly decreasing sequence of Zariski closed sets and this is impossible. So statement 1 is proven. By the previous construction, we have closed sets:

$$\emptyset = Z_{j_0+1} \subsetneq Z_{j_0} \subsetneq \cdots \subsetneq Z_1 \subsetneq \mathbb{K}^n$$

and open sets U_j's such that $Z_{j+1} = Z_j \smallsetminus (U_{j+1} \cap Z_j)$ and the map $a \mapsto \mathcal{B}_a(f)$ is constant on each $U_{j+1} \cap Z_j$. Finally

$$\mathbb{K}^n = (\bigsqcup_{j=1}^{j_0} U_{j+1} \cap Z_j) \bigsqcup U_1$$

and on each stratum the map $a \mapsto \mathcal{B}_a(f)$ is constant. □

4 Computation of Bernstein-Sato Ideals by Using Gröbner Bases

In this section, we shall recall the Malgrange point of view for defining Bernstein-Sato polynomials. This will be used to give an algorithm for computing Bernstein-Sato ideals following an idea of Briançon and Maisonobe (see [9]). This section is based on the joint article with Oaku [3].

Let us recall that a polynomial map $f = (f_1, \ldots, f_p) \in \mathbb{K}[x]^p$ is given. We still denote $\mathbf{A}_n(\mathbb{K})$ by D.

4.1 Malgrange Point of View

Let $t_1, \ldots, t_p$ be new variables. Consider the Weyl algebra $\mathbf{A}_{n+p}(\mathbb{K})$ with variables $x_1, \ldots, x_n, t_1, \ldots, t_p, \partial_1, \ldots, \partial_n, \partial_{t_1}, \ldots, \partial_{t_p}$. We shall denote it $D\langle t, \partial_t\rangle$.
The ring $D\langle t, \partial_t\rangle$ act on $\mathcal{L} = \mathbb{K}[x, \frac{1}{f_1\cdots f_p}, s]f^s$ as follows:

- $t_j \bullet (g(s)f^s) = g(s_1, \ldots, s_{j-1}, s_j + 1, s_{j+1}, \ldots, s_p)f_j f^s$
- $\partial_{t_j} \bullet (g(s)f^s) = -s_j g(s_1, \ldots, s_{j-1}, s_j - 1, s_{j+1}, \ldots, s_p)f_j^{-1} f^s$

where $g(s) \in \mathbb{K}[x, \frac{1}{f_1\cdots f_p}, s]$.

4.1.1 Remark. We have:

$$\begin{aligned} -\partial_{t_j} t_j \bullet (g(s)f^s) &= -\partial_{t_j} \bullet (g(s_1, \ldots, s_{j-1}, s_j + 1, s_{j+1}, \ldots, s_p)f_j f^s) \\ &= -(-s_j g(s)f^s) \\ &= s_j g(s)f^s. \end{aligned}$$

Thanks to this equality, we shall identify $D[s]$ with the subring $D[-\partial t, t] := D[-\partial_{t_1}t_1, \ldots, -\partial_{t_p}t_p]$ of $D\langle t, \partial_t\rangle$.

4.1.2 Lemma. *The left ideal I of $D\langle t, \partial_t\rangle$ (respectively of $D_a\langle t, \partial_t\rangle$) generated by*

$$t_j - f_j,\ j = 1, \ldots, p; \qquad \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}\partial_{t_j}, i = 1, \ldots, n$$

is the annihilating ideal of f^s.

Proof. We shall make the proof in the global case, the other case being similar. The inclusion $I \subset \mathrm{ann}_{D\langle t,\partial_t\rangle}(f^s)$ is easy an left to the reader. Conversely take $P \in D\langle t, \partial_t\rangle$ such that $P \cdot f^s = 0$. On can write it as

$$P = \sum c_{\alpha,\beta,\mu,\nu} x^\alpha \partial_t^\nu t^\mu \partial^\beta$$

with $\alpha, \beta \in \mathbb{N}^n$, $\mu, \nu \in \mathbb{N}^p$ and $c_{\alpha,\beta,\mu,\nu} \in \mathbb{K}$. Using the generators of I, we get $P = P_1 + P_2$ where $P_1 \in I$ and P_2 is written as $P_2 = \sum e_{\alpha,\nu} x^\alpha \partial_t^\nu$ with $e_{\alpha,\nu} \in \mathbb{K}$ and we have $P_2 \cdot f^s = 0$. But

$$\sum e_{\alpha,\nu} x^\alpha \partial_t^\nu \cdot f^s = \left(\sum e_{\alpha,\nu} (-1)^{\sum \nu_j} x^\alpha \prod_{j=1}^{p} s_j(s_j - 1) \cdots (s_j - (\nu_j - 1)) f_j^{-\nu_j} \right) f^s.$$

The right hand side is an element of $\mathcal{L}$ so all the $e_{\alpha,\nu}$'s are zero which proves that $P_2 = 0$ and $P \in I$. □

Now let us introduce some ideals. We fix some $a \in \mathbb{K}^n$.

- $I = \mathrm{ann}_{D\langle t, \partial_t \rangle}(f^s)$,
- $J = \mathrm{ann}_{D_a\langle t, \partial_t \rangle}(f^s)$,
- $I_1 = I \cap D[s] := I \cap D[-\partial_t t] = \mathrm{ann}_{D[s]}(f^s)$ (see the remark at the beginning of this paragraph),
- $J_1 = J \cap D_a[s] := J \cap D_a[-\partial_t t] = \mathrm{ann}_{D_a[s]}(f^s)$,
- $I_2 = (I_1 + D[s] f_1 \cdots f_p) \cap \mathbb{K}[x, s]$,
- $J_2 = (J_1 + D_a[s] f_1 \cdots f_p) \cap \mathbb{K}[x]_a[s]$,
- $I_3 = I_2 \cap \mathbb{K}[s]$,
- $J_3 = J_2 \cap \mathbb{K}[s]$.

4.1.3 Proposition. *$I_3 = \mathcal{B}_{\mathrm{glob}}(f)$ and $J_3 = \mathcal{B}_a(f)$.*

Proof. We make the proof only for I_3. It is the same proof for J_3. Take $b(s) \in \mathbb{K}[s]$. Suppose $b(s) \in \mathcal{B}_{\mathrm{glob}}(f^s)$. By definition, this is equivalent to having $b(s) f^s \in D[s] f_1 \cdots f_p f^s$ which means that there exists some $P(s) \in D[s]$ such that $b(s) - P(s) f_1 \cdots f_p \in \mathrm{ann}_{D[s]}(f^s) = I_1$. Since $b(s) \in \mathbb{K}[s]$, this is equivalent to $b(s) \in (I_1 + D[s] f_1 \cdots f_p) \cap \mathbb{K}[s] = I_3$. □

By Lemma 4.1.2, we have

$$J = D_a\langle t, \partial_t \rangle \cdot I.$$

We also have

4.1.4 Proposition. *$J_1 = D_a[s] \cdot I_1$ and $J_2 = \mathbb{K}[x]_a[s] \cdot I_2$.*

Proof. We prove the equality that concerns J_1 and I_1, the other equality can be proven in the same way. We have $I_1 \subset J_1$ then $D_a[s] \cdot I_1 \subset J_1$. Conversely take $P \in J_1$. We can write $P = \frac{1}{c} Q$ with $c \in \mathbb{K}[x]$, $c(a) \neq 0$, $Q \in D[s]$. By assumption $P f^s = 0$ thus $Q f^s = c P f^s = 0$, i.e. $Q \in I \cap D[s] = I_1$. Finally $P = \frac{1}{c} Q \in D_a[s] I_1$. □

At this step, we see that the ideals I and J (respectively I_1 and J_1; and I_2 and J_2) have the same generators. The next result shows how to obtain J_3.

4.1.5 Proposition. *Let $I_2 = \Gamma_1 \cap \cdots \cap \Gamma_r$ be a primary decomposition of I_2. Let $\sigma_a = \{i \in \{1, \ldots, r\} \mid a \in V(\Gamma_i \cap \mathbb{K}[x])\}$. If $\sigma_a = \emptyset$ then $J_3 = \mathbb{K}[s]$ otherwise $J_3 = (\bigcap_{i \in \sigma_a} \Gamma_i) \cap \mathbb{K}[s]$.*

We ended Sect. 2 by a stratification result from a theoretical point of view. The following result gives such a stratification more explicitly.

4.1.6 Corollary. *We keep the notations of Proposition 4.1.5. For any subset $\sigma \subseteq \{1, \ldots, r\}$, set*

$$W_\sigma = \begin{cases} \mathbb{K}^n \smallsetminus \bigcup_{i=1}^{r} V(\Gamma_i \cap \mathbb{K}[x]) \text{ if } \sigma = \emptyset, \\ \bigcap_{i=1}^{r} V(\Gamma_i \cap \mathbb{K}[x]) \text{ if } \sigma = \{1, \ldots, r\}, \\ \left(\bigcap_{i \in \sigma} V(\Gamma_i \cap \mathbb{K}[x]) \right) \smallsetminus \left(\bigcup_{i \notin \sigma} V(\Gamma_i \cap \mathbb{K}[x]) \right) \text{ otherwise.} \end{cases}$$

Then $\mathbb{K}^n = \bigsqcup_{\sigma} W_\sigma$ is a partition made of constructible sets such that the map $a \mapsto \mathcal{B}_a(f)$ is constant on each W_σ.

4.2 Algorithmic Point of View

In this subsection, we shall describe how we can obtain generators of the ideals I_j and J_j.

- Computing I_2 from I_1 and I_3 from I_2 is done by an elimination of (global) variables as explained at the end of paragraph 1.
- Computing J_3 is done by a primary decomposition of I_2. Primary decomposition can be done by Gröbner bases methods (see e.g., [15]).

It remains two problems:

1. How can we obtain I_1 from I?
2. How can we obtain intersection of ideals? (This is necessary for J_3.)

4.2.1 Intersection of Ideals. We shall describe it in the polynomial case but this also works in noncommutative rings. Let L_1, L_2 be two ideals of $\mathbb{K}[x] = \mathbb{K}[x_1, \ldots, x_n]$ given by finite sets of generators. Let z be a new variable then

$$L_1 \cap L_2 = \left(\mathbb{K}[x, z] z L_1 + \mathbb{K}[x, z](1 - z) L_2 \right) \cap \mathbb{K}[x].$$

Proof. Only the left-right inclusion is not trivial and needs to be proved. Let f be an element of the left hand side. Then we can write

$$f = \sum u_i(x, z) z q_i^1(x) + \sum v_i(x, z)(1 - z) q_i^2(x)$$

where $u_i, v_i \in \mathbb{K}[x, z], q_i^1 \in L_1, q_i^2 \in L_2$. Since $f \in \mathbb{K}[x]$, we have $f = f_{|z=1} \in L_1$ and $f = f_{|z=0} \in L_2$. □

Now let us discuss the last problem.

4.2.2 Computation of the Annihilator of f^s in $D[s]$. The first existing method for the computation of I_1 from I was given by Oaku and Takayama [21]. We shall use an easier method found by Briançon and Maisonobe [9].

Let us consider the subring $D\langle s, \partial_t \rangle := D\langle -\partial_t t, \partial_t \rangle$ of $D\langle t, \partial_t \rangle$. This subring can also be defined intrinsically as follows: $D\langle s, \partial_t \rangle$ is the ring

$$\mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, s_1, \ldots, s_p, \partial_{t_1}, \ldots, \partial_{t_p} \rangle$$

with the following non-trivial commutation relations

$$\partial_i x_i - x_i \partial_i = 1, \qquad s_j \partial_{t_j} - \partial_{t_j} s_j = \partial_{t_j}$$

We also define the localisation at a: $D_a\langle s, \partial_t \rangle$.

4.2.3 Proposition. The annihilating ideal I' of f^s in $D\langle s, \partial_t \rangle$ (respectively in $D_a\langle s, \partial_t \rangle$) is the left ideal generated by

$$s_j + f_j \partial_{t_j},\ j = 1, \ldots, p; \qquad \partial_i + \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \partial_{t_j}, i = 1, \ldots, n.$$

The proof is the same as that of Lemma 4.1.2. Thanks to that we can answer to the last problem. Indeed we have

$$I_1 = I' \cap D[s].$$

As a consequence, in order to compute generators of I_1, we can use the elimination of the variables $\partial_{t_1}, \ldots, \partial_{t_p}$. In Sect. 2, we introduced Gröbner bases in $D = \mathbf{A}_n(\mathbb{K})$. Here we need Gröbner bases in $D\langle s, \partial_t \rangle$. The only difference will be in the notion of admissible orders. Here an admissible order shall satisfy

$$\forall i,\ x_i \partial_i \succ 1 \quad \text{and} \quad \forall j,\ s_j \partial_{t_j} \succ \partial_{t_j}.$$

It is well-known that the computation of Gröbner bases has a double-exponential complexity with respect to the number of variables (see e.g., Dubé [14]) and this upper bound is optimal (see Mayr and Meyer [18]) but usually not reached.

5 b-Function and V-Filtration

In this last section, we want to present Bernstein-Sato polynomials (or b-function) from another point of view.
In this volume (see [20]), Nakayama and Takayama introduced the b-function in a particular situation. In this paragraph, we shall make the link between the Bernstein-Sato polynomial that we introduced in the present chapter and the b-function in [20]. Let us recall the definition of the b-function as in [20]. Let g be an operator in $\mathbf{A}_1(\mathbb{K})$ (we denote by x and ∂ the variables of this ring). Take $w = 1$. Write $g = \sum_{k,l\in\mathbb{N}} c_{k,l} x^k \partial^l$ and define

$$d := \mathrm{ord}_{(-w,w)}(g) = \max\{-k + l \mid c_{k,l} \neq 0\}.$$

If $d \geq 0$ then set $Q = x^d g$, otherwise set $Q = \partial^d g$. In any case $\mathrm{ord}_{(-w,w)}(Q) = 0$.

There exists a polynomial $b(s) \in \mathbb{K}[s]$ such that $Q = b(x\partial) + Q'$ where $\mathrm{ord}_{(-w,w)}(Q') \leq -1$. The set of these polynomials $b(s)$ is an ideal of $\mathbb{K}[s]$. The monic generator of this ideal is called the b-function of g or the indicial polynomial.

Now, let us introduce the V-filtration in $\mathbf{A}_1(\mathbb{K})$. Set, for $k \in \mathbb{Z}$,

$$V_k(\mathbf{A}_1(\mathbb{K})) = \{P \in \mathbf{A}_1(\mathbb{K}) \mid \mathrm{ord}_{(-w,w)}(P) \leq k\}$$

It is a filtration. Indeed, we have:

- $V_{k-1}(\mathbf{A}_1(\mathbb{K})) \subseteq V_k(\mathbf{A}_1(\mathbb{K}))$ for any $k \in \mathbb{Z}$.
- $V_k(\mathbf{A}_1(\mathbb{K})) \cdot V_l(\mathbf{A}_1(\mathbb{K})) \subseteq V_{k+l}(\mathbf{A}_1(\mathbb{K}))$ for any $k, l \in \mathbb{Z}$.

The associated graded ring is

$$\mathrm{gr}_V(\mathbf{A}_1(\mathbb{K})) := \bigoplus_{k\in\mathbb{Z}} V_k(\mathbf{A}_1(\mathbb{K}))/V_{k-1}(\mathbf{A}_1(\mathbb{K}))$$

and $\mathrm{gr}_V(\mathbf{A}_1(\mathbb{K}))$ is isomorphic to $\mathbf{A}_1(\mathbb{K})$. Given $P \in \mathbf{A}_1(\mathbb{K})$ with $d = \mathrm{ord}_{(-w,w))}(P)$, we set $\sigma(P) = \sigma_d(P) \in \mathrm{gr}_V(\mathbf{A}_1(\mathbb{K}))$ as its image in $V_d(\mathbf{A}_1(\mathbb{K}))/V_{d-1}(\mathbf{A}_1(\mathbb{K}))$. For a left ideal $I \subset \mathbf{A}_1(\mathbb{K})$, we define $\mathrm{gr}_V(I)$ as the ideal of $\mathrm{gr}_V(\mathbf{A}_1(\mathbb{K}))$ generated by the $\sigma(P)$, with $P \in I$. Now, let us go back to the b-function associated with some $g \in \mathbf{A}_1(\mathbb{K})$. We have:

5.0.1 Proposition. *$b(s) \in \mathbb{K}[s]$ is a b-function of g if and only if $b(x\partial) \in \mathrm{gr}_V(\mathbf{A}_1(\mathbb{K}) \cdot g)$.*

Consequently, b_g is the monic generator of $\mathrm{gr}_V(\mathbf{A}_1(\mathbb{K}) \cdot g) \cap \mathbb{K}[x\partial]$ with the identification $s = x\partial$.

Now let us compare this b-function with the Bernstein-Sato polynomial of $f \in \mathbb{K}[x] := \mathbb{K}[x_1, \ldots, x_n]$. Let us introduce the Kashiwara-Malgrange V-filtration in $\mathbf{A}_{n+1}(\mathbb{K})$ (where t is the new variable). Take $P \in \mathbf{A}_{n+1}(\mathbb{K})$. It can be written as

$P = \sum c_{\alpha,\mu,\beta,\nu} x^\alpha t^\mu \partial^\beta \partial_t^\nu$ where $\alpha, \beta \in \mathbb{N}^n$ and $\mu, \nu \in \mathbb{N}$.
Define $\mathrm{ord}_V(P) = \max\{\nu - \mu \mid c_{\alpha,\mu,\beta,\nu} \neq 0\}$. Then the V-filtration is defined as before:

$$V_k(\mathbf{A}_{n+1}(\mathbb{K})) = \{P \in \mathbf{A}_{n+1}(\mathbb{K}) \mid \mathrm{ord}_V(P) \leq k\}.$$

In the same way, we define $\mathrm{gr}_V(\mathbf{A}_{n+1}(\mathbb{K}))$ and here again it is isomorphic to $\mathbf{A}_{n+1}(\mathbb{K})$. Moreover, we can define $\sigma(P)$ for $P \in \mathbf{A}_{n+1}(\mathbb{K})$, and $\mathrm{gr}_V(I)$ for a left ideal $I \subseteq \mathbf{A}_{n+1}(\mathbb{K})$. The next proposition is similar to Proposition 5.0.1.

5.0.2 Proposition. *Let I be the annihilator ideal of f^s in $\mathbf{A}_{n+1}(\mathbb{K})$ then*

$$\mathcal{B}_{\mathrm{glob}}(f) = \mathrm{gr}_V(I) \cap \mathbb{K}[-\partial_t t],$$

where we identify s with $-\partial_t t$.

Given an arbitrary ideal J in $\mathbf{A}_{n+1}(\mathbb{K})$, we may consider $\mathcal{B}(J) = \mathrm{gr}_V(J) \cap \mathbb{K}[-\partial_t t]$ and may ask whether this ideal is zero or not. When J is holonomic then $\mathcal{B}(J)$ is not zero, see e.g., the discussion in Chap. 5 of [24].

Proof of the proposition. Take $b(s) \in \mathbb{K}[s]$. Suppose that $b(-\partial_t t) \in \mathrm{gr}_V(I) \cap \mathbb{K}[-\partial_t t]$. By definition of the graded ideal, this means that $b(-\partial_t t) \in I + V_{-1}(\mathbf{A}_{n+1}(\mathbb{K}))$. Since

$$b(-\partial_t t) \in \mathbb{K}[-\partial_t t] \subset V_0(\mathbf{A}_{n+1}(\mathbb{K})),$$

this is equivalent to having $b(-\partial_t t) \in I + V_{-1}(\mathbf{A}_{n+1}(\mathbb{K}))$. Using the fact that $t - f \in I$, this is equivalent to $b(-\partial_t t) = Q + P(-\partial_t t)t$ where $Q \in I$ and $P \in \mathbf{A}_n(\mathbb{K})[s]$. Again, thanks to the fact that $t - f \in I$ this can be rewritten in the following form: $b(-\partial_t t) = Q' + P(-\partial_t t) f$ with $Q' \in I$ and the same $P \in \mathbf{A}_n(\mathbb{K})[s]$. This is equivalent to $b(-\partial_t t) \in I \cap \mathbf{A}_n(\mathbb{K})[-\partial_t t] + \mathbf{A}_n(\mathbb{K})[-\partial_t t] f$ and then to $b(s) f^s \in \mathbf{A}_n(\mathbb{K})[s] f^{s+1}$ by using the fact that s acts as $-\partial_t t$. Finally this means that $b(s) \in \mathcal{B}_{\mathrm{glob}}(f)$. □

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Chapter 3
Introduction to Algorithms for D-Modules with Quiver D-Modules

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The goal of this expository chapter is to illustrate how to use algorithmic methods for D-modules to make mathematical experiments for D-modules and cohomology groups with examples of quiver D-modules. The first section is based on a lecture by the second author given in the Kobe-Lyon summer school 2015 *On Quivers: Computational Aspects and Geometric Applications*. The second author could attend several interesting lectures of the school and the Sects. 2 and 3 are written by an inspiration from these lectures and the interesting paper by Khoroshkin and Varchenko [5].

1 Computation of Integration Functors in One Dimensional Case

We consider D-modules on the one-dimensional space $\mathbb{C}$ in this section. We illustrate an algorithm to compute the integration functor $\pi_* = \int_\pi$ where $\pi : \mathbb{C} \to \{\mathrm{pt}\}$ and de Rham cohomology groups in the case that the D-module is singly generated. Although the one-dimensional case is special, this case illustrates essential ideas of general algorithms for the n-dimensional case, see, e.g., [1].

Let $\mathbb{K}$ be the field of complex numbers. Let $f = \sum_{j=0}^{m} f_j x^j$ be a polynomial in one variable x. Define the w degree or order of f by $\max\{j \cdot w \mid f_j \neq 0\}$ for $w \in \mathbb{Z}_{>0}$. Put $m = \mathrm{ord}_w(f)$, $w = (1)$. We use the subscript w for ord to be consistent with the notation for n-variable case. We denote by $\mathbb{K}[x]_k$ the $\mathbb{K}$ vector space of the polynomials of which degree is less than or equal to k. Multiplying x^i to f, we have

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$$x^i f(x) = \sum_{j=0}^{m} f_j x^{j+i}.$$

Then, the polynomial f defines a $\mathbb{K}$-linear map (by the correspondence $e_i \Leftrightarrow x^i$) as

$$\mathbb{K}[x]_{k-m} \simeq \mathbb{K}^{k-m+1} \ni e_i \mapsto \sum_{j=0}^{m} f_j e_{j+i} \in \mathbb{K}^{k+1} \simeq \mathbb{K}[x]_k.$$

The matrix representation of this map is denoted by $M_k(f)$ and we call it the *Macaulay-type matrix of the degree* k. Here, we regard e_i as the row vector and we multiply $M_k(f)$ from the right.

1.1 Examples. $f(x) = x^2 + 1$.

$$e_0 \mapsto 1 \cdot (x^2+1) = e_2 + e_0, e_1 \mapsto x \cdot (x^2+1) = e_3 + e_1$$

$$M_3(f) = \begin{pmatrix} 1\,0\,1\,0 \\ 0\,1\,0\,1 \end{pmatrix} \quad \text{Columns are indexed by } e_0, e_1, \ldots$$

Macaulay-type matrices constructed from differential operators will give a presentation of cohomology groups. We denote by D the ring of differential operators $D = \mathbb{K}\langle x, \partial\rangle$. Generators x and ∂ satisfy the relation $\partial x = x\partial + 1$. Put $w = 1$. We define $(-w, w)$ degree or order by $\mathrm{ord}_{(-w,w)}(x^i\partial^j) = j - i$. The differential operator $f = \sum c_{ij} x^i \partial^j$ is called $(-w, w)$ *homogeneous* when $(j - i)$'s are the same value for all the terms in f.

1.2 Lemma. *If we multiply two elements f and g in D which are $(-w, w)$ homogeneous, fg is also $(-w, w)$ homogeneous.*

This lemma is well known and is used in several works on D-modules. Since this is an expository paper and this fact is important, we give a proof.

Proof. We have, by the Leibnitz formula,

$$\partial^p x^q = x^q \partial^p + pq x^{q-1}\partial^{p-1} + \frac{p(p-1)q(q-1)}{2!} x^{q-2}\partial^{p-2} + \cdots$$

Since $\mathrm{ord}_{(-w,w)}(x^{q-i}\partial^{p-i}) = (p-i) - (q-i) = p - q$, the lemma is shown in case that $f = \partial^p$, $g = x^q$ from the Leibnitz formula. General cases are reduced to this case. □

We denote by V_k the $\mathbb{K}$ vector space spanned by $x^i\partial^j$, $j - i \leq k$. In other words, V_k is the set of the elements of which $(-w, w)$ order is less than or equal to k. Note that $V_{-1} \subseteq xD$.

1.3 Lemma. Let $g \in D$ be a differential operator of which $(-w, w)$ order is k. For $f \in V_m \cap Dg$, there exists $q \in V_{m-k}$ such that

$$f = qg$$

Proof. Since $f \in Dg$ (a left ideal generated by g), there exists q' such that $f = q'g$. Suppose that $r = \mathrm{ord}_{(-w,w)}(q') > m - k$ and $q' = q_1' + q_2'$, where q_1' is $(-w, w)$ homogeneous with the order r and the order of q_2' is less than r. We decompose g in the same way as $g = g_1 + g_2$. Then, we have $q'g = q_1'g_1 + q_1'g_2 + q_2'g = f$. Since the top degree term is $q_1'g_1$ which is $(-w, w)$ homogeneous, we have $q_1'g_1 = 0$. Then $q_1' = 0$. It is a contradiction. □

1.4 Remark. Analogous lemma holds in the $n > 1$ variables case. Fix $w \in \mathbb{R}^n_{>0}$. If the set $\{g_1, \dots, g_m\}$ is a $(-w, w)$ Gröbner basis [12, Definition 1.1.3], which is also called a $(-w, w)$ involutive basis, of a left ideal I of the ring of differential operators of n variables and f be an element of $V_m \cap I$, then there exists $q_i \in V_{m-\mathrm{ord}_{(-w,w)}(g_i)}$ such that $f = \sum_{i=1}^m q_i g_i$. See [10, Theorem 10.6].

For an element f of D, the expression as $\sum c_{ij} x^i \partial^j$ (∂'s are collected to the right) is called the normally ordered expression and is denoted by $: f :$ For example, we have $: \partial x := x\partial + 1$.

Fix a natural number k. For $g \in D$ such that $\mathrm{ord}_{(-w,w)}(g) = j$, the operator g induces a $\mathbb{K}$-linear map

$$\mathbb{K}[\partial]_{k-j} \ni \partial^i \mapsto : \partial^i g : |_{x=0} \in \mathbb{K}[\partial]_k$$

The matrix representation of this map is called the *Macaulay-type matrix for restriction of degree* k and is denoted by $M_k(g)$.

1.5 Examples. $g = x\partial^2 + x\partial$, $\mathrm{ord}_{(-\mathrm{w},\mathrm{w})}(g) = j = 1$, $k = 2$.

$$1 \mapsto \ : x\partial^2 + x\partial : \ |_{x=0} = 0$$
$$\partial \mapsto \ : \partial g : \ |_{x=0} = x\partial^3 + \partial^2 + x\partial^2 + \partial\,|_{x=0} = \partial^2 + \partial$$

$$M_2(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Here, the $\mathbb{K}$ vector space $\mathbb{K}[\partial]_{k-1}$ is regarded as a set of row vectors.

The diagram

$$C^\bullet \ : \ 0 \overset{\varphi_{m+1}}{\to} \mathbb{K}^{b_m} \overset{\varphi_m}{\to} \mathbb{K}^{b_{m-1}} \overset{\varphi_{m-1}}{\to} \cdots \to \mathbb{K}^{b_1} \overset{\varphi_1}{\to} \mathbb{K}^{b_0} \overset{\varphi_0}{\to} 0$$

where φ_i's are $\mathbb{K}$-linear maps is called a *complex of vector spaces* when $\varphi_i \circ \varphi_{i+1} = 0$ holds. Define the $-i$th cohomology group of $C^\bullet$ as

$$H^{-i}(C^\bullet) = \frac{\mathrm{Ker}\,\varphi_i}{\mathrm{Im}\,\varphi_{i+1}}$$

which is a $\mathbb{K}$-vector space.

1.6 Exercise. Let A be a $b_2 \times b_1$ matrix and B a $b_1 \times b_0$ matrix with elements in $\mathbf{Q}$ such that $AB = 0$. Develop a program on a computer algebra system to find a basis of the $\mathbf{Q}$-vector space $\frac{\mathrm{Ker}\,B}{\mathrm{Im}\,A}$.

1.7 Examples. For $g = x\partial^2 + x\partial$, consider the Macaulay-type matrix for the restriction of degree 2. The cohomology groups of the complex

$$C^\bullet \,:\, 0 \to \mathbb{K}^2 \stackrel{M_2(g)}{\longrightarrow} \mathbb{K}^3 \to 0$$

are $H^0(C^\bullet) \simeq \mathbb{K}^2$, $H^{-1}(C^\bullet) \simeq \mathbb{K}$.

Let $p(x)$ be a rational function. The action $\bullet$ of D to p is defined by

$$x^i\partial^j \bullet p = x^i \frac{\partial^j p}{\partial x^j}$$

Let g be an element of D. Suppose that the highest $(-w, w)$- order terms in g is $\sum_{i=0}^m c_i x^i \partial^{r+i}$, $r \geq 0$, we define the indicial polynomial[1] of g by the polynomial in $\theta = x\partial$ in (1) below in the proof. Note that $\mathrm{ord}_{(-w,w)}(\theta) = 0$ and θ commutes with itself in D. When the highest $(-w, w)$-order terms in g is $\sum_{i=0}^m c_i x^{i+r'}\partial^i$, $r = -r' < 0$, we define the indicial polynomial of g by the righthand side polynomial in θ in

$$\partial^{r'} \sum_{i=0}^m c_i x^{i+r'}\partial^i = \sum_{i=0}^m c_i(\theta + r')(\theta + r' - 1)\cdots(\theta - i + 1).$$

1.8 Theorem. [8]. *Suppose that $g \neq 0$ is a given element of D. Define a complex*

$$G^\bullet : \qquad 0 \longrightarrow D/xD \stackrel{\cdot g}{\longrightarrow} D/xD \longrightarrow 0$$

of $\mathbb{K}$-vector spaces. Let k_0 be the maximal integral root of the indicial polynomial of g, as the ordinary differential operator at $x = 0$. If there does not exists a nonnegative integral root, the cohomology groups $H^{-i}(G^\bullet)$ are 0. For $k \geq k_0$, define a complex of $\mathbb{K}$-vector spaces by

$$C^\bullet \,:\, 0 \to \mathbb{K}^{k+1-\mathrm{ord}_{(-w,w)}(g)} \stackrel{M_k(g)}{\longrightarrow} \mathbb{K}^{k+1} \to 0$$

[1] or the characteristic polynomial. It is sometimes called the b-function $b(\theta)$ in the theory of D-modules.

Then, we have $H^{-i}(G^\bullet) = H^{-i}(C^\bullet)$.[2]

This theorem is a special case of the fundamental theorem by Oaku in [8] for holonomic D-modules on n-dimensional space. The case of $n = 1$ can be easily proved as follows.

Proof of Theorem 1.8. We prove the case that an integer $k_0 \geq 0$ exists and $r = \mathrm{ord}_{(-w,w)}(g) \geq 0$. The complex $G^\bullet$ is rewritten as

$$G^\bullet \,:\, 0 \to D/xD \ni f \overset{\varphi}{\mapsto} :fg:\,|_{x=0} \in D/xD \simeq \mathbb{K}[\partial] \to 0$$

We will prove that $H^j(C^\bullet) \to H^j(G^\bullet)$ is an isomorphism. Consider the case H^0. Take a nonzero element $f = \partial^i + \sum_{j<i} c_j \partial^j$ of $\mathbb{K}[\partial]_k/\mathrm{Im}\, M_k(g)$ where $i \leq k$. If $f \in Dg + xD$ and $i \geq r = \mathrm{ord}_{(-w,w)}(g)$, then, by the Lemma 1.3 and Lemma 1.13 presented later, there exists $q \in V_{k-r}$, $u \in V_{k+1}$ such that $f - xu = qg$. Therefore, we have $f =: qg :|_{x=0} \in \mathrm{Im}\, M_k(g)$. It is a contradiction, then we have $f \notin Dg + xD$ and consequently it is not in $: Dg :|_{x=0}$. Since the nonzero element is sent to the nonzero element, the $\mathbb{K}$-linear map from $H^0(C^\bullet)$ to $H^0(G^\bullet)$ is injective.

Let $b(s)$ be the indicial polynomial for g. Suppose that $b(x\partial) \equiv x^r g \bmod V_{-1}$. Suppose $k' > k_0$. Applying $\partial^{k'}$ to the both sides, we have

$$b(k')\partial^{k'} + x(\cdots) \equiv \partial^{k'} x^r g \bmod V_{-1+k'}.$$

Then, we have

$$b(k')\partial^{k'} \equiv 0 \bmod V_{-1+k'} + Dg + xD.$$

It implies the surjectivity.

We have finished with the case H^0. Let us consider the case of H^{-1}.

We prove that the canonical $\mathbb{K}$-linear map from $\mathrm{Ker}\, M_k(g)$ to $\mathrm{Ker}\, \varphi$ is an isomorphism. Let $\sum_{i \leq k-r} c_i \partial^i \neq 0$ belong to the kernel of $M_k(g)$. In other words, we have $: (\sum c_i \partial^i) g :|_{x=0} = 0$. It implies that $\sum c_i \partial^i$ is a nonzero element of D/xD which belongs to the kernel of φ. Since the nonzero element is sent to the nonzero element, it is injective.

In order to prove the surjectivity, we suppose that the highest $(-w, w)$-order terms of g are $\sum_{i=0}^m c_i x^i \partial^{r+i}$. Applying ∂^j to this sum, we have

$$\partial^j g \equiv \sum_{i=0}^m c_i j(j-1)\cdots(j-i+1)\partial^{r+j} \quad \bmod V_{j+r-1} + xD$$

Suppose that $a\partial^j + \cdots$, $a \neq 0$ belongs to the kernel of φ. We have

[2] In other words, $C^\bullet$ is a subcomplex of $G^\bullet$ and they are quasi-isomorphic for $k \geq k_0$. Although we will give an elementary proof here, some part of the proof can be rewitten more cleanly by understanding in this way.

$$: (a\partial^j + \cdots)g : |_{x=0} = a(\sum_{i=0}^{m} c_i j(j-1)\cdots(j-i+1)))\partial^{r+j} + \cdots = 0.$$

On the other hand, the indicial polynomial of g is equal to

$$x^r \sum_{i=0}^{m} c_i x^i \partial^{r+i} = \sum_{i=0}^{m} c_i \theta(\theta-1)\cdots(\theta-r-i+1) \tag{1}$$

Note that we may assume that $c_0 \neq 0$ and $c_m \neq 0$. When $j > k-r$, by replacing θ by $j+r$ we have $(j+r)\cdots(j+1)(\sum_{i=0}^{m} c_i j(j-1)\cdots(j-i+1))) \neq 0$. Therefore, $a = 0$ and hence if $\partial^j + \cdots$ belongs to the kernel of φ, then $j \leq k-r$, which yields the surjectivity.

The case of $r < 0$ or of that there exists no integer $k_0 \geq 0$ can be shown analogously. □

Note that $H^0(G^\bullet) \simeq M/xM$ where $M = D/Dg$, which is called the (0th) *restriction module* of M. The $\mathbb{K}$-vector space $M/\partial M$ is called the (0th) *integration module* of M. The integration module can be constructed by the formal Fourier transform and the theorem. The formal Fourier transform of $x^i\partial^j \in D$ is defined by $(-\partial)^i x^j$. It can be extended on D.

The following Corollary can be obtained by the theorem and by specializing the Grothendieck–Deligne comparison theorem to the one variable case (see, e.g., [9]).

1.9 Corollary. *Let $p(x)$ be a square free polynomial and suppose that*

$$\mathrm{Ann}_D \frac{1}{p} = \{f \in D \mid f \bullet (1/p) = 0\}$$

is generated by $\hat{g} \in D$. Let g be the formal Fourier transform of $\hat{g}$. We suppose that g satisfies the Assumption of Theorem 1.8. Then, we have $H^i(\mathbb{C} \setminus V(p), \mathbb{C}) \simeq H^{i-1}(C^\bullet)$.

Proof. The corollary follows from the following standard arguments for D-modules. By the Grothendieck comparison theorem (the algebraic de Rham versus analytic de Rham), we have

$$H^i(\mathbb{C} \setminus V(p)) \simeq H^{i-1}(F^\bullet), \quad F^\bullet : 0 \to \mathbb{K}[x, \frac{1}{p}] \xrightarrow{d} \mathbb{K}[x, \frac{1}{p}]dx \to 0$$

Here, $\mathbb{K}[x, 1/p]$ is regarded as a left D-module by the action $\bullet$ of ∂ and d is the exterior differential. In other words, we have $\mathbb{K}[x, 1/p] \simeq D/D\hat{g} =: M$. This complex is written as

$$0 \to D \otimes_D M \overset{\mathrm{id}\otimes\partial\cdot}{\to} D \otimes_D M \to 0,$$

which also we denote by $F^\bullet$. Consider the complex of $\mathbb{K}$-vector spaces

$$0 \to D/\partial D \otimes_D D \xrightarrow{1\otimes\cdot\hat{g}} D/\partial D \otimes_D D \to 0,$$

which is denoted by $\hat{G}^\bullet$. The complex $F^\bullet$ is the vertical border complex and the complex $G^\bullet$ is the horizontal border complex of the following double complex.

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & D\otimes D & \xrightarrow{1\otimes\cdot\hat{g}} & D\otimes_D D & \xrightarrow{\mathrm{id}\otimes\mathrm{id}} & D\otimes M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial\cdot\otimes\mathrm{id}} & & \downarrow{\scriptstyle \partial\cdot\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\partial\cdot} & & \\
0 & \longrightarrow & D\otimes D & \xrightarrow{1\otimes\cdot\hat{g}} & D\otimes_D D & \xrightarrow{\mathrm{id}\otimes\mathrm{id}} & D\otimes M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}\otimes\mathrm{id}} & & \downarrow & & \\
0 & \longrightarrow & D/\partial D\otimes D & \xrightarrow{1\otimes\cdot\hat{g}} & D/\partial D\otimes_D D & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

Then, by the standard theorem of homological algebra we obtain $H^i(\hat{G}^\bullet) \simeq H^i(F^\bullet)$. From Theorem 1.8, we have $H^i(\hat{G}^\bullet) \simeq H^i(C^\bullet)$. □

1.10 Examples. When $p(x) = x(1-x)$, the formal Fourier transform of the generator of Ann $(1/p)$ is the g of our running Examples 1.5, 1.7. Since $xg = x^2\partial^2 + x^2\partial = \theta(\theta-1) + x\theta$, $\theta = x\partial$, the indicial polynomial is $s(s-1)$. Therefore, we have $k_0 = 1$.

1.11 Exercise. Compute $H^i(\mathbb{C} \setminus V(p))$ when $p(x) = x$ and $p(x) = x(x-1)(x-2)$.

All statements and algorithms in this section can be generalized to several variable cases. See [8–10], [12, Chap. 5]. Readers can try these algorithms in some computer algebra systems Macaulay 2, Singular, Risa/Asir [11]. The Chap. 7 of the book [3] contains introductory expositions to use these systems for computations in D-modules.

1.12 Exercise. Compute $H^i(\mathbb{C}^n \setminus V(p), \mathbb{C})$ for a polynomial p, e.g., $p = xy$, $n = 2$, by Risa/Asir. Is the result compatible with geometric conclusion? Hint: $S^1 \times S^1$ is the deformation retract of $\mathbb{C}^2 \setminus V(xy)$.

We close this section with a proof of the following technical lemma.

1.13 Lemma. *Let $k \geq 0$ be an integer which is larger than or equal to the maximal integral root of the indicial polynomial $b(s)$ of $g \in D$. For $f \in V_k \cap (Dg + xD)$, there exists a presentation of f as $f = xu + qg$, $u, q \in D$, such that* $\mathrm{ord}_{(-w,w)}(xu) \leq k$.

Proof. Among the presentations $f = xu + qg$, we take u such that $j = \mathrm{ord}_{(-w,w)}(u)$ is minimal. Suppose $j - 1 > k$. From the definition of $b(s)$, we have $ub(x\partial) \equiv 0 \bmod Dg \cap V_j + V_{j-1}$. It is written as

$$b(x\partial + j)u \equiv 0 \bmod Dg \cap V_j + V_{j-1}, \tag{2}$$

because we have the identity $(x\partial + j)^i x^p \partial^{p+j} = x^p \partial^{p+j}(x\partial)^i$ in D for any non-negative integers i, j, p. Exchanging x and ∂ in b, we have $b(\partial x + j - 1)u \equiv 0 \bmod Dg \cap V_j + V_{j-1}$. Since $j - 1 > k$, the highest order term in xu cancels with the highest order term in qg. Therefore, we have $xu \equiv 0 \bmod Dg \cap V_{j-1} + V_{j-2}$. Since $b(\partial x + j - 1)$ is of the form $b(j-1) + rx, r \in V_1$, we have $0 \equiv b(j-1)u + rxu \bmod Dg \cap V_j + V_{j-1}$ by (2). Note that $b(j-1) \neq 0$ and $rxu \equiv 0 \bmod Dg \cap V_j + V_{j-1}$. Then, $u \equiv 0 \bmod Dg \cap V_j + V_{j-1}$ which implies that we can take q and u such that $\mathrm{ord}_{(-w,w)}(u) < j$. It is a contradiction to the minimality of j. □

2 Quiver *D*-Modules

We present quiver D-modules introduced by Khoroshkin and Varchenko [5] by specializing to the two-dimensional case and illustrate how to apply algorithms presented in the previous section to make mathematical experiments for quiver D-modules.

Let $\{L_i\}$ be a set of linear polynomials in two variables. It defines the hyperplane arrangement $\mathcal{H}$. The 2-face of $\mathcal{H}$ is denoted by $\emptyset$. The 1-faces of $\mathcal{H}$ are $V(L_i)$'s. The 0-faces of $\mathcal{H}$ are the intersections of more than one $V(L_i)$'s. We express a face α of the arrangement by a set of indices. For example, $ij\ldots$ stands for the face defined by $L_i = L_j = \cdots = 0$, $L_k \neq 0$ for $k \notin \{i, j, \ldots\}$. The edge framing $\{\xi_{\alpha,\beta}, \omega_\alpha, f_{\alpha,\beta}\}$ is a set of differentials, differential forms, and polynomials attached to faces of $\mathcal{H}$ satisfying the conditions

$$i_{\xi_{\beta,\alpha}}(\omega_\beta) = \omega_\alpha, \quad \dim\beta = \dim\alpha + 1 \tag{3}$$

$$df_{\beta,\alpha} \wedge \omega_\beta = \omega_\alpha, \quad \dim\alpha = \dim\beta + 1 \tag{4}$$

where α, β are faces of the arrangement $\mathcal{H}$. The operator i_ξ is the Lie derivative which sends a p-form to a $p-1$ form with the rule $i_\xi c(x,y)df = c(x,y)\xi f$ and $i_\xi c(x,y)df \wedge dg = c(x,y)(\xi f \wedge dg - df \wedge \xi g)$.

The edge framing can be expressed explicitly as follows. We assume $L_i = a_i x + b_i y + c_i$. Define

$$\omega_\emptyset = dx \wedge dy$$

$$\omega_i = \frac{-b_i dx + a_i dy}{a_i^2 + b_i^2}$$

$$\omega_{ij\ldots} = 1$$

Here, following our notation for a face, we denote ω_α by ω_i for the 1-face α defined by $L_i = 0$ and denote ω_β by $\omega_{ij\ldots}$ for the 0-face β defined by $L_i = L_j = \cdots = 0$, $L_k \neq 0$ for $k \notin \{i, j, \ldots\}$. We define a polynomial $f_{\beta,\alpha}$ as

$$f_{\emptyset,i} = L_i$$
$$f_{i,ij\ldots} = \frac{-b_i x + a_i y}{a_i^2 + b_i^2} + c_{i,ij\ldots},$$

where $c_{i,ij\ldots}$ is chosen so that it vanishes at the point defined by $ij\ldots$.

Finally, we define a differential $\xi_{\beta,\alpha}$ as

$$\xi_{i,\emptyset} = \frac{a_i\partial_x + b_i\partial_y}{a_i^2 + b_i^2}$$
$$\xi_{ij\ldots,i} = -b_i\partial_x + a_i\partial_y$$

2.1 Examples. Put

$$L_1 = x, L_2 = y, L_3 = x + y - 1, L_4 = 2x + y - 1.$$

The (incidence) graph Γ associated to this arrangement is presented in Fig. 1.

$\omega_\emptyset$	ω_1	ω_2	ω_3	ω_4	ω_{12}	ω_{134}	ω_{23}	ω_{24}
$dx \wedge dy$	dy	$-dx$	$\frac{1}{2}(-dx + dy)$	$\frac{1}{5}(-dx + 2dy)$	1	1	1	1

$f_{\emptyset,1}$	$f_{\emptyset,2}$	$f_{\emptyset,3}$	$f_{\emptyset,4}$
x	y	$x + y - 1$	$2x + y - 1$

$f_{1,12}$	$f_{1,134}$	$f_{2,12}$	$f_{2,23}$	$f_{2,24}$
y	$y - 1$	$-x$	$-x + 1$	$-x + \frac{1}{2}$

$f_{3,134}$	$f_{3,23}$	$f_{4,24}$	$f_{4,134}$
$-\frac{x}{2} + \frac{y}{2} - \frac{1}{2}$	$-\frac{x}{2} + \frac{y}{2} + \frac{1}{2}$	$-\frac{x}{5} + \frac{2y}{5} + \frac{1}{10}$	$-\frac{x}{5} + \frac{2y}{5} - \frac{2}{5}$

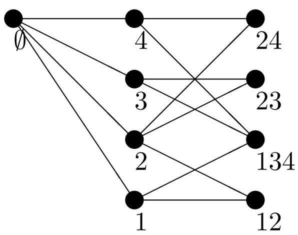


Fig. 1 Graph Γ

2.2 Examples. The Risa/Asir package `tk_edge2.rr`[3] constructs the edge framing in the two-dimensional case. The script

```
E=edge_frame2([x,y,x+y-1,2*x+y-1]);
```

outputs the table above as follows.

```
[[[[0,0],[0,1],[1,0],[1/2,0]],[x,y,x+y-1,2*x+y-1],[0], // 0-face, 1-face
  [[0,1],[0,2,3],[1,2],[1,3]]],                        // 0-fece by 1-face
 [[1,1,1,1],[dy,-dx,-1/2*dx+1/2*dy,-1/5*dx+2/5*dy],[dy*dx]], //omega
[[ y -x 0 0 ]
 [ y-1 0 -1/2*x+1/2*y-1/2 -1/5*x+2/5*y-2/5 ]
 [ 0 -x+1 -1/2*x+1/2*y+1/2 0 ]
 [ 0 -x+1/2 0 -1/5*x+2/5*y+1/10 ],  // f (0-face and 1-face)
[ x ]
[ y ]
[ x+y-1 ]
[ 2*x+y-1 ]],  // f (1-face and 2-face)
[[0,0,0,0],[dy,-dx,-dx+dy,-dx+2*dy],[dx,dy]], // T_0,T_1,T_2
[0,[dy,-dx,-dx+dy,-dx+2*dy],[dx,dy,1/2*dx+1/2*dy,2/5*dx+1/5*dy]]
   //T_i/T_point,           // T_empty/T_i
]
```

We regard the graph Γ as the bidirected graph; in other words, we regard the graph Γ as a quiver. We consider a quiver representation $(\{V_\alpha\}, \{A_{\beta\alpha}\})$ of the quiver satisfying the following condition [5].[4] Here, V_α is a vector space associated to the face α and $A_{\beta\alpha}$ is a linear map $A_{\beta\alpha} : V_\alpha \to V_\beta$.

2.3 Condition. 1. $\sum_\beta A_{\alpha\beta}A_{\beta\gamma} = 0$ for any α, γ such that the dimension of the face α and that of γ differ by 2.
2. $\sum_\beta A_{\alpha\beta}A_{\beta\gamma} = 0$ for any α and γ such that the dimension of these faces agree, $\alpha \neq \gamma$ and there exists δ such that δ is a face of the both of α and γ.

2.4 Examples. We consider the line arrangement defined by $L_1 = x = 0$ and $L_2 = y = 0$. The incidence graph is Fig. 2

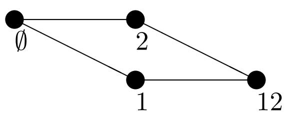
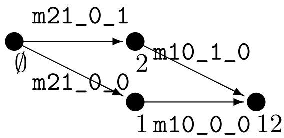


Fig. 2 Graph Γ (incidence graph) of $xy = 0$

[3]The latest version can be obtained by the command `asir_contrib_update(|update=1)`.

[4] Khoroshkin and Varchenko call the quiver representation (see, e.g., [4]) the quiver. We follow their terminology in the sequel.

Fig. 3 Representation for Γ



The condition 2.3 for this graph is

$$
\begin{aligned}
A_{\emptyset,1}A_{1,12} + A_{\emptyset,2}A_{2,12} &= 0,\\
A_{12,1}A_{1,\emptyset} + A_{12,2}A_{2,\emptyset} &= 0,\\
A_{2,12}A_{12,1} + A_{2,\emptyset}A_{\emptyset,1} &= 0,\\
A_{1,12}A_{12,2} + A_{1,\emptyset}A_{\emptyset,2} &= 0
\end{aligned}
$$

The package `tk_edge2.rr` provides some useful functions. These relations are automatically generated by the command `qrep_cond1()` and `qrep_cond2()`.

```
--> edge_frame2([x,y]);
[[[[0,0]],[x,y],[0],[[0,1]]],[[1],[dy,-dx],[dy*dx]],[[ y -x ],[ x ]
[ y ]],[[0],[dy,-dx],[dx,dy]],[0,[dy,-dx],[dx,dy]]]
// The result is stored in the global variable Edge_frame
--> genQuiver(1);  // We assume the dim of V_alpha's is 1.
[[[[ v_0_0_0 ]],
 [[ v_1_0_0 ],[ v_1_1_0 ]],
 [[ v_2_0_0 ]]],
[[ [ m01_0_0_0_0 ] [ m01_0_1_0_0 ] ],[ [ m12_0_0_0_0 ] ] [ [ m12_1_0_0_0 ] ]],
[[ [ m21_0_0_0_0 ] [ m21_0_1_0_0 ] ],[ [ m10_0_0_0_0 ] ] [ [ m10_1_0_0_0 ] ]]]
--> qrep_cond1();
[[m12_0_0_0_0*m01_0_0_0_0+m12_1_0_0_0*m01_0_1_0_0],
 [m10_0_0_0_0*m21_0_0_0_0+m10_1_0_0_0*m21_0_1_0_0]]
--> qrep_cond2();
[[m10_1_0_0_0*m01_0_0_0_0+m21_0_0_0_0*m12_1_0_0_0],
 [m10_0_0_0_0*m01_0_1_0_0+m21_0_1_0_0*m12_0_0_0_0]]
```

The symbol `mij_p_q` means that is is a linear map from the p-th i-dimensional face to the qth j-dimensional face, where p and q are numbered as $0, 1, 2, \ldots$. Note that the order of the index p, q is the reverse of the order of the index $A_{\alpha\beta}$, which is a linear map from V_β to V_α, and the index starts from 0, to follow the standard index convention of programming. See Fig. 3. The symbol `mij_p_q_s_t` stands for the (s, t)th element $(0 \le s, t)$ of the matrix for `mij_p_q`. This matrix acts from the right to the row vector of a basis of V_α for j dimensional face α. The k th element of the result is the image of the k th basis of V_β.

The level 0-quiver [5, 3.2] for a given hyperplane arrangement $\prod_{i=1}^{n} L_i = 0$ in the two-dimensional space is defined as follows. The graph Γ^0 consists of the one vertex $\emptyset$ and m-loops indexed by i standing for the hyperplane $L_i = 0$. Let us take the index i of a hyperplane $L_i = 0$ and a zero face γ in $L_i = 0$. We suppose that the linear map on the i-th loop $A^i_\emptyset : V_\emptyset \to V_\emptyset$ satisfies $[A^i_\emptyset, \sum_\delta A^\delta_\emptyset] = 0$ where $\delta \in \{1, \ldots, m\}$ runs over all δ such that $L_\delta = 0$ contains γ. The heart of [5] is that they construct an extension of this quiver for Γ^0 to a quiver on the incidence graph Γ, which they call the direct image [5, 3.4]. They prove that it is compatible with the direct image of D-modules [5, Theorem 4.6].

Let $D_2 = \mathbb{C}\langle x, y, \partial_x, \partial_y \rangle$ be the ring of differential operators in two variables with polynomial coefficient and $(\mathcal{D}_2^{alg})_a$ the ring of algebraic differential operators defined at $(x, y) = a$. In other words, it is $\mathcal{O}_a^{alg}\langle \partial_x, \partial_y \rangle$ where $\mathcal{O}_a^{alg} = \{f/g \mid f, g \in \mathbb{C}[x, y], g(a) \neq 0\}$. The quiver D-module for level 0 quiver [5, 4.4, (4.34)] is defined on $\mathbb{C}^2 \setminus V(\prod L_i)$. Let us define it when the vector space $V_\emptyset$ is one-dimensional space. We denote by $\mu_i \in \mathbb{C}$ the linear map $A^i_\emptyset$ for the loop i. Consider the left ideal I_a generated by

$$\partial_x - \sum_{i=1}^{m} \mu_i \frac{\frac{\partial L_i}{\partial x}}{L_i} \text{ and } \partial_y - \sum_{i=1}^{m} \mu_i \frac{\frac{\partial L_i}{\partial y}}{L_i} \tag{5}$$

in $(\mathcal{D}_2^{alg})_a$ where $a \in X^0 = \mathbb{C}^2 \setminus V(\prod L_i)$. The level 0-quiver D-module is the sheaf $\mathcal{D}_2^{alg}/I$ defined on X^0. Note that the function $\prod_{i=1}^m L_i^{\mu_i}$ spans the classical solution space of the D-module.

The quiver D-module for Γ obtained by taking the direct image of the level 0 quiver is defined on $\mathbb{C}^2$ and can be regarded as a left module D_2^p/Q over the Weyl algebra D_2. The generators for the D_2 module Q is given by (4.25) and (4.26) in [5, 4.2] in terms of the quiver $(\{V_\alpha\}, \{A_{\beta\alpha}\})$ and the edge framing of the arrangement. Our package `tk_edge2.rr` gives these generators by the function `eq_type1_i()` for (4.25) and the i-face α and `eq_type2_i()` for (4.26) and the i-face α. The function `qd2()` calls these two functions and merges the results. The command `map(ptomb,base_flatten(qd2()))` gives the generators of the submodule Q as we will see in Example 2.5.

2.5 Examples. Let μ_i, $(i = 1, 2, 3)$ be constants. We consider the direct image quiver on $\Gamma = \Gamma^2$ for the level 0-quiver on Γ^0 for the $L_1 L_2 L_3 = 0$, $L_1 = x$, $L_2 = y$, $L_3 = x + y - 1$ with $V_\emptyset = \mathbb{C}$ and $V^i_\emptyset = \mu_i$. The direct image quiver on $\Gamma = \Gamma^2$ is illustrated in the Fig. 4. Let V_i be the vector space over the 1-face i, which stands for $L_i = 0$ and V_{ij} the vector space over the 0-face ij, which stands for $L_i = L_j = 0$. It follows from the construction method of the direct image quiver [5, 3.2] that we have $V_i = \mathbb{C}$, $V_{ij} = \mathbb{C}$ and the linear maps $A_{i\emptyset}$ and $A_{\emptyset i}$ are μ_i and id respectively, and the linear maps $A_{k\ell,i}$ and $A_{i,k\ell}$ are given by the Fig. 5.

The quiver D-module is computed as follows.

```
--> load("tk_edge2.rr")$
```

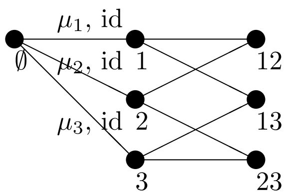


Fig. 4 A direct image quiver on the Graph $\Gamma = \Gamma^2$

Fig. 5 $A_{k\ell,i}$ and $A_{i,k\ell}$

$i, k\ell$	$i \to k\ell$, $(A_{k\ell,i})$	$i \leftarrow k\ell$, $(A_{i,k\ell})$
1,12	μ_2	1
1,13	μ_3	1
1,23	0	0
2,12	$-\mu_1$	-1
2,13	0	0
2,23	μ_3	1
3,12	0	0
3,13	$-\mu_1$	-1
3,23	$-\mu_2$	-1

```
      // We will omit this load command in the remaining examples.
--> edge_frame2([x,y,x+y-1])$
--> genQuiver([1,1,1])$
   //[dim V for 0-faces, dim V for 1-faces, dim V for 2-faces]
--> Eq=qd2();
[[om_2_0*v_2_0_0*dx-om_1_0*m21_0_0_0_0*v_1_0_0-om_1_2*m21_0_2_0_0*v_1_2_0,
 --- snip ]]
--> Eq2=map(ptomb,base_flatten(Eq));
[[dx,-m21_0_0_0_0,0,-m21_0_2_0_0,0,0,0],
[dy,0,-m21_0_1_0_0,-m21_0_2_0_0,0,0,0],
[0,dy,0,0,-m10_0_0_0_0,-m10_0_1_0_0,0],
[0,0,-dx,0,-m10_1_0_0_0,0,-m10_1_2_0_0],
[0,0,0,-dx+dy,0,-m10_2_1_0_0,-m10_2_2_0_0],
[-m12_0_0_0_0,x,0,0,0,0,0],
[-m12_1_0_0_0,0,y,0,0,0,0],
[-m12_2_0_0_0,0,0,x+y-1,0,0,0],
[0,0,m01_0_1_0_0,0,x,0,0],
[0,-m01_0_0_0_0,0,0,y,0,0],
[0,0,0,m01_1_2_0_0,0,x,0],
[0,-m01_1_0_0_0,0,0,0,x+y-1,0],
[0,0,0,-m01_2_2_0_0,0,0,y],
[0,0,m01_2_1_0_0,0,0,0,x+y-1]]
```

The last output gives the generators of Q of the quiver D-module $M = D_2^7/Q$. The symbol dx stands for ∂_x and the symbol dy stands for ∂_y. The integration module $M/(\partial_x M + \partial_y M)$ of the quiver D-module M will be calculated by the computer algebra system Risa/Asir (Example 3.2). The main theorem of [5, Theorem 4.6] claims that the integration as the D-module with respect to the variables x, y is the integration of the left D-module $D_2/\mathrm{Ann} L_1^{-\mu_1} L_2^{-\mu_2} L_3^{-\mu_3}$ (see, e.g., [9], [12, pp. 233–235]).

2.6 Examples. Let μ_i, $(i = 1, 2, 3)$ be constants. We start with the level 0-quiver on Γ^0 for the $L_1 L_2 L_3 = 0$, $L_1 = x$, $L_2 = y$, $L_3 = x - y$ with $V_\emptyset = \mathbb{C}$ and $A^i_\emptyset = \mu_i$. Even when we start with 1-dimensional $V_\emptyset$, the vector space V_α in the direct image quiver may be more than 1 dimensional space. The level 1-quiver [5, 3.2] is illustrated in Fig. 6. We denote by $\emptyset$ the 2-face, by i the 1-face $L_i = 0$ of the arrangement. We define $\dim V_\alpha = 1$ for all faces α, $A_{i\emptyset} = \mu_i$, which is a linear map from $V_\emptyset$ to V_i, $A_{\emptyset i} = \mathrm{id}$, and $A_i^{\{0\}} = \mu_j + \mu_k$ (i, j, k are different), which stands for the linear map

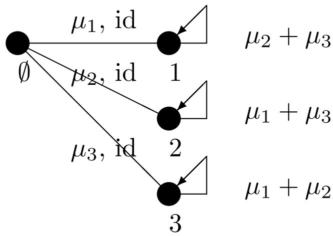


Fig. 6 A quiver on the graph $\Gamma^{(1)}$

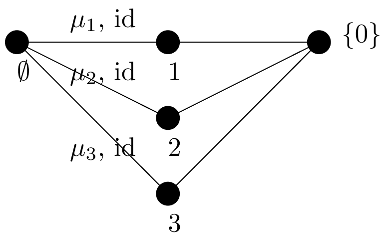


Fig. 7 A direct image quiver on the Graph $\Gamma = \Gamma^2$

for the loop at the vertex i. Note that $\emptyset \succ i$. This level 1 quiver is the direct image of the level 0 quiver defined by $(\{V_\emptyset\}, \{A^i_\emptyset\})$ such that $\dim V_\emptyset = 1$ and $A^i_\emptyset = \mu_i$. Consider the direct image of the level 1 quiver by following the construction of [5, 3.4]. The direct image quiver is illustrated in the Fig. 7. We denote by $\{0\}$ the unique 0-face of the arrangement. Let e_i be the basis of V_i. We define $V_{\{0\}}$ is the linear space spanned by $e_1 - e_2$ and $e_2 - e_3$ in the 3-dimensional space $\sum \mathbb{C}e_i$, and define a linear map from $V_{\{0\}}$ to V_i by $A_{i\{0\}}(\sum_{j=1}^3 c_j e_j) = c_i e_i$, and a linear map from V_i to $V_{\{0\}}$ by $A_{\{0\}i}(e_i) = (\mu_j + \mu_k)e_i - \mu_j e_j - \mu_k e_k$ where i, j, k are different. The quiver D-module for this quiver is generated by the output of `test_direct1()` in `tk_edge2.rr`. See Example 3.3.

3 Examples—*D*-Modules on Computer Algebra Systems

The goal of this section is to explain how to use computer algebra systems to make mathematical experiments on quiver D-modules. We use the style of Chap. 7 of the book "Dojo" [3], which is a collection of exercises and answers. In the book "Dojo", we illustrated computations of D-modules in computer algebra systems Macaulay2, Singular, and Risa/Asir. In this section, we only explain how to use Risa/Asir, for which we belong to the developing team.

3.1 Examples. Put $D_2 = \mathbb{C}[x, y]\langle\partial_x, \partial_y\rangle$, $D_1 = \mathbb{C}[y]\langle\partial_y\rangle$ and $f(x, y) = xy(x + y - 1)$. Set $I = \mathrm{Ann}_{D_2} f^\mu$ and $M = D_2/I$. Compute the 0-th integration modules $M/\partial_x M$ and $M/(\partial_x M + \partial_y M)$ when $\mu = -1/7$. Note that the 0-th integration module $M/(\partial_x M + \partial_y M)$ is $\pi^0_* M$ for $\pi : \mathbb{C}^2 \to \{0\}$.

The 0th integration module $M/\partial_x M$ can be obtained by computing the restriction module $\tilde{M}/x\tilde{M}$ for the Fourier transform $\tilde{M}$ of M by the algorithm of Oaku [8, Theorem 5.7.]. The b-function for modules can be computed by [10, Algorithm.4.6.]. Let N be a submodule of D_1 generated by $7y(1-y)\partial_y + 4y + 1$. We can see the integration module $M/\partial_x M$ is isomorphic to

$$M/\partial_x M \cong D_1/N$$

as a left D_1-module from the output of our Risa/Asir package below.

```
--> load("nk_restriction.rr")$
      // We will omit this load command in the remaining examples.
--> F=x*y*(x+y-1);
y*x^2+(y^2-y)*x
--> Ann=ann(F); // annihilating ideal of F^s
[2*y*x*dx+(-3*y*x-y^2+y)*dy+3*s*x-s,  -y*x*dx+(-y^2+y)*dy+3*s*y-s,
 (x^2+(2*y-1)*x)*dx+(-2*y*x-y^2+y)*dy]
--> I=base_replace(Ann,[[s,-1/7]]); // substitute s=-1/7 into Ann
[2*y*x*dx+(-3*y*x-y^2+y)*dy-3/7*x+1/7,  -y*x*dx+(-y^2+y)*dy-3/7*y+1/7,
 (x^2+(2*y-1)*x)*dx+(-2*y*x-y^2+y)*dy]
--> M1=nk_restriction.module_integration(I,[x,y],[dx,dy],[1,0]);
bfunction :
7*s^2+5*s
[[1,1],[s,1],[7*s+5,1]]
integer roots :
[0,0]
   *snip*
[[(-7*y^2+7*y)*dy+4*y+1]]  // the generator of N

--> nk_restriction.module_integration(M1,[y],[dy],[1]);
-- nd_weyl_gr :0sec(0.00017sec)
-- weyl_minipoly :0sec(0.0006771sec)
bfunction :
7*s^2+11*s
[[1,1],[s,1],[7*s+11,1]]
integer roots :
[0,0]
Generators:
[e1]
Relations:
[]
[]
```

The final output means that there is no relation in the one-dimensional vector space $\mathbb{C}^1$, which means that $M/(\partial_x M + \partial_y M) \simeq \mathbb{C}^1$, which is isomorphic to the twisted cohomology group

$$H^1(\mathbb{C}^2 \setminus V(xy(x+y-1)), \mathcal{L}_{-\mu})$$

where $\mathcal{L}_{-\mu}$ is the locally constant sheaf defined by $f^{-\mu}$ (see, e.g., [12, Theorem 5.5.1, pp. 233–235]).

3.2 Examples. Let $M = D_2^7/Q$ be the quiver D-module for $xy(x+y-1)$ given in Example 2.5. Compute the 0-th integration of $M/(\partial_x M + \partial_y M)$.

```
--> Eq=test_direct2();
[[dx,-mu1,0,-mu3,0,0,0],[dy,0,-mu2,-mu3,0,0,0],[0,dy,0,0,-mu2,-mu3,0],
 [0,0,-dx,0,mu1,0,-mu3],[0,0,0,-dx+dy,0,mu1,mu2],[-1,x,0,0,0,0,0],
 [-1,0,y,0,0,0,0],[-1,0,0,x+y-1,0,0,0],[0,0,-1,0,x,0,0],[0,-1,0,0,y,0,0],
 [0,0,0,-1,0,x,0],[0,-1,0,0,0,x+y-1,0],[0,0,0,1,0,0,y],[0,0,1,0,0,0,x+y-1]]

--> Eq2=base_replace(Eq,[[mu1,-1/7],[mu2,-1/7],[mu3,-1/7]])$
// In Risa/Asir, when $ is used instead of ; the return value will not
//  be printed.
--> N1=nk_restriction.module_integration(Eq2,[x,y],[dx,dy],[1,0])$
--> N=nk_restriction.module_integration(N1,[y],[dy],[1]);
 *snip*
Generators:
[e1,e2,e3,e4,e5,e6,e7]
Relations:
[-220*e1-5*e6,55*e1+5*e2,-5*e2+5*e3,e2-5*e3-e5,e2+e4,-5*e3-e4-e7]

[[0,0,0,0,0,1,1],[0,0,0,0,1,0,-1],[0,0,0,-4,0,0,1],
 [0,0,4,0,0,0,1],[0,-4,0,0,0,0,-1],[-44,0,0,0,0,0,1]]
--> length(N[0]);
7
--> matrix_rank(N);
6
```

The last two outputs mean that the integration module $M/(\partial_x M + \partial_y M)$ is isomorphic to the kernel of the matrix standing for N, which is $\mathbb{C}^7/\mathbb{C}^6$. In other words, the integration module is isomorphic to one-dimensional complex vector space. Note that the result agrees with that of Example 3.1. It is not by accident, because, roughly speaking, the D-module associated to the direct image of the level 0 quiver defined by (5) agrees with $D_2/\mathrm{Ann}\, f^{\mu}$.

3.3 Examples. In Example 3.1, set $\mu = -1$. Compare the integration of $D_2/\mathrm{Ann}\, f^{\mu}$ and the quiver D-module for $\mu_i = \mu = -1$.

The computation for the case of the annihilating ideal is as follows:

```
--> F=x*y*(x+y-1);
y*x^2+(y^2-y)*x
--> Ann=ann(F)$
--> I=base_replace(Ann, [[s, -1]])$
--> M1=nk_restriction.module_integration(I, [x,y], [dx,dy], [1,0]);
...
[[0,-y*dy-1],[(y^2-y)*dy+2*y-1,0]]
--> M2=nk_restriction.module_integration(M1, [y],[dy],[1]);
...
Generators:
[e1,e2,e1*dy,e2*dy]
Relations:
[e2*dy]

[[0,0,0,1]]  // one generator in C^4.
```

The output means that the integration module $M/(\partial_x M + \partial_y M)$ is isomorphic to $\mathbb{C}^3$. The computation for the quiver D-module is as follows.

```
--> Eq2=base_replace(Eq,[[mu1,-1],[mu2,-1],[mu3,-1]])$
--> N1=nk_restriction.module_integration(Eq2,[x,y],[dx,dy],[1,0])$
```

```
--> N=nk_restriction.module_integration(N1,[y],[dy],[1])$
--> length(N[0]);
28
--> matrix_rank(N);
25
```

The output means that the integration is isomorphic to $\mathbb{C}^{28-25} = \mathbb{C}^3$.

Problem: The case $\mu = -1$ is not strongly nonresonant in the sense of [5]. Is it by accident that both outputs are $\mathbb{C}^3$?

3.4 Examples. In Example 2.6, set $\mu = -1$. Compare the integration module of $D_2/\mathrm{Ann}(xy(x-y))^{\mu}$ and the quiver D-module for $\mu_i = \mu = -1$.

The case of the annihilating ideal is computed as follows:

```
--> F=x*y*(x-y);
y*x^2-y^2*x
-->Ann=ann(F);
[(-x^2+2*y*x)*dx+(2*y*x-y^2)*dy,-x*dx-y*dy+3*s]
--> I=base_replace(Ann, [[s, -1]]);
[(-x^2+2*y*x)*dx+(2*y*x-y^2)*dy,-x*dx-y*dy-3]
--> M1=module_integration(I, [x,y], [dx,dy], [1,0]);
...
[[0,-y*dy-1],[-y*dy-2,0]]
--> M2=module_integration(M1, [y], [dy], [1]);
...
[[0,0,0,1],[-1,0,0,0]]  // 2 dimensional vector space in C^4.
```

The output means that the integration module is $\mathbb{C}^2$. On the other hand, the integration module of the quiver D-module is computed as follows:

```
--> Eq=test_direct1()$
--> Eq2=base_replace(Eq,[[mu1,-1],[mu2,-1],[mu3,-1]])$
--> N1=module_integration(Eq2,[x,y],[dx,dy],[1,0])$
--> N=module_integration(N1,[y],[dy],[1]);
...
[[0,0,0,0,0,1],[0,0,0,0,1,0],[0,0,0,1,0,0],[0,0,-1,0,0,0],
 [0,-1,0,0,0,0],[1,0,0,0,0,0]]
--> length(N[0]);
6
--> matrix_rank(N);
6
```

The output means that the integration of the quiver D-module is 0. Note that this is a resonant case in the sense of [5].

Finally, we consider an example of a quiver D-module which is not a D-module associated to a direct image of a level 0 quiver defined by $\prod L_i^{\mu_i}$ (see (5)).

3.5 Examples. We consider the arrangement of Example 2.1. Determine all quivers when $\dim V_\alpha = 1$ for any face α and $A_{\alpha\beta} = 0$ when $\dim \beta < \dim \alpha$. Choose one of them and compute the restriction module of the quiver D-module M to $x = 0$, characteristic variety and singular locus of it. Compute also the integration module $M/(\partial_x M + \partial_y M)$.

We denote $A_{i\emptyset}$ by a_i, $A_{12,1}$ and $A_{134,1}$ by b_1 and b_2, $A_{12,2}$ and $A_{23,2}$ and $A_{24,2}$ by c_1 and c_3 and c_4, $A_{134,3}$ and $A_{23,1}$ by e_2 and e_3, $A_{134,4}$ and $A_{24,4}$ by f_2 and f_4. Then the condition 1 is written as

$$\begin{pmatrix} b_1 & c_1 & 0 & 0 \\ b_2 & 0 & e_2 & f_2 \\ 0 & c_3 & e_3 & 0 \\ 0 & c_4 & 0 & f_4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = 0 \tag{6}$$

The determinant of the matrix in (6) is $(c_3e_2f_4 + c_4e_3f_2)b_1 + b_2c_1e_3f_4$. Sets of parameters satisfying (6) = 0 are quivers satisfying the condition 2.3.

The set of parameters

$$b_1 = 1,\ b_2 = 2,\ c_1 = 3,\ c_3 = 4,\ e2 = 6,\ e3 = 7,\ f_2 = 8,\ f_4 = -140/33,$$

$$a_1 = 1,\ a_2 = -1/3,\ a_3 = 4/21,\ a_4 = -11/28,$$

satisfies the determinant condition and (6). Let M be the corresponding quiver D-module. We compute the restriction module $M' = M/xM$ by [8, Theorem 5.7.]. As we will see below, it is isomorphic to $M' = M/xM \cong (D_1)^4/N$ as D_1-module where N is generated by

$$(84\partial_y, 28, -16, 33),\ (0, y, 0, 0),\ (0, 0, y-1, 0),\ (0, 0, 0, y-1).$$

The characteristic variety and the singular locus of M' are $\mathbf{V}((y^2 - y)\xi_y)$ and $\mathbf{V}(y(y-1))$, respectively, which are computed by [7, p. 494, Algorithm].

```
--> M = test_e4c();
--> M = map(subst, M, a1, 1);
[[dx,-1,0,-4/21,11/14,0,0,0,0],[dy,0,1/3,-4/21,11/28,0,0,0,0],
--- snip
]]
--> R1=nk_restriction.module_restriction(M, [x,y], [dx,dy], [1,0]);
...
bfunction :
s^2+s
[[1,1],[s,1],[s+1,1]]
integer roots :
[-1,0]   // Integral roots are used to compute the restriction module.
...
[[0,0,0,0,0,0,0,0,-1],[0,0,0,0,0,0,0,1,0],[0,0,0,0,0,0,1,0,0],
 [0,0,0,0,0,1,0,0,0],[0,1,0,0,0,0,0,0,0],[84*dy,0,28,-16,33,0,0,0,0],
 [0,0,0,0,-y+1,0,0,0,0],[0,0,0,y-1,0,0,0,0,0],[0,0,y,0,0,0,0,0,0]]
--> nk_restriction.module_sing_locus(R1, [y,dy]);
CharId: // characteristic variety
[(-y^2+y)*dy]
Sat:   // singular locus
[-y^2+y]

[-y^2+y]
```

```
--> M1 = nk_restriction.module_integration(M, [x,y], [dx,dy], [1,0])$
--> M2 = nk_restriction.module_integration(M1, [y], [dy], [1]);
...
bfunction :
s^3+3*s^2+2*s
[[1,1],[s,1],[s+1,1],[s+2,1]]
integer roots :
[-2,0]
...
[[0,0,0,0,0,0,0,-11,-5],[0,0,0,0,0,0,66,0,-35],[0,0,0,0,0,33,0,0,35],
 [0,0,0,0,-33,0,0,0,-70],[0,0,0,11,0,0,0,0,35],[0,0,-22,0,0,0,0,0,15],
 [0,33,0,0,0,0,0,0,35],[44,0,0,0,0,0,0,0,5]]
```

The integration module $M/(\partial_x M + \partial_y M)$ is isomorphic to $\mathbb{C}^{9-8}$ from the last output.

We close this expository paper with some suggestions of research projects.

1. Develop an algorithm and a software package to compute the cohomology groups

$$H^k(\mathbb{C}^n \setminus V(\prod L_i), \mathcal{L})$$

   where L_i's are linear forms and $\mathcal{L}$ is a locally constant sheaf by utilizing quiver D-modules and the integration algorithm.
2. Construct a direct image of level 0-quiver which is compatible with the direct image of D-modules without the restrictive condition like strongly nonresonant [5, definition after Prop 3.2].
3. Develop a computer algebra system for the path algebra of a given quiver (in the standard sense). Use this system to find quivers (quiver representations) in the sense of [5]. The development should be in two steps under the theory of rewriting systems presented by Malbos in [6]. The first step is to do a rapid prototyping by a system like CafeOBJ [2]. The second step is to develop a more efficient system, which may be specialized in good classes of quivers.

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Chapter 4
Noncommutative Gröbner Bases: Applications and Generalizations

Philippe Malbos

1 Introduction

The aim of this chapter is to provide a summary of the theory of linear rewriting and the application of this theory to the construction of free resolutions for associative algebras. In Sect. 2, we present linear polygraphs as an algebraic setting for linear rewriting without a monomial order, and we review the fundamental notion of linear polygraphs. In Sect. 3, we recall several historical constructions on linear rewriting systems for associative algebras, and we show how the confluence properties are studied in these different approaches. We relate the notion of convergent linear polygraph with the notion of noncommutative Gröbner basis. In Sect. 4, we describe an algorithmic way to compute free resolutions for algebras using a method introduced by Anick. Section 5 deals with extension of linear polygraphs, seen as higher dimensional linear rewriting systems, into polygraphic resolutions for algebras. We show how to construct such a resolution starting from a convergent presentation. In the last section, we show how to relate Koszulness for algebras with the property of confluence.

1.1 Rewriting and Linear Rewriting

1.1.1 Rewriting in computer science. The notion of rewriting system comes from combinatorial algebra. It was introduced by Thue when he considered systems of transformation rules for rewriting combinatorial objects such as strings, trees or

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graphs. Its main motivation was to solve the word problem for finitely presented semigroups by using orientation of relations [64]. Afterwards, the word problem has been considered in many contexts in algebra and in computer science. Far beyond the precursor works on this decidability problem on strings, rewriting theory has been mainly developed in theoretical computer science for equational reasoning in various situations: theory of programming languages for analysis, verification and optimization, automated deduction, automated theorem proving… Rewriting theory is also present in many other computational formalisms such as Petri nets or logical systems. Depending on the context of application, rewriting theory has numerous variants corresponding to different syntaxes of the formulas being transformed: string, term, graph, circuit, term modulo, tree, λ-term, higher order term, higher dimensional term…

1.1.2 Rewriting in algebra. Rewriting appears also on various forms in algebra for universal algebras (term rewriting in Lawvere theories) [4, 43, 48, 62], monoids (string rewriting in monoids) [15, 29, 37], monoidal categories [34], linear structures, such as algebras of various type: commutative [18, 19, 21], associative [12, 13], Lie [57], D-modules [51], as well as on topological objects, such as Reidemeister moves, knots or braids [22].

This chapter focus on various aspects of rewriting in associative algebras. Rewriting theory gives algorithmic methods to study associative algebras presented by generators and defining relations. The relations are oriented as *rewriting rules* providing linear bases of normal forms with respect the defining relations. In particular, rewriting methods can be used to provide procedures for decision problems, such as the word problem, ideal membership, or to compute quadratic bases, e.g. Poincaré–Birkhoff–Witt bases, Hilbert series, syzygies of presentations, homology groups and Poincaré series.

1.1.3 We have to be careful when we rewrite over a field. Rewriting rules that relate elements in a ring or in an algebra need to be compatible with the linear structure in the following way. For a rewriting rule

$$f \to g$$

relating two elements of an algebra on a ground field $\mathbb{K}$, then for any scalar λ in $\mathbb{K}$ we would like a rewriting

$$\lambda f \to \lambda g$$

and for any other element h of the algebra, we would like a rewriting

$$f + h \to g + h.$$

Taken together, these two reductions lead to losing termination of rewriting. Indeed, it that case from the rule $f \to g$, we deduce the reductions $-f \to -g$ and $-f + (f + g) \to -g + (f + g)$. Finally, we deduce the following reduction:

$$g \to f.$$

As a consequence, the system will never terminate. Further to this remark, it is necessary to adapt the notion of rewriting system to linear situations. In the example presented above the reduction $-f + (f + g) \to -g + (f + g)$ appears as the source of the nontermination problem. In these notes, we will see two possibilities to fix this problem.

- By choosing an orientation of the rules induced by a *monomial order*, which is well founded by definition, see 2.4.1. This is the most commonly used method, in particular in the noncommutative Gröbner basis theory.
- By using the structure of *linear* 2*-polygraph* introduced in [33] and with an appropriated notion of reduction, explained in Sect. 2.2.

1.2 Noncommutative Gröbner Bases: Applications and Generalizations

1.2.1 Gröbner basis theory. Gröbner basis theory for ideals in commutative polynomial rings was introduced by Buchberger in [18]. A subset G of an ideal I in the polynomial ring $\mathbb{K}[x]$ of commutative polynomials is a *Gröbner basis* of I with respect to a given monomial order $\prec$, if the leading term ideal of I is generated by the set of leading monomials of G, that is

$$\langle\, \mathrm{lt}_{\prec}(I)\,\rangle = \langle\, \mathrm{lt}_{\prec}(G)\,\rangle.$$

Buchberger introduced the notion of *S-polynomial* to describe the obstructions to local confluence and gave an algorithm for computation of Gröbner bases [18, 21], see also [20] for a historical account. Any ideal I of a commutative polynomial ring $\mathbb{K}[x]$ has a finite Gröbner basis. Indeed, the Buchberger algorithm on a finite family of generators of an ideal I always terminates and returns a Gröbner basis of the ideal I.

Shirshov introduced in [57] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of *composition* of elements in a free Lie algebra that corresponds to the notion of S-polynomial in the work of Buchberger. He gave an algorithm to compute bases in free algebras having the computational properties of the Gröbner bases. He proved that irreducible elements for such a basis form a linear basis of the Lie algebra. This result is called now the *composition lemma* for Lie algebras.

Subsequently, the Gröbner basis theory has been developed for other types of algebras, such as associative algebras by Bokut in [13] and by Bergman in [12]. They prove Newman's Lemma for rewriting systems in free associative algebras compatible with a monomial order stating that local confluence and confluence are equivalent properties. This result was called *composition lemma* by Bokut and *diamond lemma* for ring theory by Bergman, see also [50, 66]. In general, the Buchberger algorithm does not terminate for ideals in a noncommutative polynomial ring $\mathbb{K}\langle x\rangle$. Indeed, its termination would give a decision procedure of the undecidable word problem. Even if the ideal is finitely generated it may not have a finite Gröbner basis. However, when $\mathbb{K}$ is a field an infinite Gröbner basis can be computed [50, 65]. We survey the constructions and the results of Bokut and Bergman in Sect. 3.

Note that ideas in the spirit of the Gröbner basis approach appear in several other works. Let us mention works by Hironaka in [38] and Grauert in [30] that compute bases of ideals in rings of power series having analogous properties to Gröbner bases but without a constructive method for computing such bases. In [24], Cohn gave a method to decide the word problem by a normal form algorithm based on a confluence property. Finally, Janet [41], Thomas [63] and Pommaret [54] developed the notion of involutive bases that are particular cases of Gröbner bases in the context of partial differential algebra. We refer the reader to [40] for a historical account on involutives bases and their applications to algebraic analysis of linear partial differential systems. Much more recently, Gröbner basis theory was developed in various noncommutative contexts such as Weyl algebras, see [56], or operads [27].

1.2.2 Computing normal forms. The main purpose of noncommutative Gröbner basis theory for associative algebras is to compute linear bases. Consider an algebra $\mathbf{A}$ presented by a set of generators X and a set R of defining relations, that is $\mathbf{A}$ is the quotient of the free algebra $\mathbb{K}\langle X\rangle$ by the ideal generated by R. The set of monomials on X forms a linear basis of the free algebra $\mathbb{K}\langle X\rangle$. One application of the Gröbner basis theory is to compute a basis of the algebra $\mathbf{A}$ in the form of a reduced subset of monomials. The computation is based on a monomial order on the set of monomials on X and the confluence property of a rewriting system compatible with this order. The set of monomials in normal form with respect to a Gröbner basis forms a linear basis of the algebra $\mathbf{A}$.

The Buchberger algorithm that computes Gröbner bases is the analogue of the Knuth–Bendix completion procedure in a linear setting. Several frameworks unify Buchberger and Knuth–Bendix algorithms, in particular a Gröbner basis corresponds to a confluent and terminating presentation of an algebra, see [20]. This correspondence is well known in the case of associative and commutative algebras, as recalled in the papers by Bokut [13], Bergman [12], Mora [50]. For a fuller treatment on noncommutative Gröbner bases for associative algebras, we refer the reader to the books [16, Chapter 2] and [66] and to [47, Chapter 2] and [8, Chapters 4–5] for commutative Gröbner bases.

1.2.3 Computation of free resolutions. In homological algebra, constructive methods based on noncommutative Gröbner bases were developed to compute projective resolutions for algebras. In particular, Anick and Green constructed small explicit free resolutions for algebras given by noncommutative Gröbner bases [1–3, 31]. Their constructions provide resolutions to compute homological invariants (homology groups, Hilbert and Poincaré series) of algebras presented by generators and relations given by a Gröbner basis. The chains of these resolutions are given by iterated overlaps of the leading terms of the Gröbner basis, and the differentials are constructed by Noetherian induction.

1.2.4 Linear polygraphs. All the constructions mentioned above rely on a monomial order, that is, a well-founded order of the monomials compatible with the multiplication. The termination orders in linear polygraphs introduced in [33] are less restrictive. A linear polygraph is a higher dimensional linear rewriting system for presentation of an algebra that allows more possibilities of termination orders than those associated to Gröbner bases using monomial orders. A set-theoretical 2-polygraph describes a string rewriting system, see [36]. It is defined by a data $(\Sigma_0, \Sigma_1, \Sigma_2)$ made of a 1-*polygraph* that is an oriented graph

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

where Σ_0 and Σ_1 denote, respectively, the sets of 0-cells and 1-cells and s_0, t_0 denote the source and target maps, with a *cellular extension* Σ_2 of the free category Σ_1^*, that is a set of globular 2-cells relating parallel 1-cells:

$$p \underset{g}{\overset{f}{\rightrightarrows}} q \quad (\Downarrow \varphi)$$

A 2-polygraph corresponds to a string rewriting system, where the rules are described by the globular 2-cells, see [36].

A *linear* 2-*polygraph* corresponds to the same notion for rewriting in a free algebra or a free algebroid. It is constructed in the same manner as a 2-polygraph, but the cellular extension is linear in the sense that it is constructed on 1-spheres in the free 1-algebroid over generating 1-cells. Explicitly, we define a linear 2-polygraph as a triple $(\Lambda_0, \Lambda_1, \Lambda_2)$ such that (Λ_0, Λ_1) is a 1-polygraph and Λ_2 is a cellular extension of the free algebroid Λ_1^ℓ generated by the 1-polygraph (Λ_0, Λ_1), that is given by two maps

$$\Lambda_1^\ell \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Lambda_2$$

satisfying *globular relations* $s_0 s_1 = s_0 t_1$ and $t_0 s_1 = t_0 t_1$. All the categorical background will be introduced in Sect. 2. In the free 2-algebroid Λ_2^ℓ, any 2-cell being invertible, the notion of rewriting step induced by a linear polygraph needs to be defined with attention. In Sects. 2 and 3, we recall from [33] properties of termination, confluence and local confluence for linear 2-polygraphs. We state the Newman lemma for linear 2-polygraphs in Theorem 3.2.11 showing that a terminating left-monomial linear 2-polygraph is confluent if and only if it is locally confluent. We give a formulation of a critical branching lemma for linear 2-polygraphs in Theorem 3.3.7. The formulation of this result differs from the critical branching lemma for 2-polygraphs in the sense that the termination hypothesis is required, as we will explain with several examples in Sect. 3.3. Finally, we explain how to recover non-commutative Gröbner bases as a special case of convergent linear 2-polygraphs in Sect. 3.6.

1.2.5 Polygraphic resolutions of algebroids. We recall in Sect. 5.1 the notion of linear syzygies for linear polygraphs. When the linear 2-polygraph is convergent, we show that all the syzygies can be generated by confluence diagrams induced by the critical branchings, this is the Squier theorem (Theorem 5.1.6).

In Sect. 5.2, we recall from [33] the notion of *polygraphic resolution* for an algebra giving a categorical description of higher dimensional syzygies of its presentations. A polygraphic resolution for an algebra $\mathbf{A}$ is an acyclic polygraphic extension of a presentation of $\mathbf{A}$. That is a linear ∞-polygraph, which satisfies an acyclicity condition. Theorem 5.2.6 from [33] shows that any convergent linear 2-polygraph Λ extends to an acyclic linear ∞-polygraph, presenting the same algebra and whose n-cells, for $n \geqslant 3$, are indexed by the critical $(n-1)$-fold branchings. From this point of view, this resolution is similar to Anick's resolution associated with a Gröbner basis.

Finally, we show how a polygraphic resolution of an algebra $\mathbf{A}$ induces a free resolution in the category of right modules (resp. left modules, resp. bimodules) over $\mathbf{A}$.

1.2.6 Confluence and Koszulness. In the last section of these notes, we show how Anick's resolution leads to relate the Koszul property for an associative algebra to the existence of a quadratic Gröbner basis for its ideal of relations. We also show how to prove this property using convergent linear 2-polygraphs.

In Sect. 6.1, we recall the notion of Koszulness for quadratic algebras and N-homogeneous algebras. Koszulness for quadratic algebras was introduced by Priddy [55]. A connected graded algebra $\mathbf{A}$ is *Koszul* if the Tor groups $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K}, \mathbb{K})$ vanish for $i \neq n$, where the grading n is the homological degree and the grading i corresponds to the internal grading of the algebra. This notion was generalized by Berger to the case of N-homogeneous algebras [10].

In [2], Anick showed how its resolution can be used to prove Koszulness of a quadratic algebra. Indeed, if an algebra $\mathbf{A}$ admits a presentation whose relations are defined by a quadratic Gröbner basis, then Anick's resolution associated to this Gröbner basis is concentrated in the right bidegree, and thus, the algebra $\mathbf{A}$ is Koszul, see Theorem 6.2.3. For the N-homogeneous case, a Gröbner basis concentrated in weight N is not enough to imply Koszulness: an extra condition has to be checked as shown by Berger in [10].

Finally, we present a sufficient polygraphic condition of Koszulness of graded algebras given in [33]. Using a graded version of Theorem 5.2.9, one shows that an N-homogeneous algebra having a ℓ_N-concentrated polygraphic resolution is Koszul, Theorem 6.2.7.

2 Linear Rewriting

In this section, we recall the categorical description of linear rewriting given in [33] using the notion of linear polygraph. This notion extends to associative algebras the categorical notion of 2-polygraph used to describe presentations of monoids by generators and relations. This approach is based on presentations by generators and relations of higher dimensional categories, independently introduced by Burroni and Street under the respective names of *polygraphs* in [23] and *computads* in [60, 61]. Higher dimensional rewriting has unified several paradigms of rewriting. These notes concern only rewriting in algebras, for a deeper discussion on categorical description of string rewriting systems by 2-polygraphs, we refer the reader to [36]. Note that there is a shift by 1 in the dimension: in these lecture notes the linear 2-polygraphs are linear 1-polygraphs in [36].

2.1 Linear 2-Polygraphs

2.1.1 Categories. Recall that a *(small) category* (or 1-*category*) is a data $\mathbf{C}$ made of a set $\mathbf{C}_0$, whose elements are called 0-*cells* (or *objects*) of $\mathbf{C}$, for every 0-cells p and q a set $\mathbf{C}(p, q)$, whose elements are called 1-*cells* (or *arrows*) of $\mathbf{C}$ with *source* p and *target* q, for every 0-cell p a specified 1-cell 1_p in $\mathbf{C}(p, p)$, called the *identity of* p, and for every 0-cells p, q and r a *composition* map

$$\star_0^{p,q,r} : \mathbf{C}(p, q) \times \mathbf{C}(q, r) \to \mathbf{C}(p, r),$$

that is associative and such that the identities are local units for this composition.

A monoid M with product $\cdot$ and identity element 1_M corresponds to a category $\mathbf{M}$ with only one 0-cell, denoted by $*$, and the 1-cells of $\mathbf{M}(*, *)$ are the elements of the monoid M. The identity arrow 1_* of $\mathbf{M}$ corresponds to the identity element 1_M and the composition of $u \star_0 v$ of 1-cells in $\mathbf{M}(*, *)$ corresponds to the product $u \cdot v$ in the monoid M. The associativity and unitary properties of the composition, making $\mathbf{M}$ into a category, are induced by the corresponding properties of the product $\cdot$ of the monoid. In this way, any monoid can be thought of as a one-0-cell category and a category can be thought of as a 'monoid with many 0-cells'. In a similar way, the notion of algebroid describes the concept of associative algebra with many 0-cells.

2.1.2 Algebroids. A 1-*algebroid* over a ground field $\mathbb{K}$ is a category enriched over the monoidal category of vector spaces over $\mathbb{K}$ with its usual tensor product. Explicitly, a 1-algebroid $\mathbf{A}$ is specified by the following data:

- **(i)** a set $\mathbf{A}_0$ of 0-*cells*, that we will denote by $p, q\ldots$
- **(ii)** for every 0-cells p and q, a vector space $\mathbf{A}(p, q)$, whose elements are the 1-*cells* of $\mathbf{A}$, with *source* p and *target* q, that we will denote by $f, g\ldots$
- **(iii)** for every 0-cells p, q and r, a linear map

$$\star_0 : \mathbf{A}(p,q) \otimes \mathbf{A}(q,r) \longrightarrow \mathbf{A}(p,r)$$

called the 0-*composition* of $\mathbf{A}$ and whose image on $f \otimes g$ is denoted by $f \star_0 g$ or fg. This composition is *associative*, that is the relation:

$$(f \star_0 g) \star_0 h = f \star_0 (g \star_0 h),$$

holds for any 0-composable 1-cells f, g and h, and *unitary*, that is, for any 0-cell p, there is a 1-cell 1_p such that for any 1-cell f in $\mathbf{A}(p,q)$, the following relation holds:

$$1_p \star_0 f = f \star_0 1_q = f.$$

A 1-cell f with source p and target q will be graphically represented by

$$p \xrightarrow{\ f\ } q$$

2.1.3 Remark. An 1-*algebra* is an 1-algebroid with a single one 0-cell, which can be identified to an (unital associative) algebras over $\mathbb{K}$. We will denote by **Alg** the category of algebras over $\mathbb{K}$. The notion of 1-algebroid was first introduced by Mitchell as *ring with several objects* called $\mathbb{K}$-*category* in [49]; terminology *linear category* appears also in the literature. A small $\mathbb{Z}$-category is called a *ringoid* and a one-0-cell ringoid is a ring.

2.1.4 One-dimensional polygraphs. An algebroid can be defined by generators and relations. The generators are described by one-dimensional polygraphs. A 1-*polygraph* is a directed graph

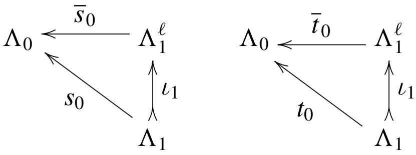
$$\Lambda_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Lambda_1$$

given by a set Λ_0 of 0-*cells*, a set Λ_1 of 1-*cells* together with two maps s_0 and t_0 sending a 1-cell x on its *source* $s_0(x)$ and its *target* $t_0(x)$. A 1-polygraph with only one 0-cell will be identified to a set.

We will denote by Λ_1^* the free 1-category generated by the 1-polygraph (Λ_0, Λ_1). Its set of 0-cells is Λ_0 and for any 0-cells p and q, the elements of the hom-set $\Lambda_1^*(p,q)$ are paths from p to q in the 1-polygraph (Λ_0, Λ_1). The composition is the concatenation of paths and the identity on a 0-cell p is the empty path with source and target p. If the 1-polygraph has only one 0-cell, Λ_1^* will be identified to the free monoid on the set Λ_1.

2.1.5 Free 1-algebroid. The free 1-algebroid on a 1-polygraph (Λ_0, Λ_1) is the 1-algebroid, denoted by Λ_1^ℓ, whose set of 0-cells is Λ_0, and for any 0-cells p and q,

$\Lambda_1^\ell(p, q)$ is the free vector space on $\Lambda_1^*(p, q)$. In other words, the space $\Lambda_1^\ell(p, q)$ has for basis the set of paths from p to q in the 1-polygraph Λ. If Λ_0 is reduced to only one 0-cell, Λ_1^ℓ is the free algebra with basis Λ_1. The source and target maps s_0 and t_0 are extended into maps on Λ_1^ℓ, denoted by $\overline{s}_0$ and $\overline{t}_0$, in a natural way making the following two diagrams commutative:



where ι_1 denotes the inclusion of 1-cells of Λ_1 in the free algebroid Λ_1^ℓ.

2.1.6 Quivers and path algebras. The terminology *directed graph* is used in graph theory. The same notion is also called *quiver* in representation theory. A linear representation of a quiver (Λ_0, Λ_1) is a functor ρ from the free category Λ_1^* to the category **Vect** of vectors spaces. The path algebra of a quiver (Λ_0, Λ_1) is the *category algebra* of the free category Λ_1^*. That is, it is the $\mathbb{K}$-algebra whose underlying space is spanned by the set of 1-cells in Λ_1^* and the product on the basis elements is defined by $u \cdot v = u \star_0 v$ if u and v are 0-composable 1-cells in Λ_1^* and $u \cdot v = 0$ otherwise. When the set Λ_0 is finite, then $\sum_{p \in \Lambda_0} 1_p$ is the identity of the path algebra. Note that we can obtain the path algebra of a quiver Λ from the free 1-algebroid Λ_1^ℓ by forgetting the 1-category structure.

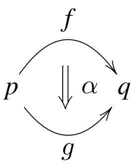
2.1.7 Linear 2-polygraph. A *cellular extension* of the 1-algebroid Λ_1^ℓ is a set Λ_2 equipped with two maps

$$\Lambda_1^\ell \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Lambda_2$$

such that, for every α in Λ_2, the pair $(s_1(\alpha), t_1(\alpha))$ is a 1-*sphere* in Λ_1^ℓ, that is, the following *globular relations* hold

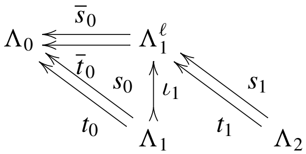
$$s_0 s_1(\alpha) = s_0 t_1(\alpha) \quad \text{and} \quad t_0 s_1(\alpha) = t_0 t_1(\alpha).$$

An element of the cellular extension Λ_2 will be graphically represented by a 2-cell with the following globular shape:



that relates parallel 1-cells f and g in Λ_1^ℓ, also denoted by $f \overset{\alpha}{\Rightarrow} g$ or by $\alpha : f \Rightarrow g$.

We define a *linear* 2*-polygraph* as a triple $(\Lambda_0, \Lambda_1, \Lambda_2)$, where (Λ_0, Λ_1) is a 1-polygraph and Λ_2 is a cellular extension of the free 1-algebroid Λ_1^ℓ:



The elements of Λ_2 are called the 2*-cells* of Λ, or the *rewriting rules* of Λ.

In the sequel, we will consider polygraphs with one 0***-cell denoted*** $*$***.***

2.1.8 Presentations of algebras by generators and relations. Given a linear 2-polygraph Λ. The *algebra presented by* Λ is the quotient algebra of the free algebra Λ_1^ℓ by the cellular extension Λ_2. That is, it is the algebra $\mathbf{A}$ obtained by identifying in Λ_1^ℓ all the 1-cells $s_1(a)$ and $t_1(a)$, for every 2-cell a in Λ_2^ℓ. We denote by $\overline{f}$ the image of a 1-cell f of Λ_1^ℓ through the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$. We say that a linear 2-polygraph Λ is a *presentation* of an algebra $\mathbf{A}$ if the algebra presented by Λ is isomorphic to $\mathbf{A}$. Two linear 2-polygraphs are said to be *Tietze equivalent* if they present isomorphic algebras.

2.1.9 First toy example. Here our first toy example that we will use through this lecture:

$$\Lambda = \langle * \mid x, y, z \mid xyz \overset{\gamma}{\Rightarrow} x^3 + y^3 + z^3 \rangle.$$

The free 1-algebroid generated by $\Lambda_1 = \{x, y, z\}$ is the free algebra $\mathbb{K}\langle x, y, z\rangle$. The algebra presented by the linear 2-polygraph Λ is the quotient of the free algebra $\mathbb{K}\langle x, y, z\rangle$ by the ideal generated by the 1-cell $xyz - x^3 - y^3 - z^3$.

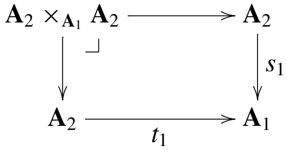
2.1.10 Other toy examples. We will consider the two following Tietze equivalent linear 2-polygraphs:

$$\Lambda = \langle * \mid x, y \mid x^2 \overset{\beta}{\Rightarrow} yx \rangle, \qquad \Lambda' = \langle * \mid x, y \mid yx \overset{\beta'}{\Rightarrow} x^2 \rangle.$$

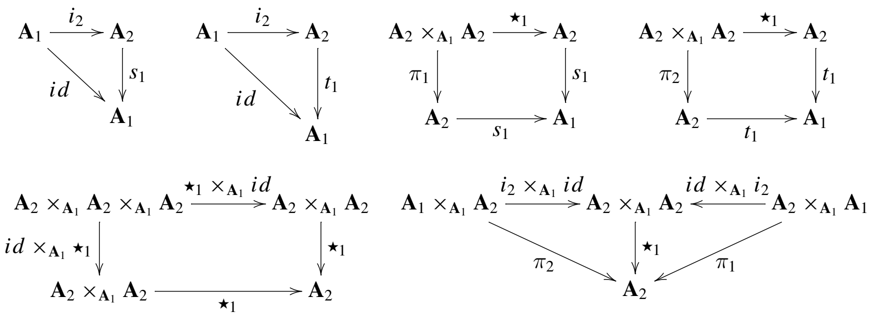
2.1.11 Two-dimensional algebras. We define a 2*-algebra* $\mathbf{A}$ as an internal 1-category in the category **Alg**. Explicitly, it is defined by a diagram

$$\mathbf{A}_1 \underset{i_2}{\overset{t_1,\ s_1}{\leftleftarrows\!\!\rightarrow}} \mathbf{A}_2 \xleftarrow{\ \star_1\ } \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \qquad (1)$$

where $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ is the algebra defined by the following pullback diagram in the category **Alg**:



Elements of the algebra $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ are pairs (a, a') of 1-*composable* 2-cells a and a', that is satisfying $t_1(a) = s_1(a')$. The morphisms of algebras s_1, t_1 and $\star_1$ satisfy the axioms in such a way that Diagram (1) defines a 1-category. Explicitly, the following diagrams commute in the category **Alg**:

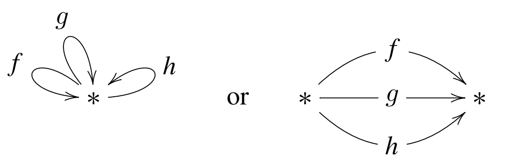


where π_1 and π_2 denote, respectively, first and second projections. Note that the linear structure and the product in the algebra $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ are given by

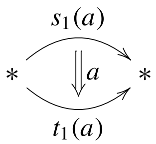
$$(a, a') + (b, b') = (a + b, a' + b'),$$
$$\lambda(a, a') = (\lambda a, \lambda a'),$$
$$(a, a')(b, b') = (ab, a'b'),$$

for all pair of 1-composable 2-cells (a, a') and (b, b') and scalar λ in $\mathbb{K}$.

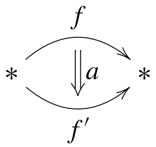
2.1.12 Notations. For a 1-cell f, the identity 2-cell $i_2(f)$ is denoted by 1_f, or f if there is no possible confusion. The 1-composite $\star_1(a, a')$ of 1-composable 2-cells a and a', will be denoted by $a \star_1 a'$. Elements of the algebra $\mathbf{A}_1$, called 1-*cells of* $\mathbf{A}$, are graphically pictured as follows:



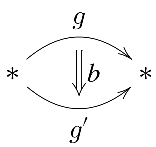
The elements of $\mathbf{A}_2$, called 2-*cells of* $\mathbf{A}$ are graphically represented by



Given 2-cells



and



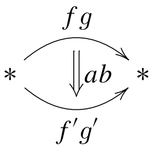
we denote by ab their product in the algebra $\mathbf{A}_2$. The source and target maps s_1 and t_1 being morphisms of algebras, we have

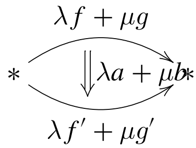
$$s_1(ab) = s_1(a)s_1(b), \qquad \text{and} \qquad t_1(ab) = t_1(a)t_1(b),$$

and for any scalars λ and μ in $\mathbb{K}$, we have

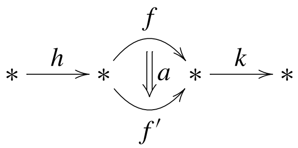
$$s_1(\lambda a + \mu b) = \lambda s_1(a) + \mu s_1(b), \qquad \text{and} \qquad t_1(\lambda a + \mu b) = \lambda t_1(a) + \mu t_1(b).$$

Hence



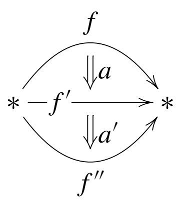


Given 1-cells h, f, f' and k in $\mathbf{A}_1$ and a 2-cell a in $\mathbf{A}_2$ such that

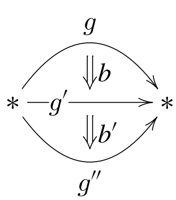


we will denote by $hak : hfk \Rightarrow hf'k$ the 0-composite $1_h \star_0 a \star_0 1_k$.

2.1.13 Properties of 1-composition. Given 1-composable 2-cells:



and



in $\mathbf{A}_2 \star_{\mathbf{A}_1} \mathbf{A}_2$, the 1-composition $\star_1$ being linear, $a \star_1 a' + b \star_1 b'$ is a 2-cell from $f + g$ to $f' + g'$ and we have

$$(a + b) \star_1 (a' + b') = a \star_1 a' + b \star_1 b'.$$

and, for any scalar λ in $\mathbb{K}$, $\lambda(a \star_1 a')$ is a 2-cell from λf to $\lambda f''$ and we have

$$(\lambda a) \star_1 (\lambda a') = \lambda(a \star_1 a').$$

Finally, the compatibility with the product induces the following relation:

$$(a \star_1 a')(b \star_1 b') = ab \star_1 a'b'. \tag{2}$$

Relation (2) corresponds to the *exchange law* in the 2-algebra $\mathbf{A}$ between the 1-composition and the product.

