Chapter 2
Calabi–Yau Manifolds with Torsion and Geometric Flows

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Abstract The main theme of these lectures is the study of Hermitian metrics in non-Kähler complex geometry. We will specialize to a certain class of Hermitian metrics which generalize Kähler Ricci-flat metrics to the non-Kähler setting. These non-Kähler Calabi–Yau manifolds have their origins in theoretical physics, where they were introduced in the works of C. Hull and A. Strominger. We will introduce tools from geometric analysis, namely geometric flows, to study this non-Kähler Calabi–Yau geometry. More specifically, we will discuss the Anomaly flow, which is a version of the Ricci flow customized to this particular geometric setting. This flow was introduced in joint works with Duong Phong and Xiangwen Zhang. Section 2.1 contains a review of Hermitian metrics, connections, and curvature. Section 2.2 is dedicated to the geometry of Calabi–Yau manifolds equipped with a conformally balanced metric. Section 2.3 introduces the Anomaly flow in the simplest case of zero slope, where the flow can be understood as a deformation path connecting non-Kähler to Kähler geometry. Section 2.4 concerns the Anomaly flow with α' corrections, which is motivated from theoretical physics and canonical metrics in non-Kähler geometry.

2.1 Review of Hermitian Geometry

We start by reviewing non-Kähler metrics in complex geometry. In particular, we study unitary connections, torsion, and curvature associated to a Hermitian metric ω.

2.1.1 Hermitian Metrics

Let X be a complex manifold of dimension n. The manifold X is covered by holomorphic charts U_μ equipped with local holomorphic coordinates $(z^1, \ldots, z^n)$

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D. Angella et al. (eds.), *Complex Non-Kähler Geometry*, Lecture Notes in Mathematics 2246, https://doi.org/10.1007/978-3-030-25883-2_2

such that $X = \bigcup_\mu U_\mu$. The complexified tangent bundle of X will be denoted TX, which splits

$$TX = T^{1,0}X \oplus T^{0,1}X.$$

Using local coordinates, a tangent vector in $T^{1,0}X$ is a combination of

$$\left\{ \frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n} \right\}$$

and a tangent vector in $T^{0,1}X$ is a combination of

$$\left\{ \frac{\partial}{\partial \bar{z}^1}, \ldots, \frac{\partial}{\partial \bar{z}^n} \right\}.$$

We will use the notation

$$\partial_k = \frac{\partial}{\partial z^k}, \quad \partial_{\bar{k}} = \frac{\partial}{\partial \bar{z}^k}.$$

Next, we will use $\Omega^{p,q}(X)$ to denote differential forms of (p,q) type. This means that in local coordinates, $\Omega^{p,q}(X)$ is generated by

$$dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots d\bar{z}^{j_q}.$$

We will use the following convention for the components $\Psi_{\bar{j}_1 \cdots \bar{j}_q i_1 \cdots i_p}$ of a differential form $\Psi \in \Omega^{p,q}(X)$

$$\Psi = \frac{1}{p!q!} \sum \Psi_{\bar{j}_1 \cdots \bar{j}_q i_1 \cdots i_p} dz^{i_p} \wedge \cdots dz^{i_1} \wedge d\bar{z}^{j_q} \wedge \cdots \wedge d\bar{z}^{j_1}. \tag{2.1}$$

The exterior derivative d decomposes into

$$d = \partial + \bar{\partial},$$

where

$$\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \quad \bar{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X),$$

are the Dolbeault operators. A Hermitian metric g on X is a smooth section $(T^{1,0}X)^* \otimes (T^{0,1}X)^*$ such that in local coordinates

$$g = g_{\bar{k}j}\, dz^j \otimes d\bar{z}^k, \tag{2.2}$$

where $g_{\bar{k}j}$ is a positive-definite Hermitian matrix at each point.

$$g_{\bar{k}j} > 0, \qquad \overline{g_{\bar{k}j}} = g_{\bar{j}k}.$$

In (2.2) we use the summation convention, which will be used throughout these notes, where we omit the summation sign for matching upper and lower indices. We use the notation $g^{j\bar{k}} = (g_{\bar{k}j})^{-1}$ for the inverse, meaning that

$$g^{i\bar{k}} g_{\bar{k}j} = \delta^i{}_j.$$

We can identify the metric g with a Hermitian form $\omega \in \Omega^{1,1}(X, \mathbf{R})$ via

$$\omega = i g_{\bar{k}j}\, dz^j \wedge d\bar{z}^k.$$

The metric g induces a metric on differential forms $\Omega^{p,q}(X)$, and we define the Hodge star operator $\star : \Omega^{p,q}(X) \to \Omega^{n-q,n-p}(X)$ by requiring

$$\alpha \wedge \star\bar{\beta} = g(\alpha, \beta)\, \frac{\omega^n}{n!}.$$

for all $\alpha, \beta \in \Omega^{p,q}(X)$.

A basic fact which will be often used in these notes is

Proposition 2.1 *Let X be a compact complex manifold with Hermitian metric g and $\partial X = \emptyset$. Let $f \in C^\infty(X, \mathbf{R})$. If*

$$g^{j\bar{k}} \partial_j \partial_{\bar{k}} f \geq 0,$$

everywhere on X, then f is a constant function.

Proof Let c denote the maximum value attained by f on X. The set

$$S = f^{-1}(c)$$

is closed. We claim that S is also open. Indeed, let $p \in S$. Let B be a ball in a local chart such that f attains a maximum in the center of B and satisfies $g^{j\bar{k}} \partial_j \partial_{\bar{k}} f \geq 0$ in B. By the Hopf strong maximum principle (e.g. Theorem 2.7 in [HL11]), we must have $f \equiv c$ in B. This shows that S is open, and hence $S = X$. □

A Hermitian metric ω is Kähler if

$$d\omega = 0.$$

Kähler manifolds are of fundamental importance as they lie at the crossroads of both Riemannian geometry and algebraic geometry. In these notes, our goal is to generalize the Kähler condition while still retaining enough structure to develop an interesting theory.

There are many ways to generalize the Kähler condition. There is the notion of a pluriclosed metric, which satisfies

$$i\partial\bar{\partial}\omega = 0.$$

There are also astheno-Kähler metrics [JY93], which satisfy

$$i\partial\bar{\partial}\omega^{n-2} = 0.$$

It was shown by Gauduchon [GA77] that every compact complex manifold admits a Gauduchon metric, which satisfies

$$i\partial\bar{\partial}\omega^{n-1} = 0.$$

More generally, Fu-Wang-Wu [FWW13] introduced the notion of k-Gauduchon, for $1 \leq k \leq n-1$, which is defined by the condition

$$i\partial\bar{\partial}\omega^k \wedge \omega^{n-k-1} = 0.$$

All these notions generalize Kähler metrics in different ways. In these notes, we will mostly focus on another notion: we say a Hermitian metric ω is balanced if

$$d\omega^{n-1} = 0. \tag{2.3}$$

The special properties of balanced metrics were noticed early in the study of Hermitian geometry, arising for examples in articles of Gauduchon [GA75]. Balanced metrics were studied systematically by Michelsohn [MI82], and these metrics were rediscovered in theoretical physics in the development of heterotic string theory [HU186, ST86, LY05]. A main theme in Michelsohn's work is that balanced metrics are in some sense dual to the Kähler condition. For example, Kähler metrics are inherited by the ambient space (via pullback) while balanced metrics can be pushed forward [MI82].

Given a Hermitian metric ω, its torsion is defined by

$$T = i\partial\omega, \quad \bar{T} = -i\bar{\partial}\omega.$$

We see that a metric is Kähler if and only if its torsion vanishes. The components of the torsion are given by

$$T = \frac{1}{2} T_{\bar{k}jm} dz^m \wedge dz^j \wedge d\bar{z}^k, \quad \bar{T} = \frac{1}{2} \bar{T}_{k\bar{j}\bar{m}} d\bar{z}^m \wedge d\bar{z}^j \wedge dz^k.$$

Explicitly, we have

$$T_{\bar{k}jm} = \partial_j g_{\bar{k}m} - \partial_m g_{\bar{k}j}, \quad \bar{T}_{k\bar{j}\bar{m}} = \partial_{\bar{j}} g_{\bar{m}k} - \partial_{\bar{m}} g_{\bar{j}k}. \tag{2.4}$$

We can raise indices using the metric, and we will write $T^k{}_{ij} = g^{k\bar{\ell}} T_{\bar{\ell}ij}$. We can also contract indices, and we will use the notation

$$T_j = g^{i\bar{k}} T_{\bar{k}ij}.$$

We will also use the 1-form τ defined by

$$\tau = T_k dz^k.$$

Taking norms, we have

$$|T|^2 = g^{m\bar{n}} g^{k\bar{\ell}} g^{j\bar{i}} T_{\bar{i}km} \bar{T}_{j\bar{\ell}\bar{n}}, \quad |\tau|^2 = g^{k\bar{\ell}} T_k \bar{T}_{\bar{\ell}}.$$

2.1.2 Connections

Let $E \to X$ be a complex vector bundle of rank r. The bundle E can be specified by an open cover $X = \bigcup_\mu U_\mu$ together with transition matrices $t_{\mu\nu} : U_\mu \cap U_\nu \to GL(r, \mathbf{C})$ satisfying

$$t_{\mu\mu}{}^\alpha{}_\beta = \delta^\alpha{}_\beta,$$

and

$$t_{\mu\nu}{}^\alpha{}_\beta t_{\nu\rho}{}^\beta{}_\gamma = t_{\mu\rho}{}^\alpha{}_\gamma \ \text{ on } U_\mu \cap U_\nu \cap U_\rho.$$

If all transition functions $t_{\mu\nu}$ are holomorphic, then E is a holomorphic bundle.

A section $s \in \Gamma(X, E)$ is given by local data $(U_\mu, s_\mu{}^\alpha)$, where

$$s = (s_\mu{}^1(z_\mu), \cdots, s_\mu{}^r(z_\mu)) \ \text{ on } U_\mu,$$

and $s_\mu : U_\mu \to \mathbf{C}^r$ is a smooth map which transforms via

$$(s_\mu)^\alpha = t_{\mu\nu}{}^\alpha{}_\beta s_\nu{}^\beta$$

on $U_\mu \cap U_\nu$. On a holomorphic bundle, we say s is holomorphic if the s_μ are holomorphic.

Let us illustrate this notation by considering the example of the holomorphic tangent bundle $T^{1,0}X$. Here the transition functions are

$$t_{\mu\nu}{}^{i}{}_{k} = \frac{\partial {z_\mu}^{i}}{\partial {z_\nu}^{k}},$$

which are defined on the intersection of coordinate patches $(U_\mu, {z_\mu}^i)$ and $(U_\nu, {z_\nu}^i)$. Sections of $T^{1,0}X$ are vector fields $V = V^i \partial_i \in \Gamma(X, T^{1,0}X)$, and on $U_\mu \cap U_\nu$,

$${V_\mu}^k = \frac{\partial {z_\mu}^k}{\partial {z_\nu}^\ell} {V_\nu}^\ell.$$

Next, we recall that from a bundle E, we can induce bundles such as E^*, $\bar{E}$, and $\det E$. If the bundle E has transition matrices $t_{\mu\nu}$, then sections $\phi \in \Gamma(X, E^*)$ are given by data $(U_\mu, \phi_{\mu\alpha})$ which transform according to

$$(\phi_\mu)_\alpha = t_{\nu\mu}{}^{\beta}{}_{\alpha} \phi_{\nu\beta}.$$

Similarly, sections $s \in \Gamma(X, \bar{E})$ transform by

$${s_\mu}^{\bar{\alpha}} = \overline{t_{\mu\nu}{}^{\alpha}{}_{\beta}}\, {s_\nu}^{\bar{\beta}},$$

and sections $\psi \in \Gamma(X, \det E)$ are given by local functions $\psi_\mu : U_\mu \to \mathbf{C}$ which transform by

$$\psi_\mu = (\det t_{\mu\nu})\, \psi_\nu.$$

To differentiate sections of a vector bundle, we use a connection ∇. Connections can be expressed locally as $\nabla = d + A_\mu$, where A_μ are local matrix-valued 1-forms $(A_\mu)_i{}^{\alpha}{}_{\beta}$ defined on U_μ. The local matrices $(A_\mu)_i$ satisfy the transformation law

$$(A_\mu)_i = t_{\mu\nu}\, (A_\nu)_i\, {t_{\mu\nu}}^{-1} - (\partial_i t_{\mu\nu}) {t_{\mu\nu}}^{-1}. \tag{2.5}$$

Here we omitted the indices for matrix multiplication. This transformation law is designed such that for any section $s \in \Gamma(X, E)$, its derivative $\nabla_i s$ is again a section. Explicitly, derivatives of s are given locally by

$$\nabla_i s^\alpha = \partial_i s^\alpha + A_i{}^{\alpha}{}_{\beta} s^\beta, \quad \nabla_{\bar{i}} s^\alpha = \partial_{\bar{i}} s^\alpha + A_{\bar{i}}{}^{\alpha}{}_{\beta} s^\beta.$$

with the notation

$$\nabla_i = \nabla_{\frac{\partial}{\partial z^i}}, \quad \nabla_{\bar{i}} = \nabla_{\frac{\partial}{\partial \bar{z}^i}}.$$

Given a connection on E, we can induce connections on E^*, $\bar{E}$, $\det E$, etc., by imposing the product rule. For example, the product rule $\partial_k(s_\alpha \phi^\alpha) = \nabla_k s_\alpha \phi^\alpha + s_\alpha \nabla_k \phi^\alpha$ leads to the definition

$$\nabla_k \phi_\alpha = \partial_k \phi_\alpha - \phi_\beta A_k{}^\beta{}_\alpha, \quad \nabla_{\bar{k}} \phi_\alpha = \partial_{\bar{k}} \phi_\alpha - \phi_\beta A_{\bar{k}}{}^\beta{}_\alpha$$

for sections $\phi \in \Gamma(X, E^*)$. Similarly, for a section $u \in \Gamma(X, \bar{E})$, the induced connection is defined by

$$\nabla_k u^{\bar{\alpha}} = \partial_k u^{\bar{\alpha}} + \overline{A_{\bar{k}}{}^\alpha{}_\beta} u^{\bar{\beta}}, \quad \nabla_{\bar{k}} u^{\bar{\alpha}} = \partial_{\bar{k}} u^{\bar{\alpha}} + \overline{A_k{}^\alpha{}_\beta} u^{\bar{\beta}},$$

and for a section $\psi \in \Gamma(X, \det E^*)$, the induced connection is

$$\nabla_i \psi = \partial_i \psi - A_i{}^\alpha{}_\alpha \psi, \quad \nabla_{\bar{i}} \psi = \partial_{\bar{i}} \psi - A_{\bar{i}}{}^\alpha{}_\alpha \psi. \tag{2.6}$$

As a final example, the induced connection on $\Gamma(X, E^* \otimes \bar{E}^*)$ is defined by

$$\nabla_k h_{\bar{\alpha}\beta} = \partial_k h_{\bar{\alpha}\beta} - \overline{A_{\bar{k}}{}^\gamma{}_\alpha} h_{\bar{\gamma}\beta} - A_k{}^\gamma{}_\beta h_{\bar{\alpha}\gamma}.$$

We now focus our attention on the holomorphic tangent bundle $T^{1,0}X$. Given a Hermitian metric $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ on X, we say a connection ∇ on $T^{1,0}X$ is unitary with respect to ω if

$$\nabla_i g_{\bar{k}j} = 0.$$

On a Hermitian manifold (X, ω), the Chern connection is the unique unitary connection on $T^{1,0}X$ such that $A_{\bar{k}} = 0$. The Chern connection acts on sections $V \in \Gamma(X, T^{1,0}X)$ by

$$\nabla_k (V^i \partial_i) = (\nabla_k V^i) \partial_i, \quad \nabla_{\bar{k}} (V^i \partial_i) = (\nabla_{\bar{k}} V^i) \partial_i,$$

where

$$\nabla_k V^i = \partial_k V^i + \Gamma^i_{k\ell} V^\ell, \quad \nabla_{\bar{k}} V^i = \partial_{\bar{k}} V^i,$$

and

$$\Gamma^i_{k\ell} = g^{i\bar{p}} \partial_k g_{\bar{p}\ell}. \tag{2.7}$$

Due to its simplicity, the Chern connection is best suited for most computations. However, in non-Kähler geometry, there are other interesting connections on $T^{1,0}X$ to consider too. We start with the Levi-Civita connection, which acts on $V \in \Gamma(X, TX)$ by

$$\nabla^g_k (V^i \partial_i + V^{\bar{i}} \partial_{\bar{i}}) = (\nabla^g_k V^i) \partial_i + (\nabla^g_k V^{\bar{i}}) \partial_{\bar{i}},$$

where

$$\nabla^g_k V^i = \partial_k V^i + \Gamma^i_{k\ell} V^\ell - \frac{T^i{}_{k\ell}}{2} V^\ell - \frac{g^{i\bar j}}{2} \bar T_{k\bar j\bar\ell} V^{\bar\ell},$$
$$\nabla^g_{\bar k} V^i = \partial_{\bar k} V^i + \frac{g^{i\bar m}}{2} \bar T_{\ell\bar k\bar m} V^\ell,$$

and

$$\nabla^g_k V^{\bar i} = \overline{\nabla^g_{\bar k} V^i}, \quad \nabla^g_{\bar k} V^{\bar i} = \overline{\nabla^g_k V^i}.$$

To be clear, we note that here, and throughout these notes, $\Gamma^i_{k\ell}$ is reserved for the expression (2.7), which is not the Christoffel symbol of the Levi-Civita connection.

This well-known connection from Riemannian geometry preserves the metric $\nabla^g g = 0$ and has zero torsion tensor $\nabla^g_X Y - \nabla^g_Y X - [X, Y]$. For Kähler metrics, $T = 0$ and we see that the Levi-Civita connection coincides with the Chern connection.

However, for general Hermitian metrics, the tensor $T_{\bar k ij}$ is nonzero and the Levi-Civita connection does not preserve the decomposition $TX = T^{1,0}X \oplus T^{0,1}X$. In particular, it does not define a connection on the holomorphic bundle $T^{1,0}X$.

We can add a correction to ∇^g to obtain a new connection which does preserve $T^{1,0}X$. We define

$$\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H, \quad H = i(\bar\partial - \partial)\omega.$$

The new connection acts on $V \in \Gamma(X, T^{1,0}X)$ by $\nabla^+_k(V^i \partial_i) = (\nabla^+_k V^i)\partial_i$ with components

$$\nabla^+_k V^i = \partial_k V^i + (\Gamma^i_{k\ell} - T^i{}_{k\ell}) V^\ell, \tag{2.8}$$
$$\nabla^+_{\bar k} V^i = \partial_{\bar k} V^i + g^{i\bar m} \bar T_{\ell\bar k\bar m} V^\ell.$$

We will call this connection the Strominger–Bismut connection [BI89, ST86]. It evidently preserves $T^{1,0}X$, and a straightforward computation shows that

$$\nabla^+ g_{\bar k j} = 0,$$

hence ∇^+ is a unitary connection. Furthermore, $\nabla^+ = \nabla^g + \frac{1}{2} g^{-1} H$ has the property that its torsion 3-form

$$\mathscr{T}(X, Y, Z) = g(\nabla^+_X Y - \nabla^+_Y X - [X, Y], Z)$$

is given by the skew-symmetric 3-form H.

