Cops and Robber on Some Families of Oriented Graphs

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Abstract. Cops and robber game on a directed graph $\overrightarrow{D}$ initiates by Player 1 placing k cops and then Player 2 placing one robber on the vertices of $\overrightarrow{D}$. After that, starting with Player 1, alternately the players may move each of their tokens to the adjacent vertices. Player 1 wins if, after a finite number of moves, a cop and the robber end up on the same vertex and Player 2 wins otherwise. However, depending on the type of moves the players make, there are three different models, namely, *the normal cop model*: both cops and robber move along the direction of the arcs; *the strong cop model*: cops can move along or against the direction of the arcs while the robber moves along them; and *the weak cop model*: the robber can move along or against the direction of the arcs while the cops move along them. A graph is *cop-win* if Player 1 playing with a single cop has a winning strategy. In this article, we study the three models on some families of oriented graphs and characterize the cop-win directed graphs for the third model.

1 Introduction

Cops and Robber is a popular two-player game introduced by Nowakowski and Winkler [22] in 1983 having applications in artificial intelligence, graph search, game development etc. [3,15,16] as well as significant implications in theory [23]. The game is extensively studied since its introduction giving rise to deep theory (see [4] for details) and its numerous variations contributed to its extent.

Recently, Nicolas Nisse[1] [8] introduced one of the variants (the strong cop model) in directed graphs (digraphs) and asked to characterize the "cop-win" graphs in two variants (the normal cop model and the strong cop model). In a seminar (summer 2018) at Simon Fraser University, another natural variant was discussed where the first author was present. In this article, we study all three variants, starting by presenting their precise definitions.

Setup and initiation: We start with an oriented[2] graph $\overrightarrow{G}$ and Player 1 places k cops on its vertices (multiple cops can be on the same vertex). After that Player 2 places the robber on one vertex of the graph.

Play: After the setup, Player 1 and 2 take turns to move their cops and robber, respectively, with Player 1 taking the first turn.

Winning: Player 1 wins if after finitely many turns the robber and a cop are on the same vertex. In this case, we say that the cop *captures* the robber. Player 2 wins if Player 1 does not win in finite number of moves.

Normal Move: Suppose uv is an arc. In a normal move, the cop/robber can move only from u to v.

Strong Move: Suppose uv is an arc. In a strong move, the cop/robber can move from u to v as well as from v to u.

Normal Cop Model: In their respective turns, Player 1 and Player 2 can perform at most one normal move on each of its cops, and the robber respectively.

Strong Cop Model: In their respective turns, Player 1 can perform at most one strong move on each of its cops, whereas Player 2 can perform at most one normal move on the robber.

Weak Cop Model: In their respective turns, Player 1 can perform at most one normal move on each of its cops, whereas Player 2 can perform at most one strong move on the robber.

Now that we have described the three models, we define a few necessary parameters. The *normal (resp., strong, weak) cop number* $c_n(\overrightarrow{G})$ (resp., $c_s(\overrightarrow{G})$, $c_w(\overrightarrow{G})$) of an oriented graph $\overrightarrow{G}$ is the minimum number of cops needed by Player 1 to have a winning strategy in the normal (resp., strong, weak) cop model. Furthermore, for a family $\mathcal{F}$ of oriented graphs

$$c_x(\mathcal{F}) = \max\{c_x(\overrightarrow{G})|\overrightarrow{G} \in \mathcal{F}\}$$

where $x \in \{n, s, w\}$. Given a fixed model, an oriented graph is *cop-win* if Player 1 has a winning strategy playing with a single cop.

Below we give a brief survey of the literature concerning the normal cop model, followed by a summary of our results.

Survey: Hamidoune [12] considered the game on Cayley digraphs. Frieze et al. [9], studied the game on digraphs and gave an upper bound of $O\left(\frac{n(\log\log n)^2}{\log n}\right)$ for cop number in digraphs. Along these lines, Loh and Oh [21] constructively proved the existence of a strongly connected planar digraph with cop number greater than three. They also prove that every n-vertex strongly connected planar digraph has cop number at most $O(\sqrt{n})$.