2.1.14 Remarkable identities in a 2-algebra. The following properties hold in a 2-algebra $\mathbf{A}$:

(i) for any 1-composable 2-cells a and a' in $\mathbf{A}$, we have

$$a \star_1 a' = a + a' - t_1(a), \tag{3}$$

(ii) any 2-cell a in $\mathbf{A}$ is invertible for the $\star_1$-composition, and its inverse is given by

$$a^- = -a + s_1(a) + t_1(a). \tag{4}$$

(iii) for any 2-cells a and b in $\mathbf{A}$, we have

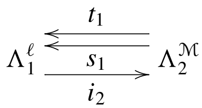
$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b) = s_1(a)b + at_1(b) - s_1(a)t_1(b). \tag{5}$$

Relation (3) is a consequence of the linearity of the 1-composition $\star_1$. Indeed, for any (a, a') in $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$, we have

$$\begin{aligned} a \star_1 a' &= (a - s_1(a') + s_1(a')) \star_1 (t_1(a) - t_1(a) + a'), \\ &= a \star_1 t_1(a) - s_1(a') \star_1 t_1(a) + s_1(a') \star_1 a', \\ &= a - t_1(a) + a'. \end{aligned}$$

2.1.15 Exercise. Show identities (4) and (5).

2.1.16 The free 2-algebra on a linear 2-polygraph. The *free* 2-*algebra over a linear* 2-*polygraph* Λ is the 2-algebra, denoted by Λ_2^ℓ, defined as follows. In dimension 1, it is the free 1-algebra Λ_1^ℓ over Λ_1. For dimension 2, we consider the following diagram in the category of Λ_1^ℓ-bimodule:



where $\Lambda_2^{\mathcal{M}}$ is the Λ_1^ℓ-bimodule $\left(\Lambda_1^\ell \otimes \mathbb{K}\Lambda_2 \otimes \Lambda_1^\ell\right) \oplus \Lambda_1^\ell$ and where the maps s_1, t_1 and i_2 are defined by

$$s_1(f\alpha g) = fs_1(\alpha)g, \qquad t_1(f\alpha g) = ft_1(\alpha)g \qquad \text{and} \qquad s_1(h) = t_1(h) = i_2(h) = h,$$

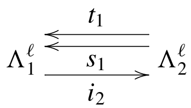
for all 2-cell α in Λ_2, and 1-cells f, g, h in Λ_1^ℓ. The quotient of the Λ_1^ℓ-bimodule $\Lambda_2^{\mathcal{M}}$ by the equivalence relation generated by

$$as_1(b) + t_1(a)b - t_1(a)s_1(b) \sim s_1(a)b + at_1(b) - s_1(a)t_1(b),$$

for all a and b in $\Lambda_1^\ell \otimes \mathbb{K}\Lambda_2 \otimes \Lambda_1^\ell$, has a structure of algebra, denoted by Λ_2^ℓ, and whose product is given by

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b).$$

We prove that the source and target maps are compatible with this quotient, so giving a structure of 2-algebra:

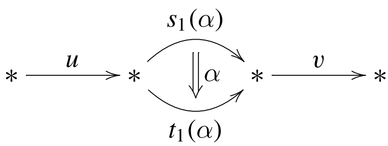


2.1.17 Monomials. A *monomial* in the free 2-algebra Λ_2^ℓ is a 1-cell of the free monoid Λ_1^* over Λ_1. The set of monomials of Λ_2^ℓ, also denoted by Λ_1^*, forms a linear basis of the free algebra Λ_1^ℓ. As a consequence, every nonzero 1-cell f of Λ_1^ℓ can be uniquely written as a linear combination of pairwise distinct monomials $u_1, \ldots, u_p$:

$$f = \lambda_1 u_1 + \ldots + \lambda_p u_p$$

with $\lambda_i \in \mathbb{K} \setminus \{0\}$, for all $i = 1, \ldots, p$. The set of monomials $\left\{u_1, \ldots, u_p\right\}$ will be called the *support* of f and denoted by $\mathrm{Supp}(f)$.

2.1.18 2-Monomials. A 2-*monomial* of a free 2-algebra Λ_2^ℓ is a 2-cell of Λ_2^ℓ with shape $u\alpha v$, where α is a 2-cell in Λ_2, and u and v are monomials in Λ_1^*:



By construction of the free 2-algebra Λ_2^ℓ, and by freeness of Λ_1^ℓ, every non-identity 2-cell a of Λ_2^ℓ can be written as a linear combination of pairwise distinct 2-monomials a_1, ..., a_p and of an 1-cell h of Λ_1^ℓ:

$$a = \lambda_1 a_1 + \ldots + \lambda_p a_p + h. \tag{6}$$

2.1.19 Exercise. Prove that the decomposition in (6) is unique up to the following relations:

$$as_1(b) + t_1(a)b - t_1(a)s_1(b) = s_1(a)b + at_1(b) - s_1(a)t_1(b), \tag{7}$$

for all 2-monomials a and b in Λ_2^ℓ.

2.1.20 Monomial linear 2-polygraphs. A linear 2-polygraph Λ is *left-monomial* if, for every 2-cell α of Λ_2, the source $s_1(\alpha)$ is a monomial in $\Lambda_1^* \setminus \mathrm{Supp}(t_1(\alpha))$. Note that a non-left-monomial linear 2-polygraph would produce useless ambiguity only due to the linear structure.

A linear 2-polygraph Λ is *monomial* if it is left-monomial and for every 2-cell α of Λ_2, $t_1(\alpha) = 0$ holds. A *monomial algebra* is an algebra admitting a presentation by a monomial linear 2-polygraph.

2.1.21 Degrees and length. For monomials u and v in Λ_1^*, we denote by $\mathrm{Occ}_v(u)$ the number of different occurrences of the monomial v in the monomial u. For instance, $\mathrm{Occ}_{x^2}(x^4) = 3$ and $\mathrm{Occ}_y(x^4) = 0$. For a subset M of monomials in Λ_1^*, we denote

$$\mathrm{Occ}_M(u) = \sum_{v \in M} \mathrm{Occ}_v(u).$$

The *length* of a monomial u in Λ_1^*, denoted by $\ell(u)$, is equal to $\mathrm{Occ}_{\Lambda_1}(u)$.

2.1.22 Exercise. Show that any linear 2-polygraph is Tietze equivalent to a left-monomial linear 2-polygraph.

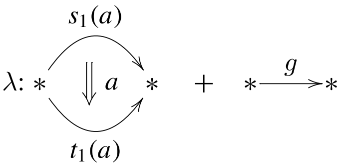
2.1.23 Examples. The linear 2-polygraph Λ given in Example 2.1.9 is left-monomial. The linear 2-polygraph $\langle * \mid x, y \mid x^2 + y^2 \Rightarrow 2xy \rangle$ is not left-monomial, but it is Tietze equivalent to the following left-monomial 2-polygraph:

$$\Lambda' = \langle * \mid x, y \mid xy \overset{\alpha'}{\Rightarrow} \frac{1}{2}(x^2 + y^2) \rangle.$$

The linear 2-polygraphs $\langle * \mid x \mid x^2 \Rightarrow 0 \rangle$ and $\langle * \mid x, y \mid xy \Rightarrow 0 \rangle$ are monomials.

2.2 Linear Rewriting Steps

2.2.1 Elementary 2-cells. Let Λ be a linear 2-polygraph. An *elementary* 2-cell of the free 2-algebra Λ_2^ℓ is a 2-cell of Λ_2^ℓ with shape



where a is a 2-monomial, g is a 1-cell of Λ_1^ℓ and λ is a nonzero scalar in $\mathbb{K}$.

2.2.2 Example. With the polygraph Λ' of Example 2.1.23, the 2-cell

$$2x\alpha' y + y^3 : 2x^2y^2 \Rightarrow x^3y + xy^3 - y^3$$

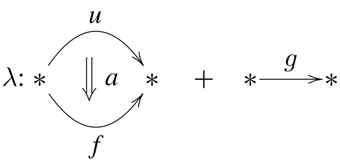
is elementary and the 2-cell

$$x\alpha' + \alpha' y : x^2y + xy^2 \Rightarrow \frac{1}{2}(x^3 + xy^2 + x^2y + y^3)$$

is not elementary.

2.2.3 Exercise. Show that any 2-cell in a free 2-algebra Λ_2^ℓ can be decomposed into a 1-composition of elementary 2-cells of Λ_2^ℓ

2.2.4 Rewriting steps. Let Λ be a left-monomial linear 2-polygraph. A *rewriting step of* Λ is an elementary 2-cell



of Λ_2^ℓ such that λ is a nonzero scalar and u is not in the support of g.

2.2.5 Examples. For the linear 2-polygraph given in Example 2.1.9, the 2-cell

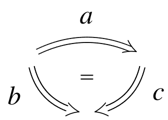
$$3x\gamma - 3xz^3 : 3x^2yz - 3xz^3 \;\Rightarrow\; 3x^4 + 3xy^3$$

is a rewriting step. For a linear 2-polygraph having a rule $\alpha : u \Rightarrow f$, the 2-cell

$$-\alpha + (u + f) : -u + (u + f) \;\Rightarrow\; -f + (u + f)$$

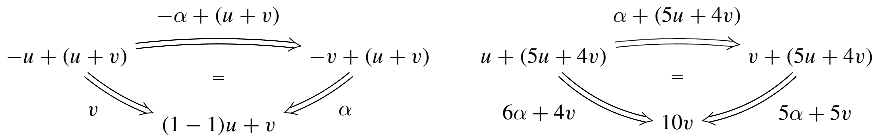
is not a rewriting step because the monomial u appears in the context $u + f$.

2.2.6 Exercise, [33, **Lemma 3.1.2**]. Let Λ be a left-monomial linear 2-polygraph and let a be an elementary 2-cell of the 2-algebra Λ_2^ℓ. Show that a can be factorized in the 2-algebra Λ_2^ℓ into



where b and c are either identities of rewriting steps.

2.2.7 Example. Let Λ be a linear 2-polygraph and let $\alpha : u \Rightarrow v$ be a 2-cell of Λ_2. The 2-cell $-\alpha + (u + v)$ and $\alpha + (5u + 4v)$ are not rewriting steps of Λ. They can be decomposed, respectively, as follows:



2.2.8 Rewriting sequences. A 2-cell a of Λ_2^ℓ is *positive*, or a *rewriting sequence*, if it is an identity or a 1-composite

$$f_0 \overset{a_1}{\Rightarrow} f_1 \Rightarrow \cdots \Rightarrow f_{k-1} \overset{a_k}{\Rightarrow} f_k$$

of rewriting steps of Λ.

2.2.9 Reduced cells. A 1-cell f of Λ_1^ℓ is called *reduced*, or *irreducible*, *with respect to* Λ_2, if there is no rewriting step of Λ with source f. As a consequence, a 1-cell is reduced if and only if it is the zero 1-cell of Λ_1^ℓ, or a linear combination of reduced monomials in Λ_1^*. The reduced 1-cells of Λ_1^ℓ form a vector subspace of Λ_1^ℓ, denoted by Λ_1^{ir}. Since Λ is left-monomial, the set of reduced monomials of Λ_1^*, denoted by Λ_1^{irm}, forms a basis of the vector space Λ_1^{ir}.

We denote by $s_1(\Lambda)$ the set of *redex* of a reduced left-monomial linear 2-polygraph Λ defined by

$$s_1(\Lambda) = \{s_1(\alpha) \mid \alpha \text{ in } \Lambda_2\}.$$

In [2], a redex is called an *obstruction*. The number of possible application of rules of Λ_2 to a monomial u is $\mathrm{Occ}_{s_1(\Lambda)}(u)$.

2.2.10 Reduced linear 2-polygraphs. We say that a linear 2-polygraph Λ is *left-reduced* if, for every 2-cell α in Λ_2, the 1-cell $s_1(\alpha)$ is reduced with respect to $\Lambda_2 \setminus \{\alpha\}$. We say that Λ is *right-reduced* if, for every 2-cell α of Λ, the 1-cell $t_1(\alpha)$ is reduced. The linear polygraph Λ is *reduced* if it is both left-reduced and right-reduced.

2.2.11 Exercise. Show that any left-monomial linear 2-polygraph is Tietze equivalent to a reduced left-monomial linear 2-polygraph.

2.2.12 Normal forms. If f is a 1-cell of Λ_1^ℓ, a *normal form for* f *with respect to* Λ_2 is a reduced 1-cell g of Λ_1^ℓ such that there exists a positive 2-cell $a : f \Rightarrow g$ in Λ_2^ℓ.

2.3 Termination of Linear 2-Polygraphs

We recall the notions of rewrite relation and termination for linear 2-polygraphs from [33, 3.2]. Let us fix a left-monomial linear 2-polygraph Λ.

2.3.1 Termination. The *rewrite relation of* Λ is the smallest transitive binary relation on Λ_1^*, denoted by $\prec_\Lambda$, such that

1. the relation $\prec_\Lambda$ is *compatible with* Λ_2, that is $w \prec_\Lambda u$ for every 2-cell $\alpha : u \Rightarrow f$ of Λ and every monomial w in $\mathrm{Supp}(f)$,
2. the relation $\prec_\Lambda$ is *compatible with products*, that is $u' \prec_\Lambda u$ implies $vu'w \prec_\Lambda vuw$ for every monomials u, u', v and w of Λ_1^*.

We say that the 2-polygraph Λ *terminates* if the rewrite relation $\prec_\Lambda$ is well founded, that is, there are no infinite descending chains in Λ_1^*:

$$u_1 \succ_\Lambda u_2 \succ_\Lambda \ldots \succ_\Lambda u_n \succ_\Lambda u_{n+1} \succ_\Lambda \ldots$$

2.3.2 Example. Consider the linear 2-polygraph $\Lambda = \langle * \mid x, y \mid xy \overset{\alpha}{\Rightarrow} x^2 + y^2 \rangle$. We have $xy \succ_\Lambda x^2$ and $xy \succ_\Lambda y^2$. Following compatibility with products, we have

$$x^2y \succ_\Lambda xy^2 \succ_\Lambda x^2y.$$

Hence, the relation $\prec_\Lambda$ is not well founded, and the polygraph Λ is not terminating. Note that we have an infinite sequence of rewriting steps:

$$x^2y \overset{x\alpha}{\Rightarrow} x^3 + xy^2 \overset{x^3 + \alpha y}{\Longrightarrow} x^3 + y^3 + x^2y \Rightarrow \ \ldots$$

2.3.3 The rewrite relation on 1-cells. The rewrite relation $\prec_\Lambda$ is extended to the 1-cells of Λ_1^ℓ by setting, for any 1-cells f and g, $g \prec_\Lambda f$ if the following two conditions hold:

1. there exists a monomial in $\mathrm{Supp}(f)$ which is not in $\mathrm{Supp}(g)$,
2. for any monomial v in $\mathrm{Supp}(g)\backslash\mathrm{Supp}(f)$, there exists a monomial u in $\mathrm{Supp}(f)\backslash\mathrm{Supp}(g)$, such that $v \prec_\Lambda u$.

2.3.4 Proposition. *The rewrite relation $\prec_\Lambda$ is well founded on* 1*-cells if and only if it is well founded on monomials.*

If Λ terminates, then for every rewriting step a of Λ, we have $t_1(a) \prec_\Lambda s_1(a)$. This implies that the 2-algebra Λ_2^ℓ contains no infinite sequence of pairwise 1-composable rewriting steps

$$f_0 \overset{a_1}{\Rightarrow} f_1 \Rightarrow \cdots \Rightarrow f_{k-1} \overset{a_k}{\Rightarrow} f_k \Rightarrow : \cdots$$

so that every 1-cell of Λ_1^ℓ admits at least one normal form with respect to Λ_2.

2.4 Monomial Orders

2.4.1 Monomial orders. A total order $\prec$ on the set of monomials Λ_1^* is a *monomial order* if the following conditions are satisfied:

(i) $\prec$ is a *well-order*, that is, there are no infinite descending chains in Λ_1^*.

$$u_1 \succ u_2 \succ u_3 \succ \ldots \succ u_n \succ u_{n+1} \succ \ldots$$

(ii) $\prec$ is *compatible with the multiplicative structure* on monomials, that is

$$u \prec u' \quad \text{implies} \quad vuw \prec vu'w,$$

for all monomials u, u', v and w in Λ_1^*.

2.4.2 Example. Given a total order relation $\prec$ on Λ_1, we define the *left degree-wise lexicographic order generated by* $\prec$, or *deglex order generated by* $\prec$, as the order $\prec_{\text{deglex}}$ on Λ_1^* that compare two monomials first by degree and then lexicographically. It is defined by

(i) $y_1 \ldots y_p \prec_{\text{deglex}} x_1 \ldots x_q$, if $p < q$,
(ii) $y_1 \ldots y_{j-1} y_j \ldots y_p \prec_{\text{deglex}} y_1 \ldots y_{j-1} x_j \ldots x_p$, if $y_j \prec x_j$.

2.4.3 Exercise. Show that the order $\prec_{\text{deglex}}$ is a monomial order.

2.4.4 Exercise. Explain why the pure lexicographic order is not a monomial order. Show that it is neither a well-order nor compatible with the product of monomials.

2.4.5 Polygraph compatible with a monomial order. A linear 2-polygraph Λ is said to be *compatible with a monomial order* $\prec$ if for every 2-cell $\alpha : u \Rightarrow f$ of Λ_2, then $w \prec u$ for any monomial w in the support of f. The monomial order $\prec$ is thus a well-founded rewrite relation for Λ. It follows that any linear 2-polygraph compatible with a monomial order is terminating. The converse is false in general as we will see in Exercise 2.4.7.

2.4.6 Example. Consider the linear 2-polygraph $\Lambda = \langle * \mid x, y \mid x^2 \overset{\alpha}{\Rightarrow} xy - y^2 \rangle$. It is Tietze equivalent to the linear 2-polygraph of Example 2.3.2, but it is terminating. Indeed, having $xy \prec x^2$ and $y^2 \prec x^2$, the linear 2-polygraph Λ is compatible with the deglex order $\prec_{\text{deglex}}$ induced by $y \prec x$; hence, it is terminating. Another way to prove that Λ is terminating is to count the number of occurrence of x in monomials. For any u in Λ_1^*, let denote by $A(u)$ the number of occurrence of x in u. To prove that the linear 2-polygraph Λ terminates, it is sufficient to check that, for every rewriting step $a : s_1(a) \Rightarrow f$, we have $A(s_1(a)) > A(v)$, for any monomial v in $\text{Supp}(f)$.

2.4.7 Exercise, [33, **Ex. 3.2.4**]. Show that the linear 2-polygraph Λ given in Example 2.1.9 is terminating. Show that Λ is not compatible with a monomial order.

2.4.8 Exercise, [12, **Ex. 5.2.1**]. Examine termination of the linear 2-polygraph $\langle * \mid x, y \mid \alpha \rangle$ in each of the following situations

$$x^2y \overset{\alpha}{\Rightarrow} yx, \qquad yx \overset{\alpha}{\Rightarrow} x^2y, \qquad x^2y^2 \overset{\alpha}{\Rightarrow} yx, \qquad yx \overset{\alpha}{\Rightarrow} x^2y^2.$$

2.4.9 Noetherian induction principle. Let us recall the principle of Noetherian induction for terminating rewriting systems, see [39] for more details. Let Λ be a left-monomial terminating linear 2-polygraph. The principle can be used to prove by induction a property formulated on the 1-cells of Λ_1^ℓ. Given a property $\mathcal{P}(f)$ of the 1-cells f of Λ_1^ℓ. In order to show that $\mathcal{P}(f)$ holds for any 1-cell f of Λ_1^ℓ, it suffices to show that

(i) $\mathcal{P}(f)$ holds for f reduced with respect to Λ_2,
(ii) $\mathcal{P}(f)$ holds under the assumption that $\mathcal{P}(g)$ holds for every $g \prec f$.

2.4.10 Leading terms. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. For a nonzero 1-cell f of Λ_1^ℓ, the *leading monomial of f with respect to* $\prec$ is the monomial of f, denoted by $\text{lm}(f)$, such that $w \prec \text{lm}(f)$, for any monomial w in the support of f. The *leading coefficient of f* is the coefficient $\text{lc}(f)$ of $\text{lm}(f)$ in f, and the *leading term of f* is the 1-cell $\text{lt}(f) = \text{lc}(f)\text{lm}(f)$ of Λ_1^ℓ. We also define $\text{lt}(0) = \text{lc}(0) = \text{lm}(0) = 0$.

Note that for any 1-cells f and g in Λ_1^ℓ, we have $f \prec g$ if and only if either $\text{lm}(f) \prec \text{lm}(g)$ or ($\text{lm}(f) = \text{lm}(g)$ and $f - \text{lt}(f) \prec g - \text{lt}(g)$). The following property

$$\text{lt}(fg) = \text{lt}(f)\text{lt}(g),$$

for any 1-cells f and g is also useful.

2.4.11. Leading polygraph. Given a monomial order $\prec$ on Λ_1^ℓ and a nonzero 1-cell g in Λ_1^ℓ, we define the 2-cell:

$$\alpha_{g,\prec} : \text{lm}(g) \Rightarrow \text{lm}(g) - \frac{1}{\text{lc}(g)}g.$$

For any set $\mathcal{G}$ of nonzero 1-cells in Λ_1^ℓ, the *leading* 2*-polygraph* associated to $\mathcal{G}$ with respect to $\prec$ is the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ whose set of 1-cells is Λ_1 and

$$\Lambda(\mathcal{G}, \prec)_2 = \{\alpha_{g,\prec} \;\mid\; g \in \mathcal{G}\}.$$

By definition, the leading polygraph $\Lambda(\mathcal{G} \prec)$ is compatible with the monomial order $\prec$.

A monomial w in Λ_1^* is $\mathcal{G}$*-reduced with respect to the monomial order* $\prec$ if it reduced with respect to $\Lambda(\mathcal{G}, \prec)_2$, that is, there is no factorization $w = u\mathrm{lm}(g)v$, with u and v monomials in Λ_1^* and g in $\mathcal{G}$. A set $\mathcal{G}$ of 1-cells is *reduced with respect to the monomial order* $\prec$ if for any 1-cell g in $\mathcal{G}$, any monomial in the support of g is $(\mathcal{G} \setminus \{g\})$-reduced.

3 Convergence in Linear Rewriting Systems

3.1 Ideal of a Linear 2-Polygraph

3.1.1 The ideal of a linear 2-polygraph. Given a linear 2-polygraph Λ. We denote by $I(\Lambda)$ the two-sided ideal of the free algebra Λ_1^ℓ generated by the following set of 1-cells

$$\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}.$$

The ideal $I(\Lambda)$ is made of the linear combinations

$$\sum_{i=1}^{p} \lambda_i u_i (s_1(\alpha_i) - t_1(\alpha_i)) v_i,$$

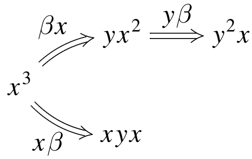
for pairwise distinct 2-monomials $u_1\alpha_1 v_1, \ldots, u_p\alpha_p v_p$ of Λ_1^ℓ, and nonzero scalars $\lambda_1, \ldots, \lambda_p$. Note that the algebra presented by Λ is isomorphic to the quotient of the free algebra Λ_1^ℓ by the ideal $I(\Lambda)$.

3.1.2 Exercise. Let Λ be a linear 2-polygraph. Given 1-cells f and g in Λ_1^ℓ, show that the 1-cell $f - g$ belongs to $I(\Lambda)$ if and only if there exists a 2-cell $a : f \Rightarrow g$ in Λ_2^ℓ.

3.1.3 Suppose that Λ is a terminating left-monomial linear 2-polygraph. Every 1-cell f of Λ_1^ℓ admits at least a normal form $\widetilde{f}$. That is, $\widetilde{f}$ is reduced and there exists a positive 2-cell $a : f \Rightarrow \widetilde{f}$ in Λ_2^ℓ. As a consequence, we have a decomposition $f =: \widetilde{f} + (f - \widetilde{f})$, with $\widetilde{f}$ in Λ_1^{ir} and $f - \widetilde{f}$ in $I(\Lambda)$ by Exercise 3.1.2. It follows that the vector space Λ_1^ℓ admits the following decomposition:

$$\Lambda_1^\ell = \Lambda_1^{ir} + I(\Lambda). \tag{8}$$

3.1.4 Example. Note that the decomposition (8) is not direct in general. Indeed, consider the linear 2-polygraph Λ from Example 2.1.10. It is terminating thanks to the deglex order generated by $x > y$. Consider the two following reduction sequences reducing the 1-cell x^3:



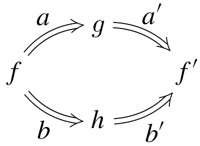
Thus, the 1-cell

$$xyx - y^2x = (x^2 - yx)x - x(x^2 - yx) + y(x^2 - yx)$$

is both in Λ_1^{ir} and $I(\Lambda)$. It follows that the sum $\Lambda_1^{ir} + I(\Lambda)$ is not direct. We will see in the next section a sufficient condition on the linear 2-polygraph Λ to have a direct decomposition.

3.2 Confluence and Convergence

3.2.1 Branchings and confluence. Let Λ be a left-monomial linear 2-polygraph. A *branching of* Λ is a non-ordered pair (a, b) of positive 2-cells of Λ_2^{ℓ} with a common *source* $s_1(a) = s_1(b)$. A branching (a, b) is *local* if both a and b are rewriting steps of Λ. A branching (a, b) of Λ is *confluent* if there exist positive 2-cells a' and b' of Λ as in the following diagram:



We say that Λ is *confluent* (resp. *locally confluent*) if every branching (resp. local branching) of Λ is confluent. An immediate consequence of the confluence property is that every 1-cell of Λ_1^{ℓ} admits at most one normal form.

Under termination hypothesis, we have the following characterization of the confluence.

3.2.2 Proposition. *Let Λ be a terminating left-monomial linear 2-polygraph. The following conditions are equivalent:*

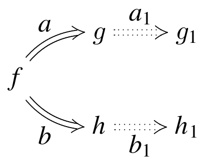
1. *Λ is confluent.*
2. *Every 1-cell of $I(\Lambda)$ admits 0 as a normal form with respect to Λ_2.*

3. *The vector space* Λ_1^ℓ *admits the direct decomposition* $\Lambda_1^\ell = \Lambda_1^{ir} \oplus I(\Lambda)$.

Proof **(i)** $\Rightarrow$ **(ii)**. Let f be a 1-cell in the ideal $I(\Lambda)$, then there exists a 2-cell $a : f \Rightarrow 0$ in Λ_2^ℓ. The polygraph Λ being confluent, the 1-cells f and 0 have the same normal form. Finally, 0 being reduced, this implies that 0 is a normal form for f.

(ii) $\Rightarrow$ **(iii)**. Prove that $\Lambda_1^{ir} \cap I(\Lambda) = 0$. If f is in Λ_1^{ir}, then $\widehat{f} = f$ is reduced and, thus, admits itself as normal form. If f is in $I(\Lambda)$, then $\widehat{f} = 0$ by **(ii)**. Hence $\Lambda_1^{ir} \cap I(\Lambda) = 0$.

(iii) $\Rightarrow$ **(i)**. Given a branching $(f \overset{a}{\Rightarrow} g, f \overset{b}{\Rightarrow} h)$. Since Λ terminates, the 1-cells g and h admit normal forms, say g_1 and h_1, respectively, and there exist positive 2-cells a_1 and b_1 in Λ_2^ℓ:



with g_1 and h_1 reduced. It follows that $g_1 - h_1$ is also reduced. Moreover, the 2-cell $(a \star_1 a_1)^- \star_1 (b \star b_1)$ has g_1 as source and h_1 as target. This implies that $g_1 - h_1$ is also in $I(\Lambda)$. As $\Lambda_1^{ir} \cap I(\Lambda) = 0$, we have $g_1 - h_1 = 0$; hence, the branching (a, b) is confluent. □

3.2.3 Convergence. We say that a left-monomial linear 2-polygraph Λ is *convergent* if it terminates, and it is confluent. In that case, every 1-cell f of Λ_1^ℓ has a unique normal form, denoted by $\widehat{f}$, such that $\overline{f} = \overline{g}$ holds in $\overline{\Lambda}$ if and only if $\widehat{f} = \widehat{g}$ holds in Λ_1^ℓ.

As a consequence, if Λ is a convergent presentation of an algebra $\mathbf{A}$, the assignment of every 1-cell f of $\mathbf{A}$ to the normal form $\widehat{f}$, defines a section $\iota : \mathbf{A} \longrightarrow \Lambda_1^\ell$ of the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$. The section ι is a linear map, i.e. it satisfies $\widehat{\lambda f + \mu g} = \lambda \widehat{f} + \mu \widehat{g}$, and it preserves the identities because Λ terminates.

3.2.4 Exercise. Show that the section ι is not a morphism of algebras in general.

3.2.5 Suppose that Λ is a convergent linear 2-polygraph. By Proposition 3.2.2, the following sequence of vector spaces is exact:

$$0 \to I(\Lambda) \to \Lambda_1^\ell \to \Lambda_1^{ir} \to 0.$$

The vector space Λ_1^{ir} admits Λ_1^{irm} as a basis, hence Λ_1^{irm} forms a linear basis of the quotient algebra $\Lambda_1^\ell / I(\Lambda)$. The polygraph Λ being convergent, any 1-cell of Λ_1^ℓ has a unique normal form; hence, the product defined by $f \cdot g = \widehat{fg}$ is associative. Indeed, for any 1-cells f, g and h, we have

$$(f \cdot g) \cdot h = \widehat{fg} \cdot h = \widehat{\widehat{fg}h} = \widehat{f\widehat{gh}} = f \cdot \widehat{gh} = f \cdot (g \cdot h).$$

It follows that this product equips Λ_1^{ir} with a structure of algebra in such a way that Λ_1^{ir} is isomorphic to the quotient algebra $\Lambda_1^{\ell}/I(\Lambda)$. We have thus proved the following result.

3.2.6 Theorem ([33, Thm 3.4.2]). *Let* **A** *be an algebra and* Λ *be a convergent presentation of* **A**. *The set* Λ_1^{irm} *of reduced monomials is a linear basis of* **A**. *Moreover, the vector space* Λ_1^{ir} *equipped with the product defined by* $f \cdot g = \widehat{fg}$, *for any* 1*-cells* f *and* g *in* Λ_1^{ir}, *is an algebra isomorphic to* **A**.

3.2.7 Exercise. Compute a linear basis of the algebra presented by

$$\langle * \mid x, y \mid xy = x^2 \rangle.$$

3.2.8 Exercise. Compute a linear basis for the symmetric algebra on k variables presented by

$$\langle x_1, \ldots, x_k \mid x_i x_j \overset{\tau_{ij}}{\Rightarrow} x_j x_i \mid 1 \leqslant i < j \leqslant k \rangle$$

and for the skew-polynomial algebra on k variables presented by

$$\langle x_1, \ldots, x_k \mid x_i x_j \overset{\tau_{ij}}{\Rightarrow} q_i^j x_j x_i \mid 1 \leqslant i < j \leqslant k \rangle,$$

where q_i^j are scalars in $\mathbb{K}$.

3.2.9 Exercise: Poincaré–Birkhoff–Witt Theorem [13, **§1**], [12, **Thm. 3.1**]. Consider an ordered basis $x_1 \prec x_2 \prec \ldots \prec x_k$ of a Lie algebra $\mathfrak{g}$. Consider the following ideals of the free tensor algebra $T(\mathfrak{g})$ over $\mathfrak{g}$:

$$I = \langle x_j x_i - x_i x_j \mid 1 \leqslant i < j \leqslant k \rangle,$$
$$J = \langle x_j x_i - x_i x_j + [x_i, x_j] \mid 1 \leqslant i < j \leqslant k \rangle.$$

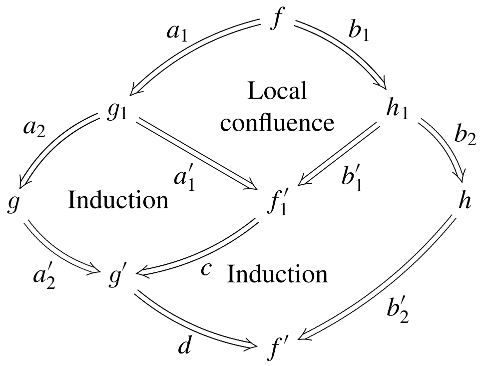
Show that the symmetric algebra $S(\mathfrak{g}) = T(\mathfrak{g})/I$ and the enveloping algebra $U(\mathfrak{g}) = T(\mathfrak{g})/J$ are isomorphic as vector spaces.

3.2.10 From local to global confluence. The Newman lemma, also called the diamond lemma, states that for terminating rewriting systems local confluence and confluence are equivalent properties. This result was proved by Newman in [52] for abstract rewriting systems. A short and simple proof of this result was given by Huet in [39] using the principle of Noetherian induction. Let us recall the arguments of this proof for linear 2-polygraphs.

3.2.11 Theorem (Newman's Lemma). *Let* Λ *be a terminating left-monomial linear* 2*-polygraph. Then* Λ *is confluent if and only if it is locally confluent.*

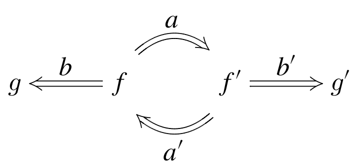
Proof The proof works as for abstract rewriting systems. One implication is trivial. Suppose Λ locally confluent and prove that it is confluent at every 1-cell f of Λ_1^ℓ. We proceed by Noetherian induction on f using the principle given in 2.4.9. If f is reduced, the only branching with source f is $(1_f, 1_f)$ which is confluent.

Suppose that f is a nonreduced 1-cell of Λ_1^ℓ and such that Λ is confluent at every 1-cell $g \prec f$. Consider a branching (a, b) of Λ with source f. If a or b is an identity, then (a, b) is confluent. Otherwise, we prove that the branching (a, b) is confluent by induction. Since a and b are not identities, they admit decompositions $a = a_1 \star_1 a_2$ and $b = b_1 \star_1 b_2$ where a_1 and b_1 are rewriting steps, and a_2 and b_2 are positive 2-cells. By local confluence, the local branching (a_1, b_1) is confluent. Hence, there exist positive 2-cells a_1' and b_1' as indicated in the following diagram:



We have $g_1 \prec_\Lambda f$ and $h_1 \prec_\Lambda f$. Then we apply the induction hypothesis on the branching (a_2, a_1') to get positive 2-cells a_2' and c, and, then, to the branching $(b_1' \star_1 c, b_2)$ to get positive 2-cells d and b_2', which complete the proof. □

3.2.12 Example, [39]. The requirement of Noetherianity is necessary to prove confluence from local confluence. Indeed, consider the 2-polygraph generated by the following four 2-cells



It is locally confluent but it is not confluent.

3.3 Critical branching lemma

3.3.1 Local branchings. A case analysis leads to a partition of the local branchings of a left-monomial linear 2-polygraph Λ into the following four families, see [33, 3.3.2] for details.

1. *Aspherical* branchings, for all 2-monomial $a : u \Rightarrow f$ of Λ_2^ℓ, nonzero scalar λ, and 1-cell h of Λ_1^ℓ such that the monomial u is not in the support of h:

$$\lambda u + h \; \overset{\lambda a + h}{\underset{\lambda a + h}{\rightrightarrows}} \; \lambda f + h$$

2. *Additive* branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : v \Rightarrow g$ of Λ_2^ℓ, nonzero scalars λ and μ, and 1-cell h of Λ_1^ℓ such that the monomials u and v are not in the support of h:

$$\lambda f + \mu v + h \; \overset{\lambda a + \mu v + h}{\Longleftarrow} \; \lambda u + \mu v + h \; \overset{\lambda u + \mu b + h}{\Longrightarrow} \; \lambda u + \mu g + h$$

3. *Peiffer* branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : v \Rightarrow g$ of Λ_2^ℓ, nonzero scalar λ, and 1-cell h of Λ_1^ℓ such that the monomial uv is not in the support of h:

$$\lambda f v + h \; \overset{\lambda a v + h}{\Longleftarrow} \; \lambda u v + h \; \overset{\lambda u b + h}{\Longrightarrow} \; \lambda u g + h$$

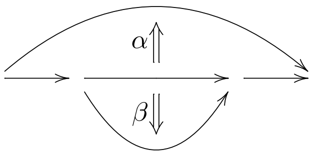
4. *Overlapping* branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : u \Rightarrow g$ of Λ_2^ℓ such that the branching (a, b) is neither aspherical nor Peiffer, and all nonzero scalar λ and 1-cell h of Λ_1^ℓ such that the monomial u is not in the support of h:

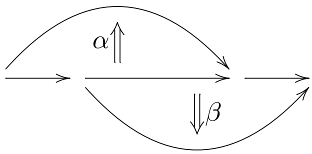
$$\lambda f + h \; \overset{\lambda a + h}{\Longleftarrow} \; \lambda u + h \; \overset{\lambda b + h}{\Longrightarrow} \; \lambda g + h$$

3.3.2 Critical branchings. A *critical branching of a left-monomial linear 2-polygraph* Λ is an overlapping branching, as defined in 3.3.1, with $\lambda = 1$ and $h = 0$, and that is minimal for the relation on branchings defined by

$$(a, b) \sqsubseteq (waw', wbw') \quad \text{for any } w \text{ and } w' \text{ in } \Lambda_1^*.$$

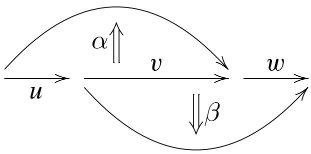
By case analysis on the source of critical branchings, they must have one of the following two shapes:





with α, β in Λ_2. When the linear 2-polygraph Λ is reduced, the first case cannot occur since, otherwise, the monomial $s_1(\alpha)$ would be reducible by β.

3.3.3 Exercise. Let Λ be a reduced linear 2-polygraph. Show that for any critical branching



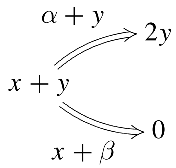
the monomial u, v and w are reduced and cannot be identities or null.

3.3.4 Critical Branching Lemma. By Newman's lemma 3.2.11, for terminating rewriting systems, local confluence and confluence are equivalent properties. It turns out that one can decide whether a rewriting system is convergent by checking local confluence. For string rewriting systems, that is, 2-polygraphs, the critical branching lemma states that local confluence is equivalent to the confluence of all critical branchings, see [36, 3.1.5] for details. For linear 2-polygraphs the critical branching lemma given in [33] differs from the case of 2-polygraphs. Indeed, in the linear setting, the termination hypothesis is required. Moreover, nonoverlapping branchings may be non-confluent as illustrated by the following example in which an additive branching is nonconfluent.

3.3.5 Example. Some local branchings can be nonconfluent without termination, even if critical confluence holds. Indeed, consider the linear 2-polygraph

$$\langle * \mid x, y \mid x \overset{\alpha}{\Rightarrow} y,\ y \overset{\beta}{\Rightarrow} -x \rangle$$

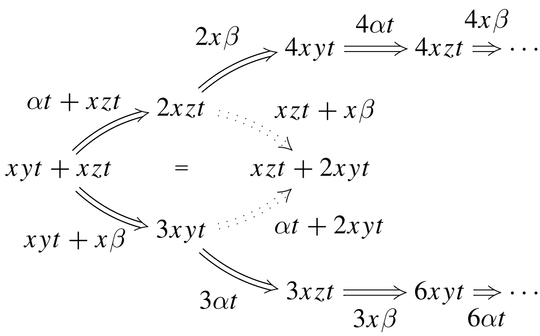
It has no critical branching, but it has a nonconfluent additive branching:



Here another example from [33, Rem. 4.2.4], for instance, the following linear 2-polygraph

$$\langle * \mid x, y, z, t \mid xy \overset{\alpha}{\Rightarrow} xz,\ zt \overset{\beta}{\Rightarrow} 2yt \rangle$$

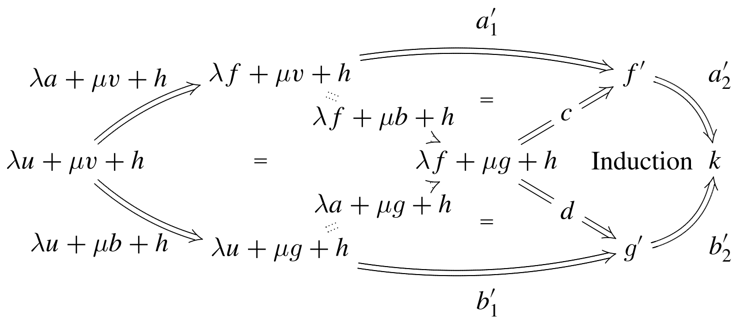
has no critical branching, but it has a nonconfluent additive branching:



3.3.6 If a linear 2-polygraph Λ is terminating and with any critical branching confluent, we can show that such an additive branching is confluent by Noetherian induction on the sources of the branchings. Let consider an additive branching $(\lambda u + \mu v + h, \lambda u + \mu g + h)$ as in 3.3.1 and suppose that Λ is locally confluent at every $g \prec_\Lambda \lambda u + \mu v + h$. By linearity of the 1-composition, the following equation

$$(\lambda a + \mu v + h) \star_1 (\lambda f + \mu b + h) = (\lambda u + \mu b + h) \star_1 (\lambda a + \mu g + h)$$

holds in the free 2-algebra Λ_2^ℓ:



Note that the dotted 2-cells $\lambda a + \mu g + h$ and $\lambda f + \mu b + h$ may be not positive in general. Indeed, the monomial u can be in the support of g or the monomial v can be in the support of f, as illustrated in Example 3.3.5. However, those 2-cells are elementary; hence, there exist, see Exercise 2.2.6, positive 2-cells a'_1, b'_1, c and d that satisfy

$$a'_1 = (\lambda f + \mu b + h) \star_1 c \qquad \text{and} \qquad b'_1 = (\lambda a + \mu g + h) \star_1 d.$$

We have $f \prec_\Lambda u$ and $g \prec_\Lambda v$, hence $\lambda f + \mu g + h \prec_\Lambda \lambda u + \mu v + h$. Thus, the branching (c, d) is confluent by induction hypothesis, yielding the positive 2-cells a'_2 and b'_2.

Under terminating hypothesis, all local branching given in 3.3.1 are confluent if all critical branchings are confluent, see [33, 4.2] for a proof of this result.

3.3.7 Theorem (Critical branching lemma *[33,* **Cor. 4.2.2]).** *A terminating left-monomial linear* 2*-polygraph is locally confluent if and only if all its critical branchings are confluent.*

As a consequence of the critical branching lemma and of Newman's lemma 3.2.11, a terminating left-monomial linear 2-polygraph is confluent if all its critical branchings are confluent. In particular, a terminating left-monomial 2-polygraph with no critical branching is convergent.

3.3.8 Example. The linear 2-polygraph given in Example 2.1.9 is terminating, see Exercise 2.4.7. Moreover, it does not have critical branching; hence, it is convergent.

3.3.9 The Knuth–Bendix completion procedure. Let us recall the completion procedure introduced in [44] to the setting of linear 2-polygraphs. Let Λ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on Λ_1^*. A *Knuth–Bendix completion* of Λ is a linear 2-polygraph $\mathcal{KB}(\Lambda)$ obtained by the following procedure that examines the confluence of the set of critical branchings.

If the procedure stops, it returns a finite convergent left-monomial linear 2-polygraph $\mathcal{KB}(\Lambda)$. Otherwise, it builds an increasing sequence of left-monomial linear 2-polygraphs, whose limit is also denoted by $\mathcal{KB}(\Lambda)$. Note that, if the starting

Input: Λ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on Λ_1^*.
$\mathcal{KB}(\Lambda) := \Lambda$
$\mathcal{C}b := \{$ critical branchings with respect to $\Lambda_2 \}$
while $\mathcal{C}b \neq \emptyset$ **do**
  Picks a branching in $\mathcal{C}b$:

$$u \overset{f}{\Rightarrow} v, \qquad u \underset{g}{\Rightarrow} w$$

  $\mathcal{C}b := \mathcal{C}b \setminus \{(f, g)\}$
  Reduce v to a normal form $\widehat{v}$ with respect to $\mathcal{KB}(\Lambda)_2$
  Reduce w to a normal form $\widehat{w}$ with respect to $\mathcal{KB}(\Lambda)_2$

$$u \overset{f}{\Rightarrow} v \Rightarrow \widehat{v}, \qquad u \underset{g}{\Rightarrow} w \Rightarrow \widehat{w}$$

  $g = \widehat{v} - \widehat{w}$
  if $g \neq 0$ **then**
    $\mathcal{KB}(\Lambda)_2 : \; :=: \mathcal{KB}(\Lambda)_2 \cup \{ \alpha_{g,\prec} : \mathrm{lm}(g) \Rightarrow \mathrm{lm}(g) - \frac{1}{\mathrm{lc}(g)} g \}$
    $\mathcal{C}b : \; :=: \mathcal{C}b \cup \{$ critical branching created by $\alpha_{g,\prec} \}$
  end
end

Fig. 1 The Knuth–Bendix completion procedure

linear 2-polygraph Λ is convergent, then the Knuth–Bendix completion of Λ is Λ itself. The linear 2-polygraph $\mathcal{KB}(\Lambda)$ obtained by this procedure depends on the order of examination of the critical branchings. Finally, since all the operations of adding new rules performed by the procedure are Tietze transformations, the linear 2-polygraph $\mathcal{KB}(\Lambda)$ is Tietze equivalent to Λ (Fig. 1).

3.3.10 Exercise, [33, **Rem. 4.2.4**]. Prove that the following linear 2-polygraph has a nonconfluent Peiffer branching:

$$\langle * \mid x, y, z \mid xy \overset{\alpha}{\Rightarrow} 2x,\; yz \overset{\beta}{\Rightarrow} z \rangle.$$

3.3.11 Weyl algebras. Let $\mathbb{K}$ be a field of characteristic zero. The *Weyl algebra* of dimension n over $\mathbb{K}$ is the algebra presented by the linear 2-polygraph whose 1-cells are

$$x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$$

and with the following 2-cells:

$$x_i x_j \Rightarrow x_j x_i, \quad \partial_i \partial_j \Rightarrow \partial_j \partial_i, \quad \partial_i x_j \Rightarrow x_j \partial_i, \qquad \text{for any } 1 \leqslant i < j \leqslant n,$$

$$\partial_i x_i \Rightarrow x_i \partial_i + 1, \qquad \text{for any } 1 \leqslant i \leqslant n.$$

This polygraph is convergent with the following six families of confluent critical branchings:

$$x_i x_j x_k \Rightarrow x_j x_i x_k \Rightarrow x_j x_k x_i \Rightarrow x_k x_j x_i, \qquad x_i x_j x_k \Rightarrow x_i x_k x_j \Rightarrow x_k x_i x_j \Rightarrow x_k x_j x_i$$

$$\partial_i \partial_j \partial_k \Rightarrow \partial_j \partial_i \partial_k \Rightarrow \partial_j \partial_k \partial_i \Rightarrow \partial_k \partial_j \partial_i, \qquad \partial_i \partial_j \partial_k \Rightarrow \partial_i \partial_k \partial_j \Rightarrow \partial_k \partial_i \partial_j \Rightarrow \partial_k \partial_j \partial_i$$

$$\partial_i x_j x_k \Rightarrow x_j \partial_i x_k \Rightarrow x_j x_k \partial_i \Rightarrow x_k x_j \partial_i, \qquad \partial_i x_j x_k \Rightarrow \partial_i x_k x_j \Rightarrow x_k \partial_i x_j \Rightarrow x_k x_j \partial_i$$

$$\partial_i \partial_j x_k \Rightarrow \partial_j \partial_i x_k \Rightarrow \partial_j x_k \partial_i \Rightarrow x_k \partial_j \partial_i, \qquad \partial_i \partial_j x_k \Rightarrow \partial_i x_k \partial_j \Rightarrow x_k \partial_i \partial_j \Rightarrow x_k \partial_j \partial_i$$

$$\partial_i x_j x_k \Rightarrow x_i \partial_i x_j + x_j \Rightarrow x_i x_j \partial_i + x_j \Rightarrow x_j x_i \partial_i + x_j, \qquad \partial_i x_j x_k \Rightarrow \partial_i x_j x_i \Rightarrow x_j \partial_i x_i \Rightarrow x_j x_i \partial_i + x_j$$

$$\partial_i \partial_j x_j \Rightarrow \partial_j \partial_i x_j \Rightarrow \partial_j x_j \partial_i \Rightarrow x_j \partial_j \partial_i + \partial_i, \qquad \partial_i \partial_j x_j \Rightarrow \partial_i x_j \partial_j + \partial_i \Rightarrow x_j \partial_i \partial_j + \partial_i \Rightarrow x_j \partial_j \partial_i + \partial_i$$

where $1 \leqslant i < j \leqslant n$.

3.3.12 Exercise. In his seminal paper on the diamond lemma, Bergman points out that he was first led to the ideas of his paper with the following American Mathematical Monthly Advanced Problem 5082, [12, 2.1].

Let R *be a ring in which, if either* $x + x = 0$ *or* $x + x + x = 0$, *it follows that* $x = 0$. *Suppose that* a, b, c *and* $a + b + c$ *are all idempotents in* R. *Does it follows that* $ab = 0$?

Solve this problem. [Hints. Consider the following linear 2-polygraph:

$$\Lambda = \langle * \mid a, b, c \mid a^2 \Rightarrow a,\ b^2 \Rightarrow b,\ c^2 \Rightarrow c,\ ba \Rightarrow -ab - bc - cb - ac - ca \rangle.$$

(1) List all critical branchings of Λ. (2) Compute a convergent left-monomial linear 2-polygraph $\mathcal{KB}(\Lambda)$ by applying the Knuth–Bendix completion procedure to Λ. (3) List all irreducible monomials with respect to $\mathcal{KB}(\Lambda)_2$. (4) Conclude that $ab \neq 0$.]

3.4 Composition Lemma

3.4.1 Compositions in free lie algebras. Shirshov introduced in [57] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of *composition* of elements in a free Lie algebra that corresponds to the notion of *S-polynomial* in the work of Buchberger [18]. This work remained unknown outside the USSR and the two theories were developed in parallel. The algorithm *completes* a given set of elements in a free algebra by adding all nontrivial compositions. This algorithm corresponds to the completion algorithm given by Knuth–Bendix for term rewriting systems [44], and by Buchberger for commutative polynomials [18]. The Shirshov completion constructs a set that may be infinite, such that every composition of its elements is trivial. Such a subset is called a *Lie Gröbner–Shirshov basis*. The key result in [57] states that the set of irreducible elements for a Gröbner–Shirshov basis $\mathcal{S}$ forms a linear basis of the Lie algebra with defining relations $\mathcal{S}$. This result is called now the *composition-diamond lemma* for Lie algebras. For a recent account of the theory of Gröbner–Shirshov, we refer the reader to [14].

In this subsection, we summarize without proofs an analogue of Shirshov's composition-diamond lemma for associative algebras given by Bokut in [13].

3.4.2 Compositions. Bokut introduced in [13] the notion of composition of elements of a free associative algebra as follows. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. Given two 1-cells f and g in Λ_1^ℓ and a monomial w in Λ_1^*. There are two kinds of compositions:

(i) if $w = \mathrm{lm}(f)v = u\mathrm{lm}(g)$ with $\ell(\mathrm{lm}(f)) + \ell(\mathrm{lm}(g)) > \ell(w)$, for some monomials u and v in Λ_1^*, then the 1-cell

$$(f, g)_w = \frac{1}{\mathrm{lc}(f)} f v - \frac{1}{\mathrm{lc}(g)} u g$$

is called the *intersection composition* of f and g with respect to w.

(ii) if $w = \mathrm{lm}(f) = u\mathrm{lm}(g)v$, for some monomials u and v in Λ_1^*, then the 1-cell

$$(f, g)_w = \frac{1}{\mathrm{lc}(f)} f - \frac{1}{\mathrm{lc}(g)} u g v$$

is called the *inclusion composition* of f and g with respect to w.

A composition $(f, g)_w$ can also be called an *S-polynomial* of f and g with respect to w. A composition $(f, g)_w$ is either zero or satisfy $(f, g)_w \prec w$. Moreover, the composition $(f, g)_w$ is in the ideal $\langle f, g\rangle$ generated by f and g. Note that a composition $(f, g)_w$ depends on the two polynomials f and g as well as the monomial w. Indeed, in some cases two polynomials f and g may overlap with different combinations creating several compositions.

3.4.3 Example. Consider the polynomial $f = x^2 - xy$. With respect to the deglex order generated by $x > y$, we have

$$(f, f)_{x^3} = x^3 - xyx - x^3 + x^2y = x^2y - xyx.$$

Compare with Example 3.1.4.

3.4.4 Gröbner–Shirshov bases. Let $\mathcal{G}$ be a set of nonzero 1-cells in Λ_1^ℓ. Given a monomial w in Λ_1^*, a 1-cell h is *trivial modulo* $(\mathcal{G}, w)$ if there exists a decomposition

$$h = \sum_{i \in I} \lambda_i u_i g_i v_i,$$

with λ_i in $\mathbb{K} \setminus \{0\}$, u_i, v_i in Λ_1^* and g_i in $\mathcal{G}$ such that $u_i \mathrm{lm}(g_i) v_i \prec w$.

A set $\mathcal{G}$ of nonzero 1-cells in Λ_1^ℓ is a *Gröbner–Shirshov basis* with respect to the monomial ordering $\prec$ if every composition $(f, g)_w$ of 1-cells in $\mathcal{G}$ is trivial modulo $(\mathcal{G}, w)$. A Gröbner–Shirshov basis $\mathcal{G}$ is *minimal* if there is no inclusion composition with elements of $\mathcal{G}$. A minimal Gröbner–Shirshov basis $\mathcal{G}$ is called *closed under composition* in [13]. Finally, a Gröbner–Shirshov basis $\mathcal{G}$ is *reduced* if the set $\mathcal{G}$ is reduced with respect to the monomial order $\prec$.

3.4.5 Exercise. Let $\mathcal{G}$ be a minimal Gröbner–Shirshov basis in a free algebra Λ_1^ℓ. Suppose that there exists a decomposition

$$w = u_1 \mathrm{lm}(g_1) v_1 = u_2 \mathrm{lm}(g_2) v_2,$$

with $u_1, v_1, u_2, v_2 \in \Lambda_1^*$ and $g_1, g_2 \in \mathcal{G}$. Show that $u_1 g_1 v_1 - u_2 g_2 v_2$ is trivial modulo $(\mathcal{G}, w)$.

3.4.6 Theorem (The composition lemma *[13,* **Prop. 1 & Cor. 1]).** *Let Λ_1^ℓ be a free algebra and let $\prec$ be a monomial order on Λ_1^ℓ. Let $\mathcal{G}$ be a set of* 1*-cells in Λ_1^ℓ and let I be the ideal generated by $\mathcal{G}$. The following conditions are equivalent:*

(i) *$\mathcal{G}$ is a Gröbner–Shirshov basis.*
(ii) *For any f in I, there exists a factorization* $\mathrm{lm}(f) = u\mathrm{lm}(g)v$ *for some u, v in Λ_1^* and g in $\mathcal{G}$.*
(iii) *The set of $\mathcal{G}$-reduced monomial forms a linear basis of the algebra given by the quotient of the free algebra Λ_1^ℓ by the ideal I.*

3.5 Reduction Operators

3.5.1 Reduction operators. Another approach of rewriting in associative algebras was developed by Bergman in [12]. With a functional description of linear rewriting reductions, he obtained an equivalent result of the composition lemma 3.4.6. Given Λ_1^ℓ a free algebra over a set Λ_1, he defines a *reduction system* as a set S of pairs $\sigma = (w_\sigma, f_\sigma)$, where w_σ is a monomial of Λ_1^ℓ and f_σ is a 1-cell of Λ_1^ℓ. Given σ in S and two monomials u, v in Λ_1^*, he considers the $\mathbb{K}$-linear map $r_{u\sigma v} : \Lambda_1^\ell \longrightarrow \Lambda_1^\ell$ defined by

$$r_{u\sigma v}(w) = \begin{cases} u f_\sigma v & \text{if } w = u w_\sigma v, \\ w & \text{otherwise.} \end{cases}$$

The endomorphism $r_{u\sigma v}$ is called *reduction by* σ. Note that this notion of reduction corresponds to the notion of rewriting step given in 2.2.4.

A 1-cell f in Σ_1^ℓ is *irreducible under* S if every reduction by elements of S acts trivially on f, that is $uw_\sigma v$ is not in the support of f, for any σ in S and monomials u, v in Σ_1^*. As in the case of linear 2-polygraphs, we denote by Λ_1^{ir} the vector subspace of Λ_1^ℓ of all irreducible 1-cells of Λ_1^ℓ.

3.5.2 Reduction-Unique. Bergman introduced the notion of confluence for reduction systems as follows. A finite sequence of reductions $r_1, \ldots, r_n$ is *final* on a 1-cell f, if the 1-cell $r_n \ldots r_1(f)$ is irreducible. A 1-cell f of Λ_1^ℓ is *reduction-finite* if for any infinite sequence $(r_n)_{n\geqslant 1}$ of reductions, r_i acts trivially on $r_{i-1} \ldots r_1(f)$ for a sufficiently large i. A 1-cell f is *reduction-unique* if it is reduction-finite and if its images under all final sequences of reduction are the same. This common image is denoted by $r_S(f)$. A reduction system S is *reduction-unique* if all 1-cells of Λ_1^ℓ are reduction-unique under S.

3.5.3 Exercise, [12, **Lemma 1.1].**

(1) Show that the set of reduction-unique 1-cells of Λ_1^ℓ forms a subspace of Λ_1^ℓ denoted by Λ_1^{ru} and that $r_S : \Lambda_1^{ru} \to \Lambda_1^{\mathrm{irr}}$ defines a linear map.

(2) Given monomials w_f, w_g and w_h in the support of the 1-cells f, g and h, respectively, such that the product $w_f w_g w_h$ is in Λ_1^{ru}. Show that for any finite composition of reductions r, then $fr(g)h$ is in Λ_1^{ru} and that $r_S(fr(g)h) = r_S(fgh)$ holds.

3.5.4 Ambiguities. A 5-tuple (σ, τ, u, v, w) with σ, τ in S and u, v, w monomials in Λ_1^*, such that $w_\sigma = uv$ and $w_\tau = vw$ (resp. $\sigma \neq \tau$, $w_\sigma = v$ and $w_\tau = uvw$) is an *overlap ambiguity* (resp. *inclusion ambiguity*) of S. Such an ambiguity is *resolvable* if there exist compositions of reductions r and r' that satisfy the *confluence condition*:

$$r(f_\sigma w) = r'(uf_\tau) \quad \big(\text{resp.} \quad r(uf_\sigma w) = r'(f_\tau)\big)\,.$$

3.5.5 Reduction system compatible with a monomia order. The diamond lemma obtained by Bergman concern reduction systems compatible with a monomial order. A reduction system S is *compatible* with a monomial order $\prec$, if for any $\sigma = (w_\sigma, f_\sigma)$ in S, we have $w \prec w_\sigma$ for any monomial w in the support of f_σ.

Given a reduction system compatible with a monomial order $\prec$. For a monomial w in Σ_1^*, we denote by $I_{\prec w}$ the subspace of Λ_1^ℓ defined by

$$I_{\prec w} = \mathrm{Span}_{\mathbb{K}}\big(u(w_\sigma - f_\sigma)v \mid (w_\sigma, f_\sigma) \in S \text{ and } uw_\sigma v \prec w\big).$$

An overlap ambiguity (resp. inclusion ambiguity) (σ, τ, u, v, w) is *resolvable relative to* $\prec$ if

$$f_\sigma w - uf_\tau \in I_{\prec uvw}, \quad \big(\text{resp. } uf_\sigma w - f_\tau \in I_{\prec uvw}\,\big).$$

Let $\mathcal{G}$ be a subset of 1-cells of Λ_1^ℓ and let $\prec$ be a monomial order on Λ_1^ℓ. We denote by $S(\mathcal{G}, \prec)$ the *reduction system generated by* $\mathcal{G}$ with respect to $\prec$ defined by

$$S(\mathcal{G}, \prec) = \{ (\mathrm{lm}(f), \mathrm{lm}(f) - \frac{1}{\mathrm{lc}(f)} f) \mid f \in \mathcal{G} \}.$$

3.5.6 Theorem (The Diamond Lemma *[12,* **Thm. 1.2]).** *Let S be a reduction system compatible with a monomial order $\prec$. The following conditions are equivalent:*

(i) *All the ambiguities of S are resolvable.*
(ii) *All the ambiguities of S are resolvable relative to $\prec$.*
(iii) *S is reduction-unique.*

A fourth equivalent condition is given in [12, Thm. 1.2] as follows. Consider the algebra $\mathbf{A}$ given as the quotient of the free algebra Λ_1^ℓ by the two-side ideal

$$I(S) = \{ w_\sigma - f_\sigma \mid \sigma \in S \}.$$

If the reduction system S is compatible with a monomial order $\prec$, the confluence conditions **(i)**–**(iii)** above hold if and only if the set Λ_1^{irm} of irreducible monomial under S is a linear basis of the algebra $\mathbf{A}$. In this case, the $\mathbb{K}$-algebra $\mathbf{A}$ is isomorphic to the $\mathbb{K}$-algebra Λ_1^{ir}, whose product is given by $f \cdot g = r_S(fg)$, for any 1-cells f and g in Λ_1^{ir}.

3.6 Noncommutative Gröbner bases

3.6.1 Noncommutative Gröbner Bases. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. A *(noncommutative) Gröbner basis* of an ideal I of Λ_1^ℓ with respect to the monomial order $\prec$ is a subset $\mathcal{G}$ of I such that the ideal generated by the leading monomials of the 1-cells of I coincides with the ideal generated by the leading monomials of the 1-cells of $\mathcal{G}$:

$$\langle \mathrm{lm}(I) \rangle = \langle \mathrm{lm}(\mathcal{G}) \rangle.$$

Equivalently, for every 1-cell f in I, there exists g in $\mathcal{G}$ with $\mathrm{lm}(f) = u\mathrm{lm}(g)v$, where u and v are monomials of Λ_1^ℓ.

The two following results show that the notion of noncommutative Gröbner basis corresponds to the notion of left-monomial convergent linear 2-polygraph compatible with a monomial order.

3.6.2 Proposition. *Let Λ be a convergent left-monomial linear 2-polygraph, compatible with a monomial order $\prec$ on Λ_1^ℓ. The set of 1-cells $\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}$ is a Gröbner basis of the ideal $I(\Lambda)$ for the monomial order $\prec$.*

3.6.3 Exercise. Prove Proposition 3.6.2.

3.6.4 Proposition. *Let I be an ideal of a free 1-algebra Λ_1^ℓ. Let $\mathcal{G}$ be a Gröbner basis for I with respect to a monomial order $\prec$. Then the leading 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is convergent and $I(\Lambda(\mathcal{G}, \prec)) = I$ holds.*

Proof Suppose that $\mathcal{G}$ is a Gröbner basis of the ideal I with respect to $\prec$. By definition, the ideal $I(\Lambda(\mathcal{G}, \prec))$ is equal to the ideal I generated by $\mathcal{G}$. Prove that the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is convergent. Its termination is a consequence of its compatibility with the monomial order $\prec$. The monomials in Λ_1^* reduced with respect to $\Lambda(\mathcal{G}, \prec)$ are the monomials that cannot be decomposed as $u\mathrm{lm}(g)v$ with g in $\mathcal{G}$ and u and v monomials in Λ_1^*. As a consequence, if a reduced 1-cell f of Λ_1^ℓ is contained in the ideal I, its leading monomial must be 0, because $\mathcal{G}$ is a Gröbner basis of I. By Proposition 3.2.2, we deduce that the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is confluent. □

The following theorem summarizes results obtained in this section. Note that some equivalences are tautological or reformulations.

3.6.5 Theorem. *Let I be an ideal of a free algebra Λ_1^ℓ over a set Λ_1. Let $\prec$ be a monomial order on Λ_1^ℓ. For a subset $\mathcal{G}$ of I, the following conditions are equivalent:*

(i) *The set $\mathcal{G}$ is a Gröbner basis with respect to $\prec$.*
(ii) *The leading polygraph $\Lambda(\mathcal{G}, \prec)$ is convergent.*
(iii) *The leading polygraph $\Lambda(\mathcal{G}, \prec)$ is confluent.*
(iv) *The leading polygraph $\Lambda(\mathcal{G}, \prec)$ is locally confluent.*
(v) *All the critical branchings of the leading polygraph $\Lambda(\mathcal{G}, \prec)$ are confluent.*
(vi) *The set $\mathcal{G}$ is a Gröbner–Shirshov basis with respect to $\prec$.*
(vii) *All the ambiguities of the reduction system $S(\mathcal{G}, \prec)$ are resolvable.*
(viii) *All the ambiguities of the reduction system $S(\mathcal{G}, \prec)$ are resolvable relative to $\prec$.*
(ix) *The reduction system $S(\mathcal{G}, \prec)$ is reduction-unique.*
(x) $\Lambda_1^\ell = \Lambda_1^{ir} \oplus I$.
(xi) *Every 1-cell of I admits 0 as a normal form with respect to $\Lambda(\mathcal{G}, \prec)_2$.*
(xii) *For any f in I, there exists a decomposition $\mathrm{lm}(f) = u\mathrm{lm}(g)v$ for some u, v in Λ_1^* and g in $\mathcal{G}$.*
(xiii) *The set of $\mathcal{G}$-reduced monomials forms a linear basis of the algebra given by the quotient of Λ_1^ℓ by the ideal I.*

3.6.6 Exercise. Prove the equivalences of Theorem 3.6.5.

3.6.7 Example. Consider the linear 2-polygraph Λ given in Example 2.1.9. For the deglex order $\prec_{\mathrm{deglex}}$ induced by the alphabetic order $x \prec y \prec z$, the leading monomial of $f = z^3 + y^3 + x^3 - xyz$ is z^3, so that

$$\Lambda(\{f\}, \prec_{\mathrm{deglex}}) = \langle * \mid x, y, z \mid z^3 \overset{\alpha_f}{\Rightarrow} xyz - x^3 - y^3 \rangle.$$

The left-monomial linear 2-polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$ is compatible with the monomial order $\prec_{\text{deglex}}$; hence, it is terminating. It is not confluent, because neither of its two critical branchings is confluent:

$$\begin{array}{l} z^4 \overset{\alpha_f z}{\Rightarrow} xyz^2 - x^3z - y^3z \\ z^4 \overset{z\alpha_f}{\Rightarrow} zxyz - zx^3 - zy^3 \end{array}$$

$$\begin{array}{l} z^5 \overset{\alpha_f z^2}{\Rightarrow} xyz^3 - x^3z^2 - y^3z^2 \overset{xy\alpha_f - x^3z^2 - y^3z^2}{\Longrightarrow} xyxyz - xy^4 - xyx^3 - x^3z^2 - y^3z^2 \\ z^5 \overset{z^2\alpha_f}{\Rightarrow} z^2xyz - z^2x^3 - z^2y^3 \end{array}$$

In particular, $\{f\}$ does not form a Gröbner basis of the ideal $I(\Lambda)$. We add to the polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$ the following 2-cell:

$$\beta : zy^3 \Rightarrow zxyz - zx^3 + y^3z + x^3z - xyz^2.$$

This new rule makes the two previous critical branchings confluent and creates a new critical branching

$$\begin{array}{l} z^3y^3 \overset{z^2\beta}{\Rightarrow} z^3xyz - z^3x^3 + z^2y^3z + z^2x^3z - z^2xyz^2 \\ z^3y^3 \overset{\alpha y^3}{\Longrightarrow} xyzy^3 - x^3y^3 - y^6 \end{array}$$

which is also confluent. Finally, the convergent linear 2-polygraph $\langle * \mid x, y, z \mid \alpha_f, \beta \rangle$ is Tietze equivalent to the initial linear 2-polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$. In particular, the set of 1-cells $\{f, s_1(\beta) - t_1(\beta)\}$ forms a Gröbner basis of the ideal $I(\Lambda)$ with respect to the order $\prec_{\text{deglex}}$.

3.6.8 Example. The algebra presented by the following linear 2-polygraph:

$$\langle * \mid x, y, z \mid x^2 = 0,\ xy = zx \rangle$$

does not have a finite Gröbner basis on three generators x, y, and z. Indeed, the first relation is oriented as $x^2 \Rightarrow 0$ and the orientation $xy \Rightarrow zx$ induces the addition of the 2-cells $xz^nx \Rightarrow 0$, for all integer $n \geqslant 1$. Another way is to orient the relation as $zx \Rightarrow xy$. But in this case, we need to add the 2-cells $xy^n x \Rightarrow 0$, for all integer $n \geqslant 1$.

3.6.9 Exercise. Show that we can compute a Gröbner basis for the algebra given in Example 3.6.8 with four generators. [Hint. Add a generator t and the relations $xy \Rightarrow t$ and $zx \Rightarrow t$.]

3.6.10 Exercise. Consider the ideal I generated by the linear 2-polygraph Λ of Example 3.1.4.

(1) Show that $\{ xy^k x - xy^{k+1} \mid k \geqslant 0 \}$ is a Gröbner basis of the ideal I with respect to a monomial order with $x \succ y$.
(2) Compute a Gröbner basis for the ideal I reduced to only one element.

4 Anick's Resolution

In two seminal papers, Anick introduced a method to compute a free resolution for an algebra starting with a Gröbner basis of its ideal of relations. First, he gave the construction for monomial algebras in [1] then for associative augmented algebras in [2]. Resolutions for path algebras using the same method were obtained by Anick and Green in [3]. For a deeper discussion on the theory of Gröbner bases for path algebras and how to apply this theory to the construction of free resolutions for path algebras, we refer the reader to [31]. Let us mention that Anick's resolution has been achieved by other methods. In particular, Anick's resolution for a homogeneous algebra can be constructed by a deformation of the resolution computed on the associated monomial algebra, see [26, Section 2.4] for details, see also the Backelin construction [5]. Anick's resolution can be also obtained using algebraic Morse theory with a Morse matching on the bar resolution, see [58, Section 3.2] for details. Morse theory allows to construct, starting from a chain complex, a new chain complex such that the homology of the two complexes coincides. This method was applied to the computation of minimal resolutions starting from Anick's resolution [42].

Note also that others constructions of free resolutions using convergent rewriting systems were obtained by several authors [17, 32, 35, 45, 46]. Finally, let us mention that noncommutative Gröbner bases were developed by Dotsenko and Khoroshkin for shuffle operads in [27], giving operadic versions of Newman's lemma and Buchberger's algorithm. Anick's resolution for shuffle operads was constructed by Dotsenko and Khoroshkin in [26, 28]. Using this construction, they prove that a shuffle operad with a quadratic Gröbner basis is Koszul [28].

The nth chains in Anick's resolution are generated by the n-fold overlaps of the leading terms of the Gröbner basis and the differentials are constructed by Noetherian induction with respect to the monomial order. The chains defined by Anick are recalled in Sect. 4.2. The construction of the resolution is given in Sect. 4.3. In the first part of this section, we briefly recall the definition of the homology of associative algebras.

4.1 Homology of an Algebra

4.1.1 Functor Tor. Let us recall the definition of the bifunctor $\mathrm{Tor}^{\mathbf{R}}$, where $\mathbf{R}$ is a fixed ring. Let M be a left $\mathbf{R}$-module and N be a right $\mathbf{R}$-module. Given a projective resolution $\mathcal{P}$ of the right $\mathbf{R}$-module N:

$$\mathcal{P}\ :\quad \cdots \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

we associate the *deleted complex*:

$$\mathcal{P}_N\ :\quad \cdots \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_0} P_0 \longrightarrow 0$$

obtained by suppressing the module N. Note that, we have not lost any information in the complex $\mathcal{P}_N$, as $N = \mathrm{Coker}\,(d_0)$ by exactness of complex $\mathcal{P}$. Then, applying the functor $-\otimes_{\mathbf{R}} M$, we form a complex of $\mathbb{Z}$-modules denoted by $\mathcal{P}_N \otimes_{\mathbf{R}} M$:

$$\cdots \longrightarrow P_n \otimes_{\mathbf{R}} M \xrightarrow{\overline{d}_{n-1}} P_{n-1} \otimes_{\mathbf{R}} M \longrightarrow \cdots \longrightarrow P_1 \otimes_{\mathbf{R}} M \xrightarrow{\overline{d}_0} P_0 \otimes_{\mathbf{R}} M \longrightarrow 0$$

where $\overline{d}_{n-1}$ denotes the map $d_{n-1} \otimes \mathrm{Id}_M$.

We defined the $\mathbb{Z}$-module $\mathrm{Tor}^{\mathbf{R}}(M, N)$ as the homology of the complex $\mathcal{P}_N \otimes_{\mathbf{R}} M$:

$$\mathrm{Tor}^{\mathbf{R}}_n(N, M) = \mathrm{H}_n(\mathcal{P}_N \otimes_{\mathbf{R}} M) = \mathrm{Ker}\,\overline{d}_{n-1}/\mathrm{Im}\,\overline{d}_n.$$

In this way, we define a bifunctor $\mathrm{Tor}^{\mathbf{R}}$ with values in the category of $\mathbb{Z}$-modules.

Following the definitions, the functor $\mathrm{Tor}^{\mathbf{R}}_0(N, -)$ is naturally equivalent to $N \otimes_{\mathbf{R}} -$ and the functor $\mathrm{Tor}^{\mathbf{R}}_n(-, M)$ is naturally equivalent to $-\otimes_{\mathbf{R}} M$. Indeed, we have $\mathrm{Tor}^{\mathbf{R}}_0(N, M) = \mathrm{Coker}\,(\overline{d}_0)$. Furthermore, the functor $N \otimes_{\mathbf{R}} -$ is right exact, hence

$$\mathrm{Coker}\,(\overline{d}_0) = P_0 \otimes_{\mathbf{R}} M/\mathrm{Im}\,(\overline{d}_0) = P_0 \otimes_{\mathbf{R}} M/\ker(\varepsilon \otimes \mathrm{Id}_M) = N \otimes_{\mathbf{R}} M.$$

This proves that

$$\mathrm{Tor}^{\mathbf{R}}_0(N, M) = N \otimes_{\mathbf{R}} M.$$

4.1.2 Contracting homotopy. Recall that a method to prove that a complex

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_0} M_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

is acyclic is to construct a *contracting homotopy*, that is, a sequence of morphisms of abelian groups

$$(\cdots) \longleftarrow M_{n+1} \xleftarrow{\iota_{n+1}} M_n \xleftarrow{\iota_n} M_{n-1} \longleftarrow (\cdots) \longleftarrow M_1 \xleftarrow{\iota_1} M_0 \xleftarrow{\iota_0} N$$

such that

$$\varepsilon\iota_0 = \mathrm{Id}_N, \qquad d_0\iota_1 + \iota_0\varepsilon = \mathrm{Id}_{M_0}, \qquad d_n\iota_{n+1} + \iota_n d_{n-1} =: \mathrm{Id}_{M_n},$$

for every $n \geqslant 1$.

4.1.3 Homology of an algebra. Let $\mathbf{A}$ be an associative algebra over a field $\mathbb{K}$. For $n \geqslant 0$, the *n-th homology space* of the algebra $\mathbf{A}$ with coefficient in a left $\mathbf{A}$-module M is defined by

$$\mathrm{H}_n(\mathbf{A}, M) = \mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K}, M).$$

In practice, to compute the nth homology spaces $\mathrm{H}_n(\mathbf{A}, \mathbb{K})$, for all $n \geqslant 0$, we construct a free resolution of $\mathbb{K}$, seen as a trivial right-$\mathbf{A}$-module:

$$\mathcal{F} \ : \cdots \longrightarrow F_n \xrightarrow{d_{n-1}} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

and we compute the homology of the complex $\mathcal{F}_{\mathbb{K}} \otimes_{\mathbf{A}} \mathbb{K}$.

4.1.4 Minimal complex. A complex of free right $\mathbf{A}$-modules

$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_n} F_n \xrightarrow{d_{n-1}} F_{n-1} \longrightarrow \cdots$$

is *minimal* if all induced maps $\overline{d}_n = d_n \otimes \mathrm{Id}_{\mathbb{K}} : F_{n+1} \otimes_{\mathbf{A}} \mathbb{K} \longrightarrow F_n \otimes_{\mathbf{A}} \mathbb{K}$ are zero. A resolution is *minimal* if the associated complex is minimal. Note that a minimal free resolution is one in which each free module has the minimal number of generators as illustrated in the following example.

4.2 Anick's Chains

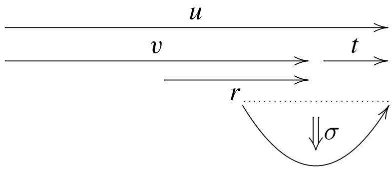
4.2.1 Anick's chains, [2]. Let Λ be a reduced left-monomial linear 2-polygraph. The *Anick n-chains* of the linear 2-polygraph Λ and their *tails* are defined by induction as follows:

- The unique (-1)-*chain* is the empty monomial, denoted by 1, it is its own tail.
- The 0-*chains* are the 1-cells in Λ_1, and the *tail of* 0-*chain* x in Λ_1 is x itself.
- For $n \geqslant 1$, suppose that the $(n-1)$-chains and their tails constructed. An *n-chain* is a monomial u in Λ_1^* of the form

$$u = vt$$

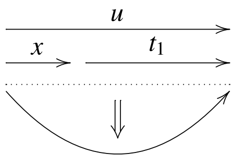
where the monomials v and t satisfy the following conditions:

(i) v is $(n-1)$-chain,
(ii) t is a reduced monomial with respect to Λ, called the *tail* of u,
(iii) if r is the tail of v, then $\mathrm{Occ}_{s_1(\Lambda)}(rt) = 1$, and
(iv) the unique reduction on rt is rightmost, that is, given by a 2-cell σ in Λ reducing the ending of the monomial rt:



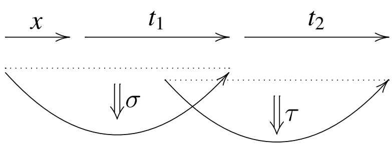
We will denote by $\Omega_n(\Lambda)$ the set of n-chains of the linear 2-polygraph Λ.

4.2.2 Anick's chains and overlapping. The linear 2-polygraph Λ being reduced, we have the following description of Anick's chains. A 1-chain u is necessarily in $s_1(\Lambda)$. Indeed, a 1-chain is a non-reduced monomial u written as $u = xt$, where x is a 1-cell in Λ_1 and t is a monomial reduced with respect to Λ:



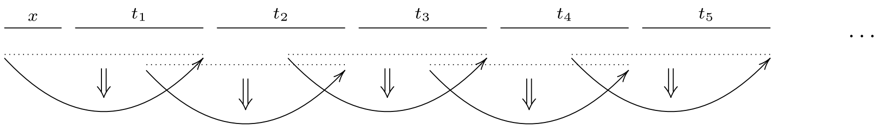
and such that there is only one 2-cell of Λ that can be applied on the monomial u.

A 2-chains u is the source of a critical branching. Indeed, $u = xt_1t_2$, where xt_1 is the source of a 2-cell σ in Λ_2 and there is a rightmost reduction τ reducing t_1t_2 and thus overlapping σ:



Moreover, u is not the source of a critical triple branching, as we have $\mathrm{Occ}_{s_1(\Lambda)}(u) = 2$. In this way, there is a 1-1 correspondence between $\Omega_2(\Lambda)$ and the set of critical branchings of the 2-polygraph Λ.

For $n \geqslant 3$, an n-chain u corresponds to an n-fold overlapping compiled by $(n-1)$ chained critical branchings. Note that it may possible that $\mathrm{Occ}_{s_1(\Lambda)}(u) > n$, see Example 4.2.5.

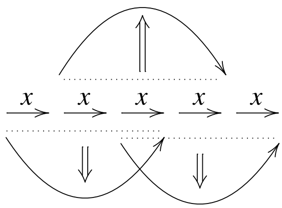


4.2.3 Proposition [2]. *Suppose $n \geqslant 1$. If $u = x_{i_1} \dots x_{i_t}$ is an n-chain, then there is a unique $s \leqslant t$ such that $x_{i_1} \dots x_{i_s}$ is an $(n-1)$-chain. Moreover, $x_{i_{s+1}} \dots x_{i_t}$ is reduced.*

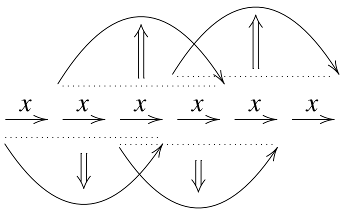
Indeed, suppose that there is two $(n-1)$-chains $x_{i_1} \dots x_{i_s}$ and $x_{i_1} \dots x_{i_{s'}}$ which factorize u. By uniqueness of the reduction on the tail, condition **(iii)** in 4.2.1, necessarily we have $s = s'$.

4.2.4 Notation. If u is an n-chain with $(n-1)$-chain v and tail t, we will denote $u = v|t$. An n-chain will be denoted by $x|t_1|t_2| \dots |t_n$.

4.2.5 Example, [2]. Let Λ be a reduced left-monomial linear 2-polygraph with $\Lambda_1 = \{x\}$ and $s_1(\Lambda) = \{x^3\}$. The 1-cell x is the unique 0-chain. The monomial $x^3 = x|x^2$ is the unique 1-chain, xx is not a 1-chain because $\mathrm{Occ}_{\Lambda_2}(x^2) = 0$. The monomial $x^4 = x^3|x$ is the unique 2-chain. Note that $x^5 = x^3x^2$ is not a 2-chain because $\mathrm{Occ}_{\Lambda_2}(x^2x^2) = 2$: on the monomial x^5, there are three possible reductions. Here, x^5 links three obstructions, with the first one intersecting with the last; hence, it forms a critical triple branching:



The monomial $x^6 = x^4|x^2$ is the unique 3-chain, note that $x^5 = x^4x$ is not a 3-chain because $\mathrm{Occ}_{\Lambda_2}(xx) = 0$. Note that there are 4-obstructions on the 3-chain x^6:



Thus, we have

$$\Omega_0(\Lambda) = \Lambda_1, \quad \Omega_1(\Lambda) = s_1(\Lambda), \quad \Omega_2(\Lambda) = \{x^4\}, \quad \Omega_3(\Lambda) = \{x^6\}.$$

More generally, we show that for any integer $n \geqslant 0$, we have

$$\Omega_{2n-1}(\Lambda) = \{x^{3n}\}, \quad \Omega_{2n}(\Lambda) = \{x^{3n+1}\}.$$

4.2.6 Example, [2]. Suppose that $\Lambda_1 = \{x, y\}$ and $s_1(\Lambda) = \{x^2yxy, xyxy^2\}$. Then we have

$$\Omega_0(\Lambda) = \{x, y\}, \quad \Omega_1(\Lambda) = \{x|xyxy, x|yxy^2\}, \quad \Omega_2(\Lambda) = \{x|xyxy|y, x|xyxy|xy^2\},$$

and $\Omega_n(\Lambda)$ is empty for $n \geqslant 3$.

4.2.7 Exercise, [1]. Let Λ be a linear 2-polygraph such that $\Lambda_1 = \{x, y, z\}$. Determine Anick's chains in the following situations:

(1) $s_1(\Lambda) = \{xyzx, zxy\}$,
(2) $s_1(\Lambda) = \{xyzx, xxy\}$. In this case, show that the number of n-chains equals the $(n+2)$nd Fibonacci number when $n \geqslant 1$.

4.3 Anick's Resolution

Let Λ be a convergent reduced left-monomial linear 2-polygraph, compatible with a monomial order $\prec$ on Λ_1^ℓ. Let denote by $\mathbf{A}$ the algebra presented by Λ. We define a section $\iota : \mathbf{A} \longrightarrow \Lambda_1^\ell$ of the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$, sending every 1-cell f of $\mathbf{A}$ to the unique normal form $\widehat{f}$ of any representative 1-cell of f in Λ_1^ℓ, as in 3.2.3. In the construction of the following resolution, the convergence hypothesis is used to guarantee the unicity of this normal form.

4.3.1 Anick's resolution. Let $\mathbf{A}[\Omega_n(\Lambda)] = \mathbb{K}[\Omega_n(\Lambda)] \otimes_{\mathbb{K}} \mathbf{A}$ be the free right $\mathbf{A}$-module over the set of n-chains $\Omega_n(\Lambda)$. We identify $\mathbf{A}[\Omega_0(\Lambda)]$ to $\mathbf{A}[\Lambda_1]$ and $\mathbf{A}[\Omega_{-1}(\Lambda)]$ to $\mathbf{A}$. Anick constructs in [2] a free resolution of right $\mathbf{A}$-modules, which we will denote by $\mathcal{A}(\Lambda)$, and defined by

$$\cdots \longrightarrow \mathbf{A}[\Omega_n(\Lambda)] \xrightarrow{d_n} \mathbf{A}[\Omega_{n-1}(\Lambda)] \longrightarrow \cdots \longrightarrow \mathbf{A}[\Omega_1(\Lambda)] \xrightarrow{d_1} \mathbf{A}[\Lambda_1] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

where the differentials d_n are constructed by induction on n together with the contracting homotopy

$$\iota_n : \operatorname{Ker} d_{n-1} \longrightarrow \mathbf{A}[\Omega_n(\Lambda)].$$

The applications d_n are morphisms of right $\mathbf{A}$-module and the applications ι_n are linear maps.

4.3.2 The applications d_n and ι_n are constructed by Noetherian induction with respect to the monomial order $\prec$. From the monomial order $\prec$ on Λ_1^ℓ, we define a partial order $\prec_{\Omega_n}$ on the set of elements $u \otimes t$ such that $u \in \Omega_n(\Lambda)$ and $t \in \Lambda_1^*$ by setting

$$u \otimes t \prec_{\Omega_n} u' \otimes t' \quad \text{if and only if} \quad \widehat{ut} \prec \widehat{u't'}.$$

This order is total on the set of n-chains. Indeed, by Proposition 4.2.3, if $ut = u't'$, then $u = u'$ and then $t = t'$.

Given a linear combination $h = \sum_{i=1}^{l} \lambda_i u_i \otimes t_i$ in $\mathbf{A}[\Omega_n(\Lambda)]$, the *leading term* of h with respect to $\prec_{\Omega_n}$ is the term $u_k \otimes t_k$ such that $u_i \otimes t_i \prec_{\Omega_n} u_k \otimes t_k$ for any $i \in \{1, \ldots, l\} \setminus \{k\}$.

4.3.3 For the first steps of the resolution

$$\mathbf{A}[\Lambda_1] \underset{\iota_0}{\overset{d_0}{\rightleftarrows}} \mathbf{A} \underset{\iota_{-1}}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{K} \longrightarrow 0$$

we set $\iota_{-1} = \eta : \mathbb{K} \hookrightarrow \mathbf{A}$ the embedding of $\mathbb{K}$ in $\mathbf{A}$ and we define the *augmentation* $\varepsilon : \mathbf{A} \to \mathbb{K}$ by $\varepsilon(x) = 0$, for all $x \in \Lambda_1$. Hence $\mathbf{A} = \mathbb{K} \oplus \operatorname{Ker} \varepsilon$ and we have $\varepsilon\iota_{-1} = \operatorname{Id}_{\mathbb{K}}$. Then we set

$$d_0(x \otimes 1) = 1 \otimes x,$$

for all x in Λ_1. For a monomial u in $\mathbf{A}$ such that the normal form with respect to Λ is written $\widehat{u} = x_1 x_2 \ldots x_k$ in Λ_1^ℓ, we define

$$\iota_0(1 \otimes u) = x_1 \otimes x_2 \ldots x_k. \tag{9}$$

Then we extend ι_0 to any element of $\mathbf{A}$ by linearity. The map ι_0 is well defined by the uniqueness of the normal form due to the convergence of the linear 2-polygraph Λ.

The exactness, $\operatorname{Im} d_0 = \operatorname{Ker} \varepsilon$, in $\mathbf{A}$ is a consequence of the two equalities:

$$\varepsilon d_0(x \otimes 1) = 0 \quad \text{and} \quad d_0 \iota_0 = \operatorname{id}_{\operatorname{Ker}(\varepsilon)}.$$

4.3.4 For $n \geqslant 0$, we define the pair (d_n, ι_n):

$$\mathbf{A}[\Omega_n(\Lambda)] \underset{\iota_n}{\overset{d_n}{\rightleftarrows}} \mathbf{A}[\Omega_{n-1}(\Lambda)] \underset{\iota_{n-1}}{\overset{d_{n-1}}{\rightleftarrows}} \mathbf{A}[\Omega_{n-2}(\Lambda)]$$

by induction on n. We suppose that the maps d_{n-1} and $\iota_{n-1} : \operatorname{Ker} d_{n-2} \longrightarrow \mathbf{A}[\Omega_{n-1}(\Lambda)]$, constructed such that the following equalities

$$d_{n-2} d_{n-1} = 0 \quad \text{and} \quad d_{n-1}\iota_{n-1} = \operatorname{Id}_{\operatorname{Ker} d_{n-2}}$$

hold. We define inductively d_n on an n-chain $u = v|t$ with tail t by

$$d_n(v|t \otimes 1) = v \otimes t - \iota_{n-1} d_{n-1}(v \otimes t). \tag{10}$$

In the right-hand side of (10), the term $v \otimes t$ will be the leading term with respect to $\prec_{\Omega_{n-1}}$.

4.3.5 Let us define recursively the map

$$\iota_n : \operatorname{Ker} d_{n-1} \longrightarrow \mathbf{A}[\Omega_n(\Lambda)].$$

Let $h \in \operatorname{Ker} d_{n-1} \subset \mathbf{A}[\Omega_{n-1}(\Lambda)]$. Denote by $u_{n-1} \otimes t$ the leading term of h with respect to $\prec_{\Omega_{n-1}}$, that is

$$h = \lambda u_{n-1} \otimes t + \text{lower terms},$$

where $\lambda \in \mathbb{K} \setminus \{0\}$. By Proposition 4.2.3, the $(n-1)$-chain u_{n-1} can be uniquely decomposed in

$$u_{n-1} = u_{n-2}|t',$$

where u_{n-2} is an $(n-2)$-chain and t' is the tail of u_{n-1}. By induction, we have

$$d_{n-1}(u_{n-1} \otimes 1) = u_{n-2} \otimes t' + \text{lower terms}.$$

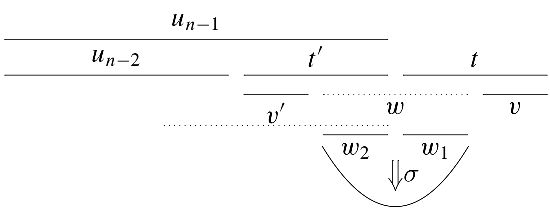
As d_{n-1} is a morphism of right $\mathbf{A}$-modules, we have

$$\begin{aligned} d_{n-1}(h) &= \lambda d_{n-1}(u_{n-1} \otimes t) + d_{n-1}(\text{lower terms}) \\ &= \lambda u_{n-2} \otimes t't + \text{lower terms}. \end{aligned}$$

Suppose that $t't$ is reduced, then $u_{n-2} \otimes t't$ remains the leading term of $d_{n-1}(h)$ and h cannot be in $\operatorname{Ker} d_{n-1}$ thus contradicting the hypothesis. It follows that $t't$ can be reduced, we set

$$t't = v'wv,$$

where w is the 1-source of the leftmost reduction σ that can be applied on the monomial $t't$:



Then $u_{n-2}v'w = u_{n-2}|t'|w_1$ forms an n-chain, it follows that $u_{n-2}v'w \otimes v \in \mathbf{A}[\Omega_n(\Lambda)]$. We set

$$\begin{aligned} \iota_n(h) &= \iota_n(\lambda u_{n-1} \otimes t + \text{lower terms}) \\ &= \lambda u_{n-2}v'w \otimes v + \iota_n(h - \lambda d_n(u_{n-2}v'w \otimes v)). \end{aligned} \tag{11}$$

This is well defined, because $h - \lambda d_n(u_{n-2}v'w \otimes v) \prec h$ by construction. Indeed

$$\begin{aligned} d_n(u_{n-2}v'w \otimes v) = d_n(u_{n-2}v'w_1w_2 \otimes v) &= u_{n-2}v'w_2 \otimes w_1v + \text{lower terms} \\ &= u_{n-1} \otimes t + \text{lower terms}. \end{aligned}$$

Moreover, $d_{n-1}(h - \lambda d_n(u_{n-2}v'w \otimes v)) = 0$.

From this construction, we deduce the following result.

4.3.6 Theorem *[2,* **Thm 1.4].** *Let* **A** *be an algebra presented by a convergent reduced left-monomial linear* 2*-polygraph* Λ*, compatible with a given monomial order* $\prec$*. The complex of right* **A***-modules* $\mathcal{A}(\Lambda)$ *defined by*

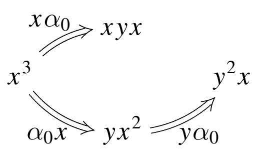
$$\cdots \longrightarrow \mathbf{A}[\Omega_n(\Lambda)] \overset{d_n}{\to} \mathbf{A}[\Omega_{n-1}(\Lambda)] \to \cdots \longrightarrow \mathbf{A}[\Omega_1(\Lambda)] \overset{d_1}{\to} \mathbf{A}[\Lambda_1] \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0$$

where, for any $n \geqslant 0$*, the morphism* d_n *is defined on a* n*-chain* $v|t$ *by*

$$d_n(v|t \otimes 1) = v \otimes t + h,$$

where $\mathrm{lt}(h) \prec v|t \otimes 1$*, if* $h \neq 0$*, is a resolution of the trivial right* **A***-module* $\mathbb{K}$*.*

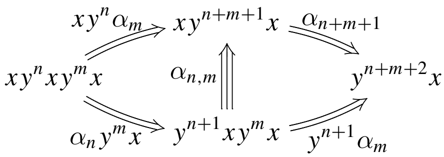
4.3.7 Example. Let consider the algebra **A** presented by the linear 2-polygraph Λ of Example 2.1.10 and denote by α_0 the 2-cell β. It appears one critical branching



We complete the linear 2-polygraph Λ with the 2-cells

$$\alpha_n : xy^n x \;\Rightarrow\; y^{n+1}x,$$

for all $n > 0$. We note that, for any integers $n, m \geqslant 0$, we have a critical branching



The linear 2-polygraph Λ', whose set of 1-cell is Λ_1 and $\Lambda'_2 = \{\alpha_n \mid n \geqslant 0\}$ is convergent, compatible with the monomial order $\prec$ and Tietze equivalent to Λ. Equivalently, the set $\{xy^nx - y^{n+1}x \mid n \geqslant 0\}$ forms a Gröbner basis for the ideal $I(\Lambda)$. Anick's 1-chains are of the form $x|y^nx$ with $n \geqslant 0$ and Anick's 2-chains are of the form $x|y^nx|y^mx$ with $n, m \geqslant 0$. More generally, for any $k \geqslant 2$, we have

$$\Omega_k(\Lambda') = \{x|y^{n_1}x|y^{n_2}x|\ldots|y^{n_k}x \;\text{ for }\; n_1, \ldots, n_k \geqslant 0\},$$

Let us compute the boundary maps d_0, d_1, d_2 and d_3. We have $d_0(x \otimes 1) = x$, $d_0(y \otimes 1) = y$ and

$$\begin{aligned} d_1(x|y^nx \otimes 1) &= x \otimes y^nx - \iota_0 d_0(x \otimes y^nx), \\ &= x \otimes y^nx - \iota_0(1 \otimes xy^nx), \\ &= x \otimes y^nx - \iota_0(1 \otimes y^{n+1}x), \\ &= x \otimes y^nx - y \otimes y^nx. \end{aligned}$$

The last equality is consequence of the definition of the map ι_0 in (9).

$$\begin{aligned} d_2(x|y^nx|y^mx \otimes 1) &= x|y^nx \otimes y^mx - \iota_1 d_1(x|y^nx \otimes y^mx), \\ &= x|y^nx \otimes y^mx - \iota_1(x \otimes y^nxy^mx - y \otimes y^nxy^mx), \\ &= x|y^nx \otimes y^mx - \iota_1(x \otimes y^{n+m+1}x - y \otimes y^{n+m+1}x), \end{aligned}$$

By (11), we have

$$\begin{aligned} \iota_1(x \otimes y^{n+m+1}x - y \otimes y^{n+m+1}x) =& x|y^{n+m+1}x \otimes 1 - \iota_1\big(x \otimes y^{n+m+1}x \\ & - y \otimes y^{n+m+1}x - x \otimes y^{n+m+1}x \\ & + y \otimes y^{n+m+1}x\big). \end{aligned}$$

Hence

$$d_2(x|y^nx|y^mx \otimes 1) = x|y^nx \otimes y^mx - x|y^{n+m+1}x \otimes 1.$$

Finally, we show that

$$\begin{aligned} d_3(x|y^nx|y^mx|y^kx \otimes 1) =& x|y^nx|y^mx \otimes y^kx - x|y^nx|y^{m+k+1}x \otimes 1 \\ & + x|y^{n+m+1}x|y^kx \otimes 1. \end{aligned}$$

4.3.8 Example. Let consider the algebra $\mathbf{A}$ given in 4.3.7, but with the presentation by the linear 2-polygraph Λ' of Example 2.1.10, compatible with the deglex order induced by the alphabetic order $x \prec y$. This polygraph does not have critical branching; thus, the sets of Anick's n-chains are empty for $n \geqslant 2$. It follows that the associated Anick's resolution is

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[y|x] \overset{d_1}{\to} \mathbf{A}[x, y] \overset{d_0}{\to} \mathbf{A} \overset{\varepsilon}{\to} \mathbb{K} \longrightarrow 0$$

with $d_0(x \otimes 1) = x$, $d_0(y \otimes 1) = y$ and $d_1(y|x \otimes 1) = y \otimes x - x \otimes x$.

4.3.9 Example. Consider the algebra $\mathbf{A}$ presented by the linear 2-polygraph Λ of Example 2.1.9. With the Gröbner basis computed in 3.6.7:

$$z^3 \overset{\alpha_f}{\Rightarrow} xyz - x^3 - y^3 \qquad zy^3 \overset{\beta}{\Rightarrow} zxyz - zx^3 + y^3z + x^3z - xyz^2$$

Anick's chains are of the form z^n and $z^n y^3$, for $n \geqslant 0$, so that Anick's resolution for the algebra $\mathbf{A}$ with respect to this Gröbner basis is infinite.

4.3.10 Exercise, ([2, **Sect. 3**]). Compute Anick's resolution for the algebra presented by the linear 2-polygraph $\langle * \mid x, y \mid xyxyx \Rightarrow xyx \rangle$.

4.4 Anick's Resolution for a Monomial Algebra

4.4.1 Anick's chains for a monomial algebra. We construct Anick's resolution in the case of a monomial algebra $\mathbf{A}$. Recall from 2.1.20, that such an algebra can be presented by a monomial linear 2-polygraph Λ, that is, left-monomial and $t_1(\alpha) = 0$ for all α in Λ_2. Note that such a presentation is always convergent. Suppose that the polygraph Λ is reduced. The sets of chains for Λ are $\Omega_0(\Lambda) = \Lambda_1$, $\Omega_1(\Lambda) = s_1(\Lambda)$ and for any $n \geqslant 2$, $\Omega_n(\Lambda)$ is the set of n-overlapping $x|t_1|\ldots|t_{n-1}|t_n$ of branchings of Λ with $x, t_1, \ldots, t_n$ in Λ_1 and xt_1, t_it_{i+1} in $s_1(\Lambda)$ for any $1 \leqslant i \leqslant n-1$. We have

$$\widehat{xt_1} = 0 \qquad \text{and} \qquad \widehat{t_{i-1}t_i} = 0, \text{ for all } 1 \leqslant i \leqslant n. \tag{12}$$

Consider the boundary map

$$d_n : \mathbf{A}[\Omega_n(\Lambda)] \longrightarrow \mathbf{A}[\Omega_{n-1}(\Lambda)]$$

defined by

$$d_n(x|t_1|\ldots|t_{n-1}|t_n \otimes 1) = x|t_1|\ldots|t_{n-1} \otimes t_n - \iota_{n-1}d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n).$$

By definition of d_{n-1}, we have

$$d_{n-1}(x|t_1|\dots|t_{n-1}\otimes t_n) = x|t_1|\dots|t_{n-2}\otimes t_{n-1}t_n - \iota_{n-2}d_{n-2}(x|t_1|\dots|t_{n-2}\otimes t_{n-1}t_n)$$

Using relation in (12), we have $d_{n-1}(x|t_1|\dots|t_{n-1}\otimes t_n) = 0$, hence

$$d_n(x|t_1|\dots|t_{n-1}|t_n \otimes 1) = x|t_1|\dots|t_{n-1}\otimes t_n.$$

As a consequence, the map $d_n \otimes_{\mathbf{A}} 1_{\mathbb{K}}$ is zero, for all $n \geqslant 0$. This proves that Anick's resolution of a monomial algebras is minimal.

4.4.2 Proposition. *Let Λ be a monomial linear 2-polygraph, and $\mathbf{A}$ be the monomial algebra presented by Λ. The following statements hold:*

- **(i)** *Anick's resolution $\mathcal{A}(\Lambda)$ is a minimal resolution.*
- **(ii)** *There is an isomorphism $\mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}\Omega_{n-1}(\Lambda)$, for all $n \geqslant 0$.*

4.5 Computing Homology with Anick's Resolution

Given an algebra $\mathbf{A}$ presented by a convergent reduced left-monomial linear 2-polygraph Λ, compatible with a monomial order, Anick's resolution $\mathcal{A}(\Lambda)$ gives a method to compute the homology groups of $\mathbf{A}$ with coefficient in a $\mathbf{A}$-module M. In particular, Anick's resolution can be used to calculate Poincaré series. In this section, we give several examples of computations of homology groups with coefficients in $\mathbb{K}$.

4.5.1 Computing Homology. From the resolution $\mathcal{A}(\Lambda)$, we compute the complex $\mathcal{A}(\Lambda)\otimes_{\mathbf{A}}\mathbb{K}$ given by

$$\cdots \longrightarrow \mathbb{K}[\Omega_n(\Lambda)] \overset{\overline{d}_n}{\longrightarrow} \mathbb{K}[\Omega_{n-1}(\Lambda)] \longrightarrow \cdots \longrightarrow \mathbb{K}[\Omega_1(\Lambda)] \overset{\overline{d}_1}{\longrightarrow} \mathbb{K}[\Lambda_1] \overset{\overline{d}_0}{\longrightarrow} \mathbb{K} \longrightarrow 0$$

where $\mathbb{K}[\Omega_n(\Lambda)]$ denotes the free vector space on $\Omega_n(\Lambda)$ and $\overline{d}_n$ denotes the map $d_n\otimes \mathrm{Id}_{\mathbb{K}}$. These maps satisfy $\overline{d}_n\overline{d}_{n+1} = 0$, for all $n \geqslant 0$, and we have

$$\mathrm{H}_0(\mathbf{A},\mathbb{K}) = \mathbb{K}, \quad \text{and} \quad \mathrm{H}_n(\mathbf{A},\mathbb{K}) = \mathrm{Ker}\,\overline{d}_{n-1}/\mathrm{Im}\,\overline{d}_n.$$

As a first application, we have the following finiteness properties.

4.5.2 Proposition. *Let $\mathbf{A}$ be an algebra presented by a finite convergent left-monomial linear 2-polygraph. The following statements hold:*

- **(i)** *$\mathbf{A}$ is of homological type right-FP_∞, that is, there exists an infinite length free finitely generated resolution of the trivial right $\mathbf{A}$-module $\mathbb{K}$.*
- **(ii)** *For any $n \geqslant 0$, the vector space $H_n(\mathbf{A},\mathbb{K})$ is finitely generated.*

(iii) *[2, Lemma 3.1] The algebra* $\mathbf{A}$ *has a Poincaré series*

$$P_{\mathbf{A}}(t) = \sum_{n=0}^{\infty} \dim_{\mathbb{K}}(H_n(\mathbf{A}, \mathbb{K}))t^n, \tag{13}$$

with exponential or slower growth, that is, there are constants $c_1, c_2 > 0$*, such that*

$$0 \leqslant \dim_{\mathbb{K}}(H_n(\mathbf{A}, \mathbb{K})) \leqslant c_2(c_1)^n.$$

Note that the finiteness conditions **(i)** and **(ii)** were obtained by Kobayashi for monoids. A monoid $\mathbf{M}$ is of *homological type right*-FP_∞ *over* $\mathbb{K}$ if the monoid algebra $\mathbb{K}\mathbf{M}$ is of homological type right-FP_∞. In [45], by constructing a resolution similar to the Anick resolution, Kobayashi shows that a monoid $\mathbf{M}$ having a presentation by a finite convergent rewriting system is of homological type FP_∞. Similar constructions of resolutions of monoids presented by convergent rewriting systems were also obtained by Brown [17] and by Groves [32]. The different constructions are based on distinct ways to describe the n-fold critical branchings of a convergent rewriting system.

4.5.3 Exercise. Prove the conditions **(i)** and **(ii)** in Proposition 4.5.2.

4.5.4 Low-dimensional homology. Let us explicit the first terms of the series (13). In the first dimensions, we have the following complex:

$$\mathbb{K}[\Omega_2(\Lambda)] \overset{\overline{d}_2}{\to} \mathbb{K}[\Omega_1(\Lambda)] \overset{\overline{d}_1}{\to} \mathbb{K}[\Lambda_1] \overset{\overline{d}_0}{\to} \mathbb{K} \longrightarrow 0$$

The map $\overline{d}_0$ is zero, hence

$$H_1(\mathbf{A}, \mathbb{K}) = \mathbb{K}[\Lambda_1]/\mathrm{Im}\,\overline{d}_1.$$

A 1-cell x of Λ_1 in $\mathrm{Im}\,\overline{d}_1$ comes from a relation with source or target x. It follows that x is a redundant generator in the presentation. Indeed, a term $x \otimes 1$, with x in Λ_1 appears in $\mathrm{Im}\,d_1$ if and only if x is the source or the target of a 2-cell in Λ_2. Let $\alpha : x \Rightarrow y_1 \dots y_k$ be a 2-cell in Λ_2, whereby hypothesis $y_1 \dots y_k$ is reduced. Thus, we have

$$d_1(x|1 \otimes 1) = x \otimes 1 - y_1 \otimes y_2 \dots y_k.$$

Hence $\overline{d}_1(x) = x$. Suppose now that $x_1 \dots x_k \overset{\alpha}{\Rightarrow} y$ is a 2-cell in Λ_2. We have

$$d_1(x_1|x_2 \dots x_k \otimes 1) = x_1 \otimes x_2 \dots x_k - y \otimes 1.$$

Hence $\overline{d}_1(x_1 \dots x_k) = -y$. Thus, we have $\overline{d}_1 = 0$ if and only if the number of generators is minimal. In this way, $\dim_{\mathbb{K}} H_1(\mathbf{A}, \mathbb{K})$ is equal to the minimal number of generators for a presentation of the algebra $\mathbf{A}$. For analogous reasons, we show that $\dim_{\mathbb{K}} H_2(\mathbf{A}, \mathbb{K})$ is the minimal required number of the defining relations, see [66, Sect. 3.9].

4.5.5 Example. Consider the algebra $\mathbf{A}$ from Example 4.3.8. Using Anick's resolution computed in 4.3.8, we deduce the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{K}[y|x] \overset{\overline{d}_1}{\longrightarrow} \mathbb{K}[x, y] \overset{\overline{d}_0}{\longrightarrow} \mathbb{K} \longrightarrow 0$$

whose boundary maps $\overline{d}_0$ and $\overline{d}_1$ are zero. We deduce

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, 2, \\ \mathbb{K}^2 & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

4.5.6 Exercise [2, **Thm 3.2**]. Let $\mathbf{A}$ be an algebra admitting a presentation by a left-monomial reduced linear 2-polygraph compatible with a monomial order and having no critical branching. Show that $\mathrm{H}_n(\mathbf{A}, \mathbb{K}) = 0$, for any $n \geqslant 3$. A presentation without critical branching is called *combinatorially free* in [2].

