Chapter 7
Rearrangement and Prékopa–Leindler Type Inequalities

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Abstract We investigate the interactions of functional rearrangements with Prékopa–Leindler type inequalities. It is shown that certain set theoretic rearrangement inequalities can be lifted to functional analogs, thus demonstrating that several important integral inequalities tighten on functional rearrangement about "isoperimetric" sets with respect to a relevant measure. Applications to the Borell–Brascamp–Lieb, Borell–Ehrhard, and the recent polar Prékopa–Leindler inequalities are demonstrated. It is also proven that an integrated form of the Gaussian log-Sobolev inequality sharpens on rearrangement.

7.1 Introduction

The Prékopa–Leindler inequality (PLI) stated below has become a useful tool in the study of log-concave distributions in probability and statistics, particularly in high dimension, and a point of interest and unification between probabilists and convex geometers.

Theorem 7.1.1 (Prékopa–Leindler) *For* $f, g : \mathbb{R}^d \to [0, \infty)$ *Borel measurable and* $t \in (0, 1)$*, define*

$$f \Box g(z) := \sup_{(1-t)x+ty=z} f^{1-t}(x) g^t(y)$$

A portion of this work relevant to information theory was announced at 56th Annual Allerton Conference on Communication, Control, and Computing [43].

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then

$$\int_{\mathbb{R}^d} f\square g(z)dz \geq \left(\int_{\mathbb{R}^d} g(z)dz\right)^{1-t} \left(\int_{\mathbb{R}^d} h(z)dz\right)^t .$$

The inequality can be motivated from a convex geometric perspective as a functional generalization of the dimension free statement of the Brunn–Minkowski inequality (BMI), which we recall as the fact that for A, B compact in $\mathbb{R}^d$ and $|\cdot|_d$ the d-dimensional Lebesgue volume,

$$|(1-t)A + tB|_d \geq |A|_d^{1-t}|B|_d^t.$$

Indeed, by taking $f = \mathbb{1}_A$ and $g = \mathbb{1}_B$, we have $f\square g = \mathbb{1}_{(1-t)A+tB}$. PLI implies that integration preserves the inequality and the result follows.

The BMI has an elegant qualitative formulation; the volume of sum-sets decreases on spherical symmetrization. More explicitly, if A and B are compact sets, with A^* and B^* Euclidean balls satisfying $|A^*|_d = |A|_d$, $|B^*|_d = |B|_d$, then

$$|A + B|_d \geq |A^* + B^*|_d. \tag{7.1}$$

Our first main result (Theorem 7.3.1) contains a functional generalization of (7.1). We will show PLI "sharpens" on rearrangement in the sense that

$$\int f\square g(z)dz \geq \int f^*\square g^*(z)dz, \tag{7.2}$$

where $*$ denotes a functional rearrangement to be defined below. In fact we will prove that for ψ increasing,

$$\int \psi(f\square g(z))dz \geq \int \psi(f^*\square g^*(z))dz. \tag{7.3}$$

Our methods are reasonably general and Theorem 7.4.6 will give a class of set theoretic inequalities that admit functional generalization in the sense of (7.3). As a consequence, we will show that analogs of (7.3) can be given to sharpen not only the PLI, but the Borell–Brascamp–Lieb inequalities [15, 18], the Borell–Ehrhard inequality in the Gaussian setting [16, 24], and a recent Polar Prékopa–Leindler [1].

These results can also be motivated from an information theoretic perspective, where the BMI can be considered a Rényi entropy power inequality. There has been considerable recent work (see [6, 7, 10, 29, 31, 33, 45]) developing Rényi entropy [46] generalizations of the classical entropy power inequality (EPI) of Shannon–Stam [47, 48]. One should compare the sharpening of PLI here to [50], where Madiman and Wang show that while spherically symmetric decreasing rearrangements of random variables preserve their Rényi entropy, they decrease the Rényi entropy of independent sums of random variables. One information theoretic application of the rearrangement result is the reduction of Rényi generalizations of the EPI to the spherically symmetric case, see for example [39] where the Madiman–Wang result is used to sharpen the Rényi EPI put forth in [40]. See [36] to find an extension and application of [50] for the ∞-Rényi entropy. It should be mentioned that the connections between BMI and entropy power inequalities are not new. The analogy between the two inequalities was first observed in [21], and a unified proof was given in [23] drawing on the work of [4, 17, 34]. The reader is directed to [35] where a further development of Rényi entropy power inequalities and their connections to convex geometry are given.

In the Gaussian case, the strict convexity of the potential gives a result stronger than PLI, and we are able to adapt the rearrangement ideas to approach the Gaussian log-Sobolev inequality. We show in Theorem 7.6.5 that for the Gaussian measure, the "integrated" log-Sobolev inequality derived in [8] by Bobkov and Ledoux, and understood as reverse hypercontractivity of the Hamilton–Jacobi equations in [12], sharpens on half space rearrangement.

An alternative motivation for this investigation is the Brascamp–Lieb–Barthe inequality's relationship to the Brascamp–Lieb–Luttinger rearrangement inequalities [19]. The Brascamp–Lieb inequality [18] enjoys the Brascamp–Lieb–Luttinger inequality as a rearrangement analog. In [2] Barthe used an optimal transport argument to prove Brascamp–Lieb and simultaneously demonstrated a dual inequality that includes PLI as a special case. It is natural to ask for a rearrangement inequality analog of Barthe's result, to provide a dual to the Brascamp–Lieb–Luttinger rearrangement inequality. This work represents a confirmation of such an inequality in the special case corresponding to PLI.

The paper is organized in the following manner; in Sect. 7.2, we will give definitions and background on a notion of rearrangement. In Sect. 7.3, we give a rearrangement inequality for PLI, before giving a general version in Sect. 7.4. In Sect. 7.5, we give applications of the theorem derived in Sect. 7.4 to special cases. In Sect. 7.6, we give a sharpening of an integrated Gaussian log-Sobolev inequality via half-space rearrangement. Finally, in Sect. 7.7, we discuss connections with the work of Barthe and Brascamp-Lieb-Luttinger closing with an open problem.

7.2 Preliminaries

For a set A, we will use the notation $\mathbb{1}_A$ to denote the indicator function of A, taking the value 1 on A, and 0 elsewhere. For $x \in \mathbb{R}^d$, $|x|$ will denote the usual Euclidean norm. We use $\mathbb{Q}_+$ to denote the non-negative rational numbers. We use γ_d to denote both the standard Gaussian measure on $\mathbb{R}^d$ and its density function

$$\gamma_d(x) = \frac{e^{-|x|^2/2}}{(2\pi)^{\frac{d}{2}}}.$$

When $d = 1$, and there is no risk of confusion, we will omit the subscript and write γ. We denote the Gaussian distribution function

$$\Phi(x) = \int_{-\infty}^{x} \gamma(y)dy$$

and its inverse Φ^{-1}.

7.2.1 *Spherically Symmetric Decreasing Rearrangements*

Given a nonempty measurable set $A \subseteq \mathbb{R}^d$, we define its spherically symmetric rearrangement A^* to be the origin centered ball of equal volume,

$$A^* := \left\{ x : |x| < (|A|_d/\omega_d)^{\frac{1}{d}} \right\},$$

where ω_d is the volume of the d-dimensional unit ball, with the understanding that $A^* = \varnothing$ in the case that $|A|_d = 0$ and $A^* = \mathbb{R}^d$ when $|A|_d = \infty$.

We can extend this notion of symmetrization to functions via the layer-cake decomposition of a non-negative function f,

$$f(x) = \int_0^{f(x)} 1dt = \int_0^{\infty} \mathbb{1}_{\{y:f(y)>t\}}(x)dt.$$

Definition 7.2.1 For a measurable non-negative function f, define its decreasing symmetric rearrangement f^* by

$$f^*(x) := \int_0^{\infty} \mathbb{1}_{\{y:f(y)>t\}^*}(x)dt. \tag{7.4}$$

Note that *decreasing* is used here in the non-strict sense, synonymous with non-increasing.

