A New Approach to Canards Chase in 3D

Elena Shchepakina

Abstract A new approach to the canards chase in 3D for some class of singularly perturbed systems is suggested. The proposed approach is discussed by the use of a competitive model of population dynamics. The presence of an exact black swan (a stable/unstable slow invariant manifold) makes it possible to find a new kind of trajectories with multiple stability changes.

1 Introduction

In this paper, a new approach to the canards chase for a class of singularly perturbed systems with two slow and one fast variables is proposed. This approach is based on the geometric theory of invariant manifolds of singularly perturbed systems [4–6]. Recall, that canards are trajectories of a singularly perturbed system which at first move along a stable slow invariant manifold and then continue for a while along an unstable slow invariant manifold. A slow invariant manifold is defined as an invariant surface of slow motions.

It should be noted that a 3D canard is a result of gluing the stable and unstable slow invariant manifolds at one point of the breakdown surface [7]. For a fixed gluing point, this is possible due to a proper choice of an additional scalar parameter of the differential system. In the proposed approach, two additional parameters are used to construct a 3D canard. Both these parameters correspond to the canards for 2D-projections of the original system. The proposed approach is illustrated via a model of competing populations.

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E. Shchepakina (✉)
Department of Differential Equations and Control Theory, Samara National Research University, Moskovskoye shosse 34, Samara 443086, Russian Federation
e-mail: shchepakina@ssau.ru

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2 A Competing Predators Model

Consider two predator species competing for a single prey in a constant and uniform environment. The singular perturbed model of the processes is the following (see [1]):

$$\dot{x} = x\left(\frac{m_1 z}{\beta_1 + z} - d_1\right), \tag{1}$$

$$\dot{y} = y\left(\frac{m_2 z}{\beta_2 + z} - d_2\right), \tag{2}$$

$$\varepsilon\dot{z} = z\left(1 - z - \frac{m_1 x}{\beta_1 + z} - \frac{m_2 y}{\beta_2 + z}\right). \tag{3}$$

Here, x and y are the dimensionless population densities of the predators; z is the dimensionless population density of the prey; $\varepsilon = 1/\gamma$, where γ is the intrinsic rate of growth of the prey; for $i = 1, 2$, $m_i > 0$ is the maximal growth or birth rate of the ith predator; $\beta_i = a_i/K$, where a_i is the half-saturation constant for the i-th predator, K is the carrying capacity of the prey; $d_i > 0$ is the death rate of the i-th predator.

3 2*D* Canards

Consider the case of the absence of one of the predators, i.e., when, for example, $y \equiv 0$. In this case the system (1)–(3) takes the form

$$\dot{x} = x\left(\frac{m_1 z}{\beta_1 + z} - d_1\right) := f(x, z), \tag{4}$$

$$\varepsilon\dot{z} = z\left(1 - z - \frac{m_1 x}{\beta_1 + z}\right) := g(x, z). \tag{5}$$

If we put $\varepsilon = 0$ into the fast subsystem, we get the *degenerate equation*

$$z\left(1 - z - \frac{m_1 x}{\beta_1 + z}\right) = 0,$$

which describes the *slow curve* S of (4) and (5); see [5, 6]. The curve S consists of the straight line $z = 0$ and the parabola. Two breakdown points,

$$A_1\left(x = \frac{\beta_1}{m_1}, z = 0\right), \quad A_2\left(x = \frac{(1 + \beta_1)^2}{4m_1}, z = \frac{1 - \beta_1}{2}\right),$$

divide S into the stable subsets (S_1^s and S_2^s) and the unstable subsets (S_1^u and S_2^u), see Fig. 1.

