# The Hull Number in the Convexity of Induced Paths of Order 3

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**Abstract.** A set $S$ of vertices of a graph $G$ is $P_3^*$*-convex* if there is no vertex outside $S$ having two non-adjacent neighbors in $S$. The $P_3^*$*-convex hull of* $S$ is the minimum $P_3^*$-convex set containing $S$. If the $P_3^*$-convex hull of $S$ is $V(G)$, then $S$ is a $P_3^*$*-hull set*. The minimum size of a $P_3^*$-hull set is the $P_3^*$*-hull number of* $G$. In this paper, we show that the problem of deciding whether the $P_3^*$-hull number of a chordal graph is at most $k$ is NP-complete and present a linear-time algorithm to determine this parameter and provide a minimum $P_3^*$-hull set for unit interval graphs.

**Keywords:** Graph convexity · Hull number · Unit interval graph. · 2-distance shortest path

## 1 Introduction

We consider finite, undirected, and simple graphs. The path with $k$ vertices is denoted by $P_k$ and an *induced path* is a path having no chords. Given a set $S$ of vertices of a graph $G$, the *interval of* $S$ *in the convexity of induced paths of order* 3, also known as the $P_3^*$ *convexity*, is the set $[S]_3^* = S \cup \{u : u$ belongs to an induced $P_3$ between two vertices of $S\}$. The set $S$ is $P_3^*$*-convex* if $S = [S]_3^*$ and is $P_3^*$*-concave* if $V(G) \setminus S$ is $P_3^*$-convex. The $P_3^*$*-convex hull of* $S$ is the minimum $P_3^*$-convex set containing $S$ and it is denoted by $\langle S \rangle_3^*$. If $\langle S \rangle_3^* = V(G)$, then $S$ is a $P_3^*$*-hull set*. The minimum size of a $P_3^*$-hull set is the $P_3^*$*-hull number* $h_3^*(G)$ *of* $G$.

Recently, the $P_3^*$ convexity has attracted attention as an alternative to other quite known convexities with different behavior despite a similar definition. It is particularly interesting in spreading dynamics which forbid the same influence by two neighbors to get spread to a common neighbor. For instance, in [4], it is shown that the problem of deciding whether the $P_3^*$-hull number of a bipartite graph is at most $k$ is NP-complete, while polynomial-time algorithms for determining this parameter for $P_4$-sparse graphs and cographs are presented. Apart from these results very little is known, as results of quite similarly defined


C. J. Colbourn et al. (Eds.): IWOCA 2019, LNCS 11638, pp. 214–228, 2019.
https://doi.org/10.1007/978-3-030-25005-8_18

well-known convexities do not help, since the proofs depend on the existence of longer shortest paths or a non-induced $P_3$.

In the well-known *geodetic convexity*, the *geodetic interval of* $S$ is $[S]_g = S \cup \{u : u$ belongs to some shortest path between two vertices of $S\}$. The terms *geodesically convex*, *geodesically concave*, *geodetic convex hull* $\langle S \rangle_g$, *geodetic hull set*, and *geodetic hull number* $h_g(G)$ are defined in a similar way to the $P_3^*$ convexity. The literature concerning the geodetic hull number is large. It is known that this problem is NP-complete for chordal graphs [5], $P_9$-free graphs [10], and partial cubes [1]; and that one can find a minimum geodetic hull set in polynomial time if the input graph is unit interval [9], $(q, q-4)$-graph [2], cobipartite [2], cactus [2], (paw, $P_5$)-free [10], $(C_3, \dots, C_{k-2}, P_k)$-free [10], $P_5$-free bipartite [3], or planar partial cube quadrangulation [1]. However, as already remarked, though an induced path of order 3 is a shortest path of length 2 between a pair of nodes, those results do not directly apply to the $P_3^*$ convexity due to the use of longer shortest paths in proofs.

Unlike the $P_3^*$ convexity, the $P_3$ *convexity* considers all paths of order 3. In this convexity, also known as irreversible 2-conversion, the problem of computing the hull number is APX-hard for bipartite graphs with maximum degree at most 4 and NP-complete for planar graphs with maximum degree at most 4 [7,15], and can be found in polynomial time for chordal graphs [6] as well as for cubic or subcubic graphs [15]. Finally, in the convexity that considers all induced paths, the *monophonic convexity*, the hull number can be computed in polynomial time for any graph [11].

The main result of this paper is a linear-time algorithm to determine both the $P_3^*$-hull number and a minimum $P_3^*$-hull set of a unit interval graph (Sect. 2). We point out that Theorem 1 is not only an explicit formula for the $P_3^*$-hull number $h_3^*(G)$ but also an almost explicit one for the minimum $P_3^*$-hull set, since one needs to compute the necessary labels before. We also show that the problem of deciding whether this parameter is at most $k$ for a chordal graph is NP-complete (Sect. 3). Remember that unit interval graphs have a variety of applications in operations research, including resource allocation problems in scheduling [13] and in genetic modeling such as DNA mapping in bioinformatics [14], where an overall agreement (on a value, a signal, a failure, a disease, a characteristic, etc.) might get forced by a minimum key set under some convexity such as the $P_3^*$.

We conclude this section giving some definitions. The distance between vertices $u$ and $v$ is denoted by $d(u, v)$ and the neighborhood of a vertex $v$ is denoted by $N(v)$. The set $\{1, \dots, k\}$ for an integer $k \geq 1$ is denoted by $[k]$. A subgraph of $G$ induced by vertex set $S$ is denoted by $G[S]$. A vertex $u$ is *simplicial* if its neighborhood induces a complete graph. Note that every $P_3^*$-hull set contains all simplicial vertices and at least one vertex of each $P_3^*$-concave set of the graph. Given an ordering $\alpha = (v_1, \dots, v_n)$ of $V(G)$ and a set $I \subseteq [n]$, the subordering $\alpha' = (v_{i_1}, \dots, v_{i_{|I|}})$ *of* $\alpha$ *induced by* $I$ is the ordering of the set $\{v_{i_j} : i_j \in I\} \subseteq V(G)$ such that $v_{i_j}$ appears before $v_{i_k}$ if and only if $i_j < i_k$. If $I = [j] \setminus [i-1]$ for $0 \leq i < j \leq n$, then the subordering of $\alpha$ induced by $I$ is denoted by $\alpha_{i,j}$.

## 2 Unit Interval Graphs

A *unit interval graph* $G$ is the intersection graph of a collection of intervals of the same size on the real line. Since one can assume that all left endpoints of the intervals of such a collection are distinct, a *canonical ordering* $\Gamma = (v_1, v_2, \dots, v_n)$ *of* $V(G)$ is defined as the one such that $i < j$ if and only if the left endpoint of the interval of $v_i$ is smaller than the left endpoint of the one of $v_j$. This ordering has the property that if $v_iv_j \in E(G)$ for $i < j$, then $\{v_i, \dots, v_j\}$ is a clique [8,16]. It is easy to see that if $v_i \in [v_j, v_k]^*_3$ for $j < k$, then $j \leq i \leq k$. In the next, we consider a canonical ordering $\Gamma = (v_1, v_2, \dots, v_n)$ of a given unit interval graph $G$.