Proposition 7.2.2 *f^* is characterized by the equality*

$$\{f^* > \lambda\} = \{f > \lambda\}^*. \tag{7.5}$$

The proof will be given in greater generality in the following section.

Corollary 7.2.3 *f^* is lower semi-continuous, spherically symmetric and non-increasing in the sense that $|x| \leq |y|$ implies $f^*(x) \geq f^*(y)$.*

Proof f^* has open super level sets by Eq. (7.5) and is thus lower semi-continuous. To prove non-increasingness observe that using the characterization above $f^*(y) > \lambda$ iff $y \in \{f > \lambda\}^*$ which implies by $|x| \leq |y|$ that $x \in \{f > \lambda\}^*$, and thus

$f^*(x) > \lambda$. Applying this to λ_n increasing to $f^*(y)$ yields our result. Observe that this implies spherical symmetry by applying the preceding argument in the opposite direction $f(x) = f(y)$ when $|x| = |y|$. □

7.2.2 More General Rearrangements

Definition 7.2.4 For Polish measure spaces (M, μ) and (N, α), with Borel σ-algebra, we will call a set map from the Borel σ-algebra of M to the Borel σ-algebra of N a *rearrangement* when it satisfies the following,

1. $*(A)$ is an open set satisfying $\alpha(*(A)) = \mu(A)$
2. $\mu(A) \leq \mu(B)$ implies $*(A) \subseteq *(B)$
3. For a sequence $A_i \subseteq A_{i+1}$, $*(\cup_{i=1}^{\infty} A_i) = \cup_{i=1}^{\infty} *(A_i)$.

Notice that in 3, $\cup_j *(A_j) \subseteq *(\cup_j A_j)$ holds from 2, so the assumption is only $\cup_j *(A_j) \supseteq *(\cup_j A_j)$. For brevity of notation, we write $A^* = *(A)$ and note the following extension to functions.

Definition 7.2.5 For a rearrangement $*$ and Borel measurable $f : M \to [0, \infty)$ define $f^* : N \to [0, \infty)$,

$$f^*(x) := \int_0^{\infty} \mathbb{1}_{\{f>t\}^*}(x) dt.$$

Rearrangement is in general non-linear, however, we do have linear behavior in the following special case.

Lemma 7.2.6 *For a simple function s, expressed as $s = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$ with $a_i > 0$ and $A_i \subsetneq A_{i-1}$,*

$$s^* = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i^*}.$$

Proof Let us give more explicit formulas for both quantities.

$$\sum_{i=1}^{n} a_i \mathbb{1}_{A_i^*}(z) = \sum_{i=1}^{m_z} a_i$$

where $m_z = \max\{i : z \in A_i^*\}$, and the formula

$$s^*(z) = \sup\{t : z \in \{s > t\}^*\},$$

which holds not just for simple functions but general f. If $z \in A^*_{m_z}$ with m_z maximal, then for $t < \sum_{i=1}^{m_z} a_i$, $A_{m_z} \subseteq \{s > t\}$, which in turn gives $A^*_{m_z} \subseteq \{s > t\}^*$. Thus $z \in \{s > t\}^*$ for all $t < \sum_{i=1}^{m_z} a_i$ and we have

$$s^*(z) = \sup_t \{z \in \{s > t\}^*\} \geq \sum_{i=1}^{m_z} a_i = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i^*}(z).$$

For the reverse inequality, assume $s^*(z) > 0$ (else there is nothing to prove) and take t such that $z \in \{s > t\}^*$. Since $\{s > t\} = A_{k_t}$ where $k_t = \min\{j : \sum_{i=1}^{j} a_i > t\}$, we have $\{s > t\}^* = A^*_{k_t}$. This implies that $\sum_{i=1} a_i \mathbb{1}_{A_i^*}(z) \geq \sum_{i=1}^{k_t} a_i > t$. Taking the supremum in t,

$$\sum_{i=1}^{n} a_i \mathbb{1}_{A_i^*}(z) \geq s^*(z).$$

□

Proposition 7.2.7 *f^* is characterized by the equality*

$$\{f^* > \lambda\} = \{f > \lambda\}^*. \tag{7.6}$$

In particular f^* is lower semi-continuous, and equi-measureable with f in that $\mu\{f > \lambda\} = \alpha\{f^* > \lambda\}$.

Proof First we prove the equality (7.6). Since $f^*(x) > \lambda$ implies $\int_0^\infty \mathbb{1}_{\{f>t\}^*}(x)\, dt > \lambda$, which in turn, by the monotonicity of $\mathbb{1}_{\{f>t\}^*}$ implies the existence of $t > \lambda$ such that $x \in \{f > t\}^*$. From this it follows that

$$\{f^* > \lambda\} \subseteq \{f > \lambda\}^*.$$

For the converse, first assume that $f = s$ is a simple function, expressed as

$$s = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i}$$

with $a_i > 0$ and $A_i \subsetneq A_{i-1}$. By Lemma 7.2.6

$$s^* = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i^*}.$$

Since $\{s > \lambda\} = A_k$ where $k = \min\{j : \sum_{i=1}^{j} a_i > \lambda\}$, $z \in \{s > \lambda\}^* = A^*_k$ implies $s^*(z) = \sum_{i=1}^{n} a_i \mathbb{1}_{A_i^*}(z) \geq \sum_{i=1}^{k} a_i > \lambda$. Thus $\{s > \lambda\}^* \subseteq \{s^* > \lambda\}$ holds for simple functions. Now take s_n to be a sequence of increasing simple functions approximating f pointwise and uniformly on sets where f is bounded. Then

$$\{f > \lambda\}^* = \left(\bigcup_{n=1}^{\infty}\{s_n > \lambda\}\right)^* = \bigcup_{n=1}^{\infty}\{s_n > \lambda\}^* = \bigcup_{n=1}^{\infty}\{s_n^* > \lambda\}.$$

where the first equality is from the assumption of increasingness of the simple functions, the second is from Definition 7.2.4 item 3, and the third follows from the characterization just proven for simple functions. Since $f_1 \leq f_2$, implies $f_1^* \leq f_2^*$ it follows that $\cup\{s_n^* > \lambda\} \subseteq \{f^* > \lambda\}$, so that $\{f > \lambda\}^* \subseteq \{f^* > \lambda\}$.

If g is another function satisfying $\{g > \lambda\} = \{f > \lambda\}^*$ for all λ, then

$$g(z) = \int_0^\infty \mathbb{1}_{\{g>\lambda\}} d\lambda = \int_0^\infty \mathbb{1}_{\{f>\lambda\}^*} d\lambda = \int_0^\infty \mathbb{1}_{\{f^*>\lambda\}} d\lambda = f^*(z).$$

The fact that f is lower semi-continuous follows from item (1) of our definition, that A^* is open. Equimeasurability is given by $\alpha\{f^* > \lambda\} = \alpha\{f > \lambda\}^* = \mu\{f > \lambda\}$. □

Proposition 7.2.8 *For an open convex set $K \subseteq \mathbb{R}^d$ with closure containing the origin. The set map $*_K$ defined by*

$$A^{*_K} := \left(\frac{|A|_d}{|K|_d}\right)^{\frac{1}{d}} K,$$

is a rearrangement with $(M, \mu) = (N, \alpha) = (\mathbb{R}^d, |\cdot|_d)$.

Proof It is immediate that A^{*_K} is open and the homogeneity of the Lebesgue measure ensures that $|A^{*_K}|_d = |A|_d$, hence (1) follows. To prove (2), note that for $0 < |A| \leq |B|$, by the definition of $*_K$, $A^{*_K} = tK$ and $B^{*_K} = sK$ for some $0 < t \leq s$. Let $x = tk$ for $k \in K$ and k_n a sequence in K converging to 0. Then

$$x = s\left(\frac{t}{s}\left(k - \left(\frac{s}{t} - 1\right)k_n\right) + \left(1 - \frac{t}{s}\right)k_n\right).$$

By K open, $k-(\frac{s}{t}-1)k_n$ belongs to K for large n, and when this holds, by convexity $(\frac{t}{s}(k - (\frac{s}{t} - 1)k_n) + (1 - \frac{t}{s})k_n) \in K$. It follows that $x \in sK$, as such, $A^{*_K} \subseteq B^{*_K}$. The continuity condition in (3) holds, since both sets are origin symmetric balls of the same volume. □

Observe that the qualitative statement of Brunn–Minkowski (7.1), for Borel A, B

$$|A + B|_d \geq |A^{*_K} + B^{*_K}|_d, \tag{7.7}$$

is preserved. In the following section, we will extend this qualitative result to the functional setting.