In an ε–neighborhood of the stable (unstable) subset S_2^s (S_2^u) of the slow curve, there exists the stable (unstable) slow invariant manifold $S_{2,\varepsilon}^s$ ($S_{2,\varepsilon}^u$). We can glue together $S_{2,\varepsilon}^s$ and $S_{2,\varepsilon}^u$ at the point A_2, using the standard procedure [4–7], to get a canard. For this, we consider d_1 as a gluing parameter. The canard and the corresponding parameter value $d_1 = d_1^c(\varepsilon)$ allow asymptotic expansions in powers of the small parameter ε:

$$z = h(x, d_1^c(\varepsilon), \varepsilon) = h_0(x, d_{10}) + \varepsilon h_1(x, d_{10}, d_{11}) + O(\varepsilon^3), \tag{6}$$

$$d_1^c(\varepsilon) = d_{10} + \varepsilon d_{11} + O(\varepsilon^2). \tag{7}$$

We can calculate the functions h_0, h_1, etc., from the *invariance equation*

$$\varepsilon \frac{\partial h}{\partial x} f\left(x, h(x, d_1^c(\varepsilon), \varepsilon)\right) = g\left(x, h(x, d_1^c(\varepsilon), \varepsilon)\right),$$

which follows from the system (4) and (5) and the asymptotic expansions (6) and (7). However, all functions in (6) have a discontinuity at the breakdown point A_2. A proper choice of d_{10}, d_{11}, etc., enables us to avoid this discontinuity. The outcome of this procedure is a canard shown by the green curve in Fig. 1. The canard corresponds to the canard point $d_1 = d_1^c$, where

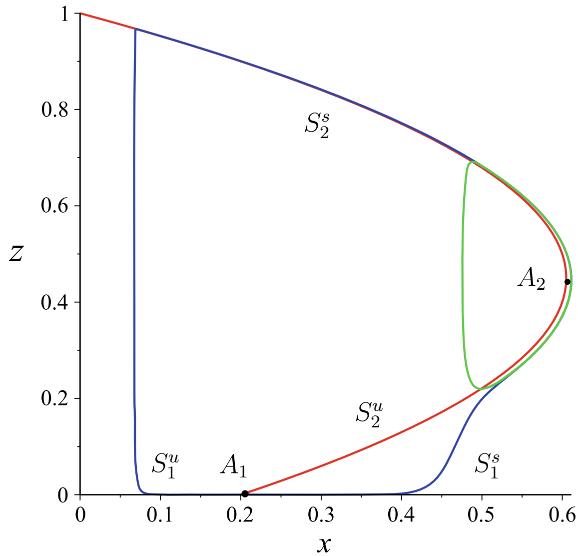

Fig. 1 The slow curve (red), the canard (green), and the canard doublet (blue) of (4) and (5)

$$d_1^c = \frac{m_1(1-\beta_1)}{1+\beta_1} - \varepsilon\frac{\beta_1^2(1+\beta_1)}{2(1-\beta_1)^2} + O(\varepsilon^2).$$

It should be noted that $z = 0$ is the exact canard of the system (4) and (5). In this special case, the trajectories of the system, starting in the basin of attraction of S_1^s, will continue their movement for a while along S_1^u. Therefore, we can transform the single canard, corresponding to the canard point $d_1 = d_1^c$, to a shape of a *canard doublet* (see the blue curve in Fig. 1) [2, 8]. Recall, that in the case of a planar system, the canards are exponentially close to each other near the slow curve and have the same asymptotic expansion (6) in powers of ε. An analogous assertion is true for corresponding parameter values (7). Namely, any two values of the parameter d_1, for which canards exist, have the same asymptotic expansions, and the difference between them is given by $\exp(-1/c\varepsilon)$, where c is some positive number. For example, for $\beta_1 = 0.1$, $m_1 = 0.5$, and $\varepsilon = 0.1$, the values of d_1 corresponding to the canard and the canard doublet in Fig. 1 are 0.408498400000 and 0.408498356366, respectively.

The results of this section can be extended to the case $x \equiv 0$ due to the competitive symmetry between the predators in (1)–(3). Similar reasoning gives the canard point d_2^c of the parameter d_2, where

$$d_2^c = \frac{m_2(1-\beta_2)}{1+\beta_2} - \varepsilon\frac{\beta_2^2(1+\beta_2)}{2(1-\beta_2)^2} + O(\varepsilon^2).$$

4 3*D* Canards

We now return to the 3D system (1)–(3). Substituting the canards points for the parameters d_1 and d_2 into the complete system (1)–(3), we get a canard in 3D. It should be noted that the discussed approach makes it possible to easily obtain various forms of 3D canards. It can be done by slightly changing the values d_1^c or/and d_2^c.