Using the Chern connection ∇ and the Strominger–Bismut connection ∇^+, we can define a line of unitary connections which preserve the complex structure.

$$\nabla^{(\kappa)} = (1-\kappa)\nabla + \kappa\nabla^+,$$

where $\kappa \in \mathbf{R}$ is a parameter. This family of connections is known as the Gauduchon line [GA97]. We note that this line collapses to a point when ω is Kähler.

There are other connections which play a role in theoretical physics which do not preserve the complex structure. One such example is the Hull connection [HU286, LE11, DS14], denoted by $\nabla^- = \nabla^g - \frac{1}{2}g^{-1}H$. Explicitly, this connection acts on $V \in \Gamma(X, TX)$ by

$$\begin{aligned}\nabla^-_k V^i &= \partial_k V^i + \Gamma^i_{k\ell} V^\ell - g^{i\bar{j}} \bar{T}_{k\bar{j}\bar{\ell}} V^{\bar{\ell}}, \\ \nabla^-_{\bar{k}} V^i &= \partial_{\bar{k}} V^i.\end{aligned} \tag{2.9}$$

Although ∇^- does not preserve $T^{1,0}X$, a direct computation shows that $\nabla^- g = 0$.

Most computations in these notes will be done using the Chern connection, and from now on we reserve ∇ to denote the Chern connection. We will use superscripts e.g. ∇^+, to denote other connections.

Next, we review integration and adjoint operators in Hermitian geometry. The first identity is the divergence theorem for Hermitian metrics.

Lemma 2.1 *Let (X, ω) be a closed Hermitian manifold. The divergence theorem for the Chern connection ∇ is given by*

$$\int_X \nabla_i V^i \, \omega^n = \int_X T_i V^i \, \omega^n, \tag{2.10}$$

for any $V \in \Gamma(X, T^{1,0}X)$.

We see that the torsion components T_i play a role when integrating by parts. The proof is similar to the Kähler case, and is omitted.

Next, we recall the L^2 pairing of differential forms, given by $\langle \phi, \psi \rangle = \int_X g(\phi, \psi)\, \omega^n$, where $g(\phi, \psi)$ is the induced metric on $\phi, \psi \in \Omega^{p,q}(X)$. For example, for $\eta, \beta \in \Omega^{1,0}(X)$, we define

$$\langle \eta, \beta \rangle = \int_X g^{j\bar{k}} \eta_j \overline{\beta_k} \, \omega^n,$$

and for $\alpha, \chi \in \Omega^{1,1}(X)$,

$$\langle \alpha, \chi \rangle = \int_X g^{j\bar{k}} g^{\ell\bar{m}} \alpha_{\bar{k}\ell} \overline{\chi_{\bar{j}m}} \, \omega^n.$$

The adjoint operators $\partial^\dagger : \Omega^{p,q}(X) \to \Omega^{p-1,q}(X)$ and $\bar\partial^\dagger : \Omega^{p,q}(X) \to \Omega^{p,q-1}(X)$ are defined by the property

$$\langle \partial\phi, \psi\rangle = \langle \phi, \partial^\dagger\psi\rangle, \quad \langle \bar\partial\phi, \psi\rangle = \langle \phi, \bar\partial^\dagger\psi\rangle.$$

We will also write $d^\dagger = \partial^\dagger + \bar\partial^\dagger$. We will need an explicit expressions for these adjoint operators in the following special case.

Lemma 2.2 *Let (X, ω) be a Hermitian manifold. The adjoint operators act on $\alpha \in \Omega^{1,1}(X)$ by*

$$(\partial^\dagger\alpha)_{\bar k} = -g^{p\bar q}\nabla_{\bar q}\alpha_{\bar k p} + g^{p\bar q}\bar T_{\bar q}\alpha_{\bar k p}. \tag{2.11}$$

$$(\bar\partial^\dagger\alpha)_k = g^{p\bar q}\nabla_p\alpha_{\bar q k} - g^{p\bar q}T_p\alpha_{\bar q k}. \tag{2.12}$$

Proof Let $\alpha \in \Omega^{1,1}(X)$ and $\beta \in \Omega^{0,1}(X)$. The components of $\partial\beta$ are

$$(\partial\beta)_{\bar k j} = \nabla_j\beta_{\bar k}.$$

The inner product $\langle\alpha, \partial\beta\rangle = \langle\partial^\dagger\alpha, \beta\rangle$ expands to

$$\int_X g^{j\bar k}g^{p\bar q}\alpha_{\bar k p}\overline{(\nabla_q\beta_{\bar j})}\,\omega^n = \int_X g^{j\bar k}(\partial^\dagger\alpha)_{\bar k}\overline{\beta_{\bar j}}\,\omega^n.$$

Applying the divergence theorem (2.10) to the left-hand side, we obtain (2.11). A similar computation leads to (2.12). □

As a corollary, if we apply these identities to $\alpha = \omega = i g_{\bar k j}dz^j \wedge d\bar z^k$, we obtain

$$(\partial^\dagger\omega)_{\bar k} = i\bar T_{\bar k}, \quad (\bar\partial^\dagger\omega)_k = -iT_k. \tag{2.13}$$

and

$$d^\dagger\omega = i(\bar\tau - \tau).$$

2.1.3 Curvature

Let $E \to X$ be a complex vector bundle. The curvature of a connection $\nabla = d + A$ on E is a 2-form valued in the endomorphisms of E given by

$$F = dA + A \wedge A,$$

with components

$$F = \frac{1}{2} F_{kj}{}^{\alpha}{}_{\beta} dz^j \wedge dz^k + \frac{1}{2} F_{\bar{k}\bar{j}}{}^{\alpha}{}_{\beta} d\bar{z}^j \wedge d\bar{z}^k + F_{\bar{k}j}{}^{\alpha}{}_{\beta}\, dz^j \wedge d\bar{z}^k.$$

The curvature form of the Chern connection of a Hermitian metric ω will be denoted Rm. In this case, one can verify that the curvature form Rm is an endomorphism-valued $(1,1)$ form

$$Rm = R_{\bar{k}j}{}^{p}{}_{q} dz^j \wedge d\bar{z}^k,$$

with components given by

$$R_{\bar{k}j}{}^{p}{}_{q} = -\partial_{\bar{k}} \Gamma^p_{jq} = -\partial_{\bar{k}}(g^{p\bar{s}} \partial_j g_{\bar{s}q}).$$

We may write this as

$$Rm = \bar{\partial}(g^{-1}\partial g), \tag{2.14}$$

which holds in a holomorphic frame on $T^{1,0}X$. We note that in general, when using unitary connections other than the Chern connection on $T^{1,0}X$, the curvature will have $(2,0)$ and $(0,2)$ components as well.

We can raise and lower indices of the curvature tensor using the metric $g_{\bar{k}j}$.

$$R_{\bar{k}j\bar{m}\ell} = g_{\bar{m}p} R_{\bar{k}j}{}^{p}{}_{\ell} = -\partial_{\bar{k}}\partial_j g_{\bar{m}\ell} + g^{s\bar{r}} \partial_{\bar{k}} g_{\bar{m}s} \partial_j g_{\bar{r}\ell}. \tag{2.15}$$

Lemma 2.3 *The curvature of the Chern connection on* (X, ω) *satisfies the following Bianchi identities*

$$R_{\bar{k}j\bar{m}\ell} = R_{\bar{m}j\bar{k}\ell} + \nabla_j \bar{T}_{\ell\bar{m}\bar{k}},$$

$$R_{\bar{k}j\bar{m}\ell} = R_{\bar{k}\ell\bar{m}j} + \nabla_{\bar{k}} T_{\bar{m}\ell j}.$$

Proof For example, we compute using the definition (2.15) and obtain

$$\begin{aligned}
R_{\bar{k}j\bar{m}\ell} - R_{\bar{m}j\bar{k}\ell} &= -\partial_{\bar{k}}\partial_j g_{\bar{m}\ell} + g^{s\bar{r}}\partial_{\bar{k}} g_{\bar{m}s}\partial_j g_{\bar{r}\ell} + \partial_{\bar{m}}\partial_j g_{\bar{k}\ell} - g^{s\bar{r}}\partial_{\bar{m}} g_{\bar{k}s}\partial_j g_{\bar{r}\ell} \\
&= \partial_j(\partial_{\bar{m}} g_{\bar{k}\ell} - \partial_{\bar{k}} g_{\bar{m}\ell}) - g^{s\bar{r}}\partial_j g_{\bar{r}\ell}(\partial_{\bar{m}} g_{\bar{k}s} - \partial_{\bar{k}} g_{\bar{m}s}) \\
&= \partial_j \bar{T}_{\ell\bar{m}\bar{k}} - \Gamma^p_{j\ell}\bar{T}_{p\bar{m}\bar{k}} \\
&= \nabla_j \bar{T}_{\ell\bar{m}\bar{k}}.
\end{aligned}$$

The other identity is derived in a similar way. □

There are four notions of Ricci curvature for the Chern connection in Hermitian geometry, and we will use the notation

$$R_{\bar{k}j} = R_{\bar{k}j}{}^{p}{}_{p}, \quad \tilde{R}_{\bar{k}j} = R^{p}{}_{p\bar{k}j}, \quad R'_{\bar{k}j} = R_{\bar{k}p}{}^{p}{}_{j}, \quad R''_{\bar{k}j} = R^{p}{}_{j\bar{k}p}.$$

From the Bianchi identity, we see that these notions of Ricci curvature are all different. We will call $R_{\bar{k}j}$ the Chern–Ricci curvature, and it is also given by

$$R_{\bar{k}j} = -\partial_{\bar{k}}\partial_j \log\det g_{\bar{p}q}.$$

The Chern–Ricci form represents the first Chern class $[\frac{i}{2\pi}\mathrm{Ric}_\omega] = c_1(X)$ and is given by

$$\mathrm{Ric}_\omega = -\partial\bar{\partial}\log\det g_{\bar{p}q} = R_{\bar{k}j}dz^j \wedge d\bar{z}^k.$$

There are two notions of scalar curvature, denoted by

$$R = g^{\ell\bar{m}}g^{j\bar{k}}R_{\bar{k}j\bar{m}\ell} = R^{p}{}_{p}{}^{j}{}_{j}, \quad R' = g^{j\bar{m}}g^{\ell\bar{k}}R_{\bar{k}j\bar{m}\ell} = R^{p}{}_{j}{}^{j}{}_{p}.$$

2.1.4 U(1) Principal Bundles

2.1.4.1 Definitions

We denote the group of complex numbers with length equal to 1 by $U(1)$. A $U(1)$ principal bundle can be specified by an open cover $X = \bigcup_\mu U_\mu$ together with smooth maps

$$g_{\mu\nu} : U_\mu \cap U_\nu \to U(1),$$

such that

$$g_{\mu\mu} = 1, \quad g_{\mu\nu}^{-1} = g_{\nu\mu},$$

and

$$g_{\mu\nu}g_{\nu\rho} = g_{\mu\rho},$$

on an non-empty overlap $U_\mu \cap U_\nu \cap U_\rho$. In this section, we review how a connection on a line bundle defines a connection on a $U(1)$ principal bundle.

Let $L \to X$ be a smooth complex line bundle with data $(U_\mu \cap U_\nu, t_{\mu\nu})$, equipped with a connection $\nabla_A = d + A$ whose curvature is $F_A = dA$. We also consider the line bundle $L' \to X$ given by the data

$$(U_\mu \cap U_\nu, e^{i\tau_{\mu\nu}}), \quad \frac{t_{\mu\nu}}{|t_{\mu\nu}|} = e^{i\tau_{\mu\nu}}.$$

To compactify the fibers, we equip L with a metric h, which is locally given by (U_μ, h_μ) where h_μ are positive functions which transforms as

$$h_\mu = \frac{1}{|t_{\mu\nu}|^2} h_\nu.$$

The metric h provides an isomorphism of the line bundles L and L', where the connection $d + A$ on L becomes the connection $d + A'$ given by

$$A' = A - \frac{1}{2} d \log h,$$

on L'. It can be checked that this expression satisfies the transformation law for a connection (2.5), which in this case becomes

$$A'_\mu = A'_\nu - i\, d\tau_{\mu\nu}. \tag{2.16}$$

Thus we have induced a connection $d + A'$ on L' with curvature

$$dA' = F_A. \tag{2.17}$$

Let $\pi : P \to X$ be the $U(1)$ bundle determined by the data $(U_\mu \cap U_\nu, e^{i\tau_{\mu\nu}})$. Locally, points in P are given by $(z_\mu, e^{i\psi_\mu})$ with projection $\pi(z_\mu, e^{i\psi_\mu}) = z_\mu$, where the coordinates $e^{i\psi_\mu}$ on the fiber transform via

$$e^{i\psi_\mu} = e^{i\tau_{\mu\nu}} e^{i\psi_\nu}.$$

In other words, on $U_\mu \cap U_\nu$, there holds

$$\psi_\mu = \psi_\beta + \tau_{\mu\nu} + 2\pi k, \tag{2.18}$$

for an integer k. Combining this with the transformation law for the connection (2.16), it follows that

$$\theta = d\psi_\mu - i A'_\mu \tag{2.19}$$

is a global 1-form on the total space of the bundle $\pi : P \to X$. We call θ the connection 1-form of the $U(1)$ bundle P. Furthermore, by (2.17), its exterior derivative is

$$d\theta = -i F_A.$$

The connection 1-form θ splits the tangent space TP of P into vertical and horizontal directions. For the vertical direction, we note that by (2.18), the expression $\frac{\partial}{\partial\psi}$ transforms as a global vector field on $\pi : P \to X$. We define the vertical subbundle V by

$$V = \ker \pi_* = \text{span}\left\{\frac{\partial}{\partial\psi}\right\}.$$

The horizontal space is given by $H = \ker\theta$. The tangent bundle of P splits as

$$TP = V \oplus H,$$

and $\pi_*|_H : H \to TX$ is isomorphism.

2.1.4.2 Non-Kähler Manifolds Constructed from Principal Bundles

Connections on $U(1)$ principal bundles can be used to construct non-Kähler complex manifolds. This idea was first used by Calabi–Eckmann [CE53], and later generalized by Goldstein–Prokushkin [GO04]. In this section, we will construct the Calabi–Eckmann manifolds.

Our first example will use $\mathbf{P}^1$ as the base manifold. We cover $\mathbf{P}^1$ by the open sets

$$U_0 = \{[Z_0, Z_1] : Z_0 \neq 0\}, \quad U_1 = \{[Z_0, Z_1] : Z_1 \neq 0\},$$

and define coordinates $\zeta = \frac{Z_1}{Z_0}$ on U_0 and $\xi = \frac{Z_0}{Z_1}$ on U_1. The line bundle $L = \mathcal{O}(-1) \to \mathbf{P}^1$ equips the covering $\{U_0, U_1\}$ with the transition function

$$t_{01} : U_0 \cap U_1 \to \mathbf{C}^*, \quad t_{01} = \frac{Z_0}{Z_1}.$$

This data defines a $U(1)$ principal bundle $\pi : P \to \mathbf{P}^1$ by the same covering $\mathbf{P}^1 = U_0 \cup U_1$ and transition function

$$\frac{Z_0}{Z_1}\frac{|Z_1|}{|Z_0|} : U_0 \cap U_1 \to S^1.$$

In the trivialisation $U_0 \times S^1$, we use coordinates $(\zeta, e^{i\psi_0})$, and in the trivialisation $U_1 \times S^1$, we use coordinates $(\xi, e^{i\psi_1})$. On the overlap,

$$e^{i\psi_0} = \frac{\xi}{|\xi|} e^{i\psi_1}.$$

In fact, the space P is diffeomorphic to the sphere S^3. If we write

$$S^3 = \{(z_0, z_1) \in \mathbf{C}^2 : |z_0|^2 + |z_1|^2 = 1\},$$

then a diffeomorphism is given by $F : S^3 \to P$, where

$$F(z_0, z_1) = \left([z_0, z_1], \frac{z_0}{|z_0|}\right) \in U_0 \times S^1, \quad z_0 \neq 0,$$
$$F(0, z_1) = ([0, 1], z_1) \in U_1 \times S^1.$$

The inverse of F is given by

$$F^{-1}(\zeta, e^{i\psi_0}) = \frac{1}{\sqrt{1+|\zeta|^2}}(e^{i\psi_0}, \zeta e^{i\psi_0}), \quad (\zeta, e^{i\psi_0}) \in U_0 \times S^1,$$
$$F^{-1}([0, 1], e^{i\psi_1}) = (0, e^{i\psi_1}), \quad ([0, 1], e^{i\psi_1}) \in U_1 \times S^1.$$

Next, we define a connection on P.

A metric on $L = \mathcal{O}(-1)$ is defined by two positive functions $h_0 : U_0 \to (0, \infty)$ and $h_1 : U_1 \to (0, \infty)$ satisfying $h_0 = \frac{h_1}{|t_{01}|^2}$. We will take

$$h_0 = 1 + |\zeta|^2, \quad h_1 = 1 + |\xi|^2.$$

The Chern connection of (L, h) is $\nabla = d + A$ with $A = \partial \log h$. As explained in (2.19), a connection on L defines a connection 1-form θ on P given by

$$\theta = d\psi - iA',$$

which satisfies

$$d\theta = -i dA' = -i\bar{\partial}\partial \log h := \omega_{FS}. \tag{2.20}$$

Next, we add a trivial fiber $S^1 = \{e^{i\phi}\}$ to our space, and consider the manifold

$$M_{1,0} = P \times S^1 \simeq S^3 \times S^1.$$

Using the connection θ, we split the tangent bundle

$$TM_{1,0} = H \oplus \left\langle \frac{\partial}{\partial \psi} \right\rangle \oplus \left\langle \frac{\partial}{\partial \phi} \right\rangle.$$

We can define an almost complex structure J on $M_{1,0}$ by identifying H with $T\mathbf{P}^1$ and using the standard complex structure on ∂_ψ and ∂_ϕ. To be precise, if j is the complex structure on $\mathbf{P}^1$, then

$$J = (\pi^*|_H j) \oplus I, \qquad I\frac{\partial}{\partial \psi} = \frac{\partial}{\partial \phi}, \ \ I\frac{\partial}{\partial \phi} = -\frac{\partial}{\partial \psi}.$$

The space $T^{1,0}M_{1,0}$ is spanned by pullbacks of $T^{1,0}\mathbf{P}^1$ and

$$\frac{\partial}{\partial \psi} - i\frac{\partial}{\partial \phi}.$$

To show J is integrable, we can apply the Newlander–Nirenberg theorem. If z denotes a local holomorphic coordinate on $\mathbf{P}^1$, then $(1,0)$-forms on $M_{1,0}$ are locally generated by

$$\{\pi^* dz, \theta + i d\phi\}.$$

We note that $\theta + i d\phi$ is a $(1,0)$ form since it sends $\partial_\psi + i\partial_\phi$ to zero and $H = \ker \theta$. For local functions f_1, f_2, then by (2.20) we compute

$$d[f_1 dz + f_2(\theta + i d\phi)] = df_1 \wedge dz + df_2 \wedge (\theta + i d\phi) + f_2 \omega_{FS}. \tag{2.21}$$

It follows that for any $\eta \in \Omega^{1,0}(M_{1,0})$, then $(d\eta)^{2,0} = 0$. By the Newlander–Nirenberg theorem, we conclude that $M_{1,0}$ is a complex manifold.