Proposition 7.2.9 *For a fixed coordinate i, the set function $*$ defined on a Polish space M with probability measure μ and $(N, \alpha) = (\mathbb{R}^d, \gamma_d)$ by*

$$A^* = \{x : x_i < \Phi^{-1}(\mu(A))\}$$

is a rearrangement.

Proof A^* is open by definition, and $\gamma_d(A^*) = \Phi(\Phi^{-1}(\mu(A))) = \mu(A)$. Conditions (2) and (3) follow from the monotonicity and continuity of Φ. □

We will not pursue examples in discrete spaces here. We direct the interested reader to [37, 38] for recent information theoretic work regarding rearrangement on discrete spaces and [25, 26, 44] for discrete PLI investigations.

7.3 Rearrangement and Prékopa–Leindler

We begin with a special case of a more general result to build some intuition for the abstractions to follow. For $f, g : \mathbb{R}^d \to [0, \infty)$ and $t \in [0, 1]$ recall

$$f \Box g(z) = \sup_{(1-t)x+ty=z} f^{1-t}(x) g^t(y). \tag{7.8}$$

Theorem 7.3.1 *For $f, g : \mathbb{R}^d \to [0, \infty)$ Borel, $t \in (0, 1)$, and $*$ denoting a rearrangement to a fixed open convex set with closure containing the origin,*

$$\int_{\mathbb{R}^d} f \Box g(z) dz \geq \int_{\mathbb{R}^d} f^* \Box g^*(z) dz \geq \left(\int f dz \right)^{1-t} \left(\int g dz \right)^t. \tag{7.9}$$

What is more, when ψ is a non-negative and non-decreasing function

$$\int_{\mathbb{R}^d} \psi(f \Box g)(z) dz \geq \int_{\mathbb{R}^d} \psi(f^* \Box g^*)(z) dz. \tag{7.10}$$

The universal measurability of $f \Box g$ will follow from the proof, which gives the universal measurability of $\psi(f \Box g)$ as a consequence.

Proof For $\lambda \in (0, \infty)$, define

$$S_0 = S_0(\lambda) = \{s \in \mathbb{Q}_+^2 : s_1^{1-t} s_2^t > \lambda\}. \tag{7.11}$$

Observe,

$$\{f \Box g > \lambda\} = \bigcup_{s \in S_0(\lambda)} (1-t)\{f > s_1\} + t\{g > s_2\}. \tag{7.12}$$

Indeed, it is routine to check that $z \in \cup_{s \in S_0} (1-t)\{f > s_1\} + t\{g > s_2\}$ implies $f \Box g(z) > \lambda$. Conversely, if $f \Box g(z) > \lambda$, then there exists a pair of x and y such that $(1-t)x + ty = z$ and $f^{1-t}(x) g^t(y) > \lambda$. By the continuity of the map $(u, v) \mapsto u^{1-t} v^t$, there exists (s_1, s_2) rational satisfying $s_1 < f(x)$, $s_2 < g(y)$, and $s_1^{1-t} s_2^t > \lambda$, which proves the claim.

Let us remark that the sum of Borel sets is universally measurable,[1] and hence $\{f \Box g > \lambda\}$ is as well. This shows we are well justified in our notation $\int_{\mathbb{R}^d} f \Box g(z) dz$. By Brunn–Minkowski and the characterizing property of rearrangements on super level sets

$$|(1-t)\{f > s_1\} + t\{g > s_2\}| \geq |(1-t)\{f > s_1\}^* + t\{g > s_2\}^*| \tag{7.13}$$

$$= |(1-t)\{f^* > s_1\} + t\{g^* > s_2\}|. \tag{7.14}$$

Now applying (7.12) to $f^* \Box g^*$ and observing that

$$(1-t)\{f^* > s_1\} + t\{g^* > s_2\}$$

is an origin centered ball in $\mathbb{R}^d$ for every $s \in S_0(\lambda)$, we see that

$$\begin{aligned} |\{f^* \Box g^* > \lambda\}| &= \left| \bigcup_{s \in S_0(\lambda)} (1-t)\{f^* > s_1\} + t\{g^* > s_2\} \right| \\ &= \sup_{s \in S_0} \left| (1-t)\{f^* > s_1\} + t\{g^* > s_2\} \right|. \end{aligned}$$

Using (7.13),

$$\left| (1-t)\{f^* > s_1\} + t\{g^* > s_2\} \right| \leq \left| \bigcup_{s \in S_0(\lambda)} (1-t)\{f > s_1\} + t\{g > s_2\} \right|.$$

Thus it follows that

$$|\{f \Box g > \lambda\}| \geq |\{f^* \Box g^* > \lambda\}|. \tag{7.15}$$

Using the layer-cake decomposition of the integral,

$$\int_{\mathbb{R}^d} \psi(f \Box g)(z) dz = \int_0^\infty |\{\psi(f \Box g) > t\}| dt.$$

Notice that by the non-decreasingness, $\psi^{-1}(\lambda, \infty)$ is an interval of the form $[x, \infty)$ or (x, ∞) for a non-negative x. From this, we can use (7.15) (and continuity of

[1]This follows from the fact that Borel sets are analytic, see [28], and analytic sets are closed under summation and universally measurable.

measure if the interval is closed) to obtain (7.10). To recover (7.9), note that the first inequality follows from setting $\psi(x) = x$, while the second is the application of PLI to f^* and g^* combined with the equimeasurability of the rearrangements ensuring $\int f^* = \int f$ and $\int g^* = \int g$. □

7.4 Functional Lifting of Rearrangements

In this section we show that in a general setting, certain set theoretic rearrangement inequalities can be extended to functional analogs, extending the rearrangement inequality proven for PLI in the previous section to more general operations than □ in (7.8). Let us make precise the set theoretic rearrangement inequality we will generalize.

Definition 7.4.1 Let $m : M^n \to M$ and $\eta : N^n \to N$ be such that $m(A_1, \dots, A_n) = \{x = m(a_1, \dots, a_n) : a_i \in A_i\}$ and $\eta(B_1, \dots, B_n) = \{y = \eta(b_1, \dots, b_n) : b_i \in B_i\}$ are universally measurable for A_i and B_j Borel. Suppose further that $\{\eta(A_1^*, \dots, A_n^*)\}_A$ indexed on n-tuples of Borel sets is totally ordered in the sense that for any Borel $A_1, \dots, A_n$ and $A_1', \dots, A_n'$ we have either $\eta(A_1^*, \dots, A_n^*) \subseteq \eta(A'^*_1, \dots, A'^*_n)$ or $\eta(A_1^*, \dots, A_n^*) \supseteq \eta(A'^*_1, \dots, A'^*_n)$ we say that $*$ satisfies a set theoretic rearrangement inequality when the following holds

$$\mu(m(A_1, \dots, A_n)) \geq \alpha(\eta(A_1^*, \dots, A_n^*)).$$

We will focus on two main examples, the rearrangement to convex sets in Euclidean space and rearrangement to half-spaces in Gaussian space.

