On Optimal Stopping and Impulse Control with Constraint

J.L. Menaldi and M. Robin

Abstract The optimal stopping and impulse control problems for a Markov-Feller process are considered when the controls are allowed only when a signal arrives. This is referred to as control problems with constraint. In [28, 29, 30], the HJB equation was solved and an optimal control (for the optimal stopping problem, the discounted impulse control problem and the ergodic impulse control problem, respectively) was obtained, under suitable conditions, including a setting on a compact metric state space. In this work, we extend most of the results to the situation where the state space of the Markov process is locally compact.

Keywords: Markov-Feller processes, information constraints, impulse control, control by interventions, ergodic control.

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1 Introduction

A considerable literature has been devoted to optimal stopping and impulse control of Markov processes (e.g., see the references in Bensoussan and Lions [3, 4], Bensoussan [2], Davis [10]). A relatively small part of this literature concerns problems where constraints are imposed on the admissible stopping times. In the present paper, we address optimal stopping and impulse control problems of a Markov process x_t when the stopping times must satisfy a constraint, namely, the control is allowed

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to take place only at the jump times of a given process y_t, these times representing the arrival of a signal.

For instance, the system evolves according to a diffusion process x_t and the signal y_t is a Poisson process as in Dupuis and Wang [11], where an optimal stopping problem is studied, with an application to finance. In this example, thanks to the memoryless property of the exponential distribution, the y_t process does not appear as such. It is interesting to notice that, in the usual (unconstrained) case, the dynamic programming leads to the variational inequality $\max\{-Au+\alpha u, u-\psi\}=0$, where A is the infinitesimal generator of x_t and ψ is the stopping cost (with running cost $f=0$). However, in the constrained case, this becomes the equation $-Au+\alpha u+\lambda[u-\psi]^+=0$, where λ is the intensity of the Poisson process (which is assumed independent from x_t). As soon as the intervals between the jumps of y_t are not exponentially distributed, the control problem must be formulated with the couple (x_t, y_t) and the generator of this two-component process intervenes in the HJB equation.

Such problems has been studied in [28, 29, 30], when the process x_t takes values in a metric compact space E and $y_t = t-\tau_n$, where $\{\tau_n\}$ is an increasing sequence of instants such that $T_n = \tau_n - \tau_{n-1}$, for $n \geq 1$ are, conditionally to x_t, IID random variables. Using an auxiliary discrete time problem in a systematic way, some results have been obtained for optimal stopping and impulse control (with discounted and ergodic cots). Several applications of optimal stopping with constraint have been studied where the decision times are related to availability of some assets (see Lempa [23] and references therein). More generally, portfolio problems with transaction costs could give rise to impulse control with constraint. Moreover, we can consider applications in simple hybrid models (with the signal being the 'discrete' variable, see last section).

The main aim of the present work is to extend the previous results to the case of a locally compact Polish space, considering the three categories of problems: optimal stopping, impulse control with discounted cost as well as ergodic cost. We also mention further extensions and how some generalizations of the present model is related to hybrid models.

Without pretending to be comprehensive, let us mention (a) that references related to optimal stopping with constraint include also Liang [25] who studied particular cases of the model considered here and (b) that other class of (analogue) constraint have been considered, e.g., in Egloff and Leippold [12]. Moreover, for impulse control with constraint, we found only a few references, Brémaud [7, 8], Liang and Wei [26], and Wang [39]. A different kind of constraint is considered in Costa et al. [9], where the constraints are written as infinite horizon expected discounted costs.

The paper is organized as follows. In section 2, we introduce notations, definitions and preliminary properties of the uncontrolled process, which is the two components process (x_t, y_t). Section 3 presents the definition of the optimal stopping problem and its solution. Section 4 describes the process controlled by impulses and the assumptions, which are used for both discounted cost and ergodic cost. In section 5, the impulse control problem with discounted cost is solved via the HJB equation. In section 6, we present the ergodic cost problem and its solution. Some extension are mentioned in section 7 and in section 8 we discuss the links with hybrid models.

2 The Uncontrolled Process

Let us begin with some notations, definitions, comments, and preliminary properties.

Basic Notations:

- $\mathbb{R}^+ = [0,\infty[$, E a locally compact, separable and complete metric space (in short, a locally compact Polish space), and also $\mathbb{N}_0 = \{0,1,\dots\}$ (i.e., natural numbers and 0), $\overline{\mathbb{N}}_0 = \mathbb{N}_0 \cup \{\infty\}$, $\overline{\mathbb{R}}^+ = [0,\infty]$;
- $\mathscr{B}(Z)$ the Borel σ-algebra of sets in Z, $B(Z)$ the space of real-valued Borel and bounded functions on Z, $C_b(Z)$ the space of real-valued continuous and bounded functions on Z, $C_0(Z)$ real-valued continuous functions vanishing at infinity on Z, i.e., a real-valued continuous function v belongs to $C_0(Z)$ if and only if for every $\varepsilon > 0$ there exists a compact set K of Z such that $|v(z)| < \varepsilon$ for every z in $Z \smallsetminus K$[1], and also, if necessary, $B^+(Z)$, $C_b^+(Z)$, $C_0^+(Z)$ for non-negative functions; usually either $Z = E$ or $Z = E \times \mathbb{R}^+$;
- the canonical space $D(\mathbb{R}^+, Z)$ of cad-lag functions, with its canonical process $z_t(\omega) = \omega(t)$ for any $\omega \in D(\mathbb{R}^+, Z)$, and its canonical filtration $\mathbb{F}^0 = \{\mathscr{F}_t^0 : t \geq 0\}$, $\mathscr{F}_t^0 = \sigma(z_s : 0 \leq s \leq t)$.

Assumption 2.1 *Let $(\Omega, \mathbb{F}, x_t, y_t, P_{xy})$ be a (realization of a) strong and normal homogeneous Markov process , on $\Omega = D(\mathbb{R}^+, E \times \mathbb{R}^+)$ with its canonical filtration universally completed $\mathbb{F} = \{\mathscr{F}_t : t \geq 0\}$ with $\mathscr{F}_\infty = \mathscr{F}$, where (x_t, y_t) is the canonical process having values in $E \times \mathbb{R}^+$, and $\mathbb{E}_{xy}$ (or $\mathbb{E}_{x,y}$ when a confusion may arrive) denotes the expectation relative to P_{xy}.*

a) *It is also assumed that x_t is a Markov process by itself (referred as the* reduced state*), with a C_0-semigroup $\Phi_x(t)$ (i.e., $\Phi_x(t)C_0(E) \subset C_0(E)$, $\forall t \geq 0$), and infinitesimal generator A_x with domain $\mathscr{D}(A_x) \subset C_0(E)$.*

b) *The process y_t (referred to as the* signal *process) has jumps to zero at times $\tau_1, \dots, \tau_n \to \infty$ and $y_t = t - \tau_n$ for $\tau_n \leq t < \tau_{n+1}$ (i.e., τ_1 is the time of the first jump –to zero– of y_t, each jump is 'the signal' and y_t is exactly the 'time elapsed since the last jump or signal'), and if $y_0 = 0$ and $\tau_0 = 0$ then it is assumed that conditionally to x_t, the intervals between jumps $T_n = \tau_n - \tau_{n-1}$ are independent, identically distributed random variables with a non-negative continuous and bounded intensity function $\lambda(x,y)$, which is such that there exists a constant $K > 0$ satisfying $\mathbb{E}_{x0}\{\tau_1\} \leq K$, for any x in E.* □

[1] Typically $E = \mathbb{R}^d$ and this means that $v(z) \to 0$ as $|z| \to \infty$.