**Lemma 1.** *If $v_j$ is simplicial, then $h^*_3(G) = h^*_3(G[\{v_1, \dots, v_j\}]) + h^*_3(G[\{v_j, \dots, v_n\}]) - 1$.*

*Proof.* Let $S, S_1$, and $S_2$ be minimum hull sets of $G, G_1 = G[\{v_1 \dots v_j\}]$, and $G_2 = G[\{v_j \dots v_n\}]$, respectively. Since $\Gamma = (v_1, v_2, \dots, v_n)$ is a canonical ordering and $v_j$ is a simplicial vertex of $G, G_1$, and $G_2$, it holds $S \supseteq \{v_j\} = S_1 \cap S_2$. It is clear that $S_1 \cup S_2$ is a hull set of $G$, and hence $h^*_3(G) \leq h^*_3(G[\{v_1, \dots, v_j\}]) + h^*_3(G[\{v_j, \dots, v_n\}]) - 1$. Now, consider an induced path $v_iv_kv_\ell$ with $v_i \in V(G_1) \setminus \{v_j\}$ and $v_\ell \in V(G_2) \setminus \{v_j\}$. If $v_k \in V(G_2)$, then $i < j < k < \ell$ and $\{v_i, \dots, v_k\}$ is a clique containing $v_j$. Since $v_j$ is simplicial, if $v_jv_\ell \in E(G)$, then $v_iv_\ell \in E(G)$, which would contradict the assumption that $v_iv_kv_\ell$ is an induced path. Therefore $v_jv_\ell \notin E(G)$ and $v_k \in [v_j, v_\ell]^*_3$. The case for $v_k \in V(G_1)$ is analogue. Hence, $S \cap V(G_i)$ is a minimum hull set of $G_i$ for $i \in [2]$, which means that $h^*_3(G) \geq h^*_3(G[v_1, \dots, v_j]) + h^*_3(G[v_j, \dots, v_n]) - 1$. $\square$

Due to Lemma 1, we can assume that $G$ has only two simplicial vertices, namely, $v_1$ and $v_n$. In the sequel, we use the geodetic interval to obtain a useful partition of the vertices of $G$. We say that $v_i$ is a *black vertex* if $v_i$ lies in a shortest $(v_1, v_n)$-path and that it is a *red vertex* otherwise. The black vertices $v$ such that $d(v_1, v) = i$ form the *black region* $B_i$. Note that all vertices of $B_i$ are consecutive in $\Gamma$. The set of vertices between the black regions $B_{i-1}$ and $B_i$ in $\Gamma$ form the *red region* $R_i$. These definitions induce a partition of $\Gamma$ into black and red regions $B_0, R_1, B_1, \dots, R_q, B_q$. Note that a red region can be empty, $B_0 = \{v_1\}$, $B_q = \{v_n\}$, and $d(v_1, v_n) = q = d(G)$, where $d(G)$ stands for the diameter of $G$. Besides, the black (and red) regions are precisely defined by the following two (not necessarily distinct) shortest $(v_1, v_n)$-paths, namely, the one whose internal vertices have all the highest possible indexes in $\Gamma$ and the one whose internal vertices have all the lowest possible indexes in $\Gamma$. Each black region $B_i$ contains precisely the vertices of those two paths having distance $i$ to $v_1$ as well as all vertices between them in the canonical ordering $\Gamma$. (See Fig. 1.)

For $i \in [q]$, denote by $r^\ell_i$ and $r^r_i$ the leftmost and rightmost vertices of $R_i$ in $\Gamma$, respectively. For $i \in [q] \cup \{0\}$, denote by $b^\ell_i$ and $b^r_i$ the leftmost and rightmost vertices of $B_i$ in $\Gamma$, respectively. If, for $i \in [q]$, $b^\ell_{i-1}b^r_i \in E(G)$, then we say that $b^\ell_{i-1}b^r_i$ is a *long edge*. For $i \in [q]$, denote by $R^r_i$ the vertices of $R_i$ having edges to $R_{i+1}$ and by $R_i^\ell$ the vertices of $R_i$ having edges to $R_{i-1}$. We say that a red vertex $v \in R_i$ is a *right vertex* if $v \notin R_i^\ell$ and that it is a *left vertex* if $v \notin R_i^r$.

Let $\Theta = (u_{k_1}, u_{k_2}, \ldots, u_{k_{|R|}})$ be the subordering of $\Gamma$ induced by the set of all red vertices $R = R_1 \cup \ldots \cup R_q$. A subordering $\Theta_{i,j} = (u_{k_i}, \ldots, u_{k_j})$ for $i \leq j$ is a *component* of $G$ if $u_{k_i}$ is a right vertex, $u_{k_j}$ is a left vertex, there is no $i' \in [j-1] \setminus [i-1]$ such that $u_{k_{i'}}$ is a left vertex and $u_{k_{i'+1}}$ is a right vertex, and it is maximal (that is, it must hold that either $u_{k_{i-1}}$ is a left vertex or $i = 1$, and additionally, that either $u_{k_{j+1}}$ is a right vertex or $j = |R|$). If, in addition, for every $v_{k'} \in \Theta_{i,j}$ there exists a long edge $v_{k''}v_{k'''} \in E(G)$ such that $k'' < k' < k'''$, then $\Theta_{i,j}$ is said to be a *covered component*. Note that a component can be a singleton. However, since we are assuming that $G$ has only two simplicial vertices $v_1$ and $v_n$, a covered component can neither be a singleton nor intersect only one red region as its elements would be simplicial vertices different from $v_1$ and $v_n$. Thus, every covered component must contain vertices of at least two red regions. (See Fig. 1.) Finally, the components of a unit interval graph can be determined in linear time [9].

Now we present some structural results. In the next, we characterize some $P_3^*$-concave sets.

**Lemma 2.** *It holds that:*

*(a) for $i \in [q-1]$, $R_i^r \cup B_i \cup R_{i+1} \cup B_{i+1}$ is a $P_3^*$-concave set;*
*(b) for $i \in [q-1]$, $R_i \cup B_i \cup R_{i+1} \cup B_{i+1}$ is a $P_3^*$-concave set;*
*(c) for $i \in \{0\} \cup [q-2]$, $B_i \cup R_{i+1} \cup B_{i+1} \cup R_{i+2}$ is a $P_3^*$-concave set; and*
*(d) for $i \in \{0\} \cup [q-1]$, if $R_{i+1}^\ell \cap R_{i+1}^r = \varnothing$, then $B_i \cup R_{i+1} \cup B_{i+1}$ is a $P_3^*$-concave set.*

*Proof.* First note that if $\Gamma_{j,k} = (v_j, \ldots, v_k)$ for $j < k$ is the ordering of a set $S = \{v_j, \ldots, v_k\}$ in $\Gamma$, then all vertices in $\Gamma_{1,j-1}$ having a common neighbor in $S$ are adjacent, and the same is true for all vertices in $\Gamma_{k+1,n}$ having a common neighbor in $S$. Thus, $S$ is always a $P_3^*$-concave set for $j = 1$. Besides, $S$ is a $P_3^*$-concave set for $j > 1$ if the distance of any vertex of $\Gamma_{1,j-1}$ to any vertex of $\Gamma_{k+1,n}$ is at least 3.

