Depinning of Traveling Waves in Ergodic Media

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Abstract We study speed of moving fronts in bistable spatially inhomogeneous media at parameter regimes where the speed tends to zero. We provide a set of conceptual assumptions under which we can prove power law asymptotics for the speed, with exponent depending a local dimension of the ergodic measure near extremal values. We also show that our conceptual assumptions are satisfied in a context of weak inhomogeneity of the medium and almost balanced kinetics, and compare asymptotics with numerical simulations. The presentation is based on a joint work with Arnd Sheel.

1 Pinning in Traveling Wave Equations

Reaction–diffusion equations describe natural phenomenon in chemistry, biology, physics, and economics and are intensively studied in the last decades. In the simplest one-dimensional form, it can be written as follows:

$$\begin{aligned} u_t &= u_{xx} + f(u), \\ u &\to U_{\pm 1}, \quad \text{as } x \to \pm\infty, \end{aligned} \tag{1}$$

where $u\colon \mathbb{R}_x \times \mathbb{R}_t \to \mathbb{R}$, $f\colon \mathbb{R} \to \mathbb{R}$, $U_{\pm 1} \in \mathbb{R}$, $f(U_{\pm 1}) = 0$. One of the most studied cases is when f is a derivative of a double-well potential $f(u) = F'(u)$ with two wells in values $u = U_{\pm 1}$. In that case, term u_{xx} in Eq. (1) pushes function u to become constant in space, whereas term $f(u)$ pushes $u(x,t) \to U_{\pm 1}$. Usually, a solution $u(x,t)$ converges, as $t \to \infty$, to a traveling wave solution $u(x,t) = v(x - ct)$. In that case, a lot of information can be picked up from a single parameter c, which describes the speed of the traveling wave. The special case when $c = 0$ corresponds to a stationary front and appears in a case of a symmetric potential F.

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The situation changes in a discrete environment. Let us consider a spatial discretization of a reaction–diffusion equation with a traveling wave solution, for instance, the discrete Nagumo/Schlogel equation

$$\dot{u}_n = \frac{1}{h^2}(u_{n+1} - 2u_n + u_{n-1}) + f(u_n), \tag{2}$$
$$f(u) = (u-a)(u^2-1),$$

where $a \in (-1, 1), a \neq 0$, which corresponds to a nonsymmetric double-well potential. If the step of discretisation h is too large, there appears a stationary front. This phenomenon is called pinning. In this research, we are interested in a bifurcation of a stationary front to a traveling wave with a change of the step of discretization h. For a review of the topic, see [6].

The pinning phenomenon is quite universal and appears in various contexts, such as periodic environments and forces [3], nonlocal interactions [1], etc. While plenty of heuristics are known near the bifurcation point (see, for instance, [2, 4]), there are only few rigorous results. In particular, the speed of a traveling wave after de-pinning is not known.

One of the known rigorous results on depinning is proved for the case of spatially periodic forces:

$$u_t = u_{xx} + (1-u)u(1+u) + \delta(l(x) + F), \tag{3}$$

where l is a 1-periodic function and $F \in \mathbb{R}$ is a parameter. In this case, there exists $F_c > 0$ such that if $F > F_c$ there exists a traveling wave, and for $F \leq F_c$ there exists a stationary solution of (3). Dirr and Yip proved asymptotics for the speed of the traveling wave for small δ and $F - F_c \sim \delta$, but still separated from 0; we refer for exact condition to [3]. In that case

$$\text{speed} \sim (F - F_c)^{1/2}. \tag{4}$$

See [9] for an overview.

2 Depinning Transition in Ergodic Media

We provide an abstract framework to study speed of a traveling wave in continuous inhomogeneous environment under depinning transition.

Consider an system in an abstract space

$$U_t = F(U, \theta; \mu), \tag{5}$$

where $U \in X$, a Banach space corresponding to $u(\cdot, t)$, $\Theta \in M$ a variable describing the environment, M is a smooth compact manifold, and μ is a depinning parameter. We assume that the system satisfies the following conditions (see [8]):

(C1) There exists a smooth flow S_ζ on M such that Eq. (5) have the symmetry $(T_\zeta U)_t = F(T_\zeta U, S_\zeta(\theta); \mu)$, where $T_\zeta : X \to X$ corresponds to a translation of U. Note that S_ζ could be interpreted as translation of the environment.

(C2) There exists a family of smooth one-dimensional manifolds $\mathcal{N}_\mu \subset X$ invariant under translation T_ζ and the flow restricted to it is generated by a C^2-vector field

$$\xi' = s(S_\xi(\theta); \mu), \tag{6}$$

where ξ is a coordinated on one-dimensional manifolds $\mathcal{N}_\mu$. Note that the form of (6) follows from condition (C1). Typically, existence of $\mathcal{N}_\mu$ can be obtained by establishing normal hyperbolicity in (5).