Remark 2.1. Actually, we begin with a realization of the reduced state process x_t on the canonical space $D(\mathbb{R}^+,E)$ and the signal process y_t is constructed based on the given intensity $\lambda(x,y)$, and this procedure yields a $C_0(E\times\mathbb{R}^+)$-semigroup denoted by $\Phi_{xy}(t)$. Thus, in view of Palczewski and Stettner [34], all this implies that both semigroups $\Phi_x(t)$ and $\Phi_{xy}(t)$ have the Feller property, i.e., $\Phi_{xy}(t)C_b(E)\subset C_b(E)$ and $\Phi_{xy}(t)C_b(E\times\mathbb{R}^+)\subset C_b(E\times\mathbb{R}^+)$, and since only a strong and normal Markov process is assumed, the semigroup $\Phi_{xy}(t)$ is (initially) acting on $B(E\times\mathbb{R}^+)$ and so, weak (or stochastic) continuity is deduced from the assumption of a cad-lag realization, which means that

$$(x,y,t)\mapsto \mathbb{E}_{xy}\{h(x_t,y_t)\} \quad \text{is a continuous function,} \tag{1}$$

for any h in $C_b(E\times\mathbb{R}^+)$. In [28, 29, 30] a probabilistic construction of the signal process y_t was described, but there are other ways to constructing $\Phi_{xy}(t)$. For instances, begin with the process $(x_t,\tilde{y}_t)$ with $\tilde{y}_t=y+t$ having infinitesimal generator $A^0=A_x+\partial_y$ and a $C_0(E\times\mathbb{R}^+)$-semigroup. Then, add the perturbation $Bh(x,y)=\lambda(x,y)[h(x,0)-h(x,y)]$, which is a bounded operator generating a $C_0(E\times\mathbb{R}^+)$-semigroup, with domain $\mathscr{D}(B)=C_0(E\times\mathbb{R}^+)$. Hence $A_{xy}=A^0+B$ generates a $C_0(E\times\mathbb{R}^+)$-semigroup, with $\mathscr{D}(A_{xy})=\mathscr{D}(A^0)$, e.g., see Ethier and Kurtz [13, Section 1.7, pp. 37–40, Thm 7.1]. Therefore A_{xy} will also denote the weak infinitesimal generator in $C_b(E\times\mathbb{R}^+)$, in several places of the following sections. $\square$

Remark 2.2. Note that Assumption 2.1 (b) on the signal process y_t means, in particular, that

$$P_{x0}\big\{T_n\in(t,t+\,dt)\,|\,x_s,\,0\le s\le t\big\}=\lambda(x_t,t)\exp\Big(-\int_0^t\lambda(x_s,s)\mathrm{d}s\Big), \tag{2}$$

and then it is deduced that $\Phi_{xy}(t)$ has an infinitesimal generator $A_{xy}=A_x+A_y$ with

$$A_y\varphi(x,y)=\partial_y\varphi(x,y)+\lambda(x,y)[\varphi(x,0)-\varphi(x,y)], \tag{3}$$

and recall that ∂_y denotes the derivative with respect to y, and that $\lambda\ge 0$ and $\lambda\in C_b(E\times\mathbb{R}^+)$. Moreover, using the law of T_1 as in (2) and the Feller property of (x_t,y_t), it is also deduced that

$$(x,y)\mapsto\mathbb{E}_{xy}\big\{\mathrm{e}^{-\alpha\tau_1}g(x_{\tau_1})\big\} \text{ belongs to } C_b(E\times\mathbb{R}^+), \tag{4}$$

for any g in $C_b(E)$ and any $\alpha\ge 0$. Note that if $y_0=y$ then τ_1 is random variable independent of $T_1,T_2,\dots$ with distribution $P_{x0}\{T_1\in\cdot\,|\,y_0=y\}$. Furthermore, in turn, by applying Dynkin's formula to $A_{xy}\varphi(x,y)+\alpha\varphi(x,y)=f(x,y)$, it follows that

$$(x,y)\mapsto\mathbb{E}_{xy}\Big\{\int_0^{\tau_1}\mathrm{e}^{-\alpha t}f(x_t,y_t)\mathrm{d}t\Big\} \text{ is in } C_b(E\times\mathbb{R}^+), \tag{5}$$

for any f in $C_b(E\times\mathbb{R}^+)$ and any $\alpha>0$. $\square$

Remark 2.3. Note that because $\lambda(x,y)$ is bounded (it suffices for y near 0), there exists a constant a such that $P_{x0}\{\tau_1 \geq a > 0\} \geq a > 0$, for any x in E. Moreover, from Assumption 2.1 (b) on the signal process y_t we have

$$\mathbb{E}_{x0}\{\tau_1\} = \mathbb{E}_{x0}\Big\{ \int_0^{\infty} t\lambda(x_t,t)\exp\Big(-\int_0^t \lambda(x_s,s)\mathrm{d}s\Big)\mathrm{d}t\Big\},$$

so if $\lambda(x,y) \leq k_1 < \infty$, for every $y \geq 0$, and $x \in E$, then $\mathbb{E}_{x0}\{\tau_1\} \geq a_1 = 1/k_1$.. Also, the condition $\mathbb{E}_{x0}\{\tau_1\} \leq a_2$ is satisfied if, for instance $\lambda(x,y) \geq k_0 > 0$ for $y \geq y_0$, $x \in E$, then $a_2 = y_0 + 1/k_0$. Moreover, since $\lambda(x,y)$ is a continuous function in $E \times \mathbb{R}^+$, the continuity of $E_{xy}\{\tau_1\}$ follows. □

Definition 2.1 (with comments). If the evolution $\dot{e} = -\alpha t$ in $[0,1]$ is added to the homogeneous Markov process $\{(x_t,y_t) : t \geq 0\}$ then the expression

$$\{(X_n,e_n) = (x_{\tau_n}, \mathrm{e}^{-\alpha\tau_n}),\ n = 0,1,\ldots\}, \tag{6}$$

with $e_0 = 1$, $\tau_0 = 0$ and $X_0 = x$, becomes a *homogeneous Markov chain* in $]0,1] \times E$ with respect to the filtration $\mathbb{G} = \{\mathscr{G}_n : n = 0,1,\ldots\}$ obtained from $\mathbb{F}$, namely, $\mathscr{G}_n = \mathscr{F}_{\tau_n}$. Note that $\{x_{\tau_n} : n \geq 0\}$ is also a Markov chain with respect to $\mathscr{G}_n$. In this context, if

$$\tau = \inf\{t > 0 : y_t = 0\}, \tag{7}$$

is considered as a functional on Ω, then the *sequence of signals* (i.e., the instants of jumps for y_t) is defined by recurrence

$$\tau_{k+1} = \inf\{t > \tau_k : y_t = 0\}, \quad \forall k = 1,2,\ldots, \tag{8}$$

with $\tau_1 = \tau$, and by convenience, set $\tau_0 = 0$. Let us also mention that Remark 2.3 yields: there exists a constant a_1 such that

$$P_{x0}\{\tau \geq a_1 > 0\} \geq a_1 > 0, \quad \forall x \in E, \tag{9}$$

and by Assumption 2.1, there exists another constant $a_2 > 0$ such that

$$\mathbb{E}_{x0}\{\tau\} \leq a_2, \quad \forall x \in E. \tag{10}$$

It is also valid,

$$0 < a_1 \leq \tau(x) := \mathbb{E}_{x0}\{\tau\} \leq a_2, \quad \forall x \in E, \tag{11}$$

for some real numbers a_1, a_2. An $\mathbb{F}$-stopping time $\theta > 0$ satisfying $y_\theta = 0$ when $\theta < \infty$ is called an *admissible stopping time*, in other words, if and only if there exists a discrete (i.e., $\overline{\mathbb{N}}_0$-valued) $\mathbb{G}$-stopping time η such that $\theta = \tau_\eta$ with the convention that $\tau_\infty = \infty$. Moreover, if the condition $\theta > 0$ (or equivalently $\eta \geq 1$) is dropped then θ is called a *zero-admissible stopping time*. □

3 Optimal Stopping with Constraint

This section is an extension of [28] to a locally compact space E.

3.1 Setting-up

The usual optimal stopping problems as presented above is well known, but our interest here is to restrict the stopping action (of the controller) to certain instants when a signal arrives. As discussed in the previous section, the state of the dynamic system is a homogeneous Markov process $\{(x_t, y_t) : t \in \mathbb{R}^+\}$ with values in the locally compact Polish space $E \times \mathbb{R}^+$, satisfying the Feller conditions (1). Suppose that

$$f \in C_b(E \times \mathbb{R}^+), \quad \psi \in C_b(E), \quad \alpha > 0, \tag{12}$$

where $f(x,y)$ is the running cost, $\psi(x)$ is the terminal cost, and α is the discount factor.