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[2] An oriented graph is a directed graph without 2-cycles i.e. each edge has a direction. For the purposes of this article, they are the same.

Goldstein and Reingold [10] proved that deciding if k cops can capture a robber is EXPTIME-complete if k is not fixed and either the initial positions are given or the graph is directed. Later Kinnersley [19] proved that determining the cop number of a graph or digraph is EXPTIME-complete. Kinnersley [20] also showed that n-vertex strongly connected cop-win digraphs can have capture time $\Omega(n^2)$.

Hahn and MacGillivray [11] gave an algorithmic characterization of the cop-win finite reflexive digraphs. They also showed that any k-cop game can be reduced to 1-cop game (resulting in an algorithmic characterization for k-cop-win finite reflexive digraphs). However, these results do not give a structural characterization of such graphs. Later Darlington et al. [6] tried to structurally characterize cop-win oriented graphs and gave a conjecture which was later disproved by Khatri et al. [17], who also study the game in oriented outerplanar graphs and line digraphs.

Recently, Hosseini and Mohar [13] (also see [14]) studied whether cop number of planar Eulerian digraphs is bounded or not, and point to evidence of the former.

Organization and Results: In Sect. 2, we compare the parameters $c_n(\cdot), c_s(\cdot), c_w(\cdot)$. The normal, strong and weak cop models are studied in Sects. 3, 4 and 5, respectively. We give an outline of our results.

1. Normal Cop Model
   – Prove a Mycielski-type result by constructing oriented graphs with high normal cop number and girth.
   – Characterize oriented triangle-free and outerplanar normal cop-win graphs.
2. Strong Cop Model
   – Find strong cop number of oriented planar graphs, oriented outerplanar graphs and oriented series-parallel graphs.
   – Prove that a specific class of oriented outerplanar graphs (whose weak dual is a collection of paths) and oriented grids are strong cop-win.
3. Weak Cop Model
   – Characterize weak cop-win oriented graphs.

Now we look into some relations between the parameters $c_n(\cdot), c_s(\cdot), c_w(\cdot)$ and some definitions.

2   Basic Results and Preliminaries

The first result follows directly from the definitions.

Proposition 1. *For any oriented graph $\overrightarrow{G}$ we have $c_s(\overrightarrow{G}) \leq c_n(\overrightarrow{G}) \leq c_w(\overrightarrow{G})$.*

Observe that there are plenty of oriented graphs, the transitive tournament for instance, where equality hold in each of the cases. However, it is interesting to study the gap between these parameters. But first we will introduce some

notations and terminologies. Let uv be an arc of an oriented graph $\overrightarrow{G}$. We say that u is an *in-neighbor* of v and v is an *out-neighbor* of u. Let $N^-(u)$ and $N^+(u)$ denote the set of in-neighbors and out-neighbors of u respectively. A vertex without any in-neighbor is a *source* and a vertex without any out-neighbor is a *sink*.

Proposition 2. *Given any $m, n \in \mathbb{N}$, there exists an oriented graph $\overrightarrow{G}$ such that $c_n(\overrightarrow{G}) - c_s(\overrightarrow{G}) = n$ and $c_w(\overrightarrow{G}) - c_n(\overrightarrow{G}) \geq m$.*

Proof. The oriented graph $\overrightarrow{G} = \overrightarrow{G}_{m,n}$ is composed of two oriented graphs $\overrightarrow{A}_n$ and $\overrightarrow{B}_m$. The oriented graph $\overrightarrow{A}_n$ is an orientation of the star graph such that its central vertex v is a sink having degree $n + 1$.