Proposition 7.4.2 *When $(M, m, \mu) = (N, \eta, \alpha) = (\mathbb{R}^d, m_t, dx)$, and $t = (t_1, \dots, t_n) \in \mathbb{R}^n$, defines a map m_t by vector space operations,*

$$x = (x_1, \dots, x_n) \mapsto \sum_{i=1}^n t_i x_i, \tag{7.16}$$

then the $*K$ rearrangement, as in Sect. 7.2 for K open, convex, and symmetric, satisfies a set theoretic rearrangement inequality. If the t_i are assumed positive, $*K$ satisfies a set theoretic rearrangement without symmetry if 0 *belongs to the closure of K.*

Proof Take $B_i = \operatorname{sgn}(t_i) A_i$ so that $t_1 A_1 + \cdots + t_n A_n = |t_1| B_1 + \cdots + |t_n| B_n$. Using the symmetry and convexity of K and the definition of our rearrangement as a scaling of K, it follows that

$$t_1 A_1^* + \cdots + t_n A_n^* = \left(\sum_{i=1}^n |t_i| |A_i|^{\frac{1}{d}} \right) K$$

and hence, the images of m_t are totally ordered. Brunn–Minkowski implies that

$$||t_1|B_1 + \cdots + |t_n|B_n| \geq ||t_1|B_1^* + \cdots + |t_n|B_n^*|.$$

It follows that

$$|t_1 A_1 + \cdots + t_n A_n| \geq |t_1 A_1^* + \cdots + A_n^*|.$$

When t_i are positive, the proof is similar and simpler. □

Proposition 7.4.3 *When (M, m, μ) is a centered Gaussian measure on a Banach space M and m defined as $x = (x_1, \ldots, x_n) \mapsto \sum_i t_i x_i$ for $t_i > 0$, $\sum_i t_i = 1$, and (N, η, α) with $N = \mathbb{R}^d$, η defined by $y \mapsto \sum_i t_i y_i$ and $\alpha = \gamma_d$ the half-space rearrangement from Proposition 7.2.9 yields a set theoretic rearrangement inequality.*

This is the content of the Borell–Ehrhard theorem, which we will discuss in more detail in Sect. 7.5.2. Now let us generalize the geometric mean used in PLI.

Definition 7.4.4 For $0 < T \leq \infty$, a function $\mathcal{M} : [0, T)^n \to [0, \infty]$ is *continuous coordinate increasing* when

1. $x, y \in \mathbb{R}^n$ satisfying $x_i > y_i$ for all i, necessarily satisfy $\mathcal{M}(x) > \mathcal{M}(y)$
2. $\mathcal{M}(x) = 0$ when $\prod_i x_i = 0$
3. $\mathcal{M}(x) = \sup_{y<x} \mathcal{M}(y)$ with the convention that $\sup_{y<x} \mathcal{M}(y) = 0$ when $\{y < x\}$ is empty.

By convention, in the case that T is finite, we extend $\mathcal{M}$ to $[0, T]^n$ by $\mathcal{M}(x) = \sup_{y<x} \mathcal{M}(y)$. It should also be assumed, all $\mathcal{M}$ that follow are defined to be zero on $\{x : \prod_i x_i = 0\}$.

7.4.1 Examples

1. For $t = (t_1, \ldots, t_n)$ with $t_i > 0$ and $p \in [-\infty, 0) \cup (0, \infty]$ take for $u \in [0, \infty)^n$

$$\mathcal{M}_p^t(u) = \left(t_1 u_1^p + \cdots + t_n u_n^p\right)^{\frac{1}{p}}. \tag{7.17}$$

   with $M_{-\infty}^t(u) = \min_i u_i$ and $M_\infty^t(u) = \max_i u_i$
2. For $t = (t_1, \cdots, t_n)$ with $t_i > 0$ and $u \in [0, \infty)^n$,

$$\mathcal{M}_0^t(u) = \prod u_i^{t_i}. \tag{7.18}$$

   Note that in the case that $\sum_i t_i = 1$, $\mathcal{M}_0^t$ is the limiting case of the previous example.

3. Define for $t_i > 0$ and $u \in (0,1)^n$,

$$\mathcal{M}_\Phi^t(u) = \Phi(t_1 \Phi^{-1}(u_1) + \cdots + t_n \Phi^{-1}(u_n))$$

Now let us define the functional operation our set theoretic rearrangement inequalities may be generalized to.

Definition 7.4.5 For $\mathcal{M}$, a continuous coordinate increasing function, $f = \{f_i\}_{i=1}^n$ with $f_i : M \to [0, T)$ and $m : M^n \to M$ define

$$\square_{\mathcal{M},m} f(z) := \sup_{m(x)=z} \mathcal{M}(f_1(x_1), \ldots, f_n(x_n)).$$

Let us further denote for a rearrangement $*$ satisfying a set theoretic rearrangement inequality, $f_* = \{f_i^*\}_{i=1}^n$, so that

$$\square_{\mathcal{M},\eta} f_*(w) = \sup_{\eta(y)=w} \mathcal{M}(f_1^*(y_1), \ldots, f_n^*(y_n)).$$

When there is no risk of ambiguity we will suppress the notation for the mapping m and write $\square_{\mathcal{M}} f$ in place of $\square_{\mathcal{M},m} f$.

Notice that Theorem 7.3.1 was the case that $m(x, y) = \eta(x, y) = (1-t)x + ty$ and $\mathcal{M}$ is the geometric mean as in (7.18).

Theorem 7.4.6 *A set theoretic rearrangement inequality,*

$$\mu(m(A_1, \ldots, A_n)) \geq \alpha(\eta(A_1^*, \ldots, A_n^*))$$

can be extended to functions in the sense that for $f = \{f_i\}_{i=1}^n$, with f_i Borel measurable from M to $[0, \infty)$, $\mathcal{M}$ a continuous coordinate increasing function, and a non-negative non-decreasing ψ,

$$\int \psi(\square_{\mathcal{M},m} f) d\mu \geq \int \psi(\square_{\mathcal{M},\eta} f_*) d\alpha.$$

Proof For $\lambda > 0$, write

$$S_{\mathcal{M}}(\lambda) = \{q \in \mathbb{Q}_+^n : \mathcal{M}(q) > \lambda\}.$$

We will prove $\mu(\square_{\mathcal{M}} f > \lambda) \geq \alpha(\square_{\mathcal{M}} f_* > \lambda)$. First observe that by arguments similar to the proof of Theorem 7.3.1

$$\{\square_{\mathcal{M}} f > \lambda\} = \bigcup_{q \in S_{\mathcal{M}}(\lambda)} m(\{f_1 > q_1\}, \ldots, \{f_n > q_n\}). \tag{7.19}$$

Indeed, suppose $\Box_{\mathcal{M}} f(z) > \lambda$. This implies the existence of some x such that $m(x) = z$ and $\mathcal{M}(f_1(x_1), \ldots, f_n(x_n)) > \lambda$. By the continuity of $\mathcal{M}$, there exists $q \in S_{\mathcal{M}}(\lambda)$ such that $\mathcal{M}(q_1, \ldots, q_n) > \lambda$ and $f(x_i) > q_i$. The opposite direction is immediate. Observe that by our measurability assumptions on m and (7.19), the superlevel sets of $\Box_{\mathcal{M},m} f$ are universally measurable. Since ψ is necessarily Borel measurable by its monotonicity, its composition with $\Box_{\mathcal{M},m} f$ is indeed universally measurable. Analogously, (note that f_i^* are Borel measurable, by lower semi-continuity),

$$\{\Box_{\mathcal{M}} f_* > \lambda\} = \bigcup_{q \in S_{\mathcal{M}}(\lambda)} \eta(\{f_1^* > q_1\}, \ldots, \{f_n^* > q_n\}). \tag{7.20}$$

This gives

$$\begin{aligned}
\mu\{\Box_{\mathcal{M}} f > \lambda\} &= \mu\left(\bigcup_{q \in S_{\mathcal{M}}(\lambda)} m(\{f_1 > q_1\}, \ldots, \{f_n > q_n\})\right). \\
&\geq \sup_{q \in S_{\mathcal{M}}(\lambda)} \mu(m(\{f_1 > q_1\}, \ldots, \{f_n > q_n\})) \\
&\geq \sup_{q \in S_{\mathcal{M}}(\lambda)} \alpha(\eta(\{f_1 > q_1\}^*, \ldots, \{f_n > q_n\}^*)) \\
&= \alpha\left(\bigcup_{q \in S_{\mathcal{M}}(\lambda)} \eta(\{f_1^* > q_1\}, \ldots, \{f_n^* > q_n\})\right) \\
&= \alpha\{\Box_{\mathcal{M}} f_* > \lambda\}.
\end{aligned}$$