Thus, for any stopping time θ

$$J_{xy}(\theta, \psi) = \mathbb{E}_{xy}\Big\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta}\psi(x_\theta)\Big\}, \tag{13}$$

is the cost function with the optimal cost

$$u(x,y) = \inf\big\{J_{xy}(\theta, \psi) : \ \theta > 0,\ y_\theta = 0\big\}, \tag{14}$$

i.e., θ is any admissible stopping time, as defined in Section 2. Also, it is defined an auxiliary problem with optimal cost

$$u_0(x,y) = \inf\big\{J_{xy}(\theta, \psi) : \ y_\theta = 0\big\}, \tag{15}$$

which provides a homogeneous Markovian model. *Since $u(x,y) = u_0(x,y)$ for any $x \in E$ and $y > 0$, it may be convenient to write $u_0(x) = u_0(x,0)$ as long as no confusion arrives.*

Remark 3.1. Both costs $u(x,y)$ and $u_0(x,y)$ represent the optimization over all stopping times that occur when the signal arrives, the difference is that for y = 0 and t=0 (i.e., when the first signal arrives at the beginning), the control action is allowed for the optimal cost $u_0(x,0)$, but it is not allowed for the optimal cost $u(x,0)$, i.e., one may say that for $u(x,0)$ the 'controller is (so to speak) always 'late' (at the beginning and arriving simultaneously with the signal) and control is not possible. One may consider even an alternative situation, where with a certain probability (independently of (x_t, y_t), for instance) the control is allowed, and therefore, the optimal cost (in the simplest case) would be a convex combination of $u(x,0)$ and $u_0(x,0)$. Clearly, all this comment will apply later, for the impulse control problem. □

The Dynamic Programming Principle shows (heuristically) that

$$u(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} \min\{\psi,u\}(x_\tau,y_\tau)\Big\}, \tag{16}$$

with $\tau = \inf\{t > 0 : y_t = 0\}$ being the first jump of y_t, and

$$\begin{aligned} u_0(x,y) &= \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau,y_\tau)\Big\}, \quad y > 0, \\ u_0(x,0) &= \min\Big\{ \mathbb{E}_{x0}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau,y_\tau)\Big\}, \psi(x)\Big\}, \end{aligned} \tag{17}$$

are the corresponding Hamilton-Jacobi-Bellman (HJB) equations, which are referred to as variational inequalities (VI) in a weak form. Also, both problems are (logically) related by the condition

$$u(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau,y_\tau)\Big\}. \tag{18}$$

Thus, $y_\tau = 0$ implies

$$\begin{aligned} u_0(x) &= \min\Big\{ \psi(x), \mathbb{E}_{x0}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau)\Big\}\Big\}, \\ u(x,y) &= \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} \min\{\psi,u\}(x_\tau,0)\Big\}, \\ u(x,y) &= \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau)\Big\}, \end{aligned}$$

i.e., if $u_0(x)$ is known then the above equalities yield $u(x,y)$ and $u_0(x,y)$.

3.2 Solving the VI

By means of (6), the continuous-time cost $J_{x0}(\theta,\psi)$ with $f = 0$ and a stopping time $\theta = \tau_\eta$ can be written as

$$\begin{aligned} J_{x0}(\theta,\psi) &= \mathbb{E}_{x0}\big\{ \mathrm{e}^{-\alpha\theta}\psi(x_\theta)\big\} \\ &= \mathbb{E}\big\{ e_\eta \psi(X_\eta) \,|\, e_0 = 1, X_0 = x\big\} := K_{1x}(\eta,\psi), \end{aligned} \tag{19}$$

for any discrete stopping time η relative to the Markov chain, i.e., where η has values in $\mathbb{N}_0 =$ and the convention $\tau_\infty = \infty$, and the last equality is the definition of the discrete cost $K_{1x}(\eta,\psi)$. This means that the optimal cost $u_0(x)$ is also the optimal cost of a discrete-time stopping time problem relative to the homogeneous Markov chain (certainly, there are several other ways of considering an equivalent problem in discrete-time), i.e, $u_0(x) = \inf\{K_{1x}(\eta,\psi) : \eta \geq 0\}$. This yields

$$u_0(x) = \min\big\{\psi(x), \mathbb{E}_{x0}\mathrm{e}^{-\alpha\tau}u_0(x_\tau)\big\} \tag{20}$$

as the HJB equation for $u_0(x)$, when $f = 0$.

Theorem 3.1. *Under* Assumption 2.1 *and* (12)*, the VI* (17) *and* (16) *have each a unique solution in* $C_b(E\times\mathbb{R}^+)$*, which are the optimal costs* (14) *and* (15)*, respectively. Moreover, the first admissible exit time of the continuation region is optimal, i.e., the discrete stopping times*

$$\begin{aligned}\hat\theta &= \inf\big\{t>0 : u(x_t,y_t)\le\psi(x_t,y_t),\ y_t=0\big\},\\ \hat\theta_0 &= \inf\big\{t\ge 0 : u_0(x_t,y_t)=\psi(x_t,y_t),\ y_t=0\big\}\end{aligned} \tag{21}$$

are optimal, namely, $u(x,y) = J_{xy}(\hat\theta,\psi)$ *and* $u_0(x,y) = J_{xy}(\hat\theta_0,\psi)$*. Furthermore, the relation* (18) *holds.* □

Proof. This result is proved in [28] when E is compact, and it is valid under the assumptions in Section 2 with the same arguments, and therefore, only the main idea and comments are presented.

First, let us mention that the translation

$$u \longmapsto u - \mathbb{E}_{xy}\Big\{\int_0^\infty \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t\Big\}$$

(and similarly with u_0) reduces to a zero running cost, i.e., in all this section we may assume $f=0$ without any lost of generality, only the terminal cost ψ is relevant. Also, Assumption 2.1(b) on the signal and the inequality

$$\begin{aligned}(1-\mathrm{e}^{-\alpha a})P_{x0}\{\tau\ge a\} &= (1-\mathrm{e}^{-\alpha a})P_{x0}\big\{1-\mathrm{e}^{-\alpha\tau}\ge 1-\mathrm{e}^{-\alpha a}\big\}\\ &\le 1-\mathbb{E}_{x0}\big\{\mathrm{e}^{-\alpha\tau}\big\},\quad \forall a>0,\end{aligned}$$

imply $\mathbb{E}_{x0}\{\mathrm{e}^{-\alpha\tau}\}\le 1-(1-\mathrm{e}^{-\alpha a_1})a_0 := k_1 < 1$. This is used to solve the VI

$$u_0(x) = \min\Big\{\mathbb{E}_{x0}\Big\{\int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau}u_0(x_\tau)\Big\},\psi(x)\Big\},$$

by means of a fixed point for a contraction operator. Then, some martingale arguments are used to establish that $u_0(x)$ is indeed the optimal cost of a discrete-time optimal stopping time problem relative to a Markov chain (6), where the first exit time of the continuation region $\{x : u_0(x) < \psi(x)\}$ is optimal. Next, this is connected with the continuous-time problem and the conclusion follows.

If the function $u_0(x,y)$ belongs to the domain $\mathscr{D}(A_{xy})$ then VI becomes

$$\begin{aligned}&A_{xy}u_0(x,y) - \alpha u_0(x) + f(x,y) = 0,\quad \forall (x,y)\in E\times]0,\infty[,\\ &\min\big\{A_{xy}u_0(x,y) - \alpha u_0(x) + f(x,y), \psi(x) - u_0(x,y)\big\} = 0,\quad \forall (x,y)\in E\times\{0\},\end{aligned}$$

where A_{xy} is the infinitesimal generator. It may be proved that this is indeed the case when ψ also belongs to $\mathscr{D}(A_{xy})$, but only continuity is usually not sufficient.

However, the optimal cost u given by (14) belongs to $\mathscr{D}(A_{xy})$ and the VI (16) is equivalent to

$$-A_{xy}u(x,y)+\alpha u(x,y)+\lambda(x,y)[u(x,0)-\psi(x,y)]^{+}=f(x,y), \tag{22}$$

for any (x,y) in $E\times[0,\infty[$, where $\lambda(x,y)$ is the jump-intensity as discussed in the previous section. Also remark $u_0=\min\{u,\psi\}$, which makes clear that $u_0(x,y)$ may not belong to the domain $\mathscr{D}(A_{xy})\subset C_b(E\times[0,\infty[)$.

Remark 3.2. The VI/HJB equation (22) is similar to the penalized equation of the unconstrained problem, e.g., see Bensoussan and Lions [3]. Similarly, using the same method as in the penalized problem, if λ goes to infinity (uniformly) then the solution u_λ converges to the solution (which is a function of x only) of the classical variational inequality of the unconstrained problem. □

There are some references regarding the stopping time problem with Poisson constraint (e.g., Dupuis and Wang [11], Lempa [23], Liang and Wei [26]), while there are many more about the usual or standard stopping times problem (e.g., the books by Bensoussan and Lions [3], Peskir and Shiryaev [35], among several others books and papers).

4 Impulse Controlled Process

This section describes the controlled process and assumptions common to both, the discounted problem and the ergodic problem, as treated in the next two sections.

4.1 Controlled Process

For a detailed construction we refer to Bensoussan and Lions [4] (see also Davis [10], Lepeltier and Marchal [24], Robin [36], Stettner [38]).