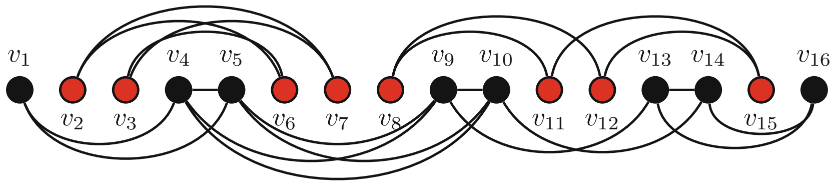


**Fig. 1.** Here there are two components: $\{v_2, v_3, v_6, v_7\}$ (covered) and $\{v_8, v_{11}, v_{12}, v_{15}\}$ (not covered), as both red vertices $v_{11}$ and $v_{12}$ are not covered by a long edge such as $v_1v_5$, $v_4v_{10}$ and $v_{13}v_{16}$. Note that only edges between black vertices and between red vertices of distinct red regions are being depicted, and that the two shortest paths $v_1$ $v_4$ $v_9$ $v_{13}$ $v_{16}$ and $v_1$ $v_5$ $v_{10}$ $v_{14}$ $v_{16}$ define precisely the black (and red) regions. (Color figure online)

(a) Take $S = R_i^r \cup B_i \cup R_{i+1} \cup B_{i+1}$. For $j > 1$, the fact that $v_{j-1}$ has no edges to $R_{i+1}$ implies that the distance of any vertex of $\Gamma_{1,j-1}$ to any vertex of $\Gamma_{k+1,n}$ is at least 3.
(b) By (a) and the fact that if $\Gamma_{j,k}$ is $P_3^*$-concave, then $\Gamma_{j',k'}$ is $P_3^*$-concave for $j' \leq j$ and $k' \geq k$.
(c) By symmetry, this case is equivalent to (b).
(d) Now take $S = B_i \cup R_{i+1} \cup B_{i+1}$. For $j > 1$, $d(v_{j-1}, v_{k+1}) \geq 3$ if $R_{i+1}^\ell \cap R_{i+1}^r = \varnothing$. $\square$

Next, we present a partition of $\Gamma$ into parts called $C$-sets and classify them into 4 types.

- If $\Theta_{i,j}(u_{k_i}, \ldots, u_{k_j})$ is a covered component, then $\Gamma_{i',j'}$ is a *$C$-set* where $v_{i'} = u_{k_i}$ and $v_{j'} = u_{k_j}$. If $\Gamma_{i',j'}$ contains an odd number of black regions, then $\Gamma_{i',j'}$ has type 1. Otherwise $\Gamma_{i',j'}$ has type 2.
- If $\Gamma_{i,j}$ is maximal having no vertex belonging to a $C$-set of type 1 or 2, then $\Gamma_{i,j}$ is a *$C$-set* as well. If $\Gamma_{i,j}$ contains an odd number of black regions, then $\Gamma_{i,j}$ has type 3. Otherwise, $\Gamma_{i,j}$ has type 4.

For example, the unit interval graph of Fig. 1 gets partitioned into exactly three $C$-sets: $\Gamma_{1,1}$ (of type 3), $\Gamma_{2,7}$ (of type 1), and $\Gamma_{8,16}$ (of type 3). It is clear that the $C$-sets always form a partition of $\Gamma$. In fact, they also induce a partition of the black regions of $G$, i.e., all vertices of any black region are contained in a unique $C$-set and every $C$-set contains at least the vertices of one black region. These facts allow us to denote by $C_{i,j}$ the $C$-set whose set of black regions is $\{B_i, \ldots, B_j\}$. If $C_{i,j} \cap R_i \neq \emptyset$, then $C_{i,j} \cap R_i = R_i^r$. Similarly, if $C_{i,j} \cap R_{j+1} \neq \emptyset$, then $C_{i,j} \cap R_{j+1} = R_{j+1}^\ell$. Therefore, from now on, if not empty, both $R_i^r$ and $R_{j+1}^\ell$ as well as $R_k$ for $i < k < j+1$ will be called the *red regions of* $C_{i,j}$. Finally, note that those red regions might be empty if $C_{i,j}$ has type 3 or 4. However, if $C_{i,j}$ has type 1 or 2, then its red regions form a covered component, and consequently, not only both $R_i^r \neq \emptyset$ and $R_{j+1}^\ell \neq \emptyset$, but also both $R_k \neq \emptyset$ and $R_k^\ell \cap R_k^r \neq \emptyset$ for $i < k < j+1$, as a covered component does not contain a left vertex followed by a right vertex.

The proposed algorithm finds a $\{0, -1\}$ labeling of the $C$-sets, which depends on the types and the relative positions of the $C$-sets. A minimum $P_3^*$-hull set is then obtained by making a *standard choice* $S$ for each $C$-set between two possibilities, which we call left and right choices, and depends on the pair type and label, as shown in Table 1. As in Figs. 2, 3, 4 and 5, for any $C$-set $C$, there is (at most) one black region $B$ of $C$ such that its standard choice alternates in containing and not containing a black vertex of each consecutive black region of $C$ from $B$ on. Besides those black vertices, the standard choice has one red vertex if $C$ contains a covered component and no red vertex otherwise.

The *left choice* $S$ for a $C$-set $C_{i,j}$ with type $t \in [4]$ and $k = j - i + 1$ black regions is defined as

$$S = \begin{cases} \{r_i^r\} \cup \{b_{i+1+2k'}^r : 0 \leq k' < \lfloor \frac{k}{2} \rfloor\} & \text{, if } t \in [2]; \\ \{b_{i+2k'}^r : 0 \leq k' < \lceil \frac{k}{2} \rceil\} & \text{, if } t \in \{3, 4\}. \end{cases}$$

**Algorithm 1.** Finding a minimum $P_3^*$-hull set.

**input:** A unit interval graph $G$ having exactly 2 simplicial vertices
$\Gamma$ canonical ordering of $V(G)$
$\mathcal{C} \leftarrow$ the partition $(C_1, \dots, C_t)$ of $\Gamma$ into $C$-sets, where $C_1$ and $C_t$ have types in $\{3, 4\}$
**if** $C_1$ *has type* 3 **then**
  label$(C_1) \leftarrow 0$ (left choice for type 3)
**else**
  label$(C_1) \leftarrow -1$ (left choice for type 4)
**for** $j$ *from* 2 *to* $q$ **do**
  **if** $C_j$ *has type* 2 *or* 3 **then**
    $label(C_j) \leftarrow -1 - label(C_{j-1})$
  **else**
    $label(C_j) \leftarrow label(C_{j-1})$
$S \leftarrow \{v_n\}$
**for** $C_i \in \mathcal{C}$ **do**
  $S \leftarrow S \cup S'$ where $S'$ is a standard choice for $C_{i,j}$ according to Table 1
**return** $S$

**Table 1.** Standard choices.

| *Type* | *Label* 0 | *Label* $-1$ |
|---|---|---|
| 1 | Right choice | Left choice |
| 2 | Left choice | Right choice |
| 3 | Left choice | Right choice |
| 4 | Right choice | Left choice |