4.5.7 Exercise. Show that the Poincaré series of the algebra $\mathbf{A}$ presented by the linear 2-polygraph $\langle * \mid x, y \mid x^2 \Rightarrow 0 \rangle$ is

$$P_{\mathbf{A}}(t) = 1 + 2t + \sum_{k=2}^{\infty} t^k.$$

4.5.8 Exercise. Let $\mathbf{B}_3^+$ be the monoid of positive braids on three strands given by the following Artin presentation:

$$\langle\, s, t \mid sts \Rightarrow tst \,\rangle.$$

Compute Anick's resolution and the Poincaré series of the monoid $\mathbf{B}_3^+$.

4.6 Minimality of Anick's Resolution

4.6.1 Example. Let $\mathbf{A}$ be the algebra presented by the linear 2-polygraph $\langle * \mid x, y \mid x \Rightarrow y \rangle$, which is compatible with the deglex order induced by $y \prec x$. The Anick resolution is

$$0 \longrightarrow \mathbf{A}[x|1] \overset{d_1}{\longrightarrow} \mathbf{A}[x, y] \overset{d_0}{\longrightarrow} \mathbf{A} \overset{\varepsilon}{\longrightarrow} \mathbb{K} \longrightarrow 0$$

with

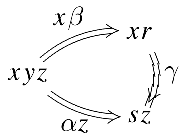
$$d_0(x \otimes 1) = x, \quad d_0(y \otimes 1) = y, \quad d_1(x|1 \otimes 1) = x \otimes 1 - 1 \otimes y.$$

This resolution is not minimal because $\overline{d}_1 \neq 0$. A minimal resolution for the algebra **A** can be constructed from the polygraph $\langle * \mid x \mid \emptyset \rangle$ with no 2-cell.

4.6.2 Example. Let consider the algebra **A** presented by the linear 2-polygraph

$$\Lambda = \langle * \mid x, y, z, r, s \mid xy \overset{\alpha}{\Rightarrow} s,\ yz \overset{\beta}{\Rightarrow} r \rangle$$

compatible with the deglex order induced by the alphabetic order $s \prec r \prec z \prec y \prec x$. There is a critical branching:



which is confluent by adding the rule $xr \overset{\gamma}{\Rightarrow} sz$. The linear 2-polygraph $\Lambda' = \langle \Lambda_1 \mid \alpha, \beta, \gamma \rangle$ is compatible with the deglex order considered above, convergent and Tietze equivalent to Λ. The induced the Anick resolution $\mathcal{A}(\Lambda')$ is

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[xy|z] \overset{d_2}{\longrightarrow} \mathbf{A}[x|y, x|r, y|z] \overset{d_1}{\longrightarrow} \mathbf{A}[x, y, z, r, s] \overset{d_0}{\longrightarrow} \mathbf{A} \overset{\varepsilon}{\longrightarrow} \mathbb{K} \longrightarrow 0$$

with

$$d_1(x|y \otimes 1) = x \otimes y - s \otimes 1, \quad d_1(x|r \otimes 1) = x \otimes r - s \otimes z,$$
$$d_1(y|z \otimes 1) = y \otimes z - r \otimes 1,$$

and $d_2(x|y|z \otimes 1) = xy \otimes z - xr \otimes 1$. This resolution is not minimal, because the maps $\overline{d}_1$ and $\overline{d}_2$ are non zero. Note that

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, \\ \mathbb{K}^3 & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 2. \end{cases}$$

and a minimal resolution for the algebra **A** can be constructed from the linear 2-polygraph $\langle * \mid x, y, z \mid \emptyset \rangle$ which produces the following resolution:

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[x, y, z] \overset{d_0}{\longrightarrow} \mathbf{A} \overset{\varepsilon}{\longrightarrow} \mathbb{K} \longrightarrow 0$$

4.6.3 Exercise. Consider the linear 2-polygraph

$$\Lambda = \langle * \mid x, y, z, r, s \mid xy \overset{\alpha}{\Rightarrow} ss,\ yz \overset{\beta}{\Rightarrow} sr \rangle.$$

(1) Complete the polygraph Λ into a convergent polygraph Λ'.
(2) Show that the Anick resolution of Λ' is not minimal.
(3) Compute the homology of the algebra **A** presented by Λ.
(4) Compute a minimal Anick's resolution of the algebra **A**.

4.6.4 Exercise. Let consider the algebra presented by

$$\langle * \mid x, y, z, r, s \mid xy = ss,\ yz = rr \rangle.$$

Show that there is no orientation of rules of this presentation giving a convergent linear 2-polygraph, and thus, there is no minimal Anick's resolution for this algebra.

4.6.5 Proposition. *Let* **A** *be an algebra and let* Λ *be a left-monomial reduced convergent linear* 2*-polygraph compatible with a monomial order that presents* **A**. *If the Anick resolution* $\mathcal{A}(\Lambda)$ *is minimal, then, for any* $n \geqslant 0$*, there is an isomorphism of spaces*

$$H_n(\mathbf{A}, \mathbb{K}) \simeq \mathbb{K}[\Omega_{n-1}(\Lambda)].$$

4.6.6 Exercise. Prove Proposition 4.6.5.

4.6.7 When Anick's resolution is minimal. We have seen in Proposition 4.4.2 that the Anick resolution $\mathcal{A}(\Lambda)$ is minimal when the presentation is monomial. Following exercise gives an other situation for which the Anick resolution is minimal.

4.6.8 Exercise. Let Λ be a left-monomial reduced linear 2-polygraph compatible with a monomial order. Suppose that Λ is convergent and quadratic, that is, any 2-cell in Λ_2 is of the form $x_{i_1}x_{i_2} \Rightarrow y_{i_1}y_{i_2}$ with $x_{i_1}, x_{i_2}, y_{i_1}, y_{i_2}$ in Λ_1. Show that the Anick resolution $\mathcal{A}(\Lambda)$ is minimal.

4.6.9 Exercise. A linear 2-polygraph is *cubical* if its 2-cells are of the form $x_{i_1}x_{i_2}x_{i_3} \Rightarrow y_{i_1}y_{i_2}y_{i_3}$. Is the result of Exercise 4.6.8 can be extended to cubical convergent linear 2-polygraphs?

4.6.10 Exercises. Compute homology spaces of the algebras presented by the following linear 2-polygraphs:

(1) $\langle * \mid x, y \mid xy \Rightarrow yx \rangle$. **2)** $\langle * \mid x, y \mid x^2 \Rightarrow 0 \rangle$. **(3)** $\langle * \mid x, y \mid x^2 \Rightarrow y^2 \rangle$.
(4) $\langle * \mid x, y \mid x^2 \Rightarrow xy \rangle$. **(5)** $\langle * \mid x, y \mid x^2 \Rightarrow xy - y^2 \rangle$. **(6)** $\langle * \mid x, y \mid xyx \Rightarrow yxy \rangle$.

5 Higher Dimensional Linear Rewriting

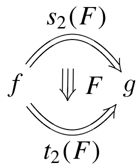
In this section, we recall the notion of coherent presentation for an algebra as a presentation of the algebra extended by a family of generating syzygies. We explain how to generate syzygies when the presentation is convergent. Finally, we recall from [33] the notion of polygraphic resolution for an algebra as an acyclic polygraphic extension of a presentation of the algebra.

5.1 Coherent Presentations of Algebras

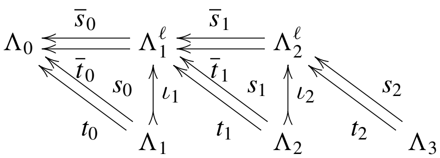
5.1.1 Linear 3-polygraph. Let Λ be a linear 2-polygraph. A *cellular extension* of the free 2-algebroid Λ_2^ℓ is a set Λ_3 equipped with maps

$$\Lambda_2^\ell \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Lambda_3$$

such that, for every F in Λ_3, the pair $(s_2(F), t_2(F))$ is a 2-sphere in Λ_2^ℓ, that is, $s_1 s_2(F) = s_1 t_2(F)$ and $t_1 s_2(F) = t_1 t_2(F)$ hold in Λ_2^ℓ. The elements of Λ_3 are the 3-*cells* of the cellular extension and graphically represented by



A *linear* 3-*polygraph* is a data $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$, where $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a linear 2-polygraph and Λ_3 is a cellular extension of the free 2-algebroid Λ_2^ℓ:



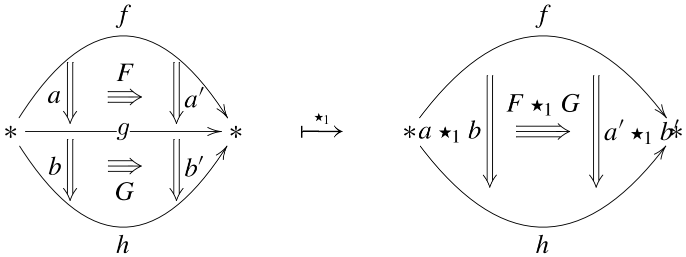
5.1.2 Three-dimensional algebras. We define a 3-*algebra* as an internal 2-category in the category **Alg**:

$$\mathbf{A}_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} \mathbf{A}_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} \mathbf{A}_3$$

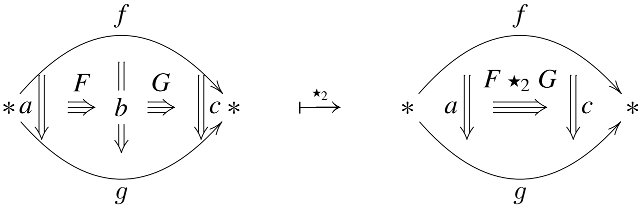
In particular, the algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ with composition $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \xrightarrow{\star_1} \mathbf{A}_2$ form a 2-algebra. The 3-cells can be composed in two different ways:

$$\mathbf{A}_3 \times_{\mathbf{A}_1} \mathbf{A}_3 \xrightarrow{\star_1} \mathbf{A}_3 \qquad \mathbf{A}_3 \times_{\mathbf{A}_2} \mathbf{A}_3 \xrightarrow{\star_2} \mathbf{A}_3$$

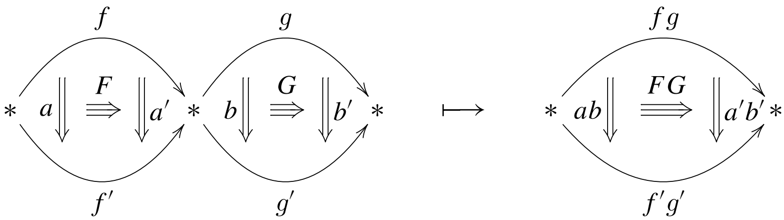
by $\star_1$, along their 1-dimensional boundary:



by $\star_2$, along their 2-dimensional boundary:



The source and target maps s_1, s_2 and t_1, t_2 being morphisms of algebras, the product of 3-cells F and G satisfies



These compositions and the product satisfy remarkable properties similar to those given in 2.1.14 for 2-algebras.

5.1.3 Free 3-algebras. The *free* 3*-algebra over a linear* 3*-polygraph* Λ is constructed similarly to the free 2-algebra given in 2.1.16. It is the 3-algebra, denoted by Λ_3^ℓ, whose underlying 2-algebra is the free 2-algebra Λ_2^ℓ, and its 3-cells are all the formal 1-composition, 2-composition and product of 3-cells of Λ_3, of identities of 2-cells, up to associativity, identity, exchange and inverse relations, see [33, 2.1.3] for more details.

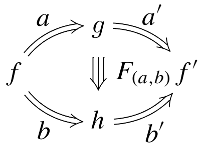
5.1.4 Coherent presentations of algebras. A *coherent presentation* of an algebra **A** is a linear 3-polygraph Λ such that

1. the linear 2-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a presentation of **A**,
2. Λ_3 is a *homotopy basis* of the free 2-algebra Λ_2^ℓ, that is, a cellular extension

$$\Lambda_2^\ell \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Lambda_3$$

   such that for every 2-sphere (a, b) of the free 2-algebra Λ_2^ℓ, there exists a 3-cell A in the free 3-algebra Λ_3^ℓ such that $s_2(A) = a$ and $t_2(A) = b$.

5.1.5 Squier's completion. Let Λ be a left-monomial linear 2-polygraph. Suppose that all critical branchings of Λ are confluent. For every critical branching (a, b) in Λ, we choose two positive 2-cells a' and b' making the branching confluent:


(14)

For any such a confluent branching, we consider a 3-cell $F_{(a,b)} : a \star_1 a' \Rrightarrow b \star_1 b'$. The set of such 3-cells

$$\Lambda_3 = \{ \ F_{(a,b)} \mid (a, b) \text{ is a critical branching } \}$$

forms a cellular extension of the free 2-algebra Λ_2^ℓ. The linear 3-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ is a *Squier's completion* of Λ. When the polygraph is confluent, there exists such a Squier's completion. However, the cellular extension Λ_3 is not unique in general. Indeed, the 3-cells can be directed in the reverse way and a branching (a, b) can have several possible positive 2-cells a' and b' making the branching confluent.

The following result is a formulation of Squier's theorem, [59], in the setting of linear 2-polygraphs.

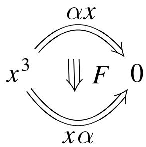
5.1.6 Theorem (Squier's Theorem *[33,* **Thm. 4.3.2]).** *Let* **A** *be an algebra and let* Λ *be a convergent left-monomial presentation of* **A**. *Any Squier's completion of* Λ *is a coherent presentation of* **A**.

5.1.7 Linear oriented syzygies. Let Λ be a presentation of an algebra **A**. Any non-trivial 2-sphere (a, b) in the free 2-algebra Λ_2^ℓ is called a *linear oriented* 3-*syzygy* of the presentation Λ. If Λ is extended into a coherent presentation (Λ, Λ_3) of the algebra **A**, the quotient 2-algebra Λ_2^ℓ/Λ_3 is *aspherical*, that is, for any 2-sphere (a, b)

in Λ_2^ℓ/Λ_3, we have $a = b$. In other words, the cellular extension Λ_3 forms a generating set of linear 3-syzygies of the presentation Λ. Theorem 5.1.6 says that when the presentation Λ is convergent, the 3-cells defined by confluence diagrams of the critical branchings, as in (14), form a family of generator for 3-syzygies.

5.1.8 Exercise. Let $\{F_1, \ldots, F_k\}$ be a generating set for linear 3-syzygies of a linear 2-polygraph Λ. Prove that $\{F_1^-, \ldots, F_k^-\}$ is also a generating set for linear 3-syzygies of Λ.

5.1.9 Example. The linear 2-polygraph $\langle * \mid x \mid x^2 \overset{\alpha}{\Rightarrow} 0 \rangle$ has one critical branching



which is confluent. The polygraph being convergent the 3-cell $F : \alpha x \Rrightarrow x\alpha$ generates all linear 3-syzygies of this presentation.

5.1.10 Example. Consider the algebra **A** presented by the linear 2-polygraph Λ given in Example 2.1.9. It does not have critical branching; hence, any Squier's completion of Λ is empty. As a consequence, Λ can be extended into a coherent presentation with an empty homotopy basis. That is, there is no 3-syzygy for this presentation.

The linear 2-polygraph $\langle * \mid x, y, z \mid \alpha_f, \beta \rangle$ considered in Example 3.6.7 is Tietze equivalent to Λ, convergent and compatible with a monomial order. It has three critical branchings, as shown in Example 3.6.7. It can be extended into a coherent presentation of **A** with three generating 3-syzygies.

5.1.11 Exercise. Give an explicit description of the 3-cells of a coherent presentation on the linear 2-polygraph Λ' of Example 5.1.10.

5.1.12 Exercise. Compute a coherent presentation for the algebras presented by the following linear 2-polygraphs

(1) $\langle * \mid x, y \mid xyx \Rightarrow y^2 \rangle$.

(2) $\langle * \mid x, y, z \mid yz \overset{\alpha}{\Rightarrow} -x^2,\ zy \overset{\beta}{\Rightarrow} -\lambda^{-1}x^2 \rangle$, where $\lambda \in \mathbb{K} \setminus \{0, 1\}$, see [53, 4.3].

5.1.13 Exercise. Compute a minimal coherent presentation for the algebra presented by the linear 2-polygraph $\langle * \mid x \mid x^3 = 0 \rangle$.

5.2 Polygraphic Resolutions of Algebras

In this subsection, we summarize the notion of polygraphic resolution for algebras as introduced in [33]. Such a resolution can be computed for an algebra given by a convergent linear 2-polygraph. The first three steps of the resolution are generated by the cells of the 2-polygraph. For $n \geqslant 3$, the n-cells are generated by confluences diagrams induced by n-fold branchings.

5.2.1 Higher dimensional algebras. Let n be a nonzero natural number. An *n-algebra* $\mathbf{A}$ is an internal $(n-1)$-category in the category **Alg**:

$$\mathbf{A}_1 \underset{t_1}{\overset{s_1}{\leftleftarrows}} \mathbf{A}_2 \underset{t_2}{\overset{s_2}{\leftleftarrows}} \mathbf{A}_3 \leftleftarrows \quad \dots \quad \leftleftarrows \mathbf{A}_{n-1} \underset{t_{n-1}}{\overset{s_{n-1}}{\leftleftarrows}} \mathbf{A}_n$$

The elements of the algebra $\mathbf{A}_k$, for $1 \leqslant k \leqslant n$, are the *$k$-cells* of the n-algebra $\mathbf{A}$. A *cellular extension* of $\mathbf{A}$ is a set Γ equipped with maps

$$\mathbf{A}_n \underset{t_n}{\overset{s_n}{\leftleftarrows}} \Gamma$$

such that, for any γ in Γ, the pair $(s_n(\gamma), t_n(\gamma))$ is an n-sphere of $\mathbf{A}$, that is, $s_{n-1}s_n(\gamma) = s_{n-1}t_n(\gamma)$ and $t_{n-1}s_n(\gamma) = t_{n-1}t_n(\gamma)$.

In these notes, we will do not develop the construction of the free k-algebra $\mathbf{A}[\Gamma]$ on a pair of a $(k-1)$-algebra $\mathbf{A}$ and a cellular extension Γ of it, for $k \geqslant 3$. The construction is the same as in the case of 2-algebras given in 2.1.5. For more details, we refer the reader to [33, 2.1.3]. It has the $(k-1)$-algebra $\mathbf{A}$ as underlying $(k-1)$-algebra and its k-cells are all formal compositions by $\star_i$ for $1 \leqslant i \leqslant k$ and product of k cells in Γ and identities of $(k-1)$-cells, up to associativity, identity, exchange and inverse relation.

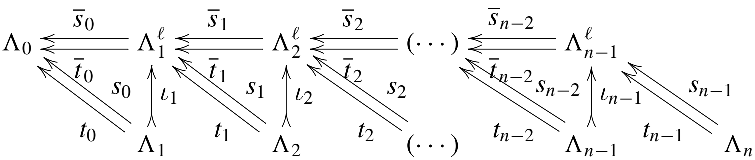
5.2.2 Linear polygraphs. A *linear n-polygraph* is a sequence $\Lambda = (\Lambda_0, \Lambda_1, \dots, \Lambda_n)$ made of

(i) a 1-polygraph (Λ_0, Λ_1),
(ii) for any $k \geqslant 2$, a cellular extension Λ_k of the free $(k-1)$-algebra

$$\Lambda_{k-1}^{\ell} = \Lambda_1^{\ell}[\Lambda_2]\cdots[\Lambda_{k-1}],$$

The elements of Λ_k are called the *k-cells* of Λ.

5.2.3 . A linear n-polygraph can be defined explicitly as a diagram



where the maps $\overline{s}_k, \overline{t}_k : \Lambda_{k+1}^{\ell} \longrightarrow \Lambda_k^{\ell}$ are the extensions of the source and target maps s_k and t_k, defined by the universal property of the free k-algebra Λ_k^{ℓ}, and such that, for any $1 \leqslant k \leqslant n-1$, the following two conditions hold:

(i) there is a structure of k-algebra on the following k-graph:

$$\Lambda_0 \underset{\overline{t}_0}{\overset{\overline{s}_0}{\leftleftarrows}} \Lambda_1^{\ell} \underset{\overline{t}_1}{\overset{\overline{s}_1}{\leftleftarrows}} \Lambda_2^{\ell} \underset{\overline{t}_2}{\overset{\overline{s}_2}{\leftleftarrows}} (\cdots) \underset{\overline{t}_{k-1}}{\overset{\overline{s}_{k-1}}{\leftleftarrows}} \Lambda_k^{\ell}$$

(ii) Λ_{k+1} is a cellular extension of the free k-algebra Λ_k^{ℓ}.

The *free n-algebra* over a linear n-polygraph Λ is the n-algebra $\Lambda_n^{\ell} = \Lambda_1^{\ell}[\Lambda_2]\cdots[\Lambda_n]$

5.2.4 Polygraphic resolutions of algebras. A *polygraphic resolution* of an algebra $\mathbf{A}$ is a linear ∞-polygraph Λ such that

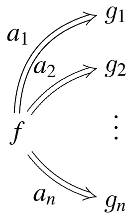
1. the linear 2-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a presentation of $\mathbf{A}$,
2. for every $n \geqslant 2$, Λ_{n+1} is a *homotopy basis* of the free n-algebra Λ_n^{ℓ}, that is a cellular extension

$$\Lambda_n^{\ell} \underset{t_n}{\overset{s_n}{\leftleftarrows}} \Lambda_{n+1}$$

   such that for every n-sphere (a, b) of Λ_n^{ℓ}, there exists an $(n+1)$-cell A in the free $(n+1)$-algebra Λ_{n+1}^{ℓ} such that $s_n(A) = a$ and $t_n(A) = b$.

As a consequence of this definition, for every $n \geqslant 2$, the quotient n-algebra $\Lambda_n^{\ell}/\Lambda_{n+1}$ of the free n-algebra Λ_n^{ℓ} by the congruence generated by the $(n+1)$-cells of Λ_{n+1} is *aspherical*, that is, any of its n-sphere γ is trivial: $s_n(\gamma) = t_n(\gamma)$. A linear ∞-polygraph satisfying this property for all n is said to be *acyclic*.

5.2.5 Higher dimensional branchings. Let Λ be a reduced linear 2-polygraph. An *n-fold branching* of Λ is a family $(a_1, \ldots, a_n)$ of positive 2-cells of Λ_2^{ℓ} with a common source:

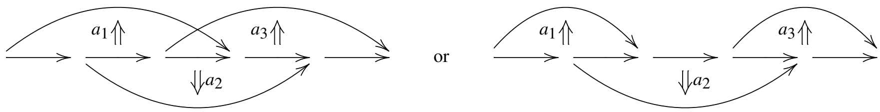


An n-fold branching $(a_1, \ldots, a_n)$ is *local* when $a_1, \ldots, a_n$ are rewriting steps. A local n-fold branching $(a_1, \ldots, a_n)$ is *aspherical* when there is $1 \leqslant i \leqslant n-1$ such that (a_i, a_{i+1}) is aspherical, (resp. *additive*) *Peiffer* when there is $1 \leqslant i \leqslant n-1$ such that (a_i, a_{i+1}) is (resp. additive) Peiffer. In all the other cases, it is said *overlapping*.

A *critical n-fold branching* of Λ is an overlapping local n-fold branching of Λ with a monomial source and that is minimal for the relation on n-fold branchings defined by

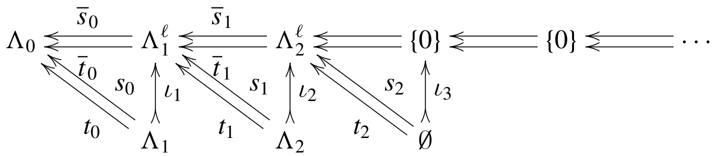
$$(a_1, \ldots, a_n) \sqsubseteq (wa_1w', \ \ldots \ , wa_nw')$$

for any monomials w, w' in Λ_1^*. For instance, a 3-fold critical branching can have two different shapes:



5.2.6 Theorem *[33,* **Thm. 6.2.4].** *Any convergent linear 2-polygraph Λ extends to a Tietze equivalent acyclic linear ∞-polygraph whose n-cells, for $n \geqslant 3$, are indexed by the critical $(n-1)$-fold branchings of Λ.*

5.2.7 Example. Consider the algebra $\mathbf{A}$ presented by the linear 2-polygraph given in Example 2.1.9. We have seen in Example 5.1.10 that any Squier's completion of Λ is empty. In particular, the polygraph Λ can be extended into a coherent presentation with empty homotopy bases, and as a consequence, into a polygraphic resolution with an empty set of k-cell, for $k \geqslant 3$:



5.2.8 A free bimodules resolution. Let Λ be a linear ∞-polygraph whose underlying 2-polygraph is a presentation of an algebra $\mathbf{A}$. For $k \geqslant 1$, we denote by $\mathbf{A}^e[\Lambda_k]$ the free $\mathbf{A}$-bimodule on Λ_k, given by the linear combinations of $f[\alpha]g$, where f and g are 1-cells in $\mathbf{A}$ and α is a k-cell in Λ_k.

The mapping of every 1-cell x in Λ_1 to the element $[x]$ in $\mathbf{A}^e[\Lambda_1]$ is uniquely extended into a derivation, denoted by $[\cdot]$, from Λ_1^ℓ with values in the $\mathbf{A}$-bimodule $\mathbf{A}^e[\Lambda_1]$, sending a 1-cell f in Λ_1^ℓ on the element $[f]$ in $\mathbf{A}^e[\Lambda_1]$, defined by linearity and by induction on the length of monomials as follows:

$$[1] = 0, \qquad [u+v] = [u] + [v], \qquad [uv] = [u]\overline{v} + \overline{u}[v], \qquad [\lambda u] = \lambda[u],$$

for any monomials u and v in Λ_1^ℓ and scalar λ in $\mathbb{K}$. We extend the bracket notation to $\mathbf{A}$-bimodules $\mathbf{A}^e[\Lambda_k]$, for $k > 1$ as follows. The mapping of every k-cell α of Λ_k to the element $[\alpha]$ in $\mathbf{A}^e[\Lambda_k]$ is extended to any k-cell a of Λ_k^ℓ by induction on the size of a. For any $(k-1)$-cell u, any k-cells a and b in Λ_k^ℓ and scalar λ, we set

$$[1_u] :=: 0, \qquad [a+b] :=: [a]+[b], \qquad [ab] :=: [a]\overline{b}+\overline{a}[b], \qquad [\lambda a] :=: \lambda[a].$$

To the linear ∞-polygraph Λ, we associate a complex of $\mathbf{A}$-bimodules

$$0 \longleftarrow \mathbf{A} \overset{\mu}{\leftarrow} \mathbf{A}^e[\Lambda_0] \overset{\delta_0}{\leftarrow} \mathbf{A}^e[\Lambda_1] \longleftarrow \cdots \longleftarrow \mathbf{A}^e[\Lambda_k] \overset{\delta_k}{\leftarrow} \mathbf{A}^e[\Lambda_{k+1}] \longleftarrow \cdots$$

where the boundary maps are defined as follows. The map μ is defined by $\mu(f\otimes g) = fg$, for any 1-cells f and g in $\mathbf{A}$. For any triple $f[x]g$ in $\mathbf{A}^e[\Lambda_1]$, we define

$$\delta_0(f[x]g) = f\otimes xg - fx\otimes g.$$

For $k \geqslant 1$, for any triple $f[\alpha]g$ in $\mathbf{A}^e[\Lambda_{k+1}]$, we define

$$\delta_k(f[\alpha]g) = f[s_k(\alpha)]g - f[t_k(\alpha)]g.$$

By induction on the length of f, we prove that $\delta_0([f]) = 1\otimes f - f\otimes 1$, for all 1-cell f in Λ_1^ℓ. We have $\mu\delta_0 = 0$, and for any k-cell α in Λ_k with $k \geqslant 2$, we have

$$\delta_{k-1}\delta_k[\alpha] = [s_{k-1}s_k(\alpha)] + [t_{k-1}s_k(\alpha)] - [s_{k-1}t_k(\alpha)] - [t_{k-1}t_k(\alpha)].$$

It follows from the globular relations that $\delta_{k-1}\delta_k = 0$. Moreover, we prove that the acyclicity of the polygraph induces the acyclicity of the complex $\mathbf{A}^e[\Lambda]$.

5.2.9 Theorem *[33,* **Thm. 7.1.3].** *If Λ is a (finite) polygraphic resolution of an algebra* $\mathbf{A}$*, then the complex* $\mathbf{A}^e[\Lambda]$ *is a (finite) free resolution of the* $\mathbf{A}$*-bimodule* $\mathbf{A}$.

5.2.10 Example. Consider the algebra $\mathbf{A}$ presented by the linear 2-polygraph given in Example 2.1.9. The resolution of $\mathbf{A}$-bimodules induced by the polygraphic resolution of Λ given in Example 5.2.7 is

$$0 \longleftarrow \mathbf{A} \overset{\mu}{\leftarrow} \mathbf{A}^e \overset{\delta_0}{\leftarrow} \mathbf{A}^e[x,y,z] \overset{\delta_1}{\leftarrow} \mathbf{A}^e[\gamma] \longleftarrow 0 \longleftarrow \cdots$$

It follows that this algebra is of cohomological dimension 2. Note that the Anick resolution for the algebra $\mathbf{A}$ computed with the same presentation is of infinite length.

5.2.11 Exercise. Consider the algebra $\mathbf{A}$ presented by the linear 2-polygraph

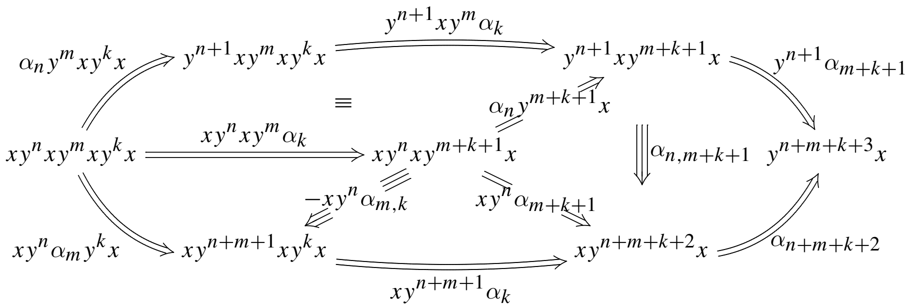
$$\Lambda = \langle * \mid x, y \mid x^2 \overset{\alpha_0}{\Rightarrow} yx \rangle.$$

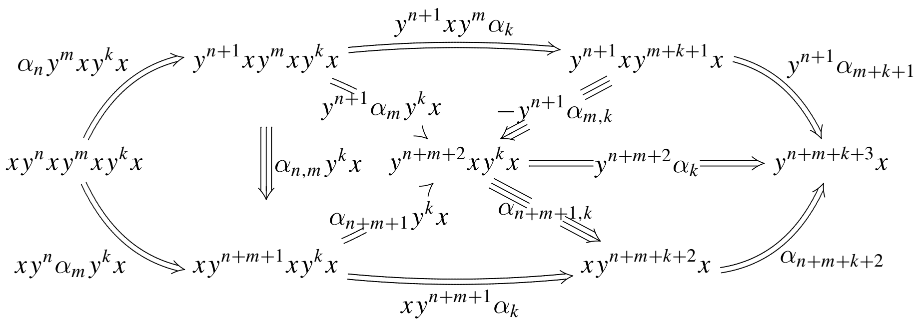
(1) Compute the first four steps of a polygraphic resolution of the algebra **A** starting with Λ.

(2) Compare the resolution of **A**-bimodules induced by this resolution with the Anick resolution $\mathcal{A}(\Lambda)$ computed in Example 4.3.7.

(3) Compute a polygraphic resolution of the algebra **A** using the presentation $\langle * \mid x, y \mid yx \Rightarrow x^2 \rangle$.

Hint. Here the source and the target of a 4-cell





6 Confluence and Koszulness

In this section, we recall the notion of Koszulness for graded associative algebras. We show how Anick's resolution leads to relate this property for an algebra to the existence of a quadratic Gröbner basis for its ideal of relations. Finally, we show how polygraphic resolutions can be used to prove this property, allowing to relate Koszulness with polygraphic convergence.

6.1 Koszulness of Associative Algebras

6.1.1 Koszulness of quadratic algebras. Recall that a connected graded algebra $\mathbf{A}$ is *Koszul* if the Tor spaces $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K},\mathbb{K})$ vanish for $i \neq n$, where the grading n is the homological degree and the grading i corresponds to the internal grading of the algebra $\mathbf{A}$. Koszul algebras were introduced by Priddy [55]. In particular, Priddy proved that quadratic algebras having a Poincaré–Birkhoff–Witt basis are Koszul [55]. The property can be also be stated in terms of existence of a linear minimal graded free resolution of $\mathbb{K}$ seen as a $\mathbf{A}$-module, see [53]. Backelin gave a characterization of the Koszul property in term of lattice [6, 7], and the Backelin condition was interpreted in terms of confluence by Berger [9], using reduction operator theory.

6.1.2 Koszulness of N-homogeneous algebras. Koszulness was generalized by Berger to the case of N-homogeneous algebras [10, Def. 2.10]. A graded N-homogeneous algebra $\mathbf{A}$, with $N \geqslant 2$, is *left-Koszul* if the ground field $\mathbb{K}$ considered as a graded left $\mathbf{A}$-module admits a graded projective resolution of the form

$$0 \longleftarrow \mathbb{K} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

such that every P_i is generated (as a graded left $\mathbf{A}$-module) by $P_i^{(\ell_N(i))}$, where $\ell_N : \mathbb{N} \longrightarrow \mathbb{N}$ is a map defined by

$$\ell_N(i) = \begin{cases} pN & \text{if } i = 2p, \\ pN+1 & \text{if } i = 2p+1. \end{cases}$$

Similarly, one can define the properties *right-Koszul* and *bi-Koszul* by considering projective resolutions of right and bimodules, respectively. The graduation on the algebra $\mathbf{A}$ induces a graduation on the vector spaces $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K},\mathbb{K})$. The spaces $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K},\mathbb{K})$ for a left-Koszul (or right-Koszul) algebra $\mathbf{A}$ vanish for $i \neq \ell_N(n)$. This property of the Tor groups is an equivalent definition of Koszul algebras, as Berger proved in [10, Thm. 2.11]. Finally, the following result shows that the Koszul property corresponds to a limit case.

6.1.3 Proposition (*[11,* **Prop. 2.1]).** *Let* $\mathbf{A}$ *be an* N*-homogeneous algebra. The graded vector space* $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K},\mathbb{K})$ *always vanish for* $i < \ell_N(n)$*, for* $n \geqslant 0$.

6.2 Confluence and Koszulness

6.2.1 Koszulness of monomial algebras. Given a monomial linear 2-polygraph Λ which is quadratic, that is its 2-cells are of the form $x_i x_j \Rightarrow 0$, with x_i, x_j in Λ_1. Then the Anick resolution $\mathcal{A}(\Lambda)$ is concentrated in the diagonal in the following sense.

The set of 0-chains is Λ_1 and they are of degree 1. The set of 1-chains is $s_1(\Lambda)$ and they are of degree 2. More generally, an n-chains $x|t_1 \dots |t_{n-1}|t_n$ is of degree $n+1$. As a consequence, we have the following result.

6.2.2 Theorem. *A quadratic monomial algebra is Koszul.*

More generally, the Anick resolution can be used to prove Koszulness of an algebra whose set of relations forms a quadratic Gröbner basis. In that case, the Anick resolution is concentrated in the right bidegree. Hence, we have the following sufficient condition for Koszulness of quadratic algebras.

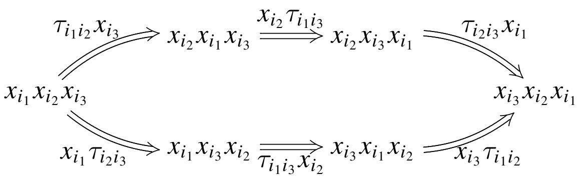
6.2.3 Theorem *[2,* **Sect. 3].** *An algebra presented by a quadratic Gröbner basis is Koszul.*

Another way to prove this result is that the existence of a quadratic Gröbner basis implies the existence of a Poincaré–Birkhoff–Witt basis of **A** [31].

6.2.4 Example. The algebra $\mathbb{K}[x_1, \dots, x_k]$ of commutative polynomials on k variables can be presented by the following linear 2-polygraph:

$$\Lambda = \langle * \mid x_1, \dots, x_k \mid x_{i_1}x_{i_2} \overset{\tau_{i_1 i_2}}{\Rightarrow} x_{i_2}x_{i_1}, \quad 1 \leqslant i_1 < i_2 \leqslant k \,\rangle.$$

For any triple (i_1, i_2, i_3) such that $1 \leqslant i_1 < i_2 < i_3 \leqslant k$, there is a critical branching on the monomial $x_{i_1}x_{i_2}x_{i_3}$ which is confluent



It follows that the linear 2-polygraph Λ is convergent and quadratic; hence, the algebra $\mathbb{K}[x_1, \dots, x_k]$ is Koszul.

6.2.5 Example, [25]. Dotsenko and Roy Chowdhury show that the algebra **A** presented by

$$\langle * \mid x, y, z \mid yx + x^2, zy, xz \,\rangle$$

is Koszul. Their proof in [25] is based on the computation of Anick's resolution with respect to the degree-lexicographic ordering induced by the alphabetic order $x > y > z$. The three quadratic relations can be completed into the following infinite Gröbner basis:

$$xz \Rightarrow 0, \quad zy \Rightarrow 0, \quad xy^kx \Rightarrow y^{k+1}x, \quad \text{for } k \geqslant 0$$

Using Anick's resolution they show that the homology of the algebra **A** is concentrated on the diagonal, proving that the algebra **A** is Koszul.

6.2.6 A sufficient polygraphic condition. In [33], a graded version of Theorem 5.2.9 is given. For that, a notion of graded linear polygraph is introduced, that generalizes in higher dimensions the notion of graded presentation for a graded algebra. As an application, one deduces the following polygraphic condition of Koszulness of graded algebras.

6.2.7 Theorem *[33,* **Prop. 7.2.2].** *Let* **A** *be an* N*-homogeneous algebra. If* **A** *has a* ℓ_N*-concentrated polygraphic resolution, then* **A** *is bi-Koszul (resp. left-Koszul, resp. right-Koszul).*

From this sufficient condition, one deduces the following consequence. Suppose that an algebra **A** has a polygraphic resolution Λ such that $(\Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1})$ is ℓ_N-concentrated, for some $k \geqslant 3$, and such that for some $i > \ell_N(k)$ the number of $(k+1)$-cells in $\Lambda_{k+1}^{(i)}$ is strictly less than the number of k-cells in $\Lambda_k^{(i)}$. Then the algebra **A** is not Koszul [33, Prop. 7.2.7].

Theorem 6.2.7 can be also used to extend the sufficient condition of Theorem 6.2.3 to linear 2-polygraph with an orientation that is not compatible with a monomial order.

6.2.8 Corollary *[33]. Let* **A** *be an algebra presented by a quadratic left-monomial convergent linear* 2*-polygraph* Λ*. Then* Λ *can be extended into a* ℓ_2*-concentrated polygraphic resolution and the algebra* **A** *is Koszul.*

6.2.9 Exercise. Let **A** be the algebra presented by $\langle * \mid x, y \mid x^2 = y^2 = xy \rangle$. Prove that **A** is not Koszul. [Hint. Consider the rules $xy \Rightarrow x^2$ and $y^2 \Rightarrow x^2$, compute a convergent presentation of **A** and its set of critical triple branchings.]

6.2.10 Remark. Note that, for an N-homogeneous algebra, that is whose relations are concentrated in degree N, the existence of a Gröbner basis concentrated in degree N is not enough to imply Koszulness. Indeed, an extra condition has to be checked as shown by Berger in [10].

6.2.11 Homogeneous coherent presentations. A coherent ℓ_N-concentrated presentation of an algebra **A** having an empty homotopy basis can be extended into a polygraphic resolution with an empty set of k-cells for $k \geqslant 3$, thus a ℓ_N-concentrated polygraphic resolution. Hence, by Theorem 6.2.7, we have the following corollary.

6.2.12 Corollary *[33]. If an* N*-homogeneous algebra has a coherent* ℓ_N*-concentrated presentation with an empty homotopy basis, then it is Koszul. In particular, an algebra having a terminating presentation by an* N*-homogeneous polygraph without any critical branching is Koszul.*

The second statement is a consequence of Squier's Theorem 5.1.6. Indeed, if Λ is a convergent left-monomial linear 2-polygraph, then it can be extended into a coherent presentation whose homotopy basis is made of generating confluences. In particular, when the polygraph Λ has no critical branching, this homotopy basis is empty, and thus trivially ℓ_N-concentrated.

6.2.13 Example, [33, **Ex. 7.2.5**]. Consider the algebra **A** presented by the linear 2-polygraph given in Example 2.1.9. From the resolution computed in Example 5.2.10, we have

$$\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}, \quad \mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3, \quad \mathrm{Tor}^{\mathbf{A}}_{2,(3)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K},$$

and $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanishes for other values of k and i. It follows that the algebra **A** is Koszul.

6.2.14 Exercise [53, **4.3**]. Show that the algebra presented by the following linear 2-polygraph, see 5.1.12,

$$\langle * \mid x, y, z \mid yz \overset{\alpha}{\Rightarrow} -x^2,\ zy \overset{\beta}{\Rightarrow} -\lambda^{-1}x^2 \rangle,$$

where $\lambda \in \mathbb{K} \setminus \{0,1\}$, is Koszul. In particular, show that $\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$, $\mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3$, $\mathrm{Tor}^{\mathbf{A}}_{2,(2)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^2$ and $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanishes otherwise.

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Chapter 5
Introduction to Computational Algebraic Statistics

Satoshi Aoki

In this paper, we introduce the fundamental notion of a Markov basis, which is one of the first connections between commutative algebra and statistics. The notion of a Markov basis is first introduced by Diaconis and Sturmfels [8] for conditional testing problems on contingency tables by Markov chain Monte Carlo methods. In this method, we make use of a connected Markov chain over the given conditional sample space to estimate the p values numerically for various conditional tests. A Markov basis plays an importance role in this argument, because it guarantees the connectivity of the chain, which is needed for unbiasedness of the estimate, for arbitrary conditional sample space. As another important point, a Markov basis is characterized as generators of the well-specified toric ideals of polynomial rings. This connection between commutative algebra and statistics is the main result of [8]. After this first paper, a Markov basis is studied intensively by many researchers both in commutative algebra and statistics, which yields an attractive field called *computational algebraic statistics*. In this paper, we give a review of the Markov chain Monte Carlo methods for contingency tables and Markov bases, with some fundamental examples. We also give some computational examples by algebraic software Macaulay2 [10] and statistical software R. Readers can also find theoretical details of the problems considered in this paper and various results on the structure and examples of Markov bases in [4].

1 Conditional Tests for Contingency Tables

A contingency table is a cross-classified table of frequencies. For example, suppose 40 students in some class took examinations of two subjects, Algebra and Statistics. Suppose that both scores are classified to one of the categories, {Excellent, Good, Fair}, and are summarized in Table 1. This is a typical example of two-way con-

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Table 1 Scores of Algebra and Statistics for 40 students (imaginary data)

Alg\Stat	Excellent	Good	Fair	Total
Excellent	11	5	2	18
Good	4	9	1	14
Fair	2	3	3	8
Total	17	17	6	40

tingency tables. Since this table has 3 rows and 3 columns, this is called a 3×3 contingency table. The two subjects, Algebra and Statistics, are called *factors* of the table, and the outcomes (i.e., scores) of each factor, {Excellent, Good, Fair}, are called *levels* of each factor. The *cells* of the $I \times J$ contingency table is the IJ possible combinations of outcomes. Three-way, four-way or higher dimensional contingency tables are defined similarly. For example, adding to the data of Table 1, if the scores of another subject (Geometry, for example) are also given, we have a three-way contingency table. An $I_1 \times \cdots \times I_m$ (m-way)contingency table has $\prod_{i=1}^{m} I_i$ cells, where I_i is the number of the levels for the ith factor, $i = 1, \ldots, m$. In statistical data analysis, the development of methods for analyzing contingency tables began in the 1960s. We refer to [3] for standard textbook in this field.

We begin with simple $I \times J$ cases, and will consider generalizations to m-way cases afterward. In statistical inference, we consider underlying random variables and statistical models for observed data such as Table 1, and treat the observed data as one realization of the random variables. In the case of Table 1, it is natural to deal with the two-dimensional discrete random variables

$$(V_1, W_1), (V_2, W_2), \ldots, (V_n, W_n), \tag{1}$$

where n is the *sample size*, ($n = 40$ for Table 1) and (V_k, W_k) is the couple of scores obtained by the kth student. The random couples (V_k, W_k) for $k = 1, \ldots, n$ are drawn independently from the same distribution

$$P(V_k = i,\ W_k = j) = \theta_{ij},\ \ i \in [I],\ j \in [J],\ k \in [n].$$

Here we use a notation $[r] = \{1, 2, \ldots, r\}$ for $r \in \mathbb{Z}_{\geqslant 0}$, where $\mathbb{Z}_{\geqslant 0}$ is the set of nonnegative integers. Note that we use appropriate coding such as 1: Excellent, 2: Good, 3: Fair. The probability $\boldsymbol{\theta} = (\theta_{ij})$ satisfies the condition

$$\sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} = 1,$$

and is called a *parameter*. The parameter space

$$\Delta_{IJ-1} = \left\{ (\theta_{11}, \ldots, \theta_{IJ}) \in \mathbb{R}^{IJ}_{\geqslant 0} : \sum_{i=1}^{I} \sum_{j=1}^{J} \theta_{ij} = 1 \right\}$$

is called an $IJ - 1$ dimensional *probability simplex*.

To consider the data in the form of a contingency table, we also summarize the underlying random variable (1) to the form of the contingency tables as

$$X_{ij} = \sum_{k=1}^{n} \mathbf{1}(V_k = i, W_k = j),$$

for $i \in [I]$, $j \in [J]$, where $\mathbf{1}(\cdot)$ is the indicator function. By this aggregation from the raw scores to the contingency table, we neglect the order of observations in (1), that is considered to have no information for estimating the parameter $\boldsymbol{\theta}$. Then the data $\mathbf{x} = (x_{ij}) \in \mathbb{Z}_{\geqslant 0}^{IJ}$ is treated as a realization of $\mathbf{x} = (X_{ij})$. The distribution of $\mathbf{x}$ is a *multinomial distribution* given by

$$p(\mathbf{x}) = P(\mathbf{X} = \mathbf{x}) = \frac{n!}{\prod_{i=1}^{I} \prod_{j=1}^{J} x_{ij}!} \prod_{i=1}^{I} \prod_{j=1}^{J} \theta_{ij}^{x_{ij}}, \quad \sum_{i=1}^{I} \sum_{j=1}^{J} x_{ij} = n. \tag{2}$$

We see that the multinomial distribution (2) is derived from the joint probability function for n individuals under the assumption that each outcome is obtained independently.

By summarizing the data in the form of contingency tables for fixed sample size n, the degree of freedom of the observed frequency $\mathbf{x}$ becomes $IJ - 1$. which coincides the degree of freedom of the parameter $\boldsymbol{\theta} \in \Delta_{IJ-1}$. Here, we use "degree of freedom" as the number of elements that are free to vary, that is a well-used terminology in statistical fields. We can see the probability simplex Δ_{IJ-1} as an example of statistical models, called a *saturated model*. Statistical model is called saturated if the degree of freedom of the parameter equals to the degree of freedom of data.

The saturated model is also characterized as the statistical model having the parameter with the largest degree of freedom. In this sense, the saturated model is the most complex statistical model. In other words, the saturated model is the statistical model that fits the observed data perfectly, i.e., fits the data *without error*. In fact, the parameter $\boldsymbol{\theta}$ in the saturated model Δ_{IJ-1} is estimated from the data as

$$\hat{\theta}_{ij} = \frac{x_{ij}}{n}, \ i \in [I], \ j \in [J], \tag{3}$$

that is also called an empirical probability of data. Because we assume that the data $\mathbf{x}$ is obtained from some probability function such as multinomial distribution (2) *with some randomness*, we want to consider more simple statistical model, i.e., a subset of the saturated model, $\mathcal{M} \subset \Delta_{IJ-1}$.

In the two-way contingency tables, a natural, representative statistical model is an *independence model*.

1.1 Definition. The independence model for $I \times J$ contingency tables is the set

$$\mathcal{M}_{indp} = \{\boldsymbol{\theta} \in \Delta_{IJ-1} \ : \ \theta_{ij} = \theta_{i+}\theta_{+j}, \forall i, \forall j\}, \tag{4}$$

where

$$\theta_{i+} = \sum_{j=1}^{J} \theta_{ij}, \ \theta_{+j} = \sum_{i=1}^{I} \theta_{ij} \ \text{ for } i \in [I], \, j \in [J].$$

1.2 Remarks. Here we consider that only the sample size n is fixed. However, several different situations can be considered for $I \times J$ contingency tables. The situation that we consider here is called a multinomial sampling scheme. For other sampling schemes such as Poisson, binomial and so on, see Chap. 2 of [3] or Chap. 4 of [13]. Accordingly, the corresponding independence model $\mathcal{M}_{indp}$ is called in a different way for other sampling schemes. For example, it is called a *common proportions model* for (product of) binomial sampling scheme where the row sums are fixed, and *main effect model* for Poisson sampling scheme where no marginal is fixed. Though there are also little differences between the descriptions of these models, we can treat these models almost in the same way by considering the *conditional probability function*, which we consider afterward. Therefore we restrict our arguments to the multinomial sampling scheme in this paper.

There are several equivalent descriptions for the independence model $\mathcal{M}_{indp}$. The most common parametric description in statistical textbooks is

$$\mathcal{M}_{indp} = \left\{\boldsymbol{\theta} \in \Delta_{IJ-1} : \ \theta_{ij} = \alpha_i \beta_j \text{ for some } (\alpha_i), \ (\beta_j)\right\}. \tag{5}$$

For other equivalent parametric descriptions or implicit descriptions, see Sect. 1 of [16], for example.

The meaning of $\mathcal{M}_{indp}$ in Table 1 is as follows. If $\mathcal{M}_{indp}$ is true, there are no relations between the scores of two subjects. Then we can imagine that the scores of two subjects follow the marginal probability functions for each score respectively, and are independent, and the discrepancy we observed in Table 1 is obtained "by chance". However, it is natural to imagine some structure between the two scores such as "there is a tendency that the students having better scores in Algebra are likely to have better scores in Statistics", because these subjects are in the same mathematical category. In fact, we see relatively large frequencies 11 and 9 in the diagonals of Table 1, which seem to indicate a positive correlation. Therefore one of the natural questions for Table 1 is "Is there some tendency between the two scores that breaks independence?". To answer this question, we evaluate the fitting of $\mathcal{M}_{indp}$ by *hypothetical testing*.

The hypothetical testing problem that we consider in this paper is as follows.

$$\mathrm{H}_0 : \ \boldsymbol{\theta} \in \mathcal{M}_{indp} \ \text{ v.s. } \ \mathrm{H}_1 : \ \boldsymbol{\theta} \in \Delta_{IJ-1} \setminus \mathcal{M}_{indp}.$$

Here we call H_0 a *null hypothesis* and H_1 an *alternative hypothesis*. The terms *null model* and *alternative model* are also used. The hypothetical testing in the above form, i.e., a null model is a subset of a saturated model, $\mathcal{M} \subset \Delta_{IJ-1}$, and the alternative model is the complementary set of $\mathcal{M}$ into the saturated model, is called a *goodness-of-fit test* of model $\mathcal{M}$. The testing procedures are composed of steps such as choosing a test statistics, choosing a significance level, and calculating the p value. We see these steps in order.

Choosing a test statistic. First we have to choose a test statistic to use. In general, the term *statistic* means a function of the random variable $\mathbf{X} = (X_{ij})$. For example, (X_{i+}) and (X_{+j}) given by

$$X_{i+} = \sum_{j=1}^{J} X_{ij}, \;\; X_{+j} = \sum_{i=1}^{I} X_{ij} \text{ for } i \in [I],\, j \in [J]$$

are examples of statistics called the row sums and the column sums, respectively. Other examples of statistics are the row mean $\bar{X}_{i+} = X_{i+}/J$ and the column mean $\bar{X}_{+j}/I$ for $i \in [I]$, $j \in [J]$. To perform the hypothetical testing, we first select an appropriate statistic, called a *test statistic*, to measure the discrepancy of the observed data from the null model. One of the common test statistic for the goodness-of-fit test is a *Pearson goodness-of-fit* χ^2 given by

$$\chi^2(\mathbf{X}) = \sum_{i=1}^{I} \sum_{j=1}^{J} \frac{(X_{ij} - \hat{m}_{ij})^2}{\hat{m}_{ij}},$$

where $\hat{m}_{ij}$ is the *fitted value* of X_{ij} under H_0, i.e., an estimator of $E(X_{ij}) = m_{ij} = n\theta_{ij}$, given by

$$\hat{m}_{ij} = n\hat{\theta}_{ij} = \frac{x_{i+}x_{+j}}{n}. \tag{6}$$

Here we use the *maximum likelihood estimate* of the parameter under the null model, $\hat{\boldsymbol{\theta}} = (\hat{\theta}_{ij})$, given by

$$\hat{\theta}_{ij} = \frac{x_{i+}x_{+j}}{n^2}, \tag{7}$$

that is obtained by maximizing the log-likelihood

$$\text{Const} + \sum_{i=1}^{I} \sum_{j=1}^{J} x_{ij} \log \theta_{ij}$$

under the constraint $\boldsymbol{\theta} \in \mathcal{M}_{indp}$. The meaning of this estimate is also clear in a parametric description (5) since the maximum likelihood estimates of (α_i), (β_j) are given by

$$\hat{\alpha}_i = \frac{x_{i+}}{n}, \;\; \hat{\beta}_j = \frac{x_{+j}}{n},$$

Table 2 The fitted value under $\mathcal{M}_{indp}$ for Table 1

Alg\Stat	Excellent	Good	Fair	Total
Excellent	7.65	7.65	2.70	18
Good	5.95	5.95	2.10	14
2–4 Fair	3.40	3.40	1.20	8
Total	17	17	6	40

respectively. The fitted value for Table 1 under $\mathcal{M}_{indp}$ is given in Table 2.

There are various test statistics other than the Pearson goodness-of-fit χ^2 that can be used in our problem. Another representative is the (twice log) likelihood ratio given by

$$2\sum_{i=1}^{I}\sum_{j=1}^{J} X_{ij}\log\frac{X_{ij}}{\hat{m}_{ij}}, \tag{8}$$

where $\hat{m}_{ij}$ is given by (6). In general, test statistic should be selected by considering their *power*, i.e., the probability that the null hypothesis is rejected if the alternative hypothesis is true. See textbooks such as [14] for the theory of the hypothetical testing, the optimality of the test statistics, examples and the guidelines for choosing test statistics for various problems.

Choosing a significance level. Once we choose a test statistic to use, as the Pearson goodness-of-fit χ^2 for example, the hypothetical testing procedure is written by

$$\chi^2(\mathbf{x}^o) \geqslant c_\alpha \ \Rightarrow\ \text{Reject H}_0,$$

where $\mathbf{x}^o$ is the observed data, and c_α is the *critical point at the significance level* α satisfying

$$P(\chi^2(X) \geqslant c_\alpha \mid \text{H}_0) \leqslant \alpha. \tag{9}$$

The probability of the left hand side of (9) is called a *type I error*. Equivalently, we define the *p-value* by

$$p = P(\chi^2(X) \geqslant \chi^2(\mathbf{x}^o) \mid \text{H}_0), \tag{10}$$

then the testing procedure is written by

$$p \leqslant \alpha \ \Rightarrow\ \text{Reject H}_0.$$

The meaning of the p-value for the data $\mathbf{x}^o$ is the conditional probability that "more or equally discrepant results are obtained than the observed data if the null hypothesis is true". Therefore, if p-value is significantly small, we conclude that null hypothesis is unrealistic, because it is doubtful that such an extreme result $\mathbf{x}^o$ is obtained. This is the idea of the statistical hypothetical testing. In this process, the significance level

α plays a threshold to decide the p-value is "significantly small" to reject the null hypothesis. In statistical and scientific literature, it is common to choose $\alpha = 0.05$ or $\alpha = 0.01$. Readers can find various topics on p-value in [19].

Calculating the p-value. Once we choose a test statistic and a significance level, all we have to do is to calculate the p-value given in (10) for observed data $\mathbf{x}^o$. The observed value of the Pearson goodness-of-fit χ^2 for Table 1 is

$$\chi^2(\mathbf{x}^o) = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{(x_{ij}^o - \hat{m}_{ij})^2}{\hat{m}_{ij}} = \frac{(11-7.65)^2}{7.65} + \cdots + \frac{(3-1.20)^2}{1.20} = 8.6687,$$

therefore the p-value for our $\mathbf{x}^o$ is

$$p = P(\chi^2(X) \geqslant 8.6687 \mid \mathrm{H}_0).$$

This probability is evaluated based on the *probability function of the test statistic* $\chi^2(\mathbf{X})$ *under* H_0, which we call a *null distribution* hereafter. Unfortunately, the null distribution depends on the unknown parameter $\boldsymbol{\theta} \in \mathcal{M}_{indp}$ and the p-values cannot be calculated in most cases in principle. One naive idea to evaluate the p-values for such cases is to calculate its supremum in $\mathcal{M}_{indp}$ and perform the test as the form

$$\sup_{\boldsymbol{\theta}\in\mathcal{M}_{indp}} P(\chi^2(X) \geqslant \chi^2(\mathbf{x}^o) \mid \mathrm{H}_0) \leqslant \alpha \;\Rightarrow\; \text{Reject } \mathrm{H}_0. \tag{11}$$

However, this idea is hard to implement in general, i.e., it is usually difficult to evaluate the left-hand side of (11) or to seek tests that are powerful under (11). Then, what should we do? We consider the following three strategies for calculating p-values in this paper.

(a) Using the asymptotic distribution of the test statistic.
(b) Exact calculation based on the conditional distribution.
(c) Estimate the p-value by the Monte Carlo method.

The aim of this paper is to introduce strategy (c). We will consider each strategy in order.

(a) Using the asymptotic distribution of the test statistic. In applications, it is common to rely on various asymptotic theories for the test statistics. As for the Pearson goodness-of-fit test χ^2 test, the following result is known.

1.3 Theorem. *Under the null model* $\mathcal{M}_{indp}$*, the Pearson goodness-of-fit* $\chi^2(X)$ *asymptotically follows the* χ^2 *distribution with* $(I-1)(J-1)$ *degree of freedom, i.e.,*

$$\lim_{n\to\infty} P(\chi^2(X) \geqslant u) = P(V \geqslant u) \ \ \textit{for } u > 0,$$

where $V \sim \chi^2_{(I-1)(J-1)}$*, i.e.,* V *is distributed to the* χ^2 *distribution with* $(I-1)(J-1)$ *degree of freedom.*

This theorem is shown as a consequence of the central limit theorem. In addition, the same asymptotic distribution is given when we consider the conditional limit, i.e., consider $n \to \infty$ under the condition that $X_{i+}/n \to a_i$ and $X_{+j}/n \to b_j$ for $i \in [I]$, $j \in [J]$ for some fixed $0 < a_i, b_j < 1$. See [5] or [17] for detail. Anyway, these asymptotic properties are the reason why we call this test as Pearson goodness-of-fit "χ^2 test". Similarly to the Pearson goodness-of-fit χ^2, there are several test statistics that have the χ^2 distribution as the asymptotic distribution. An important example is the likelihood ratio test statistic, which is given in (8) for our setting. Moreover, several asymptotic good properties of likelihood ratio test statistics are known. See [14] for details. Note also that our methods, Markov chain Monte Carlo methods, can be applicable for arbitrary type of test statistics, though we only consider the Pearson goodness-of-fit χ^2 in this paper.

Following Theorem 1.3, it is easy to evaluate the asymptotic p-value of the Pearson goodness-of-fit χ^2 test. For our data, the observed value of test statistic, $\chi^2(\mathbf{x}^o) = 8.6687$, is less than the upper 5 percent point of the χ^2 distribution with 4 degrees of freedom, $\chi^2_{4,0.05} = 9.488$. Therefore, for the significance level $\alpha = 0.05$, we cannot reject the null hypothesis H_0, i.e., we cannot say that "the fitting of the model $\mathcal{M}_{indp}$ to Table 1 is poor". Equivalently, the asymptotic p-value is calculated as the upper probability of χ^2_4, which is 0.0699 and is greater than $\alpha = 0.05$. Figure 1 presents the probability density function of the χ^2_4 distribution. The above results can be obtained numerically by the following codes of the statistical software R.

```
> x <- matrix(c(11,5,2,4,9,1,2,3,3), byrow=T, ncol=3, nrow=3)
> x
     [,1] [,2] [,3]
[1,]   11    5    2
[2,]    4    9    1
[3,]    2    3    3
> chisq.test(x)

Pearson's Chi-squared test

data:  x
X-squared = 8.6687, df = 4, p-value = 0.06994
> pchisq(8.6687,4, lower.tail=F)
[1] 0.06993543
> qchisq(0.05,4,lower.tail=F)                      # critical point
[1] 9.487729
```

As we see above, using asymptotic null distribution is an easy way to evaluate p-values, and one of the most common approaches in applications. One of the disadvantages of strategy (a) is that there might not be a good fit with the asymptotic distribution. In fact, because sample size is only $n = 40$ for Table 1, it is doubtful that we can apply the asymptotic result of $n \to \infty$. Besides, it is well known that there are cases that the fitting of the asymptotic distributions are poor for data with relatively

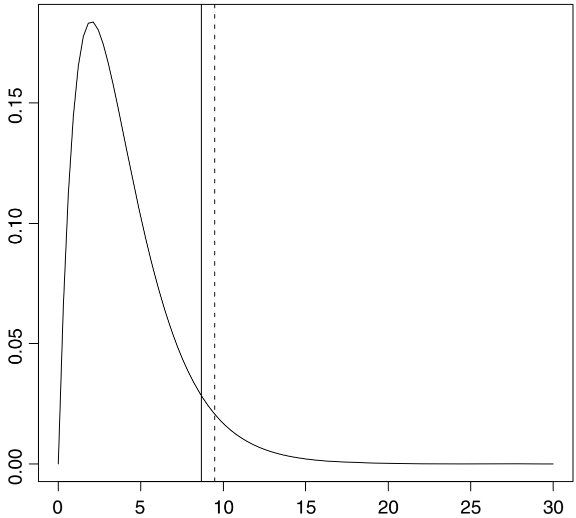

Fig. 1 χ^2 distribution with degree of freedom 4. The vertical solid line indicates the observed value $\chi^2(\mathbf{x}^o) = 8.6687$, and the dotted line indicates the critical point for the significance level $\alpha = 0.05$, $\chi^2_{4,0.05} = 9.488$

large sample sizes. One such case is sparse data case, another one is unbalanced case. See [11] for these topics.

(b) Exact calculation based on the conditional distribution. If we want to avoid asymptotic approaches as strategy (a), an alternative choice is to calculate p-values *exactly*. For the cases that the null distribution of the test statistics depend on the unknown parameters, we can formulate the exact methods based on the *conditional probability functions* for fixed *minimal sufficient statistics* under the null model $\mathcal{M}_{indp}$. The key notion here is the minimal sufficient statistics.

1.4 Definition. Let $\mathbf{X}$ be a discrete random variable with the probability function $p(\mathbf{x})$ with the parameter $\boldsymbol{\theta}$. The statistic $\mathbf{T}(\mathbf{X})$, i.e., a vector or a scalar function of $\mathbf{X}$, is called sufficient for $\boldsymbol{\theta}$ if the conditional probability function of $\mathbf{X}$ for a given $\mathbf{T}$,

$$p(\mathbf{x} \mid \mathbf{t}) = P(\mathbf{X} = \mathbf{x} \mid \mathbf{T}(\mathbf{X}) = \mathbf{t}), \tag{12}$$

does not depend on $\boldsymbol{\theta}$. The sufficient statistic $\mathbf{T}(\mathbf{X})$ is minimal if there is no other sufficient statistics that is a function of $\mathbf{T}(\mathbf{X})$.

The meaning of the minimal sufficient statistic is explained as follows. If we know the value of $\mathbf{T}$, then knowing $\mathbf{X}$ provides no further information about the parameter $\boldsymbol{\theta}$. Therefore for the parameter estimation or hypothetical testing, it is sufficient to consider the methods based on the minimal sufficient statistic. The minimal sufficient statistics for our two-way problem is as follows.

– Under the saturated model $\boldsymbol{\theta} \in \Delta_{IJ-1}$, a minimal sufficient statistic is the contingency table $\mathbf{X}$. Adding the additional information such as the scores of the kth student, (V_k, W_k), in (1) gives us no additional information on the estimation of $\boldsymbol{\theta}$. Indeed, under the saturated model, the maximum likelihood estimate

of the parameter is the empirical probability (3), that is a function of the minimal sufficient statistic.
- Under the independence model $\mathcal{M}_{indp}$, a minimal sufficient statistic is the row sums $\{X_{i+},\ i \in [I]\}$ and the column sums $\{X_{+j},\ j \in [J]\}$, as we see below. Indeed, we have already seen that the maximum likelihood estimate of the parameter under the independence model is (7), that is a function of the row sums and column sums. Note that **X** itself is also the sufficient statistic under the independence model, but is not minimal.

To see that a given statistic $\mathbf{T}(\mathbf{X})$ is sufficient for a parameter $\boldsymbol{\theta}$, a useful way is to rely on the following theorem.

1.5 Theorem. $\mathbf{T}(\mathbf{X})$ *is a sufficient statistic for* $\boldsymbol{\theta}$ *if and only if the probability function of* $\mathbf{X}$ *is factored as*

$$p(\mathbf{x}; \boldsymbol{\theta}) = h(\mathbf{x})g(T(\mathbf{x}); \boldsymbol{\theta}), \tag{13}$$

where $g(\cdot)$ *is a function that depends on the parameter* $\boldsymbol{\theta}$ *and* $h(\cdot)$ *is a function that does not.*

For the case of discrete probability function, this theorem, called a *factorization theorem*, is easily (i.e., without measure theories) proved from the definition of the sufficient statistic. Generally, to obtain such a factorization is easier than to compute explicitly the conditional distribution (12). For example, under the parametric description $\theta_{ij} = \alpha_i \beta_j$, the probability function of the multinomial distribution (2) is written as

$$p(\mathbf{x}; \boldsymbol{\theta}) = \frac{n!}{\prod\prod x_{ij}!}\left(\prod_i \alpha_i^{x_{i+}}\right)\left(\prod_j \beta_j^{x_{+j}}\right)$$

and we see that $T(\mathbf{X}) = (\{X_{i+}\}, \{X_{+j}\})$ is a sufficient statistic for the parameter $\boldsymbol{\theta} \in \mathcal{M}_{indp}$.

Here, for later generalization, we introduce a *configuration matrix* A and express a minimal sufficient statistic by A as follows. Let the number of the cells of the contingency table $\mathbf{X}$ be ν and treat $\mathbf{X}$ as a ν-dimensional column vector. Let $T(\mathbf{X})$ be a d-dimensional sufficient statistic for the parameter $\boldsymbol{\theta} \in \mathcal{M}$. For example of the independence model $\mathcal{M}_{indp}$ for $I \times J$ contingency tables, we have $\nu = IJ$, $\mathbf{X} = (X_{11}, X_{12}, \ldots, X_{IJ})'$ and

$$\mathbf{T}(\mathbf{X}) = (X_{1+}, \ldots, X_{I+}, X_{+1}, \ldots, X_{+J})'$$

and $d = I + J$. Then we see that $T(\mathbf{X})$ is written as

$$T(\mathbf{X}) = A\mathbf{X} \tag{14}$$

for $d \times \nu$ integer matrix A. For the 3×3 contingency tables, A is written as follows:

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Following the sufficiency of $T(\mathbf{X}) = A\mathbf{X}$, the conditional probability function for given $\mathbf{T} = \mathbf{t}$ does not depend on the parameter. For the case of the independence model $\mathcal{M}_{indp}$ for two-way contingency tables, it is

$$h(\mathbf{x}) = P(X = \mathbf{x} \mid A\mathbf{X} = \mathbf{t},\ \mathrm{H}_0) = \frac{\left(\prod_i x_{i+}!\right)\left(\prod_j x_{+j}!\right)}{n! \prod_{i,j} x_{ij}!}, \quad \mathbf{x} \in \mathcal{F}_{\mathbf{t}}, \tag{16}$$

where

$$\mathcal{F}_{\mathbf{t}} = \{\mathbf{x} \in \mathbb{Z}^{\nu}_{\geqslant 0} \ : \ A\mathbf{x} = \mathbf{t}\}$$

is the conditional sample space, which is called a **t**-*fiber* in the arguments of Markov bases. The conditional probability function $h(\mathbf{x})$ is called a *hypergeometric distribution*. Using this conditional probability, the conditional p-value can be defined by

$$p = E_{\mathrm{H}_0}(g(\mathbf{X}) \mid A\mathbf{X} = A\mathbf{x}^o) = \sum_{\mathbf{x} \in \mathcal{F}_{A\mathbf{x}^o}} g(\mathbf{x})h(\mathbf{x}) \tag{17}$$

for the observed table $\mathbf{x}^o$, where $g(\mathbf{x})$ is the test function

$$g(\mathbf{x}) = \begin{cases} 1, & \chi^2(\mathbf{x}) \geqslant \chi^2(\mathbf{x}^o), \\ 0, & \text{otherwise.} \end{cases}$$

Now calculate the conditional p-value exactly for Table 1. For the observed table $\mathbf{x}^o$, i.e., Table 1, we consider the independence model $\mathcal{M}_{indp}$. The configuration matrix A for $\mathcal{M}_{indp}$ is given in (15). The **t**-fiber including $\mathbf{x}^o$, i.e., $A\mathbf{x}^o$-fiber, is the set of all contingency tables that have the same value of the row sums and the column sums to $\mathbf{x}^o$,

$$\mathcal{F}_{A\mathbf{x}^o} = \left\{ \mathbf{x} \in \mathbb{Z}^9_{\geqslant 0} : \begin{array}{|c|c|c|c} \hline x_{11} & x_{12} & x_{13} & 18 \\ \hline x_{21} & x_{22} & x_{23} & 14 \\ \hline x_{31} & x_{32} & x_{33} & 8 \\ \hline \multicolumn{1}{c}{17} & \multicolumn{1}{c}{17} & \multicolumn{1}{c}{6} & 40 \end{array} \right\}.$$

There are 2366 elements in this $\mathcal{F}_{A\mathbf{x}^o}$. For each 2366 elements in $\mathcal{F}_{A\mathbf{x}^o}$, the conditional probability is given by

$$h(\mathbf{x}) = \frac{(18!14!8!)\,(17!17!6!)}{40!} \prod_{i,j} \frac{1}{x_{ij}!}, \quad \mathbf{x} \in \mathcal{F}_{A\mathbf{x}^o}.$$

Then we have the exact conditional p-value

$$p = \sum_{\mathbf{x} \in \mathcal{F}_{A\mathbf{x}^o}} g(\mathbf{x}) h(\mathbf{x}) = 0.07035480,$$

where the test function is

$$g(\mathbf{x}) = \begin{cases} 1, & \chi^2(\mathbf{x}) \geqslant 8.6687, \\ 0, & \text{otherwise.} \end{cases}$$

As a result, we cannot reject H_0 at significance level 0.05, which is the same result to strategy (a).

1.6 Example. The following toy example should help the reader in understanding the method. Let consider the 2×3 contingency table with the row sums and the column sums given as follows.

x_{11}	x_{12}	x_{13}	3
x_{21}	x_{22}	x_{23}	2
2	2	1	5

There are 5 elements in the fiber as

$$\mathcal{F}_{(3,2,2,2,1)} = \left\{ \begin{array}{|c|c|c|}\hline 2&1&0\\ \hline 0&1&1\\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 2&0&1\\ \hline 0&2&0\\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1&2&0\\ \hline 1&0&1\\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 1&1&1\\ \hline 1&1&0\\ \hline \end{array}, \begin{array}{|c|c|c|}\hline 0&2&1\\ \hline 2&0&0\\ \hline \end{array} \right\}$$
$$= \{\mathbf{x}_1, \ \mathbf{x}_2, \ \mathbf{x}_3, \ \mathbf{x}_4, \ \mathbf{x}_5\}.$$

The fitted value under the $\mathcal{M}_{indp}$ is $\begin{array}{|c|c|c|}\hline 1.2&1.2&0.6\\ \hline 0.8&0.8&0.4\\ \hline \end{array}$. Then the Pearson goodness-of-fit χ^2 for each element is calculated as

$$(\chi^2(\mathbf{x}_1), \ \chi^2(\mathbf{x}_2), \ \chi^2(\mathbf{x}_3), \ \chi^2(\mathbf{x}_4), \ \chi^2(\mathbf{x}_5)) = (2.917, \ 5, \ 2.917, \ 0.833, \ 5).$$

The conditional probabilities

$$h(\mathbf{x}) = \frac{3!2!2!2!}{5!} \prod_{i,j} \frac{1}{x_{ij}!} = \frac{2}{5} \prod_{i,j} \frac{1}{x_{ij}!}$$

for each element are calculated as

$$(h(\mathbf{x}_1),\ h(\mathbf{x}_2),\ h(\mathbf{x}_3),\ h(\mathbf{x}_4),\ h(\mathbf{x}_5)) = (0.2,\ 0.1,\ 0.2,\ 0.4,\ 0.1).$$

Therefore the conditional p-value for $\mathbf{x}_4$ is 1.0, that for $\mathbf{x}_1$ or $\mathbf{x}_3$ is 0.6, and that for $\mathbf{x}_2$ or $\mathbf{x}_5$ is 0.2.

1.7 Remarks. We briefly mention the generalization of the above method to general problems and models. First important point is the existence of the minimal sufficient statistics in the form of (14). It is known that, for the *exponential family*, well-known family of the distribution, minimal sufficient statistics exist, and for a special case of the exponential family, called the *toric model*, minimal sufficient statistics of the form (14) exist. The toric model is relatively new concept arising in the field of the computational algebraic statistics and is defined from the configuration matrix $A = (a_{ij}) \in \mathbb{Z}_{\geqslant 0}^{d\times\nu}$ as follows.

For the jth column vector $\mathbf{a}_j = (a_{1j}, \ldots, a_{dj})$ of A, $j \in [\nu]$, define the monomial

$$\boldsymbol{\theta}^{\mathbf{a}_j} = \prod_{i=1}^{d} \theta_i^{a_{ij}}, \quad j \in [\nu].$$

Then the toric model of A is the image of the orthant $\mathbb{R}_{>0}^d$ under the map

$$f : \mathbb{R}^d \to \mathbb{R}^\nu, \quad \boldsymbol{\theta} \mapsto \frac{1}{\sum_{j=1}^{\nu} \boldsymbol{\theta}^{\mathbf{a}_j}} (\boldsymbol{\theta}^{\mathbf{a}_1}, \ldots, \boldsymbol{\theta}^{\mathbf{a}_\nu}).$$

See Chap. 1.2 of [16] for detail. The toric model specified by the configuration matrix $A \in \mathbb{Z}_{\geqslant 0}^{d\times\nu}$ is also written by

$$\mathcal{M}_A = \{\boldsymbol{\theta} = (\theta_i) \in \Delta_{\nu-1} :\ \log \boldsymbol{\theta} \in \mathrm{rowspan}(A)\},$$

where $\mathrm{rowspan}(A) = \mathrm{image}(A')$ is the linear space spanned by the rows of A, and

$$\log \boldsymbol{\theta} = (\log \theta_1,\ \ldots,\ \log \theta_\nu)',$$

where $'$ is a transpose. In statistical fields, this is called a *log-linear model*. In fact, for example of the independence model $\mathcal{M}_{indp}$ of 2×3 tables, that is a log-linear model, the parametric description $\theta_{ij} = \alpha_i \beta_j$ can be written as

$$\begin{pmatrix} \log \theta_{11} \\ \log \theta_{12} \\ \log \theta_{13} \\ \log \theta_{21} \\ \log \theta_{22} \\ \log \theta_{23} \end{pmatrix} = \begin{pmatrix} 1\,0\,1\,0\,0 \\ 1\,0\,0\,1\,0 \\ 1\,0\,0\,0\,1 \\ 0\,1\,1\,0\,0 \\ 0\,1\,0\,1\,0 \\ 0\,1\,0\,0\,1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}.$$

The conditional probability function, i.e., the generalization of the hypergeometric distribution $h(\mathbf{x})$ in (16) is as follows. For the model specified by the configuration matrix A, the conditional probability function for given sufficient statistic $A\mathbf{x}^o$ is

$$P(\mathbf{X} = \mathbf{x} \mid A\mathbf{X} = A\mathbf{x}^o) = C_{A\mathbf{x}^o}^{-1} \frac{1}{\prod_{i \in [\nu]} x_i!},$$

where

$$C_{A\mathbf{x}^o} = \sum_{\mathbf{y} \in \mathcal{F}_{A\mathbf{x}^o}} \frac{1}{\prod_{i \in [\nu]} y_i!} \tag{18}$$

is a normalizing constant. Based on this conditional probability function, we can calculate the conditional p-values by (17).

Finally, we note an optimality of the method briefly. The conditional procedure mentioned above is justified if we consider the hypothetical testing to the class of *similar tests* and the minimal sufficient statistics is *complete*. For the class of the exponential family, it is known that the minimal sufficient statistic is *complete*. See Chap. 4.3 of [14] for detail.

(c) Estimate the p-value by the Monte Carlo method. The two strategies to evaluate p-values we have considered, asymptotic evaluation and exact computation, have both advantages and disadvantages. The asymptotic evaluations relying on the asymptotic χ^2 distribution are easy to carry out, especially by various packages in softwares such as R. However, poor fitting to the asymptotic distribution cannot be ignorable for sparse or unbalanced data even with relatively large sample sizes. The exact calculation of the conditional p-values is the best method if it is possible to carry out. In fact, various exact methods and algorithms are considered for problems of various types of the contingency tables, statistical models and test statistics. See the survey paper [2] for this field. However, for large size samples, the cardinality of the fiber $|\mathcal{F}_{A\mathbf{x}^o}|$ can exceed billions, making exact computations difficult to be carried out. In fact, it is known that the cardinality of a fiber increases exponentially in the sample size n. (An approximation for the cardinality of a fiber is given by [9].) For these cases, the Monte Carlo methods can be effective.

The Monte Carlo methods estimate the p-values as follows. To compute the conditional p-value (17), generate samples $\mathbf{x}_1, \ldots, \mathbf{x}_N$ from the null distribution $h(\mathbf{x})$. Then the p-value is estimated as $\hat{p} = \sum_{i=1}^{N} g(\mathbf{x}_i)/N$, that is an unbiased estimate of the p-value. We can set N according to the performance of our computer. As an advantage of the Monte Carlo method, we can also estimate the accuracy, i.e., *variance* of the estimate. For example, a conventional 95% confidence interval of p, $\hat{p} \pm 1.96\sqrt{\hat{p}(1-\hat{p})/N}$, is frequently used. The problem here is how to generate samples from the null distribution. We consider *Markov chain Monte Carlo methods*, often abbreviated as the MCMC methods, in this paper.

Following MCMC methods setup, we construct an ergodic Markov chain on the fiber $\mathcal{F} = \mathcal{F}_{A\mathbf{x}^o}$ whose stationary distribution is prescribed, given by (16). Let the elements of $\mathcal{F}$ be numbered as

$$\mathcal{F} = \{\mathbf{x}_1, \ldots, \mathbf{x}_s\}.$$

We write the null distribution on $\mathcal{F}$ as

$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_s) = (h(\mathbf{x}_1), \ldots, h(\mathbf{x}_s)).$$

Here, by standard notation, we treat $\boldsymbol{\pi}$ as a row vector. We write the transition probability matrix of the Markov chain $\{Z_t, t \in \mathbb{Z}_{\geqslant 0}\}$ over $\mathcal{F}$ as $Q = (q_{ij})$, i.e., we define

$$q_{ij} = P(Z_{t+1} = \mathbf{x}_j \mid Z_t = \mathbf{x}_i).$$

Then a probability distribution $\boldsymbol{\theta} \in \Delta_{s-1}$ is called a stationary distribution if it satisfies $\boldsymbol{\theta} = \boldsymbol{\theta} Q$. The stationary distribution uniquely exists if the Markov chain is irreducible, (i.e., connected in this case) and aperiodic. Therefore for the connected and aperiodic Markov chain, starting from an arbitrary state $Z_0 = \mathbf{x}_i$, the distribution of Z_t for large t is close to its stationary distribution. If we can construct a connected and aperiodic Markov chain with the stationary distribution $\boldsymbol{\pi}$, by running the Markov chain and discarding a large number t of initial steps (called *burn-in steps*), we can treat $Z_{t+1}, Z_{t+2}, \ldots$ to be samples from the null distribution $\boldsymbol{\pi}$ and use them to estimate p-values. Then the problem becomes *how to construct a connected and aperiodic Markov chain with the stationary distribution as the null distribution $\boldsymbol{\pi}$ over $\mathcal{F}$*. Among these conditions, the conditions for the stationary distribution can be solved easily. Once we construct an arbitrary connected chain over $\mathcal{F}$, we can modify its stationary distribution to the given null distribution $\boldsymbol{\pi}$ as follows.

1.8 Theorem (Metropolis-Hastings algorithm). *Let $\boldsymbol{\pi}$ be a probability distribution on $\mathcal{F}$. Let $R = (r_{ij})$ be the transition probability matrix of a connected, aperiodic and symmetric Markov chain over $\mathcal{F}$. Then the transition probability matrix $Q = (q_{ij})$ defined by*

$$q_{ij} = r_{ij} \min\left(1, \frac{\pi_j}{\pi_i}\right), \ i \neq j$$
$$q_{ii} = 1 - \sum_{j \neq i} q_{ij}$$

satisfies $\boldsymbol{\pi} = \boldsymbol{\pi} Q$.

This theorem is a special case of [12]. Though the symmetry assumption ($r_{ij} = r_{ji}$) can be removed easily, we only consider symmetric R for simplicity. The proof of this theorem is easy and is omitted. See [12] or Chap. 4.1 of [13], for example. Instead, we consider the algorithm for data of small size.

1.9 Example. Consider the small example in Example 1.6. As we have seen, the fiber is

$$\mathcal{F} = \{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4,\ \mathbf{x}_5\}$$

and the null distribution is

$$\boldsymbol{\pi} = (\pi_1, \ldots, \pi_5) = (h(\mathbf{x}_1), \ldots, h(\mathbf{x}_5)) = (0.2, 0.1, 0.2, 0.4, 0.1).$$

Using the Markov basis we consider in the next section, we can construct a connected, aperiodic and symmetric Markov chain with the transition probability matrix

$$R = \begin{pmatrix} 1/2 & 1/6 & 1/6 & 1/6 & 0 \\ 1/6 & 2/3 & 0 & 1/6 & 0 \\ 1/6 & 0 & 1/2 & 1/6 & 1/6 \\ 1/6 & 1/6 & 1/6 & 1/3 & 1/6 \\ 0 & 0 & 1/6 & 1/6 & 2/3 \end{pmatrix}. \tag{19}$$

Following Theorem 1.8, we modify the Markov chain to have the transition probability matrix

$$Q = \begin{pmatrix} 7/12 & 1/12 & 1/6 & 1/6 & 0 \\ 1/6 & 2/3 & 0 & 1/6 & 0 \\ 1/6 & 0 & 7/12 & 1/6 & 1/12 \\ 1/12 & 1/24 & 1/12 & 3/4 & 1/24 \\ 0 & 0 & 1/6 & 1/6 & 2/3 \end{pmatrix}.$$

We can check that the eigenvector from the left of Q with the eigenvalue 1 is $\boldsymbol{\pi}$. We can also check that each row vector of Q^T for large T converges to $\boldsymbol{\pi}$.

An important advantage of the Markov chain Monte Carlo method is that it does not require the explicit evaluation of the normalizing constant of the null distribution. As is shown in Theorem 1.8, we only need to know $\boldsymbol{\pi}$ up to a multiplicative constant, because the normalizing constant, (18) in the general form, canceled in the ratio π_j/π_i. With Theorem 1.8, the remaining problem is to construct an arbitrary connected and aperiodic Markov chain over $\mathcal{F}$, that is solved by the Gröbner basis theory.

2 Markov Bases and Ideals

As stated in the previous section, the main task for estimating p-values thanks to MCMC methods is to construct a connected and aperiodic Markov chain over $\mathcal{F} = \mathcal{F}_{A\mathbf{x}^o}$ with stationary distribution given by (16). Here, $A \in \mathbb{Z}^{d\times\nu}$ is a given configuration matrix, $\mathbf{x}^o \in \mathbb{Z}^{\nu}_{\geqslant 0}$ is the observed contingency table and $\mathcal{F}_{A\mathbf{x}^o}$, a $A\mathbf{x}^o$-fiber, is the set of all contingency tables with the same value of the minimal sufficient statistics to $\mathbf{x}^o$,

$$\mathcal{F}_{A\mathbf{x}^o} = \{\mathbf{x} \in \mathbb{Z}^{\nu}_{\geqslant 0} \ : \ A\mathbf{x} = A\mathbf{x}^o\}.$$

We write the integer kernel of A as

$$\mathrm{Ker}_{\mathbb{Z}}(A) = \mathrm{Ker}(A) \cap \mathbb{Z}^{\nu} = \{\mathbf{z} \in \mathbb{Z}^{\nu} \ : \ A\mathbf{z} = \mathbf{0}\}.$$

An element of $\mathrm{Ker}_{\mathbb{Z}}(A)$ is called a *move*. Note that $\mathbf{x} - \mathbf{y} \in \mathrm{Ker}_{\mathbb{Z}}(A)$ if and only if $\mathbf{x}, \mathbf{y} \in \mathcal{F}_{A\mathbf{x}}$. Then for a given subset $\mathcal{B} \subset \mathrm{Ker}_{\mathbb{Z}}(A)$ and $\mathbf{t} \in \mathbb{Z}^{d}_{\geqslant 0}$, we can define undirected graph $G_{\mathbf{t},\mathcal{B}} = (V, E)$ by

$$V = \mathcal{F}_{\mathbf{t}}, \quad E = \{(\mathbf{x}, \mathbf{y}) \ : \ \mathbf{x} - \mathbf{y} \in \mathcal{B} \text{ or } \mathbf{y} - \mathbf{x} \in \mathcal{B}\}.$$

2.1 Definition (**A Markov basis**). $\mathcal{B} \subset \mathrm{Ker}_{\mathbb{Z}}(A)$ is a Markov basis for A if $G_{\mathbf{t},\mathcal{B}}$ is connected for arbitrary $\mathbf{t} \in \mathbb{Z}^{d}_{\geqslant 0}$.

Once we obtain a Markov basis $\mathcal{B}$ for A, we can construct a connected Markov chain over $\mathcal{F}_{A\mathbf{x}^o}$ easily as follows. For each state $\mathbf{x} \in \mathcal{F}_{A\mathbf{x}^o}$, randomly choose a move $\mathbf{z} \in \mathcal{B}$ and a sign $\varepsilon \in \{-1, 1\}$ and consider $\mathbf{x} + \varepsilon\mathbf{z}$. If $\mathbf{x} + \varepsilon\mathbf{z} \in \mathcal{F}_{A\mathbf{x}^o}$, then $\mathbf{x} + \varepsilon\mathbf{z}$ is the next state, otherwise stay at $\mathbf{x}$. Then we have the connected Markov chain over $A\mathbf{x}^o$. We see these arguments in an example.

2.2 Example. Again we consider a small data of Example 1.6, where the fiber is redisplayed below.

$$\begin{aligned}\mathcal{F}_{(3,2,2,2,1)} &= \left\{ \begin{array}{|c|c|c|}\hline 2&1&0\\\hline 0&1&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 2&0&1\\\hline 0&2&0\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&2&0\\\hline 1&0&1\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 1&1&1\\\hline 1&1&0\\\hline\end{array},\ \begin{array}{|c|c|c|}\hline 0&2&1\\\hline 2&0&0\\\hline\end{array} \right\} \\ &= \{\mathbf{x}_1,\ \mathbf{x}_2,\ \mathbf{x}_3,\ \mathbf{x}_4,\ \mathbf{x}_5\}.\end{aligned}$$

The integer kernel for the configuration matrix

$$A = \begin{pmatrix} 1&1&1&0&0&0\\ 0&0&0&1&1&1\\ 1&0&0&1&0&0\\ 0&1&0&0&1&0\\ 0&0&1&0&0&1 \end{pmatrix} \tag{20}$$

includes moves such as

$$\mathbf{z}_1 = \begin{array}{|r|r|r|}\hline 1&-1&0\\\hline -1&1&0\\\hline\end{array},\ \mathbf{z}_2 = \begin{array}{|r|r|r|}\hline 1&0&-1\\\hline -1&0&1\\\hline\end{array},\ \mathbf{z}_3 = \begin{array}{|r|r|r|}\hline 0&1&-1\\\hline 0&-1&1\\\hline\end{array},\ \begin{array}{|r|r|r|}\hline 2&-1&-1\\\hline -2&1&1\\\hline\end{array},\ \ldots.$$

From these, we consider some sets of moves. If we consider $\mathcal{B}_1 = \{\mathbf{z}_1\}$, corresponding undirected graph $G_{(3,2,2,2,1),\mathcal{B}_1}$ is given in Fig. 2(a), which is not connected. Therefore $\mathcal{B}_1$ is not a Markov basis. If we consider $\mathcal{B}_2 = \{\mathbf{z}_1, \mathbf{z}_2\}$, corresponding undirected graph $G_{(3,2,2,2,1),\mathcal{B}_2}$ is given in Fig. 2(b), which is connected. However, $\mathcal{B}_2$ is also

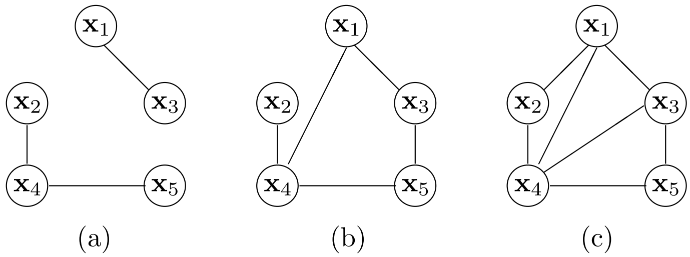

Fig. 2 Undirected graphs for $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3$ for $\mathbf{t} = (3, 2, 2, 2, 1)$

not a Markov basis, because there exists $\mathbf{t} \in \mathbb{Z}^5_{\geqslant 0}$ where $G_{\mathbf{t},\mathcal{B}_2}$ is not connected. An example of such $\mathbf{t}$ is $\mathbf{t} = (1, 1, 0, 1, 1)$, with the corresponding $\mathbf{t}$-fiber is a two-element set

$$\mathcal{F}_{(1,1,0,1,1)} = \left\{ \begin{array}{|c|c|c|}\hline 0&1&0\\\hline 0&0&1\\\hline\end{array}\,,\ \begin{array}{|c|c|c|}\hline 0&0&1\\\hline 0&1&0\\\hline\end{array} \right\}. \quad (21)$$

The above example shows that a Markov basis includes $\mathbf{z}_3$ to connect the two elements above. In fact, $\mathcal{B} = \{\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$ is a Markov basis for this A, with the corresponding undirected graph $G_{(3,2,2,2,1),\mathcal{B}_3}$ in Fig. 2(c). The transition probability matrix (19) in Example 1.9 corresponds to a Markov chain constructed from $\mathcal{B}_3$ as "in each step, choose 3 elements in $\mathcal{B}_3$ and its sign $\{-1, 1\}$ with equal probabilities".

At first sight, we may feel the cases such as (21) are trivial and may imagine that "if we only consider the cases with $\mathbf{t} \in \mathbb{Z}^d_{>0}$, i.e., cases with strictly positive minimal sufficient statistics (that may be realistic situations in the actual data analysis), it is easy to connect the fiber $\mathcal{F}_{\mathbf{t}}$". However, it is not so. We will see an example where complicated moves are needed even for the fiber with positive $\mathbf{t}$.

The connection between the Markov basis and a *toric ideal* of a polynomial ring by [8] is as follows. Let $\mathbb{K}[\mathbf{u}] = \mathbb{K}[u_1, u_2, \ldots, u_\nu]$ denote the ring of polynomials in ν variables over a field $\mathbb{K}$. Let a contingency table $\mathbf{x} \in \mathbb{Z}^\nu_{\geqslant 0}$ be mapped to the monomial $\mathbf{u}^{\mathbf{x}} \in \mathbb{K}[\mathbf{u}]$, and a move, i.e., an element of the integer kernel $\mathbf{z} = \mathbf{z}^+ - \mathbf{z}^- \in \mathrm{Ker}_{\mathbb{Z}}(A)$, be mapped to the binomial $\mathbf{u}^{\mathbf{z}^+} - \mathbf{u}^{\mathbf{z}^-} \in \mathbb{K}[\mathbf{u}]$. For the case of the independence model for the 3×3 contingency tables, examples of these correspondences are as follows.

$$\begin{array}{|r|r|r|}\hline 11&5&2\\\hline 4&9&1\\\hline 2&3&3\\\hline\end{array} \Longleftrightarrow u_{11}^{11}u_{12}^5u_{13}^2u_{21}^4u_{22}^9u_{23}u_{31}^2u_{32}^3u_{33}^3$$

$$\begin{array}{|r|r|r|}\hline 2&-1&-1\\\hline -3&1&2\\\hline 1&0&-1\\\hline\end{array} \Longleftrightarrow u_{11}^2u_{22}u_{23}^2u_{31} - u_{12}u_{13}u_{21}^3u_{33}$$

The binomial ideal in $\mathbb{K}[\mathbf{u}]$ generated by the set of binomials corresponding to the set of moves for A,

$$I_A = \left\langle \left\{ \mathbf{u}^{\mathbf{z}^+} - \mathbf{u}^{\mathbf{z}^-} \ : \ \mathbf{z}^+ - \mathbf{z}^- \in \mathrm{Ker}_{\mathbb{Z}}(A) \right\} \right\rangle,$$

is the *the toric ideal* of configuration A.

2.3 Theorem (Theorem 3.1 of [8]). $\mathcal{B} = \{\mathbf{z}_1, \ldots, \mathbf{z}_L\} \subset \mathrm{Ker}_{\mathbb{Z}}(A)$ *is a Markov basis for* A *if and only if* $\{\mathbf{u}^{\mathbf{z}_i^+} - \mathbf{u}^{\mathbf{z}_i^-},\ i = 1, \ldots, L\}$ *generates* I_A.

A proof of Theorem 2.3 is given in the original paper [8]. We can also find more detailed proof in Chap. 4 of [13]. In these proofs, the sufficiency and the necessity are shown by induction on some integer. In the proof of sufficiency, this integer represents the number of steps of the chain, and the argument is straightforward. On the other hand, in the proof of necessity, this integer represents the number of terms in the expansion that we want to show in the proof, and is not necessarily equal to the number of steps of the chain. Theorem 2.3 shows a non-trivial result on this point.

To calculate a Markov basis for a given configuration matrix A, we can use the *elimination theory*. For this purpose, we also prepare variables $\mathbf{v} = \{v_1, \ldots, v_d\}$ for the minimal sufficient statistic $\mathbf{t}$ and consider the polynomial ring $\mathbb{K}[\mathbf{v}] = \mathbb{K}[v_1, \ldots, v_d]$. The relation $\mathbf{t} = A\mathbf{x}$ can be expressed by the homomorphism

$$\begin{aligned} \psi_A \ : \ \mathbb{K}[\mathbf{u}] &\to \mathbb{K}[\mathbf{v}] \\ u_j &\mapsto v_1^{a_{1j}} v_2^{a_{2j}} \cdots v_d^{a_{dj}}. \end{aligned}$$

Then the toric ideal I_A is also expressed as $I_A = \mathrm{Ker}(\psi_A)$. We now have the following.

2.4 Corollary (Theorem 3.2 of [8]). *Let* I_A^* *be the ideal of* $\mathbb{K}[\mathbf{u}, \mathbf{v}]$ *given by*

$$I_A^* = \left\langle -\psi_A(u_j) + u_j,\ j = 1, \ldots, \nu \right\rangle \subset \mathbb{K}[\mathbf{u}, \mathbf{v}].$$

Then we have $I_A = I_A^* \cap \mathbb{K}[\mathbf{u}]$.

Corollary 2.4 suggests that we can obtain a generator of I_A as its Gröbner basis for an appropriate term order called an elimination order. For an ideal $J \in \mathbb{K}[\mathbf{u}]$ and a term order $\prec$, a set of polynomials $\{g_1, \ldots, g_s\},\ g_1, \ldots, g_s \in J$, is called a Gröbner basis of I with respect to a term order $\prec$, if $\{\mathrm{in}_\prec(g_1), \ldots, \mathrm{in}_\prec(g_s)\}$ generates an initial ideal of J defined by $\mathrm{in}_\prec(J) = \langle\{\mathrm{in}_\prec(f) \ : \ 0 \neq f \in J\}\rangle$. Here we write $\mathrm{in}_\prec(f)$ as an initial term of f with respect to a term order $\prec$. For more theories and results on Gröbner bases, see textbooks such as [6]. The elimination theory is one of the useful applications of Gröbner bases and is used for our problem as follows. For the reduced Gröbner basis G^* of I_A^* for any term order satisfying $\{v_1, \ldots, v_d\} \succ \{u_1, \ldots, u_\nu\}$, $G^* \cap \mathbb{K}[\mathbf{u}]$ is a reduced Gröbner basis of I_A. Because the Gröbner basis is a generator of I_A, we can obtain a Markov basis for A as the reduced Gröbner basis in this way.

The computations of Gröbner bases can be carried out by various algebraic softwares such as Macaulay2 [10], SINGULAR [7], CoCoA [18], Risa/Asir [15] and 4ti2 [1]. Here, we show some computations by Macaulay2, because we can also rapidly use it online at the website.[1] We start with a simple example.

[1]See macaulay2.com for detail on Macaulay2 and Macaulay2 online.

2.5 Example. In Example 2.2, we give a Markov basis for the independence model for 2×3 contingency tables without any proof or calculations. Here we check that the set of 3 moves

$$\left\{ \mathbf{z}_1 = \begin{array}{|c|c|c|} \hline 1 & -1 & 0 \\ \hline -1 & 1 & 0 \\ \hline \end{array}, \quad \mathbf{z}_2 = \begin{array}{|c|c|c|} \hline 1 & 0 & -1 \\ \hline -1 & 0 & 1 \\ \hline \end{array}, \quad \mathbf{z}_3 = \begin{array}{|c|c|c|} \hline 0 & 1 & -1 \\ \hline 0 & -1 & 1 \\ \hline \end{array} \right\}$$

constitute a Markov basis for A given in (20). In other words, we check that the corresponding toric ideal I_A is generated by 3 binomials

$$\{u_{11}u_{22} - u_{12}u_{21},\ u_{11}u_{23} - u_{13}u_{21},\ u_{12}u_{23} - u_{13}u_{22}\}. \tag{22}$$

Following Corollary 2.4, we prepare the variable $\mathbf{v} = (v_1, \ldots, v_5)$ for the row sums and column sums of $\mathbf{x}$ as

$$\begin{array}{|c|c|c|l} \hline x_{11} & x_{12} & x_{13} & v_1 \\ \hline x_{21} & x_{22} & x_{23} & v_2 \\ \hline \multicolumn{1}{c}{v_3} & \multicolumn{1}{c}{v_4} & \multicolumn{1}{c}{v_5} & \end{array}$$

and consider the homomorphism

$$\begin{aligned} u_{11} &\mapsto v_1v_3,\ u_{12} \mapsto v_1v_4,\ u_{13} \mapsto v_1v_5, \\ u_{21} &\mapsto v_2v_3,\ u_{22} \mapsto v_2v_4,\ u_{23} \mapsto v_2v_5. \end{aligned}$$

Then under the elimination order $\mathbf{v} \succ \mathbf{u}$, compute the reduced Gröbner basis of the toric ideal

$$I_A^* = \langle -v_1v_3 + u_{11},\ -v_1v_4 + u_{12}, \ldots, -v_2v_5 + u_{23} \rangle .$$

These calculations are done by Macaulay2 as follows.

```
i1 : R=QQ[v1,v2,v3,v4,v5,u11,u12,u13,u21,u22,u23,MonomialOrder=>{5,6}]

o1 = R

o1 : PolynomialRing

i2 : I=ideal(-v1*v3+u11,-v1*v4+u12,-v1*v5+u13,-v2*v3+u21,-v2*v4+u22,-v2*v5+u23)

o2 = ideal (- v1*v3 + u11, - v1*v4 + u12, - v1*v5 + u13, - v2*v3 + u21, - v2*v4
     ------------------------------------------------------------------------
     + u22, - v2*v5 + u23)

o2 : Ideal of R

i3 : G=gb(I); g=gens(G)

o4 = | u13u22-u12u23 u13u21-u11u23 u12u21-u11u22 v4u23-v5u22 v4u13-v5u12
     ------------------------------------------------------------------------
     v3u23-v5u21 v3u22-v4u21 v3u13-v5u11 v3u12-v4u11 v1u23-v2u13 v1u22-v2u12
     ------------------------------------------------------------------------
     v1u21-v2u11 v2v5-u23 v1v5-u13 v2v4-u22 v1v4-u12 v2v3-u21 v1v3-u11 |
```

```
               1       18
o4 : Matrix R  <--- R

i5 : selectInSubring(1,g)

o5 = | u13u22-u12u23 u13u21-u11u23 u12u21-u11u22 |

               1       3
o5 : Matrix R  <--- R
```

The output o4 shows the reduced Gröbner basis of I_A^* under the elimination (reverse lexicographic) order $\mathbf{v} \succ \mathbf{u}$, and the output o5 shows the reduced Gröbner basis of I_A, which we can use as a Markov basis. We have now checked a Markov basis (22).

From the Markov basis (22), we may imagine that the set of moves corresponding to the binomials

$$\{u_{ij}u_{i'j'} - u_{ij'}u_{i'j}\ ,\ \ 1 \leqslant i < i' \leqslant I,\ 1 \leqslant j < j' \leqslant J\}$$

forms a Markov basis for the independence model of the $I \times J$ contingency tables, which is actually true. This fact is given and proved as Theorem 2.1 of [4], for example.

Now we are ready to estimate p-value for our original problem of 3×3 contingency table in Table 1. The Markov basis for this problem is formed by 9 moves of the above type. Using this Markov basis, we calculate the conditional p-values for Table 1 by the Markov chain Monte Carlo method. For each step of the chain, we choose an element of the Markov basis randomly, and modify the transition probability by Theorem 1.8. We start the chain at the observed table $\mathbf{x}^o$ of Table 1, discard initial 50000 steps as the burn-in steps, and have 100000 samples of the Pearson goodness-of-fit χ^2. Figure 3 is a histogram of the sampled Pearson goodness-of-fit

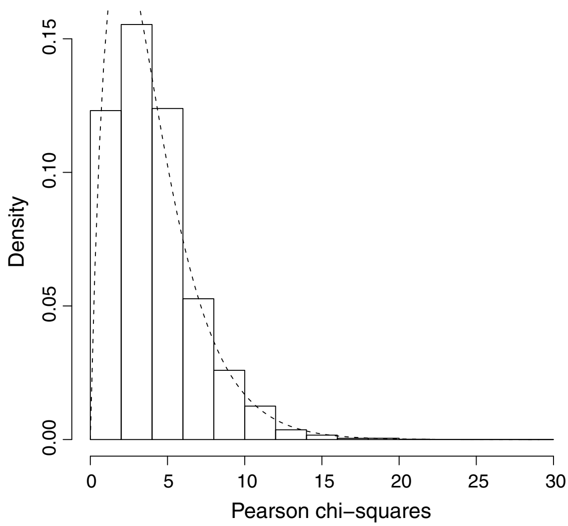

Fig. 3 A histogram of sampled Pearson χ^2 goodness-of-fit for Table 1 generated by a Markov chain Monte Carlo method. The dotted curve is the corresponding asymptotic χ_4^2 distribution

Table 3 The upper percentiles for three strategies of Pearson goodness-of-fit χ^2 for Table 1

	90%	95%	99%	99.9%
(a) Asymptotic χ^2_4 distribution	7.779	9.488	13.28	18.47
(b) Exact null distribution	7.766	9.353	12.78	17.99
(c) Monte Carlo simulated distribution	7.684	9.287	12.73	18.58

χ^2 with the asymptotic χ^2_4 distribution. In these 100000 samples, 6681 samples are larger than or equal to the observed value $\chi^2(\mathbf{x}^o) = 8.6687$, then we have the estimate $\hat{p} = 0.06681$. Therefore we cannot reject H_0 at significance level 0.05, which is the same result to the other strategies (a) and (b). Though the difference from the exact value $p = 0.07035480$ from the simulated value is slightly larger than the asymptotic estimate ($\hat{p} = 0.0699$), we may increase the accuracy of the estimates by increasing the sample sizes. To compare the three strategies for Table 1, we compute the upper percentiles of 90%, 95%, 99%, 99.9% for (a) asymptotic χ^2_4 distribution, (b) exact conditional distribution, and (c) Monte Carlo simulated distribution in Table 3.

Finally, we give an example for which the structure of the Markov basis is complicated. The model we consider is a *no three-factor interaction model* for three-way contingency tables. The parametric description of the no three-factor interaction model is given by

$$\mathcal{M}_{n3} = \{\boldsymbol{\theta} \in \Delta \ : \ \theta_{ijk} = \alpha_{ij}\beta_{ik}\gamma_{jk} \text{ for some } (\alpha_{ij}), (\beta_{ik}), (\gamma_{jk})\}.$$

This is one of the most important statistical models in the statistical data analysis of three-way contingency tables. The minimal sufficient statistics for $\mathcal{M}_{n3}$ is the two-dimensional marginals

$$\{x_{ij+}\},\ \{x_{i+k}\},\ \{x_{+jk}\},$$

where we define

$$x_{ij+} = \sum_{k=1}^{K} x_{ijk},\quad x_{i+k} = \sum_{j=1}^{J} x_{ijk},\quad x_{+jk} = \sum_{i=1}^{I} x_{ijk}.$$

We only consider $3 \times 3 \times 3$ case (i.e., $I = J = K = 3$) here. Then the configuration matrix A is 27×27 matrix written as follows.

$$
A = \begin{pmatrix}
1\;1\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 \\
0\;0\;0\;1\;1\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 \\
\vdots \\
0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;1\;1\;1 \\
1\;0\;0\;1\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 \\
0\;1\;0\;0\;1\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0 \\
\vdots \\
0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;1\;0\;0\;1 \\
1\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0 \\
0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0 \\
\vdots \\
0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;1\;0\;0\;0\;0\;0\;0\;0\;0\;1
\end{pmatrix}
$$

For this model we see that the "simplest moves", i.e., the moves with the minimum degree, correspond to the binomials of degree 4 such as

$$u_{111}u_{122}u_{212}u_{221} - u_{112}u_{121}u_{211}u_{222}, \tag{23}$$

which is called a *basic move*. There are 9 such moves for the case of $3 \times 3 \times 3$ tables. Unfortunately, however, the set of these 9 moves does not become a Markov basis. To see this consider the following example.