Let us consider $\Omega^\infty=[D(\mathbb{R}^+;E\times\mathbb{R}^+)]^\infty$, and define $\mathscr{F}_t^0=\mathscr{F}_t$ and $\mathscr{F}_t^{n+1}=\mathscr{F}_t^n\otimes\mathscr{F}_t$, for $n\geq 0$, where $\mathscr{F}_t$ is the universal completion of the canonical filtration as previously.

An *arbitrary impulse control* ν (not necessarily admissible at this stage) is a sequence $(\theta_n,\xi_n)_{n\geq 1}$, where θ_n is a stopping time of $\mathscr{F}_t^{n-1}$, $\theta_n\geq\theta_{n-1}$, and the impulse ξ_n is $\mathscr{F}_{\theta_n}^{n-1}$ measurable random variable with values in E.

The coordinate in Ω^∞ has the form $(x_t^0,y_t^0,x_t^1,y_t^1,\ldots,x_t^n,y_t^n,\ldots)$, and for any impulse control ν there exists a probability P_{xy}^ν on Ω^∞ such that the evolution of the controlled process (x_t^ν,y_t^ν) is given by the coordinates (x_t^n,y_t^n) of Ω^∞ when $\theta_n\leq t<\theta_{n+1}$, $n\geq 0$ (setting $\theta_0=0$), i.e., $(x_t^\nu,y_t^\nu)=(x_t^n,y_t^n)$ for $\theta_n\leq t<\theta_{n+1}$. Note that clearly (x_t^ν,y_t^ν) is defined for any $t\geq 0$, but (x_t^i,y_t^i) is only used for any $t\geq\theta_i$, and $(x_{\theta_i}^{i-1},y_{\theta_i}^{i-1})$ is the state at time θ_i just before the impulse (or jump) to

$(\xi_i, y^{i-1}_{\theta_i}) = (x^i_{\theta_i}, y^i_{\theta_i})$, as long as $\theta_i < \infty$. For the sake of simplicity, we will not always indicate, in the sequel, the dependency of (x^ν_t, y^ν_t) with respect to ν. A *Markov impulse control* ν is identified by a closed subset S of $E \times \mathbb{R}^+$ and a Borel measurable function $(x,y) \mapsto \xi(x,y)$ from S into $C = E \times \mathbb{R}^+ \smallsetminus S$, with the following meaning: intervene only when the the process (x_t, y_t) is leaving the continuation region C and then apply an impulse $\xi(x,y)$, while in the stopping region S, moving back the process to the continuation region C, i.e., $\theta_{i+1} = \inf\{t > \theta_i : (x^i_t, y^i_t) \in S\}$, with the convention that $\inf\{\emptyset\} = \infty$, and $\xi_{i+1} = \xi(x^i_{\theta_{i+1}}, y^i_{\theta_{i+1}})$, for any $i \geq 0$, as long as $\theta_i < \infty$.

Now, the admissible controls are defined as follows, recalling that τ_n are the arrival times of signal

Definition 4.1. (i) As mentioned earlier, a stopping time θ is called 'admissible' if almost surely there exists $n = \eta(\omega) \geq 1$ such that $\theta(\omega) = \tau_{\eta(\omega)}(\omega)$, or equivalently if θ satisfies $\theta > 0$ and $y_\theta = 0$ a.s.
(ii) An impulse control $\nu = \{(\theta_i, \xi_i), i \geq 1\}$ as above is called 'admissible', if each θ_i is admissible (i.e., $\theta_i > 0$ and $y_{\theta_i} = 0$), and $\xi_i \in \Gamma(x^{i-1}_{\theta_i})$. The set of admissible impulse controls is denoted by $\mathscr{V}$.
(iii) If $\theta_1 = 0$ is allowed, then ν is called 'zero-admissible'. The set of zero-admissible impulse controls is denoted by $\mathscr{V}_0$.
(iv) An 'admissible Markov' impulse control corresponds to a stopping region $S = S_0 \times \{0\}$ with $S_0 \subset E$, and an impulse function satisfying $\xi(x,0) = \xi_0(x) \in \Gamma(x)$, for any $x \in S_0$, and therefore, $\theta_i = \tau^i_{\eta_i}$ and $\eta_{i+1} = \inf\{k > \eta_i : x^i_{\tau^i_k} \in S_0\}$, with $\tau^0_0 = 0$, $\tau^i_k = \inf\{t > \tau^i_{k-1} : y^i_t = 0\}$, for any $k \geq i \geq 1$. □

The discrete time impulse control problem has been consider in Bensoussan [2], Stettner [37]. As seen later, it will be useful to consider an auxiliary problem in discrete time, for the Markov chain $X_n = x_{\tau_n}$, with the filtration $\mathbb{G} = \{\mathscr{G}_n, n \geq 0\}$, $\mathscr{G}_n = \mathscr{F}^{n-1}_{\tau_n}$. The impulses occurs at the stopping times η_k with values in the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ and are related to θ_k by $\eta_i = \inf\{k \geq 1 : \theta_k = \tau_k\}$ for admissible controls $\{\theta_k\}$ and similarly for zero-admissible controls. Thus,

Definition 4.2. If $\nu = \{(\eta_i, \xi_i), i \geq 1\}$ is a sequence of $\mathbb{G}$-stopping times and $\mathscr{G}_{\eta_i}$-measurable random variables ξ_i, with $\xi_i \in \Gamma(x_{\tau_{\eta_i}})$, η_i increasing and $\eta_i \to +\infty$ a.s., then ν is referred to as an 'admissible discrete time' impulse control if $\eta_1 \geq 1$. If $\eta_i \geq 0$ is allowed, this is referred as an 'zero-admissible discrete time' impulse control. □

4.2 Common Assumptions

It is assumed that there are a running cost $f(x,y)$ and a cost-per-impulse $c(x,\xi)$ satisfying

$$\begin{aligned} &f : E \times \mathbb{R}^+ \to \mathbb{R}^+ \ \text{bounded and continuous}, \quad \alpha > 0, \\ &c : E \times E \to [c_0, +\infty[, \ c_0 > 0, \ \text{bounded and continuous}, \end{aligned} \tag{23}$$

where the discount factor is not used within the ergodic contest. Moreover, for any $x \in E$, the possible impulses must be in $\Gamma(x) = \{\xi \in E : (x,\xi) \in \Gamma\}$, where Γ is a given analytic set in $E \times E$ such that for every x in E the following properties hold true

$$\begin{aligned} &\emptyset \neq \Gamma(x) \text{ is compact}^{2}, \quad \forall \xi \in \Gamma(x),\ \Gamma(\xi) \subset \Gamma(x), \quad \text{and} \\ &c(x,\xi) + c(\xi,\xi') \geq c(x,\xi'), \quad \forall \xi \in \Gamma(x),\ \forall \xi' \in \Gamma(\xi) \subset \Gamma(x). \end{aligned} \tag{24}$$

Finally, defining the operator M

$$Mv(x) = \inf_{\xi \in \Gamma(x)} \big\{ c(x,\xi) + v(\xi) \big\}, \tag{25}$$

it is assumed that

$$\begin{aligned} &M \text{ maps } C_b(E) \text{ into } C_b(E), \text{ and there exists a measurable} \\ &\text{selector } \hat{\xi}(x) = \hat{\xi}(x,v) \text{ realizing the infimum in } Mv(x),\ \forall x, v. \end{aligned} \tag{26}$$

Remark 4.1. (a) The last condition in (24) is to ensure that simultaneous impulses is never optimal. (b) (26) requires some regularity property of $\Gamma(x)$, e.g., see Davis [10]. (c) It is possible (but not necessary) that x belongs to $\Gamma(x)$, actually, even $\Gamma(x) = E$ whenever E is compact. However, an impulse occurs when the system moves from a state x to another state $\xi \neq x$, i.e., it suffices to avoid (or not to allow) impulses that moves x to itself, since they have a higher cost. □

5 Discounted Cost

This section is an extension of [29] to a locally compact space E.

5.1 HJB Equation

The *discounted* cost of an impulse control (or policy) $\nu = \{(\theta_i, \xi_i) : i \geq 1)\}$ is given by

$$J_{x,y}(\nu) = \mathbb{E}^{\nu}_{x,y} \Big\{ \int_0^{\infty} \mathrm{e}^{-\alpha t} f(x_t, y_t)\mathrm{d}t + \sum_{i=0}^{\infty} \mathrm{e}^{-\alpha\theta_i} c(x^{i-1}_{\theta_i}, \xi_i) \Big\}, \tag{27}$$

where $\mathbb{E}^{\nu}_{xy}$ is the P^{ν}_{xy}-expectation of the process under the impulse control ν with initial conditions $(x_0, y_0) = (x, y)$, and $x^{i-1}_{\theta_i}$ is the value of the process just before the impulse. Note that the process $\{y_t : t \geq 0\}$ is not subject to any impulse, and the condition $y_\theta = 0$ determines admissibility of the impulse time θ.