The complex surface $M_{1,0}$ is known as the Hopf surface. Since it is topologically $S^3 \times S^1$, we see that the second Betti number of $M_{1,0}$ is zero. Therefore $M_{1,0}$ is a non-Kähler complex surface.

This same construction can be applied to the manifold $M_{1,1} = P \times P$, which is a product of two copies of the $U(1)$ principal bundle P over $\mathbf{P}^1$. Then $M_{1,1}$ is a complex manifold of complex dimension 3, which is a fibration over $\mathbf{P}^1 \times \mathbf{P}^1$.

$$\pi : M_{1,1} \to \mathbf{P}^1 \times \mathbf{P}^1.$$

Since $M_{1,1} \simeq S^3 \times S^3$, this construction defines a non-Kähler complex structure on $S^3 \times S^3$.

In fact, the threefold $M_{1,1}$ does not even admit a balanced metric [MI82]. Suppose ω is a positive $(1,1)$ form on $M_{1,1}$ such that $d\omega^2 = 0$. Let D be a divisor on the base $\mathbf{P}^1 \times \mathbf{P}^1$. Since

$$\int_{\pi^*(D)} \omega^2 > 0,$$

it follows that the class $[\omega^2] \in H^4(M_{1,1}, \mathbf{R})$ is non-trivial. This is a contradiction, since $H^4(S^3 \times S^3, \mathbf{R}) = 0$.

The construction described above readily generalizes to $M_{p,q} = S^{2p+1} \times S^{2q+1}$, giving fibrations

$$\pi : M_{p,q} \to \mathbf{P}^p \times \mathbf{P}^q.$$

These non-Kähler complex manifolds were discovered in [CE53] and are now named Calabi–Eckmann manifolds. A variant of this construction will be revisited in Sect. 2.2.3.4 to produce T^2 fibrations over Calabi–Yau surfaces [GO04], and these manifolds will play a role as a class of solutions to the Hull–Strominger system [FY08, FY07].

2.2 Calabi–Yau Manifolds with Torsion

Let X be a compact complex manifold of complex dimension n. We assume now and henceforth in these notes that $n \geq 3$. Suppose X admits a nowhere vanishing holomorphic $(n,0)$ form Ω. Given a Hermitian metric $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$, the norm of Ω is defined by

$$\|\Omega\|_\omega^2 \frac{\omega^n}{n!} = i^{n^2} \Omega \wedge \bar{\Omega}. \tag{2.22}$$

Using a local coordinate representation $\Omega = \Omega(z)\, dz^1 \wedge \cdots \wedge dz^n$, this norm is

$$\|\Omega\|_\omega^2 = \Omega(z)\overline{\Omega}(z)(\det g_{\bar{k}j})^{-1}.$$

A Hermitian metric ω on (X, Ω) is said to be conformally balanced if it satisfies

$$d(\|\Omega\|_\omega \omega^{n-1}) = 0. \tag{2.23}$$

We see that the Hermitian metric $\chi = \|\Omega\|_\omega^{1/(n-1)} \omega$ is balanced in the sense of Michelsohn [MI82]. We will call (X, Ω, ω) a Calabi–Yau manifold with torsion.

Though Kähler manifolds provide a class of examples, Calabi–Yau manifolds with torsion need not admit a Kähler metric. We shall see that Calabi–Yau manifolds with torsion, though non-Kähler, still retain interesting structure. The geometry of Hermitian manifolds satisfying condition (2.23) belongs somewhere between Kähler geometry and the general theory of non-Kähler complex manifold described in Sect. 2.1. We note that there are other proposed generalizations of non-Kähler Calabi–Yau manifolds in the literature; see e.g. [GGP08, LE11, TO15].

It was shown by Li–Yau [LY05] that condition (2.23) is equivalent to certain $SU(n)$ structures arising in heterotic string theory [HU186, HU286, ST86, DS14, IP01, GMPW04]. In this section, we will explore the geometric implications of this condition.

2.2.1 Curvature and Holonomy

2.2.1.1 Holonomy

From the point of view of differential geometry, Calabi–Yau manifolds with torsion can be understood by imposing a holonomy constraint. While Kähler Calabi–Yau manifolds are characterized by the Levi-Civita connection having holonomy contained in $SU(n)$, here we consider the holonomy of the Strominger–Bismut connection ∇^+ instead.

Lemma 2.4 ([MI82]) *Let (X,ω) be a Hermitian manifold equipped with a nowhere vanishing holomorphic $(n,0)$ form Ω. Define $\chi = \|\Omega\|_\omega^{1/(n-1)}\omega$. Then*

$$d^\dagger_\chi \chi = i(\partial \log \|\Omega\|_\omega - \tau) - i(\bar\partial \log \|\Omega\|_\omega - \bar\tau).$$

Here τ is the torsion 1-form of ω, and $d^\dagger_\chi$ is the L^2 adjoint with respect to χ.

Proof The torsion 1-form of χ is given by

$$T^\chi_j = \|\Omega\|_\omega^{-1/(n-1)} g^{i\bar k}\Big[\partial_i(\|\Omega\|_\omega^{1/(n-1)} g_{\bar k j}) - \partial_j(\|\Omega\|_\omega^{1/(n-1)} g_{\bar k i})\Big].$$

Simplifying this expression give

$$T^\chi_j = T_j - \partial_j \log \|\Omega\|_\omega,$$

where T_j is the torsion of ω. We apply the identity (2.13) for the adjoint $\partial^\dagger_\chi$ of χ. □

Next, we interpret the conformally balanced condition in terms of a torsion constraint. This relationship between T and $\log\|\Omega\|_\omega$ will have a recurring role as the key identity in the subsequent computations.

Proposition 2.2 ([MI82]) *Let (X, ω) be a Hermitian manifold equipped with a nowhere vanishing holomorphic $(n, 0)$ form Ω. The conformally balanced condition (2.23) is equivalent to the torsion constraint*

$$T_j = \partial_j \log \|\Omega\|_\omega, \quad \bar{T}_{\bar{j}} = \partial_{\bar{j}} \log \|\Omega\|_\omega.$$

Proof Expanding the conformally balanced condition gives

$$0 = \partial \log \|\Omega\|_\omega \wedge \omega^{n-1} + (n-1)\partial\omega \wedge \omega^{n-2}.$$

A computation shows the following identity

$$(n-1)\partial\omega \wedge \omega^{n-2} = -\tau \wedge \omega^{n-1}.$$

Therefore

$$\partial \log \|\Omega\|_\omega \wedge \omega^{n-1} = \tau \wedge \omega^{n-1}.$$

It follows that $\tau = \partial \log \|\Omega\|_\omega$. □

Our first application of the torsion constraint will be to construct parallel sections of the canonical bundle.

Lemma 2.5 ([GA16]) *Let (X, ω) be a Hermitian manifold with a nowhere vanishing holomorphic $(n, 0)$ form Ω. Suppose (X, ω, Ω) satisfies the conformally balanced condition (2.23). Then $\psi = \|\Omega\|_\omega^{-1}\Omega$ satisfies*

$$\nabla^+ \psi = 0.$$

Thus $\psi \in \Gamma(X, K_X)$ is nowhere vanishing and parallel with respect to the Strominger–Bismut connection ∇^+.

Proof By (2.8) and (2.6), the induced connection ∇^+ on ψ is given by

$$\nabla^+_i \psi = \partial_i \psi - (\Gamma^\alpha_{i\alpha} - T^\alpha{}_{i\alpha})\psi, \quad \nabla^+_{\bar{i}} \psi = \partial_{\bar{i}} \psi - g^{k\bar{m}} \bar{T}_{k\bar{i}\bar{m}} \psi. \tag{2.24}$$

The unbarred derivative is

$$\nabla^+_i \psi = -\partial_i \log \|\Omega\|_\omega \psi + \|\Omega\|_\omega^{-1} \partial_i \Omega - \Gamma^\alpha_{i\alpha}\psi - T_i \psi.$$

We note that

$$2\partial_i \log \|\Omega\|_\omega = \frac{\partial_i \Omega}{\Omega} - g^{p\bar{q}} \partial_i g_{\bar{q}p} = \frac{\partial_i \Omega}{\Omega} - \Gamma^\alpha_{i\alpha}.$$

and therefore

$$\nabla_i^+ \psi = (\partial_i \log \|\Omega\|_\omega - T_i)\psi.$$

By (2.24), we also have

$$\nabla_{\bar{i}}^+ \psi = (-\partial_{\bar{i}} \log \|\Omega\|_\omega + \bar{T}_{\bar{i}})\psi.$$

If (X, ω, Ω) is conformally balanced, we may use Proposition 2.2 and substitute the torsion constraint $T_i = \partial_i \log \|\Omega\|_\omega$ to conclude $\nabla^+ \psi = 0$. □

Theorem 2.1 ([GA16]) *Let (X, ω) be a compact Hermitian manifold with nowhere vanishing holomorphic $(n, 0)$ form Ω. Then (X, ω, Ω) satisfies the conformally balanced condition (2.23) if and only if there exists $\psi \in \Gamma(X, K_X)$ which is nowhere vanishing and parallel with respect to the Strominger–Bismut connection ∇^+.*

Proof The previous lemma constructs a nowhere vanishing parallel section if (X, ω, Ω) is conformally balanced. On the other hand, suppose there exists a nowhere vanishing section $\psi \in \Gamma(X, K_X)$ such that

$$\nabla^+ \psi = 0.$$

We will follow the proof given in lecture notes of Garcia-Fernandez [GA16]. Write

$$\psi = e^{-f} \Omega,$$

for a complex function f. Since $\nabla^+ g_{\bar{k}j} = 0$, the norm of ψ is constant. Let us assume that $\|\psi\|_\omega = 1$. Then

$$1 = e^{-f-\bar{f}} \|\Omega\|_\omega^2,$$

and

$$f + \bar{f} = 2 \log \|\Omega\|_\omega.$$

By the formula (2.24), we obtain

$$\begin{aligned} 0 &= \nabla_i^+ \psi = (-\partial_i f - T_i + 2\partial_i \log \|\Omega\|_\omega)\psi, \\ 0 &= \nabla_{\bar{i}}^+ \psi = (-\partial_{\bar{i}} f + \bar{T}_{\bar{i}} f)\psi. \end{aligned} \tag{2.25}$$

We know that the real part $\mathrm{Re}\, f$ is $\log \|\Omega\|_\omega$, and we will now show that the imaginary part $\mathrm{Im}\, f$ is constant. For this, we use (2.25) to compute

$$\partial_i (f - \bar{f}) = 2(\partial_i \log \|\Omega\|_\omega - T_i),$$

$$\partial_{\bar{i}} (f - \bar{f}) = -2(\partial_{\bar{i}} \log \|\Omega\|_\omega - \bar{T}_{\bar{i}}).$$

By Lemma 2.4,

$$i d(f - \bar{f}) = 2 d_{\chi}^{\dagger} \chi,$$

for $\chi = \|\Omega\|_{\omega}^{1/(n-1)} \chi$. Therefore

$$d_{\chi}^{\dagger} d(f - \bar{f}) = 0,$$

hence $\langle d(f - \bar{f}), d(f - \bar{f}) \rangle_{\chi} = 0$ and Im f is constant. Since Re $f = \log \|\Omega\|_{\omega}$, it follows that

$$df = d \log \|\Omega\|_{\omega}$$

and (2.25) implies the torsion constraint

$$\partial \log \|\Omega\|_{\omega} = \tau.$$

By Proposition 2.2, (X, ω, Ω) is conformally balanced. □

As a consequence of the existence of parallel sections, we obtain the following interpretation of the conformally balanced condition in terms of a holonomy constraint.

Corollary 2.1 ([ST86, LY05]) *A compact Hermitian manifold with trivial canonical bundle (X, ω, Ω) satisfies the conformally balanced condition (2.23) if and only if*

$$\mathrm{Hol}(\nabla^+) \subseteq SU(n).$$

2.2.1.2 Curvature

Next, we study the structure of the curvature tensor of Calabi–Yau manifolds with torsion. We start with the curvature of the Bismut connection. By the definition (2.8), we can write $\nabla^+ = d + A$ with

$$A_j{}^p{}_q = \Gamma_{iq}^p - T^p{}_{jq}, \quad A_{\bar{j}}{}^p{}_q = g^{p\bar{k}} \bar{T}_{q\bar{j}\bar{k}}.$$

From this expression, we may compute $Rm^+ = dA + A \wedge A$. The components $(\mathrm{Tr}\, Rm^+)_{\alpha\beta} = R^+_{\alpha\beta}{}^{\gamma}{}_{\gamma}$ are

$$(\mathrm{Tr}\, Rm^+)_{kj} = \partial_j T_k - \partial_k T_j, \quad (\mathrm{Tr}\, Rm^+)_{\bar{k}\bar{j}} = -(\partial_{\bar{j}} \bar{T}_{\bar{k}} - \partial_{\bar{k}} \bar{T}_{\bar{j}}), \tag{2.26}$$

$$(\mathrm{Tr}\, Rm^+)_{\bar{k}j} = -\partial_{\bar{k}} T_j - \partial_j \bar{T}_{\bar{k}} + \partial_j \partial_{\bar{k}} \log \|\Omega\|_{\omega}^2. \tag{2.27}$$

The following characterization is due to Fino and Grantcharov, which indicates that conformally balanced metrics can be viewed as non-Kähler analogs of Kähler Ricci-flat metrics.

Theorem 2.2 ([FG04]) *Let (X, ω) be a compact Hermitian manifold with nowhere vanishing holomorphic $(n, 0)$ form Ω. Then (X, ω, Ω) is conformally balanced if and only if*

$$\mathrm{Tr}\, Rm^{+} = 0.$$

Proof From (2.26) and (2.27), we see that manifolds satisfying the torsion constraint in Proposition 2.2 satisfy $\mathrm{Tr}\, Rm^{+} = 0$. For the other direction, we note that by Lemma 2.4, we can write

$$\mathrm{Tr}\, Rm^{+} = i d d_{\chi}^{\dagger} \chi,$$

for $\chi = \|\Omega\|_{\omega}^{1/(n-1)} \omega$. It follows that if $\mathrm{Tr}\, Rm^{+} = 0$, then $\langle d_{\chi}^{\dagger} \chi, d_{\chi}^{\dagger} \chi \rangle_{\chi} = 0$ and hence $d_{\chi}^{\dagger} \chi = 0$. By Lemma 2.4, we conclude $\partial \log \|\Omega\|_{\omega} = \tau$ and hence (X, ω, Ω) is conformally balanced. □

For most subsequent computations, we will be using the Chern connection ∇, so we now turn to curvature of the Chern connection. This tensor satisfies certain useful identities on Calabi–Yau manifolds with torsion that we will now describe.

Proposition 2.3 *The Chern–Ricci curvature of a conformally balanced metric (X, ω, Ω) satisfies*

$$R_{\bar{k} j} = 2 \nabla_{\bar{k}} T_j.$$

Proof The Chern–Ricci curvature is given by

$$R_{\bar{k} j} = \partial_j \partial_{\bar{k}} \log \|\Omega\|_{\omega}^2.$$

Applying the torsion constraint (Proposition 2.2) gives the result. □

As a consequence, we obtain the following identities between Ricci curvatures of the Chern connection.

Proposition 2.4 ([PPZ318]) *A conformally balanced metric (X, ω, Ω) satisfies*

$$R'_{\bar{k} j} = R''_{\bar{k} j} = \frac{1}{2} R_{\bar{k} j},$$

$$R' = \frac{1}{2} R, \quad R = g^{j \bar{k}} \partial_j \partial_{\bar{k}} \log \|\Omega\|_{\omega}^2.$$

Proof By the Bianchi identity (Lemma 2.3),

$$R'_{\bar{k}j} = g^{p\bar{q}} R_{\bar{k}p\bar{q}j} = g^{p\bar{q}} (R_{\bar{k}j\bar{q}p} + \nabla_{\bar{k}} T_{\bar{q}jp}) = R_{\bar{k}j} - \nabla_{\bar{k}} T_j.$$

Applying the previous proposition gives $R'_{\bar{k}j} = \frac{1}{2} R_{\bar{k}j}$. The identity for $R''_{\bar{k}j}$ is derived similarly. Taking the trace gives the relation between the scalar curvatures R and R'. □

From the divergence theorem (2.10), we note in passing that the total scalar curvature of the Chern connection of a Calabi–Yau manifold with torsion is positive. In fact,

$$\int_X R\, \omega^n = \int_X (2|\tau|^2)\, \omega^n.$$

We conclude this section with the remark that in Strominger's work [ST86], the condition $d(\|\Omega\|_\omega \omega^{n-1}) = 0$ appeared in another form. The reformulation of this condition in terms of balanced metrics is due to Li and Yau [LY05].

Theorem 2.3 ([LY05]) *Let (X, ω) be a Hermitian manifold with nowhere vanishing holomorphic $(n, 0)$ form Ω. The conformally balanced condition $d(\|\Omega\|_\omega \omega^{n-1}) = 0$ is equivalent to the equation*

$$d^\dagger \omega = i(\bar{\partial} - \partial) \log \|\Omega\|_\omega.$$

Proof This follows from combining $d^\dagger \omega = i(\bar{\tau} - \tau)$ (2.13) with $\partial \log \|\Omega\|_\omega = \tau$ (Proposition 2.2). □

2.2.2 Rigidity Theorems

We note in this section some conditions under which a Calabi–Yau manifold with torsion is actually Kähler. We start with a result of Ivanov–Papadopoulos [IP01]. The proof given here follows the computation of [PPZ318].

Theorem 2.4 ([IP01]) *Let (X, ω, Ω) be a compact Calabi–Yau manifold with torsion, so that $d(\|\Omega\|_\omega \omega^{n-1}) = 0$. Suppose*

$$i\partial\bar{\partial}\omega = 0.$$

Then ω is a Kähler metric.

Proof We start by computing $i\partial\bar{\partial}\omega$. Its components are

$$i\partial\bar{\partial}\omega = \frac{1}{4} (i\partial\bar{\partial}\omega)_{\bar{i}\bar{j}k\ell}\, dz^\ell \wedge dz^k \wedge d\bar{z}^j \wedge d\bar{z}^i,$$

given explicitly by

$$(i\partial\bar{\partial}\omega)_{\bar{i}\bar{j}k\ell} = \partial_\ell \partial_{\bar{j}} g_{\bar{i}k} - \partial_\ell \partial_{\bar{i}} g_{\bar{j}k} + \partial_k \partial_{\bar{i}} g_{\bar{j}\ell} - \partial_k \partial_{\bar{j}} g_{\bar{i}\ell}.$$

Using the definition of the curvature tensor (2.15) and the torsion (2.4), we find

$$(i\partial\bar{\partial}\omega)_{\bar{i}\bar{j}k\ell} = -R_{\bar{i}k\bar{j}\ell} + R_{\bar{j}k\bar{i}\ell} - R_{\bar{j}\ell\bar{i}k} + R_{\bar{i}\ell\bar{j}k} - g^{s\bar{r}} T_{\bar{r}\ell k} \bar{T}_{s\bar{i}\bar{j}}. \tag{2.28}$$

Setting this expression to zero and contracting the indices, we see that pluriclosed metrics satisfy

$$0 = g^{\ell\bar{j}} g^{k\bar{i}} (i\partial\bar{\partial}\omega)_{\bar{i}\bar{j}k\ell} = 2R' - 2R + |T|^2.$$

Applying Proposition 2.4, we see that if we further assume that ω is conformally balanced, then

$$g^{j\bar{k}} \partial_j \partial_{\bar{k}} \log \|\Omega\|^2_\omega = |T|^2 \geq 0.$$

The maximum principle for elliptic equations (Proposition 2.1) implies that $\log \|\Omega\|^2_\omega$ must be constant, and hence $|T|^2 = 0$. □

Next, we state the result of Fino–Tomassini [FT11], which builds on work of Matsuo–Takahashi [MT01]. We follow here the computation given in [PPZ19].