The *right choice* $S$ for a $C$-set $C_{i,j}$ with type $t \in [4]$ and $k = j - i + 1$ black regions is defined as

$$S = \begin{cases} \{r\} \text{ for some } r \in R^{\ell}_{i+1} & \text{, if } t = 1 \text{ and } j = i; \\ \{r_{i+1}\} \cup \{b^r_{i+2k'} : 0 < k' < \lceil \frac{k}{2} \rceil\} \text{ for some } r_{i+1} \in R^{\ell}_{i+1} \cap R^r_{i+1} & \text{, if } t \in [2] \text{ and } j > i; \\ \{b^r_{i+1+2k'} : 0 \le k' < \lfloor \frac{k}{2} \rfloor\} & \text{, if } t \in \{3, 4\}. \end{cases}$$

The idea of the linear algorithm is to give the left choice for the first $C$-set, and then alternate the standard choice from left to right and vice-versa if and only if the preceding $C$-set had type 2 or 3. Note that label $-1$ means "missing" and indicates that no vertex of the last black region $B_j$ of $C_{i,j}$ belongs to its standard choice $S$. Note also that the first and the last $C$-sets in $\Gamma$ are not a covered component, and therefore, always have types in $\{3, 4\}$. Figures 2, 3, 4 and 5 depict the left and right choices for a $C$-set depending on its type.

In the next lemma we show that, for any $j \in [q]$, if $B_{j-1}$ and $B_j$ belong to distinct $C$-sets, then there is no vertex of $R_j$ having neighbors in both $R_{j-1}$ and $R_{j+1}$.

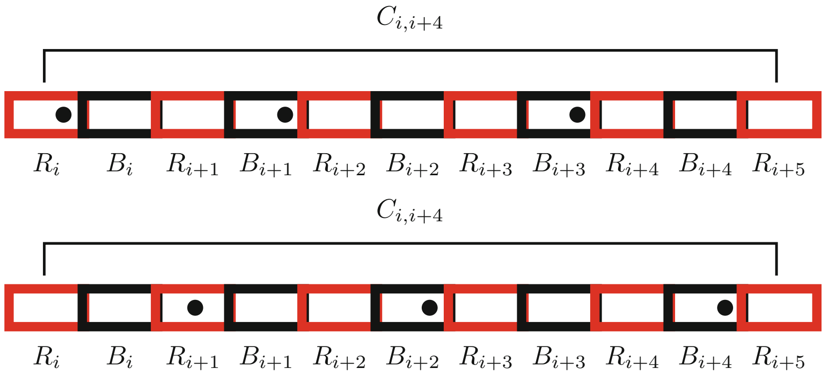


**Fig. 2.** Scheme representing the left (on top) and right (on bottom) choices of $C$-set $C_{i,i+4}$ with type 1. (Color figure online)

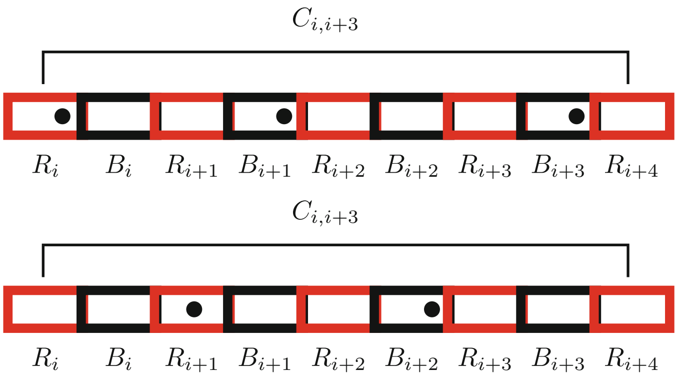


**Fig. 3.** Scheme representing the left (on top) and right (on bottom) choices of $C$-set $C_{i,i+3}$ with type 2. (Color figure online)

**Lemma 3.** *If $C_{i,j-1}$ and $C_{j,k}$ are $C$-sets, then $R_j^\ell \cap R_j^r = \varnothing$.*

*Proof.* By definition, if one of these $C$-sets has type 3 or 4, then the other one has type 1 or 2. This means that exactly one of these two $C$-sets has type 1 or 2, and therefore the red vertices of this $C$-set form a covered component. However, if $R_j^\ell \cap R_j^r \neq \varnothing$, then there would be a contradiction, as no vertex in $R_j$ could be neither a left vertex nor a right vertex, i.e., neither the first nor the last red vertex of this $C$-set of type 1 or 2. $\square$

**Lemma 4.** *Every covered component of a unit interval graph $G$ is $P_3^*$-concave.*

*Proof.* Let $C_{i,j}$ be the $C$-set of type either 1 or 2 containing the covered component, therefore, the red regions of $C_{i,j}$ form the covered component. Lemma 3 implies that, for $i < k < j+1$, each red vertex of $R_k$ neighbors only red vertices of $R_{k-1}$, $R_k$ and $R_{k+1}$, each red vertex of $R_i^r$ neighbors only red vertices

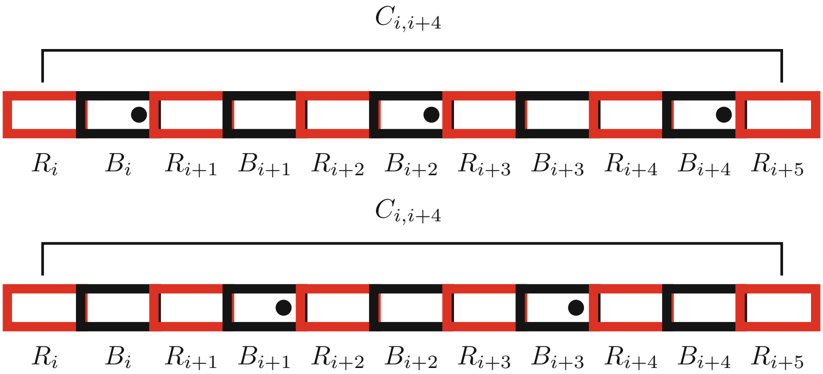


**Fig. 4.** Scheme representing the left (on top) and right (on bottom) choices of $C$-set $C_{i,i+4}$ with type 3. (Color figure online)

of $R_i$ and $R_{i+1}$ and each red vertex of $R^\ell_{j+1}$ neighbors only red vertices of $R_j$ and $R_{j+1}$. Finally, for $i \leq k \leq j+1$, since $B_{k-1} \cup R_k \cup B_k$ form a clique and $R_k$ neighbors only black vertices of $B_{k-1} \cup B_k$, each vertex of the covered component is such that all its neighbors not belonging to the covered component form a clique, meaning that the covered component is $P_3^*$-concave, and therefore, every covered component of $G$ must intersect with every $P_3^*$-hull set of $G$. (As an alternative proof, the fact that $\langle S \rangle_3^* \subseteq \langle S \rangle_g$ for every set $S \subseteq V(G)$ also implies the claim, as from [9] it is known that every covered component of $G$ is geodesically concave.) $\square$

Remember that $b^r_{i-1}$ is the rightmost vertex of $B_{i-1}$, and that $b^\ell_{i+1}$ is the leftmost vertex of $B_{i+1}$, being both $b^r_{i-1}$ and $b^\ell_{i+1}$ adjacent to every vertex in $B_i$, but not to one another. Note also that $b^\ell_{i+1}$ has no neighbor in $R^r_i$, as no vertex in that set belongs to a shortest path between $v_1$ and $v_n$, and that $b^r_{i-1}$ has no neighbor in $R^\ell_{i+1}$, as no vertex in that set has distance $i$ to $v_1$. The following property will be very useful.