[2] compactness is not really necessary, but it is convenient

Thus, the optimal cost is defined by

$$u(x,y) = \inf\{J_{x,y}(\nu) : \nu \in \mathscr{V}\}, \quad \forall (x,y) \in E \times [0,\infty[, \tag{28}$$

and its associated auxiliary impulse control problem (referred to as the 'time-homogeneous' impulse control) with an optimal cost given by

$$u_0(x,y) = \inf\{J_{x,y}(\nu) : \nu \in \mathscr{V}_0\}, \quad \forall (x,y) \in E \times [0,\infty[. \tag{29}$$

As with the optimal stopping time problems, since $u(x,y) = u_0(x,y)$ for any $x \in E$ and $y > 0$, it may be convenient to write $u_0(x) = u_0(x,0)$ as long as no confusion arrives.

The Dynamic Programming Principle shows (heuristically), see [29, Section 3] that

$$u(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} \min\{Mu,u\}(x_\tau,y_\tau)\Big\}, \tag{30}$$

and

$$\begin{aligned} u_0(x,y) &= \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau,y_\tau)\Big\}, \quad y > 0, \\ u_0(x) &= \min\Big\{\mathbb{E}_{x0}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau)\Big\}, Mu_0(x)\Big\}, \end{aligned} \tag{31}$$

are the corresponding Hamilton-Jacobi-Bellman (HJB) equations, which are referred to as quasi-variational inequalities (QVI) in a weak form. Note that M is an operator in the variable x alone, so that $Mu(x,y) = [Mu(\cdot,y)](x)$. In any case, $\min\{Mu,u\}(x_\tau,y_\tau) = \min\{Mu,u\}(x_\tau,0)$, because $y_\tau = 0$. Also, both problems are related (logically) by the condition

$$u(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau} u_0(x_\tau)\Big\}, \tag{32}$$

and so, if $u_0(x)$ is known then the last equality yields $u(x,y)$ and $u_0(x,y)$. The optimal cost $u_0(x)$ can be expressed as a discrete-time optimal impulse control similar to Bensoussan [2, Chapter 8, 89–132] (ignoring the constraint), but this not necessary for the analysis, since everything is based on the results obtained for the optimal stopping time problems discussed in section 3.

5.2 Solving the QVI

Define

$$u^0(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\infty \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t\Big\}, \quad \forall (x,y) \in E \times \mathbb{R}^+, \tag{33}$$

This function u^0 is the cost of no intervention, i.e., when the controller choose not to apply any impulse to the system. Since all cost are supposed nonnegative, the interval

$$C_b(u^0,Z) = \{v \in C_b(E \times \mathbb{R}^+) : 0 \leq v \leq u^0\}, \tag{34}$$

for either $Z = E \times \mathbb{R}^+$ or $Z = E$, contains the optimal cost either u or u_0, given by either (28) or (29).

To find a solution to the QVIs (30) and (31) set $u^0 = u_0^0 = u^0$ and consider the schemes

$$u^n(x,y) = \mathbb{E}_{xy}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau}\min\{Mu^{n-1},u^n\}(x_\tau,0)\Big\},$$
$$u_0^n(x) = \min\Big\{\mathbb{E}_{x0}\Big\{ \int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\tau}u_0^n(x_\tau)\Big\}, Mu_0^{n-1}(x)\Big\},$$

for $n \geq 1$, i.e., a sequence of optimal stopping times problems with constraint. Based on Theorem 3.1, each VI has a unique solution either $u(x,y)$ in $C_b(E \times \mathbb{R}^+)$ or u_0^n in $C_b(E)$ satisfying either/or

$$\begin{aligned} u^n(x,y) &= \inf_\theta \mathbb{E}_{xy}\Big\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta}Mu^{n-1}(x_\theta,0)\Big\},\\ u_0^n(x) &= \inf_\theta \mathbb{E}_{x0}\Big\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta}Mu_0^{n-1}(x_\theta)\Big\}, \end{aligned} \tag{35}$$

where the minimization is over all admissible (or zero-admissible) stopping times θ.

As in [29, Thms 4.2 and 4.3], we have

Theorem 5.1. *Let us suppose* Assumption 2.1 *and* (23)*,* (24)*,* (26)*. Then each of the sequences of functions* $\{u_0^n\}$ *and* $\{u^n\}$ *defined above, is monotone decreasing to the unique solution* u *in* $C_b(u^0, E \times \mathbb{R}^+)$ *and the solution* u_0 *in* $C_b(u^0,E)$*, of the QVIs* (30) *and* (31)*. Moreover, the estimate: there exist constants* $C > 0$*,* $0 < r < 1$ *such that*

$$|u^n(x,y) - u(x,y)| + |u_0^n(x,y) - u_0(x,y)| \leq Cr^n, \quad \forall (x,y) \in E \times \mathbb{R}^+,$$

for all $n \geq 1$*, as well as the relations* (32)*,*

$$u(x,y) = \inf_\theta \mathbb{E}_{xy}\Big\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta}Mu(x_\theta,0)\Big\},$$
$$u_0(x) = \inf_\theta \mathbb{E}_{x0}\Big\{ \int_0^\theta \mathrm{e}^{-\alpha t} f(x_t,y_t)\mathrm{d}t + \mathrm{e}^{-\alpha\theta}Mu_0(x_\theta)\Big\},$$

hold true, where the minimization is over (zero-)admissible stopping times θ*. Furthermore,* u^n *and* u *belong to the domain* $\mathscr{D}(A_{x,y}) \subset C_b(E \times [0,\infty[)$ *of the infinitesimal generator* $A_{x,y}$*, and* $u(x,y)$

$$-A_{x,y}u(x,y)+\alpha u(x,y)+\lambda(x,y)\big[u(x,0)-(Mu(\cdot,0))(x)\big]^{+}= \\ = f(x,y), \quad \forall (x,y)\in E\times\mathbb{R}^{+},$$

$$-A_{x,y}u^{n}(x,y)+\alpha u(x,y)+\lambda(x,y)\big[u^{n}(x,0)-(Mu^{n-1}(\cdot,0))(x)\big]^{+}= \\ = f(x,y), \quad \forall (x,y)\in E\times\mathbb{R}^{+},\ \forall n\geq 1,$$

are equivalent to the corresponding QVI and VI.

Proof. Only a short idea of the main points in the proof are mentioned. First, a decreasing and concave mapping is defined with the expressions in (35), and following an argument similar to the one used in Hanouzet and Joly [16], the exponential convergence/estimate is proved and a fixed point (solving the QVIs) is obtained. At this point, the remaining assertions are obtained with a little more work.

In the following Theorem, all assertions are written for the optimal cost (28), but a similar result holds true for the other optimal cost (29), with the zero-admissible impulse controls.

Theorem 5.2. *Under the assumptions as in* Theorem 5.1*, the unique solution of the QVI equation* (30) *is the optimal cost* (28)*, i.e.,* $u(x,y)=\inf\{J_{x,y}(\nu):\nu\in\mathscr{V}\}$*, for every* (x,y) *in* $E\times\mathbb{R}^{+}$*. Moreover, the first admissible exit time of the continuation region provides an optimal impulse control.*

Proof. The arguments are the same as in [29, Thms 4.4 & 4.5], there are no changes in assuming only E locally compact (instead of compact), only the compactness of $\Gamma(x)$ is necessary. Most of the discussion involves some martingale properties.

Note that if u is the optimal cost then (1) the continuation region $[u<Mu]$ is defined as all (x,y) in $E\times\mathbb{R}^{+}$ such that $u(x,y)<Mu(x,0)$, (2) the optimal jump-to is a Borel minimizer $\hat{\xi}(x)$ of $Mu(x,0)$, i.e., $x\mapsto\hat{\xi}(x)$ is a Borel functions from E into $\Gamma(x)$ and $c(x,\hat{\xi}(x))+u(\hat{\xi}(x),0)=Mu(x,0)$, for every x in E., and (3) the first exit time of $[u<Mu]$ is defined as

$$\hat{\theta}(x,y,s)=\inf\big\{t>s: u(x_{t-s},y_{t-s})=Mu(x_{t-s},0),\quad y_{t-s}=0\big\},$$

and $\hat{\theta}(x,y,s)=\infty$ if $u(x_{t-s},y_{t-s})<Mu(x_{t-s},0)$ for every $t>s$ such that $y_t=0$. Note that he Markov process $t\mapsto(x_{t-s},y_{t-s})$, for $t\geq s$, represents the initial condition $(x_s,y_s)=(x,y)$. Moreover, the continuity ensures that

$$u(x_{\hat{\theta}(x,y,s)-s},0)=c\big(x_{\hat{\theta}(x,y,s)-s},\hat{\xi}(x_{\hat{\theta}(x,y,s)-s})\big)+u\big(\hat{\xi}(x_{\hat{\theta}(x,y,s)-s}),0\big),$$

whenever $\hat{\theta}(x,y,s)<\infty$.

