Jørgen Hoffmann-Jørgensen (1942–2017)

Michael B. Marcus, Goran Peskir, and Jan Rosiński

Jørgen Hoffmann-Jørgensen, docent emeritus in the Department of Mathematics at Aarhus University, Denmark, died on the 8th of December 2017. He was 75 years old. He is survived by Karen, his wife of fifty years, his mother Ingeborg, his brother Bent and his niece Dorthe.

He was a devoted teacher and advisor, a wonderful, friendly person, and a very fine and prolific mathematician. His ties to Aarhus are legendary. Jørgen received his magister scientiarum degree from the Institute of Mathematics at Aarhus University in 1966. He began his research and teaching there in the previous year and continued through the academic ranks, becoming docent in 1988.

With a stroke of good luck he began his career as a probabilist under the most auspicious circumstances. Kiyoshi Itô was a professor at Aarhus from 1966 to 1969. Ron Getoor, who had been with Itô at Princeton, came to Aarhus as a visiting professor in the spring semester of 1969. Jørgen began his research career in the presence of these outstanding probabilists. He often commented that, more than any other mathematician, Itô had the greatest influence on his work.

There was widespread interest in sums of independent Banach space valued random variables at that time. The famous paper of Itô and Nisio, 'On the convergence of sums of independent Banach space valued random variables', appeared in 1968. Jean-Pierre Kahane's book, 'Some random series of functions'

M. B. Marcus
CUNY Graduate Center, New York, NY, USA
e-mail: mbmarcus@optonline.net

G. Peskir
Department of Mathematics, The University of Manchester, Manchester, UK
e-mail: goran@maths.man.ac.uk

J. Rosiński (✉)
Department of Mathematics, University of Tennessee, Knoxville, TN, USA
e-mail: jrosinsk@utk.edu

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(first edition), mostly dealing with random Fourier series, also came out in 1968. Functional analysts in the circle of Laurent Schwartz were using properties of sums of independent Banach space valued random variables to classify Banach spaces.

Engaged in this work, Jørgen published his most cited papers, 'Sums of independent Banach space valued random variables', as a publication of the Institute of Mathematics in Aarhus in 1972, and a paper with the same title, in Studia Mathematica in 1974 (cf. [9]). The two papers overlap but each has material that is not in the other. They contain the important and very useful relationship, between the norm of the maximal term in a series and the norm of the series, that is now commonly referred to as 'Hoffmann-Jørgensen's inequality'.

Continuing in this study, Jørgen collaborated on two important papers; with Gilles Pisier on the law of large numbers and the central limit theorem in Banach spaces [12], and with Richard Dudley and Larry Shepp on the lower tails of Gaussian seminorms [13]. He returned repeatedly to the topics of these and his other early papers, examining them in more general and abstract spaces. In this vein Jørgen reexamined the concept of weak convergence from a new perspective that completely changed the paradigm of its applications in statistics. He formulated his new definition of weak convergence in the 1980s[1]. This is now referred to as 'weak convergence in Hoffmann-Jørgensen's sense'.

Jørgen remained an active researcher throughout his life. He was completing a paper with Andreas Basse-O'Connor and Jan Rosiński on the extension of the Itô-Nisio theorem to non-separable Banach spaces, when he died.

Jørgen was also a very fine teacher and advisor with great concern for his students. He wrote 10 sets of lecture notes for his courses, 2,620 pages in total, and a monumental 1,184 page, two volume, 'Probability with a view toward Statistics', published by Chapman and Hall in 1994. He was the principal advisor of seven Ph.D. students.

Reflecting the interest in sums of independent Banach space valued random variables, and the related field of Gaussian processes in Europe, Laurent Schwarz and Jacques Neveu organized an auspicious conference on Gaussian Processes in Strasbourg in 1973. This stimulated research and collaborations that continue to this day. The Strasbourg conference was followed, every two or three years, by nine conferences on Probability in Banach Spaces and eight conferences on High Dimensional Probability. The last one was in Oaxaca, Mexico in 2017. The change in the conference name reflected a broadening of the interests of the participants.

Jørgen was one of a core group, many of whom attended the 1973 conference, who took part in all or most of the eighteen conferences throughout their careers, and often were the conference organizers and editors of the conference proceedings. Most significantly, Jørgen was the principal organizer of three of these conferences in the beautiful, serene, conference center in Sandbjerg, Denmark in 1986, 1993 and 2002, and was an editor of the proceedings of these conferences. Moreover, his

[1] Some authors have claimed, as we did in [14], that this definition was introduced in Jørgen's paper *Probability in Banach space* [10] in 1977. However, after a careful reading of this paper, we do not think that this is correct.

influence on the study of probability in Europe extended beyond these activities. In total, Jørgen served on the conference committees of eighteen meetings in Croatia, Denmark, Italy, France and Germany. Jørgen also served as an editor of the Journal of Theoretical Probability.

Jørgen was one of the mathematicians at Aarhus University who made Aarhus a focal point for generations of probabilists. But it was not only the research that brought them to Aarhus. Just as important was Jørgen's warmth and wit and not least of all the wonderful hospitality he and his wife Karen extended to all of them. Who can forget the fabulous Danish meals at their house, and then, sitting around after dinner, exchanging mathematical gossip and arguing politics, with the mating calls of hump backed whales playing in the background[2].