**Lemma 5.** *If $r \in R^r_i \cup R^\ell_{i+1}$, then $R^r_i \cup B_i \cup R^\ell_{i+1} \subseteq \langle\{b^r_{i-1}, r, b^\ell_{i+1}\}\rangle_3^*$.*

*Proof.* Note that $B_i \subseteq [b^r_{i-1}, b^\ell_{i+1}]_3^*$ and that $R^r_i \neq \varnothing$ if and only if $R^\ell_{i+1} \neq \varnothing$. If $r \in R^r_i$ and $v_k$ is the vertex with maximum index $k$ in $\Gamma$ belonging to $N(r) \cap R^\ell_{i+1}$, then $R^\ell_{i+1} \cap \Gamma_{1,k} \subseteq [\{r, b^\ell_{i+1}\}]_3^*$, $R^r_i \subseteq [\{b^r_{i-1}\} \cup (R^\ell_{i+1} \cap \Gamma_{1,k})]_3^*$, and $R^\ell_{i+1} \subseteq [R^r_i \cup \{b^\ell_{i+1}\}]_3^*$ as well. Similarly, in a symmetric way, if $r \in R^\ell_{i+1}$ and $v_{k'}$ is the vertex with minimum index $k'$ in $\Gamma$ belonging to $N(r) \cap R^r_i$, then $R^r_i \cap \Gamma_{k',n} \subseteq [\{r, b^r_{i-1}\}]_3^*$, $R^\ell_{i+1} \subseteq [\{b^\ell_{i+1}\} \cup (R^r_i \cap \Gamma_{k',n})]_3^*$ and $R^r_i \subseteq [R^\ell_{i+1} \cup \{b^r_{i-1}\}]_3^*$ as well. $\square$

Now we are ready to understand why the linear algorithm presented, which starts with a left choice for the first $C$-set and then flips the standard choice from left to right and vice-versa if and only if the preceding $C$-set has type 2 or

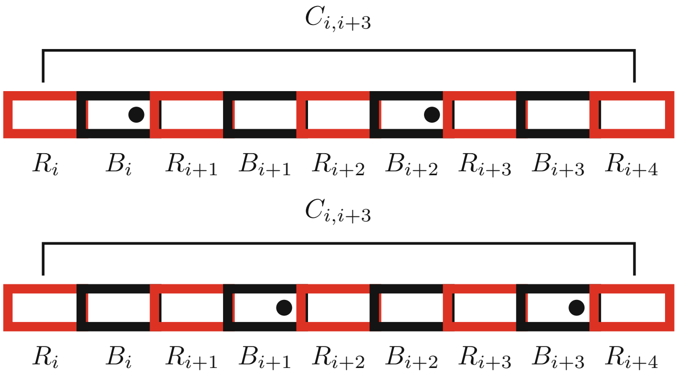


**Fig. 5.** Scheme representing the left (on top) and right (on bottom) choices of $C$-set $C_{i,i+3}$ with type 4. (Color figure online)

3, indeed provides a minimum $P_3^*$-hull set of $G$ when the choices of the $C$-sets get united together with $v_n$. The following lemma throws light on that.

**Lemma 6.** *If $S_{i,j}$ is a standard choice for a $C$-set $C_{i,j}$, then the following holds:*

*(a)* $C_{i,j} \subseteq \langle S_{i,j} \cup \{b^r_{i-1}, b^\ell_{j+1}\}\rangle^*_3$;
*(b)* $C_{i,j} \setminus (R^r_i \cup R^\ell_{i+1}) \subseteq \langle S_{i,j} \cup B_i \cup \{b^\ell_{j+1}\}\rangle^*_3$;
*(c)* $C_{i,j} \setminus (R^r_j \cup R^\ell_{j+1}) \subseteq \langle S_{i,j} \cup B_j \cup \{b^r_{i-1}\}\rangle^*_3$;
*(d)* $C_{i,j} \setminus (R^r_i \cup R^\ell_{i+1} \cup R^r_j \cup R^\ell_{j+1}) \subseteq \langle S_{i,j} \cup B_i \cup B_j\rangle^*_3$.

*Proof.* Write $S_a = S_{i,j} \cup \{b^r_{i-1}, b^\ell_{j+1}\}$, $S_b = S_{i,j} \cup B_i \cup \{b^\ell_{j+1}\}$, $S_c = S_{i,j} \cup B_j \cup \{b^r_{i-1}\}$, and $S_d = S_{i,j} \cup B_i \cup B_j$. We give only one proof for all four cases, thus let $x \in \{a, b, c, d\}$.

First consider $C_{i,j}$ with type in $\{1, 2\}$, that is, its red regions form a covered component. It is clear that $B_i \subseteq [S_x]^*_3$ for $x \in \{a, b, c, d\}$. Since $b^\ell_k b^r_{k+1} \in E(G)$ for $i - 1 \leq k \leq j$, we have $B_k \subseteq [[S_x]^*_3]^*_3 \subseteq \langle S_x\rangle^*_3$ for $x \in \{a, b, c, d\}$ and $i \leq k \leq j$. Besides, for each choice $S_{i,j}$, note that there is $r \in S_{i,j}$ such that either $r = r^r_i$ (left choice) or $r \in R^\ell_{i+1} \cap R^r_{i+1}$ (right choice), and thus, not only $[S_x]^*_3 \cap R^\ell_{i+1} \cap R^r_{i+1} \neq \varnothing$ for $x \in \{a, b, c, d\}$ and $i < j$, but also by Lemma 5 we have that $R^r_i \cup R^\ell_{i+1} \subseteq \langle S_x\rangle^*_3$ for either $x \in \{a, c\}$ and $i < j$, or $x = a$ and $i = j$. Now, since $R^\ell_{k+1} \cap R^r_{k+1} \neq \varnothing$ for $i \leq k \leq j - 1$ as the red regions of $C_{i,j}$ form a covered component, due to Lemma 5 we have for $i < j$ by forwards induction starting on $[S_x]^*_3 \cap R^\ell_{i+1} \cap R^r_{i+1} \neq \varnothing$ that $R^r_{k+1} \cup R^\ell_{k+2} \subseteq \langle S_x\rangle^*_3$ for either $x \in \{a, b\}$ and $i \leq k \leq j - 1$, or $x \in \{c, d\}$ and $i \leq k \leq j - 2$.