Therefore, the evolution under the above feedback (or Markov impulse control as in Definition 4.1-iv) and initial conditions (x,y) is as follows:

(1) first $\theta_1=\hat{\theta}(x,y,0)$ and $\xi_1=\hat{\xi}(x_{\theta_1})$ when $\theta_1<\infty$ (we may use an isolated 'coffin' state ∂ to set $x_\infty=\partial$ and $\hat{\xi}(\partial)=\partial$),

(2) next $\theta_{k+1}=\hat{\theta}(\xi_k,0,\vartheta_k)$, for any $k\geq 1$.

This is an *optimal* admissible impulse control $\hat{\nu} = \{(\theta_k, \xi_k) : k \geq 1\}$, which is proved in the same way as for the case E compact.

6 Ergodic Cost

This section is an extension of [30] to a locally compact space E.

6.1 Setting-up

We define the average cost to be minimized, as

$$\begin{aligned} J^T(0,x,y,\nu) &= \mathbb{E}^{\nu}_{xy}\Big\{\int_0^T f(x^{\nu}_s, y^{\nu}_s)\mathrm{d}s + \sum_i \mathbb{1}_{\theta_i \leq T} c(x^{i-1}_{\theta_i}, \xi_i)\Big\}, \\ J(x,y,\nu) &= \liminf_{T\to\infty} \frac{1}{T} J^T(0,x,y,\nu), \end{aligned} \tag{36}$$

the ergodic control problem is to characterize

$$\mu(x,y) = \inf_{\nu\in\mathscr{V}} J(x,y,\nu), \tag{37}$$

and to find an optimal control. The auxiliary problem is concerned with

$$\begin{aligned} \mu_0(x,y) &= \inf_{\nu\in\mathscr{V}_0} \tilde{J}(x,y,\nu), \quad \text{with} \\ \tilde{J}(x,y,\nu) &= \liminf_{n\to\infty} \frac{1}{\mathbb{E}^{\nu}_{xy}\{\tau_n\}} J^{\tau_n}(0,x,y,\nu), \end{aligned} \tag{38}$$

and $J^{\tau_n}(0,x,y,\nu)$ as in (36) with $T = \tau_n$. Actually, as seen later, $\mu(x,y) = \mu_0(x,y)$ is a constant.

The Dynamic Programming Principle shows (heuristically, see [30, Section 3] that, with $w_0(x) = w_0(x,0)$,

$$\begin{aligned} w_0(x) &= \min\Big\{\mathbb{E}_{x0}\Big\{\int_0^{\tau} [f(x_t,y_t) - \mu_0]\mathrm{d}t + w_0(x_\tau)\Big\}, Mw_0(x)\Big\}, \\ w_0(x,y) &= \mathbb{E}_{xy}\Big\{\int_0^{\tau} [f(x_t,y_t) - \mu_0]\mathrm{d}t + w_0(x_\tau)\Big\}, \end{aligned} \tag{39}$$

are the corresponding Hamilton-Jacobi-Bellman (HJB) equations in a weak form with two unknowns μ_0 and w_0. Note that M is an operator in the variable x alone, so that $Mw_0(x,y) = [Mw_0(\cdot,y)](x)$ as given by (25). Also, both problems are related (logically) by the condition

$$w(x,y) = \mathbb{E}_{xy}\Big\{\int_0^{\tau} [f(x_t,y_t) - \mu_0]\mathrm{d}t + w_0(x_\tau)\Big\}, \tag{40}$$

and so, if $w_0(x)$ is known then the last/first equality yields $w(x,y)$ and $w_0(x,y)$. Recall that τ is defined by (7) and that since $w(x,y) = w_0(x,y)$ for any $x \in E$ and $y > 0$, it may be convenient to write $w_0(x) = w_0(x,0)$ as long as no confusion arrives. Note that the functions $w(x,y)$ and $w_0(x)$ may be called *potentials*, and a priori, they are not *costs*, but they are used to determine an optimal control.

6.2 Solving the HJB

An important point to mention is to remark that the HJB equation (39) is equivalent to

$$w_0(x) = \min\big\{Mw_0(x), \ell(x) - \mu_0\tau(x) + Pw_0(x)\big\}, \tag{41}$$

where

$$\ell(x) = \mathbb{E}_{x0}\Big\{\int_0^{\tau} f(x_s,y_s)\mathrm{d}s\Big\}, \quad \tau(x) = \mathbb{E}_{x0}\{\tau\}, \tag{42}$$

with τ as in (7), and in view of the property (4),

$$Ph(x) = \mathbb{E}_{x0}\{h(x_\tau)\}, \tag{43}$$

defines the operator P on $C_b(E)$. Note that (10) yields

$$0 \le \ell(x) \le a_2\|f\|. \tag{44}$$

Moreover, from the Feller property of x_t and the law of τ, it follows that $\ell(x)$ is continuous.

In addition to the hypotheses of Sections 2 and 4, we assume that there exists a positive measure m on E such that

$$m(E) > 0 \quad \text{and} \quad P(x,U) \ge m(U), \quad \forall U \in \mathscr{B}(E), \tag{45}$$

where $P(x,U) = \mathbb{E}_{x0}\mathbb{1}_U(x_\tau)$, with τ defined by (7), and $\mathscr{B}(E)$ is the Borel σ-algebra on E.

Remark 6.1. From

$$P(x,U) = \mathbb{E}_{x0}\Big\{\int_0^{\infty} \lambda(x_t,t)\exp\Big(-\int_0^t \lambda(x_s,s)\mathrm{d}s\Big)\mathbb{1}_U(x_t)\mathrm{d}t\Big\}.$$

and Remark 2.3, one can check that (45) is satisfied when the transition probability of x_t has a density with respect to a probability on E satisfying: for every $\varepsilon > 0$ there exists $k(\varepsilon)$ such that

$$p(x,t,x') \geq k(\varepsilon) > 0, \text{ on } E \times [\varepsilon, \infty[\times E. \tag{46}$$

This is the case, for instance, for periodic diffusion processes, see Bensoussan [1], and for reflected diffusion processes with jumps, see Garroni and Menaldi [14, 15] (which is also valid for reflected diffusion processes without jumps). Furthermore, a simple example for E locally compact is provided by a pure jump process with generator

$$A_x g(x) = b(x)\Big\{ \int_E g(z) q(x,\mathrm{d}z) - g(x) \Big\}.$$

One can check that (45) is satisfied if, for instance, $0 < k_0 \leq \lambda(x,y) \leq k_1$, $0 < b_1 \leq b(x) \leq b_2$, $q(x,B) \geq m_0(B)$ for a positive measure m_0, with with $m_0(E) > 0$. □

Lemma 6.1. *Under assumption* (45)*, there exist a positive measure γ on E, and a constant $0 < \beta < 1$ such that $P(x,B) \geq \tau(x)\gamma(B)$, for every $B \in \mathscr{B}(E)$, any $x \in E$, with $\tau(x)\gamma(E) > 1 - \beta$.* □

Theorem 6.1. *Under* Assumption 2.1 *and* (23), (24), (26)*, as well as* (45)*, there exists a solution (μ_0, w_0) in $\mathbb{R}^+ \times C_b(E)$ of* (41)*, and therefore, of* (39). □

For details of the Lemma 6.1 and Theorem 6.1 proofs, note that Kurano [21, 22] results hold true for a locally compact space E, and refer to [30, Lem 4.1 and Thm 4.2]. For instance, the assumptions (45) and (11) imply

$$P\mathbb{1}_B(x) =: P(x,B) \geq \tau(x)\gamma(x), \quad \forall B \in \mathscr{B}(E),$$

with $\gamma(B) = m(B)/a_2$ and any β in $]0,1[$ such that $1 - \beta < m(E)a_1/a_2$. Now, the HJB equation (41) can be written as

$$w_0(x) = \inf_{\xi \in \Gamma(x) \cup \{x\}} \big\{ \ell(\xi) + \mathbb{1}_{\xi \neq x} c(x,\xi) - \mu_0 \tau(\xi) + P w_0(\xi) \big\}.$$

Since $P'(x,\mathrm{d}z) := P(x,\mathrm{d}z) - \tau(x)\gamma(\mathrm{d}z)$ satisfies $P'(x,E) < \beta < 1$, the operator

$$Rv(x) = \inf_{\xi \in \Gamma(x) \cup \{x\}} \Big\{ \ell(\xi) + \mathbb{1}_{\xi \neq x} c(x,\xi) + P w_0(\xi) - \tau(\xi) \int_E v(z)\gamma(\mathrm{d}z) \Big\}$$

is a contraction on $C_b(E)$ having a unique fixed point w_0, and moreover, $w_0 \geq 0$ because $\ell(x) \geq 0$ and $c(x,\xi) > 0$. Thus, (μ_0, w_0) is a solution, where $\mu_0 := \gamma(w_0)$, the integral of w_0 with respect to $\gamma(\cdot)$ on E.

