Slow Invariant Manifolds in the Problem of Order Reduction of Singularly Perturbed Systems

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Abstract The method of integral manifolds is used to study singularly perturbed systems of differential equations. The algorithms for the construction of the slow invariant manifolds in the case with different dimensions of the fast and slow variables was derived.

1 Introduction

Consider the system of differential equations

$$\dot{x} = f(x, y, \varepsilon), \tag{1}$$

$$\varepsilon\dot{y} = g(x, y, \varepsilon), \tag{2}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, ε is a small positive parameter, $0 < \varepsilon \ll 1$, functions f and g are continuous with respect to (x, y) for all $x \in \mathbb{R}^n$, $y \in D \subset \mathbb{R}^m$ $(D \subset \mathbb{R}^m)$. We will consider a situation where the system (1), (2) has an integral manifold, that is, when the following conditions are fulfilled (see [1, 4]):

(i) the equation $g(x, y, 0) = 0$ has an isolated solution $y = \psi_0(x)$ for $x \in \mathbb{R}^n$;
(ii) the functions f and g are uniformly continuous and bounded together with their partial derivatives with respect to all variables up to $(k + 2)$th order inclusively

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$(k \geq 0)$ in some region $\Omega_0 = \{(x, y, \varepsilon) : x \in \mathbb{R}^n,\ \|y - \psi_0(x)\| < \rho,\ 0 \leq \varepsilon \leq \varepsilon_0\}$;

(iii) the eigenvalues of the matrix

$$B(x) = \frac{\partial g}{\partial y}(x, \psi_0(x), 0)$$

satisfy the inequality $Re\lambda_i(x) \leq -2\gamma < 0$.

Note that some interesting aspects of the theory of slow integral manifolds and the behavior of solutions in their neighborhood were presented in [2, 3].

The degenerated system regarding to (1), (2) has a form

$$\begin{aligned} \dot{x} &= f(x, y, 0), \\ 0 &= g(x, y, 0). \end{aligned} \tag{3}$$

It should be noted that the equations of system (3) can often be either transcendental or polynomials of a high degree with respect to y. In these cases, a solution of the system cannot be found in explicit form as $y = \psi_0(x)$. In these cases, for the system-order reduction, it is possible to use a parametric form for the representation of the slow invariant manifolds [4, 5]. Below we consider the three major cases, where either fast variables, or only a fraction of the fast variables, or fast variables supplemented by a certain number of slow variables, play a role of the parameters.

2 The Case $n = m$

Consider the case of the dimensions equality of the fast and slow variables. Suppose that the system (3) can be solved with respect to x in the form $x = \varphi_0(y)$. In this case, the fast vector-variable y can play a role of a parameter for the representation of the slow invariant manifolds in the parametric form

$$x = \varphi(y, \varepsilon) = \varphi_0(y) + \varepsilon\varphi_1(y) + \cdots + \varepsilon^k\varphi_k(y) + \cdots . \tag{4}$$

The corresponding invariance equation is obtained by substituting (4) in (1):

$$\frac{\partial \varphi}{\partial y} g(\varphi, y, \varepsilon) = \varepsilon f(\varphi, y, \varepsilon). \tag{5}$$

For all functions in (5), we write the formal asymptotic expansions in the powers of the small parameter ε:

$$f\Big(\sum_{k\geq 0} \varepsilon^k \varphi_k, y, \varepsilon\Big) = \sum_{k\geq 0} \varepsilon^k f^{(k)}(\varphi_0, \ldots, \varphi_k, y),$$

$$g\Big(\sum_{k\geq 0}\varepsilon^k\varphi_k, y, \varepsilon\Big) = g^{(0)}(\varphi_0, y) + B(y)\sum_{k\geq 1}\varepsilon^k\varphi_k + \sum_{k\geq 1}\varepsilon^k g^{(k)}(\varphi_0, \dots, \varphi_{k-1}, y),$$

where $g^{(0)}(\varphi_0, y) = g(\varphi_0, y, 0)$ and the nondegenerate matrix $B(y) = g_x(\varphi_0, y, 0)$; see [1, 4]. Taking these expansions into account, the invariance equation (5) takes the form:

$$\sum_{k\geq 0}\varepsilon^k\frac{\partial\varphi_k}{\partial y}\Big(g^{(0)} + B\sum_{k\geq 1}\varepsilon^k\varphi_k + \sum_{k\geq 1}\varepsilon^k g^{(k)}\Big) = \varepsilon\sum_{k\geq 0}\varepsilon^k f^{(k)}.$$

Equating the coefficients at the same of like powers of ε in the last equation, we get the expressions, which uniquely define the coefficients in (4) when $\det(\partial\varphi_0/\partial y) \neq 0$.