2.6 Example. Consider the $3 \times 3 \times 3$ contingency tables with the fixed two-dimensional marginals

$$(x_{ij+}) = (x_{i+k}) = (x_{+jk}) = (2, 1, 1, 1, 2, 1, 1, 1, 2)'. \tag{24}$$

We write a $3 \times 3 \times 3$ table as follows.

$$
\begin{array}{|ccc|}\hline x_{111} & x_{112} & x_{113} \\ x_{121} & x_{122} & x_{123} \\ x_{131} & x_{132} & x_{133} \\ \hline \end{array}
\;
\begin{array}{|ccc|}\hline x_{211} & x_{212} & x_{213} \\ x_{221} & x_{222} & x_{223} \\ x_{231} & x_{232} & x_{233} \\ \hline \end{array}
\;
\begin{array}{|ccc|}\hline x_{311} & x_{312} & x_{313} \\ x_{321} & x_{322} & x_{323} \\ x_{331} & x_{332} & x_{333} \\ \hline \end{array}
$$

Then the fixed marginals (24) are displayed as

$$
\begin{array}{|ccc|c}\cline{1-3} x_{111} & x_{112} & x_{113} & 2 \\ x_{121} & x_{122} & x_{123} & 1 \\ x_{131} & x_{132} & x_{133} & 1 \\ \cline{1-3} 2 & 1 & 1 & 4 \end{array}
\;
\begin{array}{|ccc|c}\cline{1-3} x_{211} & x_{212} & x_{213} & 1 \\ x_{221} & x_{222} & x_{223} & 2 \\ x_{231} & x_{232} & x_{233} & 1 \\ \cline{1-3} 1 & 2 & 1 & 4 \end{array}
\;
\begin{array}{|ccc|c}\cline{1-3} x_{311} & x_{312} & x_{313} & 1 \\ x_{321} & x_{322} & x_{323} & 1 \\ x_{331} & x_{332} & x_{333} & 2 \\ \cline{1-3} 1 & 1 & 2 & 4 \end{array}
\;
\begin{array}{|ccc|c}\cline{1-3} 2 & 1 & 1 & 4 \\ 1 & 2 & 1 & 4 \\ 1 & 1 & 2 & 4 \\ \cline{1-3} 4 & 4 & 4 & 12 \end{array}
$$

where the rightmost table shows the marginal $\{x_{+jk}\}$. There are 18 elements in this fiber as follows.

$$1:\begin{array}{|ccc|}\hline 2&0&0\\0&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&0&1\\1&1&0\\\hline\end{array}\quad 2:\begin{array}{|ccc|}\hline 2&0&0\\0&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&0&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&1&0\\1&0&1\\\hline\end{array}\quad 3:\begin{array}{|ccc|}\hline 2&0&0\\0&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\0&1&1\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\1&0&0\\0&1&1\\\hline\end{array}$$

$$4:\begin{array}{|ccc|}\hline 2&0&0\\0&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\1&1&0\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\0&0&1\\1&0&1\\\hline\end{array}\quad 5:\begin{array}{|ccc|}\hline 2&0&0\\0&0&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&1&0\\1&0&1\\\hline\end{array}\quad 6:\begin{array}{|ccc|}\hline 2&0&0\\0&0&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&2&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&0&0\\0&0&2\\\hline\end{array}$$

$$7:\begin{array}{|ccc|}\hline 1&1&0\\1&0&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&2&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&0&1\\1&1&0\\\hline\end{array}\quad 8:\begin{array}{|ccc|}\hline 1&1&0\\1&0&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&1&0\\1&0&1\\\hline\end{array}\quad 9:\begin{array}{|ccc|}\hline 1&1&0\\1&0&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\0&2&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&0&1\\0&1&1\\\hline\end{array}$$

$$10:\begin{array}{|ccc|}\hline 1&1&0\\0&0&1\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&2&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\1&0&0\\0&1&1\\\hline\end{array}\quad 11:\begin{array}{|ccc|}\hline 1&1&0\\0&0&1\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&0&1\\1&1&0\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&0\\0&0&2\\\hline\end{array}\quad 12:\begin{array}{|ccc|}\hline 1&0&1\\1&0&0\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&2&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\0&0&1\\1&0&1\\\hline\end{array}$$

$$13:\begin{array}{|ccc|}\hline 1&0&1\\1&0&0\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\0&1&1\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&0\\0&0&2\\\hline\end{array}\quad 14:\begin{array}{|ccc|}\hline 1&0&1\\0&1&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&0&0\\0&0&2\\\hline\end{array}\quad 15:\begin{array}{|ccc|}\hline 1&0&1\\0&1&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&1&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&0&1\\0&1&1\\\hline\end{array}$$

$$16:\begin{array}{|ccc|}\hline 1&0&1\\0&1&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 0&1&0\\1&0&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&0\\0&0&2\\\hline\end{array}\quad 17:\begin{array}{|ccc|}\hline 0&1&1\\1&0&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&2&0\\0&0&1\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&0&1\\0&1&1\\\hline\end{array}\quad 18:\begin{array}{|ccc|}\hline 0&1&1\\1&0&0\\1&0&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&1\\0&1&0\\\hline\end{array}\begin{array}{|ccc|}\hline 1&0&0\\0&1&0\\0&0&2\\\hline\end{array}$$

Now consider connecting these elements by the set of 9 basic moves such as (23). The undirected graph we obtain is Fig. 4. Because this is not connected, the set of the basic moves is not a Markov basis. This example shows that we need moves such as

$$u_{111}u_{122}u_{133}u_{213}u_{221}u_{232} - u_{113}u_{121}u_{132}u_{211}u_{222}u_{233} \tag{25}$$

to constitute a Markov basis.

Now calculate a Markov basis by Macaulay2 for this example. Using a, b, c for the sufficient statistics instead of v, the following is the commands to calculate a reduced Gröbner basis for this problem.

```
R = QQ[a11,a12,a13,a21,a22,a23,a31,a32,a33,
       b11,b12,b13,b21,b22,b23,b31,b32,b33,
       c11,c12,c13,c21,c22,c23,c31,c32,c33,
       x111,x112,x113,x121,x122,x123,x131,x132,x133,
       x211,x212,x213,x221,x222,x223,x231,x232,x233,
       x311,x312,x313,x321,x322,x323,x331,x332,x333,
       MonomialOrder=>{27,27}]
I = ideal(x111-a11*b11*c11,x112-a11*b12*c12,x113-a11*b13*c13,
          x121-a12*b11*c21,x122-a12*b12*c22,x123-a12*b13*c23,
          x131-a13*b11*c31,x132-a13*b12*c32,x133-a13*b13*c33,
          x211-a21*b21*c11,x212-a21*b22*c12,x213-a21*b23*c13,
          x221-a22*b21*c21,x222-a22*b22*c22,x223-a22*b23*c23,
```

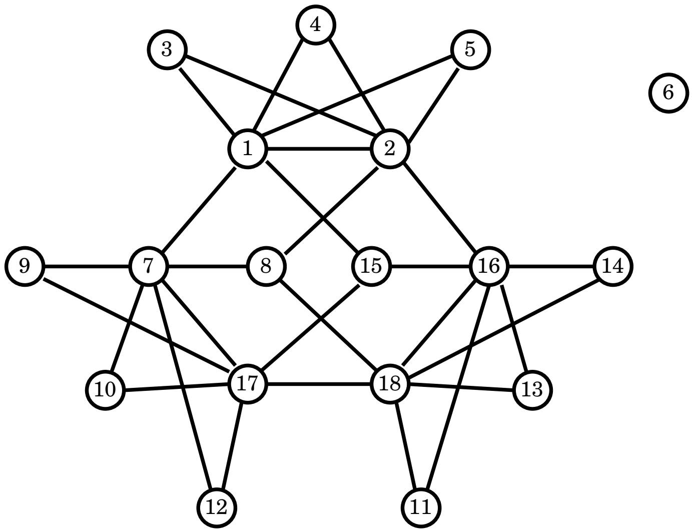

Fig. 4 Undirected graph obtained from the set of the basic moves

```
          x231-a23*b21*c31,x232-a23*b22*c32,x233-a23*b23*c33,
          x311-a31*b31*c11,x312-a31*b32*c12,x313-a31*b33*c13,
          x321-a32*b31*c21,x322-a32*b32*c22,x323-a32*b33*c23,
          x331-a33*b31*c31,x332-a33*b32*c32,x333-a33*b33*c33)
G = gb(I); g = gens(G)
selectInSubring(1,g)
```

Unfortunately, this calculation may be hard to carry out for average PC. In fact, I could not finish the above calculation within one hour by my slow laptop (with 2.80 GHz CPU, 8.00 GB RAM, running on vmware). Instead, check the calculation for $2 \times 3 \times 3$ cases. With the similar input commands, we have the output instantly in this case. From the output, we see that there are 1417 elements in the reduced Gröbner basis of I_A^*, and 15 elements in the reduced Gröbner basis of I_A as follows.

```
i10 : selectInSubring(1,g)

o10 = | x122x133x223x232-x123x132x222x233 x112x133x213x232-x113x132x212x233
      ------------------------------------------------------------------------
      x121x133x223x231-x123x131x221x233 x121x132x222x231-x122x131x221x232
      ------------------------------------------------------------------------
      x111x133x213x231-x113x131x211x233 x111x132x212x231-x112x131x211x232
      ------------------------------------------------------------------------
      x112x123x213x222-x113x122x212x223 x111x123x213x221-x113x121x211x223
      ------------------------------------------------------------------------
      x111x122x212x221-x112x121x211x222
      ------------------------------------------------------------------------
      x112x121x133x211x223x232-x111x123x132x212x221x233
      ------------------------------------------------------------------------
```

```
      x111x122x133x213x221x232-x113x121x132x211x222x233
      ------------------------------------------------------------------------
      x111x122x133x212x223x231-x112x123x131x211x222x233
      ------------------------------------------------------------------------
      x113x121x132x212x223x231-x112x123x131x213x221x232
      ------------------------------------------------------------------------
      x112x121x133x213x222x231-x113x122x131x212x221x233
      ------------------------------------------------------------------------
      x111x123x132x213x222x231-x113x122x131x211x223x232 |

             1       15
o10 : Matrix R  <--- R
```

We see that the set of the basic moves and the degree 6 moves such as (25) actually constitutes a Markov basis for $2 \times 3 \times 3$ cases.

The calculation for $3 \times 3 \times 3$ cases can be carried out by faster software such as 4ti2 [1], that can be also used in Macaulay 2 as follows.

```
loadPackage "FourTiTwo"
A = matrix "1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,1,1;
            1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,0,0,0,0,0,0,0,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,0,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1,0;
            0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,1;
            1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0;
            0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0;
            0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0;
            0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0;
            0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0;
            0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0;
            0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0;
            0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0;
            0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1"
R = QQ[x111,x112,x113,x121,x122,x123,x131,x132,x133,
       x211,x212,x213,x221,x222,x223,x231,x232,x233,
       x311,x312,x313,x321,x322,x323,x331,x332,x333]
I = toricMarkov(A,R)
```

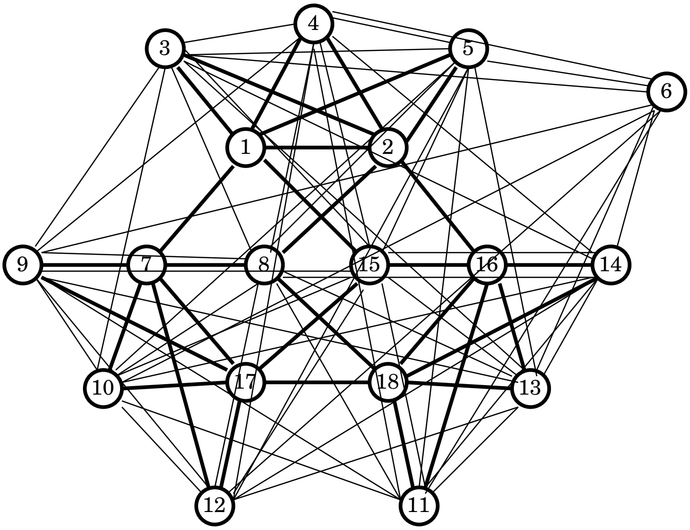

Fig. 5 Undirected graph obtained from a minimal Markov basis

This calculation is finished within 1 second by my laptop. From the output, we see that 27 basic moves such as (23) and 54 moves of degree 6 such as (25) constitute a minimal Markov basis.[2] Using this minimal Markov basis, we can construct a connected Markov chain for this fiber. The corresponding undirected graph is Fig. 5.

Interestingly, for the problems of the larger sizes, the structure of the Markov basis becomes more complicated. For example, for the no three-factor interaction model of $3 \times 3 \times 4$ tables, the set of degree 4, 6, 8 moves becomes a Markov basis, and for $3 \times 3 \times 5$ tables, the set of degree 4, 6, 8, 10 moves becomes a Markov basis. These results are summarized in Chap. 9 of [4].

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[2]The 4ti2 command `toricMarkov` gives a minimal Markov basis as the output. We can also obtain a Gröbner basis by the command `toricGroebner`.

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Part II
Second Algebraic Byway: Quivers

Chapter 6
Introduction to Representations of Quivers

Kenji Iohara

The main purpose of this lecture note is to provide a quick introduction to quivers and their representations. In particular, as there already exists several introductory and complete texts on quivers, the author tries motivating the reader to develop the theory by showing several concrete examples.

One of the excellent source for such theory is Gabriel's Bourbaki Seminar [9]. The reader may find an excellent introduction in the book [21] by Schiffler. The readers who wish to learn further topics via algebraic approach may consult some lecture notes by W. Crawley-Boevey which can be found on his webpage: http://www1.maths.leeds.ac.uk/~pmtwc/.

For more complete description of the theory, one may consult the book [1] of Assem et al.

For those who are interested in geometric approach to representation theory of quivers, one may consult Brion's lecture notes [4].

1 Quivers and Their Representations

In this section, I will introduce basic notions related to quivers and their representations. Throughout these lectures, we consider vector spaces and linear maps over a fixed algebraically closed field $\mathbb{K}$.

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1.1 Quivers and Their Representations

1.1.1 Definition. A *quiver* is a finite directed graph, possibly with multiple arrows and loops. More precisely, a quiver Q is a quadruple

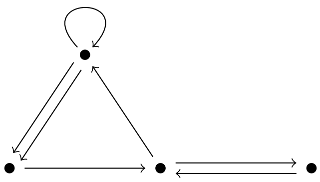
$$Q = (Q_0, Q_1, s, t),$$

where Q_0 and Q_1 are finite sets (the set of *vertices*, resp. *arrows*), and two maps $s, t : Q_1 \to Q_0$ assigning to each arrow its *source*, resp. *target*.

Some authors use out and in in place of s and t, for example, $\varphi : V_{\text{out}(\varphi)} \to V_{\text{in}(\varphi)}$ for $\varphi \in Q_1$ (cf. [11, 16, 22] in this volume).

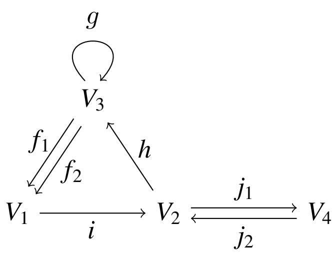
We denote the vertices by $\bullet$, numbers or letters i, j, ... An arrow with source i and target j will be denoted by $\alpha : i \to j$ or $i \xrightarrow{\alpha} j$.

Here is an example of quiver:



1.1.2 Definition. A *representation* M of a quiver Q consists of a family of vector spaces V_i ($i \in Q_0$), together with a family of linear maps $f_\alpha : V_{s(\alpha)} \to V_{t(\alpha)}$ indexed by $\alpha \in Q_1$.

For example, a representation of the above quiver is just a diagram



where V_i ($1 \leqslant i \leqslant 4$) are vector spaces, f_r, j_r ($r = 1, 2$), g, h, i are linear maps.

1.1.3 Definition. Given two representations $M = ((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$ and $N = ((W_i)_{i \in Q_0}, (g_\alpha)_{\alpha \in Q_1})$, a *morphism* $u : M \to N$ is a family of linear maps $(u_i : V_i \to W_i)_{i \in Q_0}$ such that the next diagram commutes: $\forall\, \alpha \in Q_1$,

$$\begin{array}{ccc} V_{s(\alpha)} & \xrightarrow{f_\alpha} & V_{t(\alpha)} \\ {\scriptstyle u_{s(\alpha)}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle u_{t(\alpha)}} \\ W_{s(\alpha)} & \xrightarrow{g_\alpha} & W_{t(\alpha)} \end{array}$$

For any two morphisms $u : L \to M$ and $v : M \to N$, the family of compositions $(v_i u_i)_{i \in Q_0}$ defines the *composition* $vu : L \to N$ of morphisms, which is associative and has the *identity* element $\mathrm{id}_M := (\mathrm{id}_{V_i})_{i \in Q_0}$ for each M. Hence, we may consider the *category of representations of* Q denoted by $\mathrm{Rep}(Q)$. One may check that this is a $\mathbb{K}$-*linear abelian category.*

1.1.4 Definition. A representation $M = ((V_i)_{i \in Q_0}, (f_\alpha)_{\alpha \in Q_1})$ is said to be *finite dimensional* if so are all V_i's. In such a case, we set

$$\mathbf{dim}\, M := (\dim\ V_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0},$$

and is called the *dimension vector* of M.

We denote by $(\varepsilon_i)_{i \in Q_0}$ the standard basis of $\mathbb{Z}^{Q_0}$: an element $\boldsymbol{n} = (n_i)_{i \in Q_0} \in \mathbb{Z}^{Q_0}$ is represented as $\boldsymbol{n} = \sum_{i \in Q_0} n_i \varepsilon_i$.

Notice that for every exact sequence of finite dimensional representations

$$0 \to M' \to M \to M'' \to 0,$$

one has

$$\mathbf{dim}\, M = \mathbf{dim}\, M' + \mathbf{dim}\, M''.$$

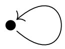
Now, a central problem in quiver theory is as follows:

> *given a quiver Q and a vector $\boldsymbol{n} \in (\mathbb{Z}_{\geqslant 0})^{Q_0}$, describe the set of isomorphism classes of representations of Q with dimension vector $\boldsymbol{n}$.*

Here are some examples from linear algebra:

1.1.5 Basic examples. **(i)** The simplest quiver $Q = (\{\bullet\}, \emptyset, s, t)$. In this case, a representation of Q is nothing but a vector space. Two representations M and N are isomorphic iff $\mathbf{dim}\, M = \mathbf{dim}\, N$. Hence, the isomorphism classes of representations of Q are parametrized by $\mathbb{N}$.

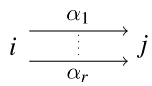
(ii) The next quiver we consider is the quiver K_1: $\bullet \longrightarrow \bullet$
In this case, its representation consists of two vector spaces V, W and a linear map $f : V \to W$. Hence, two representations $M = (f : V_1 \to V_2)$ and $N = (g : W_1 \to W_2)$ are isomorphic iff $\mathbf{dim}\, M = \mathbf{dim}\, N$, i.e., $\dim V_i = \dim W_i$ for $i \in \{1, 2\}$, and $\mathrm{rank}(f) = \mathrm{rank}(g)$.

(iii) The last quiver we consider is the quiver L_1: 

In this case, its representation consists of a vector space V with an endomorphism f on it. Hence, two representations (V, f) and (W, g) are isomorphic iff there exists an isomorphism $u : V \to W$ such that $u \circ f = g \circ u$. Fixing basis of V and W, this means that the matrix representations of f and g are conjugate. Hence, the isomorphism classes of L_1 are parametrized by Jordan normal forms.

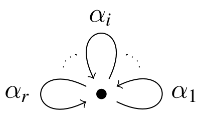
A naïve simple generalization of **(ii)** and **(iii)** provide us nontrivial examples, we shall see below:

1.1.6 Examples. **(i)** For $r \in \mathbb{Z}_{>0}$, let K_r the *r-arrows Kronecker quiver*, i.e., a quiver with vertices i, j and r-arrows $\alpha_1, ..., \alpha_r : i \to j$.

K_r: 

A representation of K_r consists of two vector spaces V and W together with r linear maps $f_i : V \to W$ $(1 \leqslant i \leqslant r)$. Hence, the isomorphism classes of representations of K_r with dimension vector (m, n) is parametrized by r-tuples of $n \times m$-matrices up to simultaneous multiplication by invertible $n \times n$-matrices from the left, and by invertible $m \times m$-matrices from the right.

(ii) For $r \in \mathbb{Z}_{>0}$, let L_r be the *r-loop* with a single vertex $\bullet$ and r-arrows $\alpha_1, ..., \alpha_r$:



The isomorphism classes of representations of L_r with the dimension vector $n \in \mathbb{Z}_{>0}$ are parametrized by r-tuples of $n \times n$-matrices up to simultaneous conjugation.

1.2 Path Algebras

As the category $\text{Rep}(Q)$ is a small abelian category, one may think of relating this category with a category of left modules over an $\mathbb{K}$-algebra (cf. the Freyd–Mitchell theorem). This can be realized by introducing

1.2.1 Definition. The *path algebra* of the quiver Q is the associative $\mathbb{K}$-algebra $\mathbb{K}Q$ generated by e_i $(i \in Q_0)$ and $\alpha \in Q_1$ subject to the relations

$$e_i^2 = e_i, \qquad e_i e_j = 0 \quad (i \neq j), \qquad e_{t(\alpha)}\alpha = \alpha e_{s(\alpha)} = \alpha.$$

In particular, e_i's are orthogonal idempotents and $\sum_{i\in Q_0} e_i = 1 \in \mathbb{K}Q$. Likewise, $e_i\alpha = 0$ unless $i = t(\alpha)$ and $\alpha e_j = 0$ unless $j = s(\alpha)$. The reader may see in the proof of Lemma 1.2.3 how this algebra was introduced.

1.2.2 Remark. The algebra $\mathbb{K}Q$ is the algebra generated by paths.
For any arrows α, β, the product $\beta\alpha$ is zero unless $s(\beta) = t(\alpha)$. Thus a product of arrows $\alpha_l\alpha_{l-1}\cdots\alpha_1$ is zero unless $\pi = (\alpha_1, ..., \alpha_l)$ is a *path*, i.e., $s(\alpha_{j+1}) = t(\alpha_j)$ for $0 < j < l$. We set $s(\pi) = s(\alpha_1)$, $t(\pi) = t(\alpha_l)$ and $l(\pi) = l$ (the *length* of the path). For any $i \in Q_0$, view e_i as a path of length 0.
Now, it is evident that the algebra with basis the set of all paths whose multiplication is given by the concatenation of paths is isomorphic to $\mathbb{K}Q^{op}$, the opposite algebra of $\mathbb{K}Q$.

1.2.3 Lemma. *For any quiver Q, the category of the left $\mathbb{K}Q$-modules and the category* Rep(Q) *are equivalent.*

Proof. (*Sketch*) Indeed, for any left $\mathbb{K}Q$-module V, the family $((V_i := e_iV)_{i\in Q_0}, (\alpha)_{\alpha\in Q_1})$ naturally has a structure of representation of Q. Conversely, for any representation $M = ((V_i)_{i\in Q_0}, (f_\alpha)_{\alpha\in Q_1})$, $V := \bigoplus_{i\in Q_0} V_i$ has a $\mathbb{K}Q$-module structure, where the actions of e_i and α are given by the compositions $V \twoheadrightarrow V_i \hookrightarrow V$ and $V \twoheadrightarrow V_{s(\alpha)} \xrightarrow{f_\alpha} V_{t(\alpha)} \hookrightarrow V$, respectively.

For a complete proof, see, e.g., [21]. □

Let us construct simple representations of a quiver $Q = (Q_0, Q_1, s, t)$. For any $i \in Q_0$, let $S(i)$ be the representation of Q defined by

$$S(i)_j := \begin{cases} \mathbb{K} & j = i, \\ 0 & j \neq i, \end{cases} \quad (j \in Q_0), \qquad f_\alpha := 0 \quad (\alpha \in Q_1).$$

Clearly, $S(i)$ is simple with dimension vector ε_i. These representations exhaust all simple representations, if $\mathbb{K}Q$ is finite dimensional:

1.2.4 Proposition. *Assume that Q has no oriented cycle. Then, any simple representation of Q is isomorphic to $S(i)$ for a unique $i \in Q_0$.*

Let $\mathbb{K}Q_{>0}$ be the ideal of $\mathbb{K}Q$ generated by the paths of positive length. It can be checked that $\mathbb{K}Q = \mathbb{K}Q_{>0} \oplus \bigoplus_{i\in Q_0} \mathbb{K}e_i$ as vector space.

Proof. Consider a simple $\mathbb{K}Q$-module M. Then, as $M \neq \mathbb{K}Q_{>0}M = \{0\}$, M may be viewed as a module over the algebra

$$\mathbb{K}Q/\mathbb{K}Q_{>0} \cong \bigoplus_{i\in Q_0} \mathbb{K}e_i \cong \prod_{i\in Q_0} \mathbb{K}.$$

As a consequence, each subspace e_iM is a $\mathbb{K}Q$-submodule of M. □

The condition that Q has no oriented cycle is essential. For example, the irreducible representations of 1-loop L_1 are exactly the spaces $S(\lambda) := \mathbb{K}[X]/(X-\lambda)\mathbb{K}[X]$ with $\lambda \in \mathbb{K}$, viewed as $\mathbb{K}L_1 = \mathbb{K}[X]$-modules.

A particular feature of $\mathbb{K}Q$ is that it is a (left) hereditary algebra, i.e., any submodule of a projective $\mathbb{K}Q$-module is projective. To show this, it is sufficient to see that its global dimension is at most 1. For $i \in Q_0$, set $P(i) := \mathbb{K}Qe_i$. As $\sum_{i\in Q_0} e_i = 1 \in \mathbb{K}Q$, one sees that $P(i)$ is a projective $\mathbb{K}Q$-module. The next proposition is known as the *standard resolution*:

1.2.5 Proposition. *For any left $\mathbb{K}Q$-module M, the next sequence is exact:*

$$0 \longrightarrow \bigoplus_{\alpha\in Q_1} P(t(\alpha)) \otimes_{\mathbb{K}} e_{s(\alpha)}M \xrightarrow{u} \bigoplus_{i\in Q_0} P(i) \otimes_{\mathbb{K}} e_iM \xrightarrow{v} M \longrightarrow 0,$$

where the maps u and v are defined as follows:

$$u(a\otimes m) := a\alpha \otimes m - a\otimes \alpha m \qquad (a \in P(t(\alpha)),\quad m \in e_{s(\alpha)}M),$$
$$v(a\otimes m) := a.m \qquad (a \in P(i),\quad m \in e_iM).$$

Here, the left $\mathbb{K}Q$-module structure on $P(i)\otimes_{\mathbb{K}} e_iM$ is defined by $a(b\otimes m) := ab\otimes m$ for $a \in \mathbb{K}Q$, $b \in P(i)$ and $m \in e_iM$.

For algebras of finite global dimension, see, e.g., [10].

1.3 Examples and Exercises

Proposition 1.2.4 shows that the classification of simple representations of the quiver Q with no oriented cycle is too simple. By the *Krull–Remak–Schmidt* theorem, any finite-dimensional representations decomposes into a direct sum of indecomposable representations. Hence, an interesting question is to *classify indecomposable representations up to isomorphism*. Now, we work on some examples.

1.3.1 Example. (First) Let us rework on the quiver $K_1 : \bullet \longrightarrow \bullet$

Recall that its representation is nothing but a linear map $f : V \to W$ between two vector spaces V and W. This representation has the isotypic decomposition:

$$V \xrightarrow{f} W = \operatorname{Ker} f \longrightarrow 0 \oplus \operatorname{Coim} f \xrightarrow{1} \operatorname{Im} f \oplus 0 \longrightarrow \operatorname{Coker} f,$$

where the indecomposable component of the each summand in the right hand side is

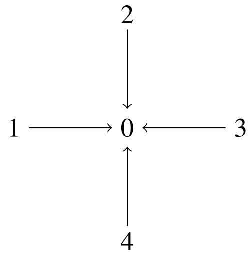
$$\mathbb{K} \longrightarrow 0 \quad , \quad \mathbb{K} \xrightarrow{1} \mathbb{K} \quad , \quad 0 \longrightarrow \mathbb{K} \quad ,$$

respectively. We remark that the dimension vectors of these representations are $\varepsilon_1, \varepsilon_1 + \varepsilon_2$ and ε_2, respectively.

1.3.2 Exercise. Determine the isomorphism classes of the indecomposable representations of the quiver $\bullet \longrightarrow \bullet \longrightarrow \bullet$. What happens if we change the orientation ?

1.3.3 Example. (Second) Consider the quiver S_r consisting of $r + 1$ vertices $0, 1, ..., r$, and r arrows with sources $1, 2, ..., r$ and common target 0, for example

S_4 :

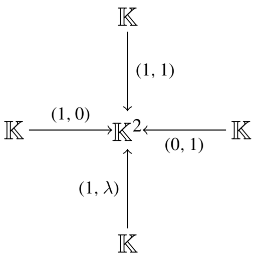


A representation M of S_r consists of $r + 1$ vector spaces $V_1, ..., V_r, W$ together with r linear maps $f_i : V_i \to W$. Consider the isomorphism classes of indecomposable representations with dimension vector $2\varepsilon_0 + \sum_{i=1}^{r} \varepsilon_i$. We may assume that every f_i is injective, since otherwise such a representation decomposes into a direct sum of (at least) two representations.

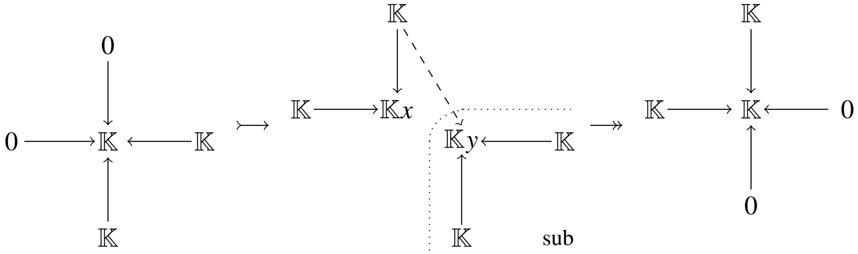
1.3.4 Exercise. For $r = 3$, check that there is only 1 isomorphism class of indecomposable representations with such dimension vector.

1.3.5 Example. (Third) But, for $r \geqslant 4$, this is no longer true. Indeed, the isomorphism classes of such representations are in bijection with the $PGL_2(\mathbb{K})$-orbits of $\mathbb{P}^1(\mathbb{K}) \times \cdots \times \mathbb{P}^1(\mathbb{K})$ (r copies of the projective line). Let us analyze the case of $r = 4$ in detail.

Suppose that $\mathrm{Im}\, f_i \neq \mathrm{Im}\, f_j$ for $i \neq j$. By an appropriate base change of $\mathbb{K}^2$, we can assume that there exists $\lambda \in \mathbb{K} \setminus \{0, 1\}$ such that the image of each f_i is generated by the vector given on each arrow:



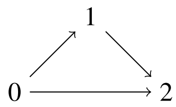
Suppose that, at least three of f_i's ($1 \leqslant i \leqslant 4$) have the same image. It can be seen that such a representation is no more indecomposable. Hence, we may assume that two of f_i's have the same image and other two have the different images. (There are $\binom{4}{2} = 6$ such possibilities.) Hence, consider the case $\lambda = \infty \in \mathbb{P}^1(\mathbb{K})$. There exists a basis $\{x, y\}$ of $\mathbb{K}^2$ such that $\mathrm{Im}\, f_1 = \mathbb{K}x$, $\mathrm{Im}\, f_2 = \mathbb{K}(x + y)$ and $\mathrm{Im}\, f_3 = \mathrm{Im}\, f_4 = \mathbb{K}y$. In such a case, there is an indecomposable sub-representation of dimension vector $\varepsilon_0 + \varepsilon_3 + \varepsilon_4$ and its quotient is an indecomposable representation of dimension vector $\varepsilon_0 + \varepsilon_1 + \varepsilon_2$:



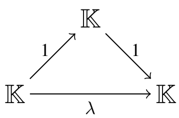
Thus, the isomorphism classes of the indecomposable representations of dimension vector $2\varepsilon_0 + \sum_{i=1}^{4} \varepsilon_i$ are parametrized by $(\mathbb{P}^1(\mathbb{K}) \setminus \{3\,\text{pts}\}) \cup \{6\,\text{pts}\}$.

In general, *the classification of isomorphism classes of representations of the quiver S_r with dimension vector $n\varepsilon_0 + m \sum_{i=1}^{r} \varepsilon_i$ is the same as the classification problem of r-subspaces of dimension m in $\mathbb{K}^n$.*

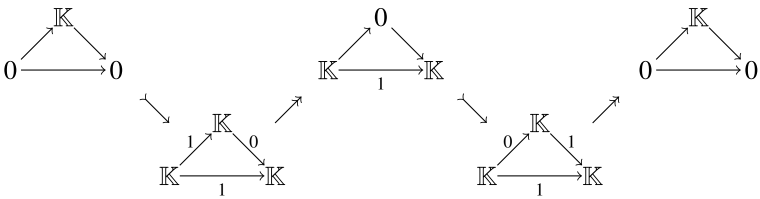
1.3.6 Exercise. Here, we consider the quiver Q:



(i) Determine the isomorphism classes of indecomposable representations of Q with dimension vector $\sum_{i=0}^{2} \varepsilon_i$. (One should find

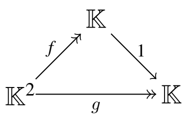


for some $\lambda \in \mathbb{K}$, and two representations at the bottom of the diagram:



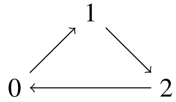
Thus, the isomorphism classes of the indecomposable representations of dimension vector $\sum_{i=0}^{2} \varepsilon_i$ are parametrized by $\mathbb{K}^* \cup \{3\,\text{pts}\}$.)

(ii) The same question with dimension vector $2\varepsilon_0 + \varepsilon_1 + \varepsilon_2$.
(Up to isomorphism, there only is 1 indecomposable representation



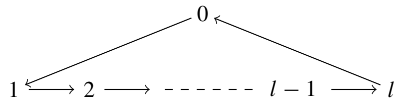
where the maps f and g satisfies $\operatorname{Ker} f \neq \operatorname{Ker} g$.)

Work out the same questions for the quiver



For information, see, e.g., the Bourbaki seminar of Gabriel [9].

1.3.7 Exercise. Let $l \in \mathbb{Z}_{>1}$. We consider the cyclic quiver Q with $l + 1$ nodes:



Classify the isomorphism classes of the indecomposable representations with dimension vector $\sum_{i=0}^{l} \varepsilon_i$. In particular, check that there are simple representations among them.

1.3.8 Exercise. Here, we consider the indecomposable representations of the 2-arrow Kronecker quiver K_2:

$$0 \rightrightarrows 1$$

(i) Show that the isomorphism classes of indecomposable representations of dimension vector $n(\varepsilon_0 + \varepsilon_1)$ for $n \in \mathbb{Z}_{>0}$ are parametrized by $\mathbb{P}^1(\mathbb{K})$.

(ii) For $n \in \mathbb{Z}_{\geqslant 0}$ and $r \in \mathbb{Z}_{>0}$, let $f, g : \mathbb{K}^n \to \mathbb{K}^{n+r}$ be the linear maps defined by

$$f(x_1, x_2, ..., x_n) := (x_1, x_2, ..., x_n, 0, ..., 0),$$
$$g(x_1, x_2, ..., x_n) := (0, ..., 0, x_1, x_2, ..., x_n).$$

Show that the representation of K_2 defined by the linear maps f and g is indecomposable if and only if $r = 1$.

(iii) For $n \in \mathbb{Z}_{\geqslant 0}$, find an indecomposable representation of K_2 whose dimension vector is $(n + 1)\varepsilon_0 + n\varepsilon_1$.

Indeed, it turns out that these representations of K_2 exhaust the isomorphism classes of indecomposable representations. (See, e.g., Kronecker's original paper [18], or Dieudonné's simplified version [5] or Benson's book [2], for detail.)

(iv) What can one say for the next quiver?

$$0 \rightleftarrows 1$$

(v) Compare the results for these two quivers.

2 Classification of Indecomposable Representations

In this section, we assume that the underlying non-oriented graph of a quiver Q is connected unless otherwise stated.

2.1 Type of Representations

A quiver Q is called of *finite type* if Q admits only finitely many indecomposable representations, up to isomorphism. For example, the quiver K_1 considered in the previous section is of finite type. More generally, we will see that the quiver

$$1 \longrightarrow 2 \longrightarrow \cdots\cdots\cdots \longrightarrow n$$

is of finite type, which is called of type A_n.

If the category $\mathrm{Rep}(Q)$ admits a full embedding of the category $\mathrm{Rep}(L_2)$, the quiver Q is called *wild* by the next reason. Indeed, the path algebra $\mathbb{K}L_2$ is isomorphic to the tensor algebra $\mathbb{K}\langle X, Y\rangle := T(\mathbb{K}X \oplus \mathbb{K}Y)$ over $\mathbb{K}$. For any $\mathbb{K}$-algebra A of finite type with generators $a_1, a_2, ..., a_n$, one can define a fully faithful functor

$$F : \mathrm{Mod}_A^f \longrightarrow \mathrm{Mod}_{\mathbb{K}\langle X,Y\rangle}^f,$$

where $\mathrm{Mod}_{\mathrm{A}}^{\mathrm{f}}$ signifies the category of $\mathbb{K}$-finite dimensional left A-modules, by setting $F(M) = M^{n+2}$ and the action of X and Y are given by

$$X = \begin{pmatrix} 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ \vdots & & & & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & \cdots & \cdots & \cdots & \cdots & 0 \\ 1 & \ddots & & & & \vdots \\ a_1 & \ddots & \ddots & & & \vdots \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_n & 1 & 0 \end{pmatrix}$$

This means a complete classification of the representations of the quiver L_2 implies, thus, a complete classification of $\mathbb{K}$-finite dimensional representations of all $\mathbb{K}$-algebras of finite type, which is somehow hopeless. If the quiver Q is neither of finite type nor wild, it is said to be *tame*.

2.2 Tits Form

For $\boldsymbol{n} = (n_i)_{i\in Q_0} \in (\mathbb{Z}_{\geqslant 0})^{Q_0}$, it is clear that the set of representations of Q of dimension vector $\boldsymbol{n}$ is in bijective correspondence with the *representation space*

$$\mathrm{Rep}_Q(\boldsymbol{n}) := \bigoplus_{\varphi\in Q_1} \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{n_{s(\varphi)}}, \mathbb{K}^{n_{t(\varphi)}}).$$

By definition, the representations of Q corresponding to two elements $(f_\alpha)_{\alpha\in Q_1}$ and $(h_\alpha)_{\alpha\in Q_1}$ are isomorphic if and only if they lie in the same $G(\boldsymbol{n})$-orbit, where the $G(\boldsymbol{n}) := \prod_{i\in Q_0} \mathrm{GL}(\mathbb{K}, n_i)$-action on $\mathrm{Rep}(Q, \boldsymbol{n})$ is given by

$$(g_i)_{i\in Q_0}.(\varphi_\alpha)_{\alpha\in Q_1} := (g_{t(\alpha)}\varphi_\alpha g_{s(\alpha)}^{-1})_{\alpha\in Q_1},$$

for $g_i \in \mathrm{GL}(n_i, \mathbb{K})$ and $\varphi_\alpha \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}^{n_{s(\varphi)}}, \mathbb{K}^{n_{t(\varphi)}})$. Clearly, this action induces an action of $PG(\boldsymbol{n}) := G(\boldsymbol{n})/\mathbb{K}^* \prod_{i\in Q_0} \mathrm{id}_{\mathbb{K}^{n_i}}$-action on $\mathrm{Rep}_Q(\boldsymbol{n})$. As $\mathbb{K}$ is infinite by assumption, if $\dim PG(\boldsymbol{n}) < \dim \mathrm{Rep}_Q(\boldsymbol{n})$, i.e.,

$$q_Q(\boldsymbol{n}) := \sum_{i\in Q_0} n_i^2 - \sum_{\alpha\in Q_1} n_{s(\alpha)}n_{t(\alpha)} \leqslant 0,$$

there are infinitely many $PG(\boldsymbol{n})$-orbits on $\mathrm{Rep}_Q(\boldsymbol{n})$, hence infinitely many isomorphism classes of representations of Q of dimension vector $\boldsymbol{n}$ which implies that there are infinitely many isomorphism classes of finite dimensional indecomposable representations of Q.
The quadratic form q_Q is called the *Tits form* associated to Q. We remark that this does not depend on a choice of Q_1, i.e., it depends only on the underlying non-oriented graph, say $|Q|$. The *Cartan matrix* C_Q describes the polarization of q_Q:

$$(\boldsymbol{m}, \boldsymbol{n})_Q = q_Q(\boldsymbol{m}+\boldsymbol{n}) - q_Q(\boldsymbol{m}) - q_Q(\boldsymbol{n}) = \boldsymbol{m}C_Q\boldsymbol{n}^T.$$

The components of $C_Q = (c_{i,j})_{i,j\in Q_0}$ have an expression in terms of $|Q|$:

$$c_{i,j} = \begin{cases} 2 - 2\cdot\sharp\{\text{loops in } i\} & i = j, \\ -\sharp\{\text{edges connecting } i \text{ and } j\} & i \neq j. \end{cases}$$

2.2.1 Lemma (cf. [15]). *Let Q be a quiver whose underlying non-oriented graph $|Q|$ is connected, q be its Tits form and C be its Cartan matrix.*

1. *q is positive definite iff $|Q|$ is a Dynkin diagram of type A_l $(l \geqslant 1)$, D_l $(l \geqslant 4)$, E_6, E_7 or E_8.*
2. *q is positive semi-definite iff either Q is the 1-loop L_1 (which is also called of type $\widetilde{A}_0$) or $|Q|$ is an extended Dynkin diagram of type $\widetilde{A}_l$ $(l \geqslant 1)$, $\widetilde{D}_l$ $(l \geqslant 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$. In this case,* $\mathrm{rank}\, C = \sharp Q_0 - 1$ *and*

$$\begin{aligned}&\{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | C\boldsymbol{m}^T \geqslant \boldsymbol{0}\} = \{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | C\boldsymbol{m}^T = \boldsymbol{0}\}\\ =&\{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | q_Q(\boldsymbol{m}) = 0\} = \mathbb{Z}_{\geqslant 0}\delta_Q,\end{aligned}$$

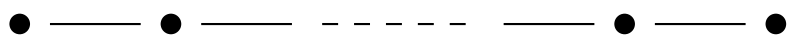
   for a unique $\delta_Q \in (\mathbb{Z}_{\geqslant 0})^{Q_0} \setminus \{\boldsymbol{0}\}$.
3. *q is indefinite iff $C\boldsymbol{m}^T \geqslant \boldsymbol{0}^T$ for $\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0}$ implies $\boldsymbol{m} = \boldsymbol{0}$, and there exists an $\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0}$ such that $\boldsymbol{m}^T > \boldsymbol{0}^T$ and $C\boldsymbol{m}^T < \boldsymbol{0}^T$.*

Here, $\boldsymbol{m}^T > \boldsymbol{0}^T$ (resp. $\boldsymbol{m}^T \geqslant \boldsymbol{0}^T$) means that $m_i > 0$ (resp. $m_i \geqslant 0$) for any $i \in Q_0$.

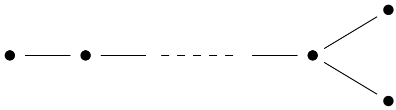
Here are the diagrams:

Dynkin diagrams (l nodes)

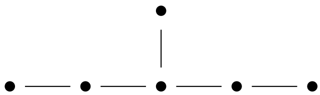
A_l ($l \geqslant 1$)



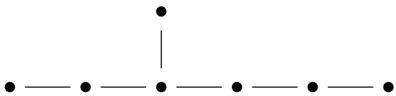
D_l ($l \geqslant 4$)



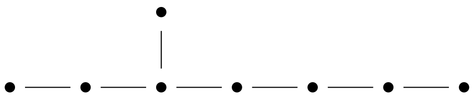
E_6



E_7

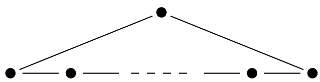


E_8

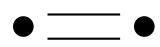


Extended Dynkin diagrams ($l+1$ nodes)

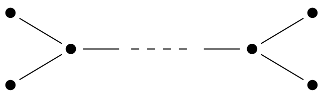
$\widetilde{A}_l$ ($l \geqslant 2$)



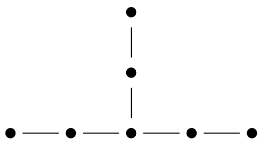
$\widetilde{A}_1$



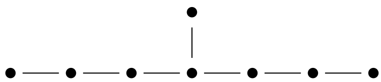
$\widetilde{D}_l$ ($l \geqslant 4$)



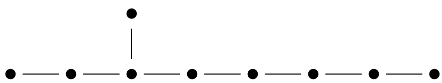
$\widetilde{E}_6$



$\widetilde{E}_7$



$\widetilde{E}_8$



We remark that the underlying non-oriented graphs of the quivers $K_1 = S_1$, K_2, S_2, S_3 and S_4 are A_2, $\widetilde{A}_1$, A_3, D_4 and $\widetilde{D}_4$, respectively.

For $\boldsymbol{n} \in (\mathbb{Z}_{\geqslant 0})^{Q_0}$, we denote by $R(Q, \boldsymbol{n})$ the set of isomorphism classes of the representations of Q of dimension vector $\boldsymbol{n}$, which is the same as the set of $PG(\boldsymbol{n})$-orbits on $\mathrm{Rep}_Q(\boldsymbol{n})$. We also denote the subset of $R(Q, \boldsymbol{n})$ consisting of the indecomposable representations by $R(Q, \boldsymbol{n})_{\mathrm{ind}}$.

2.3 Main Theorem

For $i \in Q_0$, let $r_i \in GL(\mathbb{Z}^{Q_0})$ be the reflection defined by

$$r_i(\boldsymbol{m}) = \boldsymbol{m} - (\boldsymbol{m}, \varepsilon_i)_Q \varepsilon_i$$

for $\boldsymbol{m} \in \mathbb{Z}^{Q_0}$. The subgroup $W \leqslant GL(\mathbb{Z}^{Q_0})$ generated by $\{r_i\}_{i \in Q_0}$ is called the *Weyl group* of $|Q|$. Set

$$\Pi = \{\varepsilon_i\}_{i \in Q_0} \quad \text{and} \quad K_Q := \{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | (\boldsymbol{m}, \varepsilon_i)_Q \leqslant 0 \ \forall i \in Q_0\}.$$

An element of the set $\Delta^+(Q)_{\mathrm{re}} := W.\Pi \cap (\mathbb{Z}_{\geqslant 0})^{Q_0}$ is called a *positive real root*, and an element of the set $\Delta^+(Q)_{\mathrm{im}} := W.K_Q$ is called a *positive imaginary root*. An element of $\Delta^+(Q) := \Delta^+(Q)_{\mathrm{re}} \cup \Delta^+(Q)_{\mathrm{im}}$ is just called a *positive root*. For detail, see, e.g., [15].

When $|Q|$ is a Dynkin diagram, it is clear that $\Delta^+(Q)_{\mathrm{im}} = \emptyset$, hence we have $\Delta^+(Q) = \Delta^+(Q)_{\mathrm{re}}$ in this case. We also have $\Delta^+(Q) = \{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | q_Q(\boldsymbol{m}) = 1\}$. When $|Q|$ is an extended Dynkin diagram, it is well-known (cf. [15]) that $\Delta^+(Q)_{\mathrm{im}} = \mathbb{Z}_{>0}\delta_Q$ where δ_Q is defined in Lemma 2.2.1 and $\Delta^+(Q)_{\mathrm{re}} = \{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | q_Q(\boldsymbol{m}) = 1\}$.

2.3.1 Theorem. (cf. [8, 9]). *Assume that Q is a quiver whose underlying non-oriented graph $|Q|$ is connected.*

1. *Q is of finite type iff $|Q|$ is a Dynkin diagram of type A_l $(l \geqslant 1)$, D_l $(l \geqslant 4)$, E_6, E_7 or E_8.*
2. *In this case, the map* **dim** *induces a bijection between the set of isomorphism classes of the indecomposable representations of Q and $\Delta^+(Q)$.*

The original proof of this theorem due to Gabriel [8] is rather direct computations. For more conceptual proof, see, e.g., [3] where the authors introduced the so-called *reflection functor*. Another algebraic proof is given in a lecture note by W. Crawley-Boevey mentioned at the beginning of this lecture. A proof using tilting theory can be found in [1]. A geometric proof can be found in [4].

2.3.2 Theorem. (cf. [7, 19]). *Assume that Q is a quiver whose underlying non-oriented graph $|Q|$ is connected.*

1. *Q is tame iff either Q is of type $\widetilde{A}_0$ or $|Q|$ is an extended Dynkin diagram of type $\widetilde{A}_l$ $(l \geqslant 1)$, $\widetilde{D}_l$ $(l \geqslant 4)$, $\widetilde{E}_6$, $\widetilde{E}_7$ or $\widetilde{E}_8$.*
2. *In this case, $\{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | R(Q, \boldsymbol{m})_{\mathrm{ind}} \neq \emptyset\} = \Delta^+(Q)$.*
3. *Moreover, one has $\sharp R(Q, \boldsymbol{m})_{\mathrm{ind}} = 1$ for any $\boldsymbol{m} \in \Delta^+(Q)_{\mathrm{re}}$, and $\sharp R(Q, \boldsymbol{m})_{\mathrm{ind}} = \infty$ for any $\boldsymbol{m} \in \Delta^+(Q)_{\mathrm{im}}$.*

By this theorem, Q is often said to be *affine* when it is tame and not of type $\widetilde{A}_0$. For the representation theory of tame quivers, see, e.g., [6, 20].

V. G. Kac obtained a generalization of these results to graphs with no loop, i.e., those corresponding to symmetrizable Kac–Moody algebras:

2.3.3 Theorem (cf. [12, 13]**).** *Assume that Q is a quiver whose underlying non-oriented graph $|Q|$ contains no loop.*

1. $\{\boldsymbol{m} \in (\mathbb{Z}_{\geqslant 0})^{Q_0} | R(Q, \boldsymbol{m})_{\mathrm{ind}} \neq \emptyset\} = \Delta^+(Q)$.
2. *If $\boldsymbol{m} \in \Delta^+(Q)_{\mathrm{re}}$, then $\sharp R(Q, \boldsymbol{m})_{\mathrm{ind}} = 1$.*
3. *If $\boldsymbol{m} \in \Delta^+(Q)_{\mathrm{im}}$, the number of parameters of $\mathrm{Rep}_Q(\boldsymbol{m})_{\mathrm{ind}}$ is $1 - q_Q(\boldsymbol{m})$, where $\mathrm{Rep}_Q(\boldsymbol{m})_{\mathrm{ind}} \subset \mathrm{Rep}_Q(\boldsymbol{m})$ signifies the subset of indecomposable representations.*

For $d \in \mathbb{Z}_{>0}$, let $\mathrm{Rep}_Q(\boldsymbol{m})^{(d)}_{\mathrm{ind}}$ be the subset of indecomposable representations M with $\dim \mathrm{End}(M) = d$. Then, the dimension of the orbit of $(f_\alpha)_{\alpha \in Q_1} \in \mathrm{Rep}_Q(\boldsymbol{m})^{(d)}_{\mathrm{ind}}$ is $\dim PG(\boldsymbol{m}) - d$ which implies

$$\begin{aligned}\dim \mathrm{Rep}_Q(\boldsymbol{m})^{(d)}_{\mathrm{ind}} - d =& (\dim PG(\boldsymbol{m}) - d) + 1 - (\dim G(\boldsymbol{m}) - \dim \mathrm{Rep}_Q(\boldsymbol{m})^{(d)}_{\mathrm{ind}}) \\ \leqslant& (\dim PG(\boldsymbol{m}) - d) + 1 - q_Q(\boldsymbol{m}).\end{aligned}$$

Thus, the third statement of the above theorem asserts that $\{\dim \mathrm{Rep}_Q(\boldsymbol{m})^{(d)}_{\mathrm{ind}}\}_{d>0}$ attains the maximal possible dimension, i.e., $\dim \mathrm{Rep}_Q(\boldsymbol{m})$.

For detail, the reader may consult the articles [14] by Kac, and by Kraft and Riedtmann [17] where some incorrect original statements were rectified.

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Chapter 7
Introduction to Quiver Varieties

Yoshiyuki Kimura

1 Introduction

Quiver representations and Kac–Moody Lie algebras. The interaction between quiver representations and Kac–Moody Lie algebras has an origin in Gabriel's theorem. Gabriel [15] classified the quivers which are finite representation types and showed the existence of the bijection between the set of isomorphism classes of indecomposable representations of Dynkin quivers Q and the set of positive roots of the corresponding simply laced Lie algebra $\mathfrak{g}_Q$ via dimension vectors. Bernšteĭn–Gel'fand–Ponomarev [2] gave a proof of Gabriel's theorem using reflection functors and Coxeter functors. Using the theory of species introduced by Gabriel [16], Dlab–Ringel [11] extended Gabriel's theorem to finite dimensional hereditary algebras over arbitrary fields. The classification of the indecomposable representations of affine quivers was studied by Weierstrass, Kronecker, Gel'fand–Ponomarev, Donovan–Freislich, Nazarova and Dlab–Ringel [12]. Kac [24, 25] generalized Gabriel's theorem for arbitrary quiver and related it to the symmetric Kac–Moody Lie algebras. In particular, Kac introduced the counting polynomials of absolutely indecomposable representations over finite fields and proposed the constant term conjecture [26, Conjecture 1] which relates to multiplicities of the corresponding Lie algebras and the positivity conjecture [26, Conjecture 2].

Hall algebras and canonical bases. Ringel [52] studied the Hall algebras of the abelian categories of quiver representations (or hereditary algebras) over finite fields and related them to the Drinfeld–Jimbo quantized enveloping algebras. Using the Grothendieck's function-sheaf dictionary, Lusztig geometrized Ringel's construction and invented the theory of canonical bases using perverse sheaves on the varieties of quiver representations [31]. Lusztig also studied the conormal variety of the vari-

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eties of quiver representations [32, Sect. 8] in finite representation types. In general, Lusztig [33] gave an estimate of singular support (or characteristic varieties) of the class of perverse sheaves arising in the theory of canonical bases and studied an inductive structure on the set of irreducible components of the representation variety of preprojective algebras, called Lusztig quiver varieties. He also gave a conjecture [33, 13.7 (b)] on existence and uniqueness of the bijection between the set of irreducible components of Lusztig quiver varieties and the class of perverse sheaves arising in the theory of canonical basis. Kashiwara–Saito [27] proved that the inductive structure on the set of irreducible components of Lusztig quiver varieties give rise to the crystal structure of the canonical base of quantized universal enveloping algebra, in fact they proved the existence of a bijection between the canonical base and the set of irreducible components of Lusztig quiver varieties.

Moduli spaces of instantons on ALE spaces. Kronheimer [29] gave a description of a particular family of 4-dimensional non-compact complete hyper-Kähler 4-manifolds, the so-called ALE[1] spaces (or the ALE gravitational instantons), as hyper-Kähler quotient via the McKay correspondence. In fact, Kronheimer constructed the simple singularity $\mathbb{C}^2/\Gamma$, its semi-universal deformation, and simultaneous resolution that was constructed by Brieskorn–Slodowy using an entirely different method (see also Cassens–Slodowy [3]). Kronheimer and Nakajima [30] (see also Nakajima [46]) gave the ADHM[2] description of instantons (or sheaves) on ALE spaces. Nakajima [39] introduced quiver varieties as a generalization of ADHM descriptions of the moduli spaces of instantons on ALE spaces to arbitrary quiver.

Quiver varieties and Representation theory. Nakajima [39, 41] gave a geometric construction of integrable highest weight representations of the symmetric Kac–Moody Lie algebras on constructible functions or Borel–Moore homologies of the Lagrangian quiver varieties. He also studied the representation theory of quantum affine algebras [43, 45, 48] using equivariant K-theory of the quiver varieties and perverse sheaves on (graded and cyclic) quiver varieties. Hausel [21] studied the formula of the Betti numbers of quiver varieties and proved Kac's constant term conjecture. As a generalization of Bernšteĭn–Gel'fand–Ponomarev reflection functors, the reflection functors of quiver varieties are also studied by Nakajima [39, 44], Crawley–Boevey–Holland [9], Lusztig [34] and also Maffei [35]. The Weyl group action on quiver varieties and its isotypical decomposition of the cohomology groups is used for the proof of Kac's positivity conjecture by Hausel–Letellier–Rodriguez–Villegas [22].

Plan of the chapter. The plan of the chapter is as follows. In this chapter, we will give gentle introduction of quiver varieties, that is we do not treat the applications of the quiver varieties to representation theory. In the first part, we treat the theory of stability for quiver representations. In the second part, we explain about preprojective algebras and quiver varieties as framed moduli spaces of representations of preprojective

[1] ALE stands for asymptotically locally Euclidean.

[2] ADHM stands for Atiyah–Drinfeld–Hitchin–Manin.

algebras. For the applications of the quiver varieties to representation theory, see Nakajima [40, 42], Ginzburg [20], and Schiffmann [55].

2 Moduli Spaces of Quiver Representations

In this section, we will introduce the notion of (semi)stability for quiver representations and its basic properties. We follow an approach due to King [28, Definition 1.1] and Rudakov [54].

2.1 Quivers and Their Representations

2.1.1 Definition. A *quiver* Q is a directed graph, that is a quadruple $(Q_0, Q_1, \mathrm{out}, \mathrm{in})$, where Q_0 is a set of vertices and Q_1 is a set of arrows and $\mathrm{out}\colon Q_1 \to Q_0$ is a map which assigns the outgoing vertex and $\mathrm{in}\colon Q_1 \to Q_0$ is a map which assigns the incoming vertex.

In this chapter, we assume that a quiver is finite, that is the set Q_0 of vertices and the set Q_1 of edges are finite sets. Let $\mathbb{C}$ be the field of complex numbers and we fix it as a base field for simplicity.

2.1.2 Definition. Let Q be a quiver.

(i) A quiver representation (B, V) over $\mathbb{C}$ is a pair which consists of a Q_0-graded vector space $V = \bigoplus_{i \in Q_0} V_i$ over $\mathbb{C}$ and a Q_1-tuple of $\mathbb{C}$-linear maps

$$B = \bigoplus_{h \in Q_1} B_h \in \bigoplus_{h \in Q_1} \mathrm{Hom}_{\mathbb{C}}\left(V_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}\right).$$

(ii) A homomorphism between quiver representations $\left(B^1, V^1\right)$ and $\left(B^2, V^2\right)$ is a Q_0-tuple of homomorphisms of $\mathbb{C}$-vector spaces $\varphi = (\varphi_i)_{i \in Q_0}$ with $B_h^2 \varphi_{\mathrm{out}(h)} = \varphi_{\mathrm{in}(h)} B_h^1$ for $h \in Q_1$, that is a Q_0-tuple of homomorphisms of the $\mathbb{C}$-vector spaces $\varphi_i\colon V_i^1 \to V_i^2$ such that the following diagram commutes for each $h \in Q_1$:

$$\begin{array}{ccc} V^1_{\mathrm{out}(h)} & \xrightarrow{B^1_h} & V^1_{\mathrm{in}(h)} \\ {\scriptstyle \varphi_{\mathrm{out}(h)}} \downarrow & \circlearrowright & \downarrow {\scriptstyle \varphi_{\mathrm{in}(h)}} \\ V^2_{\mathrm{out}(h)} & \xrightarrow[B^2_h]{} & V^2_{\mathrm{out}(h)} \end{array}$$

2.1.3 Definition. Let Q be a quiver and (B, V) be a quiver representation over $\mathbb{C}$.

(i) A Q_0-graded subspace $V' = \bigoplus_{i \in Q_0} V'_i$ of V is called B-adapted if $B_h V'_{\mathrm{out}(h)} \subset V'_{\mathrm{in}(h)}$ for all $h \in Q_1$.

(ii) Let V' be a B-adapted Q_0-graded subspace of V. $\left(B|_{V'}, V'\right)$ (resp. $\left(B|_{V/V'}, V/V'\right)$) is called a subrepresentation (resp. quotient representation) of (B, V), where $(B|_{V'})_h : V'_{\mathrm{out}(h)} \to V'_{\mathrm{in}(h)}$ (resp. $\left(B|_{V/V'}\right)_h : \left(V/V'\right)_{\mathrm{out}(h)} \to \left(V/V'\right)_{\mathrm{in}(h)}$) is the induced linear map from

$$V_{\mathrm{out}(h)} \xrightarrow{B_h} V_{\mathrm{in}(h)} \twoheadrightarrow V_{\mathrm{in}(h)}/V'_{\mathrm{in}(h)}.$$

Since the structure of the representations on V' and V/V' is uniquely determined by the restriction, hence V' (resp. V/V') is called subrepresentation (resp. quotient) representation. More generally, for a quiver representation (B, V) of Q and a filtration

$$0 = V^0 \subset V^1 \subset \cdots \subset V^\ell = V,$$

of B-adapted vector spaces, we consider the canonical induced structure of the representations $\left(B|_{V^i/V^{i-1}}, V^i/V^{i-1}\right)$ on V^i/V^{i-1} $(1 \leqslant i \leqslant \ell)$.

2.1.4 Examples. For quiver representations $\left(B^1, V^1\right)$, $\left(B^2, V^2\right)$ and a homomorphism φ between them, the Q_0-graded subspaces $\bigoplus_{i \in Q_0} \mathrm{Ker}\,(\varphi_i) \subset V^1$ (resp. $\bigoplus_{i \in Q_0} \mathrm{Im}\,(\varphi_i) \subset V^2$) is B^1 (resp. B^2)-adapted. So the subrepresentation $\mathrm{Ker}\,(\varphi) := \left(B^1|_{\bigoplus_{i \in Q_0} \mathrm{Ker}(\varphi_i)}, \bigoplus_{i \in Q_0} \mathrm{Ker}\,(\varphi_i)\right)$ (resp. $\mathrm{Im}\,(\varphi) := \left(B^2|_{\bigoplus_{i \in Q_0} \mathrm{Im}(\varphi_i)}, \bigoplus_{i \in Q_0} \mathrm{Im}\,(\varphi_i)\right)$) of $\left(B^1, V^1\right)$ (resp. $\left(B^2, V^2\right)$) is canonically defined.

Let $\mathsf{mod}\,(\mathbb{C}Q)$ be the category of finite dimensional representations of the quiver Q, that is each vector space V_i is finite dimensional. It is well-known that $\mathsf{mod}\,(\mathbb{C}Q)$ is a $\mathbb{C}$-linear abelian category (see also Iohara [23, Sect. 2] in the same volume for more details).

Let $\mathbb{Z}_{\geqslant 0}^{Q_0}$ and $\mathbb{Z}^{Q_0}$ be the abelian monoid (resp. the abelian group) of maps $Q_0 \to \mathbb{Z}_{\geqslant 0}$ (resp. $Q_0 \to \mathbb{Z}$) whose addition is given by its the addition of the value of maps. For $i \in Q_0$, let $\alpha_i \in \mathbb{Z}_{\geqslant 0}^{Q_0} \subset \mathbb{Z}^{Q_0}$ be the map defined by $\alpha_i : j \mapsto \delta_{i,j}$ $(i, j \in Q_0)$, where $\delta_{i,j}$ is the Kronecker's delta.

2.1.5 Definition. (**dimension vector**) Let $Q = (Q_0, Q_1)$ be a quiver and (B, V) a quiver representation over $\mathbb{C}$. We define the dimension vector $\mathbf{dim}\,(B, V) \in \mathbb{Z}_{\geqslant 0}^{Q_0} \subset \mathbb{Z}^{Q_0}$ as the map by $\mathbf{dim}\,(B, V) : i \mapsto \dim_{\mathbb{C}} V_i$, that is

$$\mathbf{dim}\,(B, V) = \sum_{i \in Q_0} (\dim_{\mathbb{C}} V_i)\, \alpha_i.$$

This induces a (surjective) group homomorphism $\mathbf{dim} \colon K_0\,(\mathsf{mod}\,(\mathbb{C}Q)) \to \mathbb{Z}^{Q_0}$, where $K_0\,(\mathsf{mod}\,(\mathbb{C}Q))$ is the Grothendieck group of the abelian category $\mathsf{mod}\,(\mathbb{C}Q)$ of representations of the quiver Q over $\mathbb{C}$, that is the abelian group which is generated by $[(B, V)]$ for all objects (B, V) of $\mathsf{mod}\,(\mathbb{C}Q)$, with a relation $\left[\left(B^1, V^1\right)\right] -$

$\left[\left(B^2, V^2\right)\right] + \left[\left(B^3, V^3\right)\right]$ for each short exact sequence $0 \to \left(B^1, V^1\right) \to \left(B^2, V^2\right) \to \left(B^3, V^3\right) \to 0$ in $\mathsf{mod}\,(\mathbb{C}Q)$.

We introduce a quadratic form and bilinear forms on $\mathbb{Z}^{Q_0}$.

2.1.6 Definition. **(i)** For $\boldsymbol{v} \in \mathbb{Z}^{Q_0}$, we define the Tits form as follows:

$$q_Q\left(\boldsymbol{v}\right) = \sum_{i \in Q_0} v_i^2 - \sum_{h \in Q_1} v_{\mathrm{out}(h)} v_{\mathrm{in}(h)}.$$

(ii) For $\boldsymbol{v}^1, \boldsymbol{v}^2 \in \mathbb{Z}^{Q_0}$, we define the Euler–Ringel form and the symmetric bilinear form associated with the Tits form as follows:

$$\left\langle \boldsymbol{v}^1, \boldsymbol{v}^2 \right\rangle_Q = \sum_{i \in Q_0} v_i^1 v_i^2 - \sum_{h \in Q_1} v^1_{\mathrm{out}(h)} v^2_{\mathrm{in}(h)},$$
$$\left(\boldsymbol{v}^1, \boldsymbol{v}^2\right) = \left\langle \boldsymbol{v}^1, \boldsymbol{v}^2 \right\rangle_Q + \left\langle \boldsymbol{v}^2, \boldsymbol{v}^1 \right\rangle_Q.$$

We note that $q_Q\left(\boldsymbol{v}\right)$ does not depend on a choice of orientation and $\langle \boldsymbol{v}, \boldsymbol{v} \rangle_Q = q_Q\left(\boldsymbol{v}\right)$ and

$$\left(\boldsymbol{v}^1, \boldsymbol{v}^2\right) = q_Q\left(\boldsymbol{v}^1 + \boldsymbol{v}^2\right) - q_Q\left(\boldsymbol{v}^1\right) - q_Q\left(\boldsymbol{v}^2\right).$$

Using the standard Ringel resolution of the path algebra, we obtain the description of the homomorphism spaces and the extension spaces.

2.1.7 Exercise. For quiver representations $\left(B^1, V^1\right)$ and $\left(B^2, V^2\right)$, we define the linear map $d_{B^1,B^2} \colon \mathbf{L}\left(V^1, V^2\right) \to \mathbf{E}_Q\left(V^1, V^2\right)$ by

$$d_{B^1,B^2}\left(\{\varphi_i\}_{i \in Q_0}\right) := \left\{B_h^2 \varphi_{\mathrm{out}(h)} - \varphi_{\mathrm{in}(h)} B_h^1\right\}_{h \in Q_1},$$

where

$$\mathbf{L}\left(V^1, V^2\right) := \bigoplus_{i \in Q_0} \mathrm{Hom}_{\mathbb{C}}\left(V_i^1, V_i^2\right),$$
$$\mathbf{E}_Q\left(V^1, V^2\right) := \bigoplus_{h \in Q_1} \mathrm{Hom}_{\mathbb{C}}\left(V^1_{\mathrm{out}(h)}, V^2_{\mathrm{in}(h)}\right).$$

Show that

$$\mathrm{Ker}\left(d_{B^1,B^2}\right) = \mathrm{Hom}_Q\left(\left(B^1, V^1\right), \left(B^2, V^2\right)\right),$$
$$\mathrm{Coker}\left(d_{B^1,B^2}\right) \simeq \mathrm{Ext}^1\left(\left(B^1, V^1\right), \left(B^2, V^2\right)\right).$$

2.2 Stability Conditions for Quiver Representations

Mumford introduced the concept of stability to construct the moduli spaces of vector bundles on algebraic curves. Rudakov [54] introduced an axiomatic approach to the stability on an abelian category also motivated by the work of King [28] for the category of quiver representations. Though the following treatment in Rudakov [54] does make sense for an (essentially small) abelian category $\mathcal{A}$ such that all objects have finite length, we restrict ourselves to the case $\mathcal{A} = \mathsf{mod}\,(\mathbb{C}Q)$ for simplicity. The following definition reduces to (a special case of) the Rudakov stability [54, Definition 3.1].

2.2.1 Definition. Fix a Q_0-tuple of real numbers $\zeta_{\mathbb{R}} = \left(\zeta_{\mathbb{R},i}\right)_{i \in Q_0} \in \mathbb{R}^{Q_0} = \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{Q_0}, \mathbb{R}\right)$, called the stability parameter. For a non-zero representation (B, V), we set

$$\langle \zeta_{\mathbb{R}}, \mathbf{dim} V \rangle := \sum_{i \in Q_0} \zeta_{\mathbb{R},i} \dim V_i,$$

$$\theta_{\zeta_{\mathbb{R}}}(B, V) = \theta_{\zeta_{\mathbb{R}}}(V) := \frac{\langle \zeta_{\mathbb{R}}, \mathbf{dim} V \rangle}{\langle 1, \mathbf{dim} V \rangle} = \frac{\sum_{i \in Q_0} \zeta_{\mathbb{R},i} \dim V_i}{\sum_{i \in Q_0} \dim V_i},$$

where $1 = (1)_{i \in Q_0} \in \mathbb{R}^{Q_0} = \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{Q_0}, \mathbb{R}\right)$. $\theta_{\zeta_{\mathbb{R}}}(B, V)$ is called the slope of the quiver representation (B, V).

(i) A non-zero representation (B, V) is said to be $\zeta_{\mathbb{R}}$-*semistable* if, for any subrepresentation $\left(B', V'\right)$ of (B, V), we have $\theta_{\zeta_{\mathbb{R}}}\left(B', V'\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(B, V)$.

(ii) A non-zero representation (B, V) is said to be $\zeta_{\mathbb{R}}$-*stable* if $\theta_{\zeta_{\mathbb{R}}}\left(B', V'\right) < \theta_{\zeta_{\mathbb{R}}}(B, V)$ for any non-zero proper subrepresentation $\left(B', V'\right)$ of (B, V).

2.2.2 Exercise. Two stability parameters $\zeta_{\mathbb{R}}$ and $\zeta'_{\mathbb{R}}$ are said to be equivalent if (B, V) is $\zeta_{\mathbb{R}}$-stable if and only if (B, V) is $\zeta'_{\mathbb{R}}$-stable for any quiver representation (B, V).

(i) For $a \in \mathbb{R}$ with $a > 0$, show that $\zeta_{\mathbb{R}}$ and $a\zeta_{\mathbb{R}} = \left(a\zeta_{\mathbb{R},i}\right)_{i \in Q_0} \in \mathbb{R}^{Q_0}$ are equivalent.

(ii) For $b \in \mathbb{R}$, show that $\zeta_{\mathbb{R}}$ and $\zeta_{\mathbb{R}} + b = \left(\zeta_{\mathbb{R},i} + b\right) \in \mathbb{R}^{Q_0}$ are equivalent.

2.2.3 Examples. **(i)** We consider a "trivial" stability $\mathbf{0} = (0) \in \mathbb{R}^{Q_0}$. By its definition, all representations are $\mathbf{0}$-semistable and $\mathbf{0}$-stable representations are simple representations.

(ii) We consider the (generalized) Kronecker quiver K_r with two vertices 1 and 2 with r-arrows from 1 to 2. It can be shown that that any (semi)stability is equivalent to the (semi)stability with respect to either $(0, 0)$, $(1, 0)$ or $(0, 1)$.

(iii) Let Q be an acyclic affine quiver, that is a quiver which does not contain oriented cycles and the underlying graph is affine (or Euclidean). Let $\delta \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ be the minimal imaginary root, that is an element $\delta \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ with $\mathrm{rad}\left(q_Q\right) = \left\{\boldsymbol{v} \in \mathbb{Z}^{Q_0} \mid \left(\boldsymbol{v}, \boldsymbol{v}'\right) = 0\ {}^{\forall}\boldsymbol{v}' \in \mathbb{Z}^{Q_0}\right\} = \mathbb{Z}\delta$.
The defect $\partial_Q \in \mathrm{Hom}_{\mathbb{Z}}\left(\mathbb{Z}^{Q_0}, \mathbb{Z}\right)$ is defined by

$$\partial_Q(B,V) = \langle \delta, \mathbf{dim}(B,V)\rangle_Q = -\langle \mathbf{dim}(B,V), \delta\rangle_Q .$$

It is introduced by Dlab–Ringel [12] for the study of the classification of indecomposable representations of affine (tame) quivers. In fact, the category of ∂-semistable representation with slope 0 characterizes the category of "regular representations" in the sense of Bernšteĭn–Gel'fand–Ponomarev [2].

2.2.4 Exercise. **(i)** Let Q be the linear quiver $1 \to 2 \to 3 \to \cdots \to n$. Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ with $\langle \zeta_{\mathbb{R}}, \alpha_k\rangle = -k$ for $1 \leqslant k \leqslant n$. Show that every indecomposable representation of Q is $\zeta_{\mathbb{R}}$-stable and $\zeta_{\mathbb{R}}$-semistables are direct sums of an indecomposable representation.

(ii) Let $K = K_2$ be the Kronecker quiver. Show that $\mathbf{dim}(B,V)$ is proportional to either $(k, k+1)$, $(1,1)$ or $(k+1,k)$ for $k \in \mathbb{Z}_{\geqslant 0}$ if (B,V) is a $(1,0)$-semistable.

The following lemma, called see-saw property, is elementary but a crucial ingredient for the study of stability for quiver representations.

2.2.5 Lemma. *Let $0 \to (B',V') \to (B,V) \to (B'',V'') \to 0$ be a short exact sequence of quiver representations. Then we have the following:*

(i) $\theta_{\zeta_{\mathbb{R}}}(B',V') \leqslant \theta_{\zeta_{\mathbb{R}}}(B,V)$ *if and only if* $\theta_{\zeta_{\mathbb{R}}}(B',V') \leqslant \theta_{\zeta_{\mathbb{R}}}(B'',V'')$ *if and only if* $\theta_{\zeta_{\mathbb{R}}}(B,V) \leqslant \theta_{\zeta_{\mathbb{R}}}(B'',V'')$.

(ii) $\min\left(\theta_{\zeta_{\mathbb{R}}}(B',V'), \theta_{\zeta_{\mathbb{R}}}(B'',V'')\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(B,V) \leqslant \max\left(\theta_{\zeta_{\mathbb{R}}}(B',V'), \theta_{\zeta_{\mathbb{R}}}(B'',V'')\right)$.

2.2.6 Exercise. Prove Lemma 2.2.5.

2.3 Harder–Narasimhan Filtration

Harder and Narasimhan introduced a "functorial" filtration of vector bundles on a Riemann surface for the study of cohomology groups of moduli spaces. In fact, they provide a recursive algorithm to compute the Betti numbers of moduli spaces using the filtration.

2.3.1 Proposition. *Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B^1,V^1) and (B^2,V^2) are $\zeta_{\mathbb{R}}$-semistable representations with $\theta_{\zeta_{\mathbb{R}}}(B^1,V^1) > \theta_{\zeta_{\mathbb{R}}}(B^2,V^2)$. Then every homomorphism φ: $(B^1,V^1) \to (B^2,V^2)$ is zero.*

Proof Let φ be a non-zero homomorphism, so $\mathrm{Im}(\varphi)$ is non-zero. We consider the following short exact sequence:

$$0 \to \mathrm{Ker}(\varphi) \to (B^1,V^1) \to \mathrm{Im}(\varphi) \to 0.$$

Assume that $\mathrm{Ker}(\varphi)$ is non-zero. Since (B^1,V^1) and (B^2,V^2) are $\zeta_{\mathbb{R}}$-semistable, we have $\theta_{\zeta_{\mathbb{R}}}(\mathrm{Ker}(\varphi)) \leqslant \theta_{\zeta_{\mathbb{R}}}(B^1,V^1)$ and $\theta_{\zeta_{\mathbb{R}}}(\mathrm{Im}(\varphi)) \leqslant \theta_{\zeta_{\mathbb{R}}}(B^2,V^2)$. But, by the see-saw property, we obtain that $\theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(\mathrm{Im}(\varphi))$. This contradicts the assumption $\theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) > \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right)$. Hence we obtain that Ker (φ) is zero. Next we assume that φ is injective, then the condition $\theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) > \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right)$ also contradicts the assumption that $\left(B^2, V^2\right)$ is a $\zeta_{\mathbb{R}}$-semistable representation. Hence we obtain the claim. □

2.3.2 Definition. Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B, V) a non-zero quiver representation. A *maximal destabilizing subrepresentation*, or *strongly contradicting semistability subrepresentation*, (with respect to the stability parameter $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$) is a non-zero B-adapted subspace W of V which satisfying the following conditions:

- **(i)** The slope $(B|_W, W)$ is maximal for all subrepresentations of (B, V), that is, for a non-zero B-adapted subspace W' of V, we have $\theta_{\zeta_{\mathbb{R}}}\left(W'\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(W)$,
- **(ii)** $(B|_W, W)$ is maximal among all subrepresentations of (B, V) of the maximal slope, that is if $\theta_{\zeta_{\mathbb{R}}}\left(W'\right) = \theta_{\zeta_{\mathbb{R}}}(W)$, we have $W' \subset W$.

2.3.3 Proposition. *Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B, V) a non-zero quiver representation.*

- **(i)** *A maximal destabilizing subrepresentation exists and it is unique.*
- **(ii)** *A maximal destabilizing subrepresentation is $\zeta_{\mathbb{R}}$-semistable.*

Proof **(i)** We prove the existence by induction on dimension vector.

Existence is clear since the set of dimension vectors of sub representations and their slopes are finite. Hence it suffices for us to prove the uniqueness of subrepresentation which satisfies the above conditions. By its definition, this subrepresentation is ζ-semistable.

Let $\theta = \max\left\{\theta_\zeta\left(B', V'\right) \mid \left(B', V'\right) \subset (B, V)\right\}$ and let $\left(B^1, V^1\right)$ and $\left(B^2, V^2\right)$ be subrepresentations which satisfy the conditions. We consider the short exact sequence:

$$0 \to V^1 \cap V^2 \to V^1 \oplus V^2 \to V^1 + V^2 \to 0.$$

Since $\left(B^1, V^1\right)$ and $\left(B^2, V^2\right)$ are semistable with $\theta = \theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) = \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right)$, it can be shown that $\left(B^1 \oplus B^2, V^1 \oplus V^2\right)$ is also semistable with $\theta = \theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) = \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right)$, then we have $\theta_{\zeta_{\mathbb{R}}}\left(V^1 \cap V^2\right) \leq \theta = \theta_{\zeta_{\mathbb{R}}}\left(V^1 \oplus V^2\right)$. By the see-saw property, we have $\theta \leqslant \theta_{\zeta_{\mathbb{R}}}\left(V^1 + V^2\right)$. By the maximality of slope, we have $\theta_{\zeta_{\mathbb{R}}}\left(V^1 + V^2\right) = \theta$. By the maximality of dimension, we have $V^1 = V^2$.

(ii) It follows from the definition of maximal destabilizing subrepresentation. □

2.3.4 Definition. Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B, V) a non-zero quiver representation. A *Harder–Narasimhan filtration* for (B, V) (with respect to the stability parameter $\zeta_{\mathbb{R}}$) is a filtration of B-adapted Q_0-graded vector spaces

$$0 = V^0 \subset V^1 \subset \cdots \subset V^\ell = V$$

satisfying the following properties:

- **(i)** The subquotients $\left(B|_{V^i/V^{i-1}}, V^i/V^{i-1}\right)$ $(1 \leqslant i \leqslant \ell)$ are $\zeta_{\mathbb{R}}$-semistable,
- **(ii)** $\theta_{\zeta_{\mathbb{R}}}\left(B|_{V^1}, V^1\right) > \theta_{\zeta_{\mathbb{R}}}\left(B|_{V^2/V^1}, V^2/V^1\right) > \cdots > \theta_{\zeta_{\mathbb{R}}}\left(B|_{V^\ell/V^{\ell-1}}, V^\ell/V^{\ell-1}\right)$.

2.3.5 Theorem. *Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B, V) a non-zero quiver representation. Then (B, V) has a unique Harder–Narasimhan filtration with respect to the stability parameter $\zeta_{\mathbb{R}}$.*

Proof We prove the existence of a Harder–Narasimhan filtration. Let $V^1 \subset V$ be the maximal destabilizing subrepresentation. By the induction on dimension, we can assume that $\left(B|_{V/V^1}, V/V^1\right)$ has a Harder–Narasimhan filtration

$$0 = W^0 \subset W^1 \subset \cdots \subset W^\ell = V/V^1.$$

Let V^{i+1} be the pre-image of W^i for the projection $V \twoheadrightarrow V/V^1$. Hence it suffices for us to prove that $\theta_{\zeta_{\mathbb{R}}}\left(V^1\right) > \theta_{\zeta_{\mathbb{R}}}\left(V^2/V^1\right)$. If this does not hold, we have $\theta_{\zeta_{\mathbb{R}}}\left(V^1\right) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(V^2/V^1\right)$. By the see-saw property, we would obtain $\theta_{\zeta_{\mathbb{R}}}\left(V^1\right) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(V^2\right)$ contradicting the maximality of V^1. Hence we have $\theta_{\zeta_{\mathbb{R}}}\left(V^1\right) > \theta_{\zeta_{\mathbb{R}}}\left(V^2/V^1\right)$ and obtain a Harder–Narasimhan filtration.

We prove the uniqueness of Harder–Narasimhan filtration by the induction on the dimension of (B, V). Let $0 = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^\ell = V$ be a filtration satisfying stated conditions and suppose that we have another filtration $0 = W^0 \subset W^1 \subset \cdots \subset W^m = V$ with the stated properties. Without losing any generality, we assume that $\theta_{\zeta_{\mathbb{R}}}\left(W^1\right) \geqslant \theta_{\zeta_{\mathbb{R}}}\left(V^1\right)$. Let n be the smallest integer such that $W^1 \subset V^n$. If $n > 1$, then the composition $W^1 \subset V^n \twoheadrightarrow V^n/V^{n-1}$ is non-zero. On the other hand, by the assumption, we have $\theta_{\zeta_{\mathbb{R}}}\left(W^1\right) \geqslant \theta_{\zeta_{\mathbb{R}}}\left(V^1\right) > \theta_{\zeta_{\mathbb{R}}}\left(V^n/V^{n-1}\right)$. By Proposition 2.3.1, this must be zero. Hence we obtain that $n = 1$. Then W^1 is a subspace of V^1. By the semistability of V^1, we have $\theta_{\zeta_{\mathbb{R}}}\left(W^1\right) = \theta_{\zeta_{\mathbb{R}}}\left(V^1\right)$. Then changing the role of V^1 and W^1, we obtain that $V^1 = W^1$. By the induction hypothesis on the dimension vector, we have the claim for $\left(B|_{V/V^1}, V/V^1\right)$, hence we obtain the claim for (B, V). □

2.3.6 Exercise. Fix $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$. For $\theta \in \mathbb{R}$ and a representation (B, V), we define

$$(B, V)^{(\theta)} = \left(B|_{V^k}, V^k\right)$$

if $\theta_{\zeta_{\mathbb{R}}}\left(B|_{V^k/V^{k-1}}, V^k/V^{k-1}\right) \geqslant \theta > \theta_{\zeta_{\mathbb{R}}}\left(B|_{V^{k+1}/V^k}, V^{k+1}/V^k\right)$, where

$$0 = V^0 \subsetneq V^1 \subsetneq V^2 \subsetneq \cdots \subsetneq V^\ell = V$$

is the Harder–Narasimhan filtration of (B, V).

(i) For $\theta \in \mathbb{R}$, for any homomorphism $\varphi\colon \left(B^1, V^1\right) \to \left(B^2, V^2\right)$, show that $\varphi\left(\left(B^1, V^1\right)^{(\theta)}\right) \subset \left(B^2, V^2\right)^{(\theta)}$.

(ii) Given $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and $\theta \in \mathbb{R}$, we define $\mathcal{T}_\theta$ as the class of all representations (B, V) with $(B, V)^{(\theta)} = (B, V)$ and we define $\mathcal{F}_\theta$ be the class of all representations (B, V) with $(B, V)^{(\theta)} = 0$. Show that $(\mathcal{T}_\theta, \mathcal{F}_\theta)$ is a torsion pair in $\mathsf{mod}(\mathbb{C}Q)$, that is $\mathrm{Hom}_Q(T, F) = 0$ for $T \in \mathcal{T}_\theta$ and $F \in \mathcal{F}_\theta$, and $\mathcal{T}_\theta$ and $\mathcal{F}_\theta$ are maximal with respect to the vanishing conditions.

(iii) For $\theta_1 < \theta_2$, show that $\mathcal{T}_{\theta_2} \subset \mathcal{T}_{\theta_2}$ and $\mathcal{F}_{\theta_2} \supset \mathcal{F}_{\theta_2}$.

2.4 Jordan–Hölder Filtration

We study the full subcategory which consists of semistable representations of a fixed slope.

2.4.1 Proposition. *For $\zeta_{\mathbb{R}} \in \mathbb{R}^I$ and $\theta \in \mathbb{R}$, let $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ be the full subcategory of* $\mathsf{mod}\,(\mathbb{C}Q)$ *which consists of $\zeta_{\mathbb{R}}$-semistable representations with $\theta_{\zeta_{\mathbb{R}}}(B, V) = \theta$ and the zero object.*

(i) *Assume that*

$$0 \to \left(B', V'\right) \to (B, V) \to \left(B'', V''\right) \to 0$$

is an exact sequence of non-zero representations of Q over $\mathbb{C}$ with

$$\theta_{\zeta_{\mathbb{R}}}(B, V) = \theta_{\zeta_{\mathbb{R}}}\left(B', V'\right) = \theta_{\zeta_{\mathbb{R}}}\left(B'', V''\right) = \theta.$$

(B, V) is $\zeta_{\mathbb{R}}$-semistable if and only if $\left(B', V'\right)$ and $\left(B'', V''\right)$ are $\zeta_{\mathbb{R}}$-semistable.

(ii) *$\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ is a weakly Serre subcategory of* $\mathsf{mod}\,(\mathbb{C}Q)$*, that is closed under extensions, kernels and cokernels. In particular, $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ is an abelian subcategory of* $\mathsf{mod}\,(\mathbb{C}Q)$.

Proof **(i)** First, we assume that $\left(B', V'\right)$ and $\left(B'', V''\right)$ are $\zeta_{\mathbb{R}}$-semistable representations with $\theta_{\zeta_{\mathbb{R}}}\left(B', V'\right) = \theta_{\zeta_{\mathbb{R}}}\left(B'', V''\right) = \theta_{\zeta_{\mathbb{R}}}(B, V) = \theta$ and show that (B, V) is $\zeta_{\mathbb{R}}$-semistable. Let $W \subset V$ be a B-adapted Q_0-graded subspace. We consider the following canonical exact sequence of Q_0-graded vector spaces:

$$0 \to W \cap V' \to W \to \left(W + V'\right)/V' \to 0.$$

We note that $W \cap V'$ is a B'-adapted Q_0-graded vector space and $\left(W + V'\right)/V'$ is a B''-adapted Q_0-graded subspace of $V'' = V/V'$ and we regard them as quiver representations. Since $\left(B', V'\right)$ and $\left(B'', V''\right)$ are $\zeta_{\mathbb{R}}$-semistable, we have $\theta_{\zeta_{\mathbb{R}}}\left(W \cap V'\right) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(V'\right)$ and $\theta_{\zeta_{\mathbb{R}}}\left(\left(W + V'\right)/V'\right) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(V''\right)$, then we obtain that

$$\begin{aligned} \theta_{\zeta_{\mathbb{R}}}(W) &\leqslant \max\left(\theta_{\zeta_{\mathbb{R}}}\left(W \cap V'\right), \theta_{\zeta_{\mathbb{R}}}\left(\left(W + V'\right)/V'\right)\right) \\ &\leqslant \max\left(\theta_{\zeta_{\mathbb{R}}}\left(V'\right), \theta_{\zeta_{\mathbb{R}}}\left(V''\right)\right) = \theta_{\zeta_{\mathbb{R}}}(V), \end{aligned}$$

that is (B, V) is $\zeta_{\mathbb{R}}$-semistable. Next we assume that (B, V) is $\zeta_{\mathbb{R}}$-semistable with

$$\theta_{\zeta_{\mathbb{R}}}\left(B', V'\right) = \theta_{\zeta_{\mathbb{R}}}\left(B'', V''\right) = \theta_{\zeta_{\mathbb{R}}}(B, V) = \theta.$$

Let $W' \subset V'$ be a $B' = B|_{V'}$-adapted Q_0-graded subspace, then it is also a B-adapted Q_0-graded subspace of V. Then we obtain

$$\theta_{\zeta_{\mathbb{R}}}\left(W''\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(V) = \theta_{\zeta_{\mathbb{R}}}\left(V''\right),$$

that is $\left(B'', V''\right)$ is $\zeta_{\mathbb{R}}$-semistable. Let $W'' \subset V''$ be a $B'' = B|_{V''}$-adapted Q_0-graded subspace and $W \subset V$ be the inverse image for the projection $V \to V''$, that is we have the following exact sequence of Q_0-graded vector spaces:

$$0 \to V' \to W \to W'' \to 0,$$

we note that W is a B-adapted Q_0-graded subspace of V. Since (B, V) is $\zeta_{\mathbb{R}}$-semistable, we have $\theta_{\zeta_{\mathbb{R}}}(W) \leqslant \theta_{\zeta_{\mathbb{R}}}(V) = \theta_{\zeta_{\mathbb{R}}}\left(V'\right)$, hence we obtain

$$\theta_{\zeta_{\mathbb{R}}}\left(W''\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(W) \leqslant \theta_{\zeta_{\mathbb{R}}}(V) = \theta_{\zeta_{\mathbb{R}}}\left(V''\right),$$

that is $\left(B'', V''\right)$ is $\zeta_{\mathbb{R}}$-semistable.

(ii) By the see-saw property and (a), it is clear that $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ is closed under extensions. So it suffices for us to prove that it is closed under kernels and cokernels for homomorphisms between semistable representations. For a homomorphism $\varphi\colon \left(B^1, V^1\right) \to \left(B^2, V^2\right)$, we consider the following canonical short exact sequences (in $\mathsf{mod}(\mathbb{C}Q)$):

$$\begin{gathered} 0 \to \operatorname{Ker}(\varphi) \to \left(B^1, V^1\right) \to \operatorname{Im}(\varphi) \to 0, \\ 0 \to \operatorname{Im}(\varphi) \to \left(B^2, V^2\right) \to \operatorname{Coker}(\varphi) \to 0. \end{gathered}$$

Since $\left(B^1, V^1\right)$ and $\left(B^2, V^2\right)$ are $\zeta_{\mathbb{R}}$-semistable, we have

$$\begin{gathered} \theta_{\zeta_{\mathbb{R}}}(\operatorname{Ker}(\varphi)) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(\operatorname{Im}(\varphi)), \\ \theta_{\zeta_{\mathbb{R}}}(\operatorname{Im}(\varphi)) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(\operatorname{Coker}(\varphi)), \end{gathered}$$

then we obtain that $\theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) \leqslant \theta_{\zeta_{\mathbb{R}}}(\operatorname{Im}(\varphi)) \leqslant \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right)$. By the assumption

$$\theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) = \theta_{\zeta_{\mathbb{R}}}\left(B^2, V^2\right) = \theta,$$

we have $\theta_{\zeta_{\mathbb{R}}}(\operatorname{Im}(\varphi)) = \theta$ and hence we obtain $\theta_{\zeta_{\mathbb{R}}}(\operatorname{Ker}(\varphi)) = \theta_{\zeta_{\mathbb{R}}}(\operatorname{Coker}(\varphi)) = \theta$ by the see-saw property. So the claim that $\operatorname{Ker}(\varphi) \in \mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ and $\operatorname{Coker}(\varphi) \in \mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ follows from (1). □

2.4.2 Remark. It is clear that the simple objects in $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ are the stable representations with $\theta_{\zeta_{\mathbb{R}}} = \theta$. Since stable representation is not necessarily a simple object in $\mathsf{mod}(\mathbb{C}Q)$ in general, the category $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ is not closed under arbitrary sub representations and quotient representations, hence $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ is not a Serre subcategory in general.

Since $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$ is an abelian category such that all representations have finite length, we obtain the following Jordan–Hölder theorem for $\mathcal{R}_{\zeta_{\mathbb{R}},\theta}$.

2.4.3 Definition. Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B, V) a $\zeta_{\mathbb{R}}$-semistable quiver representation with $\theta_{\zeta_{\mathbb{R}}}(B, V) = \theta$. A *Jordan–Hölder filtration* of (B, V) is a filtration

$$0 = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^\ell = V$$

of B-adapted Q_0-graded subspaces of V satisfying the following conditions:

(i) each subquotient $\left(B|_{V^i/V^{i-1}}, V^i/V^{i-1}\right)$ is $\zeta_{\mathbb{R}}$-stable for $1 \leqslant i \leqslant \ell$,
(ii) $\theta_{\zeta_{\mathbb{R}}}\left(B|_{V^1}, V^1\right) = \theta_{\zeta_{\mathbb{R}}}\left(B|_{V^2}, V^2\right) = \cdots = \theta_{\zeta_{\mathbb{R}}}(B, V) = \theta$ for $1 \leqslant i \leqslant \ell$.

If we consider the direct sum of two stable representation with the same slope, it can be seen that a Jordan–Hölder filtration need not be unique.

2.4.4 Theorem. *Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$. Let (B, V) be a $\zeta_{\mathbb{R}}$-semistable quiver representation. Then there exists a Jordan–Hölder filtration of (B, V) and the semisimplification or the graded quotient*

$$\mathrm{gr}^{JH}_{\zeta_{\mathbb{R}}}(B, V) := \bigoplus_{i=1}^{\ell} \left(B|_{V^i/V^{i-1}}, V^i/V^{i-1}\right)$$

does not depend on the choice of the Jordan–Hölder filtration of (B, V).

Proof If we consider any filtration of (B, V) with the same slope, then its maximal refinement of it yields a Jordan–Hölder filtration by construction. We prove the uniqueness of the graded quotient by the induction of the length of Jordan–Hölder filtrations. Let $0 = V^0 \subset V^1 \subset V^2 \subset \cdots \subset V^\ell = V$ and $0 = W^0 \subset W^1 \subset W^2 \subset \cdots \subset W^m = V$ be Jordan–Hölder filtration of (B, V). Let n_1 be the natural number such that $W^1 \subset V^{n_1}$ and $W^1 \not\subset V^{n_1-1}$. Let us consider the nontrivial composite homomorphism $W^1 \to V^{n_1} \twoheadrightarrow V^{n_1}/V^{n_1-1}$. Since $\left(B|_{W^1}, W^1\right)$ and $\left(B|_{V^{n_1}/V^{n_1-1}}, V^{n_1}/V^{n_1-1}\right)$ are $\zeta_{\mathbb{R}}$-stable with same slope, we obtain $\left(B|_{W^1}, W^1\right) \simeq \left(B|_{V^{n_1}/V^{n_1-1}}, V^{n_1}/V^{n_1-1}\right)$ and the canonical short exact sequence of quiver representations

$$0 \to \left(B|_{V^{n_1-1}}, V^{n_1-1}\right) \to \left(B|_{V^{n_1}}, V^{n_1}\right) \to \left(B|_{V^{n_1}/V^{n_1-1}}, V^{n_1}/V^{n_1-1}\right) \to 0$$

splits. Then the following filtration $0 = U^0 \subset U^1 \subset U^2 \subset \cdots \subset U^\ell = V$

$$U^i = \begin{cases} 0 & i = 0 \\ W^1 \oplus U^{i-1} & 1 \leqslant i \leqslant n_1 \\ V^i & i > n_1 \end{cases}$$

is a Jordan–Hölder filtration of (B, V).

By the non-zero homomorphism $W^1 \to V^{n_1} \twoheadrightarrow V^{n_1}/V^{n_1-1}$, it can be shown that the above short exact sequence splits as quiver representations. That is

$$\left(B|_{V^{n_1}}, V^{n_1}\right) \simeq \left(B|_{V^{n_1-1}}, V^{n_1-1}\right) \oplus \left(B|_{W^1}, W^1\right).$$

We see that

$$\bigoplus_{i=1}^{\ell} U^i/U^{i-1} \simeq \bigoplus_{i=1}^{\ell} V^i/V^{i-1}.$$

Since the filtrations $\left(W^j\right)_{1\leqslant j\leqslant m}$ and $\left(U^i\right)_{1\leqslant i\leqslant \ell}$ have the same first term, so that $\left(W^j/W^1\right)_{2\leqslant j\leqslant m}$ and $\left(U^i/W^1\right)_{2\leqslant i\leqslant \ell}$ induce two Jordan–Hölder filtrations of $\left(B|_{V/W^1}, V/W^1\right)$. By the induction on the length of filtrations, we obtain the isomorphism between graded quotients of the Jordan–Hölder filtrations $\left(W^j/W^1\right)_{2\leqslant j\leqslant m}$ and $\left(U^i/W^1\right)_{2\leqslant i\leqslant \ell}$. Hence we obtain the claim. □

2.4.5 Definition. **(i)** Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$. $\zeta_{\mathbb{R}}$-semistable quiver representations $\left(B^1, V^1\right)$ and $\left(B^2, V^2\right)$ with $\theta_{\zeta_{\mathbb{R}}}(B, V) = \theta$ are said to be S-equivalent [3] if $\operatorname{gr}^{JH}\left(B^1, V^1\right) \simeq \operatorname{gr}^{JH}\left(B^2, V^2\right)$.

(ii) A semistable representation (B, V) is said to be $\zeta_{\mathbb{R}}$-*polystable* if (B, V) is a finite direct sum of $\zeta_{\mathbb{R}}$-stable with same slope, that is we have $(B, V) \simeq \bigoplus_{j=1}^r \left(B^j, V^j\right)$ with $\left(B^j, V^j\right)$ is $\zeta_{\mathbb{R}}$-stable with $\theta_{\zeta_{\mathbb{R}}}\left(B^1, V^1\right) = \cdots = \theta_{\zeta_{\mathbb{R}}}\left(B^r, V^r\right)$.

2.4.6 Remark. **(i)** Let $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and (B, V) a $\zeta_{\mathbb{R}}$-semistable quiver representation. Then $\operatorname{gr}^{JH}(B, V)$ is $\zeta_{\mathbb{R}}$-polystable and (B, V) is S-equivalent to $\operatorname{gr}^{JH}(B, V)$.

(ii) A $\zeta_{\mathbb{R}}$-stable quiver representation $\left(B^1, V^1\right)$ is S-equivalent to $\left(B^2, V^2\right)$ if and only if $\left(B^1, V^1\right)$ is isomorphic to $\left(B^2, V^2\right)$.

2.5 Moduli Space of Quiver Representations

For a Q_0-graded vector space V, we set $\mathbf{E}_Q(V) = \mathbf{E}_Q(V, V)$ and consider the $G(V) = \prod_{i\in Q_0} \mathrm{GL}(V_i)$-action by base change, that is we consider the $G(V)$-action defined as follows:

$$g \cdot B = \left(g_{\mathrm{in}(h)} B_h g_{\mathrm{out}(h)}^{-1}\right)_{h\in Q_1}.$$

The diagonal of non-zero scaler matrices acts trivially on $\mathbf{E}_Q(V)$. So we consider the action of the quotient group $PG(V) = G(V)/\mathbb{G}_m$.

2.5.1 Theorem ([28]). *For $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ and $\boldsymbol{v} \in \mathbb{Z}_{\geqslant 0}^{Q_0}$, we set $R_Q^{\zeta_{\mathbb{R}}-ss}(\boldsymbol{v})$ (resp. $R_Q^{\zeta_{\mathbb{R}}-s}(\boldsymbol{v})$) be the subset which consists of $\zeta_{\mathbb{R}}$-semistable (resp. $\zeta_{\mathbb{R}}$-stable) representations of the quiver with dimension vector $\boldsymbol{v}$.*

(i) *The set of S-equivalence classes $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}}(\boldsymbol{v}) := R_Q^{\zeta_{\mathbb{R}}-ss}(\boldsymbol{v})/\sim_S$ of $\zeta_{\mathbb{R}}$-semistable representations has the structure of (the $\mathbb{C}$-valued point of) quasi-projective scheme over $\mathbb{C}$, where $\sim_S$ is equivalence relation defined by S-equivalence. In fact, $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}}(\boldsymbol{v})$ is a coarse moduli space of $\zeta_{\mathbb{R}}$-semistable representations modulo S-equivalence.*

[3] S stands for Seshadri.

(ii) *Let* $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}-\mathrm{reg}}(\boldsymbol{v}) := R_Q^{\zeta_{\mathbb{R}}-s}(\boldsymbol{v}) / \sim$ *be the subset of* $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}}(\boldsymbol{v})$ *which consists of isomorphism classes of* $\zeta_{\mathbb{R}}$*-stable representations, where* $\sim$ *is the equivalence relation defined by isomorphisms. Then* $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}-\mathrm{reg}}(\boldsymbol{v})$ *is a (possible empty) smooth (Zariski) open subvariety of* $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}}(\boldsymbol{v})$*. In fact,* $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}-\mathrm{reg}}(\boldsymbol{v})$ *is a fine moduli space of* $\zeta_{\mathbb{R}}$*-stable representations with*

$$\dim_{\mathbb{K}} \mathfrak{R}_Q^{\zeta_{\mathbb{R}}-\mathrm{reg}}(\boldsymbol{v}) = 1 - q_Q(\boldsymbol{v}),$$

where $q_Q(\boldsymbol{v})$ *is the Tits form (if it is not empty).*

(iii) *We have a canonical projective morphism:*

$$\pi \colon \mathfrak{R}_Q^{\zeta_{\mathbb{R}}}(\boldsymbol{v}) \to \mathfrak{R}_Q^{0}(\boldsymbol{v})$$

which assigns the semisimplification with respect to the trivial stability 0*, that is we have*

$$\pi\left(\left[\mathrm{gr}_{\zeta_{\mathbb{R}}}^{JH}(B, V)\right]\right) = \left[\mathrm{gr}_{0}^{JH}(B, V)\right].$$

It is known that $\mathfrak{R}_Q^{0}(\boldsymbol{v}) \simeq \operatorname{Spec} \Gamma\left(\mathbf{E}_Q(V), \mathcal{O}_{\mathbf{E}_Q(V)}\right)^{PG(V)}$. If Q is acyclic, that is there exist no oriented cycles, then it is known that $\mathfrak{R}_Q^{0}(\boldsymbol{v}) = \mathrm{pt}$, hence $\mathfrak{R}_Q^{\zeta_{\mathbb{R}}}(\boldsymbol{v})$ is projective.

2.6 Ice Quivers and Framed Moduli

In this section, we introduce framed representations of quivers and its moduli spaces following Nakajima [40, Sect. 3], [47, 1(ii)], see also Reineke [49] and Engel–Reineke [14]. A framed representations can be considered as a representation of "ice" quivers, that is a quiver with some "frozen" vertices.

2.6.1 Definition. An *ice quiver* is a pair $\left(\tilde{Q}, F\right)$ where $\tilde{Q}$ is a (finite) quiver and $F = F_0$ is a subset of Q_0. The subset F is called the set of frozen vertices.

For simplicity, we assume that there are no arrows $h \in \tilde{Q}_1$ with $\mathrm{out}(h), \mathrm{in}(h) \in F_0$. Let Q be the fullsubquiver on $\tilde{Q}_0 \setminus F_0$ and $F_1 := \tilde{Q}_1 \setminus Q_1$. By the assumption on F, we have the following decomposition:

$$\begin{aligned}
F_1 &= F_1^{\mathrm{in}} \sqcup F_1^{\mathrm{out}}, \\
F_1^{\mathrm{in}} &= \{h \in F_1 \mid \mathrm{in}(h) \in Q_0\} = \{h \in F_1 \mid \mathrm{out}(h) \in F_0\}, \\
F_1^{\mathrm{out}} &= \{h \in F_1 \mid \mathrm{out}(h) \in Q_0\} = \{h \in F_1 \mid \mathrm{in}(h) \in F_0\}.
\end{aligned}$$

So we have the decomposition:

$$\mathbf{E}_{(\tilde{Q},F)}(V,W) = \mathbf{E}_Q(V) \oplus \mathbf{L}_{F_1^{\mathrm{in}}}(W,V) \oplus \mathbf{L}_{F_1^{\mathrm{out}}}(V,W),$$
$$\mathbf{L}_{F_1^{\mathrm{in}}}(W,V) = \bigoplus_{h\in F_1^{\mathrm{in}}} \mathrm{Hom}_{\mathbb{C}}\left(W_{\mathrm{out}(h)}, V_{\mathrm{in}(h)}\right),$$
$$\mathbf{L}_{F_1^{\mathrm{out}}}(V,W) = \bigoplus_{h\in F_1^{\mathrm{out}}} \mathrm{Hom}_{\mathbb{C}}\left(V_{\mathrm{out}(h)}, W_{\mathrm{in}(h)}\right).$$

With respect to the above decomposition, we denote by $(B,a,b) \in \mathbf{E}_{(\tilde{Q},F)}(V,W)$.

Following Crawley–Boevey [6, Introduction], we introduce the "deframing" quiver $Q_F(\boldsymbol{w})$ associated with an ice quiver $\left(\tilde{Q},F\right)$ and $\boldsymbol{w} \in \mathbb{Z}_{\geqslant 0}^{F_0}$.

$$Q_F(\boldsymbol{w}) = Q_0 \cup \{\infty\},$$
$$Q_F(\boldsymbol{w}) = Q_1 \cup \left\{h(j) : \infty \to \mathrm{in}(h) \mid h \in F_1^{\mathrm{in}}, 1 \leqslant j \leqslant w_{\mathrm{out}(h)}\right\}$$
$$\cup \left\{h(j) : \mathrm{out}(h) \to \infty \mid h \in F_1^{\mathrm{out}}, 1 \leqslant j \leqslant w_{\mathrm{in}(h)}\right\}.$$

The following proposition can be easily checked.