Remark 6.2. When λ does not depends on x, the function $\tau(x)$ is constant and (41) is the HJB equation of a standard discrete time impulse control problem as studied in Stettner [37, Section 4] for $\Gamma(x) = \Gamma$ fixed. □

Then, we have

Theorem 6.2. *Under the assumptions as in* Theorem 6.1*, the constant μ_0 obtained in* Theorem 6.1 *satisfies*

$$\mu_0 = \inf\{\tilde{J}(x,0,\nu) : \nu \in \mathscr{V}_0\}$$

and there exists an optimal feedback control based on the exit times of the continuation region $[w_0 < Mw_0]$.

Proof. First, by means of Theorem 6.1 when $\Gamma(x) = \{x\}$ (which means 'no control'), we show that there exists $(j,h) \in \mathbb{R}^+ \times C_b(E)$ solution of

$$h(x) = \ell(x) - j\tau(x) + Ph.$$

Note that assumption (45) implies that P has a unique invariant probability denoted by $\zeta_0(\mathrm{d}x)$, see the book Hernández-Lerma [17, Section 3.3, pp. 56–61].

Thus, there are two cases: $\mu_0 = j$ and $\mu_0 < j$. First, for $\mu_0 = j$, from the equation for h and the fact that $X_n = x_{\tau_n}$ is a Markov chain, we have

$$j = \liminf_n \frac{1}{\mathbb{E}_{x0}\{\tau_n\}} \mathbb{E}_{x0}\Big\{\sum_{i=0}^{n-1} \ell(X_i)\Big\}$$
$$= \liminf_n \frac{1}{\mathbb{E}_{x0}\{\tau_n\}} \mathbb{E}_{x0}\Big\{\int_0^{\tau_n} f(x_t,y_t)\mathrm{d}t\Big\} = \tilde{J}(x,0,\nu),$$

with $\nu = 0$, i.e., no impulse at all. Then, as in [30, Thm 5.1] we have $\mu_0 \le \tilde{J}(x,0,\nu)$, for every ν in $\mathscr{V}_0$, i.e., $\mu_0 \le j$. Therefore, if $\mu_0 = j$ then

$$\mu_0 = \inf\{\tilde{J}(x,0,\nu) : \nu \in \mathscr{V}_0\} = j = \tilde{J}(x,0,0),$$

and $\nu = 0$, i.e., 'no impulses at all', is optimal.

Next, the case $\mu_0 < j$ is treated as in [30, Thm 5.1], with $\tilde{w}(x) = w_0(x) - h(x)$, $\tilde{\ell}(x) = (j-\mu_0)\tau(x)$, $\tilde{w} = \min\{M\tilde{w}, \tilde{\ell} + P\tilde{w}\}$. Indeed, using the results in Bensoussan [2, Section 7.4, pp. 74–77], we show that this discrete time problem has an optimal control $\hat{\nu} = \{(\hat{\eta}_i, \hat{\xi}_i) : i \ge 1\}$ given by

$$\hat{\eta}_i = \inf\{n \ge \hat{\eta}_{i-1} : w_0(X_n) = Mw_0(X_n)\},$$

where X_n is the controlled Markov chain and $\hat{\xi}_i = \hat{\xi}(X_{\hat{\eta}_i})$ with a measurable selector $\hat{\xi}(x)$ realizing the infimum in $Mw_0(x)$. This is translated in continuous time as $\hat{\theta}_i = \tau_{\hat{\eta}_i}$ and $\hat{\xi}_i = \hat{\xi}(x_{\hat{\theta}_i})$.

Remark 6.3. It is clear that the previous argument about (j,h) shows that the hypothesis (5.1) in our previous paper [30] is not really necessary, and therefore, it is a small improvement on it. □

Theorem 6.3. *Under the assumptions as in* Theorem 6.1, *the constant* μ_0 *obtained in* Theorem 6.1 *satisfies*

$$\mu_0 = \inf\{J(x,0,\nu) : \nu \in \mathscr{V}\} = J(x,y,\hat{\nu}),$$

where $\hat{\nu}$ *is obtained by* τ*-translations from the optimal control in Theorem 6.2.*

Proof. (sketch) The first step is to show that $w(x,y)$ defined by (40) satisfied

$$-A_{xy}w(x,y)+\lambda(x,y)[w(x,0)-Mw(x,0)]^{+}=f(x,y)-\mu_0$$

Actually, this is not surprising in view of the results for the discounted case, but the proof is somewhat cumbersome, see [30, Proposition 5.5].

This implies that the process

$$M_T=\int_0^T [f(x_t,y_t)-\mu_0]\mathrm{d}t+w(x_T,y_T), \quad T\geq 0$$

is a submartingale, and the argument is completed as in [30].

*Remark 6.4 (**Ergodic cost: A more general ergodic assumption**).* The assumption (45) is not satisfied, in general, for diffusion processes in the whole space, and thus, it is perhaps, relatively restrictive. A 'localized' substitute for (45) could be the assumption:

(i) there exist a closed set C, an open set D, $C\subset D\subset E$, and a constant $\beta_0\in]0,1[$ as well as a probability m satisfying $0<m(C)<1=m(D)$ and such that $P(x,B)\geq\beta_0 m(B)$, for every $B\in\mathscr{B}(E)$, any $x\in E$; and

(ii) there exist a continuous function $W: E\to[1,\infty[$, and constant $\beta\in]0,1[$ such that PW is continuous and $PW(x)\leq\beta W(x)+\beta_0\mathbb{1}_C\int_C W(z)m(\mathrm{d}z)$, for every $x\in E$.

An adaptation of Jaskiewicz [19] allows us to obtain a solution (μ_0,w_0) of (41), with w_0 in the weighted-space

$$C_W(E)=\left\{g \text{ continuous and } \sup_x\left\{\frac{|g(x)|}{W(x)}\right\}<\infty\right\},$$

and to obtain Theorem 6.2, under some additional technical assumptions. Also, Theorem 6.3 can be obtained under the additional assumption

$$\mathbb{E}\{\mathrm{e}^{-k_0 t}W(x_t)\}\leq W(x), \quad \forall x\in E,\ t>0,$$

where $\lambda(x,y)\geq k_0>0$ for every x,y. A detailed analysis will be in a paper in preparation [31] together with examples satisfying the various assumptions. This analysis is based on several references (e.g., Hernández-Lerma and Lasserre [18], Meyn and Tweedie [33, 32], among others). □

7 Extension

As in [28, 29, 30] let us mention some possible extensions:

- A variable discount factor $\alpha(x,y)$ instead of α constant, as well as a finite-horizon cost.