We now present some of Jørgen's better known results. This is not an attempt to place him in the history of probability but merely to mention some of his work that has been important to us and to give the reader a glimpse of his achievements.

Hoffmann-Jørgensen's Inequality Let (X_n) be a sequence of independent symmetric random variables with values in a Banach space E with norm $\|\cdot\|$. We define

$$S_n = \sum_{j=1}^{n} X_j, \quad N = \sup_n \|X_n\|, \quad M = \sup_n \|S_n\|.$$

Hoffmann-Jørgensen's inequality states that

$$\mathsf{P}(M \geq 2t + s) \leq 2\mathsf{P}(N \geq s) + 8\mathsf{P}^2(M \geq t) \tag{1.1}$$

for all $t, s > 0$.

Note that since probabilities are less than 1 and the last term in this inequality is a square it suggests that if M has sufficient regularity the distribution of M is controlled by the distribution of N. This is a remarkable result.

Jørgen gives this inequality in his famous paper [9]. He does not highlight it. It simply appears in the proof of his Theorem 3.1 which is:

Theorem 1 *Let* (X_n) *be a sequence of independent* E*-valued random variables such that*

$$\mathsf{P}(M < \infty) = 1 \quad \textit{and} \quad \mathsf{E}(N^p) < \infty$$

for some $0 < p < \infty$. *Then* $\mathsf{E}(M^p) < \infty$.

[2]The material up to this point has appeared in [14].

This is how he uses the inequality to prove this theorem. Assume that the elements of (X_n) are symmetric and let $R(t) = \mathsf{P}(M \geq t)$ and $Q(t) = \mathsf{P}(N \geq t)$ for $t \geq 0$. Using the relationship

$$\mathsf{E}(M^p) = \int_0^\infty px^{p-1}R(x)dx,$$

and similarly for N and Q, it follows from (1.1) that for $A > 0$

$$\begin{aligned}\int_0^A px^{p-1}R(x)dx &= p\,3^p \int_0^{A/3} px^{p-1}R(3x)dx \qquad (1.2)\\ &\leq 2p\,3^p \int_0^{A/3} px^{p-1}Q(x)dx + 8p\,3^p \int_0^{A/3} px^{p-1}R^2(x)dx\\ &\leq 2p\,3^p\mathsf{E}(N^p) + 8p\,3^p \int_0^{A/3} px^{p-1}R^2(x)dx.\end{aligned}$$

Choose $t_0 > 0$ such that $R(t_0) < (16p3^p)^{-1}$. The condition that $\mathsf{P}(M < \infty) = 1$ implies that $t_0 < \infty$. Then choose $A > 3t_0$. Note that

$$\begin{aligned}\int_0^{A/3} px^{p-1}R^2(x)dx &= \int_0^{t_0} px^{p-1}R^2(x)dx + \int_{t_0}^{A/3} px^{p-1}R^2(x)dx\\ &\leq t_0^p + R(t_0)\int_{t_0}^{A/3} px^{p-1}R(x)dx. \qquad (1.3)\end{aligned}$$

Combining (1.2) and (1.3) we get

$$\int_0^A px^{p-1}R(x)dx \leq 2p\,3^p\mathsf{E}(N^p) + t_0^p + \frac{1}{2}\int_0^{A/3} px^{p-1}R(x)dx. \qquad (1.4)$$

It follows from (1.4) that when the elements of (X_n) are symmetric and $\mathsf{E}(N^p) < \infty$, then $\mathsf{E}(M^p) < \infty$. Eliminating the condition that (X_n) is symmetric is routine.

Inequalities for sums of independent random variables that relate the sum to the supremum of the individual terms are often referred to as Hoffmann-Jørgensen type inequalities. Jørgen's original inequality has been generalized and extended. Many of these results are surveyed in [5] which obtains Hoffmann-Jørgensen type inequalities for U statistics. See [4] for a more recent treatment of Hoffmann-Jørgensen type inequalities in statistics.

Weak Convergence in Hoffmann-Jørgensen's Sense The classic concept of *convergence in distribution*, dating back to de Moivre's central limit theorem in 1737, admits the following well-known characterisation, traditionally referred to as *weak convergence* (cf. [3]).

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space, let S be a metric (topological) space, and let $\mathcal{B}(S)$ be the Borel σ-algebra on S. Let $X_1, X_2, \ldots$ and X be measurable functions from Ω to S with respect to $\mathcal{F}$ and $\mathcal{B}(S)$. If

$$\lim_{n\to\infty} \mathsf{E} f(X_n) = \mathsf{E} f(X) \tag{1.5}$$

for every bounded continuous function $f : S \to \mathbb{R}$, then we say that X_n *converges weakly* to X, and following Jørgen's notation, write

$$X_n \overset{\sim}{\rightarrow} X \tag{1.6}$$

as $n \to \infty$. The expectation E in (1.5) is defined as the (Lebesgue-Stieltjes) integral with respect to the (σ-additive) probability measure P.

