Breathing as a Periodic Gas Exchange in a Deformable Porous Medium

Michela Eleuteri, Erica Ipocoana, Jana Kopfová and Pavel Krejčí

Abstract We propose to model the mammalian lungs as a viscoelastic deformable porous medium with a hysteretic pressure–volume relationship described by the Preisach operator. Breathing is represented as an isothermal time-periodic process with the gas exchange between the interior and exterior of the body. The main result consists of proving the existence of a periodic solution under an arbitrary periodic forcing in suitable function spaces.

1 Introduction

As pointed out in [6], the first measurements which showed a hysteretic pressure–volume characteristic in mammalian lungs were obtained in [2] in 1913. A mechanical system combining linear viscoelasticity with the rate-independent Prandtl model of elastoplasticity was used by J. Hildebrandt in [5] to describe the breathing process of cats. Here we refer to the analysis carried out by D. Flynn in [4], where the Preisach operator is shown to be an appropriate model for the pressure–volume hysteresis relationship in lungs.

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M. Eleuteri (✉) · E. Ipocoana
Università Degli Studi di Modena e Reggio Emilia, Modena, Italy
e-mail: michela.eleuteri@unimore.it

E. Ipocoana
e-mail: 207531@studenti.unimore.it

J. Kopfová
Silesian University Opava, Opava, Czech Republic
e-mail: jana.kopfova@math.slu.cz

P. Krejčí
Faculty of Civil Engineering, Czech Technical University, Prague, Czech Republic
e-mail: krejci@math.cas.cz

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Our work focuses on representing the breathing as an isothermal, time-periodic process described by a PDE system with hysteresis. It consists of the momentum balance equation and the mass balance, similarly as in a more general study of deformable porous media in [3], but with different boundary conditions. Instead of prescribing boundary displacement as in [3], we prescribe here mechanical reaction between lungs and their surroundings. Since viscosity is present in our model, we also do not have any restriction on the input amplitude. The mathematical problem thus consists of proving that our PDE system with a degenerating Preisach operator under the time derivative admits a periodic solution for every periodic boundary forcing with a given regularity.

2 The Model

Let u denote the displacement vector in the solid, σ the stress tensor, q the gas mass flux and s the gas mass content in the pores. Similarly as in [1], we assume that the system is governed by the momentum balance equation

$$\rho u_{tt} = \mathrm{div}\sigma, \tag{1}$$

where ρ is the solid mass density, and by the gas mass balance

$$s_t + \mathrm{div} q = 0, \tag{2}$$

where q is the mass flux. Then we introduce two constitutive relations. In the first one we have

$$\sigma = \mathbf{B}\nabla_s u_t + \mathbf{A}\nabla_s u - p\ \delta, \tag{3}$$

where $\mathbf{B}$, representing viscosity, and $\mathbf{A}$, representing elasticity, are symmetric positive-definite constant tensors of order 4, the symbol ∇_s denotes the symmetric gradient, p is the air pressure, and δ is the Kronecker tensor. The second constitutive relation links pressure and volume in the form

$$f(p) + G[p] = \frac{1}{\rho_a} s - \mathrm{div}\ u, \tag{4}$$

where $\rho_a > 0$ is the referential air mass density at standard pressure, $f : \mathbb{R} \to (0, \infty)$ is an increasing function, and G is a Preisach operator.

Under the small deformation hypothesis, the term div u represents the void volume difference with respect to the reference state, it means that, at constant pressure, if div u increases, then s/ρ_a increases at the same rate. Similarly, at constant void volume, the mass content s is an increasing function (with different inflation and deflation curves) of the pressure. Eventually, at constant gas mass content, the pressure increases if the void volume decreases.

For the mass flux, we assume the Darcy law

$$q = -\rho_a \mu(x) \nabla p, \tag{5}$$

where $\mu(x) > 0$ is a permeability coefficient depending on space.

According to the previous analysis, the model reads

$$\rho u_{tt} = \operatorname{div}(\mathbf{B}\nabla_s u_t + \mathbf{A}\nabla_s u) - \nabla p\,, \tag{6}$$

$$(f(p) + G[p])_t = -\operatorname{div} u_t + \operatorname{div}\mu(x)\nabla p\,, \tag{7}$$

for x in a bounded connected Lipschitzian domain $\Omega \subset \mathbb{R}^3$ and for $t \geq 0$.

On the boundary $\partial\Omega$, we prescribe the following boundary conditions:

$$-\sigma \cdot n\big|_{\partial\Omega} = \beta(x)(\mathbf{C}u + \mathbf{D}u_t - g) + pn\,, \tag{8}$$

$$\frac{1}{\rho_a} q \cdot n\big|_{\partial\Omega} = \alpha(x)(p - h) - u_t \cdot n\,, \tag{9}$$

where n is the unit outward normal vector, $\beta \geq 0$ is the relative elasticity modulus of the boundary at the point $x \in \partial\Omega$, $\mathbf{C}$ and $\mathbf{D}$ are symmetric positive- definite 3×3 matrices, $g = g(x, t)$ is a given external force acting on the body Ω, $h = h(x, t)$ is the given outer air pressure, and $\alpha(x) \geq 0$ is the boundary permeability at the point $x \in \partial\Omega$.