Now, consider that $C_{i,j}$ has type 3 or 4. Note that $\{b^r_k | i \leq k \leq j\} \subseteq [S_x]^*_3$ and $B_j \subseteq [[S_x]^*_3]^*_3$ for $x \in \{a, b, c, d\}$, which implies, by backwards induction starting on $B_j$, that $B_k \subseteq \langle S_x\rangle^*_3$ for $x \in \{a, b, c, d\}$ and $i \leq k \leq j$. Therefore, if some red region $R_k$ for $i < k \leq j$ is not covered by a long edge, then $R_k \subseteq [b^\ell_{k-1}, b^r_k]^*_3 \subseteq \langle S_x\rangle^*_3$ for $x \in \{a, b, c, d\}$ as well. Thus, suppose that $(R_{i'}, \ldots, R_{j'})$ is a maximal sequence of covered non-empty red regions for $i + 1 \leq i' \leq k \leq j' \leq j$. Since

$C_{i,j}$ does not contain a covered component, $R_{i'-1} \supseteq R^r_{i'-1} \neq \emptyset$ with $i' > i+1$ or $R_{j'+1} \supseteq R^\ell_{j'+1} \neq \emptyset$ with $j' < j$ is not a covered red region. Without loss of generality, assume that $R_{i'-1} \supseteq R^r_{i'-1} \neq \emptyset$ with $i' > i+1$ is not a covered red region, meaning that $R_{i'-1} \subseteq \langle S_x \rangle^*_3$ for $x \in \{a,b,c,d\}$. (Otherwise, an analogous argument using Lemma 5 works with a backwards induction instead of a forwards one.) Note that Lemma 5 applied on $b^r_{i'-2}, r^r_{i'-1}, b^\ell_{i'}$ yields $R^\ell_{i'} \subseteq \langle S_x \rangle^*_3$. Now, as $C_{i,j}$ does not contain a covered component, $R^\ell_k \cap R^r_k \neq \varnothing$ for $i' \leq k < j'$, implying by forwards induction that Lemma 5 applied on $b^r_{k-1}, r, b^\ell_{k+1}$ with $r \in R^\ell_k \cap R^r_k$ yields $R^r_k \cup R^\ell_{k+1} \subseteq \langle S_x \rangle^*_3$ for either $i' \leq k \leq j'$ (if $R_{j'+1} \neq \varnothing$) or $i' \leq k < j'$ (if $R_{j'+1} = \varnothing$), that is, $R_k \subseteq \langle S_x \rangle^*_3$ for $i' \leq k \leq j'$, as either $R^\ell_{j'} \cap R^r_{j'} \neq \varnothing$ with $j' < j$ when $R_{j'+1} \neq \varnothing$ or $R_{j'} = R^\ell_{j'}$ when $R_{j'+1} = \varnothing$. Finally, it remains to show that $R^r_i \subseteq \langle S_x \rangle^*_3$ for $x \in \{a,c\}$ and that $R^\ell_{j+1} \subseteq \langle S_x \rangle^*_3$ for $x \in \{a,b\}$, but these facts are directly derived from Lemma 5, as in this case both $R_i$ and $R_{j+1}$ are covered red regions. $\square$

Let $(C_1, \ldots, C_t)$ be the $C$-sets ordered according to $\Gamma$. The set $S$ returned by Algorithm 1 is a minimum $P^*_3$-hull set of $G$ containing $v_n$ as well as the standard choices selected by the algorithm for the $C$-sets, based on both the types and the received labels. Remark that the label is applied in such a way that the algorithm gives the left choice for $C_1$, and then consecutively alternates the standard choice from left to right and vice-versa if and only if the preceding $C$-set had type 2 or 3, maintaining it otherwise. In Lemma 8 we prove that $S$ is in fact a $P^*_3$-hull set of $G$, whereas in Lemmas 9 to 11 we prove that there is no $P^*_3$-hull set of $G$ with less than $|S|$ vertices. Define $f(C_i)$ as the cardinality of the standard choice that the algorithm associated with $C_i$ and $f'(G)$ as the number of times that the labeling changes from $-1$ to 0, plus 1 if $C_1$ has type 3, and again plus 1 if $C_t$ received label $-1$. In Theorem 1 we show that $|S| = f'(G) + \sum_{1 \leq i \leq t} f(C_i)$.

The next lemma combined with the previous one is key to comprehend the correctness.

**Lemma 7.** *If $C_{i,j}$ is a $C$-set of $G$ and $S$ is the set returned by Algorithm 1, then $B_i \cup \ldots \cup B_j \subseteq \langle S \rangle^*_3$. Hence, $b^r_{i-1} \in \langle S \rangle^*_3$ for $1 \leq i \leq q$ and $b^\ell_{j+1} \in \langle S \rangle^*_3$ for $0 \leq j \leq q-1$.*

*Proof.* First, consider the case where $C_{i,j}$ has type 3 or 4. Let $S_{i,j} = S \cap C_{i,j}$. We begin assuming that $i \geq 1$ and $j \leq q-1$. Observe that if $b^r_i \in S_{i,j}$, then there is $v \in S \cap (\{b^r_{i-2}\} \cup R_{i-1})$; otherwise there is $v \in S \cap (\{b^r_{i-1}\} \cup R_i)$. Observe also that if $b^r_j \in B$, then there is $w \in S \cap (\{b^r_{j+1}, b^r_{j+2}\} \cup R_{j+2})$; otherwise there is $w \in S \cap (\{b^r_{j+1}\} \cup R_{j+1})$. In all cases, it holds that $b^r_k \in [S_{i,j} \cup \{v,w\}]^*_3$ for $i-1 \leq k \leq j+1$. Since $C_{i,j}$ has type 3 or 4, the $C$-set containing $B_{j+1}$ has type 1 or 2, which means that the edge $b^\ell_j b^r_{j+1}$ exists. Hence $b^\ell_j \in \langle S \rangle^*_3$, which implies that $B_i \cup \ldots \cup B_j \subset \langle S \rangle^*_3$. Now, if $i = 0$, then $B_i = \{v_1\}$ and $b^r_i = v_1 \in S$; and if $j = q$, then $B_j = \{v_n\}$ and $b^\ell_j = v_n \in S$, which means that $B_i \cup \ldots \cup B_j \subset \langle S \rangle^*_3$ even if for $i = 0$ or $j = q$.

Now, consider the case where $C_{i,j}$ has type 1 or 2. Note that the first $C$-set $C_1$ as well as the last $C$-set $C_t$ have both types in $\{3,4\}$. Thus, a $C$-set $C_{i,j}$ of type in $\{1,2\}$ is such that not only $0 < i \leq j < q$, but also both its preceding and subsequent $C$-sets have types in $\{3,4\}$. This fact jointly with both the previous case and (a) of Lemma 6 imply that $B_i \cup \ldots \cup B_j \subseteq \langle S \rangle_3^*$. □

**Lemma 8.** *Algorithm 1 returns a $P_3^*$-hull set of $G$.*

*Proof.* Recall that $G$ has exactly 2 simplicial vertices. Let $S$ be the set returned by Algorithm 1 and $C_{i,j}$ be a $C$-set of $G$ having type $t$. Consider first $i = 0$. If $i = 0$, $B_i = \{v_1\} \subseteq S$ and clearly by definition $R_0^r \cup R_1^\ell = \emptyset$. If $j = q$, $B_j = \{v_n\} \subseteq S$ and clearly by definition $R_q^r \cup R_{q+1}^\ell = \emptyset$. By (d) of Lemma 6, $V(G) = C_{i,j} = \langle S \rangle_3^*$. Now, consider that $j < q$. By Lemma 7, it holds $b_{j+1}^\ell \in \langle S \rangle_3^*$. Thus, by (b) of Lemma 6, $C_{i,j} \subseteq \langle S \rangle_3^*$. Next, consider $j = q$ and $i > 0$. By Lemma 7, $b_{i-1}^r \in \langle S \rangle_3^*$. By (c) of Lemma 6, $C_{i,j} \subseteq \langle S \rangle_3^*$. Finally, suppose $i > 0$ and $j < q$. By Lemma 7, $b_{i-1}^r, b_{j+1}^\ell \in \langle S \rangle_3^*$. By (a) of Lemma 6, $C_{i,j} \subseteq \langle S \rangle_3^*$. □