2.6.2 Proposition. *Let V be a Q_0-graded vector space and $\tilde{V}$ be an associated $Q_0 \cup \{\infty\}$-graded vector space with* $\dim \tilde{V}_\infty = 1$. *For an F_0-graded vector space W with* $\mathbf{dim} W = \boldsymbol{w}$*, fixing a basis of W yields a $G(V)$-equivariant isomorphism:*

$$\mathbf{E}_{(\tilde{Q},F)}(V,W) \simeq \mathbf{E}_{Q_F(\boldsymbol{w})}\left(\tilde{V}\right).$$

Using the above proposition, for $\tilde{\zeta}_{\mathbb{R}} = (\zeta_{\mathbb{R}}, \zeta_\infty) \in \mathbb{R}^{Q_0\cup\{\infty\}}$, we set

$$\tilde{\zeta}_{\mathbb{R}}(V,W) := \langle \zeta_{\mathbb{R}}, V\rangle + \zeta_\infty\left(1 - \delta_{W,0}\right),$$
$$\theta_{\tilde{\zeta}_{\mathbb{R}}}(V,W) := \frac{\tilde{\zeta}_{\mathbb{R}}(V,W)}{1 - \delta_{W,0} + \sum_{i\in Q_0} \dim V_i},$$

where $\delta_{w,0}$ is 1 if $W = 0$ and 0 otherwise, as in the case of (unframed) quiver representations.

2.6.3 Definition. **(i)** A subspace V' of V is called a submodule of (B,a,b) of type out if V' is a B-adapted subspace such that $V'_{\mathrm{out}(h)} \subset \mathrm{Ker}(b_h)$ for $h \in F_1^{\mathrm{out}}$.

(ii) A subspace V' of V is called a submodule of (B,a,b) of type in if V' is a B-adapted subspace such that $V'_{\mathrm{in}(h)} \supset \mathrm{Im}(a_h)$ for $h \in F_1^{\mathrm{in}}$.

In the first case, the subspace V' can be considered as an unframed representation $(B,0,0)|_{(V',0)}$ in $\mathbf{E}_{(\tilde{Q},F)}(V,0)$ and we can consider a framed quotient representation $(B,a,b)|_{(V/V',W)}$ in $\mathbf{E}_{(\tilde{Q},F)}\left(V/V',W\right)$. Similarly, the subspace V' can be considered as a framed representation $(B,a,b)|_{(V',W)}$ in $\mathbf{E}_{(\tilde{Q},F)}\left(V',W\right)$ and $(B,0,0)|_{(V/V',W)} \in \mathbf{E}_{(\tilde{Q},F)}\left(V/V',0\right)$.

2.6.4 Definition. **(i)** A module $(B, a, b) \in \mathbf{E}_{\tilde{Q}}(V, W)$ is said to be $\tilde{\zeta}_{\mathbb{R}}$-*semistable* if we have $\theta_{\tilde{\zeta}_{\mathbb{R}}}\left(V', \delta W\right) \leqslant \theta_{\tilde{\zeta}_{\mathbb{R}}}(V, W)$ for any non-zero submodule $\left(V', \delta W\right)$ of (B, a, b), where we mean δW is either 0 or W.

(ii) A module $(B, a, b) \in \mathbf{E}_{\tilde{Q}}(V, W)$ is said to be $\tilde{\zeta}_{\mathbb{R}}$-*stable* if it is $\tilde{\zeta}_{\mathbb{R}}$-semistable and the strict inequalities hold unless $\left(V', \delta W\right) = (V, W)$.

(iii) A module $(B, a, b) \in \mathbf{E}_{\tilde{Q}}(V, W)$ is said to be $\tilde{\zeta}_{\mathbb{R}}$-*polystable* if it is a direct sum of $\tilde{\zeta}_{\mathbb{R}}$-stable modules with the same slope.

We have the following see-saw property for framed representations

2.6.5 Proposition. *Let* $\left(V', \delta W\right)$ *be a submodule of* (B, a, b) *and* $(V, W)/\left(V', \delta W\right)$ *be its quotient. Then the following conditions are equivalent:*

(i) $\theta_{\tilde{\zeta}_{\mathbb{R}}}\left(V', \delta W\right) \leqslant$ *(resp.* $\geqslant, =$*)* $\theta_{\tilde{\zeta}_{\mathbb{R}}}(V, W)$,

(ii) $\theta_{\tilde{\zeta}_{\mathbb{R}}}(V, W) \leqslant$ *(resp.* $\geqslant, =$*)* $\theta_{\tilde{\zeta}_{\mathbb{R}}}\left((V, W)/\left(V', \delta W\right)\right)$,

(iii) $\theta_{\tilde{\zeta}_{\mathbb{R}}}\left(V', \delta W\right) \leqslant$ *(resp.* $\geqslant, =$*)* $\theta_{\tilde{\zeta}_{\mathbb{R}}}\left((V, W)/\left(V', \delta W\right)\right)$.

For $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$, we set $\zeta_{\mathbb{R},\infty} := -\sum_{i \in Q_0} \zeta_{\mathbb{R},i}$ and consider the corresponding stability condition $\tilde{\zeta}_{\mathbb{R}}$. The definition of $\tilde{\zeta}_{\mathbb{R}}$-semistable (resp. $\tilde{\zeta}_{\mathbb{R}}$-stable) representation can be phrased explicitly in the following way.

2.6.6 Definition. **(i)** A module $(B, a, b) \in \mathbf{E}_{\tilde{Q}}(V, W)$ is said to be $\tilde{\zeta}_{\mathbb{R}}$-*semistable* if the following two conditions are satisfied:

(a) For any B-adapted subspace S of V such that $S_{\mathrm{out}(h)} \subset \mathrm{Ker}\,(b_h)$ for $h \in F_1^{\mathrm{out}}$, we have $\langle \zeta_{\mathbb{R}}, \mathbf{dim} S \rangle \leqslant 0$.

(b) For any B-adapted subspace T of V such that $T_{\mathrm{in}(h)} \supset \mathrm{Im}\,(a_h)$ for $h \in F_1^{\mathrm{in}}$, we have $\langle \zeta_{\mathbb{R}}, \mathbf{dim} T \rangle \leqslant \langle \zeta_{\mathbb{R}}, \mathbf{dim} V \rangle$.

(ii) A $\tilde{\zeta}_{\mathbb{R}}$-semistable module (B, a, b) is said to be $\tilde{\zeta}_{\mathbb{R}}$-*stable* if the strict inequalities hold unless $S = 0$ or $T = V$.

As in unframed case, we can introduce Harder–Narasimhan filtrations, Jordan–Holder filtrations, and S-equivalence.

In the following we restrict ourselves to the special case of the stability parameter $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ with $\zeta_{\mathbb{R},i} > 0$ for $i \in Q_0$. For simplicity, we also assume that Q has no edge loops. We study the $G(V)$-action on $\mathbf{E}_{(\tilde{Q},F)}(V, W)$.

2.6.7 Exercise. For $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ with $\zeta_i > 0$ for $i \in Q_0$. Show that (B, a, b) is $\tilde{\zeta}_{\mathbb{R}}$-semistable if and only if (B, a, b) is $\tilde{\zeta}_{\mathbb{R}}$-stable.

2.6.8 Theorem. *For* $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$ *and* $\boldsymbol{v} \in \mathbb{Z}_{\geqslant}^{Q_0}$, *we set* $R_{(\tilde{Q},F)}^{\tilde{\zeta}_{\mathbb{R}}-ss}(\boldsymbol{v}, \boldsymbol{w})$ *and* $R_{(\tilde{Q},F)}^{\tilde{\zeta}_{\mathbb{R}}-s}(\boldsymbol{v}, \boldsymbol{w})$ *be the subset which consists of* $\tilde{\boldsymbol{\zeta}}$*-semistable (resp.* $\tilde{\boldsymbol{\zeta}}$*-stable) representations of the quiver with dimension vector* $(\boldsymbol{v}, \boldsymbol{w})$ *such that* $R_{(\tilde{Q},F)}^{\tilde{\zeta}_{\mathbb{R}}-ss}(\boldsymbol{v}, \boldsymbol{w}) = R_{(\tilde{Q},F)}^{\tilde{\zeta}_{\mathbb{R}}-s}(\boldsymbol{v}, \boldsymbol{w})$.

(i) *The set of S-equivalence classes* $\mathfrak{R}^{\tilde{\zeta}_{\mathbb{R}}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w}) := R^{\tilde{\zeta}_{\mathbb{R}}-ss}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w}) / \sim_S$ *has the structure of (the* $\mathbb{C}$*-valued point of) quasi-projective scheme over* $\mathbb{C}$*. In fact,* $\mathfrak{R}^{\tilde{\zeta}_{\mathbb{R}}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w})$ *is a fine moduli space of* $\tilde{\zeta}_{\mathbb{R}}$*-stable representations with*

$$\dim_{\mathbb{K}} \mathfrak{R}^{\tilde{\zeta}_{\mathbb{R}}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w}) = \sum_{h\in F^1_{\mathrm{in}(h)}} w_{\mathrm{out}(h)} v_{\mathrm{in}(h)} + \sum_{h\in F^1_{\mathrm{out}(h)}} v_{\mathrm{out}(h)} w_{\mathrm{in}(h)} - q_Q(\boldsymbol{v}) .$$

(ii) *We have a canonical projective morphism:*

$$\pi \colon \mathfrak{R}^{\tilde{\zeta}_{\mathbb{R}}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w}) \to \mathfrak{R}^{\tilde{0}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w}) .$$

If Q is acyclic and $F^1 = F^1_{\mathrm{out}}$, then it is known that $\mathfrak{R}^{\tilde{0}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w}) = \mathrm{pt}$, so it can be shown $\mathfrak{R}^{\tilde{\zeta}_{\mathbb{R}}}_{(\tilde{Q},F)}(\boldsymbol{v},\boldsymbol{w})$ is projective.

3 The Preprojective Algebra of a Quiver

The preprojective algebra is an associative algebra associated with a hereditary algebra, that is an algebra whose global dimension is (less than) one. Gel'fand and Ponomarev [19] introduced the preprojective algebra associated with a path algebra of a tree quiver, that is a quiver whose underlying graph is a tree in order to study the preprojective representations of a tree quiver. Dlab–Ringel [13] extended their construction to the more general setting of a valued graph or a species of modulated graph.

3.1 Reflection Functor for Quiver Representation

Bernšteĭn–Gel'fand–Ponomarev [2] have introduced so-called reflection functors to compare representations of quivers with different orientation, but also they constructed indecomposable representations inductively using the so-called Coxeter functors. In particular, they give a conceptual proof of Gabriel's theorem on the classification of indecomposable representation of quiver representations.

For a quiver Q and a vertex $k \in Q_0$, we set

$$Q_{k,\mathrm{out}} := \{h \in Q_1 \mid \mathrm{out}(h) = k\} ,$$
$$Q_{k,\mathrm{in}} := \{h \in Q_1 \mid \mathrm{in}(h) = k\} .$$

3.1.1 Definition. Let Q be a quiver and (B, V) be a quiver representation over $\mathbb{C}$. For a loop-free vertex $k \in Q_0$, let S_k be the 1-dimensional simple module corresponds to k.

(i) Let $\operatorname{soc}_k(B, V)$ be the sum of all subrepresentation of (B, V) which are isomorphic to S_k. (If there is no such submodule, then let $\operatorname{soc}_k(B, V) = 0$.) It is called *the k-socle* of (B, V).

(ii) Let $\operatorname{top}_k(B, V) = (B, V)/\operatorname{rad}_k(B, V)$, where $\operatorname{rad}_k(B, V)$ is the intersection of all representations of (B, V) whose quotient are isomorphic to S_k. If there is no such subrepresentation, then let $\operatorname{top}_k(B, V) = 0$ and $\operatorname{rad}_k(B, V) = (B, V)$. It is called *the k-top* of (B, V).

3.1.2 Exercise. Let Q be a quiver and (B, V) be a representation of quiver Q over $\mathbb{C}$.

(i) Show that we have

$$(\operatorname{soc}_k(B, V))_i = \begin{cases} 0 & i \neq k, \\ \operatorname{Ker}\left(B_{k,\mathrm{out}}\right) = \bigcap_{h \in Q_{k,\mathrm{out}}} \operatorname{Ker}(B_h) & i = k, \end{cases}$$

where $V_{k,\mathrm{out}} = \bigoplus_{h \in Q_{k,\mathrm{out}}} V_{\mathrm{in}(h)}$, $B_{k,\mathrm{out}} = \oplus_{h \in Q_{k,\mathrm{out}}} B_h \colon V_k \to V_{k,\mathrm{out}}$.

(ii) Show that we have

$$(\operatorname{rad}_k(B, V))_i = \begin{cases} V_i & i \neq k, \\ \operatorname{Im}\left(B_{k,\mathrm{in}}\right) = \sum_{h \in Q_{k,\mathrm{in}}} \operatorname{Im}(B_h) & i = k, \end{cases}$$

where $V_{k,\mathrm{in}} = \bigoplus_{h \in Q_{k,\mathrm{in}}} V_{\mathrm{out}(h)}$, $B_{k,\mathrm{in}} = \sum_{h \in Q_{k,\mathrm{in}}} B_h \colon V_{k,\mathrm{in}} \to V_k$.
In particular, we have

$$\left(\operatorname{top}_k(B, V)\right)_i = \begin{cases} 0 & i \neq k, \\ \operatorname{Coker}\left(B_{k,\mathrm{in}}\right) & i = k. \end{cases}$$

3.1.3 Definition. Let Q be a quiver. For $k \in Q_0$, let $\mathsf{mod}\,(\mathbb{C}Q)_k^+$ (resp. $\mathsf{mod}\,(\mathbb{C}Q)_k^-$) be the full subcategory which consists of representations (B, V) with $\operatorname{Hom}_Q((B, V), S_k) = 0$ (resp. $\operatorname{Hom}_Q(S_k, (B, V)) = 0$).

3.1.4 Proposition. *Let Q be a quiver. For a quiver representations (B, V) and $k \in Q_0$, the following are equivalent:*

(i) $\operatorname{Hom}_Q((B, V), S_k) = 0$ *(resp.* $\operatorname{Hom}_Q(S_k, (B, V)) = 0$*),*

(ii) $\operatorname{top}_k(B, V) = 0$ *(resp.* $\operatorname{soc}_k(B, V) = 0$*).*

3.1.5 Definition. Let $Q = (Q_0, Q_1, \mathrm{out}, \mathrm{in})$ be a quiver. $k \in Q_0$ is called a sink (resp. source) if $Q_{k,\mathrm{out}} = \emptyset$ (resp. $Q_{k,\mathrm{in}} = \emptyset$).

3.1.6 Exercise. Let Q be a finite connected quiver with Q_0 with $n = \#Q_0$. Show that the following are equivalent:

(i) k is a sink (resp. source) in Q_0,
(ii) The simple module S_k is a projective (resp. injective) module.

3.1.1 Definition (**Bernšteĭn–Gel'fand–Ponomarev reflection functors**). Let Q be a quiver. For a vertex $k \in Q_0$, we define a new quiver $\sigma_k Q$ by $(\sigma_k Q)_0 = Q_0$ and

$$(\sigma_k Q)_1 = \{h \in Q_1 \mid k \notin \{\text{out}(h), \text{in}(h)\}\} \cup \left\{h^* \mid k \in \{\text{out}(h), \text{in}(h)\}\right\},$$

where h^* is an arrow with $\text{out}(h^*) = \text{in}(h)$ and $\text{in}(h^*) = \text{out}(h)$.

(i) Let k be a sink in Q. We define a reflection functor $S_k^+ : \mathsf{mod}(\mathbb{C}Q) \to \mathsf{mod}(\mathbb{C}\sigma_k Q)$ between the categories of finite dimensional representation of Q and $\sigma_k Q$ over $\mathbb{C}$ as follows: For a representation (B, V) of Q, we define a representation $S_k^+(B, V) = \left(B', V'\right)$ of $\sigma_k Q$:

(a) Let $V_i' = V_i$ for $i \neq k$ and

$$V_k' = \text{Ker}\left(B_{k,\text{in}} : V_{k,\text{in}} \to V_k\right).$$

(b) Let $B_h' = B_h$ for all arrows $h \in Q_1 \cap (\sigma_k Q)_1$ and

$$B_{h^*} : V_k' = \text{Ker}\left(B_{k,\text{in}} : V_{k,\text{in}} \to V_k\right) \hookrightarrow V_{k,\text{in}} \to V_{\text{out}(h)} = V'_{\text{in}(h^*)},$$

where $V_{k,\text{in}} \to V_{\text{out}(h)}$ is the canonical projection to the direct summand.

For a homomorphism $\varphi = (\varphi_i) : \left(B^1, V^1\right) \to \left(B^2, V^2\right)$ in $\mathsf{mod}(\mathbb{C}Q)$, we define a homomorphism $S_k^+(\varphi) = \varphi' : S_k^+\left(B^1, V^1\right) \to S_k^+\left(B^2, V^2\right)$ in $\mathsf{mod}(\mathbb{C}\sigma_k Q)$ as follows:

(a) For $i \neq k$, we set $\varphi_i' = \varphi_i$;
(b) Let $\varphi_k' : \left(V_k^1\right)' \to \left(V_k^2\right)'$ be the linear map which makes the following diagram commutes:

$$\begin{array}{ccccc}
\left(V_k^1\right)' & \hookrightarrow & V_{k,\text{in}}^1 & \xrightarrow{B_{k,\text{in}}^1} & V_k^1 \\
\downarrow{\scriptstyle \varphi_k'} & & \downarrow{\scriptstyle \bigoplus_{h \in Q_{k,\text{in}}} \varphi_{\text{out}(h)}} & & \downarrow{\scriptstyle \varphi_i} \\
\left(V_k^2\right)' & \hookrightarrow & V_{k,\text{in}}^2 & \xrightarrow{B_{k,\text{in}}^2} & V_k^2
\end{array}$$

(ii) Let $k \in Q_0$ be a source in Q. We define a reflection functor $S_k^- : \mathsf{mod}(\mathbb{C}Q) \to \mathsf{mod}(\mathbb{C}\sigma_k Q)$ between the categories of finite dimensional representation of Q and $\sigma_i Q$ over $\mathbb{C}$ as follows:
For a representation (B, V) of a quiver Q, we define a representation $S_k^-(B, V) = \left(B', V'\right)$ of the quiver $\sigma_k Q$ by

(a) Let $V_i' = V_i$ for $i \neq k$ and we set

$$V_k' = \operatorname{Coker}\left(B_{k,\mathrm{out}} \colon V_k \to V_{k,\mathrm{out}}\right).$$

(b) Let $B_h' = B_h$ for all arrows $h \in Q_1 \cap (\sigma_k Q)_1$ and B_{h^*} be the following composite of the linear maps

$$V'_{\mathrm{out}(h^*)} = V_{\mathrm{in}(h)} \hookrightarrow V_{k,\mathrm{out}} \twoheadrightarrow \operatorname{Coker}\left(B_{k,\mathrm{out}} \colon V_k \to V_{k,\mathrm{out}}\right),$$

where $V_{\mathrm{in}(h)} \hookrightarrow V_{k,\mathrm{out}}$ be the canonical inclusion as a direct summand.

For a homomorphism $\varphi = (\varphi_i) : \left(B^1, V^1\right) \to \left(B^2, V^2\right)$ in $\mathsf{mod}\,(\mathbb{C}Q)$, we define a homomorphism $S_k^-(\varphi) = \varphi' \colon S_k^-\left(B^1, V^1\right) \to S_k^-\left(B^2, V^2\right)$ in $\mathsf{mod}\,(\mathbb{C}\sigma_k Q)$ as follows:

(a) For $i \neq k$, we set $\varphi_i' = \varphi_i$;
(b) Let $\varphi_k' \colon \left(V_k^1\right)' \to \left(V_k^2\right)'$ be the linear map which makes the following diagram commutes:

$$\begin{array}{ccccc}
V_k^1 & \longrightarrow & V_{k,\mathrm{out}}^1 & \longrightarrow\!\!\!\!\!\rightarrow & \left(V_k^1\right)' \\
\downarrow{\scriptstyle \varphi_k} & & \downarrow{\scriptstyle \bigoplus_{h \in Q_{k,\mathrm{out}}} \varphi_{\mathrm{in}}} & & \downarrow{\scriptstyle \varphi_k'} \\
V_k^2 & \longrightarrow & V_{k,\mathrm{out}}^2 & \longrightarrow\!\!\!\!\!\rightarrow & \left(V_k^2\right)'
\end{array}$$

3.1.7 Exercise. Show that above assignments define functors.

The following is clear from the definition of reflection functors and Exercise 3.1.6.

3.1.8 Proposition. *Let Q be a quiver and (B, V) a representation of Q.*

(i) *For a sink $k \in Q_0$ for Q, we have a natural (split) short exact sequence:*

$$0 \to S_k^- S_k^+ (B, V) \to (B, V) \to \mathrm{top}_k\,(B, V) \to 0,$$

where $\varepsilon_k \colon S_k^- S_k^+ (B, V) \to (B, V)$ is the canonical homomorphism defined by $(\varepsilon_k)_i = \mathrm{id}_{V_i}$ for $i \neq k$ and

$$\begin{aligned}(\varepsilon_k)_k : \left(S_k^- S_k^+ (B, V)\right)_k &= \operatorname{Coker}\left(\operatorname{Ker}\left(B_{k,\mathrm{in}} \colon V_{k,\mathrm{in}} \to V_k\right) \hookrightarrow V_{k,\mathrm{in}}\right) \\ &\simeq \operatorname{Im}\left(B_{k,\mathrm{in}} \colon V_{k,\mathrm{in}} \to V_k\right) \hookrightarrow V_k.\end{aligned}$$

(ii) *For a source $k \in Q_0$ for Q, we have a natural (split) short exact sequence:*

$$0 \to \mathrm{soc}_k\,(B, V) \to (B, V) \to S_k^+ S_k^- (B, V) \to 0,$$

where $\varepsilon_k^* \colon (B, V) \to S_k^+ S_k^- (B, V)$ is the canonical homomorphism defined by $\left(\varepsilon_k^*\right)_i = \mathrm{id}_{V_i}$ for $i \neq k$ and

$$\left(\varepsilon_k^*\right)_k : V_k \to \mathrm{Im}\left(B_{k,\mathrm{out}}\right)$$
$$\simeq \mathrm{Ker}\left(V_{k,\mathrm{out}} \twoheadrightarrow \mathrm{Coker}\left(B_{k,\mathrm{out}}\colon V_k \to V_{k,\mathrm{out}}\right)\right) = \left(S_k^+ S_k^- (B, V)\right)_k .$$

In fact, it can be shown that the pair of functors $\left(S_k^-, S_k^+\right)$ is an adjoint pair. Summarizing the above proposition, we obtain the following dichotomy for the application of reflection functors on indecomposable representation.

3.1.9 Corollary. *Let Q be a quiver.*

(i) *For a sink k for Q and (B, V) an indecomposable representation of Q over $\mathbb{C}$. We have the following two possible cases:*

(a) $S_k^+ (B, V) = 0$ *if and only if* $(B, V) \cong S_k$,

(b) $S_k^+ (B, V)$ *is indecomposable if and only if* $S_k^- S_k^+ (B, V) \cong (B, V)$ *and we have*

$$\mathbf{dim}\ S_k^+ (B, V) = \mathbf{dim}\,(B, V) - (\alpha_k, \mathbf{dim}\,(B, V))\,.$$

(ii) *For a source k for Q and (B, V) an indecomposable representation of Q over $\mathbb{C}$. We have the following two possible cases:*

(a) $S_k^- (B, V) = 0$ *if and only if* $(B, V) \cong S_k$,

(b) $S_k^- (B, V)$ *is indecomposable if and only if* $S_k^+ S_k^- (B, V) \cong (B, V)$ *and we have*

$$\mathbf{dim} S_k^- (B, V) = \mathbf{dim}\,(B, V) - (\alpha_k, \mathbf{dim}\,(B, V))\,.$$

As a corollary of the above result, we obtain the following method to construct indecomposable representation.

We study composite of reflection functors associated with acyclic quiver Q.

3.1.10 Definition. Let $r \geqslant 1$ and a sequence $\boldsymbol{k} = (k_1, \cdots, k_r) \in Q_0^r$ is called a sink (resp. source) admissible sequence if k_j is a sink (resp. source) in $\sigma_{k_{j-1}} \cdots \sigma_{k_1} Q$ for $1 \leqslant j \leqslant r$.

3.1.11 Corollary. *Let Q be a finite connected quiver and $\boldsymbol{k} = (k_1, k_2, \cdots, k_\ell) \in Q_0^\ell$ be a sink-admissible sequence (resp. source-admissible sequence).*

(i) *For $1 \leqslant j \leqslant \ell$, let $S_{k_j} \in \mathsf{mod}\left(\mathbb{C}\sigma_{k_{j-1}} \ldots \sigma_{k_1} Q\right)$. Then $S_{k_1}^- \cdots S_{k_{j-1}}^- \left(S_{k_j}\right)$ (resp. $S_{k_1}^+ \cdots S_{k_{j-1}}^+ \left(S_{k_j}\right)$) is either 0 or an indecomposable representation in mod $(\mathbb{C}Q)$.*

(ii) *Let (B, V) be an indecomposable representation of Q over $\mathbb{C}$ with $S_{k_\ell}^+ \cdots S_{k_1}^+$ $(B, V) = 0$ (resp. $S_{k_\ell}^- \cdots S_{k_1}^- (B, V) = 0$). Then we have*

$$(B, V) \cong S_{k_1}^- \cdots S_{k_{j-1}}^- \left(S_{k_j}\right) \quad \textit{(resp.} \quad (B, V) \cong S_{k_1}^+ \cdots S_{k_{j-1}}^+ \left(S_{k_j}\right) \textit{)},$$

for some $1 \leqslant j \leqslant \ell$.

3.1.12 Exercise. Let Q be a quiver with $n = \#Q_0$.

(i) A total order $(i_1, i_2, \cdots, i_n)$ on Q_0 is called sink (resp. source) admissible ordering if $(i_1, i_2, \cdots, i_n) \in Q_0^n$ is a sink (resp. source) admissible sequence. Show that $(i_1, i_2, \cdots, i_n)$ is a sink-admissible sequence if and only $(i_n, i_{n-1}, \ldots, i_1)$ is a source-admissible sequence.

(ii) Show that Q is acyclic, that is there exist no oriented cycles if and only if there exists a sink (resp. source) admissible ordering.

(iii) Let $(i_1, \ldots, i_n)$ be an admissible ordering on Q_0. Show that $\sigma_{i_n} \ldots \sigma_{i_1} Q = \sigma_{i_1} \ldots \sigma_{i_n} Q = Q$.

In the following section, we assume that Q is acyclic.

3.1.13 Definition. Let Q be a connected acyclic quiver. Let $(i_1, \ldots, i_n)$ be an sink admissible order on Q_0. The Coxeter functors with respect to this order is the composites of reflection functors:

$$C^+ := S_{i_n}^+ \ldots S_{i_1}^+ : \mathsf{mod}\,(\mathbb{C}Q) \to \mathsf{mod}\,(\mathbb{C}Q),$$
$$C^- := S_{i_1}^- \ldots S_{i_n}^- : \mathsf{mod}\,(\mathbb{C}Q) \to \mathsf{mod}\,(\mathbb{C}Q).$$

For $r \in \mathbb{Z}$, we define

$$C^r = \begin{cases} \left(C^+\right)^r & r > 0, \\ \mathrm{id} & r = 0, \\ \left(C^-\right)^{-r} & r < 0. \end{cases}$$

3.1.14 Exercise. **(i)** For $1 \leqslant k \leqslant n$, let

$$P_{i_k} = S_{i_1}^- \ldots S_{i_{k-1}}^- \left(S_{i_k}\right),$$
$$I_{i_k} = S_{i_n}^+ \ldots S_{i_{k+1}}^+ \left(S_{i_k}\right).$$

Show that P_{i_k} is a projective indecomposable representation with $\mathrm{top}_{i_k}\left(P_{i_k}\right) = S_{i_k}$ and I_{i_k} is an injective indecomposable representation with $\mathrm{soc}_{i_k}\left(I_{i_k}\right) = S_{i_k}$.

(ii) For an indecomposable representation (B, V), show that $C^+(B, V) = 0$ (resp. $C^-(B, V) = 0$) is equivalent to (B, V) is isomorphic to some P_i (resp. I_i) with $i \in Q_0$.

3.1.15 Definition. Let (B, V) be an indecomposable representation.

(i) (B, V) is called *preprojective* if $(B, V) \simeq C^r P(i)$ for $r \in \mathbb{Z}_{\leqslant 0}$,

(ii) (B, V) is called *preinjective* if $(B, V) \simeq C^r I(i)$ for $r \in \mathbb{Z}_{\geqslant 0}$,

(iii) (B, V) is called *regular* if $C^r(B, V) \neq 0$ for $r \in \mathbb{Z}$.

3.1.16 Exercise. Show that (B, V) is preprojective (resp. preinjective) if and only if $C^r(B, V) = 0$ (resp. $C^{-r}(B, V) = 0$) for some $r > 0$.

3.2 Preprojective Algebra

In this subsection, we introduce the preprojective algebra $\Lambda = \Lambda_Q$ associated with a quiver Q. For a quiver whose underlying graph is a tree, Gel'fand–Ponomarev [19] introduced the preprojective algebra (they called it "model algebra") for the study of preprojective representations of quiver Q. In fact, the aim of Gel'fand and Ponomarev was to construct an algebra Λ with the following properties:

(i) Λ contains $\mathbb{C}Q$ as a subalgebra,
(ii) As a left $\mathbb{C}Q$-module, Λ decomposes as a multiplicity-free direct sum of the all indecomposable preprojective $\mathbb{C}Q$-modules.

Later Dlab–Ringel [13] generalized the construction of preprojective algebra to the setting of species and extended the result. Also the result is also implicitly given in the work by Riedtmann in quiver case. Though there may exist several isomorphism classes of algebras Λ with the above properties. It is called "the" preprojective algebra.

Baer–Geigle–Lenzing [1] proposed another definition of the preprojective algebra as an orbit algebra associated with the inverse τ^- of the Auslander–Reiten translation. By Brenner–Butler and Gabriel's result, if the underlying graph of Q is a tree, the Auslander–Reiten translation can be identified with the Coxeter functors, so Gel'fand–Ponomarev's definition of the preprojective algebra can be identified with the Baer–Geigle–Lenzing construction. For more details, see Ringel [53, Sect. 6] and also Crawley–Boevey [4].

3.2.1 Definition. Let Q be a quiver and $\overline{Q} = \left(Q_0, H = Q_1 \sqcup Q_1^*, \mathrm{out}_{\overline{Q}}, \mathrm{int}_{\overline{Q}}\right)$ be its double which is the quiver obtained by adding opposite arrow $h^*\colon \mathrm{in}\,(h) \to \mathrm{out}\,(h)$ for each arrow $h\colon \mathrm{out}\,(h) \to \mathrm{in}\,(h)$. We extend $*\colon Q_1 \to Q_1^*$ to an involution $*\colon H \to H$ with $*\,(h^*) = h$.

We define a function $\varepsilon_Q\colon H \to \{\pm 1\}$ given by

$$\varepsilon_Q\,(h) = \begin{cases} 1 & h \in Q_1, \\ -1 & h^* \in Q_1 \end{cases}$$

and we consider an element

$$\mu := \sum_{h \in H} \varepsilon_Q\,(h)\, hh^*.$$

The algebra $\Lambda = \Lambda_Q := \mathbb{C}\overline{Q}/\left\langle \mu \right\rangle$ is called the preprojective algebra of Q.

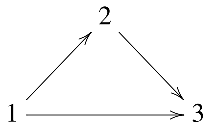
Since $\mathrm{out}\,(h^*) = \mathrm{in}\,(h)$, the algebra Λ_Q is also defined as a factor algebra by the following elements

$$\mu_i = \sum_{h \in H_{\mathrm{in},i}} \varepsilon_Q\,(h)\, hh^* = \sum_{h \in Q_{\mathrm{in},i}} hh^* - \sum_{h \in Q_{\mathrm{out},i}} h^*h.$$

3.2.2 Exercise. **(i)** Show that the isomorphism class of the algebra Λ does not depend on a choice of orientation of Q.

(ii) Let $\mu' = \sum_{h \in \overline{Q_1}} hh^*$ and $\Lambda' = \mathbb{C}\overline{Q}/\langle \mu' \rangle$. Show that $\Lambda \simeq \Lambda'$ if Q is a tree

(iii) Let Q be the following affine quiver of type $A_2^{(1)}$.



Show that Λ and Λ' are not isomorphic.

The following is due to Gel'fand–Ponomarev (see also Dlab–Ringel [13]).

3.2.3 Theorem *Let Q be a quiver whose underlying graph is a tree.*

(i) Λ_Q *is isomorphic to to the multiplicity-free direct sum of all preprojective modules as a left* $\mathbb{C}Q$*-module, that is we have*

$$\Lambda_Q \simeq \bigoplus_{r \geqslant 0} C^{-r}(\mathbb{C}Q)$$

as a left $\mathbb{C}Q$*-module.*

(ii) $\dim_{\mathbb{K}} \Lambda_Q < \infty$ *if and only if Q is a Dynkin quiver.*

3.2.4 Remarks

(i) In fact, we have $\Lambda' = \mathbb{C}\overline{Q}/\langle \mu' \rangle \simeq \bigoplus_{r \geqslant 0} C^{-r}(\mathbb{C}Q)$ in general, see [53, Sect. 6].

(ii) In general, we have

$$\Lambda_Q \simeq \bigoplus_{r \geqslant 0} \tau^{-r}(\mathbb{C}Q)$$

as a left $\mathbb{C}Q$-module, where $\tau^{-1}(B, V) = \mathrm{Ext}^1_{\mathbb{C}Q}(D(\mathbb{C}Q), (B, V))$ is the "inverse" of the Auslander–Reiten translation. For more details, see [53, Theorem A]. The right hand side is the preprojective algebra which has introduced by Baer–Geigle–Lenzing.

(iii) The Coxeter functor and the Auslander–Reiten translation are identified after twisting the sign. For more detail, see Gabriel [17, 5.5].

3.3 2-Calabi–Yau Property and the Crawley–Boevey Formula

The following Ext^1-symmetry of the preprojective algebra is called the 2-Calabi–Yau properties.

3.3.1 Theorem ([18, **Theorem 3**]). *For a quiver Q without loops, let $\Lambda = \Lambda_Q$ be the associated preprojective algebra. For finite dimensional Λ-modules $M^1 = \left(B^1, V^1\right)$ and $M^2 = \left(B^2, V^2\right)$, there is a functorial isomorphism:*

$$\mathrm{Ext}^1_\Lambda\left(M^1, M^2\right) \simeq D\mathrm{Ext}^1\left(M^2, M^1\right),$$

where $D = \mathrm{Hom}_{\mathbb{C}}(-, \mathbb{C})$ be the standard duality for the vector space over $\mathbb{C}$.

The above is proved by the analysis of (the truncation of) the bimodule projective resolution of the preprojective algebra Λ.

3.3.2 Proposition ([18, **Lemma 8.1.1**]). *Let $\Lambda = \Lambda_Q$ be the associated preprojective algebra. Let P be the $\mathbb{C}I$-bimodule spanned by the relations $\{\mu_i\}_{i\in I}$. Then*

$$\Lambda^\bullet\colon \Lambda \otimes_{\mathbb{C}I} P \otimes_{\mathbb{C}I} \Lambda \xrightarrow{d_1} \Lambda \otimes_{\mathbb{C}I} \mathbb{C}H \otimes_{\mathbb{C}I} \Lambda \xrightarrow{d_0} \Lambda \otimes_{\mathbb{C}I} \mathbb{C}I \otimes_{\mathbb{C}I} \Lambda \to \Lambda \to 0$$

is the the beginning of the bimodule projective resolution of the preprojective algebra Λ, where the map $\Lambda \otimes_{\mathbb{C}I} \mathbb{C}I \otimes_{\mathbb{C}I} \Lambda \to \Lambda$ is give by the multiplication in Λ, the map $\Lambda \otimes_{\mathbb{C}I} \mathbb{C}H \otimes_{\mathbb{C}I} \Lambda \xrightarrow{d_0} \Lambda \otimes_{\mathbb{C}I} \mathbb{C}I \otimes_{\mathbb{C}I} \Lambda$ is given by

$$d_0(1 \otimes h \otimes 1) = h \otimes e_{\mathrm{out}(h)} \otimes 1 - 1 \otimes e_{\mathrm{in}(h)} \otimes h$$

for $h \in H$ and the map $\Lambda \otimes_{\mathbb{C}I} P \otimes_{\mathbb{C}I} \Lambda \xrightarrow{d_1} \Lambda \otimes_{\mathbb{C}I} \mathbb{C}H \otimes_{\mathbb{C}I} \Lambda$ is given by

$$d_1(1 \otimes \mu_i \otimes 1) = \sum_{h\in H_{\mathrm{out},i}} \varepsilon(h)\left(h \otimes h^* \otimes 1 + 1 \otimes h \otimes h^*\right).$$

For finite dimensional Λ-modules $M^1 = \left(B^1, V^1\right)$ and $M^2 = \left(B^2, V^2\right)$, applying $\mathrm{Hom}_\Lambda\left(- \otimes_\Lambda M^1, M^2\right)$ to $P^\bullet$, we obtain the following complex:

$$0 \to \mathbf{L}\left(V^1, V^2\right) \xrightarrow{d^0_{M^1,M^2}} \mathbf{E}\left(V^1, V^2\right) \xrightarrow{d^1_{M^1,M^2}} \mathbf{L}\left(V^1, V^2\right),$$

where

$$d^0_{M^1,M^2}\colon (\varphi_i)_{i\in I} \mapsto \left(B^2_h\varphi_{\mathrm{out}(h)} - \varphi_{\mathrm{in}(h)}B^1_h\right)_{h\in H},$$

$$d^1_{M^1,M^2}\colon (C_h)_{h\in H} \mapsto \left(\sum_{h\in H_{\mathrm{out},i}} \varepsilon(h)\left(C_h B^1_{h^*} + B^2_h C_{h^*}\right)\right)_{i\in I}.$$

3.3.3 Exercise. **(i)** Show that $\operatorname{Ker}\left(d^0_{M^1,M^2}\right) = \operatorname{Hom}\left(M^1, M^2\right)$ and

$$\operatorname{Ker}\left(d^1_{M^1,M^2}\right)/\operatorname{Im}\left(d^0_{M^1,M^2}\right) \simeq \operatorname{Ext}^1_\Lambda\left(M^1, M^2\right).$$

(ii) Show that the skew-symmetric bilinear form $\omega_{\mathbf{E}} : \mathbf{E}\left(V^1, V^2\right) \times \mathbf{E}\left(V^1, V^2\right) \to \mathbb{C}$ defined by

$$\omega_{\mathbf{E}}\left(C^1, C^2\right) = \sum_{i\in I} \operatorname{Tr}\left(\sum_{h\in H_{\mathrm{out},i}} \varepsilon(h)\, C^1_h C^2_{h^*}\right)$$

induces a non-degenerate pairing between $\operatorname{Ext}^1_\Lambda\left(M^1, M^2\right)$ and $\operatorname{Ext}^1_\Lambda\left(M^2, M^1\right)$ under the identification $\operatorname{Ker}\left(d^1_{M^1,M^2}\right)/\operatorname{Im}\left(d^0_{M^1,M^2}\right) \simeq \operatorname{Ext}^1_\Lambda\left(M^1, M^2\right)$.

(iii) Show that the bilinear map $\omega_{\mathbf{L}} : \mathbf{L}\left(V^1, V^2\right) \times \mathbf{L}\left(V^2, V^1\right) \to \mathbb{C}$ defined by

$$\omega_{\mathbf{L}}\left(\varphi^1, \varphi^2\right) = \sum_{i\in I} \operatorname{Tr}\left(\varphi^1_i \varphi^2_i\right)$$

induces a non-degenerate pairing between $\operatorname{Coker}\left(d^1_{M^1,M^2}\right)$ and $\operatorname{Hom}_\Lambda\left(M^2, M^1\right)$.

The following formula can be shown easily from the above proposition.

3.3.4 Proposition (Crawley–Boevey [5, Lemma 1.1]). *We have the following formula:*

$$\left(\mathbf{dim}M^1, \mathbf{dim}M^2\right) = \dim \operatorname{Hom}_\Lambda\left(M^1, M^2\right) + \dim \operatorname{Hom}_\Lambda\left(M^2, M^1\right) - \dim \operatorname{Ext}^1_\Lambda\left(M^1, M^2\right).$$

The Crawley–Boevey formula gives a module-theoretic interpretation of the symmetric bilinear form on the root lattice.

3.3.5 Exercise. Assume that Q is non-Dynkin. Show that $P^\bullet$ is the bimodule projective resolution of Λ and

$$\left(\mathbf{dim}M^1, \mathbf{dim}M^2\right) = \dim \operatorname{Hom}_\Lambda\left(M^1, M^2\right) - \dim \operatorname{Ext}^1_\Lambda\left(M^1, M^2\right) + \dim \operatorname{Ext}^2_\Lambda\left(M^1, M^2\right).$$

3.4 McKay Correspodence

In this subsection, we study the McKay correspondence and its relation to the preprojective algebra.

Let Γ be a finite group. Since we work over a field of characteristic 0, the group algebra $\mathbb{C}\Gamma$ of Γ over $\mathbb{C}$ is semisimple. We consider the natural paring

$[\ ,\ ]: K_0(\mathbb{C}\Gamma) \otimes K_0(\mathbb{C}\Gamma) \to \mathbb{Z}$ defined by $[\rho_1, \rho_2] = \dim \mathrm{Hom}_\Gamma(\rho_1, \rho_2)$ for representations ρ_1, ρ_2 of Γ.

3.4.1 Definition (**McKay quiver**). Let ρ be a finite dimensional representation of Γ and I be a set of representatives of isomorphism classes of simple representations of Γ over $\mathbb{C}$. The McKay quiver of (Γ, ρ) is the quiver $\left(\mathrm{Irr}\Gamma, Q_\rho\right)$ with

$$\#\left\{h \in \left(Q_\rho\right)_1 \mid {}^{\mathrm{out}(h)=i,}_{\mathrm{in}(h)=j}\right\} = \dim \mathrm{Hom}\left(\rho_j, \rho \otimes \rho_i\right),$$

where $\mathrm{Irr}\Gamma$ be the set of a representatives of irreducible representation of Γ over $\mathbb{C}$ and $\rho_i, \rho_j \in \mathrm{Irr}\Gamma$.

3.4.2 Exercise. Let Γ be a subgroup of $\mathrm{SL}_2(\mathbb{C}) = \mathrm{Sp}(\rho, \omega)$ where ω is the standard symplectic (volume) form $\omega = dx \wedge dy$ and $\rho\colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be the inclusion. Show that

$$\dim \mathrm{Hom}\left(\rho_j, \rho \otimes \rho_i\right) = \dim \mathrm{Hom}\left(\rho_i, \rho \otimes \rho_j\right)$$

for $i, j \in I$.

Let us consider the symmetric affine Lie algebra of type $A_n^{(1)}, D_n^{(1)}, E_n^{(1)}$ and the corresponding Cartan matrix. Let $0 \in I$ be the vertex corresponds to the negative of the highest root of the corresponding finite dimensional simple Lie algebra of type ADE and $I_0 := I \setminus \{0\}$ be the corresponding subset for the simple Lie algebra. Let $\delta \in \mathbb{Z}_{\geqslant 0}^I$ be the minimal imaginary root, that is the vector in the kernel of the Cartan matrix for the affine Lie algebra whose 0-component is 1. The following is the observation given by McKay.

3.4.3 Theorem (**McKay**). *Let $\rho_0, \rho_1, \cdots, \rho_n$ be the isomorphism classes of irreducible representations of Γ with ρ_0 the trivial representation. Let $\mathbf{C}$ be the virtual representation*

$$\bigwedge^0 \rho - \bigwedge^1 \rho + \bigwedge^2 \rho = 2\rho_0 - \rho$$

and we set the corresponding bilinear form:

$$\left(\rho_i, \rho_j\right) := 2\left[\rho_i \otimes \rho_0\colon \rho_j\right] - \left[\rho_i \otimes \rho\colon \rho_j\right]$$

Then $(\ ,\ )$ is symmetric and the lattice $(K_0(\mathbb{C}\Gamma), (\ ,\ ))$ is isomorphic to the root lattice of an affine root system for $\widehat{\mathfrak{g}}$. We call it the McKay lattice.

Since $(\rho_i, \rho_i) = 2$, we have $(\rho, \rho) \in 2\mathbb{Z}$ for any representation ρ of Γ.

3.4.4 Exercise. Show the following results.

(i) We have $(\rho, \rho) \geqslant 0$ for $\rho \in K_0(\mathbb{C}\Gamma)$,
(ii) $(\rho, \rho) = 0$ if and only if $\rho = n(\mathbb{C}\Gamma)$, where $\mathbb{C}\Gamma = \sum_{i \in \mathrm{Irr}\Gamma} (\dim \rho_i)\, \rho_i \in K_0(\mathbb{C}\Gamma)$ is the (left) regular representation.

In the McKay lattice, the set $\{\rho_i\}_{i\in I}$ is identified with the set of simple roots of the affine root system for $\hat{\mathfrak{g}}$, and it is known that the set $\{\boldsymbol{v} \in K_0(\mathbb{C}\Gamma) \mid (\boldsymbol{v}, \boldsymbol{v}) = 2\}$ is identified with the set of real roots for $\hat{\mathfrak{g}}$ and $\{\boldsymbol{v} \in K_0(\mathbb{C}\Gamma) \mid (\boldsymbol{v}, \boldsymbol{v}) = 0\} \setminus \{\mathbf{0}\}$ is identified with the set $\{n\delta \mid n \in \mathbb{Z} \setminus \{0\}\}$ of imaginary roots of $\hat{\mathfrak{g}}$.

3.4.5 Examples. We consider the case when Γ is of type A_n, that is we consider the following subgroup

$$\Gamma = \mathbb{Z}/(n+1)\mathbb{Z} = \left\{ \begin{pmatrix} \gamma^k & 0 \\ 0 & \gamma^{-k} \end{pmatrix} \middle| 0 \leqslant k \leqslant n \right\},$$

where $\gamma = \exp\left(\dfrac{2\pi\sqrt{-1}}{n+1}\right)$. The irreducible representations ρ_k of Γ are given by the following group homomorphism defined by

$$\rho_k \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} = \gamma^k \quad (0 \leqslant k \leqslant n)$$

and we have $\rho = \rho_1 \oplus \rho_n$. Then we obtain $\rho \otimes \rho_k = \rho_{k-1} \oplus \rho_{k+1}$ where suffix is understood in $\mathbb{Z}/(n+1)\mathbb{Z}$. Hence we obtain the Cartan matrix of type $A_n^{(1)}$.

We study the skew group algebra (or smash product) of path algebras and preprojective algebras for finite groups following Demonet [10].

3.4.6 Definition. Let $\mathbb{C}$ be the field of complex numbers and Γ be a finite group. Let A be a $\mathbb{C}$-algebra and we consider a Γ-action on A, that is a group homomorphism $\Gamma \to \mathrm{Aut}_{\mathrm{alg}/\mathbb{C}}(A)$. We define a skew group algebra $A\#\Gamma$ by $A \otimes_{\mathbb{C}} \mathbb{C}\Gamma$ as $\mathbb{C}$-vector space and contains A and $\mathbb{C}\Gamma$ as $\mathbb{C}$-subalgebras with the following relations:

$$\left(a^1 \otimes \gamma^1\right)\left(a^2 \otimes \gamma^2\right) := a^1\gamma^1\left(a^2\right) \otimes \gamma^1\gamma^2,$$

where $\gamma^1\left(a^2\right)$ be the image of a^2 under the action of γ^1.

3.4.7 Exercise. **(i)** Let $e_{\mathrm{triv}} = \dfrac{1}{|\Gamma|}\sum_{\gamma\in\Gamma} \gamma \in \mathbb{C}\Gamma$. Then show that $e_{\mathrm{triv}}^2 = e_{\mathrm{triv}}$.

(ii) We consider the algebra embedding $\mathbb{C}\Gamma \hookrightarrow A\#\Gamma$ given by $\gamma \mapsto 1 \otimes \gamma$. The subspace $e_{\mathrm{triv}}(A\#\Gamma)e_{\mathrm{triv}}$ is called the spherical subalgebra (with a unit e_{triv}). Show that the map $a \mapsto ae_{\mathrm{triv}}$ gives an isomorphism $A^\Gamma = \{a \in A \mid \gamma(a) = a\} \xrightarrow{\sim} e_{\mathrm{triv}}(A\#\Gamma)e_{\mathrm{triv}}$ as unital algebras.

(iii) Show that $A^\Gamma \simeq Z(A\#\Gamma)$, where $Z(A\#\Gamma)$ is the center of $A\#\Gamma$.

We study the finite group actions on special classes of algebras, path algebras and preprojective algebras. We put an assumption on an action of Γ on the algebras.

3.4.8 Assumption. Let $Q = (Q_0, Q_1)$ be a quiver. Consider an action of Γ which permutes the set of primitive idempotents $\{e_i \mid i \in Q_0\}$ and stabilizes the vector spaces $\mathbb{C}Q_1$ spanned by edges Q_1, that is we have a pair of actions $\sigma = (\sigma_0, \sigma_1)$ where

$\sigma_0 \colon \Gamma \to \mathrm{Aut}_{\mathrm{Set}}(Q_0)$ and $\sigma_1 \colon \Gamma \to \mathrm{GL}(\mathbb{C}Q_1)$ with $\sigma_1(h) = e_{\sigma_0(\mathrm{in}(h))}\sigma_1(h)\, e_{\sigma_0(\mathrm{out}(h))}$ for $h \in Q_1$.

We note that, by [50, Proposition 2.1], without loss of generality, we can assume this assumption if A is finite dimensional. This assumption is general than considering a Γ-action which compatibly permutes Q_0 and Q_1.

Following Demonet [10], we introduce a generalized McKay quiver $Q(\Gamma, \sigma)$ associated with the group action $\sigma \colon \Gamma \to \mathrm{Aut}(\mathbb{C}Q)$.

Let $\tilde{I}$ be a set of representatives of classes of Q_0 under the group action $\rho_0 \colon \Gamma \to \mathrm{Aut}(Q_0)$. For $i \in Q_0$, let $\Gamma_i = \mathrm{Stab}_\Gamma(i)$ be the stabilizer of Γ which fixes e_i. Let $\mathcal{O}_{\bar{\imath}} = \mathcal{O}_i$ be the orbit $\bar{\imath}$ of $i \in Q_0$ under the Γ-action. For $(\bar{\imath}, \bar{\jmath}) \in \tilde{I}^2$, we we consider the diagonal Γ-action on $\mathcal{O}_{\bar{\imath}} \times \mathcal{O}_{\bar{\jmath}}$. Let $\mathcal{F}_{\bar{\imath},\bar{\jmath}}$ be a set of representatives of the classes of this action. For $i, j \in Q_0$, we denote $\mathbb{C}Q_1(i, j)$ be the vector space spanned by the $Q_1(i, j) = \{h \in Q_1 \mid \mathrm{out}(h) = i, \mathrm{in}(h) = j\}$. We consider $\Gamma_i \times \Gamma_j$ module structure on $\mathbb{C}Q_1(i, j)$. For $i' \in \mathcal{O}_{\bar{\imath}}$, we choose $\gamma_{i'} \in \Gamma$ such that $\sigma_0(\gamma_{i'})\left(i'\right) = i$. We note that $\Gamma_{i'} = \gamma_{i'}^{-1}\Gamma_i\gamma_{i'}$.

3.4.9 Definition (**Generalized McKay quiver**). Let $Q(\Gamma, \sigma)$ be the quiver defined by

$$Q(\Gamma, \rho)_0 = \left\{(\bar{\imath}, \rho) \mid \bar{\imath} \in \tilde{I}, \rho \in \mathrm{Irr}(\Gamma_i)\right\},$$

where $\mathrm{Irr}(\Gamma_i)$ is a set of representatives of isomorphism classes of irreducible representations of Γ_i and $Q(\Gamma, \sigma)_1$ be a basis of

$$\bigoplus_{(i',j') \in F_{\bar{\imath},\bar{\jmath}}} \mathrm{Hom}_{\Gamma_i \cap \Gamma_j}\left(\left(\rho^1\right)^{\gamma_{i'}}|_{\Gamma_i \cap \Gamma_j}, \left(\rho^2\right)^{\gamma_{j'}}|_{\Gamma_i \cap \Gamma_j} \otimes_{\mathbb{C}} \mathbb{C}Q_1(i, j)\right)$$

for $\left(i, \rho^1\right), \left(j, \rho^2\right) \in Q(\Gamma, \sigma)_0$, where $\left(\rho^1\right)^{\gamma_{i'}}(\gamma) = \rho^1\left(\gamma_{i'}\gamma\gamma_{i'}^{-1}\right)$ (resp. $\left(\rho^2\right)^{\gamma_{j'}}(\gamma) = \rho^2\left(\gamma_{j'}\gamma\gamma_{j'}^{-1}\right)$) for $\gamma \in \Gamma_{i'} = \gamma_{i'}^{-1}\Gamma_i\gamma_{i'}$ (resp. $\gamma \in \Gamma_{j'} = \gamma_{j'}^{-1}\Gamma_j\gamma_{j'}$). We note that this does not depend on the choice of $\gamma_{i'}$ and $\gamma_{j'}$.

3.4.10 Examples. Let Q be a quiver which consists of a single vertex and m edge-loops. By the assumption, we consider a group homomorphism $\rho \colon \Gamma \to \mathbb{C}^m$. Then the quiver $Q(\Gamma, \rho)$ is the (usual) McKay quiver.

The following is proved by Reiten–Riedtmann [50] in cyclic group case and by Demonet [10] in general.

3.4.11 Theorem (**Demonet** [10, **Theorem 1]**). *We have an equivalence of categories:*

$$\mathsf{mod}(\mathbb{C}Q(\Gamma, \sigma)) \simeq \mathsf{mod}(\mathbb{C}Q\#\Gamma).$$

Let us choose $\tilde{I}$ a set of representatives of Γ-orbits of Q_0 and let $e_{\tilde{I}} = \sum_{i \in \tilde{I}} e_i \in \mathbb{C}Q$. Then this idempotent. Then $(\mathbb{C}Q_0)\#\Gamma$ is Morita equivalent to $e_{\tilde{I}}\left((\mathbb{C}Q_0)\#\Gamma\right)e_{\tilde{I}}$. We have

$$e_{\tilde{I}}\left(\left(\mathbb{C}Q_0\right)\#\Gamma\right)e_{\tilde{I}} \simeq \prod_{i\in\tilde{I}}\mathbb{C}\left[\Gamma_i\right].$$

So let us choose a primitive idempotent $\tilde{e}_{i\rho} \in \mathbb{C}\left[\Gamma_i\right]$ associated with $\rho \in \mathrm{Irr}\Gamma_i$, that is $\rho \simeq \mathbb{C}\left[\Gamma_i\right]\tilde{e}_{i\rho}$. Let us consider the following idempotent

$$\tilde{e} = \sum_{i\in\tilde{I}}\sum_{\rho\in\mathrm{Irr}(\Gamma_i)}\tilde{e}_{i\rho}.$$

So we have $e_{\tilde{I}}\tilde{e}e_{\tilde{I}} = \tilde{e}$ and it induces a Morita equivalence between $e_{\tilde{I}}\left(\mathbb{C}Q_0\#\Gamma\right)e_{\tilde{I}}$ and

$$\tilde{e}e_{\tilde{I}}\left(\mathbb{C}Q_0\#\Gamma\right)e_{\tilde{I}}\tilde{e} = \tilde{e}\left(\mathbb{C}Q_0\#\Gamma\right)\tilde{e} = \mathbb{C}Q\left(\Gamma,\sigma\right)_0.$$

The computation of $\tilde{e}\left(\mathbb{C}Q_1\#\Gamma\right)\tilde{e}$ yields a Morita equivalence between $\mathbb{C}Q\left(\Gamma,\sigma\right) = \tilde{e}\left(\mathbb{C}Q\#\Gamma\right)\tilde{e}$ and $\mathbb{C}Q\#\Gamma$.

3.4.12 Theorem (Demonet [10, Theorem 2]). *Let $\rho\colon \Gamma \to \mathbb{C}\overline{Q}$ be an action which permutes Q_0 and stabilizes the linear subspace $\mathbb{C}\overline{Q}_1$. If $\rho(\gamma)(\mu) = \mu$ for all $\gamma \in \Gamma$, then $Q(\Gamma,\rho) \simeq \overline{Q'}$ for some quiver Q' and we have a Morita equivalence*

$$\mathsf{mod}\left(\Lambda_Q\#\Gamma\right) \simeq \mathsf{mod}\left(\Lambda_{Q'}\right),$$

in fact we have $\tilde{e}\left(\Lambda_Q\#\Gamma\right)\tilde{e} \simeq \Lambda_{Q'}$.

In the special case where Q is the quiver with a single vertex with 2 edge loops, then Q' is an affine quiver and we obtain the algebraic McKay correspondence due to Lenzing, Reiten–van den Bergh [51] and Crawley–Boevey–Holland [9, Theorem 3.4].

3.5 Moment Map and Hamiltonian Reduction

In this section, we study an algebraic (holomorphic) Hamiltonian reduction for representations of reductive groups. For more details, see Cassens–Slodowy [3, Sect. 3], Ginzburg [20, Sect. 4] (see also Yamakawa [56, Sect. 2] in this volume for more details).

3.5.1 Definition. Let $\mathbf{M}$ be a smooth algebraic variety (or complex manifold). An algebraic (holomorphic) 2-form $\omega \in \Omega^2(\mathbf{M})$ is called a symplectic form, if

(i) ω is closed 2-form, that is $d\omega = 0$,
(ii) ω is non-degenerate on $T_p\mathbf{M}$ for $p \in \mathbf{M}$.

A pair $(\mathbf{M},\omega)$ with above ω is called an algebraic (holomorphic) symplectic variety (manifold).

3.5.2 Examples. Let $\mathbf{N}$ be a smooth algebraic variety (or complex manifold). Then the cotangent bundle $\mathbf{M} = T^*\mathbf{N}$ has canonical 1-form α and canonical 2-form $\omega = -d\alpha$.

Let $\mathbf{M}$ be a smooth algebraic variety with a regular G-action. Let $\mathfrak{g}$ be the Lie algebra of G. For the regular G-action, we have an "infinitesimal" $\mathfrak{g}$-action $\mathfrak{g} \to \Gamma(\mathbf{M}, T\mathbf{M})$, $\xi \mapsto \xi_{\mathbf{M}}$

Let G be a connected reductive group, $\mathfrak{g} = \operatorname{Lie} G$ be its Lie algebra. Let $\mathbf{N}$ be a finite dimensional vector space, then it can be shown that $T^*\mathbf{N} \simeq \mathbf{N} \times \mathbf{N}^*$ and the canonical symplectic form ω is given by

$$\omega\left(\left(n^1, \nu^1\right), \left(n^2, \nu^2\right)\right) = \left\langle \nu^2, n^1 \right\rangle - \left\langle \nu^1, n^2 \right\rangle.$$

3.5.3 Proposition. *Let G be an algebraic group and $\mathbf{N}$ be a representation of G.*

(i) *The cotangent lift of the G-action on $\mathbf{N}$ is algebraic symplectic with respect to the canonical 2-form ω on $\mathbf{M}$.*

(ii) *The action has an algebraic moment map $\boldsymbol{\mu}\colon \mathbf{M} \to \mathfrak{g}^*$, that is a G-equivariant algebraic morphism given by*

$$\langle \boldsymbol{\mu}(n, \nu), \xi \rangle = \left(i_{\xi_{\mathbf{M}}} \alpha\right)_{(n,\nu)},$$

where the infinitesimal action $\xi_{\mathbf{M}} \in \Gamma(\mathbf{M}, \Theta)$ for $\xi \in \mathfrak{g}$ is defined by

$$(\xi_{\mathbf{M}})(f) = \left.\frac{d}{dt}\right|_{t=0} (\exp(-t\xi) \cdot f)$$

for $f \in \mathcal{O}_{\mathbf{M}}$.

(iii) *Fix $\xi \in (\mathfrak{g}^*)^G$. A point $(n, \nu) \in \boldsymbol{\mu}^{-1}(\xi)$ is a regular point of $\boldsymbol{\mu}^{-1}(\xi)$ if and only if G-action on (n, ν) is locally trivial, that is the stabilizer $G_{(n,\nu)}$ is a finite group.*

(iv) *If $(n, \nu) \in M$ is a smooth point, then there exists a canonical symplectic form on the vector space $T_p\left(\boldsymbol{\mu}^{-1}(\xi)\right)/\mathfrak{g}$ induced by the non-degenerate bilinear form $\omega_{(n,\nu)}$ on $T_{(n,\nu)}M$.*

Proof **(iii)** For $(n, \nu) \in \mathbf{M} = T^*\mathbf{N} = \mathbf{N} \oplus \mathbf{N}^*$, let $G_{(n,\nu)} = \operatorname{Stab}_G(n, \nu) \subset G$ be the stabilizer of G at (n, ν). For $\xi \in \mathfrak{g}$, $(\xi_{\mathbf{M}})_{(n,\nu)} \in T_{(n,\nu)}\mathbf{M}$ be the tangent vector for the generating vector field $\xi_{\mathbf{M}}$ on $\mathbf{M}$. We also regard $\xi \in \mathfrak{g}$ as a linear function on $\mathfrak{g}^*$. By the moment map equation, we have

$$\begin{aligned}\left\langle d_{(n,\nu)}\boldsymbol{\mu}(v), \xi \right\rangle &= d_{(n,\nu)}(\langle \boldsymbol{\mu}(v), \xi \rangle) \\ &= \omega\left((\xi_{\mathbf{M}})_{(n,\nu)}, v\right).\end{aligned}$$

By the moment map equation, we obtain $G_{(n,\nu)}$ is a finite group if and only if $\operatorname{Lie} G_{(n,\nu)} = 0$ if and only $(\xi_{\mathbf{M}})_{(n,\nu)} \neq 0$ for any $\xi \neq 0$. Since ω is non-degenerate, it is equivalent to non-existence of non-zero ξ such that $\left\langle d_{(n,\nu)}\boldsymbol{\mu}(v), \xi\right\rangle = 0$. Hence $d_{(n,\nu)}\boldsymbol{\mu}$ is surjective, so (n, ν) is a smooth point.

(iv) It is clear from the moment map equation. □

Let Q be a (finite) quiver (without edge loops). The reductive algebraic group $G(V) = \prod_{i \in Q_0} \mathrm{GL}(V_i)$ acts on $\mathbf{E}_Q(V, V)$ by

$$(g, x) \mapsto \left(g_{\mathrm{in}(h)} x g_{\mathrm{out}(h)}^{-1}\right),$$

so the infinitesimal action of the associated Lie algebra $\mathfrak{g}(V) = \bigoplus_{i \in Q_0} \mathfrak{gl}(V_i)$ on $\mathbf{E}_Q(V, V)$ is given by

$$(\xi, x) \mapsto \left(\xi_{\mathrm{in}(h)} x_h - x_h \xi_{\mathrm{out}(h)}\right).$$

Let ω be the $G(V)$-invariant symplectic form on

$$\mathbf{E}(V) = \mathbf{E}_Q(V, V) \oplus \mathbf{E}_{Q^{\mathrm{op}}}(V, V)$$

is defined by

$$\omega\left(B^1, B^2\right) = \sum_{h \in H} \varepsilon(h)\, B_h^1 B_{h^*}^2.$$

If we identify the dual of the Lie algebra $\mathfrak{g}(V)$ with $\mathfrak{g}(V)$ by the trace form on $\mathfrak{g}(V)$, the moment map associated with the above $G(V)$-action on the symplectic variety $(\mathbf{E}(V), \omega)$ vanishing at the origin is given by $\mu = (\mu_i)_{i \in I} : \mathbf{E}(V) \to \mathfrak{g}(V)$, where

$$\begin{aligned}\mu_i(B) &= \sum_{h \in H_{\mathrm{in},i}} \varepsilon(h)\, B_h B_{h^*} \\ &= \sum_{h \in Q_{\mathrm{in},i}} B_h B_{h^*} - \sum_{h \in Q_{\mathrm{out},i}} B_h B_{h^*}\end{aligned}$$

3.6 Quiver Varieties

In this subsection, we introduce quiver varieties using algebraic Hamiltonian reduction.

For Q_0-graded vector spaces V, W, let

$$\begin{aligned}\mathbf{M}(V, W) &= \mathbf{E}_{\overline{Q}}(V, V) \oplus \mathbf{L}(W, V) \oplus \mathbf{L}(V, W) \\ &= \mathbf{E}_Q(V, V) \oplus \mathbf{E}_{Q^{\mathrm{op}}}(V, V) \oplus \mathbf{L}(W, V) \oplus \mathbf{L}(V, W).\end{aligned}$$

The components of an element of $\mathbf{M}(V, W)$ will be denoted by (B, a, b) or (B, C, a, b) respectively. Since $\mathbf{M}(V, W)$ is identified with the cotangent bundle of $\mathbf{E}_Q(V, V) \oplus \mathbf{L}(W, V)$, so it has a natural holomorphic symplectic form given by

$$\omega\left(\left(B^1, a^1, b^1\right), \left(B^2, a^2, b^2\right)\right) = \operatorname{tr}\left(\varepsilon B^1 B^2\right) + \operatorname{tr}\left(a^1 b^2 - a^2 b^1\right),$$

where $\left(B^1, a^1, b^1\right) \in \mathbf{M}(V, W)$, $\left(B^2, a^2, b^2\right) \in \mathbf{M}(V, W)$ and

$$\begin{aligned} \varepsilon B^1 B^2 &= \sum_{h \in \overline{Q}_{i,\mathrm{in}}} \varepsilon(h) B^1_h B^2_{\overline{h}} \in \mathbf{L}\left(V^1, V^2\right) \\ a^1 b^2 &= \sum_{i \in Q_0} a^1_i b^2_i \in \mathbf{L}\left(V^1, V^2\right), \\ a^2 b^1 &= \sum_{i \in Q_0} a^2_i b^1_i \in \mathbf{L}\left(V^1, V^2\right) \end{aligned}$$

Let $G(V) = \prod_{i \in Q_0} \mathrm{GL}(V_i)$ be the product of general linear group and it acts on $\mathbf{M}(V, W)$ by

$$g \cdot (B, a, b) = \left(gBg^{-1}, ga, bg^{-1}\right)$$

and it can be checked that $G(V)$ preserves the symplectic form ω.

For $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ with $\zeta_{\mathbb{C}} \in \mathbb{C}^{Q_0}$ and $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$, we introduce quiver varieties $\mathfrak{M}_\zeta(\boldsymbol{v}, \boldsymbol{w})$ and $\mathfrak{M}^{\mathrm{reg}}_\zeta(\boldsymbol{v}, \boldsymbol{w})$.

For $(B, a, b) \in \mu^{-1}(\zeta_{\mathbb{C}})$, we can consider the following complex:

$$\mathbf{L}(V, V) \xrightarrow{\iota} \mathbf{E}(V, V) \oplus \mathbf{L}(W, V) \oplus \mathbf{L}(V, W) \xrightarrow{d\mu} \mathbf{L}(V, V)$$

where

$$\begin{aligned} \iota(\xi) &= (\xi B - B\xi, \xi a, -b\xi) \\ d\mu(C, c, d) &= \varepsilon C B + \varepsilon B C + cb + ad \end{aligned}$$

The following can be shown using the geometric invariant theory and the Hamiltonian reduction.

3.6.1 Proposition. *Let (B, a, b) be $\boldsymbol{\zeta}_{\mathbb{R}}$-stable representation. Then we have following:*

- **(i)** *the stabilizer of (B, a, b) is trivial,*
- **(ii)** *the differential $d\mu$ of μ at (B, a, b) is surjective.*

Proof **(i)** By the geometric invariant theory, it can be shown that stabilizer of a stable representation is finite. It is also well-known that the stabilizer of a quiver representation is connected, so we obtain that the stabilizer of (B, a, b) is trivial.

(ii) By Proposition 3.5.3 (3), a point (B, a, b) is a regular for $\boldsymbol{\mu}$ if and only if the stabilizer of (B, a, b) in G is finite. So we obtain the claim by the assumption that (B, a, b) is $\boldsymbol{\zeta}_{\mathbb{R}}$-stable. □

3.6.2 Theorem. *Let Q be a finite quiver without edge loops. For $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ with $\zeta_{\mathbb{C}} \in \mathbb{C}^{Q_0}$ and $\zeta_{\mathbb{R}} \in \mathbb{R}^{Q_0}$, we set $\mathfrak{M}_{\zeta}(\boldsymbol{v}, \boldsymbol{w})$ (resp. $\mathfrak{M}^{\mathrm{reg}}_{\zeta}(\boldsymbol{v}, \boldsymbol{w})$) be the set which consists of $\zeta_{\mathbb{R}}$-semistable (resp. $\zeta_{\mathbb{R}}$-stable) representations of the quiver satisfying $\mu(B, a, b) = \mu_{\mathbb{C}}$ for dimension vector $(\boldsymbol{v}, \boldsymbol{w})$, that is*

$$\mathfrak{M}_{\zeta}(\boldsymbol{v}, \boldsymbol{w}) = \left(\mu^{-1}(\zeta_{\mathbb{C}}) \cap \mathbf{M}^{\zeta_{\mathbb{R}}-sst}(\boldsymbol{v}, \boldsymbol{w})\right) // G(V),$$
$$\mathfrak{M}^{\mathrm{reg}}_{\zeta}(\boldsymbol{v}, \boldsymbol{w}) = \left(\mu^{-1}(\zeta_{\mathbb{C}}) \cap \mathbf{M}^{\zeta_{\mathbb{R}}-st}(\boldsymbol{v}, \boldsymbol{w})\right) / G(V),$$

where $\mathbf{M}^{\zeta_{\mathbb{R}}-sst}(\boldsymbol{v}, \boldsymbol{w})$ (resp. $\mathbf{M}^{\zeta_{\mathbb{R}}-st}(\boldsymbol{v}, \boldsymbol{w})$) is the subset of $\mathbf{M}(\boldsymbol{v}, \boldsymbol{w})$ which consists of $\zeta_{\mathbb{R}}$-semistable (resp. $\zeta_{\mathbb{R}}$-stable) representations and // means the S-equivalence defined by Jordan–Hölder filtrations of the $\zeta_{\mathbb{R}}$-semistable representations.

There is a canonical projective morphism $\mathfrak{M}_{\zeta}(\boldsymbol{v}, \boldsymbol{w}) \to \mathfrak{M}_{(0,\zeta_{\mathbb{C}})}(\boldsymbol{v}, \boldsymbol{w})$ which assigns the graded quotient of a Jordan–Hölder filtration with respect to 0-semistabity.

We study the sufficient condition for $\mathfrak{M}^{\mathrm{reg}}_{\zeta}(\boldsymbol{v}, \boldsymbol{w}) = \mathfrak{M}_{\zeta}(\boldsymbol{v}, \boldsymbol{w})$.

For a quiver Q, let

$$\Delta' := \left\{\beta \in \mathbb{Z}^{Q_0} \setminus \{0\} \middle| q_Q(\beta) \leqslant 1\right\}.$$

and $\Delta'_+ := \Delta' \cap \mathbb{Z}^{Q_0}_{\geqslant 0}$, then it is well known that Δ' coincides with the set of roots for the corresponding Cartan matrix of Dynkin or affine type.

For a fixed $\boldsymbol{v} \in \mathbb{Z}_{\geqslant 0}$, let

$$\Delta'_+(\boldsymbol{v}) := \left\{\beta \in \Delta'_+ \middle| \beta \leqslant \boldsymbol{v}\right\}.$$

3.6.3 Definition. For a given dimension vector $\boldsymbol{v} \in \mathbb{Z}^I_{\geqslant 0}$, $\boldsymbol{\zeta} = \left(\boldsymbol{\zeta}_{\mathbb{R}}, \boldsymbol{\zeta}_{\mathbb{C}}\right) \in (\mathbb{R} \oplus \mathbb{C})^{Q_0}$ is said to be $\boldsymbol{v}$-generic (or $\boldsymbol{v}$-regular) if

$$\boldsymbol{\zeta} \in (\mathbb{R} \oplus \mathbb{C})^{Q_0} \setminus \bigcup_{\theta \in \Delta'_+(\boldsymbol{v})} (\mathbb{R} \oplus \mathbb{C}) \otimes \beta^{\perp},$$

where $\beta^{\perp} = \left\{x = (x_i)_{i \in I} \in \mathbb{R}^I \middle| \sum_{i \in I} x_i \beta_i = 0\right\}$ for $\beta = \sum \beta_i \alpha_i \in \Delta'_+(\boldsymbol{v})$.

3.6.4 Examples. It can be checked that $(\zeta_{\mathbb{R}}, 0)$ is $\boldsymbol{v}$-generic for all $\boldsymbol{v} \in \mathbb{Z}^I_{\geqslant 0}$ if $\zeta_{\mathbb{R},i} = 1$ for $i \in I$.

3.6.5 Theorem (Demonet [39, Theorem 2]). *Let $\boldsymbol{\zeta} = \left(\boldsymbol{\zeta}_{\mathbb{R}}, \boldsymbol{\zeta}_{\mathbb{C}}\right)$ be $\boldsymbol{v}$-generic. Then the regular locus $\mathfrak{M}^{\mathrm{reg}}(\boldsymbol{v}, \boldsymbol{w})$ coincides with $\mathfrak{M}(\boldsymbol{v}, \boldsymbol{w})$. In fact, $\mathfrak{M}(\boldsymbol{v}, \boldsymbol{w})$ is smooth and it has a canonical symplectic structure induced by the Hamiltonian reduction with*

$$\dim \mathfrak{M}(\boldsymbol{v}, \boldsymbol{w}) = 2 \sum_{i \in Q_0} v_i w_i - 2 q_Q(\boldsymbol{v}).$$

Crawley–Boevey [6–8] studied connectedness and the normality and the irreducibility of the quiver varieties.

3.7 Nilpotent Orbit and Springer Resolution

In this subsection, we study an example of the quiver varieties of type A_n in special cases. For more general quiver varieties of type A, Nakajima [39, Conjecture 8.6] conjectured that it is isomorphic to intersection of a Slodowy slice and a nilpotent orbit and Maffei [36] proved the conjecture. The identification of slices in affine Grassmannian of type A with quiver varieties of type A was found by Mirković–Vybornov [37, 38]. See also Yamakawa [56, 5.1] for the description of coadjoint orbits of type A by quiver varieties of type A.

3.7.1 Definition. Let $w \geqslant 0$ be a positive integer.

- **(i)** A *composition* $\boldsymbol{r}$ of w is a sequence of (strictly) positive integers $\boldsymbol{r} = [r_1, \cdots, r_\ell]$ such that $\sum_{i=1}^{\ell} r_i = w$.
- **(ii)** A composition $\boldsymbol{r}$ is called a *partition* if $r_1 \geqslant \cdots \geqslant r_\ell$.
- **(iii)** For a partition $\boldsymbol{r}$, we define the *dual partition* $\boldsymbol{r}^\vee = \left[r_1^\vee, \cdots, r_n^\vee\right]$ by

$$r_i^\vee = \#\left\{1 \leqslant j \leq \ell \mid r_j \geqslant i\right\}.$$

For a given positive integer r, we define the following $r \times r$ matrix, called an elementary Jordan block of type r, by

$$J_0(r) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & 0 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix} \in \operatorname{End}\left(\mathbb{C}^r\right).$$

$J_0(r)$ is a nilpotent endomorphism of $\mathbb{C}^r$.

For a given partition $\boldsymbol{r} = (r_1, \cdots, r_\ell)$, we consider $J_0(\boldsymbol{r}) = J_0(r_1) \oplus J_0(r_2) \oplus \cdots \oplus J_0(r_\ell)$ which is the matrix defined by

$$J_0(\boldsymbol{r}) = \begin{pmatrix} J_0(r_1) & 0 & \cdots & 0 \\ 0 & J_0(r_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_0(r_\ell) \end{pmatrix}.$$

Then $J_0(\boldsymbol{r})$ is a nilpotent endomorphism of $\mathbb{C}^{r_1+r_2+\cdots+r_\ell} = \mathbb{C}^w$. It is well-known that any nilpotent endomorphism of $\mathbb{C}^w$ is conjugate to $J_0(\boldsymbol{r})$ for some (unique) partition $\boldsymbol{r}$ in normal Jordan block form.

Let

$$\mathcal{O}_{\boldsymbol{r}} = \mathcal{O}_{J_0(\boldsymbol{r})} = \mathrm{GL}\,(w) \cdot J_0\,(\boldsymbol{r}) = \left\{ g J_0\,(\boldsymbol{r})\, g^{-1} \mid g \in \mathrm{GL}\,(w) \right\}$$

be the $\mathrm{GL}\,(w)$-conjugacy class which contains $J_0\,(\boldsymbol{r})$.

3.7.2 Definition. (**dominance order**) For partitions $\boldsymbol{r}, \boldsymbol{r}'$ of w, we define $\boldsymbol{r} \geqslant \boldsymbol{r}'$ and say that $\boldsymbol{r}$ dominates $\boldsymbol{r}'$ if the following condition holds:

$$\sum_{k=1}^{j} r_k \geqslant \sum_{k=1}^{j} r'_k$$

for $1 \leqslant j \leqslant w$. The partial order $\geqslant$ is called the dominance order.

3.7.3 Exercise. For two given partitions $\boldsymbol{r}$ and $\boldsymbol{r}'$ of w. The following are equivalent:

(i) $\boldsymbol{r} \geqslant \boldsymbol{r}'$
(ii) $\sum_{k>j} r_k^{\vee} \geqslant \sum_{k>j} r_k^{\prime\vee}$ for all $1 \leqslant j \leqslant w$.

3.7.4 Proposition. **(i)** *Let* $X \in \mathrm{End}\,(\mathbb{C}^w)$ *be a nilpotent endomorphism.*
(ii) *Let* $\mathcal{O}_{\boldsymbol{r}}$ *and* $\mathcal{O}_{\boldsymbol{r}'}$ *be the nilpotent orbits in* $\mathrm{End}\,(\mathbb{C}^w)$ *corresponding to* $\boldsymbol{r}$ *and* $\boldsymbol{r}'$.

3.7.5 Exercise. **(i)** Prove that $\boldsymbol{r} \geqslant \boldsymbol{r}'$ if and only if $\overline{\mathcal{O}_{\boldsymbol{r}}} \supset \mathcal{O}_{\boldsymbol{r}'}$.
(ii) Let $\boldsymbol{r} = (r_1, \cdots, r_\ell)$ be a partition of w and $\boldsymbol{r}^{\vee} = \left(r_1^{\vee}, \cdots, r_n^{\vee}\right)$ be its dual. Then show that

$$\begin{aligned} \dim \mathcal{O}_{\boldsymbol{r}} &= w^2 - \sum_{i,j=1}^{\ell} \min\left(r_i, r_j\right) \\ &= w^2 - \sum_{i=1}^{n} \left(r_i^{\vee}\right)^2 = 2 \sum_{i<j} r_i^{\vee} r_j^{\vee}. \end{aligned}$$

Let W be the $\mathbb{C}$-vector space with $\dim_{\mathbb{C}} W = w$ and

$$J = J_0\,(\boldsymbol{r}) = J_0\,(r_1) \oplus J_0\,(r_2) \oplus \cdots \oplus J_0\,(r_\ell) \in \mathrm{End}_{\mathbb{C}}\,(W)$$

be the nilpotent Jordan normal form associated with $\boldsymbol{r} = (r_1, \cdots, r_\ell)$ and $\overline{\mathcal{O}_{\boldsymbol{r}}} \subset \mathrm{End}_{\mathbb{C}}\,(W)$ be the Zariski closure of associated nilpotent conjugacy class. We note that $\dim \mathrm{Ker}\left(J^i\right) = \sum_{1 \leqslant k \leqslant i} r_k^{\vee}$ and $\dim \mathrm{Im}\left(J^i\right) = \sum_{k>j} r_k^{\vee}$.

3.7.6 Exercise. **(i)** Let $\mathcal{F}l\,(v_1, \cdots, v_n; w)$ be the n-step flag variety parametrizing flags

$$0 \subset W^n \subset W^{n-1} \subset \cdots \subset W^1 \subset W$$

with $\dim W^j = v_j \;\; (1 \leqslant j \leqslant n)$. Then show that

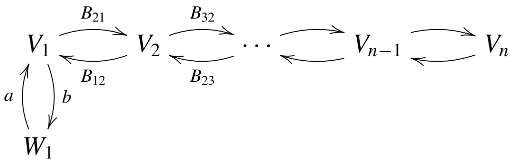
$$\begin{aligned} &T^*\mathcal{F}l\,(v_1, \cdots, v_n; w) \\ &= \left\{ (W^{\bullet}, J) \in \mathcal{F}l\,(v_1, \cdots, v_n; w) \times \mathrm{End}_{\mathbb{C}}\,(W) \mid J W^j \subset W^{j-1} \;\; (1 \leqslant j \leqslant n) \right\}. \end{aligned}$$

(ii) Show that the second projection $\mathrm{pr}_2 \colon T^*\mathcal{F}l\,(v_1, \cdots, v_n; w) \to \mathrm{End}_{\mathbb{C}}\,(W)$ is a resolution of singularity of its image $\mathrm{Im}\left(\mathrm{pr}_2\right) = \overline{\mathcal{O}_{\boldsymbol{r}}}$.

We consider an quiver Q of type A_n and dimension vectors $\boldsymbol{v} = (v_1, \cdots, v_n)$ and $\boldsymbol{w} = (w, 0, \cdots, 0)$ with the following condition:

$$w - v_1 \geqslant v_1 - v_2 \geqslant \cdots \geqslant v_{n-1} - v_n \geqslant v_n.$$

So we consider the following diagram:



Let $\varpi \colon \mathbf{M}\,(V, W) \to \mathrm{End}_{\mathbb{C}}\,(W)$ be the morphism defined by $(B, a, b) \mapsto ba$. By the construction, the morphism ϖ is $\mathrm{GL}\,(V_1)$-invariant. Then we obtain a morphism

$$\varpi_0 \colon \mathfrak{M}_0\,(V, W) \to \mathrm{End}_{\mathbb{C}}\,(W)$$

which is $G\,(W) = \mathrm{GL}\,(W_1)$-equivariant.

3.7.7 Exercise. Show that the following are equivalent:

(i) $B_{j,j+1}$ are injective for $1 \leqslant j \leqslant n-1$ and b is injective.
(ii) A data (B, a, b) is (semi)stable.

3.7.8 Theorem. *Assume that $\zeta_{\mathbb{R},i} > 0$ for $1 \leqslant i \leqslant n$ and $\zeta_{\mathbb{C}} = 0$. Then we have the following commutative diagram whose horizontal morphisms are isomorphisms:*

$$\begin{array}{ccc} \mathfrak{M}_{\zeta_{\mathbb{R}},0}\,(\boldsymbol{v}, \boldsymbol{w}) & \longrightarrow & T^*\mathcal{F}l\,(v_1, \cdots, v_n; w) \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \mathrm{pr}_2} \\ \mathfrak{M}_{0,0}\,(\boldsymbol{v}, \boldsymbol{w}) & \longrightarrow & \overline{\mathcal{O}_{\boldsymbol{r}}} \end{array},$$

where the lower horizontal morphism is induced by ϖ and the upper horizontal morphism is given by $(B, a, b) \to (W^\bullet, ba)$, where

$$W^j = \mathrm{Im}\left(bB_{1,2} \cdots B_{j,j+1}\right) \ (0 \leqslant j \leqslant n-1)$$

and we also have $\mathfrak{M}^{\mathrm{reg}}_{0,0}\,(\boldsymbol{v}, \boldsymbol{w}) \simeq \mathcal{O}_{\boldsymbol{r}}$.

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Chapter 8
On Additive Deligne–Simpson Problems

Kazuki Hiroe

In this note, we explain the additive Deligne–Simpson problem and its generalization for differential equations with unramified irregular singularities. A correspondence between spaces of solutions of these additive Deligne–Simpson problems and quiver varieties is given. As an application, the geometry of moduli spaces of meromorphic connections with unramified irregular singularities is discussed, for example, the non-emptiness of the smooth parts of moduli spaces and their connectedness. The detail of this note can also be found in [17].

1 Deligne–Simpson Problem and Riemann–Hilbert Problem

The *Deligne–Simpson problem* is the following problem:

Give a necessary and sufficient condition for the choice of the conjugacy classes $C_j \in \mathrm{GL}(n, \mathbb{C})$ *so that there exist irreducible tuples of matrices* $M_j \in C_j$ *satisfying*

$$M_0 \cdots M_p = I.$$

Here, we say that a tuple $(M_0, \ldots, M_p)$ of $n \times n$ matrices is *irreducible* if it has no nontrivial simultaneous invariant subspace of $\mathbb{C}^n$, namely, if there exists a proper subspace $W \subset \mathbb{C}^n$ such that $M_i W \subset W$ for all $i = 0, \ldots, p$, then $W = \{0\}$.

This problem was stated by P. Deligne and C. Simpson and Simpson obtained a necessary and sufficient condition under some restrictions in [31]. After that, V. Kostov studied this problem deeply and obtained many important results, see

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his survey paper [26]. The name of the *Deligne–Simpson problem* was given by Kostov.

In the series of works, Kostov introduced an additive analogy of the Deligne–Simpson problem, so-called *additive Deligne–Simpson problem*: *Give a necessary and sufficient condition for the choice of the conjugacy classes* $C_j \in M(n, \mathbb{C})$ *so that there exist irreducible tuples of matrices* $A_j \in C_j$ *satisfying*

$$A_0 + \cdots + A_p = 0.$$

The original Deligne–Simpson problem and the additive one have the following realizations as problems of monodromy and differential equations:

A tuple $(M_0, \ldots, M_p) \in \mathrm{GL}\,(n, \mathbb{C})^{p+1}$ satisfying

$$M_0 \cdots M_p = I$$

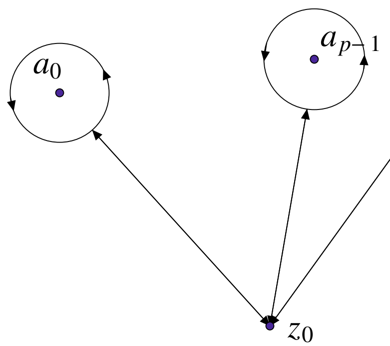
defines a linear representation

$$\rho\colon \pi_1(\mathbb{P}^1 \backslash \{a_0, \ldots, a_p\}, z_0) \longrightarrow \mathrm{GL}\,(n, \mathbb{C})$$

recalling that

$$\pi_1(\mathbb{P}^1 \backslash \{a_0, \ldots, a_p\}, z_0) \cong \langle \gamma_0, \ldots, \gamma_p \mid \gamma_0 \cdots \gamma_p = \mathrm{id} \rangle$$

where γ_i is a suitable closed path in $\mathbb{P}^1 \backslash \{a_0, \ldots, a_p\}$ encircling a_i with the base point z_0 and $\rho(\gamma_i) = M_i$ for each $i = 0, \ldots, p$.



Thus, the Deligne–Simpson problem can be seen as the problem asking the existence of irreducible monodromy representation with the prescribed local isomorphic classes of local monodromy.

On the other hand, from a tuple $(A_0, \ldots, A_p) \in M(n, \mathbb{C})$ satisfying

$$A_0 + \cdots + A_p = 0,$$

we can consider the system of linear differential equations,

$$\frac{d}{dz}Y = \sum_{i=1}^{p} \frac{A_i}{z - a_i} Y$$

called *Fuchsian* system of differential equations whose singular points are at $a_1, \ldots, a_p \in \mathbb{C}$ and $a_0 := \infty$. Here A_i is called *residue matrix* at the singular point $x = a_i$ for each $i = 0, \ldots, p$. Thus, the additive Deligne–Simpson problem asks the existence of irreducible Fuchsian differential equations with the prescribed conjugacy classes of residue matrices A_i, $i = 0, \ldots, p$.

Under these realizations, we can relate the Deligne–Simpson problem and the additive one through the *Riemann–Hilbert correspondence* as follows. There are many good references for Fuchsian differential equations and the Riemann–Hilbert problems, see, for example, [1, 15, 29, 30, 33, 34]. Let us consider a Fuchsian system

$$\frac{d}{dz}Y = \sum_{i=1}^{p} \frac{A_i}{z - a_i} Y.$$

Fix a base point $z_0 \in \mathbb{P}^1 \backslash \{a_0, \ldots, a_p\}$ and a basis $v_1, \ldots, v_n \in \mathbb{C}^n$ as a vector space. Then there is a unique collection $Y_1(z), \ldots, Y_n(z)$ of solutions defined around z_0 with $Y_i(z_0) = v_i$ and $\det(Y_1(z) \cdots Y_n(z)) \neq 0$. The square matrix $F(z) := (Y_1(z) \cdots Y_n(z))$ can be analytically continued on $\mathbb{P}^1 \backslash \{a_0, \ldots, a_p\}$ as multivalued matrix function, which is called *fundamental matrix of solutions*. Then the fundamental matrix defines a tuple of matrices $(M_0, \ldots, M_p) \in \mathrm{GL}(n, \mathbb{C})$ as follows. For a closed path γ in $\mathbb{P}^1 \backslash \{a_0, \ldots, a_p\}$ with the base point z_0, let $F_\gamma(z)$ defined around z_0 be the analytic continuation of $F(z)$ along γ. Then, we define the *local monodromy matrix*

$$M_i := F_{\gamma_i}(z)^{-1} F(z)$$

for each $i = 0, \ldots, p$. Here, it is easy to see that M_i are constant matrices. The product of paths induces the multiplication of monodromy matrices. Thus

$$M_0 \cdots M_p = F_{\gamma_0 \cdots \gamma_p}(z)^{-1} F(z) = F_{\mathrm{id}}(z)^{-1} F(z) = I.$$

Moreover if A_i are good enough, we can relate conjugacy classes of A_i and that of M_i. For example, let us assume that eigenvalues of A_i have no integer differences for each $i = 0, \ldots, p$. Then the general theory of regular singular points of differential equations shows that $\exp(2\pi\sqrt{-1}A_i)$ is conjugate with M_i for each $i = 0, \ldots, p$.

Furthermore, we can find a Fuchsian system from an irreducible monodromy representation conversely.

1.1 Theorem (Bolibruch [8], Kostov [25]). *For an irreducible representation*

$$\rho\colon \pi_1(\mathbb{P}^1\backslash\{a_0, \ldots, a_p\}, z_0) \to \mathrm{GL}(n, \mathbb{C}),$$

there exists a Fuchsian differential equation

$$\frac{d}{dz}Y = \sum_{i=1}^{p} \frac{A_i}{z - a_i}Y$$

whose monodromy matrices $(M_0, \ldots, M_p)$ *defines the representation isomorphic to* ρ.

Thus we may say that the original Deligne–Simpson problem and its additive analogue are related under the Riemann–Hilbert correspondence explained as above.

1.2 Remark. As we saw above, monodromy representations and Fuchsian differential equations are related through the Riemann–Hilbert correspondence. However, solutions of the Deligne–Simpson problem do not provide that of the additive Deligne–Simpson problem directly, vice versa. For instance, we should note that the irreducibility the tuple $(A_0, \ldots, A_p) \in M(n, \mathbb{C})^{p+1}$ does not imply the irreducibility of the tuple of monodromy matrices $(M_0, \ldots, M_p)$ of the Fuchsian equation

$$\frac{d}{dz}Y = \sum_{i=1}^{p} \frac{A_i}{z - a_i}Y$$

in general. Also, the conjugacy classes of M_i are not determined from those of A_i in general except the good case as we explained above.

2 Additive Deligne–Simpson Problem

In this note, we shall discuss the additive Deligne–Simpson problem and its generalization. For the original Deligne–Simpson problem called "multiplicative" Deligne–Simpson problem in distinction from the additive one, and its generalizations, we refer Simpson's pioneering paper [31], Kostov's paper [26], the paper of Crawley-Boevey and Shaw [12], and Boalch's paper [7].

As we saw in the previous section, the additive Deligne–Simpson problem can be seen as the problem finding an irreducible Fuchsian system with the prescribed conjugacy classes of residue matrices. Thus, it may be natural to consider similar problems for non-Fuchsian differential equations. In this note, we shall generalize the additive Deligne–Simpson problem for differential equations with at most unramified irregular singularities on the Riemann sphere and give a necessary and sufficient condition for the solvability of the problem.

To do so, we first give the definition of the generalization of the additive Deligne–Simpson problem for differential equations with at most unramified irregular singularities on the Riemann sphere. Also, we recall moduli spaces of meromorphic connections on trivial vector bundles over the Riemann sphere and moreover see that the additive Deligne–Simpson problem is related to the non-emptiness problem of the moduli spaces.

2.1 A Generalization of the Additive Deligne–Simpson Problem

As we saw in the previous section, the original additive Deligne–Simpson problem for Fuchsian differential equations consists of a collection of conjugacy classes of $M(n, \mathbb{C})$. The counterparts for irregular singular cases of the conjugacy classes are Hukuhara–Turrittin–Levelt normal forms of $M(n, \mathbb{C}(\!(z)\!))$. We shall recall the definition of Hukuhara–Turrittin–Levelt normal forms and give a definition of the additive Deligne–Simpson problem for differential equations with at most unramified irregular singularities.

Let us consider a differential equation

$$\frac{d}{dz}Y = AY, \quad (A \in M(n, \mathbb{C}(\!(z)\!))).$$

For $X \in \mathrm{GL}(n, \mathbb{C}(\!(z)\!))$, we define a new differential equation $\frac{d}{dz}\tilde{Y} = B\tilde{Y}$ by

$$B := XAX^{-1} + (\frac{d}{dz}X)X^{-1}.$$

We write $B =: X[A]$ and call this operation the *gauge transform* of A by X. Let $\mathbb{C}(\!(t)\!)$ be a finite field extension of $\mathbb{C}(\!(z)\!)$, namely, there exists $r \in \mathbb{Z}_{\geq 1}$ such that $t^r = z$. Then the differential equation $\frac{d}{dz}Y = AY$ over $\mathbb{C}(\!(z)\!)$ defines the differential equation $\frac{d}{dt}Z = \overline{A}Z$ over $\mathbb{C}(\!(t)\!)$ where $\overline{A} := rt^{r-1}AZ$.

2.1.1 Definition (HTL normal form). By *Hukuhara–Turrittin–Levelt normal form* or *HTL normal form* for short, we mean an element in $M(n, \mathbb{C}(\!(t)\!))$ of the form as a block diagonal matrix

$$\operatorname{diag}\left(q_1(t^{-1})I_{n_1}+R_1t^{-1},\ldots,q_m(t^{-1})I_{n_m}+R_mt^{-1}\right):=\begin{pmatrix}q_1(t^{-1})I_{n_1}+R_1t^{-1} & & \\ & \ddots & \\ & & q_m(t^{-1})I_{n_m}+R_mt^{-1}\end{pmatrix}$$

where $t^r=z$, $q_i(s)\in s^2\mathbb{C}[s]$ satisfying $q_i\neq q_j$ if $i\neq j$, and $R_i\in M(n_i,\mathbb{C})$. In particular when $r=1$, the normal form is said to be *unramified*.

Let us define $\mathrm{Res}_{t=0}(\sum_{i=-\infty}^{\infty}A_it^i):=A_{-1}$. For an HTL normal form $H\in M(n,\mathbb{C}(\!(t)\!))$, we call $H_{\mathrm{irr}}:=H-\mathrm{Res}_{t=0}(H)t^{-1}$ the *irregular part* of H. The following is a fundamental fact of the local formal theory of differential equations with irregular singularity.

2.1.2 Theorem (Hukuhara–Turrittin–Levelt, see [36], for instance). *For any $A\in M(n,\mathbb{C}(\!(z)\!))$, there exists a field extension $\mathbb{C}(\!(t)\!)\supset\mathbb{C}(\!(z)\!)$ with $t^r=z$, $r\in\mathbb{Z}_{\geq 1}$ and $X\in\mathrm{GL}(n,\mathbb{C}(\!(t)\!))$ such that $\overline{X[A]}$ is an HTL normal form in $M(n,\mathbb{C}(\!(t)\!))$.*

We call this $\overline{X[A]}$ a *normal form* of A.

2.1.3 Remark. We note that if two HTL normal forms $H\in M(n,\mathbb{C}(\!(t)\!))$ and $H'\in M(n,\mathbb{C}(\!(t')\!))$ ($\mathbb{C}(\!(z)\!)\subset\mathbb{C}(\!(t)\!)\subset\mathbb{C}(\!(t')\!)$) are normal forms of an $A\in M(n,\mathbb{C}(\!(z)\!))$, then there exists $g\in\mathrm{GL}(n,\mathbb{C})$ such that $g^{-1}H_{\mathrm{irr}}g=H'_{\mathrm{irr}}$ and $g^{-1}\exp(2\pi\sqrt{-1}k\mathrm{Res}_{t=0}(H))g=\exp(2\pi\sqrt{-1}k\mathrm{Res}_{t'=0}(H'))$ for some integer $k\geq 1$, see [3] for instance.

To define our generalized additive Deligne–Simpson problem for differential equations with at most unramified irregular singularities, we shall introduce coadjoint orbits of unramified HTL normal forms which play the same role as conjugacy classes in $M(n,\mathbb{C})$ in the original additive Deligne–Simpson problem.

Let us consider an unramified HTL normal form

$$B=\operatorname{diag}\left(q_1(z^{-1})I_{n_1}+R_1z^{-1},\ldots,q_m(z^{-1})I_{n_m}+R_mz^{-1}\right)$$

with the pole order

$$k:=\max_{i=1,\ldots,m}\{\deg_{\mathbb{C}[z^{-1}]}q_i(z^{-1})\}.$$

We shall consider an orbit of B under the following group action. Let $G_k:=\mathrm{GL}(n,\mathbb{C}[\![z]\!]/z^k\mathbb{C}[\![z]\!])$ which can be identified with

$$\left\{A_0+A_1z+\cdots+A_{k-1}z^{k-1}\in\sum_{i=0}^{k-1}M(n,\mathbb{C})z^i\;\middle|\;A_0\in\mathrm{GL}(n,\mathbb{C})\right\}.$$

Also define

$$\begin{aligned}\mathfrak{g}_k &:= M(n, \mathbb{C}[\![z]\!]/z^k\mathbb{C}[\![z]\!]) \\ &\cong \left\{ A_0 + A_1 z + \cdots + A_{k-1}z^{k-1} \;\middle|\; A_i \in M(n,\mathbb{C}),\ i = 0, 1, \ldots, k-1 \right\}.\end{aligned}$$

The group G_k acts on $\mathfrak{g}_k$ by the *adjoint action* $\mathrm{Ad}(g)X := gXg^{-1}$ for $g \in G_k$, $X \in \mathfrak{g}_k$. The dual vector space $\mathfrak{g}_k^*$ is identified with

$$M(n, z^{-k}\mathbb{C}[\![z]\!]/\mathbb{C}[\![z]\!]) \cong \left\{ \frac{A_k}{z^k} + \cdots + \frac{A_1}{z} \;\middle|\; A_i \in M(n,\mathbb{C}),\ i = 1, \ldots, k \right\}$$

by the bilinear form

$$\mathfrak{g}_k \times \mathfrak{g}_k^* \ni (A, B) \mapsto \mathrm{Res}(\mathrm{tr}(AB)) \in \mathbb{C}.$$

Regarding B as an element of $\mathfrak{g}_k^*$, we define the orbit of B under the coadjoint action of G_k.

2.1.4 Definition (truncated orbit). Let us regard B as an element of $\mathfrak{g}_k^*$. Then

$$\mathcal{O}_B := \{\mathrm{Ad}^*(g)B \mid g \in G_k\}$$

is called the *truncated orbit* of B.

2.1.5 Remark. If another HTL normal form $B' \in \mathfrak{g}_k^*$ is in $\mathcal{O}_B$, then there exists $g \in \mathrm{GL}(n, \mathbb{C})$ such that $g^{-1}B'g = B$ as in Remark 2.1.3. See Proposition 5 in [37] for the proof.

Let us mention a relationship between $\mathcal{O}_B$ and normal forms of $M(n, \mathbb{C}(\!(z)\!))$ under gauge transformations. Let $\iota\colon M(n, \mathbb{C}(\!(z)\!)) \to M(n, \mathbb{C}(\!(z)\!)/\mathbb{C}[\![z]\!])$ be the natural projection. Under the generic condition as we see below, B can be a normal form of $A \in M(n, z^{-k}\mathbb{C}[\![z]\!])$ with $\iota(A) \in \mathcal{O}_B$.

2.1.6 Proposition. *For an HTL normal form*

$$B = \mathrm{diag}\left(q_1(z^{-1})I_{n_1} + R_1 z^{-1}, \ldots, q_m(z^{-1})I_{n_m} + R_m z^{-1}\right)$$

of the pole order k, we assume that differences of any pairs of distinct eigenvalues of R_i never be integers for each $i = 1, \ldots, m$. Then for any $A \in M(n, z^{-k}\mathbb{C}[\![z]\!])$ with $\iota(A) \in \mathcal{O}_B$, there exists $X \in \mathrm{GL}(n, \mathbb{C}(\!(z)\!))$ such that $X[A] = B$, namely, B is a normal form of A.

This follows from the following fundamental facts.

2.1.7 Lemma (see Lemma 1 in 6.2 in [3] for instance). *Suppose $k > 1$. Let us consider*

$$C = \mathrm{diag}(\lambda_1 I_{m_1}, \ldots, \lambda_l I_{m_l}) z^{-k} + \sum_{i=-k+1}^{\infty} C_i z^i \in M(m, \mathbb{C}(\!(z)\!))$$

for distinct $\lambda_1, \ldots, \lambda_l \in \mathbb{C}$. *Then there exists* $X \in \mathrm{GL}(m, \mathbb{C}[\![z]\!])$ *such that*

$$X[C] = \mathrm{diag}\left(\lambda_1 I_{m_1} z^{-k} + \sum_{i=-k+1}^{\infty} C_i^{(1)} z^i, \ldots, \lambda_l I_{m_l} z^{-k} + \sum_{i=-k+1}^{\infty} C_i^{(l)} z^i\right)$$

where $C_i^{(j)} \in M(m_j, \mathbb{C})$ *for* $j = 1, \ldots, l$, $i = -k+1, -k+2, \ldots$.

2.1.8 Lemma (see Theorem 1 in 3.3 in [3] for instance). *Let us consider* $C = \sum_{i=-1}^{\infty} C_i z^i$ *in* $M(m, \mathbb{C}(\!(z)\!))$ *and suppose that no two distinct eigenvalues of* C_{-1} *differ by an integer. Then there exists* $X \in \mathrm{GL}(m, \mathbb{C}[\![z]\!])$ *such that* $X[C] = C_{-1} z^{-1}$.

Proof of Proposition 2.1.6 Since $\iota(A) \in \mathcal{O}_B$, there exists $\tilde{X} \in \mathrm{GL}(n, \mathbb{C}[\![z]\!])$ such that

$$\tilde{X}[A] - B = B_0 + B_1 z + B_2 z^2 + \cdots \in M(n, \mathbb{C}[\![z]\!]).$$

Using Lemma 2.1.7 repeatedly, we may assume that

$$B_i = \mathrm{diag}(B_i^{(1)}, \ldots, B_i^{(m)})$$

with $B_i^{(j)} \in M(n_j, \mathbb{C})$ for $i = 0, 1, \ldots$. Since no two eigenvalues of R_i differ by an integer, there exists $x_i \in \mathrm{GL}(n_i, \mathbb{C}[\![z]\!])$ such that $x_i[R_i z^{-1} + B_0^{(i)} + B_1^{(i)} z + \cdots] = R_i z^{-1}$ for each $i = 1, \ldots, m$ by Lemma 2.1.8. Noting that $x := \mathrm{diag}(x_1, \ldots, x_m)$ commutes with

$$B_{\mathrm{irr}} = \mathrm{diag}\left(q_1(z^{-1}) I_{n_1}, \ldots, q_m(z^{-1}) I_{n_m}\right),$$

we have $x\tilde{X}[A] = B$. □

Now we are ready to define a generalization of the additive Deligne–Simpson problem for differential equations with unramified irregular singularities. Let us consider a differential equation

$$\frac{d}{dz} Y = \left(\sum_{i=1}^{p} \sum_{j=1}^{k_i} \frac{A_{i,j}}{(z - a_i)^j} + \sum_{2 \le j \le k_0} A_{0,j} z^{j-2}\right) Y.$$

This equation has a singular point at $x = a_i$ with pole order k_i for each $i = 1, \ldots, p$. Moreover $x = a_0 = \infty$ is a singular point of pole order k_0 as well. To see this, set $z_0 = \frac{1}{z}$. Since $\frac{d}{dz} = -z_0^2 \frac{d}{dz_0}$, this differential equation can be written by

$$\frac{d}{dz_0} \tilde{Y} = \left(-\sum_{j=1}^{k_0} \frac{A_{0,j}}{z_0^j} + A_0 + A_1 z_0 + A_2 z_0^2 \cdots\right) \tilde{Y}$$

where $A_{0,1} := -\sum_{i=1}^{p} A_{i,1}$.

Let us denote the *principal term* at the singular point a_i by

$$A_i(z_i) := \sum_{j=1}^{k_i} A_{i,j} z_i^{-j}$$

for each $i = 0, \ldots, p$. Here $z_i := z - a_i$, $i = 1, \ldots, p$, $z_0 := \frac{1}{z}$. This differential equation is said to be *irreducible* if the collection of the matrices $(A_{i,j})_{\substack{0\le i\le p,\\ 1\le j\le k_i}}$ is irreducible.

2.1.9 Definition (additive Deligne–Simpson problem). Let us take $k_i \in \mathbb{Z}_{\geq 1}$ and unramified HTL normal forms $B_i \in \mathfrak{g}_{k_i}^*$ for $i = 0, 1, \ldots, p$. Then a *solution* of the additive Deligne–Simpson problem for the collection of the unramified HTL normal forms $(B_0, B_1, \ldots, B_p)$ is an irreducible differential equation

$$\frac{d}{dz}Y = \left(\sum_{i=1}^{p}\sum_{j=1}^{k_i} \frac{A_{i,j}}{(z-a_i)^j} + \sum_{2\le j\le k_0} A_{0,j} z^{j-2}\right) Y$$

such that the principal term at each singular point a_i, $i = 0, 1, \ldots, p$ satisfies

$$A_i(z) \in \mathcal{O}_{B_i}.$$

2.1.10 Remark. Let us note that if $k_0 = k_1 = \cdots = k_p = 1$, then $G_{k_i} = \mathrm{GL}(n, \mathbb{C})$ and $\mathfrak{g}_{k_i}^* = M(n, \mathbb{C})$. Thus, the truncated orbits $\mathcal{O}_{B_i}$ are just conjugacy classes of $M(n, \mathbb{C})z^{-1}$. Therefore, the additive Deligne–Simpson problem in Definition 2.1.9 contains the original additive Deligne–Simpson problem for Fuchsian differential equations.

2.2 Moduli Spaces of Meromorphic Connections and Additive Deligne–Simpson Problem

In this section, we quickly recall the definition of moduli spaces of meromorphic connections on trivial vector bundles over the Riemann sphere following [6]. The detailed treatment can be found in the original paper by Boalch [6] and we also refer [19] and their references. The solvability of the additive Deligne–Simpson problems can be seen as the problem determining the necessary and sufficient condition of the non-emptiness of the moduli spaces.