The state space S in classical examples is finite dimensional, e.g. $\mathbb{R}$ or $\mathbb{R}^n$ for $n \geq 2$. The main motivation for Jørgen's reconsideration of (1.5) and (1.6) comes from the empirical processes theory. Recall that the *empirical distribution function* is given by

$$F_n(t, \omega) := \frac{1}{n} \sum_{i=1}^{n} I(\xi_i(\omega) \leq t) \tag{1.7}$$

for $n \geq 1$, $t \in [0, 1]$ and $\omega \in \Omega$, where $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables on Ω taking values in $[0, 1]$ and having the common distribution function F. In this setting, motivated by the classical central limit theorem, one forms the *empirical process*

$$X_n(t, \omega) := \sqrt{n}\,\big(F_n(t, \omega) - F(t)\big) \tag{1.8}$$

and aims to establish that X_n converges 'weakly' to a limiting process X (of a Brownian bridge type) as $n \to \infty$. A substantial difficulty arises immediately because the mapping $X_n : \Omega \to S$ is *not measurable* when S is taken to be the set of all right-continuous functions $x : [0, 1] \to \mathbb{R}$ with left-limits, equipped with the supremum norm $\|x\|_\infty = \sup_{t\in[0,1]} |x(t)|$ as a natural choice.

Skorokhod solved this measurability problem in 1956 by creating a different metric on S, for which the Borel σ-algebra coincides with the cylinder σ-algebra, so that each X_n is measurable. For more general empirical processes

$$X_n(f, \omega) := \sqrt{n}\,\Big(\frac{1}{n} \sum_{i=1}^{n} f(X_i(\omega)) - \mathsf{E} f(X_1)\Big) \tag{1.9}$$

indexed by f belonging to a family of functions, there is no obvious way to extend the Skorokhod approach. Jørgen solved this measurability problem in the most elegant way by simply replacing the first expectation E in (1.5) by the *outer* *expectation* E^*, which is defined by

$$\mathsf{E}^* Y = \inf\{\,\mathsf{E}Z \mid Z \geq Y \text{ is measurable}\,\} \tag{1.10}$$

where Y is any (not necessarily measurable) function from Ω to $\mathbf{R}$, and leaving the second expectation E in (1.5) unchanged (upon assuming that the limit X is measurable).

This definition of *weak convergence in Hoffmann-Jørgensen's sense* is given for the first time in his monograph [11, page 149]. Although [11] was published in 1991, a draft of the monograph was available in Aarhus and elsewhere since 1984. Furthermore, the first paper [1] which uses Jørgen's new definition was published in 1985. Jørgen's definition of weak convergence became standard soon afterwards. It continues to be widely used.

It is now known that replacing the first E in (1.5) by E^* is equivalent to replacing it by E^Q where Q is any *finitely additive* extension of P from $\mathcal{F}$ to 2^Ω (see Theorem 4 in [2] for details). This revealing equivalence just adds to both simplicity and depth of Jørgen's thought when opting for E^* in his celebrated definition.

Hoffmann-Jørgensen's Work on Measure Theory As measure theory matured, difficult measurability problems arose in various areas of mathematics that could not be solved in general measure spaces. Consequently, new classes of measure spaces were introduced, such as *analytic spaces*, also called *Souslin spaces*, defined by Lusin and Souslin and further developed by Sierpiński, Kuratowski and others. For many years analytic spaces received little attention until important applications were found in potential theory by Choquet and group representation theory by Mackey. Analytic spaces were also found to be important in the theory of convex sets, and other branches of mathematics.

Stimulated by these developments, Jørgen undertook a deep study of analytic spaces early in his academic career, resulting in his monograph 'The Theory of Analytic Spaces' [7]. This monograph contains many original, and carefully presented results, that are hard to find elsewhere. For example, from Jørgen's Section Theorem, [7, Theorem 1, page 84], one can derive all of the most commonly used section and selection theorems in the literature.

The final chapter of the monograph is devoted to locally convex vector spaces, where it is shown that all of the locally convex spaces that are of interest to researchers are analytic spaces. As Jørgen wrote "The importance of analytic spaces lies in the fact that even though the category is sufficiently small to exclude all pathological examples ..., it is sufficiently large to include all (or almost all) interesting and important examples of topological measure spaces."

In one of his first papers [6] listed in Mathematical Reviews and Zentralblatt, Jørgen investigates extensions of regenerative events to continuous state spaces, a problem proposed to him by P.-A. Meyer. In his subsequent paper [8], he makes the surprising observation that the existence of a measurable modification of a stochastic process depends only on its 2-dimensional marginal distributions. He then gives necessary and sufficient conditions for the existence of such a modification for the process $(X_t)_{t\in T}$ with values in a complete separable metric space K, expressed in terms of the kernel

$$Q(s,t,A) = \mathsf{P}((X_s, X_t) \in A)$$

where T is a separable metric space, $s, t \in T$, and $A \in \mathcal{B}(K^2)$. Jørgen's interest in measure theory aspects of probability continued throughout his career.

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