Note that $z \equiv 0$ is the exact slow invariant manifold, which is divided by the line $1 - m_1x/\beta_1 - m_2y/\beta_2 = 0$ into the stable and the unstable parts. Thus, $z \equiv 0$ is the *black swan* [3, 4]. The presence of the exact black swan allows us to obtain a new kind of trajectories with multiple changes of stability, a *cascade of 3D canard doublets*.

To obtain the trajectory shown in Fig. 2a, we transform the canard on yOz-plane to a shape of a canard doublet keeping the canard on xOz-plane. A shape of this trajectory can be modified, from the cascade of 3D canards without head to the cascade of 3D canard doublets that shown in Fig. 2b.

It should be noted that the considered situation, when a differential system possesses an exact black swan is typical for many biological models with two slow and one fast variables.

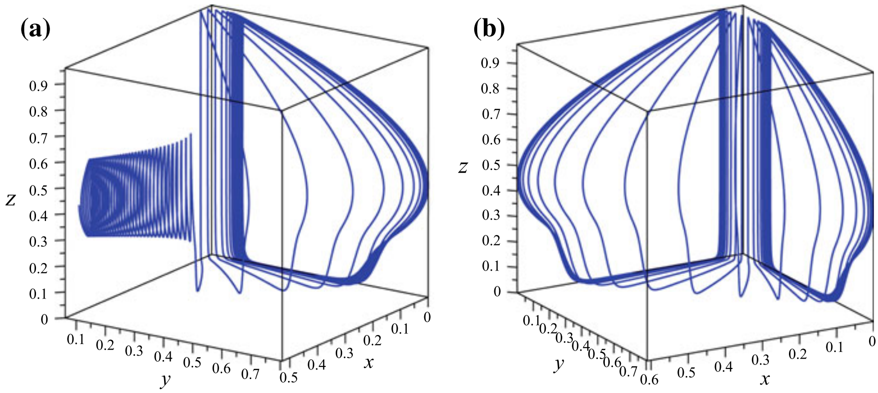

Fig. 2 The cascades of 3D canard doublets of the system (1)–(3). $\varepsilon = 0.1$, $\beta_1 = 0.1$, $\beta_2 = 0.13$, $m_1 = 0.5$, $m_2 = 0.4$, and **a** $d_1 = 0.408515869462$, $d_2 = 0.307288368584$; **b** $d_1 = 0.408498356366$, $d_2 = 0.307288368584$

References

1. W. Liu, D. Xiao, Y. Yi, Relaxation oscillations in a class of predatorprey systems. J. Differ. Equ. **188**, 306–331 (2003)
2. A. Pokrovskii, E. Shchepakina, V. Sobolev, Canard doublet in a Lotka–Volterra type model. J. Phys. Conf. Ser. **138**, 012019 (2008)
3. E. Shchepakina, Black swans and canards in self-ignition problem. Nonlinear Anal. Real World Appl. **4**, 45–50 (2003)
4. E. Shchepakina, V. Sobolev, Integral manifolds, canards and black swans. Nonlinear Anal. Theory Methods Appl. **44**, 897–908 (2001)
5. E. Shchepakina, V. Sobolev, Black swans and canards in laser and combustion models, in *Singular perturbations and hysteresis*, ed. by M.P. Mortell, R. O'Malley, A. Pokrovskii, V. Sobolev (SIAM, 2005), pp. 207–255
6. E. Shchepakina, V. Sobolev, M.P. Mortell, *Singular Perturbations: Introduction to System Order Reduction Methods with Applications*. Lecture Notes in Mathematics, vol. 2114 (Springer, Berlin, 2014)
7. V.A. Sobolev, E.A. Shchepakina, Duck trajectories in a problem of combustion theory. Differ. Equ. **32**, 1177–1186 (1996)
8. V. Sobolev, Canard cascades. Discret. Contin. Dyn. Syst. Ser. B **18**, 513–521 (2013)