Let us recall the notion of meromorphic connections and see their relationship with differential equations. For $f = \sum_{i>-\infty}^{\infty} a_i z^i \in \mathbb{C}(\!(z)\!)$, the *order* is

$$\mathrm{ord}(f) := \min\{i \mid a_i \neq 0\}.$$

If $f = 0$, we formally put $\mathrm{ord}(f) = \infty$. For a meromorphic function f locally defined near $a \in \mathbb{P}^1$, we denote the germ of f at a by f_a. We may see $f_a \in \mathbb{C}((z_a))$ by setting $z_a = z - a$ if $a \in \mathbb{C}$ and $z_a = 1/z$ if $a = \infty$, where we take z as the standard coordinate of $\mathbb{C}$. Then define

$$\mathrm{ord}_a(f) := \mathrm{ord}(f_a).$$

For a meromorphic 1-form ω defined on $\mathbb{P}^1$, the order $\mathrm{ord}_a(\omega)$ can be defined as follows. Set $U_1 = \mathbb{P}^1\backslash\{\infty\}$ and $U_2 = \mathbb{P}^1\backslash\{0\}$. Let z_i be coordinates of U_i, $i = 1, 2$, such that $z_1(0) = z_2(\infty) = 0$ and $z_2 = 1/z_1$ in $U_1 \cap U_2$. Then there exist meromorphic functions f_i on U_i such that

$$\omega = f_i\, dz_i$$

on U_i for $i = 1, 2$. Then define

$$\mathrm{ord}_a(\omega) := ord_a(f_i)$$

for $a \in U_i$, $i = 1, 2$.

Let us fix a collection of points $a_0, \ldots, a_p \in \mathbb{P}^1$ and set $S := k_0a_0 + \cdots + k_pa_p$ as an effective divisor with $k_0, \ldots, k_p > 0$. For $a \in \mathbb{P}^1$ let $S(a)$ be the coefficient of a in S, i.e.,

$$S(a) := \begin{cases} k_i & \text{if } a = a_i \text{ for } i = 0, \ldots, p, \\ 0 & \text{otherwise.} \end{cases}$$

For an open set $U \subset \mathbb{P}^1$, we define $\Omega_S(U)$ to be the set of all meromorphic 1-forms ω on U satisfying $\mathrm{ord}_a(\omega) \geq -S(a)$ for any $a \in U$. This correspondence defines the sheaf Ω_S by the natural restriction mappings.

Let $\mathcal{E}$ be a locally free sheaf of rank n on $\mathbb{P}^1$, namely a sheaf of modules over the sheaf $\mathcal{O}$ of holomorphic functions on $\mathbb{P}^1$ satisfying that for any $a \in \mathbb{P}^1$ there exists an open neighbourhood $V \subset \mathbb{P}^1$ such that $\mathcal{E}|_V \cong \mathcal{O}^n|_V$. We may sometimes regard $\mathcal{E}$ as a holomorphic vector bundle over $\mathbb{P}^1$.

2.2.1 Definition (Meromorphic connection). A *meromorphic connection* is a pair $(\mathcal{E}, \nabla)$ of a locally free sheaf $\mathcal{E}$ and a morphism $\nabla\colon \mathcal{E} \to \mathcal{E} \otimes \Omega_S$ of sheaves of $\mathbb{C}$-vector spaces satisfying

$$\nabla(fs) = df \otimes s + f \otimes \nabla(s)$$

for all $f \in \mathcal{O}(U)$, $s \in \mathcal{E}(U)$ and open subsets $U \subset \mathbb{P}^1$.

Let $U \subset \mathbb{P}^1$ be an open subset which gives a local trivialization of $\mathcal{E}$ and z a local coordinate of U. Then if we fix an identification $\mathcal{E}|_U \cong \mathcal{O}^n|_U$, we can write $\nabla = d - A\,dz$ by $A \in M(n, \mathcal{M}(U))$ on U. Note that if we write $\nabla = d - A'\,dz$ by another identification $\mathcal{E}|_U \cong \mathcal{O}^n|_U$, then A' can be obtained by a *holomorphic gauge transformation* of A, namely, there exists $X \in \mathrm{GL}(n, \mathcal{O}(U))$ such that

$$A' = X[A].$$

Thus, we may say that $(\mathcal{E}, \nabla)$ defines a holomorphic gauge equivalent class of a local differential equation

$$\frac{d}{dz}Y = AY$$

on $U \subset \mathbb{P}^1$.

In particular, suppose that $\mathcal{E}$ is *trivial*, i.e., $\mathcal{E} \cong \mathcal{O}^n$ and set $U_1 = \mathbb{P}^1 \backslash \{\infty\}$ and $U_2 = \mathbb{P}^1 \backslash \{0\}$ as before. Then if we fix a trivialization $\mathcal{E} \cong \mathcal{O}^n$, we have $\nabla = d - A(z_1)dz_1$ on U_1 with $A(z_1) = (\alpha_{i,j}(z_1))_{i,j=1,\ldots,n} \in M(n, \mathbb{C}(z))$ satisfying $ord_a(\alpha_{i,j}) \geq -S(a)$ for all $a \in U_1$. Similarly on U_2 we have $\nabla = d - B(z_2)dz_2$. Since $\mathcal{E}$ is trivial,

$$A(z_1)dz_1 = B(z_2)dz_2 \text{ on } U_1 \cap U_2.$$

Namely,

$$B(z_2) = -\frac{A(1/z_2)}{z_2^2}.$$

This is nothing but the coordinate exchange $\zeta = \frac{1}{z}$ for a differential equation

$$\frac{d}{dz}Y = A(z)Y \longmapsto -\zeta^2 \frac{d}{d\zeta}Y = A(1/\zeta)Y.$$

Thus, a meromorphic connection $(\mathcal{E}, \nabla)$ with a trivial bundle $\mathcal{E}$ on $\mathbb{P}^1$ corresponds to a meromorphic differential equation $\frac{d}{dz}Y = AY$ with $A = (\alpha_{i,j})_{i,j=1,\ldots,n} \in M(n, \mathbb{C}(z))$ satisfying $ord_a(\alpha_{i,j}\, dz) \geq -S(a)$ for all $a \in \mathbb{P}^1$, and vice versa. This correspondence is unique up to the choice of $\mathcal{E} \cong \mathcal{O}^n$, i.e., $\mathrm{GL}(n, \mathbb{C})$-action.

Let $S = k_0 a_0 + \ldots + k_p a_p$ be an effective divisor on $\mathbb{P}^1$ as before. Define a set of meromorphic connections on $\mathbb{P}^1$

$$\mathrm{Triv}_S^{(n)} := \left\{ (\mathcal{O}^n, \nabla) \,\middle|\, \nabla \colon \mathcal{O}^n \to \mathcal{O}^n \otimes \Omega_S \right\}.$$

We say $(\mathcal{O}^n, \nabla) \in \mathrm{Triv}_S^{(n)}$ is *stable* if there exists no nontrivial proper subspace $W \subset \mathbb{C}^n$ such that the subbundle $\mathcal{W} := W \otimes \mathcal{O} \subset \mathbb{C}^n \otimes \mathcal{O} = \mathcal{O}^n$ is closed under ∇, i.e.,

$$\nabla(\mathcal{W}) \subset \mathcal{W} \otimes \Omega_S.$$

Let $\mathbf{B} = (B_0, \ldots, B_p) \in M(n, \mathbb{C}(\!(z)\!))^{p+1}$ be a collection of HTL normal forms satisfying $\mathrm{ord}(B_i) = -k_i$ for all $i = 0, \ldots, p$. We write $\nabla|_{a_i} \in \mathcal{O}_{B_i}$ for a connection $(\mathcal{O}^n, \nabla)$ if there exists $A_{a_i} \in M(n, \mathbb{C}(\!(z_{a_i})\!))$ such that $\nabla = d - A_{a_i}\, dz_{a_i}$ and $\iota(A_{a_i}) \in \mathcal{O}_{B_i}$ where z_{a_i} is a local coordinate of $\mathbb{P}^1$ vanishing at a_i and $\iota \colon M(n, \mathbb{C}(\!(z_{a_i})\!)) \to M(n, \mathbb{C}(\!(z_{a_i})\!)/\mathbb{C}[\![z_{a_i}]\!])$ is the natural projection.

Then, the moduli space of stable meromorphic connections on trivial bundles is

$$\mathfrak{M}(\mathbf{B}) := \left\{ (\mathcal{O}^n, \nabla) \in \mathrm{Triv}_S^{(n)} \;\middle|\; \begin{array}{c} (\mathcal{O}^n, \nabla):\ \text{stable}, \\ \nabla|_{a_i} \in \mathcal{O}_{B_i} \text{ for all } i = 0, \ldots, p \end{array} \right\} \Big/ \mathrm{GL}(n, \mathbb{C}).$$

Here $\mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}(n, \mathcal{O}(\mathbb{P}^1))$ acts on $\mathrm{Triv}_S^{(n)}$ as the holomorphic gauge transformation.

Möbius transformation may allow us to suppose $a_0 = \infty \in \mathbb{P}^1$. Then we can identify $(\mathcal{O}^n, \nabla)$ in Triv_S^n with a meromorphic differential equation defined on $\mathbb{P}^1$,

$$\frac{d}{dz} Y = \left(\sum_{i=1}^{p} \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{(z - a_i)^\nu} + \sum_{2 \le \nu \le k_0} A_\nu^{(0)} z^{\nu-2} \right) Y$$

up to $\mathrm{GL}(n, \mathbb{C})$-action, i.e.,

$$\frac{d}{dz} Y = A(z) Y \longmapsto \frac{d}{dz} Y' = g A(z) g^{-1} Y' \quad (g \in \mathrm{GL}(n, \mathbb{C})).$$

The stability of $(\mathcal{O}^n, \nabla)$ corresponds to the irreducibility of the differential equation. Thus, we can regard $\mathfrak{M}(\mathbf{B})$ as the following moduli space of meromorphic differential equations on $\mathbb{P}^1$:

$$\left\{ \frac{d}{dz} Y = \left(\sum_{i=1}^{p} \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{(z - a_i)^\nu} + \sum_{2 \le \nu \le k_0} A_\nu^{(0)} z^{\nu-2} \right) Y \;\middle|\; \begin{array}{c} \text{irreducible}, \\ \sum_{\nu=1}^{k_i} \frac{A_\nu^{(i)}}{z^\nu} \in \mathcal{O}_{B_i}, \\ i = 0, \ldots, p \end{array} \right\} \Big/ \mathrm{GL}(n, \mathbb{C}).$$

Thus, the solvability of the additive Deligne–Simpson problem is rephrased as the non-emptiness of the moduli space.

2.2.2 Proposition. *There is a solution of the additive Deligne–Simpson problem for* $\mathbf{B}$ *if and only if* $\mathfrak{M}(\mathbf{B}) \neq \emptyset$.

Furthermore, forgetting the location of the singular points, we may regard $\mathfrak{M}(\mathbf{B})$ as a subspace of the orbit space $\prod_{i=0}^{p} \mathcal{O}_{B_i}$,

$$\mathfrak{M}(\mathbf{B}) = \left\{ \mathbf{A} = (A_i(z))_{0 \le i \le p} \in \prod_{i=0}^{p} \mathcal{O}_{B_i} \;\middle|\; \begin{array}{c} \mathbf{A} \text{ is irreducible }, \\ \sum_{i=0}^{p} \mathrm{Res}\,(A_i(z)) = 0 \end{array} \right\} \Big/ \mathrm{GL}(n, \mathbb{C})$$

which is free from locations of a_i in $\mathbb{P}^1$. Here

$$\mathrm{Res}\,(\sum_{j=1}^{k} A_j z^{-j}) := A_1$$

and we say that $\mathbf{A} = (\sum_{j=1}^{k_i} A_{i,j} z^{-1})_{0 \le i \le p}$ is *irreducible* if $(A_{i,j})_{\substack{0 \le i \le p \\ 1 \le j \le k_i}}$ is irreducible.

3 A Review of Representations of Quivers

In [10], Crawley-Boevey determined a necessary and sufficient condition of the existence of solutions of the additive Deligne–Simpson problem for Fuchsian systems. In that paper, Crawley-Boevey found a realization of a moduli space of Fuchsian systems as a quiver variety and determined the condition for the non-emptiness of the moduli space from the theory of quiver varieties. To recall his theory and generalize it to our problems for unramified differential equations, let us give a review of known results of the representation theory of quivers and theory of quiver varieties. We refer original papers for the general theory by Nakajima [28], Crawley-Boevey and Holland [11], Crawley-Boevey [9] and their references.

3.1 Representations of Quivers and Quiver Varieties

Here we recall the definition of representations of quivers and introduce quiver varieties. See also [23] in this book.

3.1.1 Quivers. A *quiver* $Q = (Q_0, Q_1, s, t)$ is the quadruple consisting of Q_0, the set of *vertices*, and Q_1, the set of *arrows* connecting vertices in Q_0, and two maps $s, t : Q_1 \to Q_0$, which associate to each arrow $\rho \in Q_1$ its *source* $s(\rho) \in Q_0$ and its *target* $t(\rho) \in Q_0$ respectively.

3.1.2 Representations of Quivers. Let Q be a finite quiver, i.e., Q_0 and Q_1 are finite sets. A *representation* M of Q is defined by the following data:

(i) To each vertex a in Q_0, a finite- dimensional $\mathbb{C}$- vector space M_a is attached.
(ii) To each arrow $\rho\colon a \to b$ in Q_1, a $\mathbb{C}$-linear map $\psi_\rho\colon M_a \to M_b$ is attached.

We denote the representation by $M = (M_a, \psi_\alpha)_{a\in Q_0, \alpha\in Q_1}$. The collection of integers defined by $\mathbf{dim}\, M = (\dim_{\mathbb{C}} M_a)_{a\in Q_0}$ is called the *dimension vector* of M.

For a fixed vector $\boldsymbol{\alpha} \in (\mathbb{Z}_{\geq 0})^{Q_0}$, the representation space is

$$\mathrm{Rep}_Q(V, \boldsymbol{\alpha}) = \bigoplus_{\rho\in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{s(\rho)}, V_{t(\rho)}),$$

where $V = (V_a)_{a\in Q_0}$ is a collection of finite dimensional $\mathbb{C}$-vector spaces with $\dim_{\mathbb{C}} V_a = \alpha_a$. If $V_a = \mathbb{C}^{\alpha_a}$ for all $a \in Q_0$, we simply write

$$\mathrm{Rep}_Q(\boldsymbol{\alpha}) = \bigoplus_{\rho\in Q_1} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho)}}, \mathbb{C}^{\alpha_{t(\rho)}}).$$

The space $\mathrm{Rep}_Q(V, \boldsymbol{\alpha})$ has an action of $\prod_{a\in Q_0} \mathrm{GL}(V_a)$. For $(\psi_\rho)_{\rho\in Q_1} \in \mathrm{Rep}_Q(V, \boldsymbol{\alpha})$ and $g = (g_a) \in \prod_{a\in Q_0} \mathrm{GL}(V_a)$, then $g \cdot (\psi_\rho)_{\rho\in Q_1} \in \mathrm{Rep}_Q(V, \boldsymbol{\alpha})$ consists of $\psi'_\rho = g_{t(\rho)}\psi_\rho g^{-1}_{s(\rho)}$ in $\mathrm{Hom}_{\mathbb{C}}(V_{s(\rho)}, V_{t(\rho)})$.

Let $M = (M_a, \psi_\rho^M)_{a\in Q_0,\rho\in Q_1}$ and $N = (N_a, \psi_\rho^N)_{a\in Q_0,\rho\in Q_1}$ be representations of a quiver Q. Then N is called the *subrepresentation* of M if we have the following:

1. For each $a \in Q_0$, $N_a \subset M_a$.
2. For each $\rho\colon a \to b \in Q_1$, $\psi_\rho^M|_{N_a} = \psi_\rho^N$.

In this case, we denote $N \subset M$. Moreover, if

(3) there exists a direct sum decomposition $M_a = N_a \oplus N_a'$ for each $a \in Q_0$,
(4) for each $\rho\colon a \to b \in Q_1$, we have $\psi_\rho^M|_{N_a'} \subset N_b'$,

then we say M has a *direct sum decomposition* $M = N \oplus N'$ where $N' = (N_a', \psi_\rho^M|_{N_a'})_{a\in Q_0,\rho\in Q_1}$.

The representation M is said to be *irreducible* if M has no subrepresentations other than M and $\{0\}$. Here $\{0\}$ is the representation of Q which consists of zero vector spaces and zero linear maps. On the other hand, if any direct sum decomposition $M = N \oplus N'$ satisfies either $N = \{0\}$ or $N' = \{0\}$, then M is said to be *indecomposable*.

Let us recall the notion of double quiver associated to a quiver.

3.1.3 Double of a Quiver. Let $Q = (Q_0, Q_1)$ be a finite quiver. Then the *double quiver* $\overline{Q}$ of Q is the quiver obtained by adjoining the reverse arrow $\rho^*\colon b \to a$ to each arrow $\rho\colon a \to b$. Namely, $\overline{Q} = (\overline{Q}_0 = Q_0, \overline{Q}_1 = Q_1 \cup Q_1^*)$ where $Q_1^* = \{\rho^*\colon t(\rho) \to s(\rho) \mid \rho \in Q_1\}$.

Here we note that the representation space $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha})$ of the double quiver $\overline{Q}$ can be regarded as the cotangent bundle of $\mathrm{Rep}_Q(\boldsymbol{\alpha})$, namely,

$$\mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha}) \cong \mathrm{Rep}_Q(\alpha) \oplus \mathrm{Rep}_Q(\boldsymbol{\alpha})^* \cong T^*\mathrm{Rep}_Q(\boldsymbol{\alpha}),$$

since we have the identification

$$\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho)}}, \mathbb{C}^{\alpha_{t(\rho)}})^* \cong \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{\alpha_{s(\rho^*)}}, \mathbb{C}^{\alpha_{t(\rho^*)}})$$

for each $\rho \in Q_1$. Then, we can regard $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha}) \cong T^*\mathrm{Rep}_Q(\boldsymbol{\alpha})$ as a symplectic manifold with the canonical symplectic form

$$\omega(x, y) := \sum_{\rho\in Q_1} (\mathrm{tr}(x_\rho y_{\rho^*}) - \mathrm{tr}(x_{\rho^*} y_\rho))$$

for $x, y \in T^*\mathrm{Rep}_Q(\boldsymbol{\alpha})$, which is invariant under the action of

$$\mathrm{GL}(\boldsymbol{\alpha}) := \prod_{a\in Q_0} \mathrm{GL}(\alpha_a, \mathbb{C}).$$

Then we define a *moment map* of the symplectic manifold $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha})$ with the $\mathrm{GL}(\boldsymbol{\alpha})$-action as follows: The map

$$\mu_{\boldsymbol{\alpha}}\colon \mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha}) \to \mathrm{Lie}\,\mathrm{GL}(\boldsymbol{\alpha}) := \prod_{a\in Q_0} M(\alpha_a, \mathbb{C})$$

is defined by

$$\mu_{\boldsymbol{\alpha}}(x)_a = \sum_{\substack{\rho \in Q_1 \\ t(\rho)=a}} x_\rho x_{\rho^*} - \sum_{\substack{\rho \in Q_1 \\ s(\rho)=a}} x_{\rho^*} x_\rho,$$

for $x = (x_\rho)_{\rho \in \overline{Q}_1} \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha})$.

Then the quiver variety is defined as the symplectic reduction of $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha})$ by the moment map $\mu_{\boldsymbol{\alpha}}$.

3.1.4 Quiver Variety. Let us take a collection of complex numbers $\lambda = (\lambda_a)_{a \in Q_0} \in \mathbb{C}^{Q_0}$ and regard $\lambda = (\lambda_a I_{\alpha_a})_{a \in Q_0} \in \prod_{a \in Q_0} M(\alpha_a, \mathbb{C})$. Then the *quiver variety* is the symplectic reduction

$$\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha}) := \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)/\mathrm{GL}(\boldsymbol{\alpha}).$$

This variety might have singularities. Thus, let us consider the (possibly empty) subspace

$$\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{irr}} := \{x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda) \mid x \text{ is irreducible}\}.$$

Then the action of $\mathrm{GL}(\boldsymbol{\alpha})/\mathbb{C}^\times$ on this space is proper and moreover free (see King [24]). Thus the quotient space

$$\mathfrak{M}_\lambda^{\mathrm{reg}}(Q, \boldsymbol{\alpha}) := \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{irr}}/\mathrm{GL}(\boldsymbol{\alpha})$$

can be seen as a complex manifold with the symplectic structure, i.e., a complex symplectic manifold. We call this manifold a quiver variety too.

3.2 Crawley-Boevey's Theorems for the Geometry of Quiver Varieties

The regular part $\mathfrak{M}_\lambda^{\mathrm{reg}}(Q, \boldsymbol{\alpha})$ may be empty as we noted above. Thus we recall a necessary and sufficient condition for the non-emptiness of $\mathfrak{M}_\lambda^{\mathrm{reg}}(Q, \boldsymbol{\alpha})$ given by Crawley-Boevey in [9].

First, let us introduce the root system of a quiver Q (cf. [20]). Let Q be a finite quiver. From the *Euler form*

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle := \sum_{a \in Q_0} \alpha_a \beta_a - \sum_{\rho \in Q_1} \alpha_{s(\rho)} \beta_{t(\rho)},$$

a symmetric bilinear form and quadratic form are defined by

$$(\boldsymbol{\alpha}, \boldsymbol{\beta}) := \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle + \langle \boldsymbol{\beta}, \boldsymbol{\alpha} \rangle,$$
$$q(\boldsymbol{\alpha}) := \frac{1}{2}(\boldsymbol{\alpha}, \boldsymbol{\alpha})$$

and set $p(\boldsymbol{\alpha}) := 1 - q(\boldsymbol{\alpha})$. Here $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{Z}^{Q_0}$.

For each vertex $a \in Q_0$, define $\boldsymbol{\varepsilon_a} \in \mathbb{Z}^{Q_0}$ $(a \in Q_0)$ so that $(\boldsymbol{\varepsilon_a})_a = 1$, $(\boldsymbol{\varepsilon_a})_b = 0$, $(b \in Q_0 \backslash \{a\})$. We call $\boldsymbol{\varepsilon_a}$ a *fundamental root* if the vertex a has no edge-loop, i.e., there is no arrow ρ such that $s(\rho) = t(\rho) = a$. Denote by Π the set of fundamental roots. For a fundamental root $\boldsymbol{\varepsilon_a}$, define the *fundamental reflection* s_a by

$$s_a(\boldsymbol{\alpha}) := \boldsymbol{\alpha} - (\boldsymbol{\alpha}, \boldsymbol{\varepsilon_a})\boldsymbol{\varepsilon_a} \text{ for } \boldsymbol{\alpha} \in \mathbb{Z}^{Q_0}.$$

The group $W \subset \operatorname{Aut} \mathbb{Z}^{Q_0}$ generated by all fundamental reflections is called *Weyl group* of the quiver Q. Note that the bilinear form $(\ ,\ )$ is W-invariant. Similarly we can define the reflection $r_a : \mathbb{C}^{Q_0} \to \mathbb{C}^{Q_0}$ by

$$r_a(\lambda)_b := \lambda_b - (\boldsymbol{\varepsilon_a}, \boldsymbol{\varepsilon_b})\lambda_a$$

for $\lambda \in \mathbb{C}^{Q_0}$ and $a, b \in Q_0$. Define the set of *real roots* by

$$\Delta^{\mathrm{re}} := \bigcup_{w \in W} w(\Pi).$$

For an element $\boldsymbol{\alpha} = (\alpha_a)_{a \in Q_0} \in \mathbb{Z}^{Q_0}$ the *support* of α is the set of $\boldsymbol{\varepsilon_a}$ such that $\alpha_a \neq 0$, and denoted by $\operatorname{supp}(\boldsymbol{\alpha})$. We say the support of $\boldsymbol{\alpha}$ is *connected* if the subquiver consisting of the set of vertices a satisfying $\boldsymbol{\varepsilon_a} \in \operatorname{supp}(\boldsymbol{\alpha})$ and all arrows joining these vertices, is connected. Define the *fundamental set* $F \subset \mathbb{Z}^{Q_0}$ by

$$F := \left\{\boldsymbol{\alpha} \in (\mathbb{Z}_{\geq 0})^{Q_0} \backslash \{0\} \mid (\boldsymbol{\alpha}, \boldsymbol{\varepsilon}) \leq 0 \text{ for all } \boldsymbol{\varepsilon} \in \Pi,\ \text{support of } \boldsymbol{\alpha} \text{ is connected}\right\}.$$

Then define the set of *imaginary roots* by

$$\Delta^{\mathrm{im}} := \bigcup_{w \in W} w(F \cup -F).$$

Then the *root system* is

$$\Delta = \Delta^{\mathrm{re}} \cup \Delta^{\mathrm{im}}.$$

Elements in $\Delta^+ := \Delta \cap (\mathbb{Z}_{\geq 0})^{Q_0}$ are called *positive roots*.

Now we are ready to see Crawley-Boevey's theorem. For a fixed $\lambda = (\lambda_a) \in \mathbb{C}^{Q_0}$, the set Σ_λ consists of the positive roots satisfying

(i) $\lambda \cdot \boldsymbol{\alpha} = \sum_{a \in Q_0} \lambda_a \alpha_a = 0$,
(ii) if there exists a decomposition $\boldsymbol{\alpha} = \boldsymbol{\beta_1} + \boldsymbol{\beta_2} + \cdots + \boldsymbol{\beta_r}$ $(r \geq 2)$, with $\boldsymbol{\beta_i} \in \Delta^+$ and $\lambda \cdot \boldsymbol{\beta_i} = 0$, then $p(\boldsymbol{\alpha}) > p(\boldsymbol{\beta_1}) + p(\boldsymbol{\beta_2}) + \cdots + p(\boldsymbol{\beta_r})$.

3.2.1 Theorem (Crawley-Boevey. Theorem 1.2 in [9]). *Let Q be a finite quiver and $\overline{Q}$ the double of Q. Let us fix a dimension vector $\boldsymbol{\alpha} \in (\mathbb{Z}_{\geq 0})^{Q_0}$ and $\lambda \in \mathbb{C}^{Q_0}$. Then $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{irr} \neq \emptyset$ if and only if $\boldsymbol{\alpha} \in \Sigma_\lambda$. Furthermore, in this case $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)$ is an irreducible algebraic variety and $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{irr}$ is dense in $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)$.*

This provides the following geometric properties of quiver varieties:

3.2.2 Theorem (Crawley-Boevey Corollary 1.4 in [9]). *If* $\boldsymbol{\alpha} \in \Sigma_\lambda$ *then the quiver variety* $\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})$ *is reduced and irreducible algebraic variety of dimension* $2p(\boldsymbol{\alpha})$.

Combining these results, we have the following non-emptiness condition of regular parts of quiver varieties.

3.2.3 Corollary. (Crawley-Boevey [9]). *The regular part of quiver variety* $\mathfrak{M}_\lambda^{reg}(Q, \boldsymbol{\alpha})$ *is nonempty if and only if* $\boldsymbol{\alpha} \in \Sigma_\lambda$. *Furthermore in this case, it is a connected symplectic complex manifold of dimension* $2p(\boldsymbol{\alpha})$.

4 A Review of Fuchsian Cases

A necessary and sufficient condition for the existence of a solution of the additive Deligne–Simpson problem for Fuchsian differential equations is determined by Crawley-Boevey in [10]. The strategy is as follows. For the additive Deligne–Simpson problem for $\mathbf{C} = (C_0, C_1, \dots, C_p)$, a collection of conjugacy classes in $M(n, \mathbb{C})$, it is shown that there exists a quiver Q, dimension vector α, and complex parameter λ such that the quiver variety $\mathfrak{M}_\lambda^{\mathrm{reg}}(Q, \boldsymbol{\alpha})$ is isomorphic to the moduli space $\mathfrak{M}(\mathbf{C})$. Thus, Theorem 3.2.1 determines the non-emptiness condition of $\mathfrak{M}(\mathbf{C})$ which is equivalent to the solvability of the additive Deligne–Simpson problem. We shall recall this correspondence between $\mathfrak{M}(\mathbf{C})$ and $\mathfrak{M}_\lambda^{\mathrm{reg}}(Q, \boldsymbol{\alpha})$.

First we construct a representation of a quiver from a conjugacy class C of $M(n, \mathbb{C})$. Let us choose complex numbers $\xi_1, \dots, \xi_d$ so that

$$\prod_{i=1}^{d}(A - \xi_i I_n) = 0 \tag{1}$$

for all $A \in C$. The minimal polynomial of C is an example of this equation. Set

$$m_k := \operatorname{rank} \prod_{i=1}^{k}(A - \xi_i I_n)$$

for $k = 1, \dots, d$. Then let us note that these $\xi_1, \dots, \xi_d$ and $m_1, \dots, m_d$ characterize C. Namely $B \in M(n, \mathbb{C})$ is contained in C if and only if B satisfies

$$\operatorname{rank} \prod_{i=1}^{k}(B - \xi_i I_n) = m_k$$

for all $k = 1, \dots, d$. This observation leads us to the following correspondence between the elements in C and some representations of a quiver.

4.0.1 Proposition (see Crawley-Boevey [10] and also Lemma A.5 in [19]). *Let us fix a conjugacy class C of $M(n,\mathbb{C})$ and choose $\xi_1,\dots,\xi_d \in \mathbb{C}$ so that the equation (1) holds for all $A \in C$. Set $m_k := \operatorname{rank} \prod_{i=1}^{k}(A-\xi_i I_n)$ for $k=1,\dots,d-1$ and $A \in C$, also set $m_0 := n$ and $\mathbf{m} := (m_i)_{i=0,\dots,d-1}$. Define a quiver Q as below.*

Also define a subspace of $\mathrm{Rep}_{\overline{Q}}(\mathbf{m})$ by

$$Z := \left\{ x=(x_\rho) \in \mathrm{Rep}_{\overline{Q}}(\mathbf{m}) \;\middle|\; \begin{array}{c} \mu_{\mathbf{m}}(x)_i = (\xi_i - \xi_{i+1}) I_{m_i} \text{ for all } i=1,\dots,d-1, \\ x_\rho : \text{ injective}, \; x_{\rho^*} : \text{ surjective for all } \rho \in Q_1, \rho^* \in Q_1^* \end{array} \right\}.$$

Then

$$\Phi_\xi \colon \{A \in C\} \longrightarrow Z \Big/ \prod_{i=1}^{d-1} \mathrm{GL}(m_i,\mathbb{C})$$

defined below is bijective. For $A \in C$, we define $(M_a,\psi_\rho)_{a\in Q_0,\rho\in\overline{Q}_1}$, a representation of $\overline{Q}$ as follows:

$$M_0 := \mathbb{C}^n, \qquad M_k := \mathrm{Im} \prod_{i=1}^{k}(A-\xi_i I_n) \text{ for all } k=1,\dots,d-1,$$
$$\psi_{\rho_i} : M_i \hookrightarrow M_{i-1} : \text{inclusion}, \qquad \psi_{\rho_i^*} = (A-\xi_i)|_{M_{i-1}}.$$

Then $\Phi_\xi(A)$ is the projection of (M_a,ψ_ρ). The inverse map is given by

$$(x_\rho)_{\rho\in\overline{Q}_1} \mapsto x_{\rho_1} x_{\rho_1^*} + \xi_1.$$

Furthermore for any $x=(x_\rho)_{\rho\in\overline{Q}_1} \in Z$ and any subspace $S \subset \mathbb{C}^n$ invariant under $x_{\rho_1}x_{\rho_1^*}+\xi_1$, there exists a subrepresentation y of x such that $x=0$ (resp. $N=M$) if and only if $S=0$ (resp. $S=\mathbb{C}^n$).

Let $\mathbf{C} := (C_0, C_1, \dots, C_p)$ be a collection of conjugacy classes in $M(n,\mathbb{C})$. As we noted before, a conjugacy class C can be seen as a truncated orbit of an HTL normal form of the case $k=1$. Thus

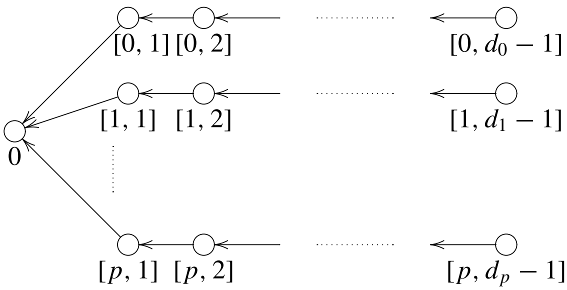
$$\mathfrak{M}(\mathbf{C}) := \left\{ (A_i)_{i=0,1,\dots,p} \in \prod_{i=0}^{p} C_i \;\middle|\; \begin{array}{c} (A_i)_{i=0,\dots,p} \text{ is irreducible}, \\ \sum_{i=0}^{p} A_i = 0 \end{array} \right\} \Big/ \mathrm{GL}(n,\mathbb{C})$$

is a moduli space of Fuchsian differential equations or equivalently that of meromorphic connections defined in Sect. 2.2. Crawely–Boevey obtained a realization of $\mathfrak{M}(\mathbf{C})$ as a quiver variety.

4.0.2 Theorem (Crawley-Boevey [10]). *Let $C_0, \ldots, C_p$ be conjugacy classes of $M(n, \mathbb{C})$. For $i = 0, \ldots, p$, choose $\xi_{[i,1]}, \ldots, \xi_{[i,d_i]} \in \mathbb{C}$ so that*

$$\prod_{j=1}^{d_i} (A_i - \xi_{[i,j]} I_n) = 0$$

for all $A_i \in C_i$. Let $\xi = (\{\xi_{[i,1]}, \ldots, \xi_{[i,d_i]}\})_{0 \le i \le p}$ be the collection of ordered sets $\{\xi_{[i,1]}, \ldots, \xi_{[i,d_i]}\}$. Set $m_0 := n$ and $m_{[i,j]} := \operatorname{rank} \prod_{k=1}^{j}(A_i - \xi_{[i,k]} I_n)$ for $j = 1, \ldots, d_i - 1$. Consider the following quiver Q.



Define $\boldsymbol{\alpha} = (\alpha_a)_{a \in Q_0} \in (\mathbb{Z}_{\ge 0})^{Q_0}$ by $\alpha_0 := m_0$ and $\alpha_{[i,j]} := m_{[i,j]}$ for $i = 0, \ldots, p$, $j = 1, \ldots, d_i - 1$. Define $\lambda = (\lambda_a)_{a \in Q_0} \in \mathbb{C}^{Q_0}$ by $\lambda_0 := -\sum_{i=0}^{p} \xi_{[i,1]}$ and $\lambda_{[i,j]} := \xi_{[i,j]} - \xi_{[i,j+1]}$ for $i = 0, \ldots, p$, $j = 1, \ldots, d_i - 1$.

Then there exists a bijection

$$\Phi_\xi : \mathfrak{M}(\mathbf{C}) \longrightarrow \mathfrak{M}_\lambda^{reg}(Q, \boldsymbol{\alpha}).$$

Thus, Theorem 3.2.1 solves additive Deligne–Simpson problem.

4.0.3 Theorem (Crawley-Boevey [10]). *Let $C_0, \ldots, C_p$ be conjugacy classes of $M(n, \mathbb{C})$. Let us choose the quiver Q, $\boldsymbol{\alpha} \in (\mathbb{Z}_{\ge 0})^{Q_0}$ and $\lambda \in \mathbb{C}^{Q_0}$ as Theorem 4.0.2. Then the additive Deligne–Simpson problem for $C_0, \ldots, C_p$ has a solution if and only if $\boldsymbol{\alpha} \in \Sigma_\lambda$.*

5 Moduli Spaces of Meromorphic Connections and Quiver Varieties

In the previous section, we saw that moduli spaces of Fuchisan differential equations are isomorphic to quiver varieties and moreover the solvability of the additive Deligne–Simpson problem for Fuchsian differential equations is determined through these isomorphisms. In this section, we shall give a generalization of this correspondence. Namely, we shall consider a collection of HTL normal forms

$\mathbf{B} = (B_0, B_1, \ldots, B_p)$, and give a correspondence between the moduli space $\mathfrak{M}(\mathbf{B})$ and a quiver variety. This is first done by Boalch in [5] when the orders of HTL normal forms $k_i = -\mathrm{ord}(B_i)$ for $i = 0, \ldots, p$ satisfy

$$k_0 \leq 3 \text{ and } k_1 = \cdots = k_p = 1.$$

This result is generalized for arbitrary k_0 by Yamakawa and the author in [19]. Thus, we could obtain isomorphisms between moduli spaces of connections and quiver varieties if the number of irregular singular points are at most 1. However in Introduction of [5], Boalch suggested that moduli spaces of meromorphic connections with more than two unramified irregular singular points might not be isomorphic to quiver varieties and gave an example.

Therefore, it may not be expected to obtain isomorphisms between arbitrary $\mathfrak{M}(\mathbf{B})$ and quiver varieties. Based on these previous results, for an arbitrary $\mathfrak{M}(\mathbf{B})$, we shall construct an injective map from $\mathfrak{M}(\mathbf{B})$ into a quiver variety which becomes an isomorphism if and only if the number of unramified irregular singular points are less than or equal to one.

5.1 A Preliminary Example: Differential Equations with Poles of Order 2 and Representations of Quivers

Before going to general cases, let us see what happens if there are many irregular singular points by the first nontrivial case $k_0 = k_1 = \cdots = k_p = 2$.

Let $B \in \mathfrak{g}_2^*$ be an HTL normal form,

$$B = \mathrm{diag}\left(c_1 I_{n_1} z^{-2} + R_1 z^{-1}, \ldots, c_m I_{n_m} z^{-2} + R_m z^{-1}\right).$$

Here $R_i \in M(n_i, \mathbb{C})$ and $c_i \in \mathbb{C}$, $i = 0, \ldots, p$ satisfying $c_i \neq c_j$ if $i \neq j$.

Let us put $B_{\mathrm{irr}} := \mathrm{diag}\left(c_1 I_{n_1}, \ldots, c_m I_{n_m}\right)$ and denote by $V(c_i) \subset \mathbb{C}^n$ the eigenspace of B_{irr} for each eigenvalue c_i, $i = 1, \ldots, m$. For each $X \in M(n, \mathbb{C})$, $X_{i,j}$ denotes the $\mathrm{Hom}_{\mathbb{C}}(V(c_j), V(c_i))$-component of X.

Then for the $G_2 = \mathrm{GL}(n, \mathbb{C}[\![z]\!]/z^2\mathbb{C}[\![z]\!])$-orbit of B, denoted by $\mathcal{O}_B$, we have the following lemma which is a direct consequence of the splitting lemma (see the Sect. 3.2 in [4] or Sect. 2.3 in [19] for example) .

5.1.1 Lemma. *Let $B \in \mathfrak{g}_2^*$ be the HTL normal form as above. Then $\mathcal{O}_B$ consists of*

$$A(x) = \sum_{i=1}^{2} A_i x^{-i} \in \mathfrak{g}_2^*$$

satisfying that there exists $G \in \mathrm{GL}(n, \mathbb{C})$ such that

$$G^{-1}A_2G = B_{irr} \quad and \quad (G^{-1}A_1G)_{i,i} \in C_{R_i}$$

where C_{R_i} are conjugacy classes of R_i for $i = 1, \ldots, m$. Moreover if $G^{(1)}, G^{(2)} \in \mathrm{GL}(n, \mathbb{C})$ satisfy $(G^{(i)})^{-1}A_2G^{(i)} = B_{irr}$, $i = 1, 2$, then $(G^{(2)})^{-1}G^{(1)} = \mathrm{diag}(h_1, \ldots, h_m)$ where $h_i \in \mathrm{GL}(n_i, \mathbb{C})$ for $i = 1, \ldots, m$.

Proof For $A = A_2z^{-2} + A_1z^{-1} \in \mathcal{O}_B$, there exists $G = G_{[0]} + G_{[1]}z \in G_2$ such that $G^{-1}AG = B$. Namely

$$(A_2z^{-2} + A_1z^{-1})(G_{[0]} + G_{[1]}z) = (G_{[0]} + G_{[1]}z)(B_{\mathrm{irr}}z^{-2} + B_{\mathrm{res}}z^{-1})$$

in $\mathfrak{g}_2^* = M(n, z^{-2}\mathbb{C}[\![z]\!]/\mathbb{C}[\![z]\!])$. Here $B_{\mathrm{res}}z^{-1} := B - B_{\mathrm{irr}}z^{-2}$. Thus, comparing the coefficients of z^{-2} and z^{-1}, we have $G_{[0]}^{-1}A_2G_{[0]} = B_{\mathrm{irr}}$ and

$$G_{[0]}^{-1}A_2G_{[1]} + G_{[0]}^{-1}A_1G_{[0]} = G_{[0]}^{-1}G_{[1]}B_{\mathrm{irr}} + B_{\mathrm{res}}.$$

Namely,

$$G_{[0]}^{-1}A_1G_{[0]} - B_{\mathrm{res}} = G_{[0]}^{-1}G_{[1]}B_{\mathrm{irr}} - B_{\mathrm{irr}}G_{[0]}^{-1}G_{[1]}.$$

Recall that $B_{\mathrm{irr}} = \mathrm{diag}(c_1I_{n_1}, \ldots, c_mI_{n_m})$. Thus $(G_{[0]}^{-1}G_{[1]}B_{\mathrm{irr}} - B_{\mathrm{irr}}G_{[0]}^{-1}G_{[1]})_{i,i}$ are zero maps for $i = 1, \ldots, m$. This means that $(G_{[0]}^{-1}A_1G_{[0]}) = R_i$ for $i = 1, \ldots, m$. □

From this lemma, we have the following one-to-one correspondence.

$$\mathcal{O}_B \longrightarrow \left\{ (G, A_1) \in \mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C}) \,\middle|\, \begin{array}{c} (G^{-1}A_1G)_{i,i} \in C_{R_i} \\ \text{for all } i = 1, \ldots, m \end{array} \right\} / \prod_{i=1}^{m} \mathrm{GL}(n_i, \mathbb{C}). \qquad (2)$$

Here $\prod_{i=1}^m \mathrm{GL}(n_i, \mathbb{C})$ acts on $\mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C})$ by

$$\begin{array}{ccc} \prod_{i=1}^m \mathrm{GL}(n_i, \mathbb{C}) \times (\mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C})) & \longrightarrow & \mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C}) \\ ((h_1, \ldots, h_m), (G, A)) & \longmapsto & (Gh, A), \end{array}$$

where $h := \mathrm{diag}(h_1, \ldots, h_m)$. The inverse map is induced by sending (G, A_1) to

$$GB_{\mathrm{irr}}G^{-1}x^{-2} + A_1x^{-1}$$

in $\mathcal{O}_B$.

5.1.2 Remark. Let us recall that $T^*\mathrm{GL}(n, \mathbb{C}) \cong \mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C})$. Then the above correspondence can be seen as a special case of the identification of G_k-orbits of HTL normal forms and a symplectic reductions of the extended orbits given by Boalch (see Lemma 2.3 in [6]).

Note that under the above identification, the adjoint action of $\mathrm{GL}(n, \mathbb{C})$ on $\mathcal{O}_B$ induces the following $\mathrm{GL}(n, \mathbb{C})$-action on $\mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C})$,

$$\begin{array}{ccc} \mathrm{GL}(n,\mathbb{C}) \times (\mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C})) & \longrightarrow & \mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C}) \\ (g,(G,A)) & \longmapsto & (g^{-1}G, g^{-1}Ag) \end{array}. \tag{3}$$

Under the above observation, let us consider the relation between $\mathfrak{M}(\mathbf{B})$ and a quiver variety. Let $B_0,\dots,B_p \in \mathfrak{g}_2^*$ be HTL normal forms written by

$$B_i = \mathrm{diag}\left(c_{[i,1]}I_{n_{[i,1]}}z^{-2} + R_{[i,1]}z^{-1},\dots,c_{[i,m_i]}I_{n_{[i,m_i]}}z^{-2} + R_{[i,m_i]}z^{-1}\right).$$

Let $V(c_{[i,j]}) \subset \mathbb{C}^n$ be the eigenspace of $(B_i)_{\mathrm{irr}}$ for each $c_{[i,j]}$, $i=0,\dots,p$, $j=1,\dots,m_i$. Let $X_{[i,j],[i'j']}$ be the $\mathrm{Hom}_{\mathbb{C}}(V(c_{[i',j']}),V(c_{[i,j]}))$-component of $X \in M(n,\mathbb{C})$. We may write $X = \left(X_{[i,j],[i'j']}\right)_{\substack{1\le j\le m_i\\ 1\le j'\le m_{i'}}}$.

First let us consider the moduli space $\overline{\mathfrak{M}(\mathbf{B})}$ without the irreducibility

$$\overline{\mathfrak{M}(\mathbf{B})} := \left\{ (A_i(z))_{i=0,\dots,p} \in \prod_{i=0}^{p} \mathcal{O}_{B_i} \;\middle|\; \sum_{i=0}^{p} \mathrm{Res}A_i(z) = 0 \right\} / \mathrm{GL}(n,\mathbb{C}).$$

5.1.3 Proposition. *The moduli space $\overline{\mathfrak{M}(\mathbf{B})}$ is isomorphic to*

$$\left\{ (G_i,A_i)_{i=0,\dots,p} \in \prod_{i=0}^{p} \mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C}) \;\middle|\; \begin{array}{l} \text{(i) } (G_i^{-1}A_iG_i)_{j,j} \in C_{R_{[i,j]}} \\ \textit{for all } i=0,\dots,p \textit{ and} \\ j=1,\dots,m_i \\ \text{(ii) } \sum_{i=0}^{p} A_i = 0 \\ \text{(iii) } G_0 = I_n \end{array} \right\} / \prod_{i=0}^{p}\prod_{j=1}^{m_i} \mathrm{GL}(n_{[i,j]},\mathbb{C}).$$

Here the action of $\prod_{i=0}^{p}\prod_{j=1}^{m_i}\mathrm{GL}(n_{[i,j]},\mathbb{C})$ is defined by $h\cdot(G_i,A_i)_{i=0,\dots,p} := (G_i',A_i')_{i=0,\dots,p}$ such that

$$G_i' = h_0^{-1}G_ih_i, \qquad A_i' := h_0^{-1}A_ih_0, \quad i=0,\dots,p$$

for $h=(h_i) \in \prod_{i=0}^{p}\prod_{j=1}^{m_i}\mathrm{GL}(n_{[i,j]},\mathbb{C})$ and $(G_i,A_i)_{i=0,\dots,p} \in \prod_{i=0}^{p}\mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C})$.

Proof Let us consider $\left(\prod_{i=0}^{p}\mathcal{O}_{B_i}\right)/\mathrm{GL}(n,\mathbb{C})$, where $\mathrm{GL}(n,\mathbb{C})$ acts diagonally. Let

$$(A_2^{(i)}z^{-2} + A^{(i)}z^{-1})_{i=0,\dots,p} \text{ and } (\tilde{A}_2^{(i)}z^{-2} + \tilde{A}^{(i)}z^{-1})_{i=0,\dots,p}$$

be representatives of X and $\tilde{X}$ in $\left(\prod_{i=0}^{p}\mathcal{O}_{B_i}\right)/\mathrm{GL}(n,\mathbb{C})$ respectively. By the action of $\mathrm{GL}(n,\mathbb{C})$, we can assume $A_2^{(0)} = \tilde{A}_2^{(0)} = (B_0)_{\mathrm{irr}}$.

Then $X=\tilde{X}$, if and only if there exists $h \in H_0 := \prod_{i=1}^{m_0}\mathrm{GL}(n_{[0,i]},\mathbb{C}) \subset \mathrm{GL}(n,\mathbb{C})$ such that $\tilde{A}_j^{(i)} = h^{-1}A_j^{(i)}h$ for $i=0,\dots,p$ and $j=1,2$ since the stabilizer of $(B_0)_{\mathrm{irr}}$ in $\mathrm{GL}(n,\mathbb{C})$ is H_0. Thus sending $X \in \left(\prod_{i=0}^{p}\mathcal{O}_{B_i}\right)/\mathrm{GL}(n,\mathbb{C})$ to

$$((I_n, A_1^{(0)}), (G^{(1)}, A_1^{(1)}), \ldots, (G^{(p)}, A_1^{(p)})) \in \prod_{i=0}^{p} \mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C})$$

such that $G^{(i)}$, $i = 1, \ldots, p$ are chosen form $A_2^{(i)} z^{-2} + A_1^{(i)} z^{-1} \in \mathcal{O}_{B_i}$ as in Lemma 5.1.1, we have a bijection from $\left(\prod_{i=0}^{p} \mathcal{O}_{B_i}\right)/\mathrm{GL}(n, \mathbb{C})$ to

$$\left\{ (G_i, A_i)_{i=0,\ldots,p} \in \prod_{i=0}^{p} \mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C}) \;\middle|\; \begin{array}{l} \text{(i) } (G_i^{-1} A_i G_i)_{j,j} \in C_{R_{[i,j]}} \\ \text{ for all } i = 0, \ldots, p \text{ and} \\ j = 1, \ldots, m_i \\ \text{(iii) } G_0 = I_n \end{array} \right\} \Big/ \prod_{i=0}^{p} \prod_{j=1}^{m_i} \mathrm{GL}(n_{[i,j]}, \mathbb{C}).$$

Here we note that from (3) and the above construction, the adjoint action of H_0 on $\left(\prod_{i=0}^{p} \mathcal{O}_{B_i}\right)$ induces

$$h \cdot (G_i, A_i)_{i=0,\ldots,p} := \left((I_n, h^{-1} A_0 h), (h^{-1} G_1, h^{-1} A_1 h), \ldots, (h^{-1} G_p, h^{-1} A_p h)\right)$$

for $h \in H_0$ and $(G_i, A_i)_{i=0,\ldots,p} \in \prod_{i=0}^{p} \mathrm{GL}(n, \mathbb{C}) \times M(n, \mathbb{C})$.

Finally, we notice that the condition $\sum_{i=0}^{p} \mathrm{Res} A_i(z) = 0$ corresponds to (ii) $\sum_{i=0}^{p} A_i = 0$. Then we have the required bijection. □

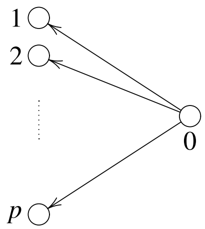
We shall give a realization of $\overline{\mathfrak{M}(\mathbf{B})}$ as a representation space of a quiver as follows.
Step 1.

Let us consider the quiver $Q^{(1)}$ defined as follows. The set of vertices is

$$Q_0^{(1)} := \{0, \ldots, p\}.$$

The set of arrows is

$$Q_1^{(1)} := \left\{ \rho_{[i]}^{[0]} : 0 \to i \;\middle|\; i = 1, \ldots, p \right\}.$$



Fix a dimension vector $\boldsymbol{\alpha}^{(1)} := (\alpha_i)_{i=0,\ldots,p}$ so that $\alpha_i := n$ for all $i = 0, \ldots, p$. Then we have a bijection,

$$\left\{(G_i, A_i)_{i=0,\dots,p} \in \prod_{i=0}^{p} \mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C}) \,\middle|\, G_0 = I_n, \sum_{i=0}^{p} A_i = 0\right\} \longrightarrow$$
$$\left\{x = (x_\rho)_{\rho \in \overline{Q^{(1)}}_1} \in \mathrm{Rep}_{\overline{Q^{(1)}}}(\boldsymbol{\alpha}^{(\mathbf{1})}) \,\middle|\, \begin{array}{l} x_{\rho^{[0]}_{[i]}} \in \mathrm{GL}(n,\mathbb{C}) \\ \text{for all } i = 1,\dots,p \end{array}\right\}$$

by setting $x_{\rho^{[0]}_{[i]}} := G_i^{-1}$ and $x_{(\rho^{[0]}_{[i]})^*} := A_i G_i$ for all $i = 1, \dots, p$. Let us note that from $x = (x_\rho)$ in the target space, setting

$$G_i := x^{-1}_{\rho^{[0]}_{[i]}}, \qquad A_i := x_{(\rho^{[0]}_{[i]})^*} x_{\rho^{[0]}_{[i]}}$$

for $i = 1, \dots, p$ and

$$A_0 := -\sum_{i=1^p} x_{(\rho^{[0]}_{[i]})^*} x_{\rho^{[0]}_{[i]}} = \mu_{\alpha^{(1)}}(x)_0,$$

we obtain the inverse map.

Step 2.

In Step 1, we could associate representations of a quiver to

$$(G_i, A_i)_{i=0,\dots,p} \in \prod_{i=0}^{p} \mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C})$$

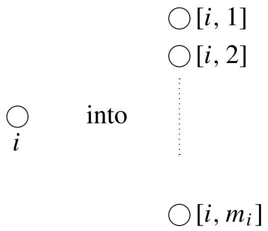
satisfying the conditions (ii) $\sum_{i=0}^{p} A_i = 0$ and (iii) $G_0 = I$. However to obtain the one-to-one correspondence with $\overline{\mathfrak{M}}(\mathbf{B})$, we need one more condition (i) $(G_i^{-1} A_i G_i)_{j,j} \in C_{R_{[i,j]}}$ for $i = 0, \dots, p$, $j = 1, \dots, m_i$. Let us recall that

$$G_i^{-1} A_i G_i = x_{\rho^{[0]}_{[i]}} x_{(\rho^{[0]}_{[i]})^*} = \mu_{\alpha^{(1)}}(x)_i$$

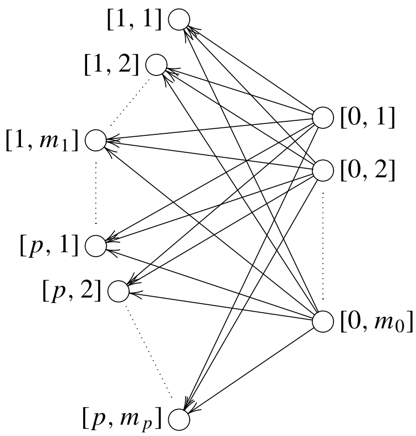
for $i = 1, \dots, p$ and

$$G_0^{-1} A_0 G_0 = A_0 = -\sum_{i=0}^{p} A_i = -\sum_{i=1^p} x_{(\rho^{[0]}_{[i]})^*} x_{\rho^{[0]}_{[i]}} = \mu_{\alpha^{(1)}}(x)_0$$

for $(G_i, A_i)_{i=0,\dots,p}$ in the domain of the isomorphism in Step 1 and its image $x \in \mathrm{Rep}_{\overline{Q^{(1)}}}(\boldsymbol{\alpha}^{(\mathbf{1})})$. To obtain block diagonal components $(G_i^{-1} A_i G_i)_{j,j}$ as images of the moment map, we shall break up the vertex



for each $i = 0, \ldots, p$ and define the following quiver $Q^{(2)}$:



Namely, the set of vertices is

$$Q_0^{(2)} := \{[i, j] \mid i = 0, \ldots, p,\ j = 1, \ldots, m_i\}.$$

The set of arrows is

$$Q_1^{(2)} := \left\{ \rho_{[i,j']}^{[0,j]} \colon [0, j] \to [i, j'] \,\middle|\, \begin{array}{l} j = 1, \ldots, m_0, \\ i = 1, \ldots, p, \\ j' = 1, \ldots, m_i \end{array} \right\}.$$

Define $\boldsymbol{\alpha}^{(2)} = (\alpha_a^{(2)})_{a \in Q_0} \in \mathbb{Z}^{Q_0}$ by $\alpha_{[i,j]}^{(2)} := \dim_{\mathbb{C}} V(c_{[i,j]}),\ i = 0, \ldots, p,\ j = 1, \ldots, m_i$. Then we have a bijection from $\overline{\mathfrak{M}(\mathbf{B})}$ to an open subset of $\mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)})/\mathrm{GL}(\boldsymbol{\alpha}^{(2)})$.

5.1.4 Proposition. *We use the same notation as above. Then there exists a bijection*

$$\Phi \colon \overline{\mathfrak{M}(\mathbf{B})} \longrightarrow \left\{ \begin{array}{l} x = (x_\rho)_{\rho \in \overline{Q^{(2)}}_1} \\ \quad \in \mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)}) \end{array} \,\middle|\, \begin{array}{l} \det\left(x_{\rho_{[i,j']}^{[0,j]}} \right)_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} \neq 0 \\ \text{for all } i = 1, \ldots, p, \\ \mu_{\boldsymbol{\alpha}^{(2)}}(x)_{[i,j]} \in C_{R_{[i,j]}},\ [i, j] \in Q_0^{(2)} \end{array} \right\} / \mathrm{GL}(\boldsymbol{\alpha}^{(2)}).$$

Proof It suffices to show that there is a bijection from

$$\left\{ (G_i, A_i)_{i=0,\dots,p} \in \prod_{i=0}^{p} \mathrm{GL}(n,\mathbb{C}) \times M(n,\mathbb{C}) \;\middle|\; \begin{array}{l} \text{(i) } (G_i^{-1}A_iG_i)_{j,j} \in C_{R_{[i,j]}} \\ \text{for all } i = 0,\dots,p \text{ and} \\ j = 1,\dots,m_i \\ \text{(ii) } \sum_{i=0}^{p} A_i = 0 \\ \text{(iii) } G_0 = I_n \end{array} \right\} \Big/ \prod_{i=0}^{p}\prod_{j=1}^{m_i} \mathrm{GL}(n_{[i,j]},\mathbb{C})$$

to the target space of the above map. Let $(G_i, A_i)_{i=0,\dots,p}$ be a representative of an element in this space. Then we define $x \in \mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)})$ as follows:

$$x_{\rho^{[0,j]}_{[i,j']}} = (G_i^{-1})_{[i,j'],[0,j]}, \qquad x_{\left(\rho^{[0,j]}_{[i,j']}\right)^*} = (A_iG_i)_{[0,j],[i,j']},$$

for $j = 1,\dots,m_0$, $i = 1,\dots,p$, $j' = 1,\dots,m_i$. Then

$$\mu_{\boldsymbol{\alpha}^{(2)}}(x)_{[i,j']} = \sum_{j=1}^{m_0} x_{\rho^{[0,j]}_{[i,j']}} x_{\left(\rho^{[0,j]}_{[i,j']}\right)^*} = (G_i^{-1}A_iG_i)_{j',j'} \in C_{R_{[i,j']}}$$

for $i = 1,\dots,p$ and $j' = 1,\dots,m_i$. Also

$$\mu_{\boldsymbol{\alpha}^{(2)}}(x)_{[0,j]} = -\sum_{i=1}^{p}\sum_{j'=1}^{m_i} x_{\left(\rho^{[0,j]}_{[i,j']}\right)^*} x_{\rho^{[0,j]}_{[i,j']}} = -\sum_{i=1}^{p}(A_i)_{j,j} = (A_0)_{j,j} \in C_{R_{[0,j]}}$$

for $j = 1,\dots,m_0$. Since this correspondence is $\prod_{i=0}^{p}\prod_{j=1}^{m_i}\mathrm{GL}(n_{[i,j]},\mathbb{C}) \cong \mathrm{GL}(\boldsymbol{\alpha}^{(2)})$-equivariant, we have the well-defined map. The inverse maps can be defined as we saw in Step 1. Thus it is bijective. □

Now we are ready to consider $\mathfrak{M}(\mathbf{B})$. Let us note that the irreducibility of differential equations does not coincide with the irreducibility of representations of quiver under the bijection in Proposition 5.1.4. Indeed, $x \in \mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)})$ without the condition $\det\left(x_{\rho^{[0,j]}_{[i,j']}}\right)_{\substack{1\le j\le m_0\\ 1\le j'\le m_i}} \neq 0$ never correspond to any elements in $\overline{\mathfrak{M}(\mathbf{B})}$. Thus, we shall introduce a weaker condition which is called $\mathcal{L}$-irreducibility in this note.

Let us define a sublattice $\tilde{\mathcal{L}}$ of $\mathbb{Z}^{Q_0^{(2)}}$ by

$$\tilde{\mathcal{L}} := \left\{ \boldsymbol{\beta} = (\beta_a)_{a\in Q_0^{(2)}} \in \mathbb{Z}^{Q_0^{(2)}} \;\middle|\; \sum_{j=1}^{m_0}\beta_{[0,j]} = \sum_{j=1}^{m_i}\beta_{[i,j]} \text{ for all } i = 1,\dots,p \right\}$$

5.1.5 Definition ($\tilde{\mathcal{L}}$-irreducible). An element in

$$\left\{ \begin{array}{l} x = (x_\rho)_{\rho \in \overline{Q^{(2)}}_1} \\ \quad \in \mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)}) \end{array} \middle| \begin{array}{l} \det \left(x_{\rho^{[0,j]}_{[i,j']}} \right)_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} \neq 0 \\ \text{for all } i = 1, \dots, p, \\ \mu_{\boldsymbol{\alpha}^{(2)}}(x)_{[i,j]} \in C_{R_{[i,j]}},\ [i,j] \in Q^{(2)}_0 \end{array} \right\} / \mathrm{GL}(\boldsymbol{\alpha})$$

is said to be $\tilde{\mathcal{L}}$*-irreducible*, if it has no proper subrepresentation y with the dimension vector $\mathbf{dim}(y) \in \tilde{\mathcal{L}}$ other than $\{0\}$.

Then, we can show that this $\tilde{\mathcal{L}}$-irreducibility of representations coincides with the irreducibility of differential equations.

5.1.6 Proposition. *Let* $\mathbf{A} \in \overline{\mathfrak{M}(\mathbf{B})}$ *and* $x \in \mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)})$ *be the corresponding elements under the map* Φ *in Proposition 5.1.4. If* $\mathbf{A}$ *is irreducible, then* x *is* $\tilde{\mathcal{L}}$*-irreducible and vice versa.*

Proof Suppose that $\mathbf{A}$ has a nontrivial invariant subspace $W \subsetneq \mathbb{C}^n$, i.e., W is invariant under all $A_j^{(i)}$. Set $W^{(i)} := (x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} W \cong W$ for $i = 1, \dots, p$ and $W^{(0)} := W$. Also set

$$\tilde{V}_{[i,j]} := W^{(i)} \cap V(c_{[i,j]}), \qquad \tilde{x}_{\rho^{[0,j]}_{[i,j']}} = x_{\rho^{[0,j]}_{[i,j']}}|_{W^{(0)}},$$
$$\tilde{x}_{\left(\rho^{[0,j]}_{[i,j']}\right)^*} = x_{\left(\rho^{[0,j]}_{[i,j']}\right)^*}|_{W^{(i)}}.$$

Then $\tilde{x} = (\tilde{V}_a, \tilde{x}_\rho)_{a \in Q_0^{(2)}, \rho \in \overline{Q^{(2)}}_1}$ defines a subrepresentation of x. Since W is $A_2^{(i)}$-invariant, $W^{(i)}$ is $G_i^{-1} A_2^{(i)} G_i = (B_i)_{\mathrm{irr}}$- invariant. Thus we have $W^{(i)} = \oplus_{j=1}^{m_i} \tilde{V}_{[i,j]}$, which shows that $\sum_{j=1}^{m_0} \dim_{\mathbb{C}} \tilde{V}_{[0,j]} = \cdots = \sum_{j=1}^{m_p} \dim_{\mathbb{C}} \tilde{V}_{[p,j]}$. Finally we need to check that $\tilde{x}_{(\rho^{[0,j]}_{[i,j']})^*}(\tilde{V}_{[i,j']}) \subset \tilde{V}_{[0,j]}$. To show this, it suffices to see that $(\tilde{x}_{(\rho^{[0,j]}_{[i,j']})^*})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} W^{(i)} \subset W$, which follows from the fact that

$$\begin{aligned} (\tilde{x}_{(\rho^{[0,j]}_{[i,j']})^*})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} W^{(i)} &= (x_{(\rho^{[0,j]}_{[i,j']})^*})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} W^{(i)} \\ &= (x_{(\rho^{[0,j]}_{[i,j']})^*})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} (x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} W \\ &= A_1^{(i)} W \subset W. \end{aligned}$$

Conversely, suppose that x has a nontrivial proper subrepresentation $\tilde{x} = (\tilde{V}_a, \tilde{x}_\rho)$ satisfying $\sum_{j=1}^{m_0} \dim_{\mathbb{C}} \tilde{V}_{[0,j]} = \cdots = \sum_{j=1}^{m_p} \dim_{\mathbb{C}} \tilde{V}_{[p,j]}$. Then $W = \bigoplus_{j=1}^{m^{(0)}} \tilde{V}_{[0,j]}$ is an $\mathbf{A}$-invariant subspace. Indeed W is $(A_1^{(0)}, A_2^{(0)})$-invariant. Also for $i = 1, \dots, p$, set $W^{(i)} := (x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} W \subset \oplus_{j=1}^{m_i} \tilde{V}_{[i,j]}$. Then we have

$$\sum_{j=1}^{m_i} \dim_{\mathbb{C}} \tilde{V}_{[i,j]} = \sum_{j=1}^{m_0} \dim_{\mathbb{C}} \tilde{V}_{[0,j]} = \dim_{\mathbb{C}} W = \dim_{\mathbb{C}} W^{(i)},$$

which implies that $W^{(i)}=\oplus_{j=1}^{m_i}\tilde{V}_{[i,j]}$. Thus since

$$(x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1\le j\le m_0\\ 1\le j'\le m_i}}(x_{(\rho^{[0,j]}_{[i,j']})^*})_{\substack{1\le j\le m_0\\ 1\le j'\le m_i}}=G_i^{-1}A_1^{(i)}G_i,$$

$W^{(i)}$ is $G_i^{-1}A_1^{(i)}G_i$-invariant, which shows that $W=G_iW^{(i)}$ is $(A_1^{(i)},A_2^{(i)})$-invariant for each $i=1,\dots,p$. □

Finally let us give a realization of $\mathfrak{M}(\mathbf{B})$ as a subset of a quiver variety $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})$ defined as below. The final step is to describe the conjugacy classes $C_{R_{[i,j]}}$ as representations of quivers by Theorem 4.0.1. And glue these quivers to $Q^{(2)}$.

For each $i=0,\dots,p$ and $j=1,\dots,m_i$, we choose $\xi_{[i,j,k]}\in\mathbb{C}, k=1,\dots,e_{[i,j]}$ so that

$$\prod_{k=1}^{e_{[i,j]}}(R_{[i,j]}-\xi_{[i,j,k]}I_{n_{[i,j]}})=0.$$

Then let us define a quiver $Q^{[i,j]}$ for each $[i,j]$ as follows:

$$Q_0^{[i,j]}:=\{[i,j,k]\mid 0\le k\le e_{[i,j]-1}\}$$
$$Q_1^{[i,j]}:=\{\rho_{[i,j,k]}\colon[i,j,k]\to[i,j,k-1]\mid 1\le k\le e_{[i,j]}-1\}$$

Set $\boldsymbol{\alpha}^{[i,j]}:=(\alpha_{[i,j,k]})_{0\le k\le e_{[i,j]}-1}$ by

$$\alpha_{[i,j,0]}=n_{[i,j]},\quad \alpha_{[i,j,k]}:=\text{rank}\prod_{l=1}^{k}(R_{[i,j]}-\xi_{[i,j,l]}I_{n_{[i,j]}}),\ k=1,\dots,e_{[i,j]}-1.$$

Then for each $\xi^{[i,j]}:=(\xi_{[i,j,1]},\dots,\xi_{[i,j,e_{[i,j]}]})$, we have the isomorphism

$$\Phi_{\xi_{[i,j]}}\colon C_{R_{[i,j]}}\to Z_{[i,j]}/\prod_{k=1}^{e_{[i,j]}-1}\mathrm{GL}(\alpha_{[i,j,k]},\mathbb{C})$$

by Theorem 4.0.1. Here

$$Z_{[i,j]}:=\left\{(x_\rho)\in\mathrm{Rep}_{\overline{Q^{[i,j]}}}(\boldsymbol{\alpha}^{[i,j]})\ \middle|\ \begin{array}{c}\mu_{\alpha_k^{[i,j]}}=(\xi_{[i,j,k]}-\xi_{[i,j,k+1]})I_{\alpha_{[i,j,k]}}\\ \text{for all } i=1,\dots,e_{[i,j]}-1,\\ x_\rho:\ \text{injective},\ x_{\rho^*}:\ \text{surjective}\\ \text{for all }\rho\in Q_1^{[i,j]},\rho^*\in(Q_1^{[i,j]})^*\end{array}\right\}.$$

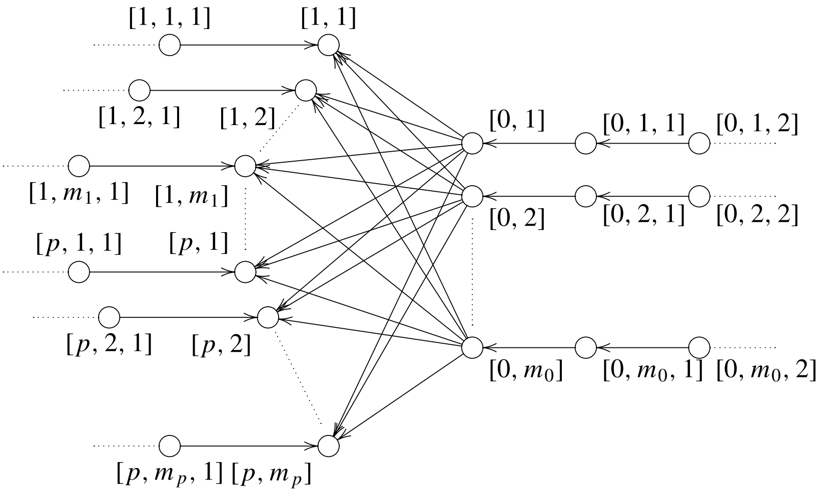
At last, let us glue $Q^{[i,j]}$ to $Q^{(2)}$ by identifying $[i,j,0]\in Q_0^{[i,j]}$ with $[i,j]\in Q_0^{(2)}$. Namely, the quiver Q is defined by the set of vertices

$$Q_0 := Q_0^{(2)} \sqcup \left\{ [i,j,k] \;\middle|\; \begin{array}{l} i = 0,\ldots,p, \\ j = 1,\ldots,m_i, \\ k = 1,\ldots,e_{[i,j]}-1 \end{array} \right\}$$

and the set of arrows

$$Q_1 := Q_1^{(2)} \bigsqcup \left\{ \rho_{[i,j,k]} \colon [i,j,k] \to [i,j,k-1] \;\middle|\; \begin{array}{l} i = 0,\ldots,p, \\ j = 1,\ldots,m_i, \\ k = 1,\ldots,e_{[i,j]}-1 \end{array} \right\}.$$

Here we set $[i,j,0] := [i,j]$.



Define the dimension vector $\boldsymbol{\alpha} = (\alpha_a)_{a \in Q_0}$ by

$$\alpha_{[i,j]} := n_{[i,j]} \qquad \alpha_{[i,j,k]} := \operatorname{rank} \prod_{l=1}^{k} (R_{[i,j]} - \xi_{[i,j,l]} I_{n_{[i,j]}}).$$

Also define $\lambda = (\lambda_a)_{a \in Q_0} \in \mathbb{C}^{Q_0}$ by

$$\lambda_{[i,j]} := -\xi_{[i,j,1]} \qquad \lambda_{[i,j,k]} := \xi_{[i,j,k]} - \xi_{[i,j,k+1]}$$

where $\xi_{[i,j,e_{[i,j]}]} := 0$. Then we can consider a sublattice $\mathcal{L} \subset \mathbb{Z}^{Q_0}$ defined by the same relation as $\tilde{\mathcal{L}} \subset \mathbb{Z}^{Q_0^{(2)}}$.

$$\mathcal{L} := \left\{ \boldsymbol{\beta} = (\beta_a)_{a \in Q_0} \in \mathbb{Z}^{Q_0} \;\middle|\; \sum_{j=1}^{m_0} \beta_{[0,j]} = \sum_{j=1}^{m_i} \beta_{[i,j]} \text{ for all } i = 1,\ldots,p \right\} \subset \mathbb{Z}^{Q_0}$$

Note that $\boldsymbol{\alpha} \in \mathcal{L}$. Then, we define a subset of the quiver variety $\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})$ as follows:

$$\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})^{\mathrm{dif}} := \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{dif}}/\mathrm{GL}(\boldsymbol{\alpha})$$

where

$$\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{dif}} := \left\{ x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda) \;\middle|\; \begin{array}{l} x \text{ is } \mathcal{L}- \text{ irreducible}, \\ \det\,(x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1\leq j\leq m_0 \\ 1\leq j'\leq m_i}} \neq 0,\ i = 1, \dots, p \end{array} \right\}.$$

Here $\mathcal{L}$-irreducibility is defined as in Definition 5.1.5. Then, from Proposition 4.0.1, 5.1.4, and 5.1.6, we obtain the following identification.

5.1.7 Theorem[17]**.** *We have a bijection*

$$\mathfrak{M}(\mathbf{B}) \longrightarrow \mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})^{dif}.$$

Proof By the isomorphism $\Phi_{\xi^{[i,j]}} : C_{R_{[i,j]}} \to Z_{[i,j]}/\prod_{k=1}^{e_{[i,j]}-1} \mathrm{GL}(\alpha_{[i,j,k]}, \mathbb{C})$ for each $[i, j] \in Q_0^{(2)}$, we can identify

$$\left\{ \begin{array}{l} x = (x_\rho)_{\rho \in \overline{Q^{(2)}}_1} \\ \quad \in \mathrm{Rep}_{\overline{Q^{(2)}}}(\boldsymbol{\alpha}^{(2)}) \end{array} \;\middle|\; \begin{array}{l} x \text{ is } \tilde{\mathcal{L}}\text{–irreducible}, \\ \det\left(x_{\rho^{[0,j]}_{[i,j']}}\right)_{\substack{1\leq j\leq m_0 \\ 1\leq j'\leq m_i}} \neq 0 \\ \text{for all } i = 1, \dots, p, \\ \mu_{\boldsymbol{\alpha}^{(2)}}(x)_{[i,j]} \in C_{R_{[i,j]}},\ [i, j] \in Q_0^{(2)} \end{array} \right\} /\mathrm{GL}(\boldsymbol{\alpha})$$

and

$$\left\{ x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{diff}} \;\middle|\; x_{\rho_{[i,j,k]}} : \text{injective},\ x_{(\rho_{[i,j,k]})^*} : \text{surjective} \right\}.$$

Then from Proposition 4.0.1, 5.1.4, and 5.1.6, it suffices to see that $x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{dif}}$ implies that $x_{\rho_{[i,j,k]}}$ are injective and $x_{(\rho_{[i,j,k]})^*}$ are surjective. This can be checked similarly to the proof of Theorem 1 in [10]. Indeed, if there exists $x_{\rho_{[i,j,k]}}$ which is not injective, then there exists a nonzero element $v \in \mathrm{Ker}(x_{\rho_{[i,j,k]}})$. Set $v_k := v$ and $v_{l+1} := \psi_{(\rho_{[i,j,l+1]})^*}(v_l)$ for $l \leq k$. Then the relation

$$x_{\rho_{[i,j,l+1]}} x_{(\rho_{[i,j,l+1]})^*} - x_{(\rho_{[i,j,l]})^*} x_{\rho_{[i,j,l]}} = \lambda_{[i,j,l]}$$

shows that $x_{\rho_{[i,j,l+1]}}(v_{l+1})$ is a multiple of v_l for $l \geq k$. Thus v_l, $l \leq k$, span a subrepresentation of x, which contradicts to the $\mathcal{L}$-irreducibility of x. A dual argument shows that $x_{(\rho_{[i,j,k]})^*}$ are surjective. □

5.1.8 Remark. In the above theorem, we obtain an isomorphism between the moduli space of meromorphic connections $\mathfrak{M}(\mathbf{B})$ and the subset $\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})^{\mathrm{dif}}$ of the quiver variety. However we should notice that $\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})^{\mathrm{dif}}$ does not coincide with $\mathfrak{M}_\lambda^{\mathrm{reg}}(Q, \boldsymbol{\alpha})$ since we imposed

$$\det\,(x_{\rho_{[i,j']}^{[0,j]}})_{\substack{1\le j\le m_0,\\ 1\le j'\le m_i}}\neq 0$$

and the $\mathcal{L}$-irreducibility does not coincide with the irreducibility in general. Thus Crawley-Boevey's theorem (see Theorem 3.2.1) is not applicable directly to our case.

5.2 Truncated Orbits and Representations of Quivers

Let us go to general cases. First recall the description of truncated orbits $\mathcal{O}_B$ of arbitrary orders k as quiver varieties. The description which will be given in this section was obtained by Boalch in [5] when $k \le 3$ and conjectured for arbitrary k and finally settled in the paper by Yamakawa and the author [19]. More detailed treatment of the materials in this section can be found in the article [38] in this book.

Fix $k > 1$ and $B = \sum_{i=1}^{k} B_i z^{-i} \in \mathfrak{g}_k^*$, an HTL normal form written by

$$B = \operatorname{diag}\left(q_1(z^{-1})I_{n_1} + R_1 z^{-1}, \ldots, q_m(z^{-1})I_{n_m} + R_m z^{-1}\right)$$

where $R_i \in M(n_i, \mathbb{C})$, $q_i(z^{-1}) \in z^{-2}\mathbb{C}[z^{-1}]$, $i = 1, \ldots, m$ and $q_i \neq q_j$ if $i \neq j$. To a pair (j, j'), $1 \le j \neq j' \le m$, we attach an integer

$$d(j, j') := \deg_{\mathbb{C}[x]}(q_j(x) - q_{j'}(x)) - 2. \tag{4}$$

Moreover we set $d(j, j) := -1$ for the latter use.

Let $\bigoplus_{j=1}^{m(s)} V_{\langle s,j\rangle}$ be the decomposition of $\mathbb{C}^n$ as simultaneous invariant spaces of

$$\{B_{s+1}, B_{s+2}, \ldots, B_k\}$$

for $s = 1, \ldots, k-1$. Especially we write $V_j := V_{\langle 1,j\rangle}$ for $j = 1, \ldots, m = m(1)$.

Let $X_{j,j'}$ be the $\mathrm{Hom}_{\mathbb{C}}(V_{j'}, V_j)$-component of $X \in M(n, \mathbb{C})$. For a power series with matrix coefficients $g(z) = \sum_{i=r}^{\infty} g_i z^i \in M(n, \mathbb{C}((z)))$, write $(g(z))_{j,j'} := \sum_{i=r}^{\infty} (g_i)_{j,j'} z^i$, $1 \le j, j' \le m$. We denote the $\mathrm{Hom}_{\mathbb{C}}(V_j, \mathbb{C}^n)$-component of $X \in M(n, \mathbb{C})$ by $X_{*,j}$ for each $i = 1, \ldots, m$. Similarly $X_{j,*}$ denote the $\mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^n, V_j)$-component. We sometimes use the notation

$$X = (X_{j,j'})_{1\le j,j'\le m} = (X_{*,j'})_{1\le j'\le m} = (X_{j,*})_{1\le j\le m}.$$

Let $\pi_s : J_s = \{1, \ldots, m(s)\} \to J_{s+1} = \{1, \ldots, m(s+1)\}$ be the natural surjection such that $V_{\langle s,j\rangle} \subset V_{\langle s+1,\pi_s(j)\rangle}$. Define the total ordering $\{1 < 2 < \cdots < m\}$ on J_1 and also define total orderings on J_s, $s = 2, \ldots, k-1$, so that

$$\text{if } j_1 < j_2, \text{ then } \pi_s(j_1) \le \pi_s(j_2), \quad j_1, j_2 \in J_s.$$

Let us define the subgroup of G_k by

$$G_k^o := \left\{ \sum_{i=0}^{k-1} A_i z^i \in G_k \,\middle|\, A_0 = I_n \right\}.$$

Similarly define the subspace $\mathfrak{g}_k^o := z\mathfrak{g}_k = M\left(n, z\mathbb{C}[\![z]\!]/z^k\mathbb{C}[\![z]\!]\right)$ of $\mathfrak{g}_k$, which can be identified with

$$\left\{ \sum_{i=0}^{k-1} A_i z^i \in \mathfrak{g}_k \,\middle|\, A_0 = 0 \right\}.$$

Then the dual space $(\mathfrak{g}_k^o)^*$ can be identified with

$$\left\{ \sum_{i=0}^{k-1} A_i z^{-i-1} \in \mathfrak{g}_k^* \,\middle|\, A_0 = 0 \right\}.$$

For $A = \sum_{i=1}^{k} A_i z^{-i} \in \mathfrak{g}_k^*$, set $A_{\mathrm{irr}} := \sum_{i=2}^{k} A_i z^{-i}$ and $A_{\mathrm{res}} := A_1$. Then we define the following two orbits

$$\mathcal{O}_B^o := \{gBg^{-1} \mid g \in G_k^o\} \subset \mathfrak{g}_k^*,$$
$$\mathcal{O}_{B_{\mathrm{irr}}} := \{gB_{\mathrm{irr}}g^{-1} \mid g \in G_k^o\} \subset (\mathfrak{g}_k^o)^*.$$

Let us define the subgroup $H \subset \mathrm{GL}(n, \mathbb{C})$ by

$$H = \{h = \mathrm{diag}(h_1, \ldots, h_m) \mid h_i \in \mathrm{GL}(n_i, \mathbb{C}),\ i = 1, \ldots, m\}.$$

The following proposition links $\mathcal{O}_B^o$ with $\mathcal{O}_B$.

5.2.1 Proposition (cf. Lemmas 2.2 and 2.4 in [6]). *Set*

$$\mathrm{Ad}_H(\mathcal{O}_B^o) := \left\{ hAh^{-1} \in \mathfrak{g}_k^* \mid h \in H,\ A \in \mathcal{O}_B^o \right\}.$$

Then we have a bijection

$$\begin{array}{ccc} \mathrm{GL}(n, \mathbb{C}) \times_H \mathrm{Ad}_H(\mathcal{O}_B^o) & \xrightarrow{\sim} & \mathcal{O}_B \\ (g, A) & \longmapsto & gAg^{-1} \end{array}.$$

Here $\mathrm{GL}(n, \mathbb{C}) \times_H \mathrm{Ad}_H(\mathcal{O}_B^o) = \left(\mathrm{GL}(n, \mathbb{C}) \times \mathrm{Ad}_H(\mathcal{O}_B^o)\right) / \sim$*, the equivalence relation* $\sim$ *is defined by* $(g, A) \sim (gh^{-1}, hAh^{-1})$ *for* $h \in H$*.*

According to the ordering on each J_s, $s = 1, \ldots, k-1$, let us define parabolic subalgebras of $M(n, \mathbb{C})$ as follows:

$$\mathfrak{p}(s)^+ := \bigoplus_{\substack{j_1, j_2 \in J_i, \\ j_1 \geq j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{\langle s, j_1 \rangle}, V_{\langle s, j_2 \rangle}),$$

$$\mathfrak{p}(s)^- := \bigoplus_{\substack{j_1, j_2 \in J_i, \\ j_1 \leq j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{\langle s, j_1 \rangle}, V_{\langle s, j_2 \rangle}),$$

and similarly nilpotent subalgebras

$$\mathfrak{u}(s)^+ := \bigoplus_{\substack{j_1, j_2 \in J_i, \\ j_1 > j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{\langle s, j_1 \rangle}, V_{\langle s, j_2 \rangle}),$$

$$\mathfrak{u}(s)^- := \bigoplus_{\substack{j_1, j_2 \in J_i, \\ j_1 < j_2}} \mathrm{Hom}_{\mathbb{C}}(V_{\langle s, j_1 \rangle}, V_{\langle s, j_2 \rangle}),$$

for $s = 1, \ldots, k-1$. Note that $\mathfrak{p}(s)^{\pm} = \mathfrak{h}(s) \oplus \mathfrak{u}(s)^{\pm}$ where

$$\mathfrak{h}(s) := \bigoplus_{j \in J_i} \mathrm{End}_{\mathbb{C}}(V_{\langle s, j \rangle}).$$

Let us define subsets of G_k^o,

$$\mathcal{P}_k^{\pm} := \left\{ \sum_{i=0}^{k-1} P_i z^i \in G_k^o \;\middle|\; P_i \in \mathfrak{p}_{i+1}^{\pm}, i = 1, \ldots, k-1 \right\},$$

$$\mathcal{U}_k^{\pm} := \left\{ \sum_{i=0}^{k-1} U_i z^i \in G_k^o \;\middle|\; U_i \in \mathfrak{u}_{i+1}^{\pm}, i = 1, \ldots, k-1 \right\},$$

and subspaces of $\mathfrak{g}_k^o$ and $(\mathfrak{g}_k^o)^*$,

$$\mathfrak{U}_k^{\pm} := \left\{ \sum_{i=1}^{k-1} U_i z^i \;\middle|\; U_i \in \mathfrak{u}_{i+1}^{\pm},\ i = 1, \ldots, k-1 \right\},$$

$$(\mathfrak{U}_k^{\mp})^* := \left\{ \sum_{i=1}^{k-1} U_i z^{-i-1} \;\middle|\; U_i \in \mathfrak{u}_{i+1}^{\pm},\ i = 1, \ldots, k-1 \right\}.$$

Here, we put $\mathfrak{p}_k^{\pm} := M(n, \mathbb{C})$ and $\mathfrak{u}_k^{\pm} := \{0\}$. Let us note that $\mathcal{P}_k^{\pm}$ are subgroups of G_k^o but $\mathcal{U}_k^{\pm}$ are not closed under the multiplication.

5.2.2 Lemma (Lemma 3.5 in [19]). *For any $g \in G_k^o$, there uniquely exist $u_- \in \mathcal{U}_k^-$ and $p_+ \in \mathcal{P}_k^+$ such that $g = u_- p_+$.*

For $A \in \mathcal{O}_{B_{\mathrm{irr}}}$, take $g \in G_k^o$ so that $g^{-1} A g = B_{\mathrm{irr}}$ and decompose $g = u_- p_+$ as above. Note that u_- does not depend on the choice of g because the stabilizer of B_{irr}

is contained in $\mathcal{P}_k^+$. Thus u_- is uniquely determined by $A \in \mathcal{O}_{B_{\mathrm{irr}}}$. Then let us put $Q = u_- - I_n$, $A' = u_-^{-1}A$ and $P = A'|_{(\mathfrak{U}_k^-)^*}$.

5.2.3 Proposition (Theorem 3.6 in [19]). *The map*

$$\begin{array}{rccc} \Phi\colon \mathcal{O}_{B_{irr}} & \longrightarrow & \mathfrak{U}_k^- \times (\mathfrak{U}_k^-)^* \\ A & \longmapsto & (Q, P) \end{array}$$

is bijective.

Now we can define a quiver Q as follows. The set of vertices is

$$Q_0 := \{0\} \cup \{1, \ldots, m\}.$$

The set of arrows is

$$Q_1 := \left\{ \rho_{i,i'}^{[j]}\colon i \to i' \;\middle|\; \begin{array}{l} 1 \le i < i' \le m, \\ 1 \le j \le d(i, i') \end{array} \right\} \cup \{\rho_i : 0 \to i \mid i = 1, \ldots, m\}.$$

Fix the dimension vector $\boldsymbol{\alpha} = (\alpha_a)_{a \in Q_0}$ defined by $\alpha_0 =: n$ and $\alpha_i := \dim_{\mathbb{C}} V_i$, $i = 1, \ldots, m$.

Let us construct a map from $\mathrm{Ad}_H(\mathcal{O}_B^o)$ to the representation space of $\overline{Q}$. For $A \in \mathcal{O}_{B'}^o$, $B' \in \mathrm{Ad}_H(B)$, we set $(Q, P) = \Phi(A_{\mathrm{irr}})$ and define the representation $x_A \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha})$ as follows:

$$\begin{array}{ll} (x_A)_{\rho_{i,i'}^{[j]}} := P_{i,i'}^{[j]}, & (x_A)_{(\rho_{i,i'}^{[j]})^*} := Q_{i',i}^{[j]}, \\ (x_A)_{\rho_i} := (I_n)_{i,*}, & (x_A)_{\rho_i^*} := (A_{\mathrm{res}})_{*,i}\,, \end{array}$$

for $i, i' = 1, \ldots, m$. Here we set $P = \sum_{i=1}^{k-2} P^{[i]} z^{-i-1}$ and $Q = \sum_{i=1}^{k-2} Q^{[i]} z^i$.

5.2.4 Proposition (Proposition 4.16 in [17]). *The following map is bijective,*

$$\tilde{\Psi}\colon \mathrm{Ad}_H(\mathcal{O}_B^o) \quad \longrightarrow \quad \left\{ x \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha}) \;\middle|\; \begin{array}{l} (\psi_{\rho_i})_{1 \le i \le m} = I_n,\ \mu_{\boldsymbol{\alpha}}(x)_i \in C_{R_i} \\ \text{for } i = 1, \ldots, m \end{array} \right\},$$

which is defined by $\tilde{\Psi}(A) := x_A$ for $A \in \mathrm{Ad}_H(\mathcal{O}_B^o)$ as above. Moreover, $\tilde{\Psi}$ preserves H-actions, i.e., $\tilde{\Psi}(hAh^{-1}) = h \cdot x_A$ for all $h \in H$.

Finally, we can obtain a correspondence between $\mathcal{O}_B$ and representations of $\overline{Q}$.

5.2.5 Proposition. *There exists a bijection*

$$\mathcal{O}_B \cong \mathrm{GL}(n, \mathbb{C}) \times_H \mathrm{Ad}_H(\mathcal{O}_B^o) \longrightarrow$$
$$\left\{ x \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha}) \;\middle|\; \begin{array}{l} \det\,(x_{\rho_i})_{1 \le i \le m} \ne 0,\ \mu_{\boldsymbol{\alpha}}(x)_i \in C_{R_i} \\ \text{for } i = 1, \ldots, m \end{array} \right\} \Big/ \prod_{i=1}^{m} \mathrm{GL}(\alpha_i, \mathbb{C}).$$

Proof Let us define a map $\overline{\Psi}$ from $\mathrm{GL}(n,\mathbb{C}) \times \mathrm{Ad}_H(\mathcal{O}_B^o)$ to

$$\left\{ x \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{\alpha}) \;\middle|\; \begin{array}{l} \det (x_{\rho_i})_{1\le i\le m} \neq 0,\ \mu_{\boldsymbol{\alpha}}(x)_i \in C_{R_i} \\ \text{for } i = 1,\dots,m \end{array} \right\}.$$

For $(g, A) \in \mathrm{GL}(n,\mathbb{C}) \times \mathrm{Ad}_H(\mathcal{O}_B^o)$, $x = \overline{\Psi}((g, A)) =$ is defined as follows:

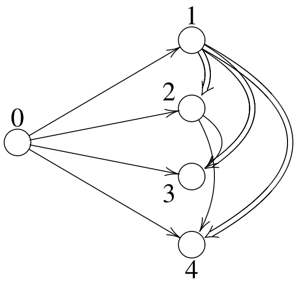
$$x_{\rho^{[j]}_{i,i'}} = P^{[j]}_{i,i'}, \qquad x_{(\rho^{[j]}_{i,i'})^*} = Q^{[j]}_{i',i},$$
$$x_{\rho_i} = (g^{-1})_{i,*}, \qquad x_{\rho_i^*} = ((gA)_{\mathrm{res}})_{*,i}\ ,$$

where $(Q, P) = \Phi(A_{\mathrm{irr}})$ and write $P = \sum_{i=1}^{k-2} P^{[i]} z^{-i-1}$, $Q = \sum_{i=1}^{k-2} Q^{[i]} z^i$. Proposition 5.2.4 shows that this map is bijective. Moreover, we can directly check that this map preserves H-actions. Thus we are done. □

For example, let us consider an HTL normal form $B = \sum_{i=1}^{4} B_i z^{-i}$ such that

$$B_4 = \mathrm{diag}\,(a_1^{(4)}, a_2^{(4)}, a_2^{(4)}, a_2^{(4)}), \qquad B_3 = \mathrm{diag}\,(*, a_1^{(3)}, a_2^{(3)}, a_2^{(3)}),$$
$$B_2 = \mathrm{diag}\,(*, *, a_1^{(2)}, a_2^{(2)}), \qquad B_1 = \mathrm{diag}\,(*, *, *, *),$$

where $a_1^{(i)} \neq a_2^{(i)}$. Then the corresponding quiver is as follows.



5.3 Quivers Associated with Differential Equations

Now we are ready to consider a correspondence between moduli spaces $\mathfrak{M}(\mathbf{B})$ of arbitrary k_i and subsets of quiver varieties $\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})$ as we saw in Sect. 5.1 under the restriction $k_0 = \cdots = k_p = 2$.

For $i = 0, \dots, p$, let us fix a collection of nonzero positive integers k_i and HTL normal forms $B_i = \sum_{j=1}^{k_i} B_j^{(i)} z^{-j} \in \mathfrak{g}^*_{k_i}$. Then write

$$B_i = \mathrm{diag}\left(q_{[i,1]}(z^{-1}) I_{n_{[i,1]}} + R_{[i,1]} z^{-1}, \dots, q_{[i,m_i]}(z^{-1}) I_{n_{[i,m_i]}} + R_{[i,m_i]} z^{-1}\right) \text{ for } i = 0,\dots,p$$

where $q_{[i,j]}(z^{-1}) \in z^{-2}\mathbb{C}[z^{-1}]$ satisfying $q_{[i,j]} \neq q_{[i,j']}$ if $j \neq j'$ and $R_{[i,j]} \in M(n_{[i,j]}, \mathbb{C})$.

For each $i = 0, \ldots, p$, decompose $\mathbb{C}^n = \bigoplus_{j=1}^{m_i(s)} V^{(i)}_{\langle s,j\rangle}$ as simultaneous $(B^{(i)}_{s+1}, \ldots, B^{(i)}_{k_i})$-invariant subspaces. In particular we write $V_{[i,j]} := V^{(i)}_{\langle 1,j\rangle}$ for $i = 0, \ldots, p$ and $j = 1, \ldots, m_i$. Here we note $m_i(1) = m_i$.

For each pair $j, j' \in \{1, \ldots, m_i\}$, attach the integer $d_i(j, j')$ defined by

$$d_i(j, j') := \deg_{\mathbb{C}[z^{-1}]}(q_{[i,j]}(z^{-1}) - q_{[i,j']}(z^{-1})) - 2$$

if $j \neq j'$ or $d_i(j, j') := -1$ if $j = j'$. Set $I_{\mathrm{irr}} := \{i \in \{0, \ldots, p\} \mid m_i > 1\} \cup \{0\}$ and $I_{\mathrm{reg}} := \{0, \ldots, p\} \backslash I_{\mathrm{irr}}$.

5.3.1 Remark. Suppose that $m_i = 1$ for some $i \in \{0, \ldots, p\}$. Then the truncated orbit of the normal form B_i is trivial, namely $\mathcal{O}_{B_i} \cong C_{R_{[i,1]}}$. Thus, I_{irr} can be seen as the set of singular points at which truncated orbits are nontrivial and we add the point 0 as a “base point” to I_{irr}.

Now consider the following quiver Q^{irr}. Let us define

$$Q_0^{\mathrm{irr}} := \{[i, j] \mid i \in I_{\mathrm{irr}},\ j = 1, \ldots, m_i\}.$$

As we saw in Sect. 4, we shall associate conjugacy classes of residue matrices of HTL normal forms to representations of quivers. For each $R_{[i,j]}$, $i = 0, \ldots, p$ and $j = 1, \ldots, m_i$, let us choose $\xi^{[i,j]}_1, \ldots, \xi^{[i,j]}_{e_{[i,j]}} \in \mathbb{C}$ so that

$$\prod_{k=1}^{e_{[i,j]}} (R_{[i,j]} - \xi^{[i,j]}_k) = 0.$$

Set

$$Q_0^{\mathrm{leg}} := \left\{ [i, j, k] \;\middle|\; \begin{array}{l} i = 0, \ldots, p, \\ j = 1, \ldots, m^{(i)}, \\ k = 1, \ldots, e_{[i,j]} - 1 \end{array} \right\}.$$

Then the set of vertices is

$$Q_0 := Q_0^{\mathrm{irr}} \sqcup Q_0^{\mathrm{leg}}.$$

Also define

$$Q_1^{0\to i_{\rm irr}} := \left\{ \rho_{[i,j']}^{[0,j]} \colon [0,j] \to [i,j'] \;\middle|\; \begin{array}{l} j=1,\dots,m_0, \\ i\in I_{\rm irr}\backslash\{0\}, \\ j=1,\dots,m_i \end{array} \right\},$$

$$Q_1^{B_i} := \left\{ \rho_{[i,j],[i,j']}^{[k]} \colon [i,j] \to [i,j'] \;\middle|\; \begin{array}{l} i\in I_{\rm irr}, \\ 1\le j<j'\le m_i, \\ 1\le k\le d_i(j,j') \end{array} \right\},$$

$$Q_1^{{\rm leg}_i} := \left\{ \rho_{[i,j,k]} \colon [i,j,k] \to [i,j,k-1] \;\middle|\; \begin{array}{l} j=1,\dots,m_i, \\ k=2,\dots,e_{[i,j]}-1 \end{array} \right\},$$

$$Q_1^{{\rm leg}_i\to B_i} := \left\{ \rho_{[i,j,1]} \colon [i,j,1] \to [i,j] \mid j=1,\dots,m_i \right\},$$

$$Q_1^{{\rm leg}_i\to 0} := \left\{ \rho_{[0,j]}^{[i,1,1]} \colon [i,1,1] \to [0,j] \mid i\in I_{\rm reg},\ j=1,\dots,m_0 \right\}.$$

The set of arrows is

$$Q_1 := Q_1^{0\to I_{\rm irr}} \sqcup \bigsqcup_{i\in I_{\rm irr}} \left(Q_1^{B_i} \sqcup Q_1^{{\rm leg}_i\to B_i} \sqcup Q_1^{{\rm leg}_i} \right) \sqcup \bigsqcup_{i\in I_{\rm reg}} \left(Q_1^{{\rm leg}_i\to 0} \sqcup Q_1^{{\rm leg}_i} \right).$$

For example, let us consider the following: $\mathbf{B} = (B_0, B_1, B_2)$.

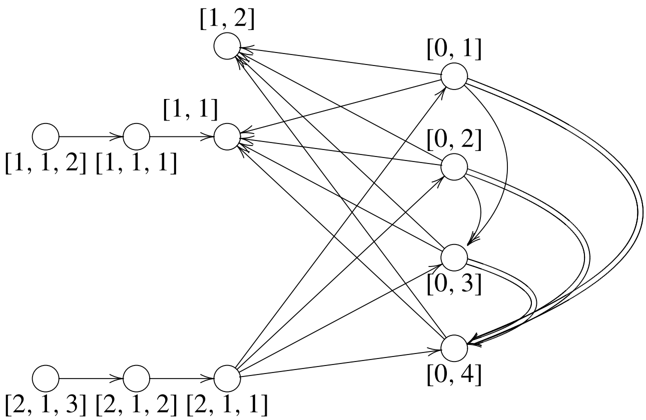
$$B^{(0)} = \begin{pmatrix} a_4^{(0)} &&& \\ & a_4^{(0)} && \\ && a_4^{(0)} & \\ &&& b_4^{(0)} \end{pmatrix} z^{-4} + \begin{pmatrix} a_3^{(0)} &&& \\ & a_3^{(0)} && \\ && b_3^{(0)} & \\ &&& c_3^{(0)} \end{pmatrix} z^{-3}$$
$$+ \begin{pmatrix} a_2^{(0)} &&& \\ & b_2^{(0)} && \\ && c_2^{(0)} & \\ &&& d_2^{(0)} \end{pmatrix} z^{-2} + \begin{pmatrix} \xi_1^{[0,1]} &&& \\ & \xi_1^{[0,2]} && \\ && \xi_1^{[0,3]} & \\ &&& \xi_1^{[0,4]} \end{pmatrix} z^{-1},$$

$$B_1 = \begin{pmatrix} a_2^{(1)} &&& \\ & a_2^{(1)} && \\ && a_2^{(1)} & \\ &&& b_2^{(1)} \end{pmatrix} z^{-2} + \begin{pmatrix} \xi_1^{[1,1]} &&& \\ & \xi_2^{[1,1]} && \\ && \xi_3^{[1,1]} & \\ &&& \xi_1^{[1,2]} \end{pmatrix} z^{-1},$$

$$B_2 = \begin{pmatrix} \xi_1^{[2,1]} &&& \\ & \xi_2^{[2,1]} && \\ && \xi_3^{[2,1]} & \\ &&& \xi_4^{[2,1]} \end{pmatrix} z^{-1}.$$

Here any distinct two of $\{a_j^{(i)}, b_j^{(i)}, c_j^{(i)}, d_j^{(i)}\}$ stand for distinct complex numbers and $\xi_k^{[i,j]} \neq \xi_{k'}^{[i,j]}$ if $k \neq k'$.

Then, we can associate the following quiver to this **B**.:



Define the dimension vector $\boldsymbol{\alpha} = (\alpha_a)_{a \in Q_0}$ by

$$\alpha_{[i,j]} := n_{[i,j]}, \qquad \alpha_{[i,j,k]} := \operatorname{rank} \prod_{l=1}^{k} (R_{[i,j]} - \xi_l^{[i,j]}).$$

Also define $\lambda = (\lambda_a)_{a \in Q_0}$ by

$$\begin{aligned} &\lambda_{[i,j]} := -\xi_1^{[i,j]}, && \text{for } i \in I_{\text{irr}} \backslash \{0\},\ j = 1, \ldots, m_i, \\ &\lambda_{[0,j]} := -\xi_1^{[0,j]} - \sum_{i \in I_{\text{reg}}} \xi_1^{[i,1]} && \text{for } j = 1, \ldots, m_0, \\ &\lambda_{[i,j,k]} := \xi_k^{[i,j]} - \xi_{k+1}^{[i,j]} && \text{for } \begin{array}{l} i = 0, \ldots, p,\ j = 1, \ldots, m_i, \\ k = 1, \ldots, e_{[i,j]} - 1. \end{array} \end{aligned}$$

Define a sublattice of $\mathbb{Z}^{Q_0}$,

$$\mathcal{L} := \left\{ \boldsymbol{\beta} \in \mathbb{Z}^{Q_0} \;\middle|\; \sum_{j=1}^{m^{(0)}} \beta_{[0,j]} = \sum_{j=1}^{m^{(i)}} \beta_{[i,j]} \text{ for all } i \in I_{\text{irr}} \backslash \{0\} \right\}.$$

Set $\mathcal{L}^+ := \mathcal{L} \cap (\mathbb{Z}_{\geq 0})^{Q_0}$.

5.3.2 Definition ($\mathcal{L}$-irreducible). If $x \in \operatorname{Rep}_{\overline{Q}}(\boldsymbol{\alpha})$ has no nontrivial proper subrepresentation $\{0\} \neq y \subsetneq x$ with $\mathbf{dim}\, y \in \mathcal{L}$, then x is said to be *$\mathcal{L}$-irreducible*.

Then we define a subset of the quiver variety $\mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})$ as follows:

$$\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{\mathrm{dif}} := \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{dif}}/\mathrm{GL}(\boldsymbol{\alpha})$$

where

$$\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{dif}} := \left\{ x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda) \;\middle|\; \begin{array}{l} x \text{ is } \mathcal{L}-\text{irreducible}, \\ \det\,(x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1\le j\le m_0 \\ 1\le j'\le m_i}} \neq 0,\ i \in I_{\mathrm{irr}}\backslash\{0\} \end{array} \right\}.$$

Also define

$$\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{det}} := \left\{ x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda) \;\middle|\; \det\,(x_{\rho^{[0,j]}_{[i,j']}})_{\substack{1\le j\le m_0 \\ 1\le j'\le m_i}} \neq 0,\ i \in I_{\mathrm{irr}}\backslash\{0\} \right\}$$

for the latter use.

5.3.3 Theorem (Theorem 4.23 in [17]). *We have a bijection*

$$\Phi_\xi \colon \mathfrak{M}(\mathbf{B}) \longrightarrow \mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{dif}.$$

6 Geometry of Moduli Spaces of Meromorphic Connections

We have seen that the moduli space of meromorphic connections $\mathfrak{M}(\mathbf{B})$ is isomorphic to a subset $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{\mathrm{dif}}$ of the quiver variety $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})$. We shall give some results for geometry of $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{\mathrm{dif}}$ and the necessary and sufficient condition of the solvability of our generalized additive Deligne–Simpson problem as a corollary of these results. These results are generalizations of previous works by Crawley-Boevey [10], Boalch [5], Yamakawa and the author [19]. The strategy to prove these results shall be given in the latter sections.

Although we defined $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{\mathrm{dif}}$ as just a quotient space, the next theorem shows that there exists an open embedding into a smooth variety $\mathfrak{M}^{\mathrm{reg}}_{\lambda'}(Q,\boldsymbol{\alpha})$.

6.0.1 Theorem (Theorem 5.14 in [17]). *If $\mathfrak{M}(\mathbf{B})$ is nonempty, there exist $\lambda' \in \mathbb{C}^{Q_0}$ and the injection*

$$\Phi \colon \mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{dif} \hookrightarrow \mathfrak{M}^{reg}_{\lambda'}(Q,\boldsymbol{\alpha})$$

whose image is $\left(\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{det} \cap \mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{irr}\right)/\mathrm{GL}(\boldsymbol{\alpha})$.

This embedding theorem and the irreducibility of $\mathfrak{M}_{\lambda'}(Q,\boldsymbol{\alpha})$ shows the connectedness of $\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{\mathrm{dif}}$.

6.0.2 Theorem (Corollary 5.15 in [17]). *If $\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{dif}$ is nonempty, then*
$\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{dif}$ has a structure of connected complex manifold.

Proof The intersection $\left(\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{det}} \cap \mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{irr}}\right)$ is an open subset of $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')$ which is irreducible space by Theorem 1.2 in [9]. Because $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{irr}}$ is open by [24] and $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{det}}$ is defined by the open condition $\det\,(x_{\rho_{[i,j']}^{[0,j]}})_{\substack{1\le j\le m_0\\ 1\le j'\le m_i}} \neq 0,\ i \in I_{\mathrm{irr}}\backslash\{0\}$. Recalling that every nonempty open subset of an irreducible space is connected, we are done. □

A necessary and sufficient condition for the non-emptiness of $\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})^{\mathrm{dif}}$ is obtained as follows.

6.0.3 Theorem (Corollary 7.13 in [17]). *The moduli space $\mathfrak{M}(\mathbf{B}) \cong \mathfrak{M}_\lambda(Q, \boldsymbol{\alpha})^{dif}$ is nonempty if and only if the following are satisfied:*

1. *$\boldsymbol{\alpha}$ is a positive root of Q and $\boldsymbol{\alpha}\cdot\lambda = \sum_{a\in Q_0}\alpha_a\lambda_a = 0$,*
2. *for any decomposition $\boldsymbol{\alpha} = \boldsymbol{\beta_1} + \cdots + \boldsymbol{\beta_r}$ where $\boldsymbol{\beta_i} \in \mathcal{L}^+$ are positive roots of Q satisfying $\boldsymbol{\beta_i}\cdot\lambda = 0$, we have*

$$p(\boldsymbol{\alpha}) > p(\boldsymbol{\beta_1}) + \cdots + p(\boldsymbol{\beta_r}).$$

Then this theorem and Proposition 2.2.2 gives the necessary and sufficient condition for the solvability of the additive Deligne–Simpson problem.

Let $\tilde{\Sigma}_\lambda$ be the set of positive roots of Q satisfying the conditions 1 and 2 in Theorem 6.0.3.