The first inequality is obvious, the second is by the assumed set theoretic rearrangement inequality, and the following equality is by the assumption of total orderedness. The last equality is the from (7.20). □

7.5 Applications

7.5.1 Borell–Brascamp–Lieb Type Inequalities

In the case that $\lambda \in (0, 1)$ and $-\infty \leq p \leq \infty$, we recall from example (1) the following continuous coordinate increasing function,

$$\mathcal{M}(u, v) = \mathcal{M}_p^{\lambda}(u, v) = \begin{cases} ((1-\lambda)u^p + \lambda v^p)^{\frac{1}{p}} & \text{if } uv \neq 0 \\ 0 & \text{if } uv = 0. \end{cases} \tag{7.21}$$

The Borell–Brascamp–Lieb inequality generalizes the PLI with the understanding that $\mathcal{M}_0^\lambda(u, v) = u^{1-\lambda}v^\lambda$. Note that $\mathcal{M}_\infty^\lambda(u, v) = \max\{u, v\}$ and $\mathcal{M}_{-\infty}^\lambda(u, v) = \min\{u, v\}$ as defined in Eq. (7.17). If we define $f \Box_{\mathcal{M}_p^\lambda} g$ using $m(x, y) = (1-\lambda)x + \lambda y$ as in Definition 7.4.5, we can state the inequality as the following.

Theorem 7.5.1 (Borell–Brascamp–Lieb [15, 18]) *For $\lambda \in (0, 1)$ and Borel functions $f, g : \mathbb{R}^n \to [0, \infty)$,*

$$\int f \Box_{\mathcal{M}_p^\lambda} g(x)\, dx \geq \mathcal{M}_{p/(np+1)}^\lambda \left(\int f(x) dx, \int g(x) dx \right)$$

when $p \geq -1/n$.

We present the following sharpening.

Theorem 7.5.2 *For Borel functions $f, g : \mathbb{R}^n \to [0, \infty)$ and $*$ a rearrangement to a convex set,*

$$\begin{aligned} \int f \Box_{\mathcal{M}_p^\lambda} g(x)\, dx &\geq \int f^* \Box_{\mathcal{M}_p^\lambda} g^*(x)\, dx \\ &\geq \mathcal{M}_{p/(np+1)}^\lambda \left(\int f(x) dx, \int g(x) dx \right) \end{aligned}$$

when $p \geq -1/n$.

Proof As described in Proposition 7.4.2, the Brunn–Minkowski inequality shows that the usual Lebesgue measure with the map $(x, y) \mapsto (1 - \lambda)x + ty)$ satisfy a set theoretic rearrangement inequality. The result then follows from Theorem 7.4.6. □

7.5.2 The Gaussian Case

For simplicity we restrict ourselves to the $\mathbb{R}^d$ case and employ the rearrangement $*$ from the Gaussian measure space $(\mathbb{R}^d, \gamma_d)$ to $(\mathbb{R}, \gamma_1)$, by

$$A^* = \{x \in \mathbb{R} : x < t\}$$

where $t = \Phi^{-1}(\gamma_d(A))$ is chosen to satisfy $\gamma_d(A) = \gamma(A^*)$. A functional half-space rearrangement by

$$f^*(x) = \int_0^\infty \mathbb{1}_{\{f>t\}^*}(x) dt.$$

The Borell–Ehrhard's inequality [16, 24] is usually stated as the assertion that $t \in (0,1)$, A, B Borel in $\mathbb{R}^d$ imply

$$\gamma_d((1-t)A + tB) \geq \Phi((1-t)\Phi^{-1}(\mu(A)) + t\Phi^{-1}(\mu(B))).$$

It can be equivalently formulated in our terminology and notation .

Theorem 7.5.3 (Borell–Ehrhard [16, 24]) *For $t \in (0,1)$, $m(x,y) = (1-t)x + ty$, $\eta(u,v) = (1-t)u + tv$, and $*$ our halfspace rearrangement from $(\mathbb{R}^d, \gamma_d)$ to $(\mathbb{R}, \gamma)$, satisfy a the set theoretic rearrangement inequality, explicitly for Borel A and B*

$$\gamma_d((1-t)A + tB) \geq \gamma((1-t)A^* + tB^*).$$

We will extend Theorem 7.5.3 to a functional inequality by Theorem 7.4.6. However, it should be mentioned that the semigroup proof of Borell actually gave a functional inequality already. The argument was streamlined by Barthe and Huet, and it is their generalization below that we will sharpen.

Theorem 7.5.4 (Barthe–Huet [3]) *Fix a set $I \subseteq \{1, 2, \ldots, n\}$ and positive numbers $\lambda_1, \ldots, \lambda_n$ satisfying $\sum \lambda_i \geq 1$ and $\lambda_j - \sum_{i \neq j} \lambda_i \leq 1$ for $j \notin I$. Then for Borel $f_1, \ldots, f_n$ from $\mathbb{R}^d$ to $[0,1]$ such that $\Phi^{-1} \circ f_i$ is concave for $i \in I$, and a Borel h satisfying $h(\sum_i \lambda_i x_i) \geq \Phi(\sum_i \lambda_i \Phi^{-1}(f_i(x_i)))$, then*

$$\int h d\gamma_d \geq \Phi\left(\lambda_1 \Phi^{-1}\left(\int f_1 d\gamma_d\right) + \cdots + \lambda_n \Phi^{-1}\left(\int f_n d\gamma_d\right)\right).$$

A consequence of Theorem 7.5.4 (and actually proven equivalent to Theorem 7.5.4 in the same paper) is the following.

Corollary 7.5.5 *Fix a set $I \subseteq \{1, 2, \ldots, n\}$ and set of positive numbers $\lambda_1, \ldots, \lambda_n$ satisfying $\sum \lambda_i \geq 1$ and $\lambda_j - \sum_{i \neq j} \lambda_i \leq 1$ for $j \notin I$. Then for Borel A_j,*

$$\begin{aligned}\gamma_d(\lambda_1 A_1 + \cdots + \lambda_n A_n) &\geq \Phi(\lambda_1 \Phi^{-1}(\gamma_d(A_1)) + \cdots + \lambda_n \Phi^{-1}(\gamma_d(A_n))) \\ &= \gamma(\lambda_1 A_1^* + \cdots + \lambda_n A_n^*)\end{aligned}$$

holds, provided A_i are convex when $i \in I$.

Strictly speaking, unless I is empty, the half-line rearrangement does not yield a set theoretic rearrangement inequality with the maps $m_\lambda(x) = \lambda_1 x_1 + \cdots + \lambda_n x_n$ and $\eta_\lambda(y) = \lambda_1 y_1 + \cdots + \lambda_n y_n$. However the proof of Theorem 7.4.6 can be adapted to achieve the following refinement of Barthe-Huet.

Theorem 7.5.6 *For Borel $f_1, \dots, f_n$ from $\mathbb{R}^d$ to $[0, 1]$ such that $\Phi^{-1} \circ f_i$ is concave for $i \in I$ and*

$$\int \square_{\mathcal{M}_\Phi^\lambda} f d\gamma_d \geq \int \square_{\mathcal{M}_\Phi^\lambda} f_* d\gamma$$
$$\geq \mathcal{M}_\Phi^\lambda \left(\int f_1^* d\gamma, \dots, \int f_n^* d\gamma \right)$$
$$= \mathcal{M}_\Phi^\lambda \left(\int f_1 d\gamma, \dots, \int f_n d\gamma \right).$$

In analyzing the proof of Theorem 7.5.6, a loosening of the hypothesis can be achieved, requiring only that for $i \in I$, f_i is quasi-concave and $\Phi^{-1} \circ f_i^*$ concave.