(C3) The function s is nondegenerate in the following sense: there is a unique $\theta_* \in \mathcal{M}$ such that $s(\theta; 0) > 0$ for $\theta \neq \theta_*$, $s(\theta_*; 0) = 0$, $\partial_\mu s(\theta_*; 0) > 0$, and $D^2_\theta s(\theta_*; 0) > 0$. This condition is the most difficult to be verified.

(C4) The flow S_ζ is ergodic with respect to an invariant measure ν on $\mathcal{M}$ with local dimension κ at point θ_*. In case when ν is the Lesbeugue measure, κ coincides with the dimension of M.

The term *ergodic media* refers to condition (C4).

Theorem 1 (Scheel–Tikhomirov, [8]) *If conditions (C1)–(C4) are satisfied, then for ν-almost $\theta \in M$ and small enough $|\mu|$ solution is pinned for $\mu < 0$ (i.e., $\xi(t)$ is bounded), solution is depinned for $\mu > 0$ (i.e., $\xi(t) \to \infty$ as $t \to \infty$), the speed $c(\mu) = \lim_{t\to\infty} \xi(t)/t$ have the following asymptotics:*

$$c(\mu) \sim \begin{cases} \mu^{1-\kappa/2}, & \kappa < 2, \\ (|\log(\mu)|^{-1}, & \kappa = 2, \\ 1, & \kappa > 2. \end{cases} \tag{7}$$

The proof is based on a skew product structure and notion of relative equilibria [5, 7].

The easiest example of an ergodic media, satisfying assumptions of Theorem 1, is a quasiperiodic media. Consider a modified Nagumo/Schlogel equation

$$u_t = u_{xx} + (u + \mu)(1 - u^2) + \varepsilon\alpha(x; \theta)g(u) \tag{8}$$

with a quasiperiodic inhomogeneity

$$\alpha(y; \theta) = \sum_{j=1}^{\kappa} \alpha_j \cos(\omega_j y + 2\pi\theta_j)$$

with rationally independent $(\omega_j)_{j=1,\dots,\kappa}$ and $\alpha_j \neq 0$. The function $g(u)$ satisfies technical assumptions $g(\pm 1) = g'(\pm 1) = 0$, $g \in C^2$; see [8].

3 Depinning Conjecture in Discrete Quasiperiodic Media

Note that results of a previous section do not provide a rigorous proof of depinning speed asymptotic in discrete media. The author was not able to find an asymptotic behavior $c(h)$ of a traveling wave solution $u_n(t) = v(n - ct)$ of (2) near the depinning transition either.

Analogy between pinning in discrete media and continuous periodic media and asymptotics (7) suggests to study depinning in nonhomogeneous discrete media. Consider nonhomogeneous discrete Nagumo–Schlogel equation

$$\begin{aligned} \dot{u}_n &= d(u_{n+1} - 2u_n + u_{n-1}) + (u_n - a_n)(1 - u_n^2), \\ u_n &\to \pm 1, \quad \text{as } n \to \pm 1, \end{aligned} \tag{9}$$

where $a_n = a + \varepsilon \sum_{j=1}^{k} b_n \cos(2\pi\omega_j n + \theta_j)$ is a quasiperiodic sequence, with ω_j rationally independent with $1, b_j, \theta_j \in \mathbb{R}$ and k is a number of additional frequencies. Results of numerical simulations (se Fig. 1 and [8]) show that the behavior of speed of wave propagation strongly depends on the value of k, where speed is defined as

$$\text{speed} = \frac{1}{2} \lim_{t\to\infty} \frac{1}{t} \sum_{n=-\infty}^{+\infty} u_n(t) - u_n(0).$$

While it is hard to make a good conjecture based only on numerical simulations, Theorem 1 suggests the following.

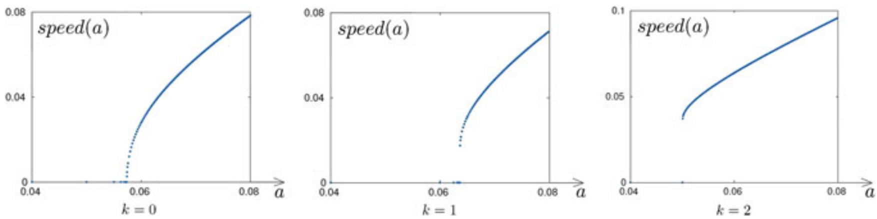

Fig. 1 Speed of wave propagation as a function of a for $k = 0, 1, 2$

Conjecture 2 *There exists $a_c > 0$ such that if $a \in (0, a_c)$ then Eq.* (9) *admits a stationarily solution. If $a > a_c$, then there exists a moving solution with an average speed behavior as $a \to a_c$*

$$speed(a) \sim \begin{cases} (a - a_c)^{1/2}, & k = 0, \\ (|\log(a - a_c)|^{-1}, & k = 1, \\ 1, & k \geq 2. \end{cases} \tag{10}$$

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