6.0.4 Theorem (Theorem 7.12 in [17]). *Let us consider the additive Deligne–Simpson problem for $k_0, \ldots, k_p$ and the HTL normal forms $B_i \in \mathfrak{g}_{k_i}^*$ for $i = 0, \ldots, p$. Then the problem has a solution if and only if $\boldsymbol{\alpha} \in \widetilde{\Sigma}_\lambda$.*

7 Outline of the Proofs of the Main Theorems

The remaining of this note is devoted to give an outline of the proofs of Theorems 6.0.1 and 6.0.3.

7.1 A Review of Middle Convolutions

Let us give a review of middle convolutions on differential equations with irregular singular points. The middle convolution is originally defined by N. Katz in [21] and reformulated as an operation on Fuchsian systems by Dettweiler–Reiter [14], see also [13] and Völklein's paper [35]. There are several studies to generalize the middle convolution to non-Fuchsian differential equations, see [2, 22, 32, 37] for example. Among them, we shall give a review of middle convolutions following [37].

From $\mathbf{A} = (\sum_{j=1}^{k_i} A_j^{(i)} z^{-j})_{0\le i\le p} \in \prod_{i=0}^{p} \mathcal{O}_{B_i}$, let us construct a 5-tuple (V, W, T, Q, P) consisting of $\mathbb{C}$-vector spaces V, W and $T \in \mathrm{End}_{\mathbb{C}}(W)$, $Q \in \mathrm{Hom}_{\mathbb{C}}(W, V)$, $P \in \mathrm{Hom}_{\mathbb{C}}(V, W)$. Set $V := \mathbb{C}^n$ and $\widehat{W}_i := V^{\oplus k_i}$ for $i = 0, \dots, p$. Then define

$$\widehat{Q}_i := (A_{k_i}^{(i)}, A_{k_i-1}^{(i)}, \dots, A_1^{(i)}) \in \mathrm{Hom}_{\mathbb{C}}(\widehat{W}_i, V),$$

$$\widehat{P}_i := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathrm{Id}_V \end{pmatrix} \in \mathrm{Hom}_{\mathbb{C}}(V, \widehat{W}_i),\ \widehat{N}_i := \begin{pmatrix} 0 & \mathrm{Id}_V & & 0 \\ & 0 & \ddots & \\ & & \ddots & \mathrm{Id}_V \\ 0 & & & 0 \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(\widehat{W}_i).$$

Setting

$$\widehat{W} := \bigoplus_{i=0}^{p} \widehat{W}_i,$$

$$\widehat{T} := (\widehat{N}_i)_{0\le i\le p} \in \bigoplus_{i=0}^{p} \mathrm{End}_{\mathbb{C}}(\widehat{W}_i) \subset \mathrm{End}_{\mathbb{C}}(\widehat{W}),$$

$$\widehat{Q} := (\widehat{Q}_i)_{0\le i\le p} \in \bigoplus_{i=0}^{p} \mathrm{Hom}_{\mathbb{C}}(\widehat{W}_i, V) = \mathrm{Hom}_{\mathbb{C}}(\widehat{W}, V),$$

$$\widehat{P} := (\widehat{P}_i)_{0\le i\le p} \in \bigoplus_{i=0}^{p} \mathrm{Hom}_{\mathbb{C}}(V, \widehat{W}_i) = \mathrm{Hom}_{\mathbb{C}}(V, \widehat{W}),$$

we have a 5-tuple $(V, \widehat{W}, \widehat{T}, \widehat{Q}, \widehat{P})$. Further setting

$$\widehat{A}_i := \begin{pmatrix} A_{k_i}^{(i)} & A_{k_i-1}^{(i)} & \cdots & A_1^{(i)} \\ & A_{k_i}^{(i)} & \ddots & \vdots \\ & & \ddots & A_{k_i-1}^{(i)} \\ 0 & & & A_{k_i}^{(i)} \end{pmatrix} \in \mathrm{End}_{\mathbb{C}}(\widehat{W}_i),$$

we define $W_i := \widehat{W}_i / \mathrm{Ker}\, \widehat{A}_i$ and $W := \bigoplus_{i=0}^{p} W_i$. Then T, Q, P are the maps induced from $\widehat{T}, \widehat{Q}, \widehat{P}$ respectively.

7.1.1 Definition (Yamakawa [37]). The 5-tuple (V, W, T, Q, P) given above is called the *canonical datum* for $\mathbf{A} \in \prod_{i=0}^{p} \mathcal{O}_{B_i}$.

Fix $t \in \{0, \dots, p\}$, take a polynomial $p_t(z^{-1}) = \sum_{j=1}^{k_t} p_j^{(t)} z^{-j} \in z^{-1}\mathbb{C}[z^{-1}]$ and define an operation, called *addition*, as follows. For an element $\mathbf{A} = (A_i(x^{-1}))_{0\le i\le p} \in \prod_{i=0}^{p} \mathcal{O}_{B_i}$, we define $\mathrm{Add}_{p_t(z^{-1})}^{(t)}(\mathbf{A}) := (A_i'(z^{-1}))_{0\le i\le p}$ by

$$A_i'(z^{-1}) := \begin{cases} A_i(z^{-1}) & \text{if } i \neq t, \\ A_t(z^{-1}) - p_t(x^{-1}) & \text{if } i = t. \end{cases}$$

Then $\mathrm{Add}^{(t)}_{p_t(x^{-1})}(\mathbf{A}) \in \prod_{i=0}^{p} \mathcal{O}_{B_i'}$ where

$$B_i' := \begin{cases} B_i & \text{if } i \neq t, \\ B_t - p_t(z^{-1}) & \text{if } i = t. \end{cases}$$

Set

$$\mathcal{J}_i := \{[i, j] \mid j = 1, \ldots, m_i\}$$

for $i = 0, \ldots, p$ and

$$\mathcal{J} := \prod_{i=0}^{p} \mathcal{J}_i.$$

Then let us define

$$\mathrm{Add}_{\mathbf{i}} := \prod_{i=0}^{p} \mathrm{Add}^{(i)}_{q_{[i,j_i]}(z^{-1}) + \xi_1^{[i,j_i]} z^{-1}},$$

for $\mathbf{i} = ([i, j_i])_{0 \le i \le p} \in \mathcal{J}$. Here we use the notation $\prod_{i \in \{a,b,\ldots\}} f_i = f_a \circ f_b \circ \cdots$ and note that the operators $\mathrm{Ad}^{(i)}_{q_{[i,j_i]}(z^{-1}) + \xi_1^{[i,j_i]} z^{-1}}$ for $i \in 0, \ldots, p$ are commutative.

Take $\mathbf{A} = (A_i(z^{-1}))_{0 \le i \le p} \in \prod_{i=0}^{p} \mathcal{O}_{B_i}$ satisfying $\sum_{i=0}^{p} \mathrm{Res} A_i(z^{-1}) = 0$. Suppose that we can choose $\mathbf{i} \in \mathcal{J}$ so that

$$\xi_{\mathbf{i}} := \sum_{i=0}^{p} \xi_1^{[i,j_i]} \neq 0.$$

Let (V, W, T, Q, P) be the canonical datum of $\mathrm{Add}_{\mathbf{i}}(\mathbf{A})$. Following Example 3 in [37], we construct a new 5-tuple (V', W, T, Q', P') as follows. Note that $QP = -\xi_{\mathbf{i}} \mathrm{Id}_V$. Thus Q and P are surjective and injective, respectively. Let us set $V' := \mathrm{Coker}\, P$ and $Q' \colon W \to V'$, the natural projection. Then, we have the split exact sequence

$$0 \longrightarrow V \xrightarrow{P} W \xrightarrow{Q'} V' \longrightarrow 0$$

with the left splitting $(-\xi_{\mathbf{i}}^{-1} Q)P = \mathrm{Id}_V$. Then from the splitting, we can define $P' \colon V' \to W$ be the injection such that $Q'(\xi_{\mathbf{i}}^{-1} P') = \mathrm{Id}_{V'}$. Then we have a 5-tuple (V', W, T, Q', P').

Next we set Q_i' (resp. P_i') to be the $\mathrm{Hom}_{\mathbb{C}}(W_i, V)$ (resp. $\mathrm{Hom}_{\mathbb{C}}(V, W_i)$) component of Q' (resp. P'). Also set N_i to be the $\mathrm{End}_{\mathbb{C}}(W_i)$-component of T. Define

$$(A')_j^{(i)} := Q_i' N_i^{j-1} P_i'$$

and $\mathbf{A}' := (A_i'(z^{-1}))_{0\le i\le p}$ where $A_i'(z^{-1}) := \sum_{j=1}^{k_i}(A')_j^{(i)}z^{-j}$. We note that

$$\sum_{i=0}^{p}(A')_1^{(i)} = Q'P' = \xi_{\mathbf{i}}\mathrm{Id}_{V'}.$$

Finally, let us set

$$\mathbf{A}'' := \mathrm{Add}_{\mathbf{i}}^{-1} \circ \mathrm{Add}^{(0)}_{2\xi_{\mathbf{i}}z^{-1}}(\mathbf{A}').$$

Then $\mathbf{A}'' = (A_i''(x^{-1}))_{0\le i\le p}$ satisfies that $\sum_{i=0}^{p}\mathrm{Res}A_i''(x^{-1}) = 0$. Let us denote $\mathbf{A}''$ by $\mathrm{mc}_{\mathbf{i}}(\mathbf{A})$ and call the operator $\mathrm{mc}_{\mathbf{i}}$ the *middle convolution* at $\mathbf{i}$.

Then we can see that the middle convolution gives an analogue of reflection functors on $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{\mathrm{dif}}$ by translating the computations by Yamakawa in [37] to our setting.

7.1.2 Proposition (Proposition 5.5 in [17]). *Let ξ and $\mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{dif}$ be same as in Theorem 5.3.3. Suppose that we can choose $\mathbf{i} = ([i, j_i])_{0\le i\le p} \in \mathcal{J}$ so that*

$$\lambda_{\mathbf{i}} := \sum_{i\in I_{irr}}\lambda_{[i,j_i]} = -\xi_{\mathbf{i}} \neq 0.$$

Define $\mathrm{mc}_{\mathbf{i}}(\boldsymbol{\alpha}) := (\alpha_a')_{a\in Q_0} \in \mathbb{Z}^{Q_0}$ *and* $\mathrm{mc}_{\mathbf{i}}(\lambda) := (\lambda_a')_{a\in Q_0} \in \mathbb{C}^{Q_0}$ *by*

$$\alpha'_{[i,j]} := \begin{cases}\alpha_{[i,j]} & \text{if } j\neq j_i,\\ \alpha_{[i,j_i]} + n_{\mathbf{i}} & \text{if } j = j_i,\end{cases}$$

$$\alpha'_{[i,j,k]} := \alpha_{[i,j,k]},$$

$$\lambda'_{[i,j]} := \begin{cases}\lambda_{[i,j_i]} & \text{if } [i,j]=[i,j_i] \text{ and } i\neq 0,\\ \lambda_{[0,j_0]} - 2\lambda_{\mathbf{i}} & \text{if } [i,j]=[0,j_0],\\ \lambda_{[i,j]} + (d_i(j,j_i)+2)\lambda_{\mathbf{i}} & \text{if } i\neq 0 \text{ and } j\neq m_i,\\ \lambda_{[0,j]} + d_0(j,j_0)\lambda_{\mathbf{i}} & \text{if } i=0 \text{ and } j\neq m_0,\end{cases}$$

$$\lambda'_{[i,j,k]} := \begin{cases}\lambda_{[i,j,k]} & \text{if } [i,j,k]\neq[i,j_i,1],\\ \lambda_{[i,j_i,1]} + \lambda_{\mathbf{i}}.\end{cases}$$

Here

$$n_{\mathbf{i}} := \sum_{i\in I_{irr}}\sum_{j=1}^{m^{(i)}}(d_i(j,j_i)+1)\alpha_{[i,j]} + \sum_{i\in I_{irr}}((n-\alpha_{[i,j_i]})+\alpha_{[i,j_i,1]}) + \sum_{i\in I_{reg}}\alpha_{[i,1,1]} - 2n.$$

Then there exists a bijection

$$\mathrm{mc}_{\mathbf{i}}\colon \mathfrak{M}_\lambda(Q,\boldsymbol{\alpha})^{dif} \longrightarrow \mathfrak{M}_{\mathrm{mc}_{\mathbf{i}}(\lambda)}(Q,\mathrm{mc}_{\mathbf{i}}(\boldsymbol{\alpha}))^{dif}.$$

Let us see that $\mathrm{mc}_{\mathbf{i}}(\boldsymbol{\alpha})$ is obtained by a reflection. For $\mathbf{i} = ([i, j_i])_{0\le i\le p} \in \mathcal{J}$, let us define $\varepsilon_{\mathbf{i}} \in \mathbb{Z}^{Q_0}$ by

$$(\varepsilon_{\mathbf{i}})_a := \begin{cases} 1 & \text{if } a = [i, j_i],\ i \in I_{\mathrm{irr}}, \\ 0 & \text{otherwise.} \end{cases}$$

We note that $\varepsilon_{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{J}$ are positive real roots of Q. Let us define

$$s_{\mathbf{i}}(\boldsymbol{\beta}) := \boldsymbol{\beta} - (\boldsymbol{\beta}, \varepsilon_{\mathbf{i}})\varepsilon_{\mathbf{i}}$$

for $\mathbf{i} \in \mathcal{J}$ and $\boldsymbol{\beta} \in \mathbb{Z}^{Q_0}$.

Let us see this reflection $s_{\mathbf{i}}$ can be obtained by a product of simple reflections.

7.1.3 Lemma. *Let us take* $\mathbf{i} = ([i, j_i])_{0\le i\le p} \in \mathcal{J}$. *Then we have*

$$\left(\prod_{i\in I_{irr}\setminus\{0\}} s_{[i,j_i]}\right) \circ s_{[0,j_0]} \circ \left(\prod_{i\in I_{irr}\setminus\{0\}} s_{[i,j_i]}\right)(\boldsymbol{\beta}) = s_{\mathbf{i}}(\boldsymbol{\beta})$$

for any $\boldsymbol{\beta} \in \mathbb{Z}^{Q_0}$.

Proof Set $r := \prod_{i\in I_{\mathrm{irr}}\setminus\{0\}} s_{[i,j_i]}$ for short. Note that r is an involution and $\varepsilon_{\mathbf{i}} = r(\varepsilon_{\mathbf{[0,j_0]}})$. Then

$$\begin{aligned} r \circ s_{[0,j_0]} \circ r(\boldsymbol{\beta}) &= r(r(\boldsymbol{\beta}) - (r(\boldsymbol{\beta}), \varepsilon_{\mathbf{[0,j_0]}})\varepsilon_{\mathbf{[0,j_0]}}) \\ &= r^2(\boldsymbol{\beta}) - (\boldsymbol{\beta}, r^{-1}(\varepsilon_{\mathbf{[0,j_0]}}))r(\varepsilon_{\mathbf{[0,j_0]}}) \\ &= \boldsymbol{\beta} - (\boldsymbol{\beta}, r(\varepsilon_{\mathbf{[0,j_0]}}))r(\varepsilon_{\mathbf{[0,j_0]}}) \\ &= \boldsymbol{\beta} - (\boldsymbol{\beta}, \varepsilon_{\mathbf{i}})\varepsilon_{\mathbf{i}} \\ &= s_{\mathbf{i}}(\boldsymbol{\beta}). \end{aligned}$$

□

This lemma tells us that $\mathrm{mc}_{\mathbf{i}}$ can be regarded as a reflection and a product of simple reflections as follow.

7.1.4 Proposition. *Retain the notation in Proposition 7.1.2. Then we have*

$$\begin{aligned} \mathrm{mc}_{\mathbf{i}}(\boldsymbol{\alpha}) &= s_{\mathbf{i}}(\boldsymbol{\alpha}) \\ &= \left(\prod_{i\in I_{irr}\setminus\{0\}} s_{[i,j_i]}\right) \circ s_{[0,j_0]} \circ \left(\prod_{i\in I_{irr}\setminus\{0\}} s_{[i,j_i]}\right)(\boldsymbol{\alpha}). \end{aligned}$$

Proof From the definition of $\mathrm{mc}_{\mathbf{i}}(\boldsymbol{\alpha})$ given in Proposition 7.1.2, it suffices to show

$$n_{\mathbf{i}} = -(\boldsymbol{\alpha}, \varepsilon_{\mathbf{i}}).$$

Indeed

$$
\begin{aligned}
(\boldsymbol{\alpha}, \boldsymbol{\varepsilon}_{\mathbf{i}}) &= \sum_{i \in I_{\mathrm{irr}}} (\boldsymbol{\alpha}, \boldsymbol{\varepsilon}_{[i,j_i]}) \\
&= \sum_{i \in I_{\mathrm{irr}} \backslash \{0\}} \left(2\alpha_{[i,j_i]} - \sum_{\substack{1 \le j \le m_i \\ j \ne j_i}} d_i(j, j_i)\alpha_{[i,j]} - \alpha_{[i,j_i,1]} - \sum_{j=1}^{m_0} \alpha_{[0,j]} \right) \\
&\quad + 2\alpha_{[0,j_0]} - \sum_{\substack{1 \le j \le m_0 \\ j \ne j_0}} d_0(j, j_0)\alpha_{[0,j]} - \alpha_{[0,j_0,1]} - \sum_{i \in I_{\mathrm{irr} \backslash \{0\}}} \sum_{j=1}^{m_i} \alpha_{[i,j]} \\
&\quad - \sum_{i \in I_{\mathrm{reg}} \alpha_{[i,1,1]}} \alpha_{[i,j]}.
\end{aligned}
$$

Recalling that $d_i(j, j) = -1$ and $\sum_{j=1}^{m_i} \alpha_{[i,j]} = n$, we can continue the above computation,

$$
\begin{aligned}
(\boldsymbol{\alpha}, \boldsymbol{\varepsilon}_{\mathbf{i}}) &= -\sum_{i \in I_{\mathrm{irr}}} \left(\sum_{j=1}^{m_i} (d_i(j, j_i)\alpha_{[i,j]}) + (n - \alpha_{[i,j_i]}) + \alpha_{[i,j_i,1]} \right) \\
&\quad - \sum_{i \in I_{\mathrm{reg}}} \alpha_{[i,1,1]} - (\# I_{\mathrm{irr}} - 2)n \\
&= -\sum_{i \in I_{\mathrm{irr}}} \left(\sum_{j=1}^{m_i} (d_i(j, j_i + 1)\alpha_{[i,j]}) + (n - \alpha_{[i,j_i]}) + \alpha_{[i,j_i,1]} \right) \\
&\quad - \sum_{i \in I_{\mathrm{reg}}} \alpha_{[i,1,1]} + 2n \\
&= -n_{\mathbf{i}}.
\end{aligned}
$$

□

7.2 Irreducibility and $\mathcal{L}$-Irreducibility

The $\mathcal{L}$-irreducibility is a weaker condition than the usual irreducibility. We shall show that if we shift the parameter λ by using the operation Add, then these two irreducibility can be identical.

Fix $i_0 \in I_{\mathrm{irr}} \backslash \{0\}$ and define an operation on

$$
\mathcal{S} := \left\{ (\beta, \nu) \in \mathcal{L} \times \mathbb{C}^{Q_0} \,\middle|\, \beta \cdot \nu = \sum_{a \in Q_0} \beta_a \nu_a = 0 \right\}
$$

as an analogue of $\mathrm{Add}_{z^{-1}}^{(i_0)} \circ \mathrm{Add}_{-z^{-1}}^{(0)}$ as follows. Let us define $z^{(i_0)} = (z_a^{(i_0)})_{a \in Q_0} \in \mathbb{C}^{Q_0}$ by

$$z^{(i_0)}_{[i,j]} = \begin{cases} 1 & \text{if } i = i_0, \\ -1 & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$
$$z^{(i_0)}_{[i,j,k]} = 0.$$

Then let us define
$$\begin{array}{rccc} \mathrm{add}^{(i_0)}_{\gamma}\colon & \mathcal{S} & \longrightarrow & \mathcal{S} \\ & (\boldsymbol{\beta}, \nu) & \longmapsto & (\boldsymbol{\beta}, \nu + \gamma z^{(i_0)}) \end{array}$$

for $i_0 \in I_{\mathrm{irr}} \backslash \{0\}$ and $\gamma \in \mathbb{C}$.

For $\nu \in \mathbb{C}^{Q_0}$, let R^+_ν be the set of positive roots $\boldsymbol{\beta}$ of Q satisfying $\boldsymbol{\beta} \cdot \nu = 0$. Denote $\mathcal{L} \cap R^+_\nu$ by $\widetilde{R}^+_\nu$. The subset Σ_ν of R^+_ν consists of $\boldsymbol{\beta}$ satisfying that $p(\boldsymbol{\beta}) > \sum_t p(\boldsymbol{\beta_t})$ for any decomposition $\boldsymbol{\beta} = \boldsymbol{\beta_1} + \cdots + \boldsymbol{\beta_r}$ with $r \geq 2$ and $\boldsymbol{\beta_t} \in R^+_\nu$. Similarly define $\widetilde{\Sigma}_\nu$ consisting of $\boldsymbol{\beta} \in \widetilde{R}^+_\nu$ satisfying that $\boldsymbol{\beta} \cdot \nu = 0$ and $p(\boldsymbol{\beta}) > \sum_t p(\boldsymbol{\beta_t})$ for any decomposition $\boldsymbol{\beta} = \boldsymbol{\beta_1} + \cdots + \boldsymbol{\beta_r}$ with $r \geq 2$ and $\boldsymbol{\beta_t} \in \widetilde{R}^+_\nu$.

If $\boldsymbol{\beta}, \boldsymbol{\beta}' \in (\mathbb{Z}_{\geq 0})^{Q_0}$ satisfy that $\beta'_a \leq \beta_a$ for all $a \in Q_0$, then we write $\boldsymbol{\beta}' \leq \boldsymbol{\beta}$.

7.2.1 Lemma. *Fix $(\boldsymbol{\beta}, \nu) \in \mathcal{S}$. There exist $\gamma_i \in \mathbb{C}$ for $i \in I_{irr} \backslash \{0\}$ such that*

$$\nu' = \nu + \sum_{i \in I_{irr} \backslash \{0\}} \gamma_i z^{(i)}$$

satisfies the following. If $\boldsymbol{\beta}' \in (\mathbb{Z}_{\geq 0})^{Q_0}$ satisfies that $\boldsymbol{\beta}' \leq \boldsymbol{\beta}$ and $\boldsymbol{\beta}' \cdot \nu' = 0$, then $\boldsymbol{\beta}' \in \mathcal{L}$.

Proof Let F_β be the set of all elements $\boldsymbol{\beta}'$ in $(\mathbb{Z}_{\geq 0})^{Q_0}$ satisfying $\boldsymbol{\beta}' \leq \boldsymbol{\beta}$ and $\boldsymbol{\beta}' \notin \mathcal{L}$. Note that F_β is a finite set. Define a closed subset of $\mathbb{C}^{Q_0}$ by

$$V_\beta := \bigcup_{\boldsymbol{\beta}' \in F_\beta} \{\eta \in \mathbb{C}^{Q_0} \mid \boldsymbol{\beta}' \cdot \eta = 0\}.$$

Namely, if $\nu \notin V_\beta$, then $\boldsymbol{\beta}' \cdot \nu = 0$ and $\boldsymbol{\beta}' \leq \boldsymbol{\beta}$ imply $\boldsymbol{\beta}' \in \mathcal{L}$. Thus let us suppose $\nu \in V_\beta$. Consider the affine space $W_\nu := \{\nu + \sum_{i \in I_{\mathrm{irr}} \backslash \{0\}} t_i z^{(i)} \mid t_i \in \mathbb{C}\}$. Then $W_\nu \cap \{\eta \in \mathbb{C}^{Q_0} \mid \boldsymbol{\beta}' \cdot \eta = 0\}$ is a proper closed subset of W_μ for any $\boldsymbol{\beta}' \in F_\beta$. Indeed, since $\boldsymbol{\beta}' \notin \mathcal{L}$ there exists $i_0 \in I_{\mathrm{irr}} \backslash \{0\}$ such that $\sum_{j=1}^{m^{(0)}} \beta'_{[0,j]} \neq \sum_{j=1}^{m^{(i_0)}} \beta'_{[i_0,j]}$. Then the line $\{\nu + t z^{(i_0)} \mid t \in \mathbb{C}\} \subset W_\nu$ is not contained in the hyperplane $\{\eta \in \mathbb{C}^{Q_0} \mid \boldsymbol{\beta}' \cdot \eta = 0\}$. Thus $\dim W_\nu > \dim \left(W_\nu \cap \{\eta \in \mathbb{C}^{Q_0} \mid \boldsymbol{\beta}' \cdot \eta = 0\}\right)$ for any $\boldsymbol{\beta}' \in F_\beta$ since W_μ is an irreducible algebraic set. This shows the inequality

$$\dim W_\nu \cap V_\beta = \max_{\boldsymbol{\beta}' \in F_\beta} \left\{\dim \left(W_\nu \cap \{\eta \in \mathbb{C}^{Q_0} \mid \boldsymbol{\beta}' \cdot \eta = 0\}\right)\right\} < \dim W_\nu.$$

Hence, there exists $\nu' \in W_\nu$ which is not contained in V_β as required. □

This connects Σ_ν and $\widetilde{\Sigma}_{\nu'}$ as follows.

7.2.2 Proposition. *For any $\boldsymbol{\beta} \in \widetilde{\Sigma}_\nu$, there exist $\gamma_i \in \mathbb{C}$ for $i \in I_{irr}\backslash\{0\}$ such that*

$$\boldsymbol{\beta} \in \Sigma_{\nu+\sum_{i\in I_{irr}\backslash\{0\}} \gamma_i z^{(i)}}.$$

Proof For $\boldsymbol{\beta} \in \widetilde{\Sigma}_\nu$, let us choose γ_i as in Lemma 7.2.1 and set $\nu' = \nu + \sum_{i\in I_{\mathrm{irr}}\backslash\{0\}} \gamma_i z^{(i)}$. Then Lemma 7.2.1 shows that $\boldsymbol{\beta} \in \Sigma_{\nu'}$. □

Then we can show a direction of the statement of Theorem 6.0.3.

7.2.3 Theorem. *If $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{dif} \neq \emptyset$, then $\boldsymbol{\alpha} \in \widetilde{\Sigma}_\lambda$.*

Proof Let us suppose that there exists an $\mathcal{L}$-irreducible representation $x \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{det}}$. Choose $\gamma_i \in \mathbb{C}$ for $i \in I_{\mathrm{irr}}\backslash\{0\}$ as in Lemma 7.2.1 and put $\lambda' = \lambda + \sum_{i\in I_{\mathrm{irr}}\backslash\{0\}} \gamma_i z^{(i)}$. Then the operation $\prod_{i\in I_{\mathrm{irr}}\backslash\{0\}} \mathrm{add}_{\gamma_i}^{(i)}$ sends x to the $\mathcal{L}$-irreducible element $x' \in \mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{det}}$. However, Lemma 7.2.1 shows that if an element in $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')$ is $\mathcal{L}$-irreducible, then it is irreducible. Thus x' is irreducible, which shows $\boldsymbol{\alpha} \in \mathcal{L} \cap \Sigma_{\lambda'}$ by Crawley-Boevey's result (see Theorem 3.2.1). Hence $\boldsymbol{\alpha} \in \mathcal{L} \cap \Sigma_{\lambda'} \subset \widetilde{\Sigma}_{\lambda'} = \widetilde{\Sigma}_\lambda$. □

Finally with the lemma below, we can give a proof of Theorem 6.0.1.

7.2.4 Lemma. *Suppose that $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{dif} \neq \emptyset$. Fix $i_0 \in I_{irr}$ and $\gamma \in \mathbb{C}$. Then there exists a $\mathrm{GL}(\boldsymbol{\alpha})$-equivariant analytic bijection*

$$\mathrm{add}_\gamma^{(i_0)} : \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{dif} \longrightarrow \mu_{\boldsymbol{\alpha}}^{-1}(\lambda + \gamma z^{(i_0)})^{dif}.$$

Proof The required map is obtained by $\Phi_{\xi'} \circ \mathrm{Add}_{-\gamma z^{-1}}^{(i_0)} \circ \mathrm{Add}_{\gamma z^{-1}}^{(0)} \circ \Phi_\xi^{-1}$ with suitable ξ and ξ'. Thus it follows that the map preserves the $\mathcal{L}$-irreducibility since Add preserves the irreducibility of differential equations.

We can directly check that for $x \in \mu_\alpha^{-1}(\lambda)^{\mathrm{dif}}$, its image $x' := \mathrm{add}_\gamma^{(i_0)}(x) \in \mu_\alpha^{-1}(\lambda + \gamma z^{(i_0)})$ is written as follows. Set

$$x_{\rho_{i_0}} := \left(x_{\rho_{[i_0,j']}^{[0,j]}}\right)_{\substack{1\le j\le m_0 \\ 1\le j'\le m_{i_0}}}, \qquad x_{\rho_{i_0}^*} := \left(x_{(\rho_{[i_0,j']}^{[0,j]})^*}\right)_{\substack{1\le j\le m_0 \\ 1\le j'\le m_{i_0}}}.$$

Then

$$x'_{(\rho_{[i_0,j']}^{[0,j]})^*} = \left(x_{\rho_{i_0}^*} + \gamma \cdot x_{\rho_{i_0}}^{-1}\right)_{[0,j],[i_0,j']}$$

for $1 \le j \le m_0$ and $1 \le j' \le m_{i_0}$ and

$$x'_\rho = x_\rho$$

for the remaining $\rho \in \overline{Q}_1$, which tells us that the map is analytic. □

Proof of Theorem 6.0.1 Let us choose λ' as in the proof of Theorem 7.2.3. Then

$$\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{dif}} = \mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{irr}}$$

by Lemma 7.2.1. Thus lemma 7.2.4 shows that

$$\begin{aligned}\mathfrak{M}(\mathbf{B}) &\cong \mathfrak{M}_{\lambda}(Q,\boldsymbol{\alpha})^{\mathrm{dif}}\\ &\cong \mathfrak{M}_{\lambda'}(Q,\boldsymbol{\alpha})^{\mathrm{dif}} = \left(\mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{det}} \cap \mu_{\boldsymbol{\alpha}}^{-1}(\lambda')^{\mathrm{irr}}\right)/\mathrm{GL}(\boldsymbol{\alpha}) \subset \mathfrak{M}_{\lambda'}^{\mathrm{reg}}(Q,\boldsymbol{\alpha}). \quad \square\end{aligned}$$

7.3 $\mathcal{L}$-Fundamental Set

What is left to be proved is the converse of Theorem 7.2.3. Namely if $\boldsymbol{\alpha} \in \widetilde{\Sigma}_{\lambda}$, then we want to show that $\mu_{\boldsymbol{\alpha}}^{-1}(\lambda)^{\mathrm{dif}} \neq \emptyset$. We shall give an outline of the proof of this statement only when $\boldsymbol{\alpha}$ is an element in an analogue of the fundamental set F. As we see in the proof of Kac's theorem for the existence of indecomposable representations of quivers [20], we can show that the middle convolution which plays a role of the reflection functor enable us to reduce the argument to the case $\boldsymbol{\alpha} \in \tilde{F}$ or $\boldsymbol{\alpha} = \boldsymbol{\varepsilon}_{\mathbf{i}}$ for some $\mathbf{i} \in \mathcal{J}$, see [17].

7.3.1 Definition **($\mathcal{L}$-fundamental set).** Let us define the subset of $\mathcal{L}$ by

$$\tilde{F} := \left\{ \boldsymbol{\beta} \in \mathcal{L}^{+}\backslash\{0\} \;\middle|\; \begin{array}{c}(\boldsymbol{\beta}, \boldsymbol{\varepsilon}_{\boldsymbol{a}}) \leq 0 \text{ for all } a \in \mathcal{J} \cup Q_0^{\mathrm{leg}},\\ \text{support of } \boldsymbol{\beta} \text{ is connected}\end{array} \right\}$$

and call *$\mathcal{L}$-fundamental set*.

This $\tilde{F}$ may be regarded as an analogue of the fundamental set of imaginary roots in $\mathcal{L}$. However, $\mathcal{L}$ is just a sublattice of $\mathbb{Z}^{Q_0}$ and it does not necessarily have the structure of a Kac–Moody root lattice. Thus we shall give a lift of $\mathcal{L}$ to a Kac–Moody root lattice. This lift enable us to treat elements in $\tilde{F}$ as in the fundamental set of the imaginary roots.

Let us note that $\mathcal{L}$ is generated by

$$\left\{ \boldsymbol{\varepsilon}_{\boldsymbol{a}} \mid a \in \mathcal{J} \cup Q_0^{\mathrm{leg}} \right\}.$$

Then we can verify that

$$(\boldsymbol{\varepsilon}_{\mathbf{i}}, \boldsymbol{\varepsilon}_{\mathbf{i}'}) = 2 - \sum_{\substack{0 \leq i \leq p \\ j_i \neq j_i'}} (d_i(j_i, j_i') + 2), \tag{5}$$

$$(\boldsymbol{\varepsilon}_{\mathbf{i}}, \boldsymbol{\varepsilon}_{[i,j,k]}) = \begin{cases} -1 & \text{if } j = j_i \text{ and } k = 1, \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

$$(\varepsilon_{[i,j,k]}, \varepsilon_{[i',j',k']}) = \begin{cases} 2 & \text{if } [i, j, k] = [i', j', k'], \\ -1 & \text{if } (i, j) = (i', j') \text{ and } |k - k'| = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (7)$$

Cf. Sect. 3.2 in [16]. Here $\mathbf{i} = ([i, j_i])_{0\leq i\leq p}$, $\mathbf{i}' = ([i, j_i'])_{0\leq i\leq p} \in \mathcal{J}$. Thus, we consider the new lattice $\mathcal{M}$ generated by the set of indeterminate

$$\mathcal{C} := \left\{c_a \mid a \in \mathcal{J} \cup Q_0^{\text{leg}}\right\},$$

namely all $c_a \in \mathcal{C}$ have no relations, and define a symmetric bilinear form (,) on $\mathcal{M}$ in accordance with Eqs. (5), (6) and (7).

We can attach $\mathcal{M}$ to a diagram, called *Dynkin diagram*, regarding elements in $\mathcal{C}$ as vertices and connecting $c, c' \in \mathcal{C}$ by $|(c, c')|$ edges if $c \neq c'$. We say $c, c' \in \mathcal{C}$ are *connected* if there exists a sequence $c_0 = c, c_1, \dots, c_r = c'$ in $\mathcal{C}$ such that $(c_{i-1}, c_i) \neq 0$ for all $i = 1, \dots, r$. Then we may define *Dynkin diagram* of $\gamma \in \mathcal{M}$ which is a subdiagram obtained by connecting the vertices in $\text{supp}(\boldsymbol{\beta})$ in the same manner.

Also we can define reflections s_a on $\mathcal{M}$ by

$$s_a(\gamma) := \gamma - (\gamma, c_a)c_a$$

for $a \in \mathcal{J} \cup Q_0^{\text{leg}}$ and $\gamma \in \mathcal{M}$. Let us denote the set of all positive elements in $\mathcal{M}$ by $\mathcal{M}^+$.

Then the inclusion $\mathcal{L} \hookrightarrow \mathbb{Z}^{Q_0}$ induces

$$\Xi \colon \mathcal{M} \longrightarrow \mathbb{Z}^{Q_0}$$

where for $\gamma = \sum_{c\in\mathcal{C}} \gamma_c c \in \mathcal{M}$, the image $\Xi(\gamma) = (\beta_a)_{a\in Q_0}$ is given by

$$\beta_{[i,j]} := \sum_{\{\mathbf{i}=([i,j_i])\in\mathcal{J} \mid j_i=j\}} \gamma_{c_{\mathbf{i}}},$$
$$\beta_{[i,j,k]} := \gamma_{c_{[i,j,k]}}.$$

7.3.2 Proposition (Theorem 3.6 in [16]). *We have the following.*

1. *We have $(\gamma, \gamma') = (\Xi(\gamma), \Xi(\gamma'))$ for any $\gamma, \gamma' \in \mathcal{M}$.*
2. *The image of Ξ is $\mathcal{L}$.*
3. *The map Ξ is injective if and only if*

$$\#\{i \in \{0, \dots, p\} \mid m_i > 1, \ i = 0, \dots, p\} \leq 1.$$

4. *For $\gamma \in \mathcal{M}$ and $a \in \mathcal{J} \cup Q_0^{leg}$, we have*

$$\Xi(s_a(\gamma)) = s_a(\Xi(\gamma)).$$

From this proposition, $\mathcal{M}$ can be seen as a "lift" of $\mathcal{L}$ to a Kac–Moody root lattice in which $s_{\mathbf{i}}$ for $\mathbf{i} \in \mathcal{J}$ are simple reflections.

The kernel of Ξ is a big space in general. Thus if we consider the inverse image of an element $\boldsymbol{\beta} \in \mathcal{L}$, it is convenient to restrict Ξ to some smaller space as follows. Fix $\boldsymbol{\beta} \in \mathcal{L}$. Define $\mathcal{J}_{\boldsymbol{\beta}} := \{([i, j_i])_{i=0,\dots,p} \in \mathcal{J} \mid \beta_{[i,j_i]} \neq 0 \text{ for all } i \in I_{\mathrm{irr}}\}$, $Q_0^{\mathrm{leg}}(\boldsymbol{\beta}) := \mathrm{supp}(\boldsymbol{\beta}) \cap Q_0^{\mathrm{leg}}$, and a sublattice $\mathcal{M}_{\boldsymbol{\beta}} := \sum_{\{a \in \mathcal{J}_{\boldsymbol{\beta}} \cup Q_0^{\mathrm{leg}}(\boldsymbol{\beta})\}} \mathbb{Z} c_a$. Denote the set of all positive elements in $\mathcal{M}_{\boldsymbol{\beta}}$ by $\mathcal{M}_{\boldsymbol{\beta}}^{+}$. We write the restriction of Ξ on $\mathcal{M}_{\boldsymbol{\beta}}$ by $\Xi_{\boldsymbol{\beta}}$. The following lemma shows that if $\boldsymbol{\beta} \in \mathcal{L}^{+}$, then $\Xi_{\boldsymbol{\beta}}^{-1}(\boldsymbol{\beta}) \cap \mathcal{M}_{\boldsymbol{\beta}}^{+} \neq \emptyset$. Namely, there exist at least one positive element in the inverse image of a positive element in $\mathcal{L}$.

7.3.3 Lemma (Lemma 3 in [18]). *Take $\boldsymbol{\beta} \in \mathcal{L}^{+} \backslash \{0\}$ and set*

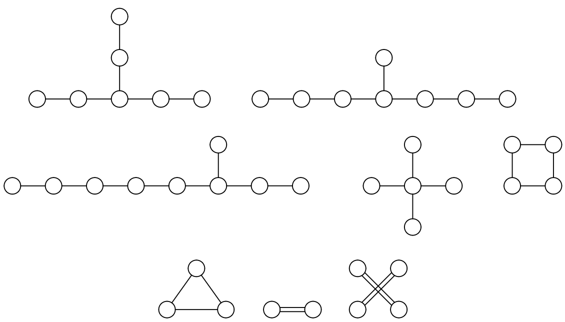
$$\overline{m}_i := \max\{j \in \{1, \dots, m_i\} \mid \beta_{[i,j]} \neq 0\},$$
$$\underline{m}_i := \min\{j \in \{1, \dots, m_i\} \mid \beta_{[i,j]} \neq 0\},$$

for $i \in I_{irr}$. Further set $\bar{\mathbf{i}} := ([i, \overline{m}_i])_{0 \leq i \leq p}$ and $\underline{\mathbf{i}} := ([i, \underline{m}_i])_{0 \leq i \leq p}$ where we put $\overline{m}_i = \underline{m}_i := 1$ for $i \in I_{reg}$.

Then there exists $\tilde{\beta} \in \mathcal{M}_{\boldsymbol{\beta}}^{+}$ such that $\Xi(\tilde{\beta}) = \boldsymbol{\beta}$ and $\tilde{\beta}_{c_{\bar{\mathbf{i}}}} \cdot \tilde{\beta}_{c_{\underline{\mathbf{i}}}} \neq 0$.

For the support of this positive element $\tilde{\beta}$, we can show the following.

7.3.4 Proposition (Theorem 9 in [18]). *Let us consider $\boldsymbol{\beta} \in \widetilde{F}$. Then if $q(\boldsymbol{\beta}) < 0$, the support of $\tilde{\beta}$ is connected. If $q(\boldsymbol{\beta}) = 0$, then the Dynkin diagram of the support of $\tilde{\beta}$ is one of the following.*



Now let us give a strategy to show that if $\boldsymbol{\beta} \in \widetilde{\Sigma}_{\nu} \cap \widetilde{F}$, then $\mu_{\boldsymbol{\beta}}^{-1}(\nu)^{\mathrm{dif}} \neq \emptyset$.

We consider first the wild case, i.e., $q(\boldsymbol{\beta}) < 0$.

7.3.5 Proposition. *Let $\boldsymbol{\beta} = \boldsymbol{\gamma_1} + \cdots + \boldsymbol{\gamma_r} \in \tilde{F}$ with $q(\boldsymbol{\beta}) < 0$, $r \geq 2$ and $\boldsymbol{\gamma_1}, \dots, \boldsymbol{\gamma_r} \in \mathcal{L}^{+} \backslash \{0\}$ then $q(\boldsymbol{\beta}) < q(\boldsymbol{\gamma_1}) + \cdots + q(\boldsymbol{\gamma_r})$.*

Proof For the above $\boldsymbol{\gamma_1}, \dots, \boldsymbol{\gamma_r}$, take $\tilde{\gamma}_1, \dots, \tilde{\gamma}_r \in \mathcal{M}^{+} \backslash \{0\}$ as in Lemma 7.3.3 and define $\tilde{\beta} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_r$. Then $\tilde{\beta}$ satisfies conditions in Lemma 7.3.3. Thus, the support of $\tilde{\beta}$ is connected from Proposition 7.3.4. Recall that $(\tilde{\beta}, c_a) \leq 0$ for all

$a \in \mathcal{J} \cup Q_0^{\mathrm{leg}}$ and the assumption $q(\boldsymbol{\beta}) < 0$. Then the standard argument (see Lemma 2 in [27] for example) shows that $q(\boldsymbol{\beta}) = (\tilde{\beta}, \tilde{\gamma}) < \sum_{i=1}^{r} (\tilde{\gamma}_i, \tilde{\gamma}_i) = \sum_{i=1}^{r} q(\boldsymbol{\gamma}_i)$. □

Let us fix $\boldsymbol{\beta} \in \widetilde{F}$ with $q(\boldsymbol{\beta}) < 0$. Define a nonempty open subset of $\mathrm{Rep}_Q(\boldsymbol{\beta})$ by

$$\mathrm{Rep}_Q(\boldsymbol{\beta})^{\mathrm{det}} := \left\{ x \in \mathrm{Rep}_Q(\boldsymbol{\beta}) \,\middle|\, \det\left(x_{\rho^{[0,j]}_{[i,j']}} \right)_{\substack{1 \le j \le m_0 \\ 1 \le j' \le m_i}} \neq 0,\ i \in I_{\mathrm{irr}} \backslash \{0\} \right\}.$$

7.3.6 Lemma. *If $x \in \mathrm{Rep}_Q(\boldsymbol{\beta})^{det}$ is decomposed as $x = x_1 \oplus \cdots \oplus x_r$ in $\mathrm{Rep}_Q(\boldsymbol{\beta})$, then $\mathbf{dim}\, x_i \in \mathcal{L}^+$ for all $i = 1, \ldots, r$.*

Proof Since $x \in \mathrm{Rep}_Q(\boldsymbol{\beta})^{\mathrm{det}}$, subrepresentations x_t with $\boldsymbol{\gamma_t} = \mathbf{dim}\, x_t$ for $t = 1, \ldots, r$ satisfy $\sum_{j=1}^{m_0} (\boldsymbol{\gamma_t})_{[0,j]} \le \sum_{j=1}^{m_i} (\boldsymbol{\gamma_t})_{[i,j]}$ for all $i \in I_{\mathrm{irr}} \backslash \{0\}$. If there exists "<" among these inequalities, then $\boldsymbol{\beta} = \boldsymbol{\gamma_1} + \cdots + \boldsymbol{\gamma_r} \notin \mathcal{L}^+$ which contradicts to the assumption $\boldsymbol{\beta} \in \widetilde{F} \subset \mathcal{L}^+$. Thus $\boldsymbol{\gamma_t} \in \mathcal{L}^+$ for all $t = 1, \ldots, r$. □

Let us recall the notion of *generic decomposition*. A decomposition $\boldsymbol{\beta} = \boldsymbol{\gamma_1} + \cdots + \boldsymbol{\gamma_r}$, $\boldsymbol{\gamma_t} \in (\mathbb{Z}_{\ge 0})^{Q_0} \backslash \{0\}$, is called the generic decomposition if

$$\mathrm{Ind}(Q; \boldsymbol{\gamma_1}, \ldots, \boldsymbol{\gamma_r}) = \left\{ x_1 \oplus \cdots \oplus x_r \in \mathrm{Rep}_Q(\boldsymbol{\beta}) \,\middle|\, \begin{array}{c} \mathbf{dim}\, x_t = \boldsymbol{\gamma_t} \text{ and } x_t \text{ are} \\ \text{indecomposable for } t = 1, \ldots, r \end{array} \right\}$$

contains a nonempty open dense subset of $\mathrm{Rep}_Q(\boldsymbol{\beta})$. It is known that the generic decomposition uniquely exists for any $\boldsymbol{\beta}' \in (\mathbb{Z}_{\ge 0})^{Q_0} \backslash \{0\}$, see Proposition 2.7 in [27] for example.

7.3.7 Proposition. *Let us take $\boldsymbol{\beta} \in \widetilde{F}$ with $q(\boldsymbol{\beta}) < 0$. If $\boldsymbol{\beta} = \boldsymbol{\gamma_1} + \cdots + \boldsymbol{\gamma_r}$ is the generic decomposition of $\boldsymbol{\beta} \in \widetilde{F}$, then $r = 1$.*

Proof If $\boldsymbol{\beta} = \boldsymbol{\gamma_1} + \cdots + \boldsymbol{\gamma_r}$ is the generic decomposition, then

$$\mathrm{Ind}(Q; \boldsymbol{\gamma_1}, \ldots, \boldsymbol{\gamma_r}) \cap \mathrm{Rep}_Q(\boldsymbol{\beta})^{\mathrm{det}} \neq \emptyset.$$

Thus $\boldsymbol{\gamma_i} \in \mathcal{L}^+$ for $i = 1, \ldots, r$ by Lemma 7.3.6. Then Proposition 7.3.5 shows that $q(\boldsymbol{\beta}) < q(\boldsymbol{\gamma_1}) + \cdots + q(\boldsymbol{\gamma_r})$ if $r \ge 2$. This contradicts to that $\boldsymbol{\beta} = \boldsymbol{\gamma_1} + \cdots + \boldsymbol{\gamma_r}$ is the generic decomposition by the standard argument, see Theorem 3.3 in [27] for example. Thus $r = 1$. □

7.3.8 Corollary. *If $\boldsymbol{\beta} \in \widetilde{F}$ and $q(\boldsymbol{\beta}) < 0$, then $\boldsymbol{\beta}$ is a positive root of Q.*

Proof Proposition 7.3.7 shows that $\mathrm{Rep}_Q(\boldsymbol{\beta})$ contains a indecomposable representation. Then Kac's theorem (Theorem 1.10 in [20]) tells us that $\boldsymbol{\beta}$ is a positive root of Q. □

7.3.9 Corollary. *Let us take $\boldsymbol{\beta} \in \widetilde{\Sigma}_\nu$ and suppose that $\boldsymbol{\beta} \in \tilde{F}$ and $q(\boldsymbol{\beta}) < 0$. Then $\mu_{\boldsymbol{\beta}}^{-1}(\nu)^{dif} \neq \emptyset$.*

Proof Let us take ν' as in Lemma 7.2.1 and show that $\mu_{\boldsymbol{\beta}}^{-1}(\nu')^{\mathrm{dif}} \neq \emptyset$. Let us note that $\sum_{j=1}^{m_i} \beta_{[i,j]} \geq 1$ for all $i \in I_{\mathrm{irr}}$ since $\mathrm{supp}(\boldsymbol{\beta})$ is connected. Then Proposition 7.3.7 shows that the subset Z of $\mathrm{Rep}_Q(\boldsymbol{\beta})$ consisting of all indecomposable representations is a dense subset. Thus $Z \cap \mathrm{Rep}_Q(\boldsymbol{\beta})^{\mathrm{det}} \neq \emptyset$. Then Theorem 3.3 in [9] shows that $\mu_{\boldsymbol{\beta}}^{-1}(\nu')^{\mathrm{det}} \neq \emptyset$. Moreover Theorem 1.2 in [9] says that the set of all irreducible representations in the irreducible topological set $\mu_{\boldsymbol{\beta}}^{-1}(\nu')$ is a dense subset. Here we note that $\boldsymbol{\beta} \in \Sigma_{\nu'}$. Thus, the nonempty open subset $\nu_{\boldsymbol{\beta}}^{-1}(\nu')^{\mathrm{det}}$ contains a irreducible representation x. Thus $\mu_{\boldsymbol{\beta}}^{-1}(\nu')^{\mathrm{dif}} \neq \emptyset$ which shows that $\mu_{\boldsymbol{\beta}}^{-1}(\nu)^{\mathrm{dif}} \neq \emptyset$. □

For the tame case, i.e., $q(\boldsymbol{\beta}) = 0$, by using the classification given in Proposition 7.3.4, we can show the non-emptiness. Thus we obtain the following.

7.3.10 Theorem (Theorem 6.21 in [17]). *For $\boldsymbol{\beta} \in \widetilde{\Sigma}_\nu \cap \tilde{F}$, we have $\mu_{\boldsymbol{\beta}}^{-1}(\nu)^{dif} \neq \emptyset$.*

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Chapter 9
Applications of Quiver Varieties to Moduli Spaces of Connections on $\mathbb{P}^1$

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1 Introduction

The aim of this lecture is to explain the main result of [13], which affirmatively solves Boalch's conjecture (proposed in [3]) on some relationship between meromorphic connections on the Riemann sphere $\mathbb{P}^1$ and the quiver varieties introduced by Nakajima [18].

Such a relationship was first found by Crawley-Boevey. Let $O_0, O_1, \ldots, O_m$ be conjugacy classes of $n \times n$ matrices. We say an element $(A_i)_{i=0}^m$ of the direct product $\boldsymbol{O} := \prod_{i=0}^m O_i$ to be *stable* if there is no non-zero proper vector subspace of $\mathbb{C}^n$ preserved by all A_i. Let $\boldsymbol{O}^s \subset \boldsymbol{O}$ be the open subset consisting of all stable points and put

$$\mathcal{M}^s = \left\{ (A_i)_{i=0}^m \in \boldsymbol{O}^s \mid \sum_{i=0}^m A_i = 0 \right\} / \mathrm{GL}(n, \mathbb{C}),$$

where $\mathrm{GL}(n, \mathbb{C})$ acts on $\boldsymbol{O}^s$ by simultaneous conjugation. In [8], he constructed a bijection between $\mathcal{M}^s$ and a quiver variety associated to a "star-shaped" quiver. The space $\mathcal{M}^s$ is sometimes called a *residue manifold* from the following reason: Fixing $(m+1)$ points $t_0, t_1, \ldots, t_m$ on the complex plane $\mathbb{C}$, associate to each $(A_i)_{i=0}^m \in M_n(\mathbb{C})^{m+1}$ the logarithmic connection

$$d - \sum_{i=0}^m \frac{A_i}{x - t_i}\, dx$$

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on the trivial vector bundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ over the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$. It is nonsingular away from $\{t_0, t_1, \ldots, t_m\}$ if and only if $\sum_{i=0}^{m} A_i = 0$. Since every automorphism of $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ is given by an element of $\mathrm{GL}(n, \mathbb{C})$, the space $\mathcal{M}^s$ may be embedded into the set of gauge equivalence classes of logarithmic connections on $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ holomorphic away from $\{t_0, t_1, \ldots, t_m\}$.

One can check that the space $\mathcal{M}^s$ is the *Hamiltonian reduction* of $\boldsymbol{O}^s$ by the action of $\mathrm{GL}(n, \mathbb{C})$, so has a structure of complex symplectic manifold, and that Crawley-Boevey's bijection is in fact an isomorphism of complex symplectic manifolds.

The residue manifolds are contained in a larger class of complex symplectic manifolds, called the *polar-parts manifolds* (named by Boalch in his thesis). Roughly speaking, they are obtained by the Hamiltonian reduction of some complex symplectic manifolds $\boldsymbol{O}^s$ contained in $M_n(\mathbb{C}[z^{-1}])^{m+1}$, and may be embedded into the set of gauge equivalence classes of meromorphic connections on $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ holomorphic away from $\{t_0, t_1, \ldots, t_m\}$ with prescribed pole order at each t_i. Based on his study on the geometric structure of polar-parts manifolds (see [2]), Boalch extended Crawley-Boevey's result to some class of polar-parts manifolds in [3, 4] and conjectured that a further extension is possible (see [3, Appendix C]).

The organization of this lecture is as follows. In Sect. 2, we review the Hamiltonian geometry, especially the Hamiltonian reduction procedure. In Sects. 3 and 4, following [2, 3] we introduce the main objects of this lecture: the *open/closed* quiver varieties and the polar-parts manifolds. The relationship between polar-parts manifolds and meromorphic connections on $\mathbb{P}^1$ is explained in Sect. 4, II. In Sect. 5 we review the theorem of Crawley-Boevey [8] on the residue manifolds and star-shaped quiver varieties. Section 6 is devoted to prove the main result of [13] (the proof given here is a slight modification of the original one).

2 Hamiltonian Geometry

In this section we introduce the notions of Hamiltonian space and Hamiltonian reduction.

2.1 Hamiltonian Spaces

Let G be a complex algebraic group with Lie algebra $\mathfrak{g}$. A G-action on a complex manifold M is an abstract group action of G on M such that the action map $G \times M \to M$, $(g, p) \mapsto g \cdot p$ is holomorphic. If G acts on a complex manifold M, each $\xi \in \mathfrak{g}$ defines the *fundamental vector field* ξ_M:

$$\xi_{M,p} = \frac{d}{dt} \exp(t\xi) \cdot p \bigg|_{t=0} \quad (p \in M).$$

Note that for each $p \in M$ the map

$$\varphi_p : \mathfrak{g} \to T_p M; \quad \xi \mapsto \xi_{M,p}$$

is exactly the differential of the evaluation map $g \mapsto g \cdot p$ at the identity, so its kernel

$$\mathfrak{g}_p := \mathrm{Ker}\varphi_p = \left\{ \xi \in \mathfrak{g} \mid \xi_{M,p} = 0 \right\}$$

coincides with the Lie algebra of the stabilizer G_p of p and the image

$$\mathfrak{g}_{M,p} := \mathrm{Im}\varphi_p = \left\{ \xi_{M,p} \mid \xi \in \mathfrak{g} \right\}$$

coincides with the tangent space $T_p(G \cdot p)$ of the orbit of p.

For $g \in G$, we denote by $\mathrm{Ad}_g \in \mathrm{Aut}(\mathfrak{g})$ (resp. $\mathrm{Ad}^*_g \in \mathrm{Aut}(\mathfrak{g}^*)$) the adjoint (resp. coadjoint) action of g.

Recall that a *(complex) symplectic form* on a complex manifold M is a closed holomorphic two-form ω on M such that for any $p \in M$ the bilinear form $\omega_p : T_p M \times T_p M \to \mathbb{C}$ on the (holomorphic) tangent space is non-degenerate. For each $p \in M$, a symplectic form ω defines an isomorphism

$$\omega_p^\sharp : T_p M \xrightarrow{\simeq} T_p^* M; \quad v \mapsto \iota(v)\omega_p = \omega_p(v, \cdot).$$

For a vector subspace $W \subset T_p M$, the preimage under $\omega_p^\sharp$ of the annihilator

$$W^\perp := \left\{ \alpha \in T_p^* M \mid \alpha|_W = 0 \right\}$$

is denoted by W^ω:

$$W^\omega = \left\{ v \in T_p M \mid \omega_p(v, w) = 0 \ (w \in W) \right\}.$$

A complex manifold M equipped with a symplectic form ω is called a *(complex) symplectic manifold*, which we denote by (M, ω) when we want to emphasize the symplectic form. A biholomorphism $M \to N$ between symplectic manifolds is called a *symplectomorphism* if the pull-back of the symplectic form on N coincides with that on M. A G-action on a complex symplectic manifold is said to be *symplectic* if the symplectic form is G-invariant.

2.1.1 Definition. Let (M, ω) be a complex symplectic manifold equipped with a symplectic G-action. A G-equivariant holomorphic map $\mu : M \to \mathfrak{g}^*$ (where G acts on $\mathfrak{g}^*$ by the coadjoint action) is called a *moment map* if it satisfies

$$d\langle \mu, \xi \rangle = \iota_{\xi_M}\omega = \omega(\xi_M, \cdot), \qquad \xi \in \mathfrak{g}.$$

If a moment map exists, the action is said to be *Hamiltonian*. A complex symplectic manifold M equipped with a Hamiltonian G-action and a moment map is called a *(complex) Hamiltonian G-space*, which we denote by (M, ω, μ) when we want to emphasize the symplectic form ω and moment map μ. A G-equivariant symplectomorphism $M \to N$ between Hamiltonian G-spaces is called an *isomorphism* of Hamiltonian G-spaces if the pull-backs of the symplectic form and moment map for N coincide with those for M.

Note that if $\mu_1, \mu_2 : M \to \mathfrak{g}^*$ are two moment maps for a Hamiltonian G-action, then the difference $\mu_1 - \mu_2$ is locally constant with values in $(\mathfrak{g}^*)^G$.

If G is a complex reductive group, then $\mathfrak{g}$ has a non-degenerate Ad-invariant symmetric bilinear form. Using it we frequently identify $\mathfrak{g}^*$ with $\mathfrak{g}$, so that moment maps take values in $\mathfrak{g}$. For instance, if G is a general linear group, we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ using the trace pairing $(X, Y) = \mathrm{tr}(XY)$.

2.1.2 Exercise. (1) Let (M, ω, μ) be a Hamiltonian G-space, H a complex algebraic group with Lie algebra $\mathfrak{h}$ and $\rho : H \to G$ be a homomorphism (of complex algebraic groups). Let H act on M through ρ. Show that $(M, \omega, \rho^* \circ \mu)$ is a Hamiltonian H-space, where $\rho^* : \mathfrak{g}^* \to \mathfrak{h}^*$ is the map induced from ρ.

(2) Let (M, ω, μ) be a Hamiltonian G-space and K a closed normal subgroup of G with Lie algebra $\mathfrak{k}$. Suppose that K acts trivially on M and $\mu(M) \subset (\mathfrak{g}/\mathfrak{k})^*$. Show that (M, ω, μ) is a Hamiltonian G/K-space with respect to the induced action of G/K.

(3) For $i = 1, 2$, let G_i be a complex algebraic group and (M_i, ω_i, μ_i) a Hamiltonian G_i-space. Let $\pi_i : M_1 \times M_2 \to M_i$, $i = 1, 2$, be the projections. Show that the triple

$$(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2, (\pi_1^*\mu_1, \pi_2^*\mu_2))$$

is a Hamiltonian $G_1 \times G_2$-space, and that if $G_1 = G_2 = G$, the triple

$$(M_1 \times M_2, \pi_1^*\omega_1 + \pi_2^*\omega_2, \pi_1^*\mu_1 + \pi_2^*\mu_2)$$

is a Hamiltonian G-space with respect to the diagonal action of G.

The most familiar example of Hamiltonian spaces is a coadjoint orbit.

2.1.3 Examples (Coadjoint orbits). Let $O \subset \mathfrak{g}^*$ be a G-coadjoint orbit (it is a complex submanifold of $\mathfrak{g}^*$ since G is algebraic, see e.g., [6, Proposition 1.8]). It is an easy exercise to show that the following two-form ω is well-defined, G-invariant and symplectic:

$$\omega_\alpha(\xi_O|_\alpha, \eta_O|_\alpha) = \langle \alpha, [\xi, \eta] \rangle \qquad (\alpha \in O,\ \xi, \eta \in \mathfrak{g}).$$

This is called the *Kirillov–Kostant–Souriau symplectic form*. Let us show that the inclusion map $\iota : O \hookrightarrow \mathfrak{g}^*$ is a moment map. Clearly it is equivariant. Also, for $\xi, \eta \in \mathfrak{g}$ and $\alpha \in O$, we have

$$\begin{aligned}(\iota(\xi_O)\omega)_\alpha(\eta_O|_\alpha) = \omega_\alpha(\xi_O|_\alpha, \eta_O|_\alpha) &= \langle \alpha, [\xi, \eta] \rangle \\ &= \frac{d}{dt}\langle \alpha,\ \mathrm{Ad}_{\exp(-t\eta)}(\xi) \rangle \Big|_{t=0} \\ &= \frac{d}{dt}\langle\ \mathrm{Ad}^*_{\exp(t\eta)}(\alpha), \xi \rangle \Big|_{t=0} = \langle \eta_O|_\alpha, \xi \rangle.\end{aligned}$$

Since $\mathfrak{g} \to T_\alpha O$, $\eta \mapsto \eta_O|_\alpha$ is surjective, the above shows that ι is a moment map. Hence (O, ω, ι) is a Hamiltonian G-space.

The following well-known fact enables us to construct many examples of Hamiltonian spaces.

2.1.4 Proposition. *Let M be a complex manifold equipped with a G-action and θ be a G-invariant holomorphic one-form on M such that $d\theta$ is symplectic. Define a map $\mu : M \to \mathfrak{g}^*$ by*

$$\langle \mu, \xi \rangle = -\theta(\xi_M) \qquad (\xi \in \mathfrak{g}).$$

Then $(M, d\theta, \mu)$ is a Hamiltonian G-space.

Proof. Cartan's formula implies

$$d\langle \mu, \xi \rangle = -d\iota_{\xi_M}\theta = (\iota_{\xi_M} d - L_{\xi_M})\theta,$$

where L_{ξ_M} is the Lie derivation. Since θ is G-invariant, we have $L_{\xi_M}\theta = 0$. □

2.1.5 Examples (Cotangent bundles). Recall that the cotangent bundle T^*X of any complex manifold X has a canonical symplectic form. This is defined by $\omega = -d\theta_X$, where θ_X is the *canonical one-form*:

$$(\theta_X)_\alpha(v) = \langle \alpha, \pi_*(v) \rangle, \qquad \alpha \in T^*X,\ v \in T_\alpha(T^*X).$$

($\pi : T^*X \to X$ is the projection.) If G acts on X, the one-form θ_X is invariant under the induced G-action on T^*X. Hence the above proposition shows that the map $\mu : T^*X \to \mathfrak{g}^*$ defined by $\langle \mu, \xi \rangle = \theta_X(\xi_{T^*X})$ ($\xi \in \mathfrak{g}$) is a moment map and the triple (T^*X, ω, μ) is a Hamiltonian G-space.

2.1.6 Exercise. Let V, W be finite-dimensional complex vector spaces and put $X = \mathrm{Hom}(W, V)$. The group $G := \mathrm{GL}(V) \times \mathrm{GL}(W)$ acts on X by

$$(g, h) \cdot Q = gQh^{-1}, \qquad (g, h) \in G,\ Q \in X.$$

Since X is a vector space, the cotangent bundle T^*X may be identified with the direct sum $X \oplus X^*$ of X with its dual X^*. Moreover, the bilinear form

$$\mathrm{Hom}(W, V) \times \mathrm{Hom}(V, W) \to \mathbb{C}; \quad (Q, P) \mapsto \mathrm{tr}(QP)$$

is non-degenerate and so we may identify X^* with $\mathrm{Hom}(W, V)$. Show that θ_X and $\omega = -d\theta_X$ are then expressed as

$$\theta_X = \mathrm{tr}(P dQ), \qquad \omega = \mathrm{tr}(dQ \wedge dP),$$

where $Q : X \oplus X^* \to X$ is the first projection regarded as a $\mathrm{Hom}(W, V)$-valued function on T^*X, and $P : X \oplus X^* \to X^*$ is the second projection (regarded as a $\mathrm{Hom}(V, W)$-valued function similarly). Also show that the moment map $\mu : T^*X \to \mathfrak{g} = \mathfrak{gl}(V) \oplus \mathfrak{gl}(W)$ for the induced G-action

$$(g, h) \cdot (Q, P) = (gQh^{-1}, hPg^{-1}), \qquad (g, h) \in G,\ (Q, P) \in T^*X.$$

with $\mu(0, 0) = 0$ is given by

$$\mu(Q, P) = (QP, -PQ), \qquad (Q, P) \in X \oplus X^*.$$

2.1.7 Examples (Cotangent bundles of Lie groups). Consider the cotangent bundle T^*G of a complex algebraic group G. We identify it with the trivial vector bundle $G \times \mathfrak{g}^* = G \times T_e^*G$ using the isomorphism

$$G \times \mathfrak{g}^* \to T^*G; \quad (g, \alpha) \mapsto L_{g^{-1}}^* \alpha,$$

where $L_{g^{-1}} : G \to G$ is the left translation by g^{-1}: $L_{g^{-1}}(a) := g^{-1}a$. Let us first calculate the canonical symplectic form on T^*G. The canonical one-form θ_G is described as

$$(\theta_G)_{(g,\alpha)}(v, \beta) = \langle L_{g^{-1}}^* \alpha, v\rangle = \langle \alpha, (L_{g^{-1}})_* v\rangle,$$
$$(g, \alpha) \in T^*G,\ (v, \beta) \in T_gG \oplus \mathfrak{g}^* = T_{(g,\alpha)}(T^*G).$$

Recall the *Maurer–Cartan form* Θ on G; it is the $\mathfrak{g}$-valued one-form on G defined by $\Theta_g(v) = (L_{g^{-1}})_* v$ ($g \in G$, $v \in T_gG$). Using it we can write

$$\theta_G = \langle \alpha, \pi^* \Theta\rangle,$$

where $\alpha\colon G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is the second projection regarded as a $\mathfrak{g}^*$-valued function on T^*G. Hence

$$\begin{aligned}\omega = -d\theta_G &= -\langle d\alpha \wedge \pi^*\Theta\rangle - \langle \alpha, \pi^* d\Theta\rangle \\ &= -\langle d\alpha \wedge \pi^*\Theta\rangle + \frac{1}{2}\langle \alpha, \pi^*[\Theta \wedge \Theta]\rangle,\end{aligned}$$

where we have used the Maurer–Cartan equation: $d\Theta = -[\Theta \wedge \Theta]/2$. The product $G \times G$ acts on G by the left and right translations:

$$(G \times G) \times G \to G; \quad (a, b, g) \mapsto L_a R_b^{-1}(g) = agb^{-1}.$$

The induced action on $T^*G = G \times \mathfrak{g}^*$ is given by

$$(G \times G) \times T^*G \to T^*G; \quad (a, b, g, \alpha) \mapsto (agb^{-1}, L_{b^{-1}}^* R_b^* \alpha) = (agb^{-1}, \mathrm{Ad}_b^*(\alpha)).$$

2.1.8 Exercise. Show that the map

$$\mu\colon T^*G \to \mathfrak{g}^* \oplus \mathfrak{g}^*; \quad \mu(g, \alpha) = (\mathrm{Ad}_g^*(\alpha), -\alpha)$$

is a moment map for the $G \times G$-action.

2.2 Hamiltonian Reduction

Let G be a complex algebraic group with Lie algebra $\mathfrak{g}$ and (M, ω, μ) be a Hamiltonian G-space.

2.2.1 Proposition. *For any $p \in M$, the following equalities hold:*

$$\mathrm{Ker}(d\mu)_p = \mathfrak{g}_{M,p}^{\omega}, \quad (\mathrm{Im}(d\mu)_p)^{\perp} = \mathfrak{g}_p,$$

where $S^{\perp} := \{\, \xi \in \mathfrak{g} \mid \langle \alpha, \xi \rangle = 0 \ (\alpha \in S) \,\}$ *for* $S \subset \mathfrak{g}^*$.

Proof. Let $p \in M$. By the definition of the moment map, we have

$$\omega_p(\xi_M, v) = \langle d\mu(v), \xi \rangle$$

for $v \in T_pM$ and $\xi \in \mathfrak{g}$. Therefore $v \in \mathrm{Ker}(d\mu)_p$ if and only if $\omega_p(\xi_M, v) = 0$ for all $\xi \in \mathfrak{g}$, or equivalently $v \in \mathfrak{g}_{M,p}^{\omega}$. Also, $\xi \in \mathfrak{g}_p$ (i.e., $\xi_M|_p = 0$) if and only if $\langle d\mu(v), \xi \rangle = 0$ for all $v \in T_pM$, or equivalently $\xi \in (\mathrm{Im}(d\mu)_x)^{\perp}$. □

Note that for any $\alpha \in \mathfrak{g}^*$ the level set $\mu^{-1}(\alpha)$ is preserved by the action of the stabilizer G_α of α for the coadjoint action (because μ is equivariant).

2.2.2 Corollary. *Take $\alpha \in \mathfrak{g}^*$ and suppose that G_α acts freely on $\mu^{-1}(\alpha)$. Then $\mu^{-1}(\alpha)$ is a complex submanifold of codimension* $\dim G$.

Proof. The equivariance of moment map implies that the stabilizer G_p of any $p \in M$ is contained in $G_{\mu(p)}$. Therefore the assumption implies that $G_p = \{1\}$ for all $p \in \mu^{-1}(\alpha)$, which together with Proposition 2.2.1 shows that $\mu^{-1}(\alpha)$ is a complex submanifold of codimension $\dim G$. □

Recall that a G-action on a complex manifold M is said to be *proper* if the map $G \times M \to M \times M$, $(g, p) \mapsto (g \cdot p, p)$ is a proper map of topological spaces. See e.g., [10, Appendix B] for basic properties of proper actions. If a G-action on M is proper, then its restriction to any closed subgroup $H \subset G$ is a proper H-action on M, and its restriction to any G-invariant subset $N \subset M$ is a proper G-action on N.

2.2.3 Theorem (Marsden–Weinstein [17]). Take $\alpha \in \mathfrak{g}^*$ and suppose that the G_α-action on $\mu^{-1}(\alpha)$ is free and proper (so in particular $\mu^{-1}(\alpha)$ is a complex submanifold). Then the following hold:

(1) The orbit space $\mu^{-1}(\alpha)/G_\alpha$ has a unique structure of complex manifold such that the quotient map $\pi\colon \mu^{-1}(\alpha) \to \mu^{-1}(\alpha)/G_\alpha$ is a principal G_α-bundle.

(2) There exists a unique symplectic form $\overline{\omega}$ on $\mu^{-1}(\alpha)/G_\alpha$ such that

$$\pi^*\overline{\omega} = \omega|_{\mu^{-1}(\alpha)}.$$

Proof. Assertion (1) follows from a general fact on proper actions, see e.g., [10, Corollary B.32].

Let us prove assertion (2). The uniqueness is clear since π is a submersion. To show the existence, for simplicity, put $Z_\alpha = \mu^{-1}(\alpha)$ and $M_\alpha = Z_\alpha/G_\alpha$. For $p \in M_\alpha$ and $v, w \in T_pM_\alpha$, take any $\tilde{p} \in Z_\alpha$, $\tilde{v}, \tilde{w} \in T_{\tilde{p}}Z_\alpha$ with $\pi(\tilde{p}) = p$, $\pi_*(\tilde{v}) = v$, $\pi_*(\tilde{w}) = w$ and put

$$\overline{\omega}_p(v, w) = \omega_{\tilde{p}}(\tilde{v}, \tilde{w}).$$

We must check that the right hand side does not depend on the choice of $\tilde{p}, \tilde{v}, \tilde{w}$. Let $\tilde{p}', \tilde{v}', \tilde{w}'$ be another choice. Then there is $g \in G_\alpha$ such that $\tilde{p}' = g \cdot \tilde{p}$, and the invariance of ω implies

$$\omega_{\tilde{p}'}(\tilde{v}', \tilde{w}') = \omega_{\tilde{p}}(g_*^{-1}\tilde{v}', g_*^{-1}\tilde{w}').$$

Since $\pi_*(g_*^{-1}\tilde{v}') = v = \pi_*(\tilde{v})$, we have

$$g_*^{-1}\tilde{v}' - \tilde{v} \in \operatorname{Ker}(\pi_*)_{\tilde{p}} = T_{\tilde{p}}(G_\alpha \cdot \tilde{p}).$$

Similarly $g_*^{-1}\tilde{w}' - \tilde{w} \in T_{\tilde{p}}(G_\alpha \cdot \tilde{p})$. Thus we can take $\xi, \eta \in \mathfrak{g}_\alpha = \operatorname{Lie} G_\alpha$ so that

$$\xi_{M,\tilde{p}} = g_*^{-1}\tilde{v}' - \tilde{v}, \quad \eta_{M,\tilde{p}} = g_*^{-1}\tilde{w}' - \tilde{w}.$$

Then

$$\begin{aligned}\omega_{\tilde{p}}(g_*^{-1}\tilde{v}', g_*^{-1}\tilde{w}') - \omega_{\tilde{p}}(\tilde{v}, \tilde{w}) &= \omega_{\tilde{p}}(\xi_M, g_*^{-1}\tilde{w}') + \omega_{\tilde{p}}(\tilde{v}, \eta_M) \\ &= \langle d\mu(g_*^{-1}\tilde{w}'), \xi\rangle - \langle d\mu(\tilde{v}), \eta\rangle.\end{aligned}$$

The two terms on the most right hand side are both zero because $\tilde{v}, g_*^{-1}\tilde{w}'$ are contained in $T_{\tilde{p}}Z_\alpha = \operatorname{Ker}(d\mu)_{\tilde{p}}$. Therefore $\overline{\omega}_p$ is a well-defined bilinear form on T_pM_α. Using local trivializations of the principal bundle $Z_\alpha \to M_\alpha$, one can easily check that $\overline{\omega}_p$, $p \in M_\alpha$ define a holomorphic two-form $\overline{\omega}$ on M_α satisfying $\pi^*\overline{\omega} = \omega|_{Z_\alpha}$. Since ω is closed and π is a submersion, $\overline{\omega}$ is also closed. To check the non-degeneracy of $\overline{\omega}$, take any $p \in M_\alpha$, $v \in T_pM_\alpha$ and suppose

$$\overline{\omega}_p(v, w) = 0 \quad (w \in T_p M_\alpha).$$

If we take $\tilde{p} \in Z_\alpha$, $\tilde{v} \in T_{\tilde{p}} Z_\alpha$ as above, then it implies

$$\omega_{\tilde{p}}(\tilde{v}, \tilde{w}) = 0 \quad (\tilde{w} \in T_{\tilde{p}} Z_\alpha).$$

By Proposition 2.2.1, we thus obtain $\tilde{v} \in (T_{\tilde{p}} Z_\alpha)^\omega = \mathrm{Ker}(d\mu)^\omega_{\tilde{p}} = \mathfrak{g}_{M,\tilde{p}}$, so we can take $\xi \in \mathfrak{g}$ so that $\xi_{M,\tilde{p}} = \tilde{v}$. Since $\tilde{v} \in T_{\tilde{p}} Z_\alpha$, we have

$$0 = (d\mu)_{\tilde{p}}(\tilde{v}) = (d\mu)_{\tilde{p}}(\xi_M) = \xi_{\mathfrak{g}^*,\alpha},$$

i.e., $\xi \in \mathfrak{g}_\alpha$. Hence $\tilde{v}$ is tangent to the G_α-orbit and $v = \pi_*(\tilde{v}) = 0$. □

Theorem 2.2.3 can be extended as follows:

2.2.4 Theorem. *Let G, H be complex algebraic groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, respectively, and (M, ω, μ) be a Hamiltonian $G \times H$-space with moment map $\mu = (\mu_G, \mu_H) \colon M \to \mathfrak{g}^* \oplus \mathfrak{h}^*$. Take $\alpha \in \mathfrak{g}^*$ and suppose that the G_α-action on $\mu_G^{-1}(\alpha)$ is free and proper. Then the orbit space $\mu_G^{-1}(\alpha)/G_\alpha$ has a unique structure $(\overline{\omega}, \overline{\mu}_H)$ of Hamiltonian H-space for the induced H-action such that*

$$\pi^*\overline{\omega} = \omega|_{\mu_G^{-1}(\alpha)}, \quad \pi^*\overline{\mu}_H = \mu_H|_{\mu_G^{-1}(\alpha)}.$$

2.2.5 Exercise. Prove Theorem 2.2.4.

2.2.6 Definition. For a Hamiltonian $G \times H$-space M with moment map (μ_G, μ_H), the (topological) orbit space $\mu_G^{-1}(\alpha)/G_\alpha$ equipped with the induced H-action and the map $\overline{\mu}_H \colon \mu_G^{-1}(\alpha)/G_\alpha \to \mathfrak{h}^*$ induced from μ_H is denoted by $M /\!\!/_\alpha G$ and called the *Hamiltonian reduction* of M by G at the level α. When the G_α-action on $\mu_G^{-1}(\alpha)$ is free and proper, we regard $M /\!\!/_\alpha G$ as a Hamiltonian H-space using Theorem 2.2.4.

2.2.7 Examples. Recall that the cotangent bundle $T^*G \simeq G \times \mathfrak{g}^*$ is a Hamiltonian $G \times G$-space, see Example 2.1.7. Note that the two G-actions on T^*G are both free and proper. Take any $\alpha \in \mathfrak{g}^*$ and consider the Hamiltonian reduction $T^*G /\!\!/_\alpha G$ by the left translation:

$$T^*G /\!\!/_\alpha G = \left\{ (g, \beta) \in G \times \mathfrak{g}^* \mid \mathrm{Ad}_g^*(\beta) = \alpha \right\} / G_\alpha \subset G/G_\alpha \times \mathfrak{g}^*.$$

Let O be the G-coadjoint orbit of $-\alpha$. Then the map

$$T^*G /\!\!/_\alpha G \to O; \quad [g, \beta] \mapsto -\beta$$

is a biholomorphism intertwining the G-moment maps. Since the G-action on O is transitive, the symplectic form is characterized by the moment map. Hence the above map is an isomorphism of Hamiltonian G-spaces.

2.2.8 Exercise. In the situation of Theorem 2.2.4, let $O \subset \mathfrak{g}^*$ be the G-coadjoint orbit of $-\alpha$. We let H act on O trivially (so the zero map $O \to \mathfrak{h}^*$ is a moment map) and endow the direct product $M \times O$ with a Hamiltonian $G \times H$-structure. Show that the map

$$M /\!\!/_{\alpha} G \to (M \times O) /\!\!/ G; \quad [p] \mapsto [p, -\alpha]$$

is an isomorphism of Hamiltonian H-spaces. This is called the *shifting trick*.

2.2.9 Exercise. Let (M, ω, μ) be a Hamiltonian G-space. Let G act on T^*G by the right translation. Show that the diagonal G-action on $T^*G \times M$ is free and proper, and the map

$$M \to (T^*G \times M) /\!\!/ G; \quad p \mapsto [1, \mu(p), p]$$

is an isomorphism of Hamiltonian G-spaces, where G acts on $(T^*G \times M) /\!\!/ G$ by $a \cdot [g, \beta, p] = [ag, \beta, p]$.

2.2.10 Exercise. Let M be a Hamiltonian $G \times H$-space with moment map $\mu = (\mu_G, \mu_H)$. Take $(\alpha, \beta) \in \mathfrak{g}^* \oplus \mathfrak{h}^*$ and put $M_\alpha = M /\!\!/_{\alpha} G$, $M_\beta = M /\!\!/_{\beta} H$. Then $M_\alpha /\!\!/_{\beta} H$, $M_\beta /\!\!/_{\alpha} G$ make sense as topological spaces. Show that there are natural homeomorphisms

$$M_\alpha /\!\!/_{\beta} H \simeq M /\!\!/_{(\alpha,\beta)} (G \times H) \simeq M_\beta /\!\!/_{\alpha} G$$

and they are isomorphisms of symplectic manifolds if the $G_\alpha \times H_\beta$-action on $\mu^{-1}(\alpha, \beta)$ is free and proper.

3 Quiver Varieties

Roughly speaking, a quiver variety is defined as the Hamiltonian reduction of the cotangent bundle of the vector space of representations of a quiver with prescribed dimension vector by the action of base changes at all vertices (at some level). In this lecture it is useful to consider also the Hamiltonian reduction by the action of base changes on some prescribed subset of vertices. Then we obtain an *open* quiver variety.

3.1 Open/Closed Quiver Varieties

Recall that a quiver is a (finite) directed graph. For a quiver Q, we denote

- the set of vertices by Q_0,
- the set of arrows by Q_1,
- the map taking the source vertex by $s\colon Q_1 \to Q_0$,
- the map taking the target vertex by $t\colon Q_1 \to Q_0$.

For $\boldsymbol{n} = (n_p)_{p\in Q_0} \in \mathbb{Z}_{\geqslant 0}^{Q_0}$, let

$$V^{\boldsymbol{n}} = \bigoplus_{p\in Q_0} V_p^{\boldsymbol{n}}, \quad V_p^{\boldsymbol{n}} = \mathbb{C}^{n_p}$$

be the standard Q_0-graded $\mathbb{C}$-vector space with dimension vector $\boldsymbol{n}$. Set

$$\mathrm{Rep}_Q(\boldsymbol{n}) = \bigoplus_{a\in Q_1} \mathrm{Hom}(V_{s(a)}^{\boldsymbol{n}}, V_{t(a)}^{\boldsymbol{n}}).$$

The complex reductive group

$$\mathrm{GL}(\boldsymbol{n}) := \prod_{p\in Q_0} \mathrm{GL}(V_p^{\boldsymbol{n}})$$

with Lie algebra $\mathfrak{gl}(\boldsymbol{n}) := \bigoplus_{p\in Q_0} \mathfrak{gl}(V_p^{\boldsymbol{n}})$ acts on $\mathrm{Rep}_Q(\boldsymbol{n})$ by base change, and the orbit space $\mathrm{Rep}_Q(\boldsymbol{n})/\mathrm{GL}(\boldsymbol{n})$ parameterizes the isomorphism classes of representations of Q with dimension vector $\boldsymbol{n}$ (see [14] for basic notions in the representation theory of quivers). Since each piece $\mathrm{Hom}(V_{s(a)}^{\boldsymbol{n}}, V_{t(a)}^{\boldsymbol{n}})$ is dual to $\mathrm{Hom}(V_{t(a)}^{\boldsymbol{n}}, V_{s(a)}^{\boldsymbol{n}})$, we have an identification

$$T^*\mathrm{Rep}_Q(\boldsymbol{n}) \simeq \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}),$$

where $\overline{Q}$ is the *double* of Q, the quiver obtained by adjoining a reverse arrow $a^*\colon q \to p$ for each arrow $a\colon p \to q$ in Q. It is useful to extend the map $a \mapsto a^*$ to an involution $*$ of $\overline{Q}_1$ in the obvious way and define a map $\varepsilon\colon \overline{Q}_1 \to \{\pm 1\}$ by $\varepsilon|_{Q_1} = 1$, $\varepsilon|_{Q_1^*} = -1$. The symplectic form on $T^*\mathrm{Rep}_Q(\boldsymbol{n}) = \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ is expressed as

$$\omega = \sum_{a\in Q_1} \mathrm{tr}(d\,\Xi_a \wedge d\,\Xi_{a^*}) = \frac{1}{2} \sum_{a\in \overline{Q}_1} \varepsilon(a)\mathrm{tr}(d\,\Xi_a \wedge d\,\Xi_{a^*}),$$

where $\Xi_a\colon \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}) \to \mathrm{Hom}(V_{s(a)}^{\boldsymbol{n}}, V_{t(a)}^{\boldsymbol{n}})$ $(a \in \overline{Q}_1)$ is the projection regarded as a $\mathrm{Hom}(V_{s(a)}^{\boldsymbol{n}}, V_{t(a)}^{\boldsymbol{n}})$-valued function. Example 2.1.5 and Exercise 2.1.2, (3) show that the map

$$\mu = (\mu_p)_{p\in Q_0}\colon \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}) \to \mathfrak{gl}(\boldsymbol{n}); \quad \mu_p\colon \Xi = (\Xi_a)_{a\in\overline{Q}_1} \mapsto \sum_{\substack{a\in\overline{Q}_1 \\ t(a)=p}} \varepsilon(a)\,\Xi_a\,\Xi_{a^*},$$

is a moment map for the induced $\mathrm{GL}(\boldsymbol{n})$-action, where we identify each $\mathfrak{gl}(V_p^{\boldsymbol{n}})$ with its dual using the trace pairing.

Now choose a subset I of Q_0 and say the vertices in I to be *closed* and the others to be *open*. We put $\boldsymbol{n}_I = (n_p)_{p\in I} \in \mathbb{Z}^I$, and for a Q_0-graded vector space $V = \bigoplus_{p\in Q_0} V_p$, set $V_I = \bigoplus_{p\in I} V_p$ (which is an I-graded vector space). According to the decomposition $Q_0 = I \sqcup I^c$, the group $\mathrm{GL}(\boldsymbol{n})$ and its Lie algebra $\mathfrak{gl}(\boldsymbol{n})$ are decomposed as

$$\mathrm{GL}(\boldsymbol{n}) = \mathrm{GL}(\boldsymbol{n}_I) \times \mathrm{GL}(\boldsymbol{n}_{I^c}), \quad \mathfrak{gl}(\boldsymbol{n}) = \mathfrak{gl}(\boldsymbol{n}_I) \oplus \mathfrak{gl}(\boldsymbol{n}_{I^c}).$$

We denote by $\mu = (\mu_I, \mu_{I^c})$ the according decomposition of the moment map and consider the Hamiltonian $\mathrm{GL}(\boldsymbol{n}_I)$-space $(\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}), \omega, \mu_I)$.

In what follows we assume $\boldsymbol{n}_I \neq 0$ (otherwise the group $\mathrm{GL}(\boldsymbol{n}_I)$ is trivial). Let $J \subset I$ be the support of $\boldsymbol{n}_I$ and

$$J = J_1 \sqcup J_2 \sqcup \cdots \sqcup J_k$$

be the decomposition into the connected components. For $i = 1, 2, \ldots, k$, define a subquiver $Q^{(i)}$ of Q with $Q_0^{(i)} = J_i \cup J^c$ by

$$Q_1^{(i)} = \left\{ a \in Q_1 \mid (s(a), t(a)) \in J_i^2 \cup (J_i \times J^c) \cup (J^c \times J_i) \right\}.$$

Let Q' be the maximal subquiver of Q with vertices J^c. Then we have a decomposition

$$\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}) = \mathrm{Rep}_{\overline{Q'}}(\boldsymbol{n}_{J^c}) \times \prod_{i=1}^{k} \mathrm{Rep}_{\overline{Q^{(i)}}}(\boldsymbol{n}_{J_i \cup J^c}). \tag{1}$$

Observe that $\mathrm{GL}(\boldsymbol{n}_I) = \mathrm{GL}(\boldsymbol{n}_J) = \prod_i \mathrm{GL}(\boldsymbol{n}_{J_i})$ acts on $\mathrm{Rep}_{\overline{Q'}}(\boldsymbol{n}_{J^c})$ trivially and on each $\mathrm{Rep}_{\overline{Q^{(i)}}}(\boldsymbol{n}_{J_i \cup J^c})$ through the projection $\mathrm{GL}(\boldsymbol{n}_J) \to \mathrm{GL}(\boldsymbol{n}_{J_i})$. In fact, the above describes the Hamiltonian $\mathrm{GL}(\boldsymbol{n}_I)$-space $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ as the direct product of the Hamiltonian $\{1\}$-space $\mathrm{Rep}_{\overline{Q'}}(\boldsymbol{n}_{J^c})$ and the Hamiltonian $\mathrm{GL}(\boldsymbol{n}_{J_i})$-spaces $\mathrm{Rep}_{\overline{Q^{(i)}}}(\boldsymbol{n}_{J_i \cup J^c})$.

The kernel of the $\mathrm{GL}(\boldsymbol{n}_I)$-action (the subgroup consisting of all elements acting trivially) can be easily described as follows:

3.1.1 Proposition. *Assume that the support of $\boldsymbol{n}_I$ is connected.*

(1) *If some $p \in I$ and $q \in I^c$ are connected by arrow in $\overline{Q}$ and both contained in the support of $\boldsymbol{n}$, then the $\mathrm{GL}(\boldsymbol{n}_I)$-action on $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ is effective.*

(2) *Otherwise, the kernel of the action is the image K of the map*

$$\mathbb{C}^\times \to \mathrm{GL}(\boldsymbol{n}_I); \quad c \mapsto (c\,\mathrm{Id}_{V_p^{\boldsymbol{n}}})_{p \in I}$$

and the image of μ_I is contained in $(\mathrm{Lie}K)^\perp$.

If the support of $\boldsymbol{n}_I$ is disconnected, the kernel of the $\mathrm{GL}(\boldsymbol{n}_I)$-action is equal to the product $\prod_{i=1}^k K_i$, where K_i is the kernel of the $\mathrm{GL}(\boldsymbol{n}_{J_i})$-action on the factor $\mathrm{Rep}_{\overline{Q^{(i)}}}(\boldsymbol{n}_{J_i \cup J^c})$ of the decomposition (1).

3.1.2 Exercise. Prove the above proposition.

We want to take the Hamiltonian reduction of $(\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}), \omega, \mu_I)$ to get a Hamiltonian $\mathrm{GL}(\boldsymbol{n}_{I^c})$-space. However, the action on a level set of the moment map is not proper in general. So we will choose some nice $\mathrm{GL}(\boldsymbol{n}_I)$-invariant open subset of $(\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}), \omega, \mu_I)$ on which the action is proper.

3.1.3 Definition. For $\Xi = (\Xi_a)_{a\in\overline{Q}_1} \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$, a Q_0-graded subspace $W = \bigoplus_{p\in Q_0} W_p$ of V is said to be Ξ-*invariant* if $\Xi_a(W_{s(a)}) \subset W_{t(a)}$ holds for any $a \in \overline{Q}_1$.

When the support of $\boldsymbol{n}_I$ is connected, a point $\Xi = (\Xi_a)_{a\in\overline{Q}_1} \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ is said to be *stable for the action of* $\mathrm{GL}(\boldsymbol{n}_I)$ (or $\mathrm{GL}(\boldsymbol{n}_I)$-*stable* for short) if there exists no non-zero proper I-graded subspace $W = \bigoplus_{p\in I} W_p$ of $V_I^{\boldsymbol{n}}$ satisfying at least one of the following two conditions:

(S1) The Q_0-graded subspace $\widetilde{W} \subset V^{\boldsymbol{n}}$ with $\widetilde{W}_I = W$, $\widetilde{W}_{I^c} = \{0\}$ is Ξ-invariant.
(S2) The Q_0-graded subspace $\widetilde{W} \subset V^{\boldsymbol{n}}$ with $\widetilde{W}_I = W$, $\widetilde{W}_{I^c} = V_{I^c}^{\boldsymbol{n}}$ is Ξ-invariant.

When the support of $\boldsymbol{n}_I$ is disconnected, $\Xi = (\Xi_a)_{a\in\overline{Q}_1} \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ is said to be $\mathrm{GL}(\boldsymbol{n}_I)$-*stable* if its $\mathrm{Rep}_{\overline{Q}^{(i)}}(\boldsymbol{n}_{J_i\cup J^c})$-component $\Xi^{(j)}$ with respect to the decomposition (1) is $\mathrm{GL}(\boldsymbol{n}_I)$-stable for any $i \in \{1, 2, \ldots, k\}$.

3.1.4 Remark. The above stability condition comes from Mumford's geometric invariant theory. See [15] and references therein for stability conditions on the representation spaces of quivers.

It is clear from the definition that if a point $\Xi \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ is $\mathrm{GL}(\boldsymbol{n}_I)$-stable, then it is $\mathrm{GL}(\boldsymbol{n}_J)$-stable for any $J \subset I$.

For a $\mathrm{GL}(\boldsymbol{n}_I)$-invariant subset Z of $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$, we denote by $Z^{s,I}$ the set consisting of all $\mathrm{GL}(\boldsymbol{n}_I)$-stable points in Z. It is a $\mathrm{GL}(\boldsymbol{n}_I)$-open subset of Z.

3.1.5 Proposition. *Let $K \subset \mathrm{GL}(\boldsymbol{n}_I)$ be the kernel of the $\mathrm{GL}(\boldsymbol{n}_I)$-action on $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$. Then the induced $\mathrm{GL}(\boldsymbol{n}_I)/K$-action on $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I}$ is free and proper.*

Proof. We may assume that the support J of $\boldsymbol{n}_I$ is connected. To check that the $\mathrm{GL}(\boldsymbol{n}_I)/K$-action on $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I}$ is free, take any $\Xi \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I}$ and suppose that it is fixed by some $g = (g_p) \in \mathrm{GL}(\boldsymbol{n}_I)$. If K is non-trivial, choose any $p \in J$ and let λ be an eigenvalue of g_p. If K is trivial, put $\lambda = 1$. Then define a non-zero I-graded subspace $W = \bigoplus W_p$ of $V_I^{\boldsymbol{n}}$ by

$$W_p = \begin{cases} \mathrm{Ker}(g_p - \lambda\,\mathrm{Id}_{V_p^{\boldsymbol{n}}}) & (p \in J), \\ V_p^{\boldsymbol{n}} = \{0\} & (p \in I \setminus J). \end{cases}$$

One can easily check that W satisfies (S2). Hence $W = V_I^{\boldsymbol{n}}$, which implies $g \in K$.

Next we show that the $\mathrm{GL}(\boldsymbol{n}_I)/K$-action is proper. It suffices to show that for

$$g(n) \in \mathrm{GL}(\boldsymbol{n}_I), \quad \Xi(n) \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I} \qquad (n = 1, 2, \ldots),$$

if both $\Xi(n)$ and $\widetilde{\Xi}(n) := g(n) \cdot \Xi(n)$ converge as $n \to \infty$ in $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I}$, then $(g(n))_{n=1}^{\infty}$ has a convergent subsequence. Recall that any invertible matrix can be expressed as the product of a unitary matrix and a Hermitian matrix with positive eigenvalues. Since every Hermitian matrix is diagonalizable by a unitary matrix, we can write

$$g(n) = u(n)h(n)v(n),$$

where $u(n)$, $v(n)$ are tuples of unitary matrices and $h(n)$ is a tuple of diagonal matrices with positive diagonal entries. Since unitary groups are compact, we may assume that $u(n)$, $v(n)$ converge (replacing them with subsequences if necessary). We then replace $\Xi(n)$ and $\widetilde{\Xi}(n)$ with $v(n) \cdot \Xi(n)$ and $u(n)^{-1} \cdot \widetilde{\Xi}(n)$, respectively, so that $\widetilde{\Xi}(n) = h(n) \cdot \Xi(n)$. Let us show that the sequence in $\mathrm{GL}(\boldsymbol{n}_I)/K$ represented by $(h(n))_{n=1}^{\infty}$ has a convergent subsequence. Put

$$\mathcal{J} = \left\{ (p, i) \in J \times \mathbb{Z}_{>0} \mid i \leqslant n_p \right\}.$$

Write $h(n) = (h_p(n))_{p \in I}$ and

$$h_p(n) = \operatorname{diag}(h_{p;1}(n), h_{p;2}(n), \dots, h_{p;n_p}(n)) \quad (p \in J).$$

Since $h_{p;i}(n) \in \mathbb{R}_{>0}$, we may assume that all the sequences

$$(h_{p;i}(n))_{n=1}^{\infty}, \quad (h_{q;j}(n)/h_{p;i}(n))_{n=1}^{\infty}, \qquad (p, i), (q, j) \in \mathcal{J}$$

have limits in $[0, \infty]$. Now we divide into two cases.

First, consider the case where $K = \{1\}$. Set

$$\mathcal{J}_0 = \left\{ (p, i) \in \mathcal{J} \mid \lim_{n \to \infty} h_{p;i}(n) = 0 \right\}.$$

For $(p, i) \in \mathcal{J}$ let $e_{p;i} \in V_p^{\boldsymbol{n}}$ be the i-th coordinate vector, and put

$$W = \bigoplus_{(p,i) \in \mathcal{J}_0} \mathbb{C}e_{p,i} \subset V_J^{\boldsymbol{n}},$$

which we regard as an I-graded subspace of $V_I^{\boldsymbol{n}}$ in the obvious way. We claim that W satisfies (S1) for $\Xi := \lim_{n \to \infty} \Xi(n)$. For an arrow a connecting vertices in J, write

$$\Xi_a(n) = (\Xi_{a;ij}(n))_{1 \leqslant j \leqslant n_{s(a)}}^{1 \leqslant i \leqslant n_{t(a)}}, \quad \widetilde{\Xi}_a(n) = (\widetilde{\Xi}_{a;ij}(n))_{1 \leqslant j \leqslant n_{s(a)}}^{1 \leqslant i \leqslant n_{t(a)}}.$$

Then we have

$$\widetilde{\Xi}_{a;ij}(n) = \frac{h_{t(a);i}(n)}{h_{s(a);j}(n)} \, \Xi_{a;ij}(n).$$

If $(s(a), j) \in \mathcal{J}_0$ and $(t(a), i) \notin \mathcal{J}_0$, then we must have $\lim_{n \to \infty} \Xi_{a;ij}(n) = 0$ because the left hand side converges by the assumption. It follows that $\Xi_a(W_{s(a)}) \subset W_{t(a)}$, where $\Xi_a := \lim_{n \to \infty} \Xi_a(n)$. For an arrow a with $s(a) \in J$ and $t(a) \notin I$, write

$$\Xi_a(n) = (\Xi_{a;1}(n), \dots, \Xi_{a;n_{s(a)}}(n)), \quad \Xi_{a;i}(n) \in V^{\boldsymbol{n}}_{t(a)},$$

and similarly for $\widetilde{\Xi}_a$. Then we have

$$\widetilde{\Xi}_{a;i}(n) = h_{s(a);i}(n)^{-1}\Xi_{a;i}(n).$$

If $(s(a), i) \in \mathcal{J}_0$, then we must have $\lim_{n\to\infty} \Xi_{a;i}(n) = 0$. Hence $\Xi_a(W_{s(a)}) = \{0\}$. Therefore W satisfies (S1). Since $K = \{1\}$, Proposition 3.1.1 shows that there exists an arrow a with $s(a) \in J$, $t(a) \in I^c$ and $n_{t(a)} \neq 0$. Therefore the stability condition implies $W = \{0\}$, i.e., $\mathcal{J}_0 = \emptyset$. If we set

$$\mathcal{J}_{<\infty} = \left\{ (p, i) \in \mathcal{J} \mid \lim_{n\to\infty} h_{p;i}(n) < \infty \right\}, \quad W' = \bigoplus_{(p,i)\in\mathcal{J}_{<\infty}} \mathbb{C}e_{p,i},$$

then a similar argument shows that W' satisfies (S2) and hence that $W' = V^{\boldsymbol{n}}_I$, i.e., $\mathcal{J}_{<\infty} = \mathcal{J}$. Hence $h(n)$ has a limit in $\mathrm{GL}(\boldsymbol{n}_I)$.

Next consider the case where $K \neq \{1\}$. For $(p, i), (q, j) \in \mathcal{J}$, write $(p, i) \sim (q, j)$ if $\lim_{n\to\infty} h_{q,j}(n)/h_{p,i}(n) \in (0, \infty)$. Then $\sim$ is an equivalence condition. Furthermore, the following total ordering $\leqslant$ on $\mathcal{J}/\!\sim$ is well-defined:

$$[p, i] \leqslant [q, j] \overset{\text{def}}{\iff} \lim_{n\to\infty} h_{q,j}(n)/h_{p,i}(n) > 0.$$

Let $[p_0, i_0] \in \mathcal{J}/\!\sim$ be the minimal element and set

$$W = \bigoplus_{(p,i)\in[p_0,i_0]} \mathbb{C}e_{p,i} \subset V^{\boldsymbol{n}}_I.$$

Then $\Xi_a(W_{s(a)}) \subset W_{t(a)}$ for any arrow a connecting vertices in J because if $(s(a), j) \in [p_0, i_0]$ and $(t(a), i) \notin [p_0, i_0]$ then $\Xi_{a;ij}(n)$ must tend to zero. Furthermore, since $K \neq \{1\}$, Proposition 3.1.1 shows that any vertex in I^c connected to a vertex in J by an arrow does not support $\boldsymbol{n}$. Hence W satisfies both (S1) and (S2). Note that W is non-zero by the construction. Therefore the stability condition implies $W = V^{\boldsymbol{n}}_I$, i.e., $\lim_{n\to\infty} h_{q,j}(n)/h_{p,i}(n) \in (0, \infty)$ for all $(p, i), (q, j) \in \mathcal{J}$. Therefore we can find a sequence $(c_n)_{n=1}^{\infty}$ in $\mathbb{C}^\times$ so that $c_n h(n) = (c_n h_p(n))_{p\in I}$ has a limit in $\mathrm{GL}(\boldsymbol{n}_I)$. □

3.1.6 Exercise. Assume that the support of $\boldsymbol{n}_I$ is connected and that the kernel K is trivial. Define a quiver Q' with vertices $I \sqcup \{\infty\}$ by

$$\#\left\{a \in Q_1' \mid a\colon p \to q\right\} = \begin{cases} \#\{a \in Q_1 \mid a\colon p \to q\} & (p, q \in I), \\ \displaystyle\sum_{\substack{a \in Q_1 \\ s(a)=p, t(a) \in I^c}} n_p n_{t(a)} & (p \in I,\ q = \infty), \\ \displaystyle\sum_{\substack{a \in Q_1 \\ s(a) \in I^c, t(a)=q}} n_{s(a)} n_q & (p = \infty,\ q \in I), \\ \displaystyle\sum_{\substack{a \in Q_1 \\ s(a), t(a) \in I^c}} n_{s(a)} n_{t(a)} & (p, q = \infty). \end{cases}$$

Also define $\boldsymbol{n}' = (n'_p) \in \mathbb{Z}_{\geqslant 0}^{Q'_0}$ by

$$n'_p = n_p \quad (p \in I), \qquad n'_\infty = 1.$$

Then we may identify $\mathrm{GL}(\boldsymbol{n}')/\mathbb{C}^\times$ with $\mathrm{GL}(\boldsymbol{n}_I)$. Show that there exists an isomorphism

$$\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}) \xrightarrow{\simeq} \mathrm{Rep}_{\overline{Q'}}(\boldsymbol{n}')$$

of Hamiltonian $\mathrm{GL}(\boldsymbol{n}_I)$-spaces intertwining the $\mathrm{GL}(\boldsymbol{n}_I)$-stability and $\mathrm{GL}(\boldsymbol{n}')$-stability conditions.

Note that the center $\mathfrak{gl}(\boldsymbol{n}_I)^{\mathrm{GL}(\boldsymbol{n}_I)}$ coincides with the image of the map

$$\mathbb{C}^I \to \mathfrak{gl}(\boldsymbol{n}_I); \quad (\lambda_p)_{p \in I} \mapsto (\lambda_p \operatorname{Id}_{V_p^{\boldsymbol{n}}})_{p \in I}.$$

Let $\zeta \in \mathbb{C}^I$. We use the same letter ζ for its image under the above map. Proposition 3.1.5 and Theorem 2.2.4 imply that $\mu_I^{-1}(\zeta)^{s,I}$ is non-empty, its orbit space

$$\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \zeta) := \mu_I^{-1}(\zeta)^{s,I}/\mathrm{GL}(\boldsymbol{n}_I) = \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I} /\!\!/_{\zeta} (\mathrm{GL}(\boldsymbol{n}_I)/K)$$

is a Hamiltonian $\mathrm{GL}(\boldsymbol{n}_{I^c})$-space.

3.1.7 Definition. The space $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \zeta)$ is called a *quiver variety*. It is sometimes called an *open* quiver variety (with I^c open) if $I \neq Q_0$, and a *closed* quiver variety if $I = Q_0$. The closed quiver variety is simply denoted by $\mathfrak{M}^s_Q(\boldsymbol{n}, \zeta)$.

3.2 Closing, Gluing and Blowing Up

We introduce some basic operations on quivers with open/closed vertices following [3]. Suppose that an open quiver variety $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \zeta)$ is given.

(i) Closing vertices. For a subset J of I^c, let $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \zeta)^{s,J}$ be the set consisting of all $[B] \in \mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \zeta)$ such that some (and hence any) representative $B \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})^{s,I}$ is $\mathrm{GL}(\boldsymbol{n}_{I \cup J})$-stable. Then the following is clear.

3.2.1 Proposition. *For any $\zeta' \in \mathbb{C}^J$, the orbit space $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})^{s,J} /\!\!/_{\zeta'} \mathrm{GL}(\boldsymbol{n}_J)$ has a structure of Hamiltonian $\mathrm{GL}(\boldsymbol{n}_{(I\cup J)^c})$-space and naturally isomorphic to the quiver variety $\mathfrak{M}^s_{Q,I\cup J}(\boldsymbol{n}, \boldsymbol{\zeta} \oplus \boldsymbol{\zeta}')$.*

In particular, we have

$$\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})^{s,J} /\!\!/_{\zeta'} \mathrm{GL}(\boldsymbol{n}_J) \simeq \mathfrak{M}^s_{Q,I\cup J}(\boldsymbol{n}, \boldsymbol{\zeta} \oplus \boldsymbol{\zeta}') \simeq \mathfrak{M}^s_{Q,J}(\boldsymbol{n}, \boldsymbol{\zeta}')^{s,I} /\!\!/_{\zeta} \mathrm{GL}(\boldsymbol{n}_I).$$

(ii) Gluing open vertices. Given two open vertices $p, q \in I^c$ with $n_p = n_q$, we have the diagonal subgroup

$$G = \left\{ g \in \mathrm{GL}(\boldsymbol{n}_{I^c}) \mid g_p = g_q \right\} \subset \mathrm{GL}(\boldsymbol{n}_{I^c}).$$

If we restrict the action of $\mathrm{GL}(\boldsymbol{n}_{I^c})$ on $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$ to that of G, then the resulting Hamiltonian G-space may be regarded as a quiver variety for the quiver obtained by "gluing" p and q together. The precise statement is as follows.

Let $\varphi\colon Q \to Q'$ be a morphism of quivers, i.e., a pair of maps $\varphi_0\colon Q_0 \to Q'_0$ and $\varphi_1\colon Q_1 \to Q'_1$ such that $s \circ \varphi_1 = \varphi_0 \circ s$, $t \circ \varphi_1 = \varphi_0 \circ t$. The morphism φ is called a *gluing* if φ_0 is surjective and φ_1 is bijective (then Q' may be viewed as the quiver obtained by an iteration of "gluing" some vertices in Q together).

Let $\varphi\colon Q \to Q'$ be a gluing and suppose that the glued vertices are all open, i.e., $\varphi_0^{-1}(\varphi_0(I)) = I$. Take $I' := \varphi_0(I)$ to be the set of closed vertices for Q'. Then φ_0 restricts to a bijection $I \to I'$. Let $(\varphi_0)_*\colon \mathbb{C}^I \to \mathbb{C}^{I'}$ be the induced isomorphism. Also let $\varphi_0^*\colon \mathbb{Z}^{Q'_0} \to \mathbb{Z}^{Q_0}$ be the injection induced from φ_0.