Theorem 2.5 ([FT11, MT01]) *Let (X, Ω, ω) be a compact Calabi–Yau manifold with torsion of dimension $n \geq 3$, so that $d(\|\Omega\|_\omega \omega^{n-1}) = 0$. Suppose*

$$i\partial\bar{\partial}\omega^{n-2} = 0.$$

Then ω is a Kähler metric.

Proof We assume that $n \geq 4$, since the statement follows from the previous theorem when $n = 3$. Expanding derivatives,

$$i\partial\bar{\partial}\omega^{n-2} = (n-2) i\partial\bar{\partial}\omega \wedge \omega^{n-3} + i(n-2)(n-3) T \wedge \bar{T} \wedge \omega^{n-4}.$$

We will wedge this expression with ω to obtain an equation on top forms. For this, we use the general identities

$$\Phi \wedge \omega^{n-2} = \frac{1}{2n(n-1)} \left\{ g^{i\bar{j}} g^{k\bar{\ell}} \Phi_{\bar{\ell}\bar{j}ki} \right\} \omega^n, \tag{2.29}$$

and

$$\Psi \wedge \omega^{n-3} = -\frac{i}{6n(n-1)(n-2)} \left\{ g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} \Psi_{\bar{n}\bar{\ell}\bar{j}mki} \right\} \omega^n, \tag{2.30}$$

for any $\Phi \in \Omega^{2,2}(X, \mathbf{R})$ and $\Psi \in \Omega^{3,3}(X, \mathbf{R})$, where we use the component convention (2.1). Applying these identities gives

$$\omega \wedge i\partial\bar{\partial}\omega^{n-2} = \left[\frac{(n-2)}{2n(n-1)} g^{i\bar{j}} g^{k\bar{\ell}} (i\partial\bar{\partial}\omega)_{\bar{\ell}\bar{j}ki} + \frac{(n-3)}{6n(n-1)} g^{i\bar{j}} g^{k\bar{\ell}} g^{m\bar{n}} (T \wedge \bar{T})_{\bar{n}\bar{\ell}\bar{j}mki}\right]\omega^n. \tag{2.31}$$

Symmetrizing the components of the torsion tensor T, we see that

$$(T \wedge \bar{T})_{\bar{n}\bar{\ell}\bar{j}mki} = T_{\bar{j}mi}\bar{T}_{k\bar{n}\bar{\ell}} + T_{\bar{\ell}mi}\bar{T}_{k\bar{j}\bar{n}} + T_{\bar{n}mi}\bar{T}_{k\bar{\ell}\bar{j}} + T_{\bar{j}km}\bar{T}_{i\bar{n}\bar{\ell}} + T_{\bar{\ell}km}\bar{T}_{i\bar{j}\bar{n}} + T_{\bar{n}km}\bar{T}_{i\bar{\ell}\bar{j}} + T_{\bar{j}ik}\bar{T}_{m\bar{n}\bar{\ell}} + T_{\bar{\ell}ik}\bar{T}_{m\bar{j}\bar{n}} + T_{\bar{n}ik}\bar{T}_{m\bar{\ell}\bar{j}}. \tag{2.32}$$

Setting (2.31) to zero and substituting the expression (2.28) for $i\partial\bar{\partial}\omega$ and (2.32) for $T \wedge \bar{T}$, we obtain the following identity

$$0 = \frac{(n-2)}{2n(n-1)}(2R' - 2R + |T|^2) + \frac{(n-3)}{6n(n-1)}(6|\tau|^2 - 3|T|^2),$$

satisfied by any astheno-Kähler metric ω. We now use the conformally balanced condition by applying Proposition 2.4, which gives $2R' - 2R = -g^{j\bar{k}}\partial_j\partial_{\bar{k}} \log \|\Omega\|_\omega$. Simplifying, we obtain

$$(n-2)g^{j\bar{k}}\partial_j\partial_{\bar{k}} \log \|\Omega\|_\omega = |T|^2 + 2(n-3)|\tau|^2 \geq 0.$$

By the maximum principle for elliptic equations (Proposition 2.1) we must have $|T|^2 + 2(n-3)|\tau|^2 = 0$. Hence $|T|^2 = 0$ and ω is Kähler. □

There are more theorems of this nature; for other conditions on balanced metrics which imply that it is Kähler, see [FIUV09, LY12, LY17].

A folklore conjecture in the field (e.g. [FV16]) speculates that if a Calabi–Yau with torsion (X, Ω, ω) admits another metric ω_2 which is pluriclosed, then X must be a Kähler. If ω_2 is instead assumed to be astheno-Kähler, then X need not be Kähler [FGV, LU17].

2.2.3 Examples

2.2.3.1 Kähler Calabi–Yau

We have already seen that conformally balanced metrics generalize Kähler Ricci-flat metrics, since they are characterized by vanishing of the Ricci curvature of ∇^+, and ∇^+ coincides with the Levi-Civita connection for Kähler metrics. We note here a simple direct proof that Kähler Ricci-flat metrics are conformally balanced.

Let (X, Ω) be a Kähler Calabi–Yau manifold. By Yau's theorem [YA78], there exists a Kähler metric ω with zero Ricci curvature. In this case, $\|\Omega\|_\omega$ is constant, since

$$i\partial\bar{\partial}\log\|\Omega\|^2_\omega = i\partial\bar{\partial}\log\Omega(z)\overline{\Omega(z)} - i\partial\bar{\partial}\log\det g_{\bar{k}j} = 0,$$

and hence $g^{j\bar{k}}\partial_j\partial_{\bar{k}}\log\|\Omega\|^2_\omega = 0$. By the maximum principle, $\|\Omega\|_\omega$ is constant. Since ω is Kähler, we have $d\omega^{n-1} = 0$, and hence $d(\|\Omega\|_\omega\omega^{n-1}) = 0$.

2.2.3.2 Complex Lie Groups

Next, we study invariant metrics on complex Lie groups, which provide a class of natural non-Kähler metrics. Let G be a complex Lie group. Choose a positive definite inner product on the Lie algebra $\mathbf{g}$, and let $e_1, \ldots, e_n \in \mathbf{g}$ be an orthonormal frame of left-invariant holomorphic vector fields on G. The structure constants of the Lie algebra $\mathbf{g}$ in this basis will be denoted

$$[e_a, e_b] = c^d{}_{ab}e_d.$$

Taking the dual frame $e^1, \ldots, e^n$, we may define a left-invariant metric ω by

$$\omega = i\sum_a e^a \wedge \bar{e}^a.$$

We note that this metric cannot be Kähler unless G is trivial. Indeed, taking the exterior derivative gives

$$\partial e^a = \frac{1}{2}c^a{}_{bd}e^d \wedge e^b. \tag{2.33}$$

Therefore

$$\begin{aligned} i\bar{\partial}\omega &= \frac{1}{2}\overline{c^a{}_{bd}}\, e^a \wedge \bar{e}^d \wedge \bar{e}^b, \\ i\partial\bar{\partial}\omega &= \frac{1}{4}\overline{c^a{}_{bd}}c^a{}_{rs}\, e^s \wedge e^r \wedge \bar{e}^d \wedge \bar{e}^b, \end{aligned} \tag{2.34}$$

so this invariant metric is not Kähler or pluriclosed in general. We take the Calabi–Yau form to be

$$\Omega = e^1 \wedge \cdots \wedge e^n.$$

which is a nowhere vanishing holomorphic $(n, 0)$ form. Using (2.22), we see that

$$\|\Omega\|_\omega = 1.$$

Checking whether ω is conformally balanced reduces to verifying that $d\omega^{n-1} = 0$. This implies a condition of the structure constants, which does not hold for arbitrary Lie groups, but still admits plenty of examples. We say that G is unimodular if its structure constants satisfy

$$\sum_p c^p{}_{pa} = 0.$$

This condition is well-defined on G and does not depend on the choice of frame. It was noted by Abbena and Grassi [AG86] that $d\omega^{n-1} = 0$ if and only if G is unimodular. Indeed, from (2.33) we see that $T^a{}_{bd} = c^a{}_{bd}$. Hence G is unimodular if and only if $T_j = 0$, which holds if and only if ω is conformally balanced by Proposition 2.2.

Thus unimodular complex Lie groups admit left invariant conformally balanced metrics. An explicit example is given by $SL(2, \mathbf{C})$. To obtain a compact threefold, we may quotient out by a discrete group and let $X = SL(2, \mathbf{C})/\Lambda$.

We claim that X does not admit a Kähler metric. For this, we use the fact that $SL(2, \mathbf{C})$ admits a basis e^a such that $c^a{}_{bd} = \epsilon_{abd}$ the Levi-Civita symbol. Let $\omega = i\delta_{ba}\, e^a \wedge \bar{e}^b$, and compute

$$(\omega^2)_{\bar{b}\bar{d}rs} = 2(\delta_{\bar{d}s}\delta_{\bar{b}r} - \delta_{\bar{d}r}\delta_{\bar{b}s}).$$

In dimension 3, we have the contracted epsilon identity

$$\epsilon_{ars}\epsilon_{abd} = \delta_{rb}\delta_{sd} - \delta_{rd}\delta_{bs}. \tag{2.35}$$

Therefore, by (2.34),

$$(i\partial\bar{\partial}\omega)_{\bar{b}\bar{d}rs} = \delta_{ds}\delta_{br} - \delta_{dr}\delta_{bs}.$$

We see that ω^2 and $i\partial\bar{\partial}\omega$ are proportional to each other.

$$i\partial\bar{\partial}\omega = \frac{1}{2}\omega^2. \tag{2.36}$$

This in particular illustrates another difference with Kähler geometry, where ω^2 always represents a non-zero cohomology class. Now suppose X admits a Kähler metric χ. Then

$$0 = \int_X i\partial\bar{\partial}\omega \wedge \chi = \frac{1}{2}\int_X \omega^2 \wedge \chi \tag{2.37}$$

which is a contradiction since $\omega^2 \wedge \chi > 0$.

For more examples of complex Lie groups, Fei–Yau [FY15, Proposition 3.7] classify complex unimodular Lie algebras of dimension 3 and study the Hull–Strominger system in each case. A theorem of Wang [WA54] states that the only compact parallelizable manifolds admitting Kahler metrics are the complex tori.

2.2.3.3 Iwasawa Manifold

We consider the action of $a, b, c \in \mathbf{Z}[i]$ on $\mathbf{C}^3$ given by

$$(x, y, z) \mapsto (x + a, y + c, z + \bar{a}y + b). \tag{2.38}$$

Let X be the quotient of $\mathbf{C}^3$ under this action. The manifold X is an example of an Iwasawa manifold. We have a projection

$$\pi : X \to T^4 = \mathbf{C}/\Lambda \times \mathbf{C}/\Lambda, \quad \pi(x, y, z) = (x, y).$$

Here Λ is the lattice generated by $1, i$. The fibers are isomorphic to tori $\pi^{-1}(x, y) = T^2$. Hence M is a torus fibration over T^4. The form

$$\Omega = dz \wedge dx \wedge dy,$$

is defined on X, and is holomorphic nowhere vanishing. We define

$$\theta = dz - \bar{x}dy.$$

This form on $\mathbf{C}^3$ is invariant under the action (2.38), and is thus well-defined on X. Consider the family of metrics

$$\omega_u = e^u \hat{\omega} + i\theta \wedge \bar{\theta}, \quad \hat{\omega} = i dx \wedge d\bar{x} + i dy \wedge d\bar{y},$$

where $u : T^4 \to \mathbf{R}$ is an arbitrary function on the base T^4. A computation shows that

$$\|\Omega\|_{\omega_u} = e^{-u},$$

and

$$d(\|\Omega\|_{\omega_u} \omega_u^2) = 0.$$

Thus (X, ω_u, Ω) is conformally balanced. However, X does not admit a Kähler metric. Let ω_0 be metric with $u = 0$. Direct computation gives

$$i\partial\bar{\partial}\omega_0 = \frac{\hat{\omega}^2}{2}.$$

We can rule out the existence of a Kähler metric χ by considering $\int_X i\partial\bar{\partial}\omega_0 \wedge \chi$ as in the previous section, see (2.37).

2.2.3.4 Goldstein–Prokushkin Fibrations

In this section, we describe a construction of Goldstein–Prokushkin [GO04] which utilizes $U(1)$ principal bundles to generalize the previous example. Let $(S, \hat{\omega}, \Omega)$ be a Kähler Calabi–Yau surface equipped with two $(1,1)$ form $\omega_1, \omega_2 \in 2\pi H^2(S, \mathbf{Z})$, which are anti-self-dual with respect to $\hat{\omega}$.

$$\star\omega_1 = -\omega_1, \quad \star\omega_2 = -\omega_2.$$

There exists line bundles L_1, L_2 over S with connections A_1, A_2 whose curvature iF_{A_1}, iF_{A_2} is equal to ω_1, ω_2. As detailed in Sect. 2.1.4, the line bundles L_1, L_2 can be compactified to form S^1 principal bundles $P_1 \to S$, $P_2 \to S$ equipped with connection 1-forms θ_1, θ_2 satisfying

$$d\theta_i = -\omega_i.$$

Let X denote the total space of the $S^1 \times S^1$ principal bundle $\pi : X \to S$ whose fibers are the product of the fibers of P_1, P_2. Locally, points of X are given by $(z, e^{i\psi_1}, e^{i\psi_2})$. As we discussed in Sect. 2.1.4, we have the global vector fields

$$\frac{\partial}{\partial\psi_1}, \quad \frac{\partial}{\partial\psi_2},$$

which span the vertical space $V = \ker \pi_*$, and satisfy

$$\theta_1\left(\frac{\partial}{\partial\psi_1}\right) = 1, \quad \theta_2\left(\frac{\partial}{\partial\psi_2}\right) = 1.$$

The horizontal space is given by

$$H = \ker\theta_1 \cap \ker\theta_2,$$

and the tangent space admits the decomposition

$$TX = H \oplus V.$$

Furthermore

$$\pi_\star|_H : H \to TS$$

is an isomorphism. It follows that the complex structure j_S on S induces an almost complex structure on H. We define on X the almost complex structure

$$J = (\pi^*|_H j_S) \oplus I, \quad I\frac{\partial}{\partial \psi_1} = \frac{\partial}{\partial \psi_2}, \; I\frac{\partial}{\partial \psi_2} = -\frac{\partial}{\partial \psi_1}.$$

We define the 1-form

$$\theta = -\theta_1 - i\theta_2.$$

Since $\theta|_H = 0$ and $\theta(\partial_{\psi_1} + i\partial_{\psi_2}) = 0$, we see that $\theta(V) = 0$ for any $V \in T^{0,1}X$. Thus θ is a $(1, 0)$ form. Furthermore,

$$d\theta = \pi^*(\omega_1 + i\omega_2).$$

Similarly to our discussion of Eq. (2.21) in Sect. 2.1.4.2, we can use that $(1, 0)$ forms are locally generated by $\{\pi^* dz^1, \pi^* dz^2, \theta\}$ to apply the Newlander–Nirenberg theorem and establish that J integrable. Thus X is a compact complex manifold of dimension 3.

In fact, X is a Calabi–Yau manifold with torsion. Let

$$\Omega = \theta \wedge \pi^*\Omega_S,$$

which is a nowhere vanishing $(3, 0)$ form. The form Ω is holomorphic since $d\Omega = 0$.

For $u \in C^\infty(S, \mathbf{R})$, we consider the family of metrics

$$\omega_u = \pi^*(e^u\hat{\omega}) + i\theta \wedge \bar{\theta}.$$

These metrics will be revisited, as they form the Fu–Yau ansatz of solutions to the Hull–Strominger system [FY08]. We compute

$$i\Omega \wedge \bar{\Omega} = i\theta \wedge \bar{\theta} \wedge \pi^*(\Omega_S \wedge \overline{\Omega_S}) = i\theta \wedge \bar{\theta} \wedge \pi^*\left(\|\Omega_S\|^2_{\hat{\omega}} \frac{\hat{\omega}^2}{2}\right),$$

$$\omega_u^2 = \pi^*(e^{2u}\hat{\omega}^2) + 2\pi^*(e^u\hat{\omega}) \wedge i\theta \wedge \bar{\theta}, \quad \omega_u^3 = 3\pi^*(e^{2u}\hat{\omega}^2) \wedge i\theta \wedge \bar{\theta}.$$

Since $(S, \hat{\omega})$ is Kähler Ricci-flat, then $\|\Omega_S\|_{\hat{\omega}}$ is constant, which we may normalize such that

$$\|\Omega\|_{\omega_u} = e^{-u}. \tag{2.39}$$

We can now compute

$$\begin{aligned} d(\|\Omega\|_{\omega_u}\omega_u^2) &= d(\pi^*(e^u\hat{\omega}^2) + 2\pi^*\hat{\omega}\wedge i\theta\wedge\bar{\theta}) \\ &= 2\pi^*\hat{\omega}\wedge i\pi^*(\omega_1 + i\omega_2)\wedge\bar{\theta} - 2\pi^*\hat{\omega}\wedge i\theta\wedge\pi^*(\omega_1 - i\omega_2) \\ &= 0, \end{aligned}$$

since

$$\hat{\omega}\wedge\omega_1 = \hat{\omega}\wedge\omega_2 = 0,$$

as ω_1, ω_2 are anti-self-dual. Thus (X, ω_u, Ω) is Calabi–Yau with torsion. In fact, X is non-Kähler unless $\omega_1 = \omega_2 = 0$. To see this, we compute

$$i\partial\bar{\partial}\omega_0 = -\bar{\partial}\theta\wedge\partial\bar{\theta} = -(\pi^*\omega_1 + i\pi^*\omega_2)(\pi^*\omega_1 - i\pi^*\omega_2) = -\pi^*(\omega_1^2 + \omega_2^2).$$

Since ω_1, ω_2 are anti-self-dual,

$$i\partial\bar{\partial}\omega_0 = \pi^*(\omega_1\wedge\star\omega_1 + \omega_2\wedge\star\omega_2).$$

If X admits a Kähler metric χ, then

$$0 = \int_X i\partial\bar{\partial}\omega_0\wedge\chi = \int_X \pi^*(\omega_1\wedge\star\omega_1 + \omega_2\wedge\star\omega_2)\wedge\chi,$$

which is strictly positive unless $\|\omega_1\|^2_{\hat{\omega}} = \|\omega_2\|^2_{\hat{\omega}} = 0$.

2.2.3.5 Fei Twistor Space

As our last example, we outline a construction of Fei [FE16, FE15] which generalizes earlier constructions of Calabi [CA58] and Gray [GR69]. The example will be a T^4 fibration over a Riemann surface.

We first describe the base of the fibration. Let (Σ, φ) be a Riemann surface equipped with a nonconstant holomorphic map $\varphi : \Sigma \to \mathbf{P}^1$ satisfying $\varphi^*\mathcal{O}(2) = K_\Sigma$. This condition is known to imply that the genus of Σ must be at least three. As a concrete example, we may take Σ to be a minimal surface in T^3 with φ being the Gauss map [FHP17]. By the work of Meeks [ME90] and Traizet [TR08], there exists minimal surfaces of genus $g \geq 3$ in T^3.