The physical meaning of the first boundary condition in (8) is that on the part of the boundary where β is positive, the body Ω interacts with the exterior, which is viscoelastic with stiffness $\mathbf{C}$, viscosity $\mathbf{D}$, and active component g. There is no mechanical interaction with the exterior on the part of boundary where β vanishes. Similarly, the second boundary condition in (8) reflects the assumption that gas exchange proportional to the inner and outer pressure difference takes place on the part of the boundary where α is positive.

We now write Problems (6)–(7) in variational form for all test functions $\phi \in W^{1,2}(\Omega; \mathbb{R}^3)$ and $\psi \in W^{1,2}(\Omega)$ as follows:

$$\int_\Omega \big(\rho u_{tt}\phi + (\mathbf{B}\nabla_s u_t + \mathbf{A}\nabla_s u) : \nabla_s\phi + \nabla p\, \phi\big)\mathrm{d}x + \int_{\partial\Omega} \beta(x)(\mathbf{C}u + \mathbf{D}u_t - g)\phi\,\mathrm{d}s(x) = 0\,, \tag{10}$$

$$\int_\Omega \big((f(p) + G[p])_t\psi + (\mu(x)\nabla p - u_t)\nabla\psi\big)\mathrm{d}x + \int_{\partial\Omega} \alpha(x)(p - h)\psi\,\mathrm{d}s(x) = 0\,, \tag{11}$$

and the identities (10)–(11) are supposed to hold for a. e. $t > 0$.

2.1 Setting

Before presenting the main result of the present work, we must introduce the setting we need to study our problem. In particular, we have to make appropriate mathematical assumptions about different terms involved in the model.

- *Preisach operator*: Let $\gamma \in L^\infty((0,\infty)\times\mathbb{R})$ be a given function, $\gamma(r,v)\geq 0$ a.e., and there exists $B>0$ such that $\gamma(r,v)=0$ for $r+|v|\geq B$. We define

$$G[p]=\int_0^\infty\int_0^{\xi_r}\gamma(r,v)\mathrm{d}v\mathrm{d}r,$$

where $\xi_r=\mathfrak{p}_r[p]$ is the output of the play operator applied to p. Note that for a fixed initial distribution of the play operators and input $p\in L^q(\Omega;C_T)$, the output $G[p]$ of the Preisach operator is T-periodic for $t\geq T$, so that we can consider G as a (Lipschitz continuous) mapping $L^q(\Omega;C_T)\to L^q(\Omega;C_T)$.
- *Periodic spaces*: We fix a period $T>0$ and denote by L^q_T the L^q-space of T-periodic functions $v\colon\mathbb{R}\to\mathbb{R}$ for $q\geq 1$, and by C_T the space of continuous real T-periodic functions on $\mathbb{R}$. It is now quite natural to introduce $L^q_T(W^{1,2}(\Omega))$ and $L^q_T(W^{1,2}(\Omega,\mathbb{R}^3))$ of T-periodic L^q-functions $v\colon\mathbb{R}\to W^{1,2}(\Omega)$ and $v\colon\mathbb{R}\to W^{1,2}(\Omega,\mathbb{R}^3)$, respectively, as well as with the spaces $L^q(\Omega;C_T)$.
- *Nonlinearity*: $f:\mathbb{R}\to\mathbb{R}$ is a C^1-function such that there exist $0<f_0<f_1$ and $\omega\geq 0$ with the property

$$\frac{f_0}{1+p^2}\leq f'(p)\leq f_1(1+p^2)^\omega,$$

for all $p\in\mathbb{R}$.

Moreover, we assume that

(i) the permeability coefficient μ belongs to $L^\infty(\Omega)$ and there exists a constant $\mu_0>0$ such that $\mu(x)\geq\mu_0$ a.e.;
(ii) the nonnegative functions α and β belong to $L^\infty(\partial\Omega)$ and do not identically vanish, that is, $\int_{\partial\Omega}\beta(x)\mathrm{d}s(x)>0$, $\int_{\partial\Omega}\alpha(x)\mathrm{d}s(x)>0$;
(iii) the functions g, g_t belong to $L^2_T(L^2(\partial\Omega;\mathbb{R}^3))$, h, h_t belong to $L^2_T(L^2(\partial\Omega))$, $h\in L^\infty(\partial\Omega\times(0,T))$;
(iv) the symmetric positive-definite constant tensors **A**, **B** and symmetric positive-definite constant matrices **C**, **D** are given.

2.2 Main Result

Theorem 1 *Let the assumptions from Sect. 2.1 hold. Then system (10)–(11) has a solution* (u, p) *such that* $u, u_t, \nabla_s u, \nabla_s u_t \in L^2_T(L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(T, 2T; L^2(\Omega;\mathbb{R}^3))$, $u_{tt} \in L^2_T(L^2(\Omega;\mathbb{R}^3))$, $p_t, \nabla p \in L^2_T(L^2(\Omega))$, $p \in L^\infty(\Omega \times (T, 2T))$.

Proof The main ideas of the proof are the following:

(i) We take a cut-off function for f;
(ii) We use the Galerkin method, both in space and time, so we take suitable orthonormal bases;
(iii) As a result, we get an algebraic system which has a solution by a homotopy argument;
(iv) We derive a priori estimates with the help of Preisach energy inequality and Korn and Poincaré inequalities;
(v) In order to get uniform estimates in time, we test the cut-off equations by u_t and u_{tt}, regularizing in time when needed;
(vi) Eventually, we are able to remove the cut-off parameter using the Moser method. □

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