3.2.2 Proposition. *Suppose that $\boldsymbol{n} \in \mathrm{Im}\varphi_0^*$, i.e., $n_p = n_q$ if $\varphi_0(p) = \varphi_0(q)$. Take $\boldsymbol{n}' \in (\varphi_0^*)^{-1}(\boldsymbol{n})$ and put $\boldsymbol{\zeta}' = (\varphi_0)_*(\boldsymbol{\zeta})$. Let $\mathrm{GL}(\boldsymbol{n}'_{(I')^c})$ act on the quiver variety $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$ through the diagonal embeddings*

$$\mathrm{GL}(n'_q, \mathbb{C}) \hookrightarrow \prod_{p \in \varphi_0^{-1}(q)} \mathrm{GL}(n_p, \mathbb{C}), \qquad q \in (I')^c.$$

Then there exists a canonical isomorphism

$$\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta}) \simeq \mathfrak{M}^s_{Q',I'}(\boldsymbol{n}', \boldsymbol{\zeta}')$$

of Hamiltonian $\mathrm{GL}(\boldsymbol{n}'_{(I')^c})$-spaces.

Proof. Observe that φ_1 induces a linear isomorphism $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}) \xrightarrow{\simeq} \mathrm{Rep}_{\overline{Q'}}(\boldsymbol{n}')$ preserving the symplectic structure. Exercise 2.1.2, (3) shows that it is an isomorphism of Hamiltonian $\mathrm{GL}(\boldsymbol{n}'_{I'})$-spaces, and since $\varphi_0^{-1}(\varphi_0(I)) = I$, it intertwines the $\mathrm{GL}(\boldsymbol{n}_I)$-stability on $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ and the $\mathrm{GL}(\boldsymbol{n}'_{I'})$-stability on $\mathrm{Rep}_{\overline{Q'}}(\boldsymbol{n}')$. Hence it induces the desired isomorphism. □

We mainly use this fact in the following situation. Suppose we are given some open quiver varieties $\mathfrak{M}^s_{Q^{(j)},I_j}(\boldsymbol{n}^{(j)}, \boldsymbol{\zeta}^{(j)})$, $j = 1, 2, \ldots, k$ and injections $\iota_j\colon \Sigma \hookrightarrow$

I_j^c, $j = 1, 2, \dots, k$ from a common finite set Σ to the sets of open vertices such that

$$\boldsymbol{n}^{(1)}_{\iota_1(\lambda)} = \boldsymbol{n}^{(2)}_{\iota_2(\lambda)} = \cdots = \boldsymbol{n}^{(k)}_{\iota_k(\lambda)}, \qquad \lambda \in \Sigma.$$

Consider the "direct sum" $\widetilde{Q} := \bigsqcup_{j=1}^k Q^{(j)}$ of the quivers (obtained by taking the disjoint union of the vertex/arrow sets), and glue the k vertices $\iota_1(\lambda), \iota_2(\lambda) \dots, \iota_k(\lambda)$ in $\widetilde{Q}$ all together for each $\lambda \in \Sigma$. Then we obtain a new quiver Q (which we also denote by $\bigcup_\Sigma Q^{(j)}$) with the vertices labeled by

$$Q_0 = \Sigma \sqcup \bigsqcup_{j=1}^k (Q_0^{(j)} \setminus \iota_j(\Sigma)).$$

The obvious morphism $\varphi\colon \widetilde{Q} \to Q$ sending $\iota_j(\lambda)$ to λ is a gluing. By the assumption there exists $\boldsymbol{n} \in \mathbb{Z}^{Q_0}$ such that $\varphi_0^*(\boldsymbol{n}) = \oplus_j \boldsymbol{n}^{(j)}$. By Proposition 3.2.2 we thus obtain an isomorphism

$$\prod_{j=1}^k \mathfrak{M}^s_{Q^{(j)}, I_j}(\boldsymbol{n}^{(j)}, \zeta^{(j)}) = \mathfrak{M}_{\widetilde{Q}, \sqcup_j I_j}(\oplus_j \boldsymbol{n}^{(j)}, \oplus_j \zeta^{(j)}) \simeq \mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$$

of Hamiltonian $\mathrm{GL}(\boldsymbol{n})$-spaces, where $I := \varphi_0(\sqcup_j I_j)$ and $\boldsymbol{\zeta} := (\varphi_0)_*(\oplus_j \boldsymbol{\zeta}^{(j)})$.

(iii) Blowing up open vertices. Given a decomposition $n_p = \sum_{i=1}^k n'_i$ of n_p into smaller positive integers for some open vertex $p \in I^c$, we have the block-diagonal subgroup

$$G = \prod_{q \in I^c \setminus \{p\}} \mathrm{GL}(n_q) \times \prod_{i=1}^k \mathrm{GL}(n'_i) \subset \mathrm{GL}(\boldsymbol{n}_{I^c}).$$

If we restrict the action of $\mathrm{GL}(\boldsymbol{n}_{I^c})$ on $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$ to that of G, then the resulting Hamiltonian G-space may be regarded as a quiver variety for the quiver obtained by "blowing up" p into k vertices. The precise statement is as follows.

A morphism $\varphi\colon Q' \to Q$ of quivers is called a *blow-up* if φ_0 is surjective and for each $p, q \in Q'_0$ the restriction

$$\left\{ a \in Q'_1 \mid s(a) = p,\ t(a) = q \right\} \xrightarrow{\varphi_1} \{ b \in Q_1 \mid s(b) = \varphi_0(p),\ t(b) = \varphi_0(q) \}$$

is bijective (then Q' may be viewed as the quiver obtained by "blowing up" some vertices in Q). For a blow-up $\varphi\colon Q' \to Q$, we define a surjective map $(\varphi_0)_*\colon \mathbb{Z}^{Q'_0} \to \mathbb{Z}^{Q_0}$ by

$$(\varphi_0)_*(\boldsymbol{n}') = \boldsymbol{n}, \quad n_p = \sum_{q \in \varphi_0^{-1}(p)} n'_q.$$

If $\boldsymbol{n}' \in \mathbb{Z}_{\geqslant 0}^{Q'_0}$ and $\boldsymbol{n} = (\varphi_0)_*(\boldsymbol{n}')$, for each $p \in Q_0$ we may identify $V_p^{\boldsymbol{n}}$ with the direct sum $\bigoplus_{q \in \varphi_0^{-1}(p)} V_q^{\boldsymbol{n}'}$. Accordingly for each $p \in Q_0$ we may regard $\prod_{q \in \varphi_0^{-1}(p)} \mathrm{GL}(n'_q, \mathbb{C})$ as a block-diagonal subgroup of $\mathrm{GL}(n_p, \mathbb{C})$; in particular $\mathrm{GL}(\boldsymbol{n}'_{\varphi_0^{-1}(J)}) \subset \mathrm{GL}(\boldsymbol{n}_J)$ for any $J \subset Q_0$.

Let $\varphi\colon Q' \to Q$ be a blow-up and suppose that the "blown-up" vertices are all open, i.e., $I' := \varphi_0^{-1}(I) \xrightarrow{\varphi_0} I$ is bijective. Then φ_0 induces a bijection $\varphi_0^*\colon \mathbb{C}^I \to \mathbb{C}^{I'}$.

3.2.3 Proposition. *Take* $\boldsymbol{n}' \in (\varphi_0)_*^{-1}(\boldsymbol{n})$ *and put* $\boldsymbol{\zeta}' = \varphi_0^*(\boldsymbol{\zeta})$. *Let* $\mathrm{GL}(\boldsymbol{n}'_{(I')^c})$ *act on* $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$ *through the inclusion* $\mathrm{GL}(\boldsymbol{n}'_{(I')^c}) \subset \mathrm{GL}(\boldsymbol{n}_{I^c})$. *Then there is a natural isomorphism*

$$\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta}) \simeq \mathfrak{M}^s_{Q',I'}(\boldsymbol{n}', \boldsymbol{\zeta}')$$

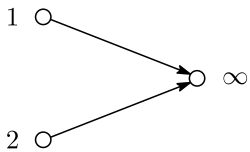
of Hamiltonian $\mathrm{GL}(\boldsymbol{n}'_{(I')^c})$*-spaces.*

As a typical example of blow-ups, consider an arbitrary quiver Q and a finite set Σ. Take any vertex $p_0 \in Q_0$ and define a new quiver Q' with vertices $\Sigma \sqcup (Q_0 \setminus \{p_0\})$ by

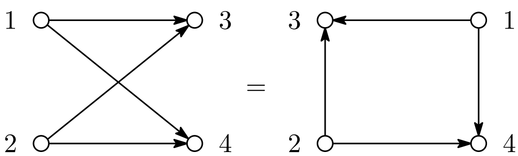
$$\#\left\{ a \in Q'_1 \mid a\colon p \to q \right\} = \begin{cases} \#\{ a \in Q_1 \mid a\colon p \to q \} & (p, q \in Q_0 \setminus \{p_0\}), \\ \#\{ a \in Q_1 \mid a\colon p \to p_0 \} & (p \in Q_0 \setminus \{p_0\},\ q \in \Sigma), \\ \#\{ a \in Q_1 \mid a\colon p_0 \to q \} & (p \in \Sigma,\ q \in Q_0 \setminus \{p_0\}), \\ 0 & (p, q \in \Sigma). \end{cases}$$

Then there exists a blow-up $\varphi\colon Q' \to Q$, which we call the *blow-up of* Q at p_0 by Σ.

For instance, if Q is the quiver with two vertices $0, \infty$ and one arrow $0 \to \infty$, its blow-up at 0 by $\{1, 2\}$ is given as follows.



If we further blow-up it at ∞ by $\{3, 4\}$, then we obtain the following quiver.



4 Polar-Parts Manifolds

In this section we define the polar-parts manifolds and explain their relation to meromorphic connections on the Riemann sphere.

4.1 Definition

From now on, we fix $n \in \mathbb{Z}_{>0}$ and put $G = \mathrm{GL}(n, \mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$. Consider the infinite-dimensional complex Lie algebras

$$\mathfrak{g}[\![z]\!] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[\![z]\!], \quad \mathfrak{g}[z^{-1}] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[z^{-1}].$$

Both are contained in the larger Lie algebra $\mathfrak{g}(\!(z)\!) := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}(\!(z)\!)$, which may be identified with the matrix ring $M_n(\mathbb{C}(\!(z)\!))$ over $\mathbb{C}(\!(z)\!)$ (as a complex vector space) in the obvious way. We have a bilinear form

$$\mathfrak{g}[\![z]\!] \times \mathfrak{g}[z^{-1}] \to \mathbb{C}; \quad (X, A) \mapsto \operatorname*{res}_{z=0} \mathrm{tr} \left(XA \, \frac{dz}{z} \right),$$

where $\mathrm{res}_{z=0} \colon \mathbb{C}(\!(z)\!)dz \to \mathbb{C}$ is the formal residue (taking the coefficient of dz/z). It enables us to embed $\mathfrak{g}[z^{-1}]$ into the dual vector space of $\mathfrak{g}[\![z]\!]$. Heuristically $\mathfrak{g}[\![z]\!]$ is the Lie algebra of the infinite-dimensional group

$$G[\![z]\!] := \{ g(z) \in M_n(\mathbb{C}[\![z]\!]) \mid \det g(0) \neq 0 \},$$

and the adjoint action of $G[\![z]\!]$ is given by the conjugation of matrices over $\mathbb{C}[\![z]\!]$:

$$\mathrm{Ad}_g(X) = gXg^{-1}, \qquad g \in G[\![z]\!], \ X \in \mathfrak{g}[\![z]\!].$$

Through the bilinear form, it induces the "coadjoint action" of $G[\![z]\!]$ on $\mathfrak{g}[z^{-1}]$, which is explicitly given by

$$\mathrm{Ad}^*_g(A) = (gAg^{-1})_{\leqslant 0}, \qquad g \in G[\![z]\!], \ A \in \mathfrak{g}[z^{-1}],$$

where $(\,\cdot\,)_{\leqslant 0} \colon \mathfrak{g}(\!(z)\!) = z\mathfrak{g}[\![z]\!] \oplus \mathfrak{g}[z^{-1}] \to \mathfrak{g}[z^{-1}]$ is the map forgetting the $z\mathfrak{g}[\![z]\!]$-part. If we write

$$g = \sum_{j \geqslant 0} g_j z^j, \quad g^{-1} = \sum_{j \geqslant 0} g'_j z^j, \quad A = \sum_{j \geqslant 0} A_j z^{-j},$$

then

$$\mathrm{Ad}^*_g(A) = \sum_{k\geqslant 0}\left(\sum_{i,j\geqslant 0} g_i A_{i+j+k} g'_j\right) z^{-k}.$$

In particular, the coadjoint action preserves the degree in z^{-1}. So for each $r \in \mathbb{Z}_{\geqslant 0}$ the action preserves the subspace

$$\mathfrak{g}^*_{r+1} := \sum_{j=0}^{r} z^{-j}\mathfrak{g} \subset \mathfrak{g}[z^{-1}],$$

on which it reduces to an action of the complex algebraic group

$$G_{r+1} := \left\{ g(z) \in M_n(\mathbb{C}[\![z]\!]/(z^{r+1})) \mid \det g(0) \neq 0 \right\}.$$

Observe that its Lie algebra may be identified with the quotient Lie algebra $\mathfrak{g}_{r+1} := \mathfrak{g}[\![z]\!]/z^{r+1}\mathfrak{g}[\![z]\!]$, which is dual to $\mathfrak{g}^*_{r+1}$ via the bilinear form on $\mathfrak{g}[\![z]\!]$. So the induced action of G_{r+1} is the coadjoint action.

In what follows a $G[\![z]\!]$-*coadjoint orbit* always means an orbit for the $G[\![z]\!]$-action on $\mathfrak{g}[z^{-1}]$. The above observation shows that any $G[\![z]\!]$-coadjoint orbit is a finite-dimensional complex symplectic manifold.

Now let $O_0, O_1, \ldots, O_m \subset \mathfrak{g}[z^{-1}]$ be $G[\![z]\!]$-coadjoint orbits. The general linear group G acts on each O_i by conjugation. It is Hamiltonian since the action is induced from the natural embedding $G \hookrightarrow G[\![z]\!]$ (with the image consisting of constant matrices). A moment map is given by the restriction to O_i of the map $\pi_0 : \mathfrak{g}[z^{-1}] \to \mathfrak{g} \simeq \mathfrak{g}^*$ taking the constant term. Hence the G-action on the direct product $\boldsymbol{O} := \prod_{i=0}^m O_i$ defined by the simultaneous conjugation is Hamiltonian with moment map

$$\mu\colon \boldsymbol{O} \to \mathfrak{g} \simeq \mathfrak{g}^*; \quad (A_i)_{i=0}^m \mapsto \sum_{i=0}^{m} \pi_0(A_i).$$

We are interested in the Hamiltonian reduction of $\boldsymbol{O}$ by G at the level 0. Observe first that the center $\mathbb{C}^\times \subset G$ acts trivially on $\boldsymbol{O}$ and the level set $\mu^{-1}(0)$ is empty unless $\sum_{i=0}^m \mathrm{tr}\pi_0(O_i) = 0$ (the map $\mathrm{tr} \circ \pi_0\colon \mathfrak{g}[z^{-1}] \to \mathbb{C}$ is $G[\![z]\!]$-invariant and hence $\mathrm{tr}\pi_0(O_i) \in \mathbb{C}$ makes sense). Therefore if $\mu^{-1}(0)$ is non-empty then the map μ takes values in the dual of the Lie algebra of $G/\mathbb{C}^\times$ and is a moment map for the induced $G/\mathbb{C}^\times$-action.

As in the case of quiver varieties we need a stability condition.

4.1.1 Definition. A tuple $(A_i)_{i=0}^m \in \mathfrak{g}[z^{-1}]^{m+1}$, $A_i = \sum_{j\geqslant 0} A_{i,j} z^{-j}$ is said to be *stable* if there exists no non-zero proper vector subspace of $\mathbb{C}^n$ preserved by all $A_{i,j}$.

For a G-invariant subset $Z \subset \mathfrak{g}[z^{-1}]^{m+1}$, we denote by Z^s the set consisting of all stable points in Z. It is a G-invariant open subset of Z.

4.1.2 Proposition. *The $G/\mathbb{C}^\times$-action on $\boldsymbol{O}^s$ is free and proper.*

4.1.3 Exercise. Deduce the above proposition from Proposition 3.1.5.

Therefore Theorem 2.2.3 shows that if $\mu^{-1}(0)^s$ is non-empty, its orbit space

$$\mathcal{M}^s(O_0, O_1, \dots, O_m) := \mu^{-1}(0)^s/G = \boldsymbol{O}^s /\!\!/ (G/\mathbb{C}^\times)$$

has a symplectic structure.

4.1.4 Definition. The complex symplectic manifold $\mathcal{M}^s(O_0, O_1, \dots, O_m)$ is called the *polar-parts manifold* associated to $O_0, O_1, \dots, O_m$.

4.2 Relation to Stable Meromorphic Connections on $\mathbb{P}^1$

Let Σ be a compact Riemann surface and $D \subset \Sigma$ a finite subset. Define $\Omega^1_\Sigma(*D)$ to be the sheaf of germs of meromorphic one-forms on Σ holomorphic away from D.

4.2.1 Definition. Let $\mathcal{V}$ be a holomorphic vector bundle on Σ, i.e., a locally free $\mathcal{O}_\Sigma$-module. A *meromorphic connection on* $\mathcal{V}$ *with poles on* D is a morphism

$$\nabla \colon \mathcal{V} \to \Omega^1_\Sigma(*D) \otimes_{\mathcal{O}_\Sigma} \mathcal{V}$$

of sheaves of vector spaces satisfying the Leibniz rule:

$$\nabla(fv) = df \otimes v + f\nabla(v) \qquad (f \in \mathcal{O}_\Sigma,\ v \in \mathcal{V}).$$

A pair $(\mathcal{V}, \nabla)$ of such $\mathcal{V}$ and ∇ is called a *meromorphic connection on* (Σ, D). The *rank*, *degree* of $(\mathcal{V}, \nabla)$ are those of $\mathcal{V}$.

Two meromorphic connections $(\mathcal{V}, \nabla)$, $(\mathcal{V}', \nabla')$ on (Σ, D) are said to be *isomorphic* if there exists an isomorphism $\varphi \colon \mathcal{V} \xrightarrow{\simeq} \mathcal{V}'$ such that $\varphi \circ \nabla = \nabla' \circ \varphi$.

Meromorphic connections have a natural notion of stability:

4.2.2 Definition. Let $(\mathcal{V}, \nabla)$ be a meromorphic connection on (Σ, D). A subbundle $\mathcal{W} \subset \mathcal{V}$ is said to be ∇-*invariant* if it satisfies

$$\nabla(\mathcal{W}) \subset \Omega^1_\Sigma(*D) \otimes_{\mathcal{O}_\Sigma} \mathcal{W}.$$

The meromorphic connection $(\mathcal{V}, \nabla)$ is said to be *stable* if the inequality

$$\frac{\deg \mathcal{W}}{\operatorname{rank} \mathcal{W}} < \frac{\deg \mathcal{V}}{\operatorname{rank} \mathcal{V}}$$

holds for any non-zero proper ∇-invariant subbundle $\mathcal{W} \subset \mathcal{V}$.

Take $(m+1)$ mutually distinct points $t_0, t_1, \dots, t_m$ on the complex plane $\mathbb{C}$ and put $D = \{t_0, t_1, \dots, t_m\}$. For each $(A_i)_{i=0}^m \in \mathfrak{g}[z^{-1}]^{m+1}$, write $A_i = \sum A_{i,j} z^{-j}$ and define a $\mathfrak{g}$-valued meromorphic one-form A on the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ by

$$A = \sum_{i=0}^{m} \sum_{j \geqslant 0} \frac{A_{i,j}}{(x - t_i)^{j+1}} \, dx.$$

Then $\nabla := d - A$ is a meromorphic connection on the trivial vector bundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$, and it has no pole at ∞ if and only if

$$\sum_{i=0}^{m} \pi_0(A_i) = \sum_{i=0}^{m} A_{i,0} = 0.$$

Since any automorphism of $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ is given by an element of $G = \mathrm{GL}(n, \mathbb{C})$, two connections $(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, d - A)$, $(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, d - A')$ associated to (A_i), $(A'_i) \in \mathfrak{g}[z^{-1}]^{m+1}$, respectively, are isomorphic if and only if there is $g \in G$ such that $g A_i g^{-1} = A'_i$ for all i. Furthermore one can show the following:

4.2.3 Proposition. *Take $(A_i)_{i=0}^m \in \mathfrak{g}[z^{-1}]^{m+1}$ and let $\nabla = d - A$ be the associated meromorphic connection on $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$. Then (A_i) is stable if and only if $(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, \nabla)$ is stable.*

Proof. Since any line bundle on $\mathbb{P}^1$ of negative degree has no global section, the trivial bundle $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ has no subbundle of positive degree and any subbundle of degree 0 is trivial. Hence a meromorphic connection $(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, \nabla)$ is stable if and only if there is no non-zero proper ∇-invariant trivial subbundle. Since any trivial subbundle of $\mathcal{O}_{\mathbb{P}^1}^{\oplus n}$ is of the form $\mathcal{O}_{\mathbb{P}^1} \otimes W$ for some vector subspace $W \subset \mathbb{C}^n$, the stability for $(\mathcal{O}_{\mathbb{P}^1}^{\oplus n}, \nabla)$ is equivalent to that for (A_i). □

Hence the polar-parts manifolds $\mathcal{M}^s(O_0, \dots, O_m)$ are embedded into the set of isomorphism classes of stable meromorphic connections on $(\mathbb{P}^1, D)$ of rank n whose underlying bundle is trivial.

In terms of meromorphic connections the condition $A_i \in O_i$ is rephrased as follows. Let $(\mathcal{V}, \nabla)$ be a meromorphic connection on $(\mathbb{P}^1, D)$ of rank n. For each $i = 0, 1, \dots, m$, take a trivialization $\psi_i \colon \mathcal{V}|_{U_i} \xrightarrow{\simeq} \mathcal{O}_{U_i}^{\oplus n}$ on a small neighborhood U_i of t_i and define $A_i = \sum_{j \geqslant 0} A_{i,j} z^{-j} \in \mathfrak{g}[z^{-1}]$ by the condition

$$\psi_i \circ \nabla \circ \psi_i^{-1} + \sum_{j \geqslant 0} \frac{A_{i,j}}{(x - t_i)^{j+1}} dx \text{ is a holomorphic connection on } \mathcal{O}_{U_i}^{\oplus n}.$$

One can check that the $G[\![z]\!]$-orbit of each A_i is independent of the choice of ψ_i (though A_i itself is not). Furthermore, if two meromorphic connections $(\mathcal{V}, \nabla)$, $(\mathcal{V}', \nabla')$ are isomorphic then the associated $G[\![z]\!]$-orbits coincide for each i. Therefore the condition $A_i \in O_i$ makes sense for all isomorphism classes of meromorphic connections on $(\mathbb{P}^1, D)$ of rank n.

Let $\overline{\mathcal{M}}^s(O_0, \ldots, O_m)$ be the set of isomorphism classes of stable meromorphic connections $(\mathcal{V}, \nabla)$ on $(\mathbb{P}^1, D)$ of rank n and degree 0 such that each A_i is contained in O_i. Using the above embedding we may regard the polar-parts manifold $\mathcal{M}^s(O_0, \ldots, O_m)$ as a subset of $\overline{\mathcal{M}}^s(O_0, \ldots, O_m)$. It is an open subset, and if $\mathcal{M}^s(O_0, \ldots, O_m) \neq \emptyset$ and $\overline{\mathcal{M}}^s(O_0, \ldots, O_m)$ is connected, then $\mathcal{M}^s(O_0, \ldots, O_m)$ is furthermore dense (see [19, Theorem 5.3]).

5 Residue Manifolds and Star-Shaped Quiver Varieties

The space $\mathcal{M}^s(O_0, \ldots, O_m)$ is called a *residue manifold* if $\deg_{1/z}(O_i) = 0$ for all i, i.e., all O_i are G-coadjoint orbits. Crawley-Boevey [8] found that every residue manifold is isomorphic to a quiver variety associated to a "star-shaped" quiver, which we explain in this section.

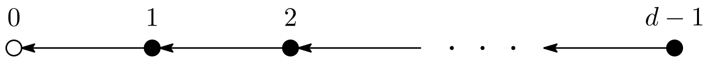
5.1 Coadjoint Orbits of Type A and Quiver Varieties

We first relate G-coadjoint orbits with quivers of type A.

5.1.1 Definition. A *marking* of a matrix $A \in \mathfrak{g}$ (or its conjugacy class O) is an ordered tuple $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ of complex numbers such that

$$\prod_{j=1}^{d} (A - \lambda_j I_n) = 0.$$

Let $O \subset \mathfrak{g}^* = \mathfrak{g}$ be a G-coadjoint orbit. Fix a marking $(\lambda_1, \lambda_2, \ldots, \lambda_d)$ of O. Define a quiver Q with vertices $Q_0 = \{0, 1, \ldots, d-1\}$ by drawing one arrow from each $k \in Q_0$ $(k \geqslant 1)$ to $k-1$, and choose $I = \{1, \ldots, d-1\} \subset Q_0$ as the closed vertices. In the following picture of Q the closed vertices are painted black.



Define $\boldsymbol{n} = (n_k) \in \mathbb{Z}_{\geqslant 0}^{Q_0}$ by

$$n_0 = n, \quad n_k = \operatorname{rank} \prod_{j=1}^{k} (A - \lambda_j I_n) \ \ (k \in I),$$

where A is an arbitrary element of O, and consider the Hamiltonian $\mathrm{GL}(\boldsymbol{n})$-space $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$. By the definition of Q, a point Ξ in $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ is a tuple consisting of linear maps

$$\Xi_{k-1,k}\colon V_k^{\boldsymbol{n}} \to V_{k-1}^{\boldsymbol{n}}, \quad \Xi_{k,k-1}\colon V_{k-1}^{\boldsymbol{n}} \to V_k^{\boldsymbol{n}} \qquad (k \in I).$$

The moment map $\mu_I = (\mu_k)_{k\in I}$ for the $\mathrm{GL}(\boldsymbol{n}_I)$-action is given by

$$\mu_k(\Xi) = \Xi_{l,k+1}\Xi_{k+1,l} - \Xi_{k,k-1}\Xi_{k-1,k} \qquad (k \in I),$$

where we use the convention $\Xi_{d-1,d} = 0$, $\Xi_{d,d-1} = 0$. Also the moment map for the action of $\mathrm{GL}(V_0^{\boldsymbol{n}}) = G$ is $\mu_0(\Xi) = \Xi_{0,1}\Xi_{1,0}$.

5.1.2 Proposition. *Define $\boldsymbol{\zeta} = (\zeta_k) \in \mathbb{C}^I$ by $\zeta_k = \lambda_k - \lambda_{k+1}$. Then the G-moment map*

$$\bar{\mu}_0 + \lambda_1 I_n \colon \mathfrak{M}^s_{Q,Q_0\setminus\{0\}}(\boldsymbol{n}, \boldsymbol{\zeta}) \to \mathfrak{g}$$

induces a G-equivariant isomorphism $\mathfrak{M}^s_{Q,Q_0\setminus\{0\}}(\boldsymbol{n}, \boldsymbol{\zeta}) \xrightarrow{\simeq} O$ of symplectic manifolds.

We first construct a map from O to the quiver variety $\mathfrak{M}^s_{Q,Q_0\setminus\{0\}}(\boldsymbol{n}, \boldsymbol{\zeta})$. Suppose we are given an element $A \in O$. For each $k \in I$, taking a basis we identify the vector space $\prod_{j=1}^{k}(A - \lambda_k I_n)$ with $V_k^{\boldsymbol{n}}$. We have the inclusion map $\Xi_{k-1,k}\colon V_k^{\boldsymbol{n}} \to V_{k-1}^{\boldsymbol{n}}$ and the surjection $\Xi_{k,k-1}\colon V_{k-1}^{\boldsymbol{n}} \to V_k^{\boldsymbol{n}}$ induced from $A|_{V_{k-1}^{\boldsymbol{n}}} - \lambda_k \mathrm{Id}_{V_{k-1}^{\boldsymbol{n}}}$. They gives a point Ξ in $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$.

5.1.3 Lemma. $\mu_I(\Xi) = \boldsymbol{\zeta}$.

Proof. By the definition we have

$$\Xi_{k-1,k}\Xi_{k,k-1} = A|_{V_{k-1}^{\boldsymbol{n}}} - \lambda_k \mathrm{Id}_{V_{k-1}^{\boldsymbol{n}}}, \quad \Xi_{k,k-1}\Xi_{k-1,k} = A|_{V_k^{\boldsymbol{n}}} - \lambda_k \mathrm{Id}_{V_k^{\boldsymbol{n}}} \qquad (2)$$

for $k \in I$. Since $A|_{V_{d-1}^{\boldsymbol{n}}} = \lambda_d \mathrm{Id}_{V_{d-1}^{\boldsymbol{n}}}$, the assertion follows. □

Furthermore, since $\Xi_{k-1,k}$ is injective and $\Xi_{k,k-1}$ is surjective for any $k \in I$, the following lemma shows that the point Ξ is stable for the $\mathrm{GL}(\boldsymbol{n}_I)$-action.

5.1.4 Lemma. *A point $\Xi \in \mu_I^{-1}(\boldsymbol{\zeta})$ is $\mathrm{GL}(\boldsymbol{n}_I)$-stable if and only if $\Xi_{k-1,k}$ is injective and $\Xi_{k,k-1}$ is surjective for any $k \in I$.*

Proof. Suppose that $\Xi_{k-1,k}$ is injective and $\Xi_{k,k-1}$ is surjective for any $k \in I$. If an I-graded subspace $W \subset V_I^{\boldsymbol{n}}$ satisfies condition (S1), then the injectivity of each $\Xi_{k-1,k}$ implies

$$0 \geqslant \dim W_1 \geqslant \dim W_2 \geqslant \cdots \geqslant \dim W_{d-1},$$

and hence $W = \{0\}$. If an I-graded subspace $W \subset V_I^{\boldsymbol{n}}$ satisfies condition (S2), then each $\Xi_{k,k-1}$ induces a surjection $V_{k-1}^{\boldsymbol{n}}/W_{k-1} \to V_k^{\boldsymbol{n}}/W_k$, which implies

$$0 \geqslant \operatorname{codim} W_1 \geqslant \operatorname{codim} W_2 \geqslant \cdots \geqslant \operatorname{codim} W_{d-1},$$

and hence $W = V_I^{\boldsymbol{n}}$. Therefore Ξ is $\mathrm{GL}(\boldsymbol{n}_I)$-stable.

Conversely, suppose that $\Xi \in \mu_I^{-1}(\zeta)$ is $\mathrm{GL}(\boldsymbol{n}_I)$-stable. For $k, l \in Q_0$ with $k < l$, we put

$$\Xi_{k,l} = \Xi_{k,k+1}\,\Xi_{k+1,k+2} \cdots \Xi_{l-1,l}, \quad \Xi_{l,k} = \Xi_{l,l-1}\,\Xi_{l-1,l-2} \cdots \Xi_{k+1,k}.$$

By induction one can then deduce the following relations from the equality $\mu_I(\Xi) = \zeta$:

$$\Xi_{0,k+1}\,\Xi_{k+1,k} = \left(\Xi_{0,1}\,\Xi_{1,0} + \sum_{j=1}^{k} \zeta_j \operatorname{Id}_{V_0^{\boldsymbol{n}}}\right) \Xi_{0,k} \quad (k \in I), \tag{3}$$

$$\Xi_{k,k+1}\,\Xi_{k+1,0} = \Xi_{k,0}\left(\Xi_{0,1}\,\Xi_{1,0} + \sum_{j=1}^{k} \zeta_j \operatorname{Id}_{V_0^{\boldsymbol{n}}}\right) \quad (k \in I). \tag{4}$$

Now define two I-graded subspaces $W, W' \subset V_I^{\boldsymbol{n}}$ by

$$W_k = \operatorname{Ker} \Xi_{0,k}, \quad W'_k = \operatorname{Im} \Xi_{k,0} \qquad (k \in I).$$

Then relation (3) implies that W satisfies condition (S1), and (4) implies that W' satisfies condition (S2). Since $n > 0$, the stability of Ξ implies $W = \{0\}$, $W' = V_I^{\boldsymbol{n}}$. Hence each $\Xi_{k-1,k}$ is injective and each $\Xi_{k,k-1}$ is surjective. □

Thus we obtain a map

$$O \to \mathfrak{M}^s_{Q, Q_0 \setminus \{0\}}(\boldsymbol{n}, \zeta); \quad A \mapsto [\Xi],$$

and $\mu_0(\Xi) = \Xi_{0,1}\,\Xi_{1,0} = A - \lambda_1 I_n$ by the definition.

Conversely, for $\Xi \in \mu_I^{-1}(\zeta)^{s,I}$, consider the element

$$A := \mu_0(\Xi) + \lambda_1 I_n = \Xi_{0,1}\,\Xi_{1,0} + \lambda_1 I_n \in \mathfrak{g}.$$

Put $V_k = \operatorname{Im} \Xi_{0,k} \subset \mathbb{C}^n$ $(k \in I)$ and $V_d = \{0\}$. Then relation (3) together with Lemma 5.1.4 implies that each V_k is an A-invariant subspace of dimension n_k and

$$(A - \lambda_{k+1} I_n)(V_k) = V_{k+1} \qquad (k \in I).$$

Also we have $\operatorname{Im}(A - \lambda_1 I_n) = \operatorname{Im}(\Xi_{0,1}\,\Xi_{1,0}) = V_1$. Thus we obtain

$$\prod_{j=1}^{d}(A - \lambda_j I_n) = 0, \qquad \operatorname{rank} \prod_{j=1}^{k}(A - \lambda_j I_n) = n_k \quad (k \in I),$$

which uniquely determines the conjugacy class of A (we leave the proof of this fact as an exercise; see Exercise 5.1.5). On the other hand any element of O also satisfies the above. Hence $A \in O$.

5.1.5 Exercise. Let $(\lambda_1, \lambda_2, \dots, \lambda_d)$ be a marking of a matrix $A \in \mathfrak{g}$ and set

$$n_0 = n, \qquad n_k = \operatorname{rank} \prod_{j=1}^{k} (A - \lambda_j I_n) \quad (k = 1, 2, \dots, d).$$

Fix an eigenvalue λ of A, and let $W \subset \mathbb{C}^n$ be the generalized λ-eigenspace of A. Take a Jordan normal form N of the nilpotent endomorphism $A|_W - \lambda \operatorname{Id}_W$. Write $\{ k \mid \lambda_k = \lambda \} = \{k_1 < \cdots < k_l\}$. Then show that N has no Jordan block of size $> l$, and for $j = 1, 2, \dots, l$, the number of Jordan blocks in N of size $\geqslant j$ is equal to $n_{k_j - 1} - n_{k_j}$.

5.1.6 Exercise. Take $[\Xi] \in \mathfrak{M}^s_{Q, Q_0 \setminus \{0\}}(\boldsymbol{n}, \boldsymbol{\zeta})$ and let $A \in O$ be the corresponding element under the isomorphism given in Proposition 5.1.2.

(1) Show that if $W \subset V^{\boldsymbol{n}}$ is a Ξ-invariant Q_0-graded subspace then $W_0 \subset \mathbb{C}^n$ is A-invariant, and that $W = \{0\}$ (resp. $W = V^{\boldsymbol{n}}$) if and only if $W_0 = \{0\}$ (resp. $W_0 = \mathbb{C}^n$).

(2) Show that if $V \subset \mathbb{C}^n$ is A-invariant, then the Q_0-graded subspace

$$W = \bigoplus_{k \in Q_0} W_k, \quad W_k := \begin{cases} V & (k = 0), \\ \Xi_{k,0}(V) & (k \neq 0) \end{cases}$$

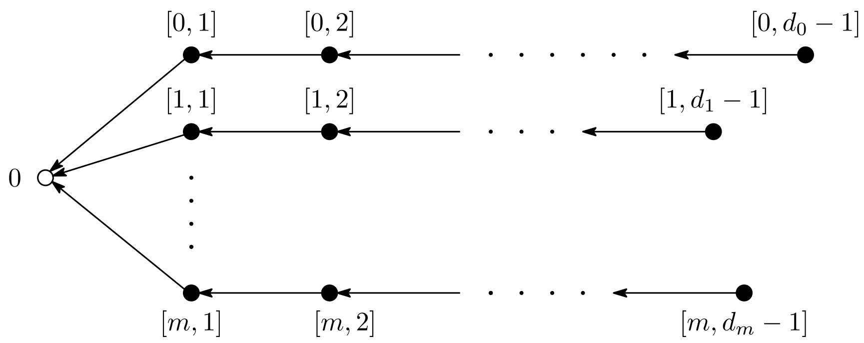
is Ξ-invariant, and $W = \{0\}$ (resp. $W = V^{\boldsymbol{n}}$) if and only if $V = \{0\}$ (resp. $V = \mathbb{C}^n$).

5.2 Residue Manifolds and Star-Shaped Quiver Varieties

Let $O_0, O_1, \dots, O_m \subset \mathfrak{g}$ be G-coadjoint orbits and consider the associated residue manifold. For each $i = 0, 1, \dots, m$, take a marking $(\xi^i_1, \dots, \xi^i_{d_i})$ of O_i. Let $Q^{(i)}$, $\boldsymbol{n}^{(i)}$, $\boldsymbol{\zeta}^{(i)}$ be as in Proposition 5.1.2 for O_i and write $Q^{(i)}_0 = \{ [i, 0], [i, 1], \dots, [i, d_i - 1] \}$, so the map

$$\mathfrak{M}^s_{Q^{(i)}, Q^{(i)}_0 \setminus \{[i,0]\}}(\boldsymbol{n}^{(i)}, \boldsymbol{\zeta}^{(i)}) \to O_i; \quad [\Xi] \mapsto \mu_{[i,0]}(\Xi) + \xi^i_0 I_n$$

induces an isomorphism $\mathfrak{M}^s_{Q^{(i)}, Q^{(i)}_0 \setminus \{[i,0]\}}(\boldsymbol{n}^{(i)}, \boldsymbol{\zeta}^{(i)}) \simeq O_i$. Recall that the residue manifold is the Hamiltonian reduction of $\boldsymbol{O}^s = (\prod_{i=0}^m O_i)^s$ by G. So let us describe the Hamiltonian G-space $\boldsymbol{O}$ as a quiver variety using the gluing method mentioned just after Proposition 3.2.2. Glue the open vertices $[i, 0]$, $i = 0, 1, \dots, m$ in the direct sum $\widetilde{Q} := \bigsqcup_{i=0}^m O_i$ all together and denote the resulting vertex by 0. Then we obtain the following *star-shaped* quiver $Q = \bigcup_0 Q^{(i)}$ with closed vertices $I = Q_0 \setminus \{0\}$.



The obvious morphism $\varphi\colon \widetilde{Q} \to Q$ is a gluing, and since $\boldsymbol{n}^{(i)}_{[i,0]} = n$ for all $i = 0, 1, \ldots, m$ there exists a unique $\boldsymbol{n} \in \mathbb{Z}^{Q_0}_{\geqslant 0}$ such that $\varphi_0^*(\boldsymbol{n}) = \oplus_i \boldsymbol{n}^{(i)}$. Set $\boldsymbol{\zeta} = (\varphi_0)_*(\oplus_i \boldsymbol{\zeta}^{(i)}) \in \mathbb{C}^I$, i.e.,

$$\boldsymbol{\zeta} = (\zeta_{[i,j]}), \quad \zeta_{[i,j]} = \xi^i_j - \xi^i_{j+1}.$$

Then Proposition 3.2.2 implies that the G-moment map

$$\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta}) \to \mathfrak{g}; \quad [\Xi] \mapsto \mu_0(\Xi) + \sum_{i=0}^{m} \xi^i_1 I_n \tag{5}$$

induces a G-equivariant symplectomorphism from $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$ to $\boldsymbol{O}$.

5.2.1 Lemma. *The above isomorphism maps $\boldsymbol{O}^s$ onto $\mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})^{s,0}$.*

Proof. Each $\Xi \in \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ consists of linear maps

$$\Xi^{(i)}_{j,j-1}\colon V^{\boldsymbol{n}}_{[i,j-1]} \to V^{\boldsymbol{n}}_{[i,j]}, \qquad \Xi^{(i)}_{j-1,j}\colon V^{\boldsymbol{n}}_{[i,j]} \to V^{\boldsymbol{n}}_{[i,j-1]}$$
$$(i = 0, 1, \ldots, m, \ \ j = 1, \ldots, d_i - 1),$$

where we have used the convention $V^{\boldsymbol{n}}_{[i,0]} \equiv V^{\boldsymbol{n}}_0$ ($i = 0, 1, \ldots, m$). For $[\Xi] \in \mathfrak{M}^s_{Q,I}(\boldsymbol{n}, \boldsymbol{\zeta})$, the corresponding point $(A_i)_{i=0}^m \in \prod_{i=0}^m O_i$ is given by $A_i = \Xi^{(i)}_{0,1}\Xi^{(i)}_{1,0} + \xi^i_1 \mathrm{Id}_{V^{\boldsymbol{n}}_0}$. First, suppose that Ξ is $\mathrm{GL}(\boldsymbol{n})$-stable and that a subspace $W_0 \subset \mathbb{C}^n = V^{\boldsymbol{n}}_0$ is preserved by all A_i. Then we extend it to a Q_0-graded subspace $W \subset V^{\boldsymbol{n}}$ by

$$W_{[i,j]} = (\Xi^{(i)}_{j,j-1} \cdots \Xi^{(i)}_{2,1}\Xi^{(i)}_{1,0})(W_0), \qquad [i,j] \in I.$$

Then relation (4) shows that W is Ξ-invariant, and hence it is $\{0\}$ or $V^{\boldsymbol{n}}$ since Ξ is stable. In particular $W_0 = \{0\}$ or $W_0 = \mathbb{C}^n$. Therefore (A_i) is stable. Conversely, suppose that (A_i) is stable and that a Q_0-graded subspace $W \subset V^{\boldsymbol{n}}$ is Ξ-invariant. Then $W_0 \subset V^{\boldsymbol{n}}_0$ is preserved by all $A_i = \Xi^{(i)}_{0,1}\Xi^{(i)}_{1,0} + \xi^i_1 \mathrm{Id}_{V^{\boldsymbol{n}}_0}$, and hence $W_0 = \{0\}$

or $W_0 = V_0^{\boldsymbol{n}}$ since (A_i) is stable. If $W_0 = \{0\}$, the $\mathrm{GL}(\boldsymbol{n}_I)$-stability of Ξ implies $W_I = \{0\}$, so $W = \{0\}$. Similarly, if $W_0 = V_0^{\boldsymbol{n}}$, the $\mathrm{GL}(\boldsymbol{n}_I)$-stability of Ξ implies $W_I = V_I^{\boldsymbol{n}}$, so $W = V^{\boldsymbol{n}}$. Therefore Ξ is $\mathrm{GL}(\boldsymbol{n})$-stable. □

Note that under the isomorphism (5) the moment map $\boldsymbol{O} \to \mathfrak{g}$, $(A_i) \mapsto \sum A_i$ corresponds to the map $\mu_0 + \sum \xi_1^i I_n$. Hence the following holds:

5.2.2 Theorem (Crawley-Boevey [8]). Define $\boldsymbol{\zeta} \in \mathbb{C}^{Q_0}$ by

$$\zeta_0 = -\sum_{i=0}^{m} \xi_1^i, \quad \zeta_{[i,j]} = \xi_j^i - \xi_{j+1}^i.$$

Then $\mathcal{M}^s(O_0, \dots, O_m) \simeq \mathfrak{M}_Q^s(\boldsymbol{n}, \boldsymbol{\zeta})$.

6 Higher Order Pole Case

In the previous section we see that any G-coadjoint orbit can be described as an open quiver variety of type A. In fact, even in the higher order pole case, some good class of $G[\![z]\!]$-coadjoint orbits is related to quivers. Using this fact we can describe some class of polar-parts manifolds as quiver varieties, extending Theorem 5.2.2.

6.1 Normal Forms and $G[\![z]\!]_0$-Coadjoint Orbits

First, we introduce a good class of $G[\![z]\!]$-coadjoint orbits. Let $\mathfrak{t} \subset \mathfrak{g}$ be the standard maximal torus.

6.1.1 Definition. A(n unramified) *irregular type* is an element Λ of $z^{-1}\mathfrak{t}[z^{-1}]$.

A *normal form* with irregular type Λ is an element of $\mathfrak{g}[z^{-1}]$ of the form

$$\delta\Lambda + L,$$

where $\delta = \delta_z$ is the degree operator $\delta(Cz^i) = iCz^i$ ($C \in \mathfrak{g}$) and L is an element of $\mathfrak{g}$ satisfying $[\Lambda(z), L] \equiv 0$. L is called the *matrix of exponents of formal monodromy*.

An irregular type Λ is a diagonal matrix with entries in $z^{-1}\mathbb{C}[z^{-1}]$. So we can write

$$\Lambda(z) = \bigoplus_{\lambda \in z^{-1}\mathbb{C}[z^{-1}]} \lambda(z)\,\mathrm{Id}_{V_\lambda} \tag{6}$$

for some vector space decomposition $\mathbb{C}^n = \bigoplus_\lambda V_\lambda$. Note that

$$\{ L \in \mathfrak{g} \mid [\Lambda(z), L] \equiv 0 \} = \bigoplus_{\lambda \in z^{-1}\mathbb{C}[z^{-1}]} \mathfrak{gl}(V_\lambda).$$

6.1.2 Remark. Note that the system of linear ordinary differential equations

$$z \frac{dv}{dz} = (\delta\Lambda + L)v$$

has the fundamental matrix $e^\Lambda z^L$. The classical Hukuhara–Turrittin–Levelt theory shows that any system of linear ordinary differential equations

$$z \frac{dv}{dz} = A(z)v, \qquad A(z) \in \mathfrak{g}(\!(z)\!)$$

with formal Laurent series coefficients has a fundamental matrix of the form

$$g(w)e^{\Lambda(w)}w^L,$$

where $z = w^r$ for some $r \in \mathbb{Z}_{>0}$, $g(w) \in G(\!(w)\!)$ and $\delta_w \Lambda + L \in \mathfrak{g}[w^{-1}]$ is a normal form (see e.g., [1]). In particular, one has

$$rg^{-1}Ag - g^{-1}\delta_w g = \delta_w \Lambda + L.$$

For any $r \in \mathbb{Z}_{\geqslant 0}$ a generic element in $\mathfrak{g}_{r+1}^*$ is equivalent to some normal form under the $G[\![z]\!]$-coadjoint action.

6.1.3 Proposition. *Let $A = \sum_{j=0}^r A_j z^{-j} \in \mathfrak{g}[z^{-1}]$. Assume that the top coefficient A_r is regular semisimple. Then the $G[\![z]\!]$-coadjoint orbit of A contains a normal form.*

This is a corollary of the following "block-diagonalization" theorem:

6.1.4 Theorem. *Let $A = \sum_{j=-\infty}^r A_j z^{-j} \in \mathfrak{g}(\!(z)\!)$, $r > 0$ with A_r semisimple and put $\mathfrak{h} = \mathrm{Ker}\,\mathrm{ad}_{A_r}$. Then there exists $g(z) \in G[\![z]\!]$ with $g(0) = 1$ such that $g[A] := gAg^{-1} + (\delta g)g^{-1} \in \mathfrak{h}(\!(z)\!)$.*

Note that the centralizer $\mathrm{Ker}\,\mathrm{ad}_T$ of any semisimple element $T \in \mathfrak{g}$ is the direct sum $\bigoplus \mathfrak{gl}(W_i)$, where W_i are the eigenspaces of T. If T is regular semisimple then the centralizer is a maximal torus, so Proposition 6.1.3 follows from Theorem 6.1.4.

Proof. Observe first that for any $X_1 \in \mathfrak{g}$ the gauge transform

$$A^{(1)} = \sum_{j=-\infty}^{r} A_j^{(1)} z^{-j} := e^{X_1 z}[A]$$

satisfies

$$A_r^{(1)} = A_r, \quad A_{r-1}^{(1)} = [X_1, A_r] + A_{r-1}.$$

Since A_r is semisimple, the Lie algebra $\mathfrak{g}$ has the decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \operatorname{Im} \operatorname{ad}_{A_r}.$$

Hence there exists a unique $X_1 \in \operatorname{Im} \operatorname{ad}_{A_r}$ such that $A_{r-1}^{(1)} \in \mathfrak{h}$. Next, for $X_2 \in \mathfrak{g}$ the gauge transform

$$A^{(2)} = \sum_{j=-\infty}^{r} A_j^{(2)} z^{-j} := e^{X_2 z^2}[A^{(1)}]$$

satisfies

$$A_r^{(2)} = A_r^{(1)} = A_r, \quad A_{r-1}^{(2)} = A_{r-1}^{(1)} \in \mathfrak{h}, \quad A_{r-2}^{(2)} = [X_2, A_r] + A_{r-2}^{(1)}.$$

Hence there exists a unique $X_2 \in \operatorname{Im} \operatorname{ad}_{A_r}$ such that $A_{r-2}^{(2)} \in \mathfrak{h}$. Iterating this argument, we can find $X_k \in \operatorname{Im} \operatorname{ad}_{A_r}$ $(k = 1, 2, \dots)$ such that if we set

$$A^{(0)} = A, \qquad A^{(k)} = \sum_{j=-\infty}^{r} A_j^{(k)} z^{-j} = e^{X_k z^k}[A^{(k-1)}] \quad (k > 0)$$

inductively, then

$$A_{r-j}^{(k)} \in \mathfrak{h} \qquad (j = 0, 1, \dots, k).$$

We leave the rest of the proof as an exercise, see below. □

6.1.5 Exercise. Complete the above proof by showing that

$$g(z) := \lim_{k \to \infty} e^{X_k z^k} \cdots e^{X_2 z^2} e^{X_1 z} \in G[[z]]$$

is well-defined and satisfies

$$g(0) = 1, \qquad g[A] = \sum_{j=0}^{\infty} A_{r-j}^{(j)} z^{j-r} \in \mathfrak{h}((z)).$$

To examine the structure of $G[[z]]$-coadjoint orbits containing normal forms, we introduce the following subgroup:

$$G[[z]]_0 := \{ g(z) \in G[[z]] \mid g(0) = 1 \}.$$

Heuristically the Lie algebra of $G[[z]]_0$ is $z\mathfrak{g}[[z]] \subset \mathfrak{g}[[z]]$. We have a semi-direct product decomposition $G[[z]] = G \ltimes G[[z]]_0$, which induces a direct sum decomposition

$$\mathfrak{g}[\![z]\!] = \mathfrak{g} \oplus z\mathfrak{g}[\![z]\!]$$

(as vector spaces), and accordingly,

$$\mathfrak{g}[z^{-1}] = \mathfrak{g} \oplus z^{-1}\mathfrak{g}[z^{-1}].$$

Note that the first projection $\mathfrak{g}[z^{-1}] \to \mathfrak{g}$ coincides with the map π_0. Let $\pi_{\mathrm{irr}} \colon \mathfrak{g}[z^{-1}] \to z^{-1}\mathfrak{g}[z^{-1}]$ be the second projection. The adjoint action of $G[\![z]\!]_0$ is just the restriction of that of $G[\![z]\!]$, and the induced "coadjoint action" on $z^{-1}\mathfrak{g}[z^{-1}]$ is given by

$$\mathrm{Ad}^{\sharp}_b(B) := \pi_{\mathrm{irr}}(\mathrm{Ad}^*_b(B)) \qquad (b \in G[\![z]\!]_0,\ B \in z^{-1}\mathfrak{g}[z^{-1}]),$$

where Ad^*_b on the right hand side is the $G[\![z]\!]$-coadjoint action by b. Note that if $\deg_{1/z}(B) = 1$ then $\mathrm{Ad}^{\sharp}_b(B) = B$ for all $b \in G[\![z]\!]_0$.

Suppose that a normal form $\delta\Lambda + L \in \mathfrak{g}[z^{-1}]$ is given. Then consider first the $G[\![z]\!]_0$-coadjoint orbit O_B through $\delta\Lambda$. If $\deg_{1/z}(\Lambda) = r$, then O_B is contained in the finite-dimensional vector space

$$\mathfrak{b}^*_{r+1} := \left\{ B \in \mathfrak{g}^*_{r+1} \mid \pi_0(B) = 0 \right\}$$

dual to the Lie algebra $\mathfrak{b}_{r+1} := \{ Y \in \mathfrak{g}_{r+1} \mid Y(0) = 0 \}$. The action of $G[\![z]\!]_0$ on $\mathfrak{b}^*_{r+1}$ reduces to the coadjoint action of the algebraic group

$$B_{r+1} := \{ b \in G_{r+1} \mid b(0) = 1 \}$$

as in the case of $G[\![z]\!]$-coadjoint orbits. In particular it is a (finite-dimensional) complex symplectic manifold. In fact, using the following proposition one can check that it is affine:

6.1.6 Proposition. (1) *If an element B of O_B lies in $z^{-1}\mathfrak{t}'[z^{-1}]$ for some maximal torus $\mathfrak{t}'$, then $B = \delta\Lambda$.*

(2) *Write $\Lambda = \sum_{j=1}^{r} \Lambda_j z^{-j}$. Then the stabilizer $(B_{r+1})_{\delta\Lambda}$ of $\delta\Lambda$ coincides with the subset*

$$\left\{ \sum_{j=0}^{r} b_j z^j \in B_{r+1} \mid b_j \in \bigcap_{i>j} \mathrm{Ker}\, \mathrm{ad}_{\Lambda_i}\ (j = 1, 2, \ldots, r-1) \right\}.$$

Proof. Suppose that $B = \mathrm{Ad}^{\sharp}_b(\delta\Lambda) \in O_B$ lies in $z^{-1}\mathfrak{t}'[z^{-1}]$. Write $B = \sum_{j=1}^{r} B_j z^{-j}$, $b = \sum_{j=0}^{r} b_j z^j$. Then

$$\sum_{j=0}^{k} B_{r-k+j} b_j = \sum_{j=0}^{k} (k - j - r) b_j \Lambda_{r-k+j} \qquad (k = 0, \ldots, r-1).$$

Looking at the equality for $k = 0$, we obtain $B_r = -r\Lambda_r$ (because $b_0 = 1$). Next look at the equality for $k = 1$:

$$B_{r-1} + B_r b_1 = (1-r)\Lambda_{r-1} - rb_1\Lambda_r,$$

or equivalently

$$B_{r-1} - (1-r)\Lambda_{r-1} = B_r b_1 - rb_1\Lambda_r = [-r\Lambda_r, b_1].$$

The left hand side lies in Ker ad_{Λ_r} while the right hand side lies in Im ad_{Λ_r}. Since Λ_r is semisimple, we have

$$\mathfrak{g} = \mathrm{Ker}\ \mathrm{ad}_{\Lambda_r} \oplus \mathrm{Im}\ \mathrm{ad}_{\Lambda_r}.$$

Hence $B_{r-1} = (1-r)\Lambda_{r-1}$ and $[\Lambda_r, b_1] = 0$. Further looking at the equality for $k = 2$ and using the decomposition

$$\mathfrak{g} = (\mathrm{Ker}\ \mathrm{ad}_{\Lambda_r} \cap \mathrm{Ker}\ \mathrm{ad}_{\Lambda_{r-1}}) \oplus \mathrm{Im}\left(\mathrm{ad}_{\Lambda_{r-1}}|_{\mathrm{Ker}\ \mathrm{ad}_{\Lambda_r}}\right) \oplus \mathrm{Im}\ \mathrm{ad}_{\Lambda_r},$$

we obtain $B_{r-2} = (2-r)\Lambda_{r-1}$ and $[\Lambda_{r-1}, b_1] = [\Lambda_r, b_2] = 0$. Iterating this argument yields $B = \delta\Lambda$ and $b_j \in \bigcap_{i>j} \mathrm{Ker}\ \mathrm{ad}_{\Lambda_i}$, $j = 1, \ldots, r-1$. □

6.1.7 Examples. If $\deg_{1/z}(\Lambda) = 2$, for $b = 1 + b_1 z + b_2 z^2$ we have

$$\mathrm{Ad}_b^\sharp(\delta\Lambda) = -2\Lambda_2 z^{-2} - (\Lambda_1 + 2[b_1, \Lambda_2])z^{-1}.$$

Hence

$$O_B = \left\{ -2\Lambda_2 z^{-2} - (\Lambda_1 + 2[X, \Lambda_2])z^{-1} \mid X \in \mathfrak{g} \right\} \simeq \mathrm{Im}\ \mathrm{ad}_{\Lambda_2}.$$

Define

$$H = \{ h \in G \mid h\Lambda h^{-1} = \Lambda \} \subset G$$

and let $\mathfrak{h}$ be the Lie algebra of H. Using expression (6), we have

$$H = \prod_{\lambda \in z^{-1}\mathbb{C}[z^{-1}]} \mathrm{GL}(V_\lambda), \quad \mathfrak{h} = \bigoplus_{\lambda \in z^{-1}\mathbb{C}[z^{-1}]} \mathfrak{gl}(V_\lambda).$$

In particular we may identify $\mathfrak{h}^*$ with $\mathfrak{h}$ using the restriction of the pairing on $\mathfrak{g}$. Let H act on O_B by $h \colon B \mapsto hBh^{-1}$; it is well-defined because if $h \in H$ and $B = \mathrm{Ad}_b^\sharp(\delta\Lambda) \in O_B$ (where $b \in G[\![z]\!]_0$) then $hbh^{-1} \in G[\![z]\!]_0$ and

$$hBh^{-1} = h\,\mathrm{Ad}_b^\sharp(\delta\Lambda)h^{-1} = \mathrm{Ad}_{hbh^{-1}}^\sharp(h\delta\Lambda h^{-1}) = \mathrm{Ad}_{hbh^{-1}}^\sharp(\delta\Lambda) \in O_B.$$

Note that $L \in \mathfrak{h}$ by the definition of normal form. The following fact due to Boalch [2] relates the $G[\![z]\!]$-coadjoint orbit of $\delta\Lambda + L$ to O_B:

6.1.8 Proposition. (1) *The H-action on O_B is Hamiltonian with a moment map $\mu_H\colon O_B \to \mathfrak{h}$ satisfying $\mu_H(\delta\Lambda) = 0$.*

(2) *Endow T^*G with a Hamiltonian $H \times G$-structure by restricting the action of $G \times G$ to the subgroup $H \times G$. Then the map*

$$\varphi\colon T^*G \times O_B \to \mathfrak{g}[z^{-1}];\quad (u, R, B) \mapsto u^{-1}Bu - R$$

induces an isomorphism of Hamiltonian G-spaces from the Hamiltonian reduction

$$(T^*G \times O_B)/\!\!/_{-L} H$$

to the $G[\![z]\!]$-coadjoint orbit O of $\delta\Lambda + L$.

Proof. Let $\deg_{1/z}(\Lambda) = r$. By Example 2.2.7, the B_{r+1}-coadjoint orbit O_B is isomorphic to the Hamiltonian reduction $T^* B_{r+1}/\!\!/_{-\delta\Lambda}\, B_{r+1}$ by the left translation at the level $-\delta\Lambda$. Furthermore, one can observe that $T^* B_{r+1}$ is isomorphic to the Hamiltonian reduction $T^*G_{r+1}/\!\!/ G$ by the right translation. Thus we obtain an isomorphism

$$\chi\colon T^*G_{r+1}/\!\!/_{(0,-\delta\Lambda)}(G \times B_{r+1}) \xrightarrow{\simeq} O_B;\quad [g, A] \mapsto -g(0)A\, g(0)^{-1}$$

of symplectic manifolds. Let H act on T^*G_{r+1} by the left translation. Then it descends to an action on $T^*G_{r+1}/\!\!/_{(0,-\delta\Lambda)}(G \times B_{r+1})$ because H normalizes $(B_{r+1})_{\delta\Lambda}$, and the above isomorphism is equivariant. Since the H-action on T^*G_{r+1} is Hamiltonian the induced action on the Hamiltonian reduction is also Hamiltonian; a moment map is given by

$$T^*G_{r+1}/\!\!/_{(0,-\delta\Lambda)}(G \times B_{r+1}) \to \mathfrak{h};\quad [g, A] \mapsto (\mathrm{Ad}_g^* A)|_{\mathfrak{h}},$$

where $(\,\cdot\,)|_{\mathfrak{h}}\colon \mathfrak{g} \to \mathfrak{h}$ is the transpose of the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$ (with respect to the pairings). Therefore assertion (1) follows.

Let $Z \subset T^*G_{r+1}$ be the level set of the moment map for the $G \times B_{r+1}$-action:

$$Z = \left\{ (g, A) \in T^*G_{r+1} \mid \pi_{\mathrm{irr}}(\mathrm{Ad}_g^* A) = -\delta\Lambda,\ \pi_0(A) = 0 \right\}.$$

Put $\widetilde{H} = H(B_{r+1})_{\delta\Lambda} \subset G_{r+1}$, and define an action of $G \times \widetilde{H}$ on $T^*G \times Z$ by

$$(p, h)\colon (u, R, g, A) \mapsto (h(0)u, R, hgp^{-1}, pAp^{-1}).$$

Note that the subgroup $G \times (B_{r+1})_{\delta\Lambda}$ trivially acts on the first factor T^*G and the orbit space $(T^*G \times Z)/(G \times (B_{r+1})_{\delta\Lambda})$ is isomorphic to $T^*G \times O_B$ via χ. Also the $G \times \widetilde{H}$-invariant map

$$\widetilde{\varphi}\colon T^*G \times Z \to \mathfrak{g}_{r+1}^*;\quad (u, R, g, A) \mapsto -u^{-1}g(0)A\, g(0)^{-1}u - R$$

descends to φ, and the map

$$\nu\colon T^*G \times Z \to \mathfrak{h};\quad (u, R, g, A) \mapsto \pi_0(\mathrm{Ad}_g^* A)|_{\mathfrak{h}} + (uRu^{-1})|_{\mathfrak{h}}$$

descends to the moment map of $T^*G \times O_B$ for the H-action.

6.1.9 Claim. Any non-empty fiber of $\widetilde{\varphi}$ is a single $G \times \widetilde{H}$-orbit and $\widetilde{\varphi}$ maps the level set $\nu^{-1}(-L)$ onto O.

Note that the orbit space $\nu^{-1}(-L)/(G \times H_L(B_{r+1})_{\delta\Lambda})$ is biholomorphic to the Hamiltonian reduction $(T^*G \times O_B)/\!\!/_{-L} H$. From the claim we thus obtain a biholomorphism $(T^*G \times O_B)/\!\!/_{-L} H \xrightarrow{\simeq} O$. By a direct calculation one can check that it intertwines the Hamiltonian G-structures. We leave it as an exercise.

We show the claim. Suppose that $(u, R, g, A), (\tilde{u}, \widetilde{R}, \tilde{g}, \widetilde{A}) \in T^*G \times Z$ are contained in a common fiber of $\widetilde{\varphi}$. Define

$$B = -g(0)A\, g(0)^{-1},\ \widetilde{B} = -\tilde{g}(0)\widetilde{A}\,\tilde{g}(0)^{-1} \in O_B.$$

Then we have $R = \widetilde{R}$ and $u^{-1}Bu = \tilde{u}^{-1}\widetilde{B}\tilde{u}$. So $h_0 := \tilde{u}u^{-1} \in G$ satisfies $h_0 B h_0^{-1} = \widetilde{B}$. Since G normalizes B_{r+1} we have $h_0 \delta\Lambda h_0^{-1} \in O_B$, which together with Proposition 6.1.6, (1) implies $h_0 \in H$. Put $p = \tilde{g}(0)^{-1} h_0\, g(0) \in G$. Then

$$pAp^{-1} = \tilde{g}(0)^{-1} h_0 B h_0^{-1} \tilde{g}(0)^{-1} = \widetilde{A},$$

and hence

$$(p, h_0) \cdot (u, R, g, A) = (\tilde{u}, R, h_0 g p^{-1}, \widetilde{A}) = (\tilde{u}, R, b\tilde{g}, \widetilde{A}),$$

where

$$b := h_0 g p^{-1} \tilde{g}^{-1} = h_0 (g\, g(0)^{-1}) h_0^{-1} (\tilde{g}\, \tilde{g}(0)^{-1})^{-1} \in B_{r+1}.$$

We have

$$\mathrm{Ad}_b^\sharp(\delta\Lambda) = \mathrm{Ad}^\sharp_{h_0(g\,g(0)^{-1})h_0^{-1}}(\widetilde{B}) = \mathrm{Ad}^\sharp_{h_0(g\,g(0)^{-1})}(B) = \mathrm{Ad}^\sharp_{h_0}(\delta\Lambda) = \delta\Lambda,$$

i.e., $b \in (B_{r+1})_{\delta\Lambda}$. Hence $(u, R, g, A), (\tilde{u}, \widetilde{R}, \tilde{g}, \widetilde{A})$ are contained in a common orbit. Finally, for any $g \in G_{r+1}$ the element

$$(g(0)^{-1}, -\pi_0(\mathrm{Ad}_g^*(\delta\Lambda + L)), g^{-1}, -\pi_{\mathrm{irr}}(\mathrm{Ad}_g^* \delta\Lambda)) \in T^*G \times Z$$

lies in $\nu^{-1}(-L)$ and is mapped to $\mathrm{Ad}_g^*(\delta\Lambda + L)$ by $\widetilde{\varphi}$. □

6.2 Triangular Decomposition of $G[\![z]\!]_0$-Coadjoint Orbits

We still fix a normal form $\delta\Lambda + L \in \mathfrak{g}[z^{-1}]$ and retain the notation used in the previous section. We will show that the $G[\![z]\!]_0$-coadjoint orbit O_B is equivariantly symplectomorphic to the representation space $\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ for some Q and $\boldsymbol{n}$.

Write $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$, $\lambda_i \in z^{-1}\mathbb{C}[z^{-1}]$. Without loss of generality we may assume that Λ satisfies the following condition:

$$i < j < k \Longrightarrow \deg_{1/z}(\lambda_i - \lambda_j) \leqslant \deg_{1/z}(\lambda_i - \lambda_k), \tag{7}$$

where we use the convention $\deg_{1/z}(0) = -\infty$. If one writes

$$\Lambda = \sum_{j=1}^{r} \Lambda_j z^{-j}, \quad \Lambda_j \in \mathfrak{t},$$

then the condition says that for any $k \in \{1, 2, \dots, r\}$, one has

$$\sum_{j=k}^{r} \Lambda_j z^{-j} = \begin{pmatrix} \sigma_1(z)\, I_{m_1} & & & 0 \\ & \sigma_2(z)\, I_{m_2} & & \\ & & \ddots & \\ 0 & & & \sigma_l(z)\, I_{m_l} \end{pmatrix}$$

for some $\sigma_1, \sigma_2, \dots, \sigma_l \in z^{-k}\mathbb{C}[z^{-1}]$ and $m_1, m_2, \dots, m_l \in \mathbb{Z}_{>0}$ with $\sum m_i = n$.

6.2.1 Exercise. Show that for any irregular type Λ there exists a permutation matrix P such that $P\Lambda P^{-1}$ satisfies the above condition.

Let G' be the centralizer of Λ_r and $\mathfrak{g}'$ its Lie algebra. By condition (7), we may write

$$G' = \{\, g = \mathrm{diag}(g_1, g_2, \dots, g_l) \mid g_i \in \mathrm{GL}(n_i, \mathbb{C}) \,\} \subset G,$$

where $n_1, n_2, \dots, n_l$ are the multiplicities of the eigenvalues of Λ_r. Using the block triangular decomposition

$$\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{u}_+ \oplus \mathfrak{u}_-,$$

define

$$\mathfrak{U}_\pm = \left\{ \sum_{i=1}^{r-1} X_i z^i \mid X_i \in \mathfrak{u}_\pm\ (1 \leqslant i < r) \right\} \subset \mathfrak{b}_r, \qquad \mathcal{U}_\pm = \exp(\mathfrak{U}_\pm),$$

$$\mathfrak{b}'_r = \left\{ \sum_{i=1}^{r-1} X_i z^i \mid X_i \in \mathfrak{g}'\ (1 \leqslant i < r) \right\} \subset \mathfrak{b}_r, \qquad B'_r = \exp(\mathfrak{b}'_r).$$

Observe that if we write $B_r = B_r(n)$, $\mathfrak{b}_r = \mathfrak{b}_r(n)$ to emphasize the size of matrices, then we have $B'_r = \prod_{i=1}^{l} B_r(n_i)$, $\mathfrak{b}'_r = \bigoplus_{i=1}^{l} \mathfrak{b}_r(n_i)$ according to the eigenspace decomposition for Λ_r.

6.2.2 Lemma. *Any $b \in B_r$ is uniquely decomposed as $b = u_- u_+ b'$, where $u_\pm \in \mathcal{U}_\pm$ and $b' \in B'_r$.*

Proof. Write $b = e^X$, $b' = e^{X'}$, $u_\pm = e^{Y_\pm}$ and

$$X = \sum_{i=1}^{r-1} X_i z^i, \quad X' = \sum_{i=1}^{r-1} X'_i z^i, \quad Y_\pm = \sum_{i=1}^{r-1} Y_i^\pm z^i.$$

Then the equality $b = u_- u_+ b'$ may be written as

$$X_i = Y_i^- + Y_i^+ + X'_i + R_i \qquad (i = 1, 2, \ldots, r-1),$$

where $R_i \in \mathfrak{g}$ is some noncommutative polynomial expression of $X_j, X'_j, Y_j^\pm, j < i$. Thanks to the triangular decomposition, the above equalities uniquely determine $Y_i^\pm, X'_i$ inductively. □

Note that any element in B_{r+1} of the form e^{Xz^r}, $X \in \mathfrak{g}$ acts trivially on O_B. Hence the coadjoint action on O_B reduces to an action of B_r.

6.2.3 Proposition. *For each $B \in O_B$ take any $b \in B_r$ so that $\mathrm{Ad}_b^\sharp(\delta\Lambda) = B$ and decompose $b = u_- u_+ b'$ using the previous lemma. Put $\Lambda' = \Lambda - \Lambda_r z^{-r}$ and let O'_B be the B'_r-coadjoint orbit of $\delta\Lambda'$. Define*

$$B' = \mathrm{Ad}_{b'}^\sharp(\delta\Lambda') \in (\mathfrak{b}'_r)^*, \quad Y = \mathrm{Ad}_{u_-^{-1}}^\sharp(B) - \mathrm{Ad}_{b'}^\sharp(\delta\Lambda) \in \mathfrak{b}_r^*.$$

Then $Y \in \mathfrak{U}_-^* \subset \mathfrak{b}_r^*$ and the following is a well-defined symplectomorphism:

$$O_B \to T^*\mathcal{U}_- \times O'_B; \quad B \mapsto (u_-, Y, B').$$

Furthermore, if we let H act on $\mathcal{U}_+$, O'_B by conjugation, then the above map is H-equivariant.

Proof. Proposition 6.1.6 implies $(B_r)_{\delta\Lambda} \subset B'_r$. Hence $u_\pm$, $\mathrm{Ad}_{b'}^\sharp(\delta\Lambda)$ and $B' = \mathrm{Ad}_{b'}^\sharp(\delta\Lambda) + r\Lambda_r z^{-r}$ are all independent of the choice of b. Put

$$\overline{B} = B' - r\Lambda_r z^{-r} = \mathrm{Ad}_{b'}^\sharp(\delta\Lambda), \quad \overline{Y} = Y + \overline{B} = \mathrm{Ad}_{u_-^{-1}}^\sharp(B) = \mathrm{Ad}_{u_+}^\sharp(\overline{B}).$$

Since $\overline{B}$ has coefficients in $\mathfrak{g}'$ and $u_+ - 1$ has coefficients in $\mathfrak{u}_+$, we see that $Y = \mathrm{Ad}_{u_+}^\sharp(\overline{B}) - \overline{B}$ has coefficients in $\mathfrak{u}_+$, i.e., $Y \in \mathfrak{U}_-^*$. We will construct an inverse of the map $B \mapsto (u_-, Y, B')$. Given $(u_-, Y, B') \in T^*\mathcal{U}_- \times O'_B$, put $\overline{B}, \overline{Y}$ as above and we claim that there exists a unique $u_+ \in \mathcal{U}_+$ such that $\overline{Y} = \mathrm{Ad}_{u_+}^\sharp(\delta\Lambda)$. Then

$B := \mathrm{Ad}^\sharp_{u_-}(\overline{Y})$ is contained in O_B and the map $(u_-, Y, B') \mapsto B$ gives an inverse. To find such u_+, we have to solve the system of equations

$$\sum_{j=0}^{k} \overline{Y}_{r-k+j} u_j^+ = \sum_{j=0}^{k} u_j^+ \overline{B}_{r-k+j} \qquad (k = 0, 1, \ldots, r-1),$$

where we write $u_+ = \sum u_j^+ z^j$, $\overline{Y} = \sum \overline{Y}_j z^{-j}$, e.t.c. The equation for $k = 1$ reads

$$\overline{Y}_{r-1} - \overline{B}_{r-1} = u_1^+ \overline{B}_r - \overline{Y}_r u_1^+ = [u_1^+, -r\Lambda_r].$$

Since the left hand side is equal to Y_{r-1} and hence lies in $\mathfrak{u}_+$, there exists a unique $u_1^+ \in \mathfrak{u}_+$ satisfying the equality. Next the equation for $k = 2$ reads

$$\overline{Y}_{r-2} - \overline{B}_{r-2} + \overline{Y}_{r-1} u_1^+ - u_1^+ \overline{B}_{r-1} = [u_2^+, -r\Lambda_r].$$

Since the left hand side is contained in $\mathfrak{u}_+$, there exists a unique $u_2^+ \in \mathfrak{u}_+$ satisfying the equality. Iterating this argument, we see the unique existence of $u_+ \in \mathcal{U}_+$ with $\overline{Y} = \mathrm{Ad}^\sharp_{u_+}(\delta\Lambda)$.

Finally we show that the map $B \mapsto (u_-, Y, B')$ preserves the symplectic structure. The pull-back of the symplectic form on O_B via the map

$$B_r \to O_B; \quad b \mapsto B = \mathrm{Ad}^\sharp_b(\delta\Lambda)$$

is described as

$$\omega = (B, db\, b^{-1} \wedge db\, b^{-1}) = (\delta\Lambda, b^{-1}db \wedge b^{-1}db),$$

where $(\cdot, \cdot)$ is the dual pairing between $\mathfrak{b}_{r+1}^*$ and $\mathfrak{b}_{r+1}$ (so in the above we take any lift of $b^{-1}db \in \mathfrak{b}_r$ to $\mathfrak{b}_{r+1}$). In terms of the decomposition $b = u_- v$, $u_- \in \mathcal{U}_-$, $v \in \mathcal{U}_+ B_r'$, we can express the Maurer–Cartan form $b^{-1}db$ as

$$b^{-1}db = v^{-1}(u_-^{-1}du_-)v + v^{-1}dv,$$

and accordingly,

$$\begin{aligned}
\omega &= (\delta\Lambda, v^{-1}(u_-^{-1}du_-) \wedge (u_-^{-1}du_-)v) + (\delta\Lambda, v^{-1}(u_-^{-1}du_-) \wedge dv) \\
&\quad + (\delta\Lambda, v^{-1}dv \wedge v^{-1}(u_-^{-1}du_-)v] + (\delta\Lambda, v^{-1}dv \wedge v^{-1}dv) \\
&= (\overline{Y}, u_-^{-1}du_- \wedge u_-^{-1}du_-) + (\overline{Y}, u_-^{-1}du_- \wedge dv\, v^{-1}) \\
&\quad + (\overline{Y}, dv \wedge v^{-1}u_-^{-1}du_-) + (\delta\Lambda, v^{-1}dv \wedge v^{-1}dv).
\end{aligned}$$

On the other hand, the symplectic form $-d\,\mathrm{res}_{z=0}\,\mathrm{tr}(Y\, u_-^{-1}du_-)$ on $T^*\mathcal{U}_-$ is equal to $-d(\overline{Y}, u_-^{-1}du_-)$ because $\overline{B}\, u_-^{-1}du_-$ is strictly block upper-triangular and is traceless.

Since $\overline{Y} = \mathrm{Ad}^\sharp_v(\delta\Lambda)$ under the pull-back via the map $b = u_- v \mapsto B \mapsto (u_-, Y, B')$, we have

$$\begin{aligned} d(\overline{Y}, u_-^{-1}du_-) &= (d\overline{Y} \wedge u_-^{-1}du_-) - (\overline{Y}, u_-^{-1}du_- \wedge u_-^{-1}du_-) \\ &= -(\overline{Y}, u_-^{-1}du_- \wedge dv\, v^{-1}) - (\overline{Y}, dv\, v^{-1} \wedge u_-^{-1}du_-) \\ &\quad - (\overline{Y}, u_-^{-1}du_- \wedge u_-^{-1}du_-). \end{aligned}$$

Hence

$$\omega = -d(Y, u_-^{-1}du_-) + (\delta\Lambda, v^{-1}dv \wedge v^{-1}dv).$$

Also, in terms of the decomposition $v = u_+ b'$, $u_+ \in \mathcal{U}_+$, $b' \in B_r'$ we have

$$v^{-1}dv = (b')^{-1}(u_+^{-1}du_+)b' + (b')^{-1}db'.$$

Since $(b')^{-1}(u_+^{-1}du_+)b'$ is strictly block upper-triangular, we obtain

$$(\delta\Lambda, v^{-1}dv \wedge v^{-1}dv) = (\delta\Lambda, (b')^{-1}db' \wedge (b')^{-1}db'),$$

which is the symplectic form on O_B' because $(b')^{-1}db'$ commutes with Λ_r and hence

$$\mathrm{tr}[\Lambda_r\, (b')^{-1}db' \wedge (b')^{-1}db'] = 0.$$

We are done. □

Note that we have a decomposition $O_B' = \prod_{i=1}^l O_B^{(i)}$, where $O_B^{(i)}$ is the $B_r(n_i)$-coadjoint orbit through the i-th block of $\delta\Lambda'$. Since $\delta\Lambda'$ has degree less than r, we can recursively apply the above proposition.

Let us consider the Hamiltonian H-structure on $T^*\mathcal{U}_-$. Recalling expression (6), set

$$\Sigma = \left\{ \lambda \in z^{-1}\mathbb{C}[z^{-1}] \mid V_\lambda \neq \{0\} \right\}.$$

We have the surjective map $\{1, 2, \ldots, n\} \to \Sigma$, $i \mapsto \lambda_i$ taking the diagonal entries of Λ. Condition (7) implies that for each $\lambda \in \Sigma$ the subspace $V_\lambda \subset \mathbb{C}^n$ is canonically identified with $\mathbb{C}^{n_\lambda}$, where $n_\lambda := \dim V_\lambda$, and that Σ has a total ordering such that $\lambda_i \leqslant \lambda_j$ $(i < j)$. Under this identification we have $H = \prod_{\lambda\in\Sigma} \mathrm{GL}(n_\lambda, \mathbb{C})$.

6.2.4 Proposition. *Define a quiver Q with vertices Σ by drawing $(r-1)$ arrows from λ to λ' for each $\lambda, \lambda' \in \Sigma$ with $\lambda < \lambda'$, $\deg_{1/z}(\lambda - \lambda') = r$. Put $\boldsymbol{n}^\Lambda = (n_\lambda)_{\lambda\in\Sigma}$, so $\mathrm{GL}(\boldsymbol{n}) = H$. Then there exists an isomorphism of Hamiltonian H-spaces*

$$T^*\mathcal{U}_- \xrightarrow{\simeq} \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n}).$$

Proof. The biholomorphism $\mathcal{U}_- \xrightarrow{\simeq} \mathfrak{U}_-$, $u_- \mapsto u_- - 1$ induces an isomorphism $T^*\mathcal{U}_- \xrightarrow{\simeq} T^*\mathfrak{U}_- \simeq \mathfrak{U}_- \times \mathfrak{U}_-^*$ of Hamiltonian H-spaces. Taking the blocks of coefficients yields a linear isomorphism $T^*\mathfrak{U}_- \xrightarrow{\simeq} \mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})$ of Hamiltonian H-spaces. □

The map from Σ onto the set of eigenvalues of Λ_r taking the coefficient in degree r induces a decomposition $\Sigma = \bigsqcup_{i=1}^{l} \Sigma_i$ (into fibers of the map). If we apply Propositions 6.2.3 and 6.2.4 to each $O_B^{(i)}$, then we obtain an isomorphism

$$O_B^{(i)} \simeq \mathrm{Rep}_{\overline{Q^{(i)}}}(\boldsymbol{n}^{(i)}) \times (O_B^{(i)})',$$

where $\boldsymbol{n}^{(i)} = \boldsymbol{n}_{\Sigma_i}$ and $Q^{(i)}$ is the quiver with vertices Σ_i obtained by drawing $(r-2)$ arrows from λ to λ' for each $\lambda, \lambda' \in \Sigma_i$ with $\lambda < \lambda'$, $\deg_{1/z}(\lambda - \lambda') = r - 1$. Iterating this argument, we obtain the following theorem conjectured by Boalch [3]:

6.2.5 Theorem ([13]). Define a quiver $Q(\Lambda)$ with vertices Σ by

$$\#\left\{ a \in Q(\Lambda)_1 \mid a\colon \lambda \to \lambda' \right\} = \begin{cases} \deg_{1/z}(\lambda - \lambda') - 1 & (\lambda < \lambda'), \\ 0 & (\lambda \geqslant \lambda'). \end{cases}$$

Then the iterative application of Propositions 6.2.3 and 6.2.4 yields an isomorphism of Hamiltonian H-spaces

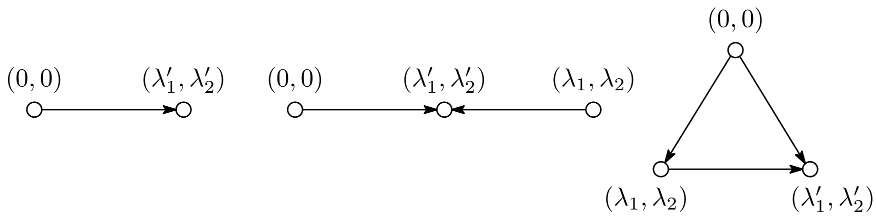
$$O_B \xrightarrow{\simeq} \mathrm{Rep}_{\overline{Q(\Lambda)}}(\boldsymbol{n}).$$

6.2.6 Example. (1) If $\deg_{1/z}(\Lambda) = 1$, then $Q(\Lambda)$ has no arrow and the vertices are parameterized by the eigenvalues of Λ_1.

(2) If $\deg_{1/z}(\Lambda) = 2$, then any two vertices in $Q(\Lambda)$ are connected by at most one arrow, and the vertices are parameterized by the joint eigenvalues of (Λ_1, Λ_2). Two joint eigenvalues (λ_1, λ_2), (λ_1', λ_2') are connected by an arrow if and only if $\lambda_2 \neq \lambda_2'$. For instance, if

$$\Lambda = \mathrm{diag}(0, \lambda_1 z^{-1} + \lambda_2 z^{-2}, \lambda_1' z^{-1} + \lambda_2' z^{-2})$$

with $\lambda_2'(\lambda_2 - \lambda_2') \neq 0$, then the quiver $Q(\Lambda)$ is one of the following three.



The first case occurs when $\lambda_2 = \lambda_1 = 0$, the second case occurs when $\lambda_2 = 0$ and $\lambda_1 \neq 0$, and the third case occurs when $\lambda_2 \neq 0$. See [3, 5] for more examples.

Using the isomorphism $O_B \xrightarrow{\simeq} \mathrm{Rep}_{\overline{Q(\Lambda)}}(\boldsymbol{n})$ the subspaces of $\mathbb{C}^n$ preserved by all the coefficients of given $B \in O_B$ are characterized as follows:

6.2.7 Proposition. *Take* $B = \sum_{i=1}^{r} B_i z^{-i} \in O_B$ *and let* $\Xi \in \mathrm{Rep}_{\overline{Q(\Lambda)}}(\boldsymbol{n})$ *be the corresponding element under the isomorphism given in Theorem 6.2.5. Then a subspace* $W \subset \mathbb{C}^n$ *is preserved by all* B_i *if and only if it is homogeneous with respect to the decomposition* $\mathbb{C}^n = \bigoplus_{p \in Q(\Lambda)_0} V_p^{\boldsymbol{n}}$ *and* Ξ*-invariant as a* $Q(\Lambda)_0$*-graded subspace of* $V^{\boldsymbol{n}}$*.*

This follows from the following lemma:

6.2.8 Lemma. *Take* $B = \sum_{i=1}^{r} B_i z^{-i} \in O_B$ *and let* $(u_-, Y, B') \in T^*\mathcal{U}_- \times O'_B$ *be the corresponding element under the isomorphism given in Proposition 6.2.3. Then a subspace* $W \subset \mathbb{C}^n$ *is preserved by all* B_i *if and only if it is* Λ_r*-invariant and preserved by all the coefficients of* u_-*,* Y*,* B'*.*

Proof. We follow the notation used in the proof of Proposition 6.2.3. Suppose that a subspace $W \subset \mathbb{C}^n$ is preserved by all B_i. Since $B_r = -r\Lambda_r$, it is Λ_r-invariant. To show that it is preserved by all the coefficients of u_-, Y, B', look at the equality $B = \mathrm{Ad}^{\sharp}_{u_-}(\overline{Y})$:

$$\sum_{j=0}^{k} B_{r-k+j} u_j^- = \sum_{j=0}^{k} u_j^- \overline{Y}_{r-k+j} \qquad (k = 0, 1, \ldots, r-1).$$

The equality for $k = 0$ implies that W is preserved by $\overline{Y}_r = B_r$. The equality for $k = 1$ reads

$$B_{r-1} = \overline{Y}_{r-1} + u_1^- \overline{Y}_r - B_r u_1^- = Y_{r-1} + \overline{B}_{r-1} + [u_1^-, B_r].$$

Since W is Λ_r-invariant and the terms on the right hand side are contained in $\mathfrak{u}_+$, $\mathfrak{g}'$, $\mathfrak{u}_-$, respectively, we see that W is preserved by Y_{r-1}, $\overline{B}_{r-1}$ and u_1^-. Next the equality for $k = 2$ reads

$$B_{r-2} - u_1^- \overline{Y}_{r-1} + B_{r-1} u_1^- = Y_{r-2} + \overline{B}_{r-2} + [u_2^-, B_r].$$

Since W is preserved by the left hand side and the right hand side respects the decomposition $\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{g}' \oplus \mathfrak{u}_-$, we see that W is preserved by Y_{r-2}, $\overline{B}_{r-2}$ and u_2^-. Iterating this argument we finally see that W satisfies the desired condition. □

6.3 Polar-Parts Manifolds and Quiver Varieties

Let $O_0 \equiv O$ be the $G[\![z]\!]$-coadjoint orbit through a normal form $\delta\Lambda + L$ and $O_1, O_2, \ldots, O_m$ be G-coadjoint orbits. We describe the polar-parts manifold $\mathcal{M}^s(O_0, \ldots, O_m)$ for such $O_0, \ldots, O_m$ as a quiver variety.

Put $\boldsymbol{O}^{\mathrm{rs}} = \prod_{i=1}^{m} O_i$. Then Proposition 6.1.8 enables us to describe the product $\boldsymbol{O} = O \times \boldsymbol{O}^{\mathrm{rs}}$ as

$$\boldsymbol{O} \simeq \big((T^*G \times O_B)/\!\!/_{-L} H\big) \times \boldsymbol{O}^{\mathrm{rs}} \simeq \big(T^*G \times O_B \times \boldsymbol{O}^{\mathrm{rs}}\big) /\!\!/_{-L} H,$$

where $\boldsymbol{n}^{\mathrm{irr}} := (n_\lambda)_{\lambda\in\Sigma}$. Using the shifting trick we thus obtain

$$\boldsymbol{O} \simeq \big(T^*G \times O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big) /\!\!/ H,$$

where O^{fm} is the H-coadjoint orbit of L. The isomorphism is defined by

$$\big(T^*G \times O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big) /\!\!/ H \to \boldsymbol{O}; \quad [u, R, B, A^{\mathrm{rs}}, S] \mapsto (u^{-1}Bu - R, A^{\mathrm{rs}}),$$

and the $H \times G$-action on $T^*G \times O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}$ is given by

$$(h, g)\cdot(u, R, B, A^{\mathrm{rs}}, S) = (hug^{-1}, \mathrm{Ad}^*_g R, hBh^{-1}, \mathrm{Ad}^*_h S).$$

Now Exercises 2.2.9 and 2.2.10 imply that there are homeomorphisms

$$\begin{aligned}\boldsymbol{O}/\!\!/ G &\simeq \big(T^*G \times O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big) /\!\!/ H \times G\\ &\simeq \big(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big) /\!\!/ H,\end{aligned}$$

where on the most right hand side H acts on $\boldsymbol{O}^{\mathrm{rs}}$ as the restriction of the G-action to H. The map is explicitly given by

$$\big(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big) /\!\!/ H \to \boldsymbol{O}/\!\!/ G; \quad [B, (A_i)_{i=1}^m, S] \mapsto [B - \sum_{i=1}^m A_i, (A_i)_{i=1}^m].$$

So the following is clear:

6.3.1 Proposition. *A point $[B, (A_i)_{i=1}^m, S] \in \big(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big) /\!\!/ H$ corresponds to a point of $\boldsymbol{O}^s /\!\!/ G$ if and only if it satisfies the following condition: there is no non-zero proper subspace V of $\mathbb{C}^n$ preserved by S, A_i, $i = 1, \ldots, m$ and all the coefficients of B.*

Let $\big(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big)^s$ be the subset of all elements satisfying the above condition. Then the H-action on $\big(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big)^s$ reduces to the action of $H/\mathbb{C}^\times$ which is free and proper, and the above map induces a symplectomorphism

$$\mathcal{M}^s(O_0, O_1, \ldots, O_m) \simeq \big(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\big)^s /\!\!/ H.$$

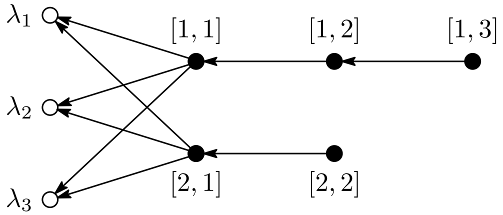
Let us describe the Hamiltonian H-space $O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}$ as an open quiver variety. First, for each $i = 1, 2, \ldots, m$ take a marking $(\xi^i_1, \ldots, \xi^i_{d_i})$ of O_i and let $Q^{(i)}, \boldsymbol{n}^{(i)}, \boldsymbol{\zeta}^{(i)}$ be as in Proposition 5.1.2 for O_i. Then the arguments in Sect. 5, II show that the Hamiltonian G-space $\boldsymbol{O}^{\mathrm{rs}}$ is isomorphic to the open quiver variety $\mathfrak{M}^s_{Q^{\mathrm{rs}}, Q_0^{\mathrm{rs}}\setminus\{0\}}(\boldsymbol{n}^{\mathrm{rs}}, \boldsymbol{\zeta}^{\mathrm{rs}})$ for the star-shaped quiver Q^{rs} with m legs obtained from $Q^{(i)}$, $i = 1, 2, \ldots, m$ by gluing, where the moment map of $\mathfrak{M}^s_{Q^{\mathrm{rs}}, Q_0^{\mathrm{rs}}\setminus\{0\}}(\boldsymbol{n}^{\mathrm{rs}}, \boldsymbol{\zeta}^{\mathrm{rs}})$ is shifted by $-\sum_{i=1}^m \xi^i_1 I_n$. To restrict the action to H, let $\varphi\colon \widetilde{Q}^{\mathrm{rs}} \to Q^{\mathrm{rs}}$ be the blow-up of Q^{rs} at the central vertex 0 by $Q(\Lambda)_0 = \Sigma$. Then Proposition 3.2.3 implies that

$$\mathfrak{M}^s_{Q^{\mathrm{rs}}, Q_0^{\mathrm{rs}} \setminus \{0\}}(\boldsymbol{n}^{\mathrm{rs}}, \boldsymbol{\zeta}^{\mathrm{rs}}) \simeq \mathfrak{M}^s_{\widetilde{Q}^{\mathrm{rs}}, \widetilde{Q}_0^{\mathrm{rs}} \setminus \Sigma}(\widetilde{\boldsymbol{n}}^{\mathrm{rs}}, \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}})$$

as Hamiltonian H-spaces, where $\widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}} := \varphi_0^*(\boldsymbol{\zeta})$ and $\widetilde{\boldsymbol{n}}^{\mathrm{rs}} = (\widetilde{n}_p^{\mathrm{rs}})$ is defined by

$$\widetilde{n}_\lambda^{\mathrm{rs}} = n_\lambda \ (\lambda \in \Sigma), \quad \widetilde{n}_p^{\mathrm{rs}} = n_p^{\mathrm{rs}} \ (p \in Q_0' \setminus \Sigma = Q_0^{\mathrm{rs}} \setminus \{0\}).$$

6.3.2 Examples. The following quiver is an example of $\widetilde{Q}^{\mathrm{rs}}$ with $\Sigma = \{\lambda_1, \lambda_2, \lambda_3\}$ and $m = 2$.



Next, since $H = \prod_{\lambda \in \Sigma} \mathrm{GL}(n_\lambda, \mathbb{C})$ we have $O^{\mathrm{fm}} = \prod_{\lambda \in \Sigma} O_\lambda$, where O_λ is the $\mathrm{GL}(n_\lambda, \mathbb{C})$-coadjoint orbit through the $\mathfrak{gl}(n_\lambda, \mathbb{C})$-component of L. For each $\lambda \in \Sigma$, take a marking $(\xi_1^\lambda, \ldots, \xi_{d_\lambda}^\lambda)$ of O_λ and let $Q^{(\lambda)}, \boldsymbol{n}^{(\lambda)}, \boldsymbol{\zeta}^{(\lambda)}$ be as in Proposition 5.1.2 for O_λ. We embed Σ into the vertex set of the direct sum $Q^{\mathrm{fm}} := \coprod_{\lambda \in \Sigma} Q^{(\lambda)}$ by sending each λ to the unique open vertex of $Q^{(\lambda)}$. Then

$$O^{\mathrm{fm}} \simeq \mathfrak{M}^s_{Q^{\mathrm{fm}}, Q_0^{\mathrm{fm}} \setminus \Sigma}(\boldsymbol{n}^{\mathrm{fm}}, \boldsymbol{\zeta}^{\mathrm{fm}}),$$

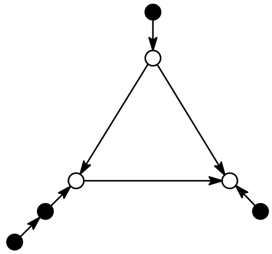
where $\boldsymbol{n}^{\mathrm{fm}} := \oplus_\lambda \boldsymbol{n}^{(\lambda)}$, $\boldsymbol{\zeta}^{\mathrm{fm}} := \oplus_\lambda \boldsymbol{\zeta}^{(\lambda)}$, and the moment map of the Hamiltonian H-space $\mathfrak{M}^s_{Q^{\mathrm{fm}}, Q_0^{\mathrm{fm}} \setminus \Sigma}(\boldsymbol{n}^{\mathrm{fm}}, \boldsymbol{\zeta}^{\mathrm{fm}})$ is shifted by $-(\xi_1^\lambda)_{\lambda \in \Sigma} \in \mathbb{C}^\Sigma$. By Theorem 6.2.5, we have $O_B \simeq \mathrm{Rep}_{\overline{Q(\Lambda)}}(\boldsymbol{n}^{\mathrm{irr}})$ as Hamiltonian H-spaces, where $\boldsymbol{n}^{\mathrm{irr}} = (n_\lambda)_{\lambda \in \Sigma}$. Thus we obtain

$$\begin{aligned} &O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}} \\ &\qquad \simeq \mathrm{Rep}_{\overline{Q(\Lambda)}}(\boldsymbol{n}^{\mathrm{irr}}) \times \mathfrak{M}^s_{Q^{\mathrm{fm}}, Q_0^{\mathrm{fm}} \setminus \Sigma}(\boldsymbol{n}^{\mathrm{fm}}, \boldsymbol{\zeta}^{\mathrm{fm}}) \times \mathfrak{M}^s_{\widetilde{Q}^{\mathrm{rs}}, \widetilde{Q}_0^{\mathrm{rs}} \setminus \Sigma}(\widetilde{\boldsymbol{n}}^{\mathrm{rs}}, \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}}). \end{aligned}$$

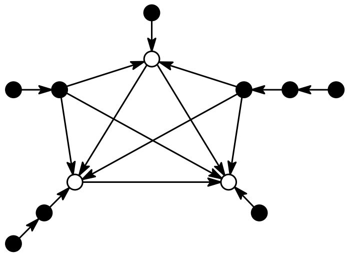
Finally, put

$$Q = \left(Q(\Lambda) \cup_\Sigma Q^{\mathrm{fm}}\right) \cup_\Sigma \widetilde{Q}^{\mathrm{rs}}.$$

6.3.3 Examples. The following quiver is an example of $Q(\Lambda) \cup_\Sigma Q^{\mathrm{fm}}$ with $Q(\Lambda)$ a triangle.



If $\widetilde{Q}^{\mathrm{rs}}$ is as in the previous example, then $\left(Q(\Lambda) \cup_\Sigma Q^{\mathrm{fm}}\right) \cup_\Sigma \widetilde{Q}^{\mathrm{rs}}$ becomes the following quiver.



Since $\boldsymbol{n}^{\mathrm{fm}}_\Sigma = \widetilde{\boldsymbol{n}}^{\mathrm{rs}}_\Sigma = \boldsymbol{n}^{\mathrm{irr}}$, there exists a unique $\boldsymbol{n} \in \mathbb{Z}^{Q_0}$ such that

$$\boldsymbol{n}_\Sigma = \boldsymbol{n}^{\mathrm{irr}}, \quad \boldsymbol{n}_{Q_0^{\mathrm{fm}}} = \boldsymbol{n}^{\mathrm{fm}}, \quad \boldsymbol{n}_{\widetilde{Q}_0^{\mathrm{rs}}} = \widetilde{\boldsymbol{n}}^{\mathrm{rs}}.$$

Therefore Proposition 3.2.2 implies that

$$O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}} \simeq \mathfrak{M}^s_{Q, Q_0 \setminus \Sigma}(\boldsymbol{n}, \boldsymbol{\zeta}^{\mathrm{fm}} \oplus \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}}),$$

where the moment map of the Hamiltonian H-space $\mathfrak{M}^s_{Q, Q_0 \setminus \Sigma}(\boldsymbol{n}, \boldsymbol{\zeta}^{\mathrm{fm}} \oplus \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}})$ is shifted by

$$-\boldsymbol{\zeta}^{\mathrm{irr}} = -(\zeta_\lambda) \in \mathbb{C}^\Sigma, \quad \zeta_\lambda := -\xi_1^\lambda - \sum_{i=1}^m \xi_1^i.$$

6.3.4 Lemma. *The above isomorphism maps* $\left(O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\right)^s$ *onto the* $\mathrm{GL}(\boldsymbol{n}_\Sigma)$*-stable locus* $\mathfrak{M}^s_{Q, Q_0 \setminus \Sigma}(\boldsymbol{n}, \boldsymbol{\zeta}^{\mathrm{fm}} \oplus \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}})^{s,\Sigma}$.

Proof. Take $(B, (A_i), S) \in O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}$ and let $[\Xi] \in \mathfrak{M}^s_{Q, Q_0 \setminus \Sigma}(\boldsymbol{n}, \boldsymbol{\zeta}^{\mathrm{fm}} \oplus \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}})$ be the corresponding point. First suppose that $(B, (A_i), S)$ lies in $\boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}$ and let $W \subset V^{\boldsymbol{n}}$ be a Ξ-invariant Q_0-graded subspace. Then Proposition 6.2.7 implies that $W_\Sigma \subset \mathbb{C}^n$ is preserved by all the coefficients of B. Also, by Exercise 5.1.6 it is preserved by S and all A_i. Hence $W_\Sigma = \{0\}$ or $W_\Sigma = \mathbb{C}^n$, which implies $W = \{0\}$ or $W = V^{\boldsymbol{n}}$ by Exercise 5.1.6 again. Next suppose that $[\Xi]$ lies in $\mathfrak{M}^s_{Q, Q_0 \setminus \Sigma}(\boldsymbol{n}, \boldsymbol{\zeta}^{\mathrm{fm}} \oplus \widetilde{\boldsymbol{\zeta}}^{\mathrm{rs}})^{s,\Sigma}$ and let $V \subset \mathbb{C}^n$ be a subspace preserved by S, all A_i and all the coefficients of B. Then Proposition 6.2.7 shows that V is homogeneous with respect to the decomposition $\mathbb{C}^n = \bigoplus_{\lambda \in \Sigma} \mathbb{C}^{n_\lambda}$ and preserved by the $\mathrm{Rep}_{\overline{Q(\Lambda)}}(\boldsymbol{n}^{\mathrm{irr}})$-component of Ξ as a Σ-graded subspace of $V^{\boldsymbol{n}}_{\Sigma}$. Furthermore, since V is preserved by S and all A_i, Exercise 5.1.6 implies that V extends to a Ξ-invariant Q_0-graded subspace $W \subset V^{\boldsymbol{n}}$. By the stability we have $W = \{0\}$ or $W = V^{\boldsymbol{n}}$, in particular $V = \{0\}$ or $V = \mathbb{C}^n$. □

Thus we obtain the following, which is the main theorem in this lecture:

6.3.5 Theorem ([13]). Set $\zeta = \zeta^{\mathrm{irr}} \oplus \zeta^{\mathrm{fm}} \oplus \widetilde{\zeta}^{\mathrm{rs}} \in \mathbb{C}^{Q_0}$. Then there exists a symplectomorphism

$$\mathcal{M}^s(O_0, O_1, \ldots, O_m) \xrightarrow{\simeq} \mathfrak{M}^s_Q(\boldsymbol{n}, \zeta).$$

6.3.6 Remark. The theorem in the case of $\deg_{1/z}(O_0) = 1$ was first pointed out by Boalch in [4, Exercise 3]. He then proved the case of $\deg_{1/z}(O_0) \leqslant 2$ in [3, 5].

6.4 Further Directions

The problem asking when the polar-parts manifold $\mathcal{M}^s(O_0, O_1, \ldots, O_m)$ is non-empty is called the *(generalized) additive Deligne–Simpson problem*, originally proposed by Kostov for residue manifolds (see [16]). Theorem 6.3.5 gives an answer to this problem when O_0 has a normal form and $\deg_{1/z}(O_i) = 0$ for $i \geqslant 1$, since we have a necessary and sufficient condition for the emptiness of quiver variety $\mathfrak{M}^s_Q(\boldsymbol{n}, \zeta)$ (obtained by Crawley-Boevey; see [7]).

To solve the generalized additive Deligne–Simpson problem when every O_i has a normal form (without any assumption for pole order), Hiroe described the polar-parts manifold as an analogue of quiver variety. The idea is as follows (for further details, see [12]).

Recall that if O_0 has a normal form we have an isomorphism

$$\boldsymbol{O} \simeq \left(T^*G \times O_B \times \boldsymbol{O}^{\mathrm{rs}} \times O^{\mathrm{fm}}\right) /\!\!/ H,$$

which holds without any assumption for O_i, $i \geqslant 1$. Now assume further that O_i has a normal form $\delta\Lambda_i + L_i$ for $i \geqslant 1$. Then Proposition 6.1.8 and the shifting trick implies

$$O_i \simeq (T^*G \times O^{(i)}_B \times O^{\mathrm{fm}}_i) /\!\!/ H_i \qquad (i = 1, 2, \ldots, m),$$

where the definitions of $O^{(i)}_B$, O^{fm}_i, H_i are similar to those of $O^{(0)}_B := O_B$, $O^{\mathrm{fm}}_0 := O^{\mathrm{fm}}$, $H_0 := H$. Thus we obtain

$$\boldsymbol{O} \simeq \left\{ (T^*G)^{m+1} \times \prod_{i=0}^{m} O^{(i)}_B \times \prod_{i=0}^{m} O^{\mathrm{fm}}_i \right\} /\!\!/ \prod_{i=0}^{m} H_i,$$

which induces a homeomorphism

$$\boldsymbol{O}/\!\!/G \simeq \left\{ (T^*G)^m \times \prod_{i=0}^{m} O_B^{(i)} \times \prod_{i=0}^{m} O_i^{\mathrm{fm}} \right\} /\!\!/ \prod_{i=0}^{m} H_i$$

thanks to Exercises 2.2.9 and 2.2.10. Since $G = \mathrm{GL}_n(\mathbb{C})$, we have an open embedding $T^*G \hookrightarrow T^*\mathfrak{g}$ of Hamiltonian $G \times G$-spaces. Observe that $T^*\mathfrak{g}$ is the vector space of representations of the double of the quiver $\bullet \to \bullet$ with dimension vector (n, n). Also each $O_B^{(i)}$ is a vector space of representations of some double quiver and each O_i^{fm} is an H_i-coadjoint orbit. Therefore we can embed $\boldsymbol{O}/\!\!/G$ as an open subset into the Hamiltonian reduction

$$\mathrm{Rep}_{\overline{Q}}(\boldsymbol{n})/\!\!/_{\zeta}\mathrm{GL}(\boldsymbol{n})$$

for some $Q, \boldsymbol{n}, \boldsymbol{\zeta}$. Unfortunately the image of $\mathcal{M}^s(O_0, O_1, \ldots, O_m)$ under this embedding does not coincide with the quiver variety $\mathfrak{M}_Q^s(\boldsymbol{n}, \boldsymbol{\zeta})$ in general. Nevertheless, in [11] Hiroe successfully obtained a necessary and sufficient condition on $Q, \boldsymbol{n}, \boldsymbol{\zeta}$ for the emptiness of $\mathcal{M}^s(O_0, O_1, \ldots, O_m)$.

On the other hand, the polar-parts manifolds lead us to a generalization of quiver varieties.

Each finite graph with no edge-loop (an edge joining a vertex with itself) defines a symmetric Kac–Moody Lie algebra, and vice versa. Thus to each quiver variety $\mathfrak{M}_Q^s(\boldsymbol{n}, \boldsymbol{\zeta})$ with Q having no edge-loop we can associate a symmetric Kac–Moody Lie algebra. An important issue in geometric representation theory is to find a nice generalization of quiver varieties related to symmetrizable (possibly non-symmetric) Kac–Moody Lie algebras.

Motivated by this problem, the author introduced the notion of quiver variety with multiplicities [20], which generalizes quiver variety and relates to some class of symmetrizable Kac–Moody Lie algebras. A remarkable fact is that some polar-parts manifolds having several higher order poles may be described as quiver varieties with multiplicities. Recently, the author's definition was improved by Geiss–Leclerc–Schröer so that all symmetrizable Kac–Moody Lie algebras appear (see [9]).

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