Using stereographic coordinates, we may write $\varphi = (\alpha, \beta, \gamma)$ with $(\alpha, \beta, \gamma) \in S^2 \subseteq \mathbf{R}^3$. Fixing the Fubini-Study metric ω_{FS} on $\mathbf{P}^1$, we pullback via φ an orthonormal basis of sections of $\mathcal{O}(2)$ to obtain 1-forms μ_1, μ_2, μ_3. We then equip Σ with the metric

$$\hat{\omega} = i\mu_1\wedge\bar{\mu}_1 + i\mu_2\wedge\bar{\mu}_2 + i\mu_3\wedge\bar{\mu}_3.$$

This metric has Gauss curvature κ given by

$$\kappa\hat{\omega} = -\varphi^*\omega_{FS},$$

hence $\kappa \leq 0$ and κ vanishes at branch points of φ.

We now describe the fibers. Let (T^4, g) be the 4-torus with flat metric, which we will view as a hyperkähler manifold with complex structures I, J, K satisfying $IJ = K$ and $I^2 = J^2 = K^2 = -1$, and corresponding Kähler metrics ω_I, ω_J, ω_K. At each $z \in \Sigma$, we use the map $\varphi = (\alpha, \beta, \gamma)$ to equip T^4 with the complex structure

$$\alpha I + \beta J + \gamma K.$$

If j_Σ denotes the complex structure on Σ, we may form the product $X = \Sigma \times T^4$ and equip it with the complex structure

$$J_0 = j_\Sigma \oplus (\alpha I + \beta J + \gamma K).$$

This complex structure is integrable, thus X is a compact complex manifold of dimension 3. In fact, X has trivial canonical bundle, and we can give an explicit expression for a nowhere vanishing holomorphic (3, 0) form

$$\Omega = \mu_1 \wedge \omega_I + \mu_2 \wedge \omega_J + \mu_3 \wedge \omega_K.$$

Let

$$\omega' = \alpha\omega_I + \beta\omega_J + \gamma\omega_K$$

be the Kähler metric corresponding to the complex structure $\alpha I + \beta J + \gamma K$ on T^4. The Fei ansatz ω_f on X is the following family of conformally balanced metrics.

Proposition 2.5 ([FE16, FE15]) *Given any $f \in C^\infty(\Sigma, \mathbf{R})$, the Hermitian metric given by*

$$\omega_f = e^{2f}\hat{\omega} + e^f\omega',$$

is conformally balanced. Furthermore, $\|\Omega\|_{\omega_f} = e^{-2f}$.

Thus X is Calabi–Yau with torsion, and in fact, it is non-Kähler.

2.2.3.6 Other Examples

We have now discussed many examples of Calabi–Yau manifolds with balanced metrics, many of which were already listed in the pioneering work of Michelsohn [MI82]. There are also example which will not be studied in these notes. For example, there is the construction of Fu et al. [FLY12] on connected sums of $S^3 \times S^3$. There are parallelizable examples on nilmanifolds and solvmanifolds [UG07, OUV17, FIUV09, FG04, UV14, UV15]. Non-compact examples are constructed in [FY09, FE17, FIUV14]. There are also examples from the physics literature, e.g. [BD02, BBDG03, DRS99, HIS16, MS11].

2.3 Anomaly Flow with Zero Slope

In this section, we will discuss a geometric flow which preserves the geometry described in Sect. 2.2. The material in this section can be found in joint work with Phong and Zhang [PPZ218, PPZ318, PPZ19].

A central problem in complex geometry is to detect when a given complex manifold admits a Kähler metric. We would like to study this question on Calabi–Yau manifolds with torsion. Motivated by Sect. 2.2.2, we will deform conformally balanced metrics towards astheno-Kähler ($i\partial\bar\partial\omega^{n-2} = 0$).

Together with Phong and Zhang [PPZ19], we introduce the flow

$$\frac{d}{dt}(\|\Omega\|_\omega \omega^{n-1}) = i\partial\bar\partial\omega^{n-2},$$
$$d(\|\Omega\|_{\omega(0)}\omega(0)^{n-1}) = 0. \tag{2.40}$$

We call this evolution equation the Anomaly flow with zero slope. The name comes from an extension of the flow which adds higher order correction terms proportional to a parameter α', which is used to study the Hull–Strominger system and the cancellation of anomalies in theoretical physics. We will discuss the Anomaly flow when α' terms are included in Sect. 2.4.

The first thing to note is that the conformally balanced property is preserved by the flow

$$d(\|\Omega\|_{\omega(t)}\omega(t)^{n-1}) = 0,$$

which follows from taking the exterior derivative of (2.40). In fact, the balanced class of the initial metric

$$[\|\Omega\|_{\omega(0)}\omega(0)^{n-1}] \in H_{BC}^{n-1,n-1}(X, \mathbf{R})$$

is also preserved, since

$$\frac{d}{dt}[\|\Omega\|_\omega \omega^{n-1}] = [i\partial\bar\partial\omega^{n-2}] = 0. \tag{2.41}$$

Here $H_{BC}^{n-1,n-1}(X)$ is the Bott–Chern cohomology of X, given by

$$H_{BC}^{n-1,n-1}(X) = \frac{\{\alpha \in \Omega^{n-1,n-1}(X) : d\alpha = 0\}}{\{i\partial\bar{\partial}\beta : \beta \in \Omega^{n-2,n-2}(X)\}}.$$

Stationary points ω_∞ of the flow satisfy both

$$d(\|\Omega\|_{\omega_\infty}\omega_\infty^{n-1}) = 0, \;\; i\partial\bar{\partial}\omega_\infty^{n-2} = 0,$$

hence by Theorem 2.5, they are Kähler. The Anomaly flow with zero slope thus deforms balanced metrics to a Kähler metric in a given balanced class.

2.3.1 Evolution of the Metric

The first question to ask about the flow (2.40) is whether it exists for a short-time, and if so, we would like an explicit expression for the evolution equation of the metric $\omega = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$.

We begin by deriving the evolution of the determinant of the metric.

Lemma 2.6 *Suppose $\omega(t) = ig_{\bar{k}j}dz^j \wedge d\bar{z}^k$ satisfies the evolution equation*

$$\frac{d}{dt}(\|\Omega\|_\omega\omega^{n-1}) = \Psi(t), \tag{2.42}$$

for $\Psi(t) \in \Omega^{n-1,n-1}(X, \mathbf{R})$. Then the norm of Ω evolves by

$$\frac{d}{dt}\|\Omega\|_\omega = -\frac{n}{(n-2)}\frac{\Psi \wedge \omega}{\omega^n},$$

which follows from the identity

$$\mathrm{Tr}\,\dot{\omega} = \frac{2n}{(n-2)\|\Omega\|_\omega}\frac{\Psi \wedge \omega}{\omega^n}.$$

From now on, traces will always be taken with respect to the evolving metric ω. Explicitly,

$$\mathrm{Tr}\,\alpha = i^{-1}g^{j\bar{k}}\alpha_{\bar{k}j},$$

for a $(1, 1)$ form $\alpha = \alpha_{\bar{k}j}dz^j \wedge d\bar{z}^k$.

Proof Using the well-known formula

$$\delta \det g_{\bar{k}j} = (\det g_{\bar{k}j}) g^{j\bar{k}} (\delta g)_{\bar{k}j},$$

we differentiate

$$\frac{d}{dt} \|\Omega\|_\omega = \frac{d}{dt} (\Omega \bar{\Omega})^{1/2} (\det g)^{-1/2} = -\frac{1}{2} \|\Omega\|_\omega \mathrm{Tr}\, \dot{\omega}.$$

Expanding (2.42), we obtain

$$\left(\frac{d}{dt} \|\Omega\|_\omega \right) \omega^{n-1} + (n-1) \|\Omega\|_\omega \dot{\omega} \wedge \omega^{n-2} = \Psi.$$

Substituting the variation of $\|\Omega\|_\omega$ gives

$$-\frac{1}{2} \|\Omega\|_\omega (\mathrm{Tr}\, \dot{\omega}) \omega^{n-1} + (n-1) \|\Omega\|_\omega \dot{\omega} \wedge \omega^{n-2} = \Psi. \tag{2.43}$$

Next, we wedge this equation with ω to obtain the following equation of top forms.

$$-\frac{1}{2} \|\Omega\|_\omega (\mathrm{Tr}\, \dot{\omega}) \omega^n + (n-1) \|\Omega\|_\omega \frac{(\mathrm{Tr}\, \dot{\omega})}{n} \omega^n = \Psi \wedge \omega.$$

From this equation we can solve for $\mathrm{Tr}\, \dot{\omega}$. □

Lemma 2.7 *Suppose $\omega(t)$ satisfies*

$$\frac{d}{dt} (\|\Omega\|_\omega \omega^{n-1}) = \Psi(t),$$

for $\Psi(t) \in \Omega^{n-1,n-1}(X, \mathbf{R})$. Then the metric evolves by

$$\partial_t \omega = \left[\frac{n}{(n-2)\|\Omega\|_\omega} \frac{\Psi \wedge \omega}{\omega^n} \right] \omega - \frac{1}{(n-1)! \|\Omega\|_\omega} \star \Psi.$$

Proof To extract $\partial_t \omega$, we will apply the Hodge star operator $\star$ with respect to ω to the expanded equation (2.43).

$$-\frac{(n-1)!}{2} \|\Omega\|_\omega (\mathrm{Tr}\, \dot{\omega}) \omega + (n-1)! \|\Omega\|_\omega (-\partial_t \omega + (\mathrm{Tr}\, \dot{\omega}) \omega) = \star \Psi$$

Here we used the identities $\star \omega^{n-1} = (n-1)! \omega$ and

$$[\star(\alpha \wedge \omega^{n-2})]_{\bar{q}p} = -(n-2)! \alpha_{\bar{q}p} + i(n-2)! (\mathrm{Tr}\, \alpha) g_{\bar{q}p}, \tag{2.44}$$

for any $\alpha \in \Omega^{1,1}(X)$. This last identity can be found in e.g. [HU305, PPZ318]. Therefore

$$\partial_t \omega = \frac{1}{2}(\mathrm{Tr}\,\dot{\omega})\omega - \frac{1}{(n-1)!\|\Omega\|_\omega} \star \Psi.$$

Substituting the previous lemma gives the desired expression. □

For the Anomaly flow with zero slope, the form Ψ is given by

$$\Psi = i\partial\bar{\partial}\omega^{n-2} = (n-2)i\partial\bar{\partial}\omega \wedge \omega^{n-3} + i(n-2)(n-3)T \wedge \bar{T} \wedge \omega^{n-4}. \tag{2.45}$$

To obtain an explicit expression for the evolution of the metric, we must expand the torsion terms.

Theorem 2.6 ([PPZ19]) *Suppose $\omega(t)$ solves the Anomaly flow*

$$\frac{d}{dt}(\|\Omega\|_\omega \omega^{n-1}) = i\partial\bar{\partial}(\omega^{n-2}), \quad d(\|\Omega\|_{\omega(0)}\omega(0)^{n-1}) = 0.$$

If $n = 3$, then the metric evolves via

$$\partial_t g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega}\Big[- \tilde{R}_{\bar{k}j} + g^{m\bar{\ell}} g^{s\bar{r}} T_{\bar{r}mj} \bar{T}_{s\bar{\ell}\bar{k}} \Big],$$

and if $n \geq 4$, then

$$\partial_t g_{\bar{k}j} = \frac{1}{(n-1)\|\Omega\|_\omega}\Big[- \tilde{R}_{\bar{k}j} + \frac{1}{2(n-2)}(|T|^2 - 2|\tau|^2)\, g_{\bar{k}j} - \frac{1}{2} g^{q\bar{p}} g^{s\bar{r}} T_{\bar{k}qs} \bar{T}_{j\bar{p}\bar{r}} + g^{s\bar{r}}(T_{\bar{k}js}\bar{T}_{\bar{r}} + T_s \bar{T}_{j\bar{k}\bar{r}}) + T_j \bar{T}_{\bar{k}} \Big]. \tag{2.46}$$

The metric evolution can be compared with other flows in Hermitian geometry, e.g. [ST10, ST11, TW15, US16, ZH16]. The expression when $n = 3$ is similar to the metric evolution in the Streets–Tian pluriclosed flow [ST10], though they differ by the presence of the determinant of the metric $\|\Omega\|_\omega$. We note that the Anomaly flow is a flow of balanced metrics while the pluriclosed flow is a flow of pluriclosed metrics, so these flows exist in different realms of Hermitian geometry. Such torsion-type terms appearing in (2.46) also appear in other Ricci flows preserving other types of geometry, such as for example the metric evolution in the G2 Laplacian flow [KA09, BR05].

Proof We will derive the expression assuming that $n \geq 4$, as the case $n = 3$ is easier and follows a similar argument. We use the notation

$$\mathrm{Tr}\,\Phi = i^{-2} g^{p\bar{q}} g^{j\bar{k}} \Phi_{\bar{k}j\bar{q}p}, \quad \mathrm{Tr}\,\Psi = i^{-3} g^{j\bar{k}} g^{p\bar{q}} g^{s\bar{r}} \Psi_{\bar{r}s\bar{q}p\bar{k}j},$$

for $\Phi \in \Omega^{2,2}(X)$ and $\Psi \in \Omega^{3,3}(X)$. We begin by computing

$$\begin{aligned}(\star\, i\partial\bar\partial\omega^{n-2})_{\bar q p} &= (n-2)[\star\,(i\partial\bar\partial\omega\wedge\omega^{n-3})]_{\bar q p} + i(n-2)(n-3)[\star\,(T\wedge\bar T\wedge\omega^{n-4})]_{\bar q p}\\ &= i(n-2)!g^{s\bar r}(i\partial\bar\partial\omega)_{\bar r s\bar q p} + i\frac{(n-2)!}{2}(\mathrm{Tr}\, i\partial\bar\partial\omega)g_{\bar q p}\\ &\quad + i\frac{(n-2)!}{2}g^{i\bar j}g^{s\bar r}(T\wedge\bar T)_{\bar r s\bar j i\bar q p} - \frac{(n-2)!}{6}(\mathrm{Tr}\, T\wedge\bar T)g_{\bar q p}.\end{aligned} \tag{2.47}$$

This follows from (2.45) and the following identities for the Hodge star operator

$$\begin{aligned}[\star(\Phi\wedge\omega^{n-3})]_{\bar q p} &= i(n-3)!g^{s\bar r}\Phi_{\bar r s\bar q p} + i\frac{(n-3)!}{2}(\mathrm{Tr}\,\Phi)g_{\bar q p},\\ [\star(\Psi\wedge\omega^{n-4})]_{\bar q p} &= \frac{(n-4)!}{2}g^{i\bar j}g^{s\bar r}\Psi_{\bar r s\bar j i\bar q p} + i\frac{(n-4)!}{6}(\mathrm{Tr}\,\Psi)g_{\bar q p},\end{aligned} \tag{2.48}$$

which hold for any $\Phi \in \Omega^{2,2}(X, \mathbf{R})$ and $\Psi \in \Omega^{3,3}(X, \mathbf{R})$. For a proof of these Hodge star identities, see [PPZ19].

Next, we compute using (2.29) and (2.30),

$$\begin{aligned}\frac{i\partial\bar\partial\omega^{n-2}\wedge\omega}{\omega^n} &= (n-2)\frac{i\partial\bar\partial\omega\wedge\omega^{n-2}}{\omega^n} + i(n-2)(n-3)\frac{T\wedge\bar T\wedge\omega^{n-3}}{\omega^n}\\ &= \frac{(n-2)}{2n(n-1)}\mathrm{Tr}\,(i\partial\bar\partial\omega) + \frac{i(n-3)}{6n(n-1)}\mathrm{Tr}\,(T\wedge\bar T).\end{aligned} \tag{2.49}$$

We now substitute (2.47) and (2.49) into Lemma 2.7. The $\mathrm{Tr}\,(i\partial\bar\partial\omega)$ terms cancel exactly, and we are left with

$$\begin{aligned}\partial_t g_{\bar q p} = &-\frac{1}{(n-1)\|\Omega\|_\omega}g^{s\bar r}(i\partial\bar\partial\omega)_{\bar r s\bar q p} - \frac{1}{2(n-1)\|\Omega\|_\omega}g^{i\bar j}g^{s\bar r}(T\wedge\bar T)_{\bar r s\bar j i\bar q p}\\ &-\frac{i}{6(n-1)(n-2)\|\Omega\|_\omega}\mathrm{Tr}\,(T\wedge\bar T)\, g_{\bar q p}.\end{aligned} \tag{2.50}$$

By identity (2.28), we have

$$g^{s\bar r}(i\partial\bar\partial\omega)_{\bar r\bar q sp} = -\tilde R_{\bar q p} + R'_{\bar q p} - R_{\bar q p} + R''_{p\bar q} - g^{s\bar r}g^{n\bar m}T_{\bar m ps}\bar T_{n\bar r\bar q}.$$

We now use that the evolving metrics are conformally balanced. In this case, by Proposition 2.4, we have

$$g^{s\bar r}(i\partial\bar\partial\omega)_{\bar r s\bar q p} = \tilde R_{\bar q p} - g^{s\bar r}g^{n\bar m}T_{\bar m sp}\bar T_{n\bar r\bar q}. \tag{2.51}$$

Substituting (2.51) and (2.32) into (2.50) and expanding the torsion terms gives the explicit expression for $\partial_t g_{\bar{q}p}$. □

As a consequence of Theorem 2.6, the Anomaly flow with zero slope exists for a short-time from any initial metric. Indeed, from (2.15) we have

$$\tilde{R}_{\bar{m}\ell} = -g^{j\bar{k}}\partial_j\partial_{\bar{k}}g_{\bar{m}\ell} + g^{j\bar{k}}g^{s\bar{r}}\partial_{\bar{k}}g_{\bar{m}s}\partial_j g_{\bar{r}\ell}, \tag{2.52}$$

and so $\tilde{R}_{\bar{m}\ell}(g)$ is an elliptic operator in g. There is a slight subtlety, which is that the proof of Theorem 2.6 only shows that the Anomaly flow with zero slope is parabolic when restricted to variations in the space of conformally balanced metrics. One way to resolve this issue is by using the Hamilton–Nash–Moser [HA82] implicit function theorem, and we refer to [PPZ116, PPZ19] for details.

Corollary 2.2 ([PPZ19]) *Let ω_0 be a conformally balanced Hermitian metric. There exists an $\epsilon > 0$ such that Anomaly flow with zero slope admits a unique solution on $[0, \epsilon)$ with $\omega(0) = \omega_0$.*

2.3.2 Non-Kähler Examples

We outline here some simple examples to illustrate possible behaviors of the flow.