We now define a lower bound, proved in Lemma 9, for the number of vertices that any $P_3^*$-hull set contains from a $C$-set $C_{i,j}$ as a function of its type $t$.

$$f(C_{i,j}) = \begin{cases} \frac{j-i+1}{2} & \text{, if } t \in \{2,4\}; \\ \frac{j-i+2}{2} & \text{, if } t = 1; \\ \frac{j-i}{2} & \text{, if } t = 3. \end{cases}$$

Let $S_{i,j}$ be a standard choice of $C_{i,j}$. Note that $f(C_{i,j}) = |S_{i,j}|$ if $t \in \{1,4\}$ or $S_{i,j}$ is a right choice; otherwise, $f(C_{i,j}) = |S_{i,j}| - 1$.

**Lemma 9.** *If $S$ is a $P_3^*$-hull set and $C_{i,j}$ is a $C$-set of a unit interval graph $G$, then $|S \cap C_{i,j}| \geq f(C_{i,j})$.*

*Proof.* The number of black regions contained in $C_{i,j}$ is $j-i+1$. By Lemma 2 (a), $R_i^r \cup B_i \cup R_{i+1} \cup B_{i+1}$ is a $P_3^*$-concave set and $R_k \cup B_k \cup R_{k+1} \cup B_{k+1}$ is a $P_3^*$-concave set for $i+1 \leq k \leq j-1$. Therefore, $C_{i,j}$ contains at least $\lfloor \frac{j-i+1}{2} \rfloor$ disjoint $P_3^*$-concave sets, which implies the result if the type of $C_{i,j}$ is 2 or 4 as $j-i+1$ is even or if its type is 3 as $j-i$ is not only even but also smaller than $j-i+1$.

Now, consider that $C_{i,j}$ has type 1 and let $S$ be a $P_3^*$-hull set of $G$. By definition, $j-i+1$ is odd. Since $C_{i,j}$ contains a covered component, Lemma 4 implies that at least one vertex of a red region of $C_{i,j}$ belongs to $S$. Now, due to this fact, if $|S \cap C_{i,j}| < \lceil \frac{j-i+1}{2} \rceil$, then there are four consecutive regions of $\Gamma$, w.l.o.g. say $V' = R_{i'} \cup B_{i'} \cup R_{i'+1} \cup B_{i'+1}$ for $i \leq i' < j$ such that $V' \cap S = \varnothing$. By Lemma 2, $V'$ is a $P_3^*$-concave set, which is a contradiction. Thus, the result also holds for type 1. □

**Lemma 10.** *Let $S$ be a $P_3^*$-hull set and let $C_{i,j}$ be a $C$-set of $G$ such that $|S \cap C_i| = f(C_i)$.*

*(a) If $C_{i,j}$ has type 2, then $S \cap (R_i^r \cup B_i \cup B_j \cup R_{j+1}^\ell) = \varnothing$;*
*(b) If $C_{i,j}$ has type 1 or 4, then $S \cap (R_i^r \cup B_i) = \varnothing$ or $S \cap (B_j \cup R_{j+1}^\ell) = \varnothing$;*
*(c) If $C_{i,j}$ has type 3, then $S \cap (R_i^r \cup B_i \cup R_{i+1} \cup R_j \cup B_j \cup R_{j+1}^\ell) = \varnothing$.*

*Proof.* We first count the number of regions of $C_{i,j}$ in terms of $f(C_{i,j})$. (a) Suppose for contradiction that $v \in S \cap (R_i^r \cup B_i \cup B_j \cup R_{j+1}^\ell)$. By symmetry, assume that $S \cap (R_i^r \cup B_i) \neq \varnothing$. The number of black regions of $C_{i,j}$ is $j-i+1$, which means that $C_{i,j}$ has $2(j-i+1)+1 = 4f(C_{i,j})+1$ regions, namely, $R_i^r, B_i, R_{i+1}, B_{i+1}, \ldots, R_j, B_j, R_{j+1}^\ell$. (b) First, consider that $C_{i,j}$ has type 1. Suppose for contradiction that $S \cap (R_i^r \cup B_i) \neq \varnothing$ and $S \cap (B_j \cup R_{j+1}^\ell) \neq \varnothing$. Then, $C_{i,j}$ has $2(j-i+1)+1 = 2(j-i+2)-1 = 4f(C_{i,j})-1$ regions. Now, consider that $C_{i,j}$ has type 4. Suppose for contradiction that $S \cap (R_i^r \cup B_i) \neq \varnothing$ and $S \cap (B_j \cup R_{j+1}^\ell) \neq \varnothing$. Then, $C_{i,j}$ has $2(j-i+1)+1 = 4f(C_{i,j})+1$ regions. (c) Suppose for contradiction that $v \in S \cap (R_i^r \cup B_i \cup R_{i+1} \cup R_j \cup B_j \cup B_{j+1}^\ell)$. By symmetry, assume $v \in R_{i-1}^r \cup B_i \cup R_i$. Then, $C_{i,j}$ has $2(j-i+1)+1 = 2(j-i)+3 = 4f(C_{i,j})+3$ regions.

Besides, by Lemma 4, $S$ contains a vertex of a red region of $C_{i,j}$ if its type is either 1 or 2 (that is, if it contains a covered component). Now, using the pigeonhole principle in all (a), (b) and (c) items, we conclude that in all cases there are four consecutive regions of $C_{i,j}$ having no vertices of $S$. By Lemma 2, these four regions form a $P_3^*$-concave set, which implies that $S$ is not a $P_3^*$-hull set of $G$, a contradiction. $\square$

**Lemma 11.** *Consider the labeling obtained by Algorithm 1 and let $S$ be a minimum $P_3^*$-hull set of $G$. The following sentences hold:*

*(a) If $(C_i, \ldots, C_j)$ is a maximal sequence of $C$-sets such that label$(C_j) = 0$ and label$(C_k) = -1$ for $i \leq k < j$, then $|S \cap (C_i \cup \ldots \cup C_j)| \geq f(C_i) + \ldots + f(C_j) + 1$; and*
*(b) If $C_{j-1}$ and $C_j = C_{\ell_j, d(G)}$ are $C$-sets and label$(C_j) = -1$, then $|S \cap (C_{j-1} \cup C_j)| \geq f(C_{j-1}) + f(C_j) + 1$.*