- Letting the discount factor $\alpha \to 0$ in the optimal discounted costs $u^\alpha(x,y)$ and $u^\alpha(x) = u^\alpha(x,0) = u_0^\alpha(x,0)$ we expect to obtain ergodic costs, e.g., if $\mu_\alpha = \alpha u^\alpha(x)$ and $w_0^\alpha(x) = u^\alpha(x) - u^\alpha(x_0)$ then $\mu_\alpha \to \mu_0$ and $w_0^\alpha(x) \to w_0(x)$, but this is still something to be properly shown, when $\Gamma(x)$ is not reduced to a fixed compact.
- A quantify signal, e.g, y_t has jumps back to $\{0,1,2\}$ instead of only $\{0\}$ with the following meaning: there are three classes of impulse controls $\mathscr{V}_0 \subset \mathscr{V}_1 \subset \mathscr{V}_2$ that are enabled only and accordingly to the value of y_t (some more details are necessary for a convenient example). In this case, instead (7), the signals are given by the functional

$$\tau = \inf\{t > 0 : y_t \in I\}, \tag{47}$$

  where a prototype is $I = \{0,1,2\}$. In this case, the Markov chain will include also y_{τ_n}, i.e., $(Z_n, e_n) = (x_{\tau_n}, y_{\tau_n}, \mathrm{e}^{-\alpha\tau_n})$. We may think that as the waiting-time passes (indicated or represented by the process y_t) the necessity of 'controlling' increases and impulses to other regions (that previously were not allowed) becomes enabled, i.e., when $i < y_t < i+1$ then only the class $\mathscr{V}_i$ of impulse controls is available, which produces an impulse back to some $y = j < i+1$. Actually, a detailed example may be needed, and this is not discussed here.
  In this case, jumps should be always backward, i.e., y_t may jumps only to the values $0,1$ or 2 that are smaller that the value of y_t. Certainly, what is accomplished for three values could be applied for any finite number of values, and perhaps 'extrapolate' to infinite many values (as long as they are isolated values). Thus, $\psi(x,y)$ makes sense for the optimal stopping time problem (without any changes!) but the analysis within the impulse control could give some interesting surprises.
- For stopping time problems, recall that several extensions are possible, in particular the use of data with polynomial growth (instead of bounded). However, there are some extra complications for the impulse control problems.

8 Hybrid Models

The state of a continuous-time hybrid model has a continuous-type variable x (with cad-lag paths) and a discrete-type variable n (with cad-lag piecewise constant paths). The 'signal' is represented by the 'jumps' of n_t, and in general, this signal enable any possible change in the setting of the model, not only the 'possibility of controlling' as studied in this paper (an others). The general idea is that the usual evolution of the system is described by the component x_t, and 'once in a while' (or under some specific conditions) a discrete transition (i.e., a jump of n_t occurs) and everything may change, and the evolution continues thereafter. With this in mind, the signal (to act, e.g., to control the system as in our model) is given by the 'hitting time' of a set of states S, i.e., $\tau = \inf\{t > 0 : (x_t, n_t) \in S\}$, and this set S plays the role of a 'set-interface', where the continuous-type and discrete-type variables exchange information. This set-interface may be given a priori or used as part of the parameters of control. In our previous 'control with constraint' presentation, the discrete-type component n_t was ignored (because there are only on/off possibilities) and the continuous-type component x_t is actually composed by two parts (x_t, y_t), as they were called, the reduced state x_t and the signal process y_t. Thus, in our model, the set-interface S is $E \times \{0\}$, the same for every n (which is ignored, as mentioned earlier).

To present the problem studied in this paper as a hybrid model the 'details' (a) and (b) of Assumption 2.1 are not mentioned, and instead, assumptions directly on the functional (7) and the signal (8) are imposed, e.g., at least it is assumed (9), but for ergodic cost, the condition (10) is required. Also, some continuity is needed, i.e.,

$$(x,y) \mapsto \mathbb{E}_{xy}\{\mathrm{e}^{-\alpha\tau}\varphi(x_\tau)\} \text{ and } (x,y) \mapsto \mathbb{E}_{xy}\Big\{\int_0^\tau \mathrm{e}^{-\alpha t} f(x_t,y_t)\Big\}\mathrm{d}t \tag{48}$$

are continuous functions, for every φ in $C_b(E)$ and f in $C_b(E \times [0,\infty[)$. Most of the results in previous section are valid under these 'more general' assumptions, except those involving the specific form of infinitesimal generator A_y (3). To be more specific, the following results can be extended under these more general hypotheses: Theorem 3.1, without (22), for optimal stopping; Theorem 5.1 (without the formula regarding the generator), and Theorem 5.2 for the discounted cost; Theorem 6.1 and Theorem 6.2 (but not Theorem 6.3) for the ergodic cost. For instance, if we assume (9) and that signals given by (8) then define the time-interval between jumps $T_n = \tau_n - \tau_{n-1}$, which (conditionally to x_t) forms an independent, identically distributed sequence of random variables with a non-negative and bounded intensity $\Lambda(x,y)$. Hence, the initial 'signal process' (which is not necessarily equal to the time elapsed since the last signal) can be replaced to obtain an equivalent (in most aspects) model as the one in this paper.

Indeed, let us make an example of a similar situation, i.e., a signal process $\tilde{y}_t$ which is not equal to the process y_t, the 'time elapsed since the last signal'. In this example, the state is $(x,\tilde{y})$, the controller is allowed to 'control' (via an impulse) when $\tilde{y} = 0$, however, $\tilde{y}_t$ has

$$A_z\varphi(z) = \partial_z\varphi(z) + \Lambda(z)[q\varphi(0) + (1-q)\varphi(z/2) - \varphi(z)],$$

as its infinitesimal generator, with $0 < q < 1$, i.e., the process z_t jumps at s_n, $\sigma_n = s_{n+1} - s_n$ are IID having an intensity $\Lambda(z)$, and at the jump-times, $z_{s_n} = 0$ with probability q and $z_{s_n} = z_{s_n-}/2$ with probability $1-q$. In this case, the functional of interest is always the same (7), namely, $\tau = \inf\{t > 0 : \tilde{y}_t = 0\}$, with the sequence of signals $\tau_{k+1} = \inf\{t > \tau_k : \tilde{y}_t = 0\}$, $\tau_0 = 0$, which are not necessarily the sequence of jump-times $\{s_n\}$ of the process $\tilde{y}_t$. However, if

$$F(t) = 1 - \exp\Big(-\int_0^t \Lambda(s)\mathrm{d}s\Big)$$

is the law of σ_1 for P_0 then the convolution $F^{*n}(t)$ is the law of $s_n = \sum_n \sigma_n$ and the law of $\tau = \tau_1$ (when $\tilde{y}_0 = 0$) is given by

$$P_0\{\tau \leq t\} = \sum_n F^{*n}(t) q(1-q)^{n-1} := G(t),$$

and the sequence of signals $\{\tau_k\}$ define another sequence $T_k = \tau_{k+1} - \tau_k$ of IID random variables with law $G(t)$. Hence, if we take $\lambda(t) = G'(t)/(1-G(t))$ then the control problem for $(x_t, \tilde{y}_t)$ and $f(x)$ (i.e., independent of $\tilde{y}$) should be equivalent to the problem (x_t, y_t), with y_t constructed from $\lambda(y)$, since the discrete problems are identical for $(x_t, \tilde{y}_t)$ and (x_t, y_t). It is clear that theses considerations can be extended to a similar model (x_t, z_t)

$$A_z\varphi(z) = b(z)\partial_z\varphi(z) + \Lambda(z) \int_{\mathbb{R}^+} \big(\varphi(\zeta) - \varphi(z)\big) m(z, \mathrm{d}\zeta),$$

under suitable assumptions on the drift b and the probability kernel $m(\mathrm{d}\zeta, z)$.

Another kind of problem could have the constraint 'control is allowed at any jump of z_t', with x_t as the reduced state process and z_t as the signal process. For this model, the condition, 'when the process z_t jumps' is not exactly the same as 'when z_t vanishes'. In other words, technically speaking, the full state of the system needs something else that the knowledge of (x,z), i.e., we need to know z_t and z_{t-} to check if a jump has really occurred. Thus, if $z_t = \tilde{y}_t$ above, then we would have $\tau_n = s_n$, the jump-times of z_t. For the (x_t, y_t) model (as well as for the hybrid model) presented in the above sections, the constraint "control is allowed only ..." 'when y_t jumps' is exactly the same as saying 'when y_t vanishes'. Nevertheless, we may have an infinitesimal generator like A_z (of the piecewise deterministic process z_t – or something else–) with a $b(z) > 0$ and $m(\varphi, z) = \varphi(0)$, which is not exactly the process y_t (the time elapsed since the last signal), but it has the property that $z_t = 0$ iff $y_t = 0$. Thus, for those type of processes, the constraint "control is allowed only when z_t vanishes" is equivalent to "control is allowed only when y_t vanishes".

Because of the particular meaning of our signal process y_t as the 'time elapsed since last signal', we obtain more detailed results than in the general hybrid model. Therefore, there are many generalization in various directions, e.g., in between to consecutive signals some other type of control could be allowed, signals of various types enabling particular types of controls may be given, and many other ways on how a continuous-type and a discrete-type variables may interact. Actually, much more details on the (hybrid) model are necessary to advance further in this discussion, and this is part of our book in preparation Jasso-Fuentes et al. [20], which follows some the problems discussed in Bensoussan and Menaldi [5, 6] and [27].

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