2.3.2.1 Iwasawa Manifold

Let $\pi : X \to T^4$ be the Iwasawa manifold considered in Sect. 2.2.3.3 with ansatz $\omega_u = e^u\hat{\omega} + i\theta \wedge \bar{\theta}$, where

$$\hat{\omega} = idx \wedge d\bar{x} + idy \wedge d\bar{y}, \quad \theta = dz - \bar{x}dy,$$

and $u(x, y)$ is a smooth function $u : T^4 \to \mathbf{R}$. We will show that this ansatz is preserved by the Anomaly flow. We previously computed that $\|\Omega\|_{\omega_u} = e^{-u}$, and so

$$\|\Omega\|_{\omega_u}\omega_u^2 = e^u\hat{\omega}^2 + 2i\hat{\omega} \wedge \theta \wedge \bar{\theta}.$$

Furthermore,

$$i\partial\bar{\partial}\omega_u = i\partial\bar{\partial}e^u \wedge \hat{\omega} + \frac{\hat{\omega}^2}{2}.$$

The Anomaly flow with zero slope $\partial_t(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega$ reduces to

$$\partial_t e^u = \frac{1}{2}(\Delta_{\hat{\omega}} e^u + 1). \tag{2.53}$$

The flow exists for all time by linear parabolic theory. The functional defined by

$$M(\omega(t)) = \int_X \|\Omega\|_{\omega(t)}\, \omega(t)^3,$$

satisfies in this case

$$\begin{aligned}\frac{d}{dt}M(t) &= \frac{d}{dt}\int_X 3e^u \hat{\omega}^2 \wedge i\theta \wedge \bar{\theta} \\ &= 3\int_X i\partial\bar{\partial}(e^u \hat{\omega} \wedge i\theta \wedge \bar{\theta}) + \frac{3}{2}\int_X \hat{\omega}^2 \wedge i\theta \wedge \bar{\theta} \\ &= \frac{1}{2}\int_X (\hat{\omega} + i\theta \wedge \bar{\theta})^3 > 0.\end{aligned}$$

It follows that $M(t) \to \infty$ linearly as $t \to \infty$. The functional $M(\omega)$ is sometimes called the dilaton functional, and was introduced in [GRST18] to develop a variational formulation of the Hull–Strominger system.

Since (2.53) is a linear parabolic equation and $\int e^u \to \infty$ as $t \to \infty$, we also have that $e^u \to \infty$ everywhere on T^4 as $t \to \infty$. The geometric statement is that $\|\Omega\|_{\omega_u} \to 0$ everywhere on the base T^4. The flow cannot converge in this case since the Iwasawa manifold does not admit a Kähler metric.

2.3.2.2 Compact Quotients of $SL(2, \mathbf{C})$

Next, we study quotients of $SL(2, \mathbf{C})$ by a lattice Λ as described in Sect. 2.2.3.2. Let $\{e_a\}$ be a left-invariant basis of holomorphic vector fields with $[e_a, e_b] = \epsilon_{abd} e_d$. We will study the ansatz

$$\omega = \rho\hat{\omega}, \quad \hat{\omega} = ie^a \wedge \bar{e}^a,$$

where $\rho > 0$ is a constant. This ansatz was used by Fei–Yau to solve the Hull–Strominger system on complex Lie groups [FY15].

As computed in (2.36),

$$i\partial\bar{\partial}\omega = \rho\frac{\hat{\omega}^2}{2}.$$

Next, we compute using the definition of the norm (2.22) and obtain

$$\|\Omega\|_\omega = \rho^{-3/2}.$$

Thus

$$\|\Omega\|_\omega \omega^2 = (\rho^{-3/2}\rho^2)\hat{\omega}^2.$$

Using the ansatz $\omega = \rho\hat{\omega}$ on $X = SL(2, \mathbf{C})/\Lambda$, the Anomaly flow with zero slope becomes the ODE

$$\frac{d}{dt}(\rho^{1/2}) = \frac{1}{2}\rho,$$

whose solution is given by

$$\rho(t) = \frac{1}{(\rho(0)^{-1/2} - \frac{t}{2})^2}.$$

We see that the flow develops a singularity as $\rho \to \infty$ in finite time. In particular, there exists $T < \infty$ such that $\|\Omega\|_\omega \to 0$ as $t \to T$. The flow cannot converge since X does not admit a Kähler metric.

2.3.3 Kähler Manifolds

The previous two examples illustrate how the Anomaly flow can develop singularities on non-Kähler manifolds. If the manifold is already known to admit a Kähler metric, the flow should detect it. Since there are many different Kähler metrics on a given Kähler manifold, the flow must select a single one in the limit. We will explain this mechanism in this section and explain how the flow may provide insight in studying the relation between the Kähler cone and the balanced cone.

Let X be a compact complex manifold with Kähler metric $\hat{\chi} = i\hat{\chi}_{\bar{k}j} dz^j \wedge d\bar{z}^k$ and nowhere vanishing holomorphic $(n, 0)$ form Ω. We will start the Anomaly flow with zero slope with the initial data

$$\|\Omega\|_{\omega(0)}\omega(0)^{n-1} = \hat{\chi}^{n-1}. \tag{2.54}$$

This equation determines the initial metric $\omega(0)$, which is manifestly conformally balanced and is explicitly given by the following lemma.

Lemma 2.8 *Let $\chi \in \Omega^{1,1}(X, \mathbf{R})$ be a Hermitian metric and $\Omega \in \Omega^{n,0}(X)$ be nowhere vanishing. The equation*

$$\|\Omega\|_\omega \omega^{n-1} = \chi^{n-1} \tag{2.55}$$

admits a unique Hermitian metric solution ω *given by*

$$\omega = \|\Omega\|_\chi^{-2/(n-2)} \chi.$$

Proof We let

$$\omega = \|\Omega\|_\omega^{-1/(n-1)} \chi, \tag{2.56}$$

and so we only need to solve for the determinant. Taking the determinant of both sides of (2.55) and raising to the power of $\frac{-1}{(n-1)}$ gives

$$\|\Omega\|_\omega^{-n/(n-1)} (\det \omega)^{-1} = (\det \chi)^{-1}.$$

Recall that $\|\Omega\|_\omega^2 = \Omega\bar{\Omega}(\det \omega)^{-1}$. Multiplying both sides by $\Omega\bar{\Omega}$, we obtain

$$\|\Omega\|_\omega^2 \|\Omega\|_\omega^{-n/(n-1)} = \|\Omega\|_\chi^2.$$

Therefore

$$\|\Omega\|_\omega^{1/(n-1)} = \|\Omega\|_\chi^{2/(n-2)}, \tag{2.57}$$

and the existence result follows from (2.56). For uniqueness, suppose ω and $\tilde{\omega}$ solve (2.55). Then (2.57) determines $\|\Omega\|_\omega = \|\Omega\|_{\tilde{\omega}}$ and so $\tilde{\omega}^{n-1} = \omega^{n-1}$, from which it follows [MI82] that $\omega = \tilde{\omega}$. □

We claim that the solution to the Anomaly flow with zero slope and initial data (2.54) is given by

$$\|\Omega\|_{\omega(t)} \omega(t)^{n-1} = \chi(t)^n, \tag{2.58}$$

where

$$\chi = \hat{\chi} + i\partial\bar{\partial}\varphi > 0,$$

and the scalar potential φ satisfies

$$\dot{\varphi} = e^{-f} \frac{(\hat{\chi} + i\partial\bar{\partial}\varphi)^n}{\hat{\chi}^n}, \quad \varphi(x, 0) = 0,$$

(we use the notation $\dot{\varphi} = \partial_t \varphi$), with

$$e^{-f} = \frac{1}{(n-1)\|\Omega\|_{\hat{\chi}}^2}.$$

Indeed, the ansatz (2.58) solves the equation of the flow. To see this, we compute

$$\begin{aligned}\frac{d}{dt}\|\Omega\|_\omega \omega^{n-1} &= (n-1)\dot{\chi} \wedge \chi^{n-2} \\ &= (n-1) i\partial\bar{\partial}\dot{\varphi} \wedge \chi^{n-2}.\end{aligned}$$

The equation for $\dot{\varphi}$ can be rearranged as

$$\dot{\varphi} = \frac{1}{(n-1)\|\Omega\|_\chi^2}.$$

Therefore

$$\frac{d}{dt}\|\Omega\|_\omega \omega^{n-1} = i\partial\bar{\partial}(\|\Omega\|_\chi^{-2}) \wedge \chi^{n-2}.$$

On the other hand, by Lemma 2.8, we have

$$\begin{aligned}i\partial\bar{\partial}\omega^{n-2} &= i\partial\bar{\partial}(\|\Omega\|_\chi^{-2}\chi^{n-2}) \\ &= i\partial\bar{\partial}(\|\Omega\|_\chi^{-2}) \wedge \chi^{n-2}.\end{aligned}$$

It follows that the ansatz (2.58) satisfies

$$\frac{d}{dt}\|\Omega\|_\omega \omega^{n-1} = i\partial\bar{\partial}\omega^{n-2}.$$

By uniqueness of solutions, the ansatz (2.58) is preserved by the Anomaly flow with zero slope. To summarize our discussion, we state the following result.

Theorem 2.7 ([PPZ19]) *Let X be a compact complex manifold of dimension n with a nowhere vanishing holomorphic $(n, 0)$ form Ω. Suppose X admits a Kähler metric $\hat{\chi}$. Then the Anomaly flow $\frac{d}{dt}\|\Omega\|_\omega \omega^{n-1} = i\partial\bar{\partial}\omega^{n-2}$ with initial metric satisfying*

$$\|\Omega\|_{\omega(0)}\omega(0)^{n-1} = \hat{\chi}^{n-1} \tag{2.59}$$

reduces to the following scalar flow of potentials

$$\dot{\varphi} = e^{-f}\frac{\det(\hat{\chi}_{\bar{k}j} + \varphi_{\bar{k}j})}{\det \hat{\chi}_{\bar{k}j}}, \quad \varphi(x,0) = 0, \tag{2.60}$$

with the positivity condition $\hat{\chi} + i\partial\bar{\partial}\varphi > 0$, where $e^f = (n-1)\|\Omega\|_{\hat{\chi}}^2$. The evolving metric in the Anomaly flow is given by

$$\omega(t) = \|\Omega\|_{\chi(t)}^{-2/(n-2)}\chi(t), \quad \chi(t) = \hat{\chi} + i\partial\bar{\partial}\varphi. \tag{2.61}$$

The Monge–Ampère flow (2.60) arising here shares similarities with the Kähler–Ricci flow and the MA^{-1} flow. The Kähler–Ricci flow was introduced by Cao [CA852] and has since been an area of active research in Kähler geometry (e.g. [CSW18, DL17, GZ17, PS06, PT17, ST07, SW13, TZ06, TZ16]). The MA^{-1} flow was recently introduced by Collins–Hisamoto–Takahashi [CHT18], and is expected to produce optimal degenerations on Fano manifolds which do not admit Kähler-Einstein metrics.

Unlike the Kähler–Ricci flow, the logarithm does not appear in the speed of evolution $\dot{\varphi}$, and unlike the MA^{-1} flow, the determinant of χ appears in the numerator instead of the denominator. For general parabolic equations, changes in speed can have major implications in the analysis, see [FGP18] for a recent example of this phenomenon in Kähler geometry. Though the analysis of (2.60) does differ from the Kähler–Ricci flow and MA^{-1} flow, in [PPZ19] we show that a smooth solution to the flow exists for all time t.

In contrast to the previous examples in section Sect. 2.3.2, in this case we can easily show that $\|\Omega\|_\omega$ stays bounded above and below along the flow. Differentiating (2.60),

$$\partial_t \dot{\varphi} = e^{-f} \left\{ \frac{\det \chi_{\bar{k}j}}{\det \hat{\chi}_{\bar{k}j}} \right\} \chi^{j\bar{k}} \partial_j \partial_{\bar{k}} \dot{\varphi}.$$

This is a linear parabolic equation for $\dot{\varphi}$. It follows from the maximum principle for parabolic equations (e.g. Proposition 1.7 in [SW13]) that

$$\inf_X \dot{\varphi}(x,0) \leq \dot{\varphi}(x,t) \leq \sup_X \dot{\varphi}(x,0).$$

Since $\varphi(x,0) = 0$, we have

$$\inf_X e^{-f} \leq \dot{\varphi}(x,t) \leq \sup_X e^{-f}.$$

By (2.60), we have

$$e^f \inf_X e^{-f} \leq \frac{\det \chi_{\bar{k}j}}{\det \hat{\chi}_{\bar{k}j}} \leq e^f \sup_X e^{-f}.$$

By (2.57),

$$\|\Omega\|_{\omega(t)} = \|\Omega\|_\chi^{2(n-1)/(n-2)} = \|\Omega\|_{\hat{\chi}}^{2(n-1)/(n-2)} \left(\frac{\det \hat{\chi}}{\det \chi} \right)^{(n-1)/(n-2)}.$$

Therefore

$$C^{-1} \leq \|\Omega\|_{\omega(t)} \leq C,$$

along the flow, where $C > 0$ only depends on $\|\Omega\|_{\hat{\chi}}$ and n. The degeneration of $\|\Omega\|_\omega$ exhibited for non-Kähler examples in Sect. 2.3.2 does not occur in this case.

Estimating $\|\Omega\|_{\omega(t)}$ is only the first step in the study of the flow. From here, we can use a priori estimates and techniques from fully nonlinear PDE to establish long-time existence and convergence. We refer to [PPZ19] for full details. The result is

Theorem 2.8 ([PPZ19]) *Let X be a compact complex manifold of dimension n with a nowhere vanishing holomorphic $(n, 0)$ form Ω. Suppose X admits a Kähler metric $\hat{\chi}$. Then the Anomaly flow $\frac{d}{dt}\|\Omega\|_\omega \omega^{n-1} = i\partial\bar{\partial}\omega^{n-2}$ with initial metric satisfying*

$$\|\Omega\|_{\omega(0)}\omega(0)^{n-1} = \hat{\chi}^{n-1}$$

exists for all time, and smoothly converges to a Kähler metric ω_∞.

In fact, ω_∞ is given explicitly by

$$\omega_\infty = \|\Omega\|_{\chi_\infty}^{-2/(n-2)}\chi_\infty,$$

where χ_∞ is the unique Kähler Ricci-flat metric in the cohomology class $[\hat{\chi}]$, and

$$\|\Omega\|^2_{\chi_\infty} = \frac{n!}{[\hat{\chi}]^n}\int_X i^{n^2}\Omega \wedge \bar{\Omega}.$$

To conclude this section, we note that we cannot expect the Anomaly flow on Kähler manifolds to converge starting from an arbitrary metric. This is due to the relationship between the Kähler cone and the balanced cone. Indeed, an initial conformally balanced metric determines a balanced class

$$[\|\Omega\|_{\omega(0)}\omega(0)^{n-1}] \in H^{n-1,n-1}_{BC}(X),$$

and the evolving metric $\omega(t)$ remains in this class (2.41). Stationary points of the flow are Kähler metrics, so convergence of the flow would produce a Kähler metric in the balanced class of the initial metric. However, there exists Kähler manifolds with balanced classes which do not admit any Kähler metric [FX14, TO09]. Understanding which balanced classes come from Kähler classes is an interesting problem in Hermitian geometry [FX14], and we hope that future work studying the Anomaly flow and its singularities will provide insight.

2.4 Anomaly Flow with α' Corrections

We will now restrict our attention to Calabi–Yau threefolds. In this section, we modify the Anomaly flow (2.40) by adding α' correction terms. The parameter $\alpha \in \mathbf{R}$ will be referred to as the slope parameter.

Let X be a compact complex manifold of dimension $n = 3$. Suppose X admits a nowhere vanishing holomorphic $(3, 0)$ form Ω. We first study the case of threefolds with vanishing second Chern class, so we assume that $c_1(X) = c_2(X) = 0$. Consider the flow

$$\frac{d}{dt}(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}\mathrm{Tr}\, Rm \wedge Rm, \tag{2.62}$$
$$d(\|\Omega\|_{\omega(0)}\omega(0)^{n-1}) = 0.$$

Recall that we use the notation Rm for the endomorphism-valued $(1,1)$ form which is the curvature of the Chern connection of ω. When $\alpha' = 0$ and $n = 3$, this flow becomes (2.40) from Sect. 2.3. Stationary points ω_∞ satisfy

$$\frac{\alpha'}{4}\mathrm{Tr}\, Rm \wedge Rm = i\partial\bar{\partial}\omega_\infty, \quad d(\|\Omega\|_{\omega_\infty}\omega_\infty^2) = 0,$$

which can be viewed as a sort of non-Kähler analog of the Kähler–Einstein equation

$$\mathrm{Tr}\, Rm = \lambda\omega, \quad d\omega = 0.$$

More generally, if $c_2(X) \neq 0$, we can add a cancellation term $\Phi \in \Omega^{2,2}(X, \mathbf{R})$ with $[\Phi] = c_2(X)$, and consider the flow

$$\frac{d}{dt}(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}\, Rm \wedge Rm - \Phi(t)), \tag{2.63}$$
$$d(\|\Omega\|_{\omega(0)}\omega(0)^2) = 0.$$

Flows of type (2.63) are called Anomaly flows, as introduced in joint work with Phong and Zhang [PPZ218, PPZ318]. The motivation for studying this evolution equation comes from theoretical physics, which we describe next.

2.4.1 *Hull–Strominger System*

Our motivation for adding the α' correction terms comes from heterotic string theory. The Hull–Strominger system [HU186, ST86] is the following system of equations on a Calabi–Yau threefold

$$F \wedge \omega^2 = 0, \quad F^{0,2} = F^{2,0} = 0, \tag{2.64}$$

$$i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}\, Rm \wedge Rm - \mathrm{Tr}\, F \wedge F) = 0, \tag{2.65}$$

$$d(\|\Omega\|_\omega \omega^2) = 0. \tag{2.66}$$

The system is a coupled equation for a Hermitian meric ω on X and a metric h on a given holomorphic vector bundle $E \to X$. Here Rm, F are the curvature forms of unitary connections of ω, h, viewed as endomorphism valued 2-forms.

Equation (2.64) is the Hermitian-Yang-Mills equation, which admits solutions as long as E is stable of degree zero with respect to ω by the Donaldson-Uhlenbeck-Yau theorem [DO85, UY85] (see [LY86, BU88] for its extension to the Hermitian setting). Equation (2.65) is the Green-Schwarz anomaly cancellation equation from theoretical physics [GS87]. All together, the system was introduced by Hull and Strominger as a model for the heterotic string admitting non-zero torsion, generalizing the equation proposed by Candelas–Horowitz–Strominger–Witten [CA851] where the threefold is required to be Kähler with Ricci-flat metric.

For example, Kähler Calabi–Yau threefolds provide solutions to the Hull–Strominger system. In this case, we take the gauge bundle E to be the tangent bundle $E = T^{1,0}X$, and $h = \omega$ to be Kähler Ricci-flat. Then (2.64) and (2.65) hold automatically, and by the argument in Sect. 2.2.3.1, we see that ω is conformally balanced.

Going beyond Kähler geometry, there are many diverse examples of solutions using various gauge bundles E. The first solutions in the mathematics literature were obtained by Li and Yau [LY05] by perturbing the Kähler solutions, and the first solutions on non-Kähler manifolds were obtained by Fu and Yau [FY08]. Since then, there have been constructions of parallelizable examples [FIUV14, FIUV14, FY15, OUV17, GR11], solutions on Kähler manifolds for arbitrary admissible gauge bundles [AG121, AG122], solutions on fibrations over a Riemann surface [FHP17], and non-compact examples [FY09, FE17, HIS16].

The Hull–Strominger system is interesting from the point of view of canonical metrics on non-Kähler Calabi–Yau threefolds, as it is a curvature constraint (2.65) combined with a closedness condition (2.66). There are also other proposed optimal metrics in non-Kähler complex geometry: e.g. constant Chern scalar curvature [ACS17], vanishing Chern–Ricci curvature [TW10, TW17, STW17], Chern–Ricci flat balanced [FE17], just to name a few.