*Proof.* (*a*) Suppose that $|S \cap (C_i \cup \ldots \cup C_j)| \leq f(C_i) + \ldots + f(C_j)$. If $j = 1$, then $C_j$ has type 3. By Lemma 10, $S \cap B_0 = \varnothing$, which is a contradiction since $B_0 = \{v_1\}$ and $v_1$ is a simplicial vertex. Then consider $j > 1$. Remember that a $C$-set has label different of its predecessor if and only if its type is 2 or 3. Hence, $C_j = C_{j',j''}$ has type 2 or 3. By Lemma 10, it holds $S \cap (R_{j'}^r \cup B_{j'}) = \varnothing$. If $i = 1$, then $C_1 = C_{0,\ell_2-1}$ has type 4. Since $v_1$ is a simplicial vertex, $v_1 \in S$, then, by Lemma 10, $S \cap (B_{\ell_2-1} \cup R_{\ell_2}^\ell) = \varnothing$. If $i \geq 2$, then $C_i = C_{\ell_i, \ell_{i+1}-1}$ has type 2 or 3. By Lemma 10, $S \cap (B_{\ell_2-1} \cup R_{\ell_2}^\ell) = \varnothing$. In both cases, $C_k = C_{\ell_k, \ell_{k+1}-1}$ has type 1 or 4 for $i+1 \leq k < j$. This means by Lemma 10 that $S \cap (R_{\ell_k}^r \cup B_{\ell_k}) = \varnothing$ or $S \cap (B_{\ell_{k+1}-1} \cup R_{\ell_{k+1}}^\ell) = \varnothing$ for $i+1 \leq k < j$. Therefore, by the pigeonhole principle, there is some $p \in \{i+1, \ldots, j\}$ such that $B_{\ell_p} \cup R_{\ell_p+1}^r \subset C_{\ell_{p-1}, \ell_p-1}$ and $R_{p+1}^\ell \cup B_{p+1} \subset C_{\ell_p, \ell_{p+1}-1}$ such that $S \cap (B_p \cup R_{p+1} \cup B_{p+1}) = \varnothing$. By Lemmas 2 (d) and 3, $B_p \cup R_{p+1} \cup B_{p+1}$ is a $P_3^*$-concave set, which contradicts the assumption that $S$ is a $P_3^*$-hull set.

($b$) Suppose that $|S \cap (C_{j-1} \cup C_j)| \leq f(C_{j-1}) + f(C_j)$. We know that $C_j$ has type $t \in \{3, 4\}$, $C_{j-1}$ has type 1 or 2, $B_{d(G)} = \{v_n\}$, and $R_{d(G)+1} = \varnothing$. If $t = 3$, then Lemma 10 (c) implies that $S \cap (R_i^r \cup B_i \cup R_{i+1} \cup R_{d(G)} \cup B_{d(G)}) = \varnothing$. But this is a contradiction because $v_n \in S$. Then consider $t = 4$. By Lemma 10 (b), $S \cap (R_{\ell_j}^r \cup B_{\ell_j}) = \varnothing$ or $S \cap B_{d(G)} = \varnothing$. Since $v_n \in S$, it holds $S \cap (R_{\ell_j}^r \cup B_{\ell_j}) = \varnothing$. Note that $label(C_{j-1}) = -1$. Let $(C_i, \ldots, C_j)$ be the maximal sequence of $C$-sets such that $label(C_k) = -1$ for $i \leq k \leq j$. Note that $C_k$ has type 1 or 4 for $i < k \leq j$. Write $C_k = C_{\ell_k, \ell_{k+1}-1}$ for $i \leq k < j$. Consider first $i = 1$. By the algorithm, $C_1 = C_{0,\ell_2-1}$ has type 4. By Lemma 10 (b), $B_0 \cap S = \varnothing$ or $(B_{\ell_2-1} \cup R_{\ell_2}^\ell) \cap S = \varnothing$. Since $v_1 \in B_0$ is a simplicial vertex, it holds $(B_{\ell_2-1} \cup R_{\ell_2}^\ell) \cap S = \varnothing$. Now consider $i > 1$. The algorithm implies that $C_i$ has type 2 or 3. Lemmas 10 (a) and (c) imply that $(B_{\ell_{i+1}-1} \cup R_{\ell_{i+1}}^\ell) \cap S = \varnothing$. Thus, in any case, Lemma 10 (b) implies that $S \cap (R_{\ell_k}^r \cup B_{\ell_k}) = \varnothing$ or $S \cap (B_{\ell_{k+1}-1} \cup R_{\ell_{k+1}}^\ell) = \varnothing$ for $i < k < j$. This means that there is some $p \in \{i+1, \ldots, j\}$ such that $S \cap (B_p \cup R_{p+1} \cup B_{p+1}) = \varnothing$, which is a contradiction by Lemmas 2 and 3. □

**Theorem 1.** *If $G$ is a unit interval graph with exactly two simplicial vertices, then $h_3^*(G) = f'(G) + \sum_{1 \leq i \leq t} f(C_i)$. Besides, the $P_3^*$-hull number of a unit interval graph $G$ can be found in linear time.*

*Proof.* Consequence of Lemmas 8, 9, 10, and 11. Besides, a canonical ordering of a unit interval graph can be found in linear time [8,16], and thus, its simplicial vertices as well. Since the components of a unit interval graph can be determined in linear time [9], the result follows due to Lemma 1. □

## 3 Chordal Graphs

We conclude by pointing out the succeeding NP-completeness for the superclass of chordal graphs.

**Theorem 2.** *Given a chordal graph $G$ and an integer $k$, it is NP-complete to decide whether $h_3^*(G) \leq k$.*

The main idea behind the NP-completeness proof (omitted here due to lack of space) is a polynomial reduction from a restricted version of SATISFIABILITY which is NP-complete [10,12] . Let $\mathcal{C}$ be an instance of SATISFIABILITY consisting of $m$ clauses $C_1, \ldots, C_m$ over $n$ boolean variables $x_1, \ldots, x_n$ such that every clause in $\mathcal{C}$ contains at most three literals and, for every variable $x_i$, there are exactly two clauses in $\mathcal{C}$, say $C_{j_i^1}$ and $C_{j_i^2}$, that contain the literal $x_i$, and exactly one clause in $\mathcal{C}$, say $C_{j_i^3}$, that contains the literal $\bar{x}_i$, and these three clauses are distinct.

Let the graph $G$ be constructed as follows starting with the empty graph. For every $j \in [m]$, add a vertex $c_j$. For every $i \in [n]$, add 10 vertices $x_i, y_i, z_i, x_i^1, x_i^2, w_i^1, w_i^2, \bar{x}_i, \bar{y}_i, \bar{w}_i$ and 17 edges to obtain the subgraph indicated in Fig. 6. Add a vertex $z$ and the edges to make a clique of $C \cup Z \cup \{z\}$, where

$C = \{c_j | j \in [m]\}$ and $Z = \{z_i | i \in [n]\}$. Setting $k = 4n + 1$, we show in the full version of the paper that $\mathcal{C}$ is satisfiable if and only if $G$ contains a $P_3^*$-hull set of order at most $k$.

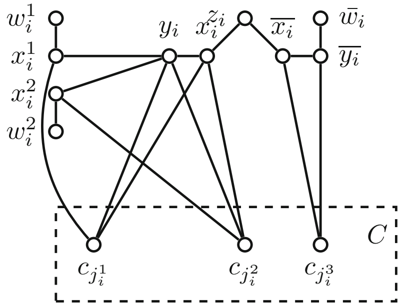


**Fig. 6.** When the construction of $G$ ends, $z_i$ will belong to the clique $C \cup \{z_1, \ldots, z_n\} \cup \{z\}$.

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