Indeed, for ε^0 we have $g(\varphi_0, y, 0) = 0$, which give the function $\varphi_0(y)$. For ε^1, we get

$$\varphi_1 = \Big(\frac{\partial\varphi_0}{\partial y}B\Big)^{-1}\Big(f^{(0)} - \frac{\partial\varphi_0}{\partial y}g^{(1)}\Big).$$

Likewise, for ε^k we obtain

$$\varphi_k = \Big(\frac{\partial\varphi_0}{\partial y}B\Big)^{-1}\Big[f^{(k-1)} - \frac{\partial\varphi_0}{\partial y}g^{(k)} - \sum_{i=1}^{k-1}\frac{\partial\varphi_i}{\partial y}\big(B\varphi_i + g^{(k-i)}\big)\Big].$$

Thus, the parametric representation of the slow invariant manifold of (1), (2) is found in the form (4).

3 The Case $n < m$

Consider the case where the number of fast variables in the system (1), (2) exceeds the number of slow variables. Then, the system (3) contains m equations for n unknowns and $n < m$. We take all components of vector x ($\dim(x) = n$) complemented by $m - n$ components of vector y, as the unknowns. Thereby, the number of equations and unknowns in the system (3) will coincide.

Suppose that the solution of (3) can be written in the form

$$x = \varphi_0(y_2), \quad y_1 = \psi_0(y_2),$$

with a parameter y_2, where $y = (y_1, y_2)^T$, $\dim y_1 = m - n$, $\dim y_2 = n$. The system (1), (2) start the sentence with this can be rewritten in a more convenient form:

$$\dot{x} = f(x, y_1, y_2, \varepsilon), \tag{6}$$

$$\varepsilon\dot{y}_1 = g_1(x, y_1, y_2, \varepsilon), \tag{7}$$

$$\varepsilon \dot{y}_2 = g_2(x, y_1, y_2, \varepsilon). \tag{8}$$

We will find the slow integral manifold in the form

$$x = \varphi(y_2, \varepsilon), \tag{9}$$

$$y_1 = \psi(y_2, \varepsilon). \tag{10}$$

Substituting (9), (10) into (6) and (7), and taking into account (8), we obtain the invariance equations

$$\frac{\partial \varphi}{\partial y_2} g_2(\varphi, \psi, y_2, \varepsilon) = \varepsilon f(\varphi, \psi, y_2, \varepsilon),$$

$$\frac{\partial \psi}{\partial y_2} g_2(\varphi, \psi, y_2, \varepsilon) = g_1(\varphi, \psi, y_2, \varepsilon).$$

For the functions $\varphi(y_2, \varepsilon)$, and $\psi(y_2, \varepsilon)$ we write the formal asymptotic expansions:

$$\varphi(y_2, \varepsilon) = \varphi_0(y_2) + \varepsilon\varphi_1(y_2) + \cdots + \varepsilon^k \varphi_k(y_2) + \cdots, \tag{11}$$

$$\psi(y_2, \varepsilon) = \psi_0(y_2) + \varepsilon\psi_1(y_2) + \cdots + \varepsilon^k \psi_k(y_2) + \cdots \tag{12}$$

Equating the coefficients of the same powers of ε in the invariance equations, we get the expressions, which uniquely define the coefficients in (11) and (12) when $\det \partial\varphi_0/\partial y_2 \neq 0$ and $\det \partial\psi_0/\partial y_2 \neq 0$.

4 The Case $n > m$

Consider the case when the dimension of slow variables is greater than the dimension of fast variables. We call attention to the degenerate subsystem (3). It contains m equations for n unknowns, where $n > m$. To find the parametric representation of the slow invariant manifold of (1), (2), we take all components of y complemented by $n - m$ components of the vector x, as the parameters. Then a solution of the system (3) can be written in the parametric form $x_1 = \varphi_0(x_2, y)$, where $x = (x_1, x_2)^T$, $\dim x_1 = m$, $\dim x_2 = n - m$.

The system (1), (2) in this case can be rewritten in more convenient form as

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, y, \varepsilon), \\ \dot{x}_2 &= f_2(x_1, x_2, y, \varepsilon), \\ \varepsilon \dot{y} &= g_2(x_1, x_2, y, \varepsilon). \end{aligned} \tag{13}$$

We will find the slow integral manifold in the form

$$x_1 = \varphi(x_2, y, \varepsilon) = \varphi_0(x_2, y) + \varepsilon\varphi_1(x_2, y) + \cdots + \varepsilon^k \varphi_k(x_2, y) + \cdots . \quad (14)$$

The invariance equation

$$\varepsilon \frac{\partial \varphi}{\partial x_2} f_2(\varphi, x_2, y, \varepsilon) + \frac{\partial \varphi}{\partial y} g(\varphi, x_2, y, \varepsilon) = \varepsilon f_1(\varphi, x_2, y, \varepsilon) \quad (15)$$

is yielded from (13) and (14). Equating the coefficients at the same powers of ε in the last equation, we get the expressions, which uniquely define the coefficients in (14) for the case when $\det\Big(\partial\varphi_0/\partial y \;\; G\Big) \neq 0$.

Thus, formula (14) defines the slow integral manifold of the system in the parametric form.

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