As a system of partial differential equations, the Hull–Strominger system is fully nonlinear. It can be viewed as an analog of the σ_2 equation, but as a full system for the metric tensor $g_{\bar{k}j}$. There has been much progress in the study of scalar σ_k-type equations in complex geometry e.g. [BL05, CJY15, DDT17, DL15, DK17, DPZ18, HMW10, PPZ116], but very little is known about PDE systems which are nonlinear in second derivatives.

To study the Hull–Strominger system, it was proposed in [PPZ218] to use the Anomaly flow with $\Phi = \mathrm{Tr}\, F \wedge F$ coupled to the Donaldson heat flow [DO85].

$$h^{-1}\partial_t h = -\Lambda_\omega F,$$
$$\frac{d}{dt}(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}\, Rm \wedge Rm - \mathrm{Tr}\, F \wedge F),$$
$$d(\|\Omega\|_{\omega(0)}\omega(0)^2) = 0.$$

Stationary points solve the Hull–Strominger system. The Anomaly flow, when restricted to certain ansatzes, provides new nonlinear equations arising naturally from geometry and physics [PPZ217, PPZ317]. We will describe some of these new equations in the following sections.

2.4.2 Evolution of the Metric

We now derive the evolution of the metric tensor $\omega = i g_{\bar{k}j} dz^j \wedge d\bar{z}^k$ under the Anomaly flow (2.63). The argument given here is similar to the one from Sect. 2.3.1. We write

$$\frac{d}{dt}(\|\Omega\|_\omega \omega^2) = \Psi,$$

with

$$\Psi = \left[i\partial\bar{\partial}\omega - \frac{\alpha'}{4}(\mathrm{Tr}\, Rm \wedge Rm - \Phi) \right].$$

By Lemma 2.6, we already know that the trace of the evolution of the metric is given by

$$\mathrm{Tr}\,\dot{\omega} = \frac{6}{\|\Omega\|_\omega} \frac{\Psi \wedge \omega}{\omega^3},$$

which combined with identity (2.29) is

$$\mathrm{Tr}\,\dot{\omega} = \frac{1}{2\|\Omega\|_\omega} \mathrm{Tr}\,\Psi. \tag{2.67}$$

As in (2.43), we expand the flow to the following expression

$$-\frac{1}{2}(\mathrm{Tr}\,\dot{\omega})\omega^2 + 2\dot{\omega} \wedge \omega - \frac{1}{\|\Omega\|_\omega}\Psi = 0. \tag{2.68}$$

We apply the Hodge star operator $\star$ with respect to ω to both sides of the equation. By identities (2.44), (2.48), and $\star\omega^2 = 2\omega$, the components of the resulting $(1,1)$ form are given by

$$\begin{aligned} 0 &= \star\left[-\frac{1}{2}(\mathrm{Tr}\,\dot{\omega})\omega^2 + 2\dot{\omega} \wedge \omega - \frac{1}{\|\Omega\|_\omega}\Psi \right]_{\bar{k}j} \\ &= -2i\partial_t g_{\bar{k}j} + i(\mathrm{Tr}\,\dot{\omega}) g_{\bar{k}j} - \frac{1}{\|\Omega\|_\omega}\left[-i g^{s\bar{r}} \Psi_{\bar{r}\bar{k}sj} + \frac{i}{2}(\mathrm{Tr}\,\Psi) g_{\bar{k}j} \right]. \end{aligned} \tag{2.69}$$

Substituting the expression for $\mathrm{Tr}\,\dot{\omega}$ (2.67) into (2.69), we see that the $\mathrm{Tr}\,\Psi$ terms cancel and the evolution of the metric is

$$\frac{d}{dt} g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega} g^{s\bar{r}} \Psi_{\bar{r}\bar{k}sj}.$$

From here, we can derive an explicit expression for the evolution of the metric.

Theorem 2.9 ([PPZ318]) *Suppose $\omega(t)$ solves the Anomaly flow (2.63). Then the metric evolves by*

$$\frac{d}{dt} g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega} \Big[-\tilde{R}_{\bar{k}j} + g^{s\bar{r}} g^{n\bar{m}} T_{\bar{m}sj} \bar{T}_{n\bar{r}\bar{k}} - \frac{\alpha'}{4} g^{s\bar{r}} (R_{[\bar{k}s}{}^{\alpha}{}_{\beta} R_{\bar{r}j]}{}^{\beta}{}_{\alpha} - \Phi_{\bar{r}\bar{k}sj}) \Big], \tag{2.70}$$

where $[,]$ *denotes antisymmetrization in both barred and unbarred indices.*

Proof We have already established

$$\frac{d}{dt} g_{\bar{k}j} = \frac{1}{2\|\Omega\|_\omega} \Big[-g^{s\bar{r}} (i\partial\bar{\partial}\omega)_{\bar{r}s\bar{k}j} - \frac{\alpha'}{4} g^{s\bar{r}} (\mathrm{Tr}\, Rm \wedge Rm - \Phi)_{\bar{r}\bar{k}sj} \Big].$$

By (2.51), we have an expression for $g^{s\bar{r}}(i\partial\bar{\partial}\omega)_{\bar{r}\bar{k}sj}$ in terms of Ricci curvature and torsion. This gives the desired expression. □

We note that (2.70) is a fully nonlinear system, as it is quadratic in the curvature. For other geometric flows which are quadratic in the curvature, see e.g. [FR85, GGI13, OL09]. Since the flow is fully nonlinear, we cannot expect short-time existence for arbitrary initial data. However, from (2.70), we see that the right-hand side is parabolic if the α' correction terms are small. The full details are provided in [PPZ218].

Theorem 2.10 ([PPZ218]) *Let ω_0 be a conformally balanced Hermitian metric on X satisfying $|\alpha' Rm| < \frac{1}{2}$. Then there exists $T > 0$ such that the Anomaly flow (2.63) admits a unique solution $\omega(t)$ on $[0, T)$ with $\omega(0) = \omega_0$.*

Given any metric $g_{\bar{k}j}$, we can find $\lambda \gg 1$ so that $\lambda g_{\bar{k}j}$ satisfies $|\alpha' Rm| \ll 1$. This is simply because $Rm(\lambda g) = Rm(g)$ (with Rm defined as in (2.14)). Thus to guarantee short-time existence starting from a given metric, we can rescale the size of the manifold, or choose a small value for α'. For several examples [FHP17, PPZ418], the condition $|\alpha' Rm| \ll 1$ is preserved along the flow, which suggests that it is a natural condition.

2.4.3 Anomaly Flow with Fu–Yau Ansatz

2.4.3.1 Scalar Reduction

In this section, we return to the construction of Goldstein–Prokushkin described in Sect. 2.2.3.4. We first recall the setup.

The base of the fibration $(S, \hat{\omega}, \Omega_S)$ is a Calabi–Yau surface with Kähler Ricci-flat metric $\hat{\omega}$ and nowhere vanishing holomorphic $(2, 0)$ form Ω_S. Let $\omega_1, \omega_2 \in 2\pi H^2(S, \mathbf{Z})$ be anti-self-dual $(1, 1)$ forms. Using this data, Goldstein and Prokushkin [GO04] constructed a T^2 fibration $\pi : X \to S$ which is non-Kähler but admits conformally balanced metrics. Their construction builds on earlier ideas of Calabi and Eckmann [CE53], which we discussed in detail in Sect. 2.1.4.2.

We recall that the connections of the $U(1)$ principal bundles forming the S^1 fibers of X define $\theta \in \Omega^{1,0}(X)$ satisfying

$$\partial\theta = 0, \quad \bar{\partial}\theta = \omega_1 + i\omega_2.$$

Furthermore,

$$\Omega = \Omega_S \wedge \theta$$

is a nowhere vanishing holomorphic $(3, 0)$ form on X, and the family of metrics

$$\omega_u = e^u \hat{\omega} + i\theta \wedge \bar{\theta}, \tag{2.71}$$

is conformally balanced for any $u : S \to \mathbf{R}$. These metrics were used by Fu and Yau [FY08, FY07] to solve the Hull–Strominger system on the threefold X.

In this section, we will start the Anomaly flow with a metric of this form, and check whether the ansatz is preserved. For this, we compute (see (2.39))

$$\|\Omega\|_{\omega_u} = e^{-u}, \quad \|\Omega\|_{\omega_u}\omega_u^2 = e^u\hat{\omega}^2 + 2\hat{\omega} \wedge i\theta \wedge \bar{\theta}, \tag{2.72}$$

and

$$i\partial\bar{\partial}\omega_u = i\partial\bar{\partial}e^u \wedge \hat{\omega} - \bar{\partial}\theta \wedge \partial\bar{\theta} = i\partial\bar{\partial}e^u \wedge \hat{\omega} - (\omega_1^2 + \omega_2^2). \tag{2.73}$$

Next, we must compute the curvature terms. This calculation was done by Fu and Yau in [FY08].

Theorem 2.11 ([FY08]) *The curvature of the Chern connection of ω_u satisfies*

$$\mathrm{Tr}\, Rm(\omega_u) \wedge Rm(\omega_u) = \mathrm{Tr}\, Rm(\hat{\omega}) \wedge Rm(\hat{\omega}) + 2\partial\bar{\partial}u \wedge \partial\bar{\partial}u + 4i\partial\bar{\partial}(e^{-u}\rho),$$

where $\rho \in \Omega^{1,1}(S, \mathbf{R})$ *is given by* $\rho = \rho_{\bar{k}j}\, dz^j \wedge d\bar{z}^k$ *with*

$$\rho_{\bar{k}j} = \frac{i}{2}\hat{g}^{p\bar{q}}(\omega_1 - i\omega_2)_{\bar{q}j}(\omega_1 + i\omega_2)_{\bar{k}p}. \tag{2.74}$$

Proof We work in a local coordinate chart. Since $\bar{\partial}(\omega_1 + i\omega_2) = 0$, there are local functions φ_1, φ_2 such that

$$\bar{\partial}(\varphi_i dz^i) = \omega_1 + i\omega_2, \tag{2.75}$$

where z^1, z^2 are local holomorphic coordinates on the base S . Define

$$\theta_0 = \theta - \varphi_1 dz^1 - \varphi_2 dz^2.$$

Then $\{dz^1, dz^2, \theta_0\}$ is a local holomorphic frame of $\Omega^{1,0}(X)$. The metric can be written as

$$\begin{aligned}\omega_u = {} & (e^u \hat{g}_{\bar{k}j} + \overline{\varphi_k}\varphi_j) i dz^j \wedge d\bar{z}^k \\ & + \overline{\varphi_k}\, i\theta_0 \wedge d\bar{z}^k + \varphi_k\, i dz^k \wedge \overline{\theta_0} + i\theta_0 \wedge \overline{\theta_0}.\end{aligned}$$

Let $B = (\varphi_1, \varphi_2)$. Then the metric in this local frame is given by

$$g = \begin{bmatrix} (e^u \hat{g} + B^* B) & B^* \\ B & 1 \end{bmatrix}.$$

Its inverse is

$$g^{-1} = \begin{bmatrix} e^{-u}\hat{g}^{-1} & -e^{-u}\hat{g}^{-1}B^* \\ -e^{-u}B\hat{g}^{-1} & 1 + e^{-u}B\hat{g}^{-1}B^* \end{bmatrix}.$$

The curvature in this frame is $Rm = \bar{\partial} g^{-1}\partial g$. Computing at a point $p \in X$, we may assume that $p = 0$ and $B(0) = 0$. The curvature at p is then

$$Rm = \begin{bmatrix} R_{\bar{1}1} & R_{\bar{1}2} \\ R_{\bar{2}1} & R_{\bar{2}2} \end{bmatrix},$$

with

$$\begin{aligned} R_{\bar{1}1} &= \bar{\partial}\partial u \cdot I + \hat{Rm} - e^{-u}\hat{g}^{-1}\partial B^* \wedge \bar{\partial} B \\ R_{\bar{2}1} &= -\bar{\partial} B \wedge \partial u - \bar{\partial} B \hat{g}^{-1}\partial\hat{g} + \bar{\partial}\partial B \\ R_{\bar{1}2} &= \bar{\partial}(e^{-u}\hat{g}^{-1}\partial B^*) \\ R_{\bar{2}2} &= -e^{-u}\bar{\partial} B\hat{g}^{-1}\partial B^*. \end{aligned}$$

We must compute

$$\mathrm{Tr}\, Rm \wedge Rm = \mathrm{Tr}\, R_{\bar 1 1} R_{\bar 1 1} + \mathrm{Tr}\, R_{\bar 1 2} R_{\bar 2 1} + \mathrm{Tr}\, R_{\bar 2 1} R_{\bar 1 2} + \mathrm{Tr}\, R_{\bar 2 2} R_{\bar 2 2}.$$

Expanding this out, we obtain the following expression.

$$\begin{aligned}
&\mathrm{Tr} Rm \wedge Rm \\
&= 2(\bar\partial\partial u)^2 + \mathrm{Tr}\, \hat{Rm}^2 + e^{-2u}\mathrm{Tr}\,(\hat g^{-1}\partial B^*\bar\partial B \hat g^{-1}\partial B^*\bar\partial B) \\
&\quad + 2\partial\bar\partial u \mathrm{Tr}\, \hat{Rm} - 2e^{-u}\bar\partial\partial u \mathrm{Tr}\, \hat g^{-1}\partial B^*\bar\partial B - 2e^{-u}\mathrm{Tr}(\hat{Rm}\hat g^{-1}\partial B^*\bar\partial B) \\
&\quad - 2\mathrm{Tr}(\bar\partial(e^{-u}\hat g^{-1}\partial B^*)\bar\partial B\partial u) - 2\mathrm{Tr}(\bar\partial(e^{-u}\hat g^{-1}\partial B^*)\bar\partial B\hat g^{-1}\partial\hat g) \\
&\quad + 2\mathrm{Tr}(\bar\partial(e^{-u}\hat g^{-1}\partial B^*)\bar\partial\partial B) + e^{-2u}\bar\partial B\hat g^{-1}\partial B^*\bar\partial B g^{-1}\partial B^*.
\end{aligned}$$

Using the identities

$$\begin{aligned}
-2\mathrm{Tr}\,\bar\partial(e^{-u}\hat g^{-1}\partial B^*)\bar\partial B\hat g^{-1}\partial\hat g = &-2\bar\partial\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^*\bar\partial B\hat g^{-1}\partial\hat g) \\
&+2\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^*\bar\partial B\, \hat{Rm}),
\end{aligned}$$

and

$$\begin{aligned}
-2e^{-u}\bar\partial\partial u\, \mathrm{Tr}(\hat g^{-1}\partial B^*\bar\partial B) = &-2\bar\partial\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^*\bar\partial B\partial u) \\
&+2\mathrm{Tr}\,\bar\partial(e^{-u}\hat g^{-1}\partial B^*)(\bar\partial B\partial u),
\end{aligned}$$

as well as $\mathrm{Tr}\, \hat{Rm} = 0$, we cancel a few terms and are left with

$$\begin{aligned}
\mathrm{Tr}\, Rm \wedge Rm = 2(\bar\partial\partial u)^2 + \mathrm{Tr}\hat{Rm}^2 &- 2\bar\partial\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^*\bar\partial B\hat g^{-1}\partial\hat g) \\
&-2\bar\partial\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^*\bar\partial B\partial u) + 2\bar\partial\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^*\bar\partial\partial B).
\end{aligned}$$

Using $\partial\hat g^{-1} = -\hat g^{-1}\,\partial\hat g\,\hat g^{-1}$, this expression simplifies to

$$\mathrm{Tr}\, Rm \wedge Rm = 2(\bar\partial\partial u)^2 + \mathrm{Tr}\, \hat{Rm} \wedge \hat{Rm} + 2\bar\partial\partial\mathrm{Tr}(e^{-u}\hat g^{-1}\partial B^* \wedge \bar\partial B).$$

We have by definition

$$\partial B^* \wedge \bar\partial B = \begin{pmatrix} \partial_i\overline{\varphi_1}\partial_{\bar k}\varphi_1 & \partial_i\overline{\varphi_1}\partial_{\bar k}\varphi_2 \\ \partial_i\overline{\varphi_2}\partial_{\bar k}\varphi_1 & \partial_i\overline{\varphi_2}\partial_{\bar k}\varphi_1 \end{pmatrix} dz^i \wedge d\bar z^k.$$

Using (2.75), we obtain (2.74). □

We now add a gauge bundle to the system. Let E_S be a stable vector bundle of degree zero over the base Kähler surface $(S, \hat\omega)$. By the Donaldson-Uhlenbeck-Yau theorem [DO85, UY85], we may equip E_S with a metric H_S satisfying

$$F(H_S) \wedge \hat\omega = 0.$$

On the threefold, we consider the bundle $E = \pi^* E_S \to X$ with metric $H = \pi^* H_S$. This metric is Hermitian–Yang–Mills with respect to the Fu–Yau ansatz ω_u, since

$$F(H) \wedge \omega_u^2 = 0$$

for any $u \in C^\infty(S, \mathbf{R})$.

Putting together everything computed so far, we have

$$\begin{aligned} & i\partial\bar{\partial}\omega_u - \frac{\alpha'}{4}(\mathrm{Tr} Rm(\omega_u) \wedge Rm(\omega_u) - \mathrm{Tr}\, F(H) \wedge F(H)) \\ & = i\partial\bar{\partial}(e^u \hat{\omega} - \alpha' e^{-u}\rho) - \frac{\alpha'}{2}(\partial\bar{\partial}u) \wedge (\partial\bar{\partial}u) + \mu, \end{aligned} \tag{2.76}$$

where $\mu \in \Omega^{2,2}(S, \mathbf{R})$ is given by

$$\mu = \frac{\alpha'}{4}(\mathrm{Tr}\, F(H_S) \wedge F(H_S) - \mathrm{Tr} Rm(\hat{\omega}) \wedge Rm(\hat{\omega})) - (\omega_1^2 + \omega_2^2).$$

Combining (2.72) and (2.76), we see that the Anomaly flow reduces to the following scalar fully nonlinear PDE on the base manifold S.

$$\frac{d}{dt} e^u\, \hat{\omega}^2 = i\partial\bar{\partial}(e^u \hat{\omega} - \alpha' e^{-u}\rho) + \frac{\alpha'}{2}(i\partial\bar{\partial}u)^2 + \mu. \tag{2.77}$$

This evolution equation can also be written as

$$\frac{d}{dt} e^u = \frac{1}{2}\left[\Delta_{\hat{\omega}} e^u - \alpha' \frac{i\partial\bar{\partial}(e^{-u}\rho)}{\hat{\omega}^2/2!} + \alpha' \hat{\sigma}_2(i\partial\bar{\partial}u) + \frac{\mu}{\hat{\omega}^2/2!}\right].$$

Here $\hat{\sigma}_2(i\partial\bar{\partial}u) = (i\partial\bar{\partial}u)^2 \hat{\omega}^{-2}$ is the determinant of the complex Hessian of u with respect to $\hat{\omega}$.