Proof Once it is observed that $\Phi^{-1} \circ f_i$ concave ensures $\{f_i > q_i\}$ is a convex set, so that one can apply Corollary 7.5.5, the first inequality can be derived following the proof of Theorem 7.4.6. The equality is immediate as well, following from our definition of rearrangement. Thus, to prove the result, we need only justify the second inequality, which follows from Theorem 7.5.4 once we know that the concavity of $\Phi^{-1} \circ f_i$ implies the concavity of $\Phi^{-1} \circ f_i^*$ as well. For this, we prove a general result below. □

Definition 7.5.7 For a fixed $t \in (0, 1)$ and a convex set K we will call $f : K \to \mathbb{R}$, Ψ_t-*concave* when there exists a continuous coordinate increasing function Ψ_t such that

$$f((1-t)x_1 + tx_2) \geq \Psi_t(f(x_1), f(x_2)).$$

Notice that the concavity of $\Phi^{-1} \circ f$ is equivalent to the statement that f is Ψ_t-concave with $\Psi_t(u_1, u_2) = \mathcal{M}_\Phi^t(u_1, u_2) = \Phi((1-t)\Phi^{-1}(u_1) + t\Phi^{-1}(u_2))$ for $t \in (0, 1)$.

Proposition 7.5.8 *Suppose that f, g, h are Borel functions on a space (M, μ) satisfying*

$$h((1-t)x + ty) \geq \Psi_t(f(x), g(y)) \tag{7.22}$$

for $x, y \in M$, and that $*$ is a rearrangement from (M, μ) to a space (N, α) satisfying

$$\mu((1-t)A + tB) \geq \alpha((1-t)A^* + tB^*). \tag{7.23}$$

Additionally assume that the space of rearranged sets has a total ordering that respects Minkowski summation in the sense that $(1-t)A^* + tB^*$ and C^* satisfy either

$$(1-t)A^* + tB^* \subseteq C^* \text{ or } (1-t)A^* + tB^* \supseteq C^* \tag{7.24}$$

then

$$h^*((1-t)x+ty) \geq \Psi_t(f^*(x), g^*(y)) \tag{7.25}$$

holds for $x, y \in N$.

Note that Theorem 7.5.6 follows from the proposition by taking $f = g = h$ and $\Psi_t = \mathcal{M}_\Phi^t$. Indeed, since the half-line rearrangement satisfies (7.24), as half-lines are stable under convex combination, it follows that f^* to be $\mathcal{M}_\Phi^t$-concave if f is.

Proof Observe that inequality (7.22) can be equivalently stated as $\lambda_i \in \mathbb{R}$ implies

$$(1-t)\{f > \lambda_1\} + t\{g > \lambda_2\} \subseteq \{h > \Psi_t(\lambda_1, \lambda_2)\}. \tag{7.26}$$

which can be easily verified using our assumptions of continuity and monotonicity. Indeed, if (7.22) holds, then for $z = (1-t)x+ty$ for $x \in \{f > \lambda_1\}$ and $y \in \{g > \lambda_2\}$ we have $h(z) \geq \Psi_t(f(x), g(y)) > \Psi_t(\lambda_1, \lambda_2)$. For the converse, given x, y take $\lambda_1 < f(x)$ and $\lambda_2 < g(y)$, then $z = (1-t)x + ty \in (1-t)\{f > \lambda_1\} + t\{g > \lambda_2\}$. By (7.26), $h(z) > \Psi_t(f(x), g(y))$, and by the continuity assumption on Ψ_t, $\Psi_t(f(x), g(y)) = \sup_\lambda \Psi_t(\lambda_1, \lambda_2) \leq h(z)$. Thus we will prove $(1-t)\{f^* > \lambda_1\} + t\{g^* > \lambda_2\} \subseteq \{h^* > \Psi_t(\lambda_1, \lambda_2)\}$, or equivalently

$$(1-t)\{f > \lambda_1\}^* + t\{g > \lambda_2\}^* \subseteq \{h > \Psi_t(\lambda_1, \lambda_2)\}^*.$$

By (7.24), it is enough to show

$$\alpha((1-t)\{f > \lambda_1\}^* + t\{g > \lambda_2\}^*) \leq \alpha(\{h > \Psi_t(\lambda_1, \lambda_2)\}^*).$$

By our assumptions (7.23) and (7.26),

$$\begin{aligned}\alpha((1-t)\{f > \lambda_1\}^* + t\{g > \lambda_2\}^*) &\leq \mu((1-t)\{f > \lambda_1\} + t\{g > \lambda_2\}) \\ &\leq \mu(\{h > \Psi_t(\lambda_1, \lambda_2)\}).\end{aligned}$$

Our result follows since

$$\mu(\{h > \Psi_t(\lambda_1, \lambda_2)\}) = \alpha(\{h > \Psi_t(\lambda_1, \lambda_2)\}^*).$$

□

Observe that Proposition 7.5.8 gives another proof of Theorem 7.3.1. Indeed, since $f \square g((1-t)x+ty) \geq f^{1-t}(x)g^t(y)$ holds for all x, y, $(f \square g)^*((1-t)x + ty)) \geq (f^*)^{1-t}(x)(g^*)^t(y)$ holds as well. This implies $(f \square g)^* \geq f^* \square g^*$ and hence

$$|\{f \square g > \lambda\}| = \left|\left\{(f \square g)^* > \lambda\right\}\right| \geq \left|\left\{f^* \square g^* > \lambda\right\}\right|.$$

Let us also point out the corollary obtained by taking $f = g = h$, as it is of interest independent of the application to Theorem 7.5.6.

Corollary 7.5.9 *If* $f : \mathbb{R}^d \to [0, \infty)$ *is* Ψ_t*-concave, and* $*$ *implies* f^* *is as well.*

It follows immediately that the class of d-dimensional s-concave measures is stable under (convex set) rearrangement. See [11, 13] for background and [30, 32] for recent connections between s-concave measures and information theory.

7.5.3 Polar Prékopa–Leindler

For fixed $t, \lambda \in (0, 1)$, define $\mathcal{M} : [0, \infty)^2 \to [0, \infty)$ by

$$\mathcal{M}(u, v) = \min\left\{u^{\frac{1-t}{1-\lambda}}, v^{\frac{t}{\lambda}}\right\},$$

and for $x, y \in \mathbb{R}^d$ define $m(x, y) = (1-t)x + ty$ so that

$$f \Box_{\mathcal{M}} g(z) = \sup_{m(x,y)=z} \min\left\{f(x)^{\frac{1-t}{1-\lambda}}, g(y)^{\frac{t}{\lambda}}\right\}.$$

We can state the recent polar analog of Prékopa–Leindler due to Artstein-Avidan, Florentin, and Segal.

Theorem 7.5.10 (Artstein-Avidan et al. [1]) *For* $f, g : \mathbb{R}^d \to [0, \infty)$ *Borel, and* μ *log-concave*

$$\int f \Box_{\mathcal{M}} g(x) d\mu(x) \geq \mathcal{M}_{-1}^{\lambda}\left(\int f(x) d\mu(x), \int g(x) d\mu(x)\right).$$

In the case that μ is Lebesgue (with $*$ rearrangement to a convex set) or Gaussian (with $*$ rearrangement to a half-space), and $\eta(x, y) = (1-t)x + ty$, this can be sharpened to the following.