By standard parabolic theory, this equation admits a short-time solution as long as

$$\omega' = e^u \hat{\omega} + \alpha' e^{-u}\rho + \frac{\alpha'}{2} i\partial\bar{\partial}u > 0.$$

2.4.3.2 Stationary Points

For stationary points of (2.77) to exist, integrating both sides shows that we require

$$\int_S \mu = 0,$$

which is the cohomological constraint

$$\frac{\alpha'}{4}\int_S \left[\mathrm{Tr} Rm(\hat{\omega}) \wedge Rm(\hat{\omega}) - \mathrm{Tr}\, F(H_S) \wedge F(H_S)\right] = \int_S [|\omega_1|^2 + |\omega_2|^2]\frac{\hat{\omega}^2}{2!}.$$

It is possible to construct data $(S, E_S, \omega_1, \omega_2, \alpha')$ satisfying this condition. Indeed, since we assume $c_1(S) = c_1(E_S) = 0$, the constraint is

$$\frac{\alpha'}{4}[c_2(S) - c_2(E_S)] = \int_S \left[\left|\frac{\omega_1}{2\pi}\right|^2_{\hat{\omega}} + \left|\frac{\omega_2}{2\pi}\right|^2_{\hat{\omega}}\right]\frac{\hat{\omega}^2}{2}.$$

Note that when seeking solutions to the Hull–Strominger system, after rescaling $\omega_u \mapsto \lambda\omega_u$ in (2.65) we can assume that $\frac{\alpha'}{4} \in \mathbf{Z}$. Explicit examples are exhibited in [FY08, FY07]; when $\alpha' > 0$, we may take S to be a $K3$ surface and use the theory of stable bundles over $K3$ surfaces to construct E_S, and when $\alpha' < 0$ we may take S to be either a torus T^4 or a $K3$ surface.

The main theorem of Fu–Yau guarantees the existence of smooth solutions to the Hull–Strominger system when the cohomological condition $\int_S \mu = 0$ is satisfied.

Theorem 2.12 ([FY08, FY07]) *Let $(S, \hat{\omega})$ be a Kähler surface, $\alpha' \in \mathbf{R}$, $\rho \in \Omega^{1,1}(S, \mathbf{R})$, and $\mu \in \Omega^{2,2}(S, \mathbf{R})$. Suppose μ satisfies the condition $\int_S \mu = 0$. Then there exists a smooth function $u : S \to \mathbf{R}$ solving*

$$0 = i\partial\bar{\partial}(e^u\hat{\omega} - \alpha' e^{-u}\rho) + \frac{\alpha'}{2}(i\partial\bar{\partial}u)^2 + \mu,$$

such that $\omega' = e^u\hat{\omega} + \alpha' e^{-u}\rho + \frac{\alpha'}{2}i\partial\bar{\partial}u > 0$.

For further work relating to the Fu-Yau solutions, we refer to [CHZ118, CHZ218, GA40, LE11, PPZ117, PPZ116, PPZ216, PPZ118].

2.4.3.3 Long-Time Existence

The first observation in the Anomaly flow with Fu-Yau ansatz is the following conserved quantity.

Lemma 2.9 *Let $\omega(t) = e^{u(t)}\hat{\omega} + i\theta \wedge \bar{\theta}$ be a solution to the Anomaly flow with the cohomology condition $\int_S \mu = 0$ satisfied. Then the conservation law*

$$\frac{d}{dt}\int_X \|\Omega\|_\omega \omega^3 = 0,$$

holds along the flow.

Proof In the case of the Fu-Yau ansatz $\omega = e^u \hat{\omega} + i\theta \wedge \bar{\theta}$, by (2.72) we have

$$\int_X \|\Omega\|_\omega \omega^3 = \int_X 3e^u \hat{\omega}^2 \wedge i\theta \wedge \bar{\theta}.$$

Using $\int_S \mu = 0$, from (2.77) we see that

$$\frac{d}{dt} \int_S e^u \, \hat{\omega}^2 = 0$$

is a conserved quantity. □

Together with D.H. Phong and X.-W. Zhang, we prove the following result.

Theorem 2.13 ([PPZ418]) *There exists $L_0 \gg 1$ depending only on $(S, \hat{\omega})$, μ, ρ, α' with the following property. Suppose $\int_S \mu = 0$. Start the Anomaly flow on the fibration $\pi : X \to S$ with initial data*

$$\omega(0) = L\hat{\omega} + i\theta \wedge \bar{\theta},$$

for any constant $L \geq L_0$. Then the flow exists for all time, and converges to a solution to the Hull–Strominger system.

For initial data with small L, we suspect that the flow will develop singularities. We will discuss in Sect. 2.4.4.1 an example of the Anomaly flow over Riemann surfaces where this behavior is observed.

Different choices of L correspond to different balanced classes of the stationary point. We know that the balanced class $[\|\Omega\|_\omega \omega^2] \in H^4(X, \mathbf{R})$ is preserved by the Anomaly flow, and in this case

$$[\|\Omega\|_\omega \omega^2] = [e^u \hat{\omega}^2] + 2[\hat{\omega} \wedge i\theta \wedge \bar{\theta}].$$

The class $[e^u \hat{\omega}^2] \in H^4(S, \mathbf{R})$ is a top cohomology class on the Kähler surface S, and is therefore parametrized by the integrals

$$\int_S e^u \hat{\omega}^2 \in \mathbf{R}.$$

Therefore the choice of $\int_S e^u \hat{\omega}^2$ in the initial data is related to the choice of balanced class of the evolving metric.

As an aside, we note that in general, the conservation of the balanced class $[\|\Omega\|_\omega \omega^2] \in H^{2,2}_{BC}(X)$ along the Anomaly flow should lead to conserved quantities, which may also be useful when studying the flow beyond the Fu–Yau ansatz. The Bott–Chern cohomology of complex manifolds differs in general from the de Rham cohomology, and we refer to [AT13, AN13, ADT16] for recent progress on computing Bott–Chern cohomology.

2.4.4 *Nonlinear Blow-Up*

In this section, we briefly describe a few more examples and illustrate some of the nonlinear phenomena which can occur.

2.4.4.1 Fibrations over Riemann Surfaces

We return to the construction of fibrations $p : X \to \Sigma$ over a Riemann surface $(\Sigma, \hat{\omega})$ of genus $g \geq 3$ described in Sect. 2.2.3.5. We recall that these were non-Kähler threefolds, and the Fei ansatz metrics

$$\omega_f = e^{2f}\hat{\omega} + e^f \omega',$$

are conformally balanced for any smooth function $f : \Sigma \to \mathbf{R}$.

It is not immediately clear that this family of metrics will be preserved by the Anomaly flow. It turns out that this is indeed the case, and the flow reduces to a single scalar parabolic PDE for f on the base Σ of the fibration. The key computation in [FE15, FHP17] gives the identity

$$i\partial\bar{\partial}\omega_f - \frac{\alpha'}{4}\mathrm{Tr}\, Rm(\omega_f) \wedge Rm(\omega_f) = (i\partial\bar{\partial}u - \kappa u\hat{\omega}) \wedge \omega',$$

where

$$u = e^f + \frac{\alpha'}{2}\kappa e^{-f}.$$

and $\kappa \leq 0$ is the Gauss curvature of the background metric $\hat{\omega}$. Since

$$\|\Omega\|_{\omega_f}\omega_f^2 = 2\mathrm{vol}_{T^4} + 2e^f\hat{\omega} \wedge \omega',$$

we can factor out ω' in the formulation of the Anomaly flow as (2, 2) forms, and the flow reduces to

$$\partial_t e^f = \frac{1}{2}\left[\hat{g}^{z\bar{z}}\partial_z\partial_{\bar{z}}\left(e^f + \frac{\alpha'}{2}\kappa e^{-f}\right) - \kappa\left(e^f + \frac{\alpha'}{2}\kappa e^{-f}\right)\right], \tag{2.78}$$

on the Riemann surface $(\Sigma, \hat{\omega})$. The flow admits a short-time solution as long as

$$e^f - \frac{\alpha'}{2}\kappa e^{-f} > 0,$$

which is automatic if $\alpha' > 0$. In [FHP17], together with T. Fei and Z. Huang, we study the asymptotics of the flow.

Theorem 2.14 ([FHP17]) *There exists $L_0 \gg 1$ depending on $(\Sigma, \hat{\omega})$ and α' with the following property. Start Anomaly flow with initial data*

$$\omega(0) = L^2\hat{\omega} + L\omega',$$

for any constant $L \geq L_0$. Then the flow exists for all time and

$$\frac{\omega_f}{\frac{1}{3!}\int_X \|\Omega\|_{\omega_f}\omega_f^3} \to p^*\omega_\Sigma,$$

where $\omega_\Sigma = q_1^2\hat{\omega}$ is a smooth metric on Σ, and $q_1 > 0$ is the first eigenfunction of the operator $-\Delta_{\hat{\omega}} + 2\kappa$.

In the above theorem, we have long-time existence, but unlike Theorem 2.13, $\|\Omega\|_{\omega_f} \to 0$ as $t \to \infty$. This can be understood by the fact that there are no stationary points in the large radius regime $e^f \gg 1$. We note that the result in [FHP17] is more general than the one stated above; the asymptotic behavior holds if the initial data satisfies $u(x, 0) \geq 0$.

For initial data with small L, finite-time blow-up can occur. Indeed, following [FHP17], we consider the case when $\alpha' > 0$. If

$$L^2 < \frac{8\alpha'\pi^2(g-1)^2}{\|\kappa\|_{L^\infty(\Sigma)}\mathrm{Vol}(\Sigma, \hat{\omega})^2}, \tag{2.79}$$

then the flow encounters a singularity in finite time. To see this, we compute using the evolution equation (2.78), and use that $\kappa \leq 0$ and that the Laplacian integrates to zero.

$$\frac{d}{dt}\int_\Sigma e^f\hat{\omega} = \frac{1}{2}\int_\Sigma |\kappa| e^f\hat{\omega} - \frac{\alpha'}{4}\int_\Sigma \kappa^2 e^{-f}\hat{\omega}.$$

By the Cauchy–Schwarz inequality and the Gauss–Bonnet theorem,

$$(4\pi(g-1))^2 = \left(\int_\Sigma |\kappa|\hat{\omega}\right)^2 \leq \left(\int_\Sigma e^f\hat{\omega}\right)\left(\int_\Sigma \kappa^2 e^{-f}\hat{\omega}\right).$$

Therefore

$$\frac{d}{dt}\left[\int_\Sigma e^f\hat{\omega}\right] \leq \frac{\|\kappa\|_{L^\infty(\Sigma)}}{2}\left[\int_\Sigma e^f\hat{\omega}\right] - \frac{\alpha'}{4}(4\pi(g-1))^2\left[\int_\Sigma e^f\hat{\omega}\right]^{-1}.$$

The ODE for $A(t) = \int e^f$ is then

$$\frac{d}{dt}A^2 \leq \|\kappa\|_{L^\infty}A^2 - 8\alpha'\pi^2(g-1)^2,$$

which can be rearranged as

$$\frac{d}{dt}\left((\|\kappa\|_{L^\infty} A^2 - 8\alpha'\pi^2(g-1)^2)e^{-\|\kappa\|_\infty t} \right) \leq 0.$$

Therefore

$$\begin{aligned} &|\kappa\|_{L^\infty} A(t)^2 \\ &\quad \leq 8\alpha'\pi^2(g-1)^2 - \left[8\alpha'\pi^2(g-1)^2 - \|\kappa\|_{L^\infty} \mathrm{Vol}(\Sigma)^2 L^2 \right] \exp(\|\kappa\|_{L^\infty} t), \end{aligned}$$

and we see that the flow must terminate in finite time if (2.79) holds. In fact, $\|\Omega\|_{\omega_f} \to \infty$ in finite time.

2.4.4.2 Lie Groups

For our final example, we will study the Anomaly flow using unitary connections beyond the Chern connection. Let X be a complex Lie group of dimension $n = 3$, and let $\{e_1, e_2, e_3\}$ be a frame of holomorphic vector fields. Let $\{e^1, e^2, e^3\}$ be the dual frame of holomorphic $(1, 0)$ forms. Denote the structure constants by

$$[e_a, e_b] = c^d{}_{ab} e_d.$$

Consider the Hermitian metric

$$\hat{\omega} = i \sum_a e^a \wedge \bar{e}^a.$$

A section of $T^{1,0}X$ can be expressed as $V = V^a e_a$. By definition (2.8), Strominger–Bismut connection ∇^+ of $\hat{\omega}$ acts in the frame $\{e_a\}$ by

$$\nabla^+_b V^a = \nabla^C_b V^a - T^a{}_{bc} V^c, \quad \nabla^+_{\bar{b}} V^a = \nabla^C_{\bar{b}} V^a + \bar{T}_{c\bar{b}\bar{a}} V^c,$$

where we now denote the Chern connection by ∇^C for clarity. Since $g_{\bar{a}b} = \delta_{ab}$ in this frame, $\nabla^C = d$. Furthermore,

$$T = i\partial\omega = -\frac{1}{2} c^a{}_{bd} e^d \wedge e^b \wedge \bar{e}^a.$$

Therefore

$$\nabla^+_b V^a = \partial_b V^a + c^a{}_{bd} V^d, \quad \nabla^+_{\bar{b}} V^a = \partial_{\bar{b}} V^a - \overline{c^d{}_{ba}} V^d.$$

Along the Gauduchon line $\nabla^{(\kappa)} = (1-\kappa)\nabla^C + \kappa\nabla^+$, we have

$$\nabla^{(\kappa)}_b V^a = \partial_b V^a + A^{(\kappa)}{}_b{}^a{}_c V^c, \quad \nabla^{(\kappa)}_{\bar b} V^a = \partial_{\bar b} V^a + A^{(\kappa)}{}_{\bar b}{}^a{}_c V^c,$$

with

$$A^{(\kappa)}{}_b{}^a{}_d = \kappa\, c^a{}_{bd}, \quad A^{(\kappa)}{}_{\bar b}{}^a{}_d = -\kappa\, \overline{c^d{}_{ba}}.$$

The curvature form is defined by $Rm = dA + A \wedge A$. More specifically,

$$Rm = \frac{1}{2} R_{kj}{}^a{}_b\, e^j \wedge e^k + \frac{1}{2} R_{\bar k \bar j}{}^a{}_b\, \bar e^j \wedge \bar e^k + R_{\bar k j}{}^a{}_b\, e^j \wedge \bar e^k,$$

where the components are

$$\begin{aligned}
R_{kj}{}^a{}_b &= \partial_{e_j} A_k{}^a{}_b - \partial_{e_k} A_j{}^a{}_b - c^r{}_{jk} A_r{}^a{}_b + A_j{}^a{}_c A_k{}^c{}_b - A_k{}^a{}_c A_j{}^c{}_b,\\
R_{\bar k \bar j}{}^a{}_b &= \partial_{\bar e_j} A_{\bar k}{}^a{}_b - \partial_{\bar e_k} A_{\bar j}{}^a{}_b - \overline{c^r{}_{jk}} A_{\bar r}{}^a{}_b + A_{\bar j}{}^a{}_c A_{\bar k}{}^c{}_b - A_{\bar k}{}^a{}_c A_{\bar j}{}^c{}_b,\\
R_{\bar k j}{}^a{}_b &= \partial_{e_j} A_{\bar k}{}^a{}_b - \partial_{\bar e_k} A_j{}^a{}_b + A_j{}^a{}_c A_{\bar k}{}^c{}_b - A_{\bar k}{}^a{}_c A_j{}^c{}_b.
\end{aligned}$$

Using the expression for the connection $A^{(\kappa)}$ on the Gauduchon line, the components are explicitly

$$\begin{aligned}
R_{kj}{}^p{}_q &= -\kappa c^r{}_{jk} c^p{}_{rq} + \kappa^2 c^p{}_{jr} c^r{}_{kq} - \kappa^2 c^p{}_{kr} c^r{}_{jq},\\
R_{\bar k \bar j}{}^p{}_q &= \kappa \overline{c^r{}_{jk} c^q{}_{rp}} + \kappa^2 \overline{c^r{}_{jp} c^q{}_{kr}} - \kappa^2 \overline{c^r{}_{kp} c^q{}_{jr}},\\
R_{\bar k j}{}^p{}_q &= \kappa^2 (-c^p{}_{jr} \overline{c^q{}_{kr}} + \overline{c^r{}_{kp}} c^r{}_{jq}).
\end{aligned}$$

The surprising computation of Fei–Yau [FY15] shows that $\mathrm{Tr}\, Rm \wedge Rm$ is actually a (2, 2) form, and its (2, 2) part is given by

$$(\mathrm{Tr}\, Rm \wedge Rm)_{\bar k \bar \ell i j} = 2\kappa^2 (2\kappa - 1) \overline{c^r{}_{k\ell} c^s{}_{rp}} c^q{}_{ij} c^s{}_{qp}.$$

We refer to [FY15] for the full calculation.

We now specialize to the Lie group $SL(2, \mathbf{C})$ with structure constants $c^i{}_{jk} = \epsilon_{ijk}$ the Levi-Civita symbol. Let $\Omega = e^1 \wedge e^2 \wedge e^3$. We also fix $\kappa = 1$ for simplicity, so that we only consider the Strominger–Bismut connection ∇^+. In this case, by two applications of the contracted epsilon identity (2.35), we derive

$$\begin{aligned}
(\mathrm{Tr}\, Rm^+ \wedge Rm^+)_{\bar k \bar \ell i j} &= 2\overline{c^r{}_{k\ell}} c^q{}_{ij}\, [\, \overline{c^s{}_{rp}} c^s{}_{qp}\,]\\
&= 2\overline{c^r{}_{k\ell}} c^q{}_{ij}\, [2\delta_{rq}]\\
&= 4(\delta_{ki}\delta_{\ell j} - \delta_{kj}\delta_{\ell i}).
\end{aligned}$$

Since $\hat{\omega} = i\delta_{ik} e^k \wedge \bar{e}^i$, we have

$$(\mathrm{Tr}\, Rm^+ \wedge Rm^+)_{\bar{k}\bar{\ell}ij} = 2(\hat{\omega}^2)_{\bar{k}\bar{\ell}ij}.$$

By (2.36), we know $i\partial\bar{\partial}\hat{\omega}$ is also proportional to $\hat{\omega}^2$.

$$i\partial\bar{\partial}\hat{\omega} = \frac{1}{2}\hat{\omega}^2.$$

By scaling the metric $\hat{\omega}$, we see that the diagonal ansatz

$$\omega(t) = \lambda^2(t)\hat{\omega},$$

is preserved by the Anomaly flow

$$\frac{d}{dt}(\|\Omega\|_\omega \omega^2) = i\partial\bar{\partial}\omega - \frac{\alpha'}{4}\mathrm{Tr}\, Rm^+ \wedge Rm^+,$$

and becomes the ODE

$$\frac{d}{dt}\lambda = \frac{1}{2}(\lambda^2 - \alpha').$$

In the large radius regime, if we start with

$$\omega(0) = L\hat{\omega}$$

where $L \gg 1$, then $\|\Omega\|_{\omega(t)} \to 0$ in finite-time. Outside of this region, the behavior is sensitive to initial data and sign of α'. For example, if $\alpha' > 0$, then for small initial λ, we may have that $\|\Omega\|_{\omega(t)} \to \infty$ in finite-time.

Acknowledgements I would like to first thank D.H. Phong, my former Ph.D. advisor, for guiding me through this material over the course of many years, and whose style shaped the presentation of this course. I thank Xiangwen Zhang and Teng Fei, whose joint work is discussed here, for countless inspiring discussions on the content of these notes. I also thank Daniele Angella, Giovanni Bazzoni, Slawomir Dinew, Kevin Smith, Freid Tong, and Yuri Ustinovskiy for valuable comments and corrections. These lecture notes were prepared for a course given at the CIME Summer School on complex non-Kähler geometry in 2018, and I would like to thank D. Angella, L. Arosio and E. Di Nezza for the invitation and for organizing a wonderful conference.

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