Theorem 7.5.11 *For* $f, g : \mathbb{R}^d \to [0, \infty)$ *Borel, and* μ *either Gaussian, with* $*$ *the half space rearrangement, or Lebesgue with* $*$ *a convex set rearrangement, then*

$$\begin{aligned} \int f \Box_{\mathcal{M}} g d\mu &\geq \int f^* \Box_{\mathcal{M}} g^* d\mu \\ &\geq \mathcal{M}_{-1}^{\lambda}\left(\int f d\mu, \int g d\mu\right). \end{aligned}$$

Proof As we have seen, the map $(x, y) \mapsto (1-t)x + ty$ satisfies a set theoretic rearrangement inequality by Brunn–Minkowski with respect to Lebesgue measure and rearrangement to a convex set, and by Borell–Ehrhard with respect to Gaussian measure and rearrangement to a halfspace. The map $\mathcal{M}(u, v) = \min\{u^{\frac{1-t}{1-\lambda}}, v^{\frac{t}{\lambda}}\}$ is clearly continuous and coordinate increasing for $\lambda, t \in (0, 1)$. Thus in both cases, Gaussian and Lebesgue, we can invoke Theorem 7.4.6 to obtain the first inequality. The second inequality is obtained from the application of Theorem 7.5.10 to f^* and g^*, and the equimeasurability of rearrangements. □

7.6 Gaussian Log-Sobolev Inequality

For a probability measure μ define the entropy functional[2] for a non-negative f by

$$H_\mu(f) = \int f \log f d\mu - \int f d\mu \log \int f d\mu.$$

One formulation of the Gaussian log-Sobolev inequality is the following.

Theorem 7.6.1 (Gaussian Log-Sobolev) *For positive smooth f,*

$$H_{\gamma_d}(f) \leq \frac{1}{2} \int \frac{|\nabla f|^2}{f} d\gamma_d.$$

In this form the inequality is due to Gross [27]. Carlen [20] showed it to be equivalent to the earlier information theoretic Blachman–Stam inequality [5, 48]. The Gaussian log-Sobolev inequality was shown to be a consequence of a strengthened PLI for strongly log-concave measures by Bobkov–Ledoux [8], and it is this perspective that we now develop to motivate the main result of this section, a rearrangement sharpening of an integrated Gaussian log-Sobolev inequality. In this direction, let us recall that the PLI can be easily extended to the log-concave case.

Theorem 7.6.2 (Log-Concave PLI) *For measure μ with density φ satisfying*

$$\varphi((1-t)x + ty) \geq \varphi^{1-t}(x)\varphi^t(y),$$

the inequality for non-negative functions u, v, w

$$u((1-t)x + ty) \geq v^{1-t}(x)w^t(y)$$

implies

$$\int u d\mu \geq \left(\int v d\mu\right)^{1-t} \left(\int w d\mu\right)^t. \tag{7.27}$$

[2]Note that when $f = \frac{d\nu}{d\mu}$ is the density function of a probability measure ν with respect to μ, $H_\mu(f)$ is the Kullback–Liebler divergence $D(\nu||\mu)$ or relative entropy [22].

Proof Observing that the functions $\tilde{u}(z) = u(z)\varphi(z)$, $\tilde{v}(z) = v(z)\varphi(z)$, and $\tilde{w}(z) = w(z)\varphi(z)$ satisfy

$$\tilde{u}((1-t)x + ty) \geq \tilde{v}^{1-t}(x)\tilde{w}^t(y)$$

so that applying the ordinary PLI, we have

$$\int \tilde{u}(z)dz \geq \left(\int \tilde{v}(z)dz\right)^{1-t} \left(\int \tilde{w}(z)dz\right)^t,$$

which is exactly (7.27). □

The log-concave case corresponds to the case when the measure is given by a density corresponding to a convex potential, that is, $\varphi(x) = e^{-V(x)}$ with V is convex. For the Gaussian measure something stronger is true. In this case, V satisfies

$$V((1-t)x + ty) \leq (1-t)V(x) + tV(y) - t(1-t)|x-y|^2/2. \tag{7.28}$$

Note that in the case that V is smooth, log-concavity is exactly $V'' \geq 0_d$ in the sense of positive semi-definite matrices, while (7.28) is $V'' \geq I_d$. Under these assumptions, Theorem 7.6.2 admits the following strengthening.

Theorem 7.6.3 (Curved Prékopa–Leindler) *For $t \in (0,1)$, μ strongly log-concave in the sense of* (7.28)*, and $u, v, w : \mathbb{R}^d \to [0,\infty)$ satisfying*

$$u((1-t)x + ty) \geq e^{-t(1-t)|x-y|^2/2} v^{1-t}(x) w^t(y),$$

for all $x, y \in \mathbb{R}^d$, then

$$\int u d\mu \geq \left(\int v\, d\mu\right)^{1-t} \left(\int w d\mu\right)^t.$$

Proof The proof follows again from applying the Euclidean PLI to $\tilde{u}(z) = u(z)\varphi(z)$, $\tilde{v}(z) = v(z)\varphi(z)$. □

Following arguments of Bobkov–Ledoux [8] we pursue a specialization of Theorem 7.6.3 to a single function, revealing a log-Sobolev inequality as a consequence of a strengthened PLI. For a fixed $t \in (0,1)$, and a strongly log-concave probability measure μ, and f, take $w = f^{\frac{1}{t}}$, $v = 1$, then for any u, satisfying

$$u((1-t)x + ty) \geq e^{-t(1-t)|x-y|^2/2} f(y)$$

we have from Theorem 7.6.3

$$\int u\, d\mu \geq \left(\int f^{\frac{1}{t}} d\mu\right)^t.$$

With the interest of determining the optimal such u achievable through the methods of PLI, it is natural to consider

$$u(z) = \sup_{\{(x,y):(1-t)x+ty=z\}} e^{-t(1-t)|x-y|^2/2} f(y).$$

Writing $\lambda = \frac{1-t}{t}$, note that the constraint on x, y is equivalent to $y = z + \lambda(z - x)$, so that the $u(z)$ above can be expressed as $Q_\lambda f(z)$ in the following definition.

Definition 7.6.4 For $\lambda \in (0, \infty)$ and f non-negative and Borel measurable, define

$$\begin{aligned} Q_\lambda f(z) &= \sup_w f(z + \lambda w) e^{-\lambda |w|^2/2} \\ &= \sup_w f(z + w) e^{-|w|^2/2\lambda}. \end{aligned}$$

Writing $\|f\|_p = \left(\int |f|^p d\mu\right)^{\frac{1}{p}}$ we can collect the above as the following.

Theorem 7.6.5 (Integrated Log-Sobolev) *For μ a strongly log-concave probability measure, $\lambda \in (0, \infty)$ and f non-negative and Borel measurable,*

$$\|Q_\lambda f\|_1 \geq \|f\|_{1+\lambda}.$$

The log-Sobolev inequality for strongly log-concave probability measures can be recovered as a corollary.

Corollary 7.6.6 (Log-Sobolev Inequality) *For μ strongly log-concave probability measure, and f a positive smooth function*

$$H_\mu(f) \leq \frac{1}{2} \int \frac{|\nabla f|^2}{f} d\mu$$

A proof is given in [8] where the expressions are given in terms of f^2 rather than f. It follows as a limiting case of Theorem 7.6.5 with $\lambda \to 0$.

Sketch of Proof For smooth positive functions constant outside of a compact set, one observes that equality holds when $\lambda = 0$. Then the Taylor series expansion,

$$\|f\|_{1+\lambda} = \|f\|_1 + \lambda H_\mu(f) + o(\lambda)$$

and a derived inequality

$$\|Q_\lambda f\|_1 \leq \|f\|_1 + \frac{\lambda}{2} \int \frac{|\nabla f|^2}{f} d\mu + o(\lambda)$$

deliver the conclusion. A limiting argument gives the result for general functions. □

To state our main result of the section, let $\mu = \gamma_d$ a standard Gaussian and $*$ be the half-space rearrangement of a set under γ_d, as in Proposition 7.4.3.

Theorem 7.6.7 *For non-negative Borel f and $\lambda, s > 0$,*

$$\gamma_d(\{Q_\lambda f > s\}) \geq \gamma(\{Q_\lambda f^* > s\})$$

where f^* is the Gaussian half-line rearrangement of f.

It will be a consequence of the proof that $Q_\lambda f$ is universally measurable.

Proof We first express $\{Q_\lambda f > s\}$ as the union of simpler sets. Denoting

$$S = S(s, q_1, q_2) = \{q = (q_1, q_2) \in \mathbb{Q}_+^2 : q_1 q_2 > s\},$$

it is straight forward to verify

$$\{Q_\lambda f > s\} = \bigcup_{q \in S} \left(\{x \in \mathbb{R}^d : f(x) > q_1\} + \left\{ y \in \mathbb{R}^d : |y| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\} \right). \tag{7.29}$$

Indeed, for z belonging to the union, there exists rational q_i, and x, y satisfying $f(x) > q_1$, $|y| < \sqrt{2\lambda \ln \frac{1}{q_2}}$, and $x + y = z$. Taking $w = -x = y - z$,

$$f(w) e^{-|w|^2/2\lambda} > q_1 q_2 > s,$$

so that $z \in \{Q_\lambda f > s\}$. Conversely if there exists a w such that $f(z+w)e^{-|w|^2/2\lambda} > s$ then by continuity there exist rational q_i satisfying $f(z+w) > q_1$, $e^{-|w|^2/2\lambda} > q_2$, and $q_1 q_2 > s$. Taking $x = z + w$ and $y = -w$ we see that $(q_1, q_2) \in S$ and

$$z \in \{f > q_1\} + \left\{ |y| < \sqrt{2\lambda \ln \frac{1}{q_2}} \right\}.$$

Notice that this gives $\{Q_\lambda f > s\}$ as a countable union of Minkowski sums of analytic sets. Since analytic sets are closed under such operations, $\{Q_\lambda f > s\}$ is an analytic set as well, and the universal measurability of $Q_t f$ follows.

Applying the Gaussian isoperimetric inequality [14, 49], which in our preferred formulation states that $\gamma_d(A + B_d) \geq \gamma(A^* + B_1)$ where B_d and B_1 are origin symmetric Euclidean balls of equal radius (in $\mathbb{R}^d$ and $\mathbb{R}$ respectively), we have

$$\begin{aligned}\gamma_d(\{Q_\lambda f > s\}) &= \gamma_d\left(\bigcup_{q\in S}\{f > q_1\} + \left\{w \in \mathbb{R}^d : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}\right) \\ &\geq \sup_{q\in S} \gamma_d\left(\{f > q_1\} + \left\{w \in \mathbb{R}^d : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}\right) \\ &\geq \sup_{q\in S} \gamma\left(\{f > q_1\}^* + \left\{w \in \mathbb{R} : |w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}\right).\end{aligned}$$

But $\{f > q_1\}^* = \{f^* > q_1\}$ is a half-line and hence the sets $\{f^* > q_1\} + \left\{|w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}$, indexed by $S(\lambda, q_1, q_2)$, are totally ordered. Thus,

$$\sup_{q\in S} \gamma\left(\{f > q_1\}^* + \left\{|w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}\right) = \gamma\left(\bigcup_{q\in S}\{f^* > q_1\} + \left\{|w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}\right).$$

Applying (7.29),

$$\gamma\left(\bigcup_{q\in S}\{f^* > q_1\} + \left\{|w| < \sqrt{2\lambda \ln \frac{1}{q_2}}\right\}\right) = \gamma(\{Q_\lambda f^* > \lambda\}),$$

and our theorem follows. □

As an immediate consequence, we have a sharpening of Theorem 7.6.5.

Corollary 7.6.8 *For f non-negative and Borel, and norms taken with respect to γ,*

$$\int Q_\lambda f d\gamma \geq \int Q_\lambda f^* d\gamma \geq \|f^*\|_{1+\lambda} = \|f\|_{1+\lambda}.$$

Proof The first inequality is a consequence of Theorem 7.6.7, while the second is from Theorem 7.6.5. □

We also direct the reader to the articles [41, 42] of Martín and Milman, whose work on symmetrization, isoperimetry, and log-Sobolev inequalities the author learned of during the revision of this paper.

7.7 Barthe, Brascamp, Lieb and Rearrangement

The Brascamp–Lieb inequality is the following.

Theorem 7.7.1 (Brascamp–Lieb [17]) *For natural numbers $n \leq m$, and $\{n_i\}_{i=1}^m$ with $n_i \leq n$ and $\{c_i\}_{i=1}^m$ a sequence of positive numbers such that $\sum_{i=1}^m c_i n_i = n$ then for surjective linear maps $B_i : \mathbb{R}^n \to \mathbb{R}^{n_i}$, with $\cap_i \ker(B_i) = 0$ and transposes denoted B_i' satisfy the following,*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i^{c_i}(B_i x) dx \leq C^{-1/2} \prod \left(\int_{\mathbb{R}^{n_i}} f_i \right)^{c_i}$$

for $f_i : \mathbb{R}^{n_i} \to [0, \infty)$ integrable, and

$$C = \inf \left\{ \frac{\det(\sum_{i=1} c_i B_i' A_i B_i)}{\prod \det^{c_i} A_i} : A_i \text{ positive definite} \right\}.$$

The theorem enjoys a qualitative analog in the case that $n_i = d$, so that $n = md$ and $x \in \mathbb{R}^n$ can be expressed as $x = (x_1, \ldots, x_m)$ for $x_j \in \mathbb{R}^d$ and B_i are of the form

$$B_i x = \sum_{j=1}^m B_{ij} x_j. \tag{7.30}$$

Theorem 7.7.2 (Braskamp et al. [19]) *For B_i satisfying (7.30),*

$$\int_{\mathbb{R}^n} \prod_{i=1}^m f_i(B_i x) dx \leq \int_{\mathbb{R}^n} \prod_{i=1}^m f_i^*(B_i x) dx,$$

where $*$ represents the spherically symmetric decreasing rearrangement.

Notice that when Theorem 7.7.2 applies, it gives an intermediary inequality to Theorem 7.7.1. Indeed, since $(f^{c_i})^* = (f^*)^{c_i}$, applying first Theorem 7.7.2, and then 7.7.1, gives

$$\begin{aligned} \int_{\mathbb{R}^n} \prod_{i=1}^m f^{c_i}(B_i x) dx &\leq \int_{\mathbb{R}^n} \prod_{i=1}^m (f^*)^{c_i}(B_i x) dx \\ &\leq C^{-1/2} \prod_{i=1}^m \left(\int_{\mathbb{R}^{n_i}} f \right)^{c_i}. \end{aligned}$$

Barthe gave the following reversal of Brascamp–Lieb, which serves as a dual inequality.

Theorem 7.7.3 (Barthe [2]) *For n, m, $\{n_i\}_{i=1}^m$, $\{c_i\}_{i=1}^m$, B_i, and C as in Theorem 7.7.1 then the inequality*

$$C^{1/2} \prod_{i=1}^{m} \left(\int_{\mathbb{R}^{n_i}} f_i \right)^{c_i} \leq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i^{c_i}(y_i) : \sum_i c_i B_i' y_i = x \right\} dx,$$

holds for $f_i : \mathbb{R}^{n_i} \to [0, \infty)$ integrable.

Taking $m = 2$, $c_1 = (1-t)$, $c_2 = t$ and $n_i = n$ and B_i to be the identity map yields $C = 1$ and we recover the Prekopa–Liendler inequality. We ask if further extensions of our work here exist.

Question 7.7.4 Suppose that B_i are of the form (7.30), and $f_i : \mathbb{R}^d \to [0, \infty)$, when is it true that

$$\int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i(y_i) : \sum_i B_i' y_i = x \right\} dx \geq \int_{\mathbb{R}^n} \sup \left\{ \prod_{i=1}^{m} f_i^*(y_i) : \sum_i B_i' y_i = x \right\} dx \tag{7.31}$$

holds?

The results presented here verify the inequality for general Borel f_i in the case that B_i are scalar multiples of the identity. Note that the case $f_i = \mathbb{1}_{A_i}$ asks if the following generalization of BMI holds

$$\left| \sum_i B_i' A_i \right|_n \geq \left| \sum_i B_i' A_i^* \right|_n, \tag{7.32}$$

where

$$\sum_i B_i' A_i = \left\{ z = \sum_i B_i' x_i : x_i \in A_i \right\}.$$

In the case that $B_i' : \mathbb{R} \to \mathbb{R}^d$, inequality (7.32) was proven by Zamir and Feder [51].

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