We know that there exist graphs with arbitrarily high cop number in undirected case [1]. Let B_m be a connected undirected graph with cop number at least m. Let $\overrightarrow{B}_n$ be such an orientation of B_m that it is a directed acyclic graph having a single source u. The graph $\overrightarrow{G}_{m,n}$ is obtained by merging vertices u and v (call this vertex v_{merge}). Note that $c_s(\overrightarrow{G}_{m,n}) = 1$ as Player 1 can place one cop on v_{merge} and capture the robber in one move if it is in $\overrightarrow{A}_n$ or capture the robber in a finite number of moves if it is in $\overrightarrow{B}_m$.

On the other hand, $c_n(\overrightarrow{G}_{m,n}) = n + 1$, as Player 1 must keep a cop on each source to win, and since $\overrightarrow{B}_m$ is a directed acyclic graph, one of the cops reaches v_{merge} and then captures the robber in $\overrightarrow{B}_m$. Also, $c_w(\overrightarrow{G}_{m,n}) \geq m + n + 1$ as Player 1 needs to place $n+1$ cops at sources in $\overrightarrow{A}_n$ and it needs at least as many cops as the cop number of B_m. □

We end this section with some general notations and terminologies. The *out-degree* of v is $d^+(v) = |N^+(v)|$ and its *in-degree* is $d^-(v) = |N^-(v)|$. Let $N^+[v] = N^+(v) \cup \{v\}$ denote the *closed out-neighbourhood* of v.

In the rest of this article, we refer to the robber as $\mathcal{R}$; and to the cop, only in case of cop-win graphs, as $\mathcal{C}$.

If a cop moves to an in-neighbour of the robber $\mathcal{R}$, then we say that the cop *attacks* the robber. The robber is on a *safe* vertex from a cop if it cannot be captured by the cop in the next turn of Player 1. The robber *evades* capture if every time the cop attacks it, $\mathcal{R}$ can move to a safe vertex.

3   Normal Cop Model

In the context of cops and robbers on oriented graphs, the weakly connected case reduces to solving the strongly connected case [9]. Hence it suffices to consider strongly connected oriented graphs. We begin by constructing strongly connected oriented graphs with arbitrarily high normal cop number and girth (length of a smallest cycle in the graph).

Theorem 1. *Given any $g \geq 5$ and $c \geq 3$, there exists a strongly connected oriented graph $\overrightarrow{G}_{g,c}$ with girth at least g having $c_n(\overrightarrow{G}_{g,c}) \geq c + 1$.*

Proof. We borrow a construction to form *regular expander graphs* with high girth [18]. For sake of completeness, we present their complete construction. (Also see [2].) Let G be a simple graph and let L be a set. Define the graph G^L to be a *L-lift* of G if $V(G^L) = V(G) \times L$, and for every edge $uv \in E(G)$, the sets $F_u = \{(u, l_i)\}_{l_i \in L}$ and $F_v = \{(v, l_i)\}_{l_i \in L}$ induce a perfect matching in G^L. Here G is the base graph and G^L depends on the matching between F_u and F_v assigned to each edge uv. Observe that the lifts of $k-$regular graphs are also $k-$regular. From Amit and Nilial [2], it follows that there are lifts of G which are $\delta-$connected for $\delta \geq 3$, where δ is the minimum degree of G. For path $u - v - \cdots - w$ in G, G^L will have a unique path $(u, l_{i_1}) - (v, l_{i_2}) - \cdots - (w, l_{i_k})$, for some $l_{i_1}, l_{i_2}, \ldots, l_{i_k} \in L$. The path $u - v - \cdots - w$ is called as the *projection* of $(u, l_{i_1}) - (v, l_{i_2}) - \cdots - (w, l_{i_k})$.

Now consider a graph G with a fixed ordering π of $m = |E(G)|$ edges, and let $L = \{0, 1\}^m$, that is, the set of all possible m-tuples of 0's and 1's. For $u, v \in V(G)$ and $l_i, l_j \in L$, we connect (u, l_i) with (v, l_j) in G^L if $uv \in E(G)$ and l_i and l_j differ only at the index of edge uv in the ordering π. So (u, l_i) is adjacent to (v, l_j) and (u, l_j) is adjacent to (v, l_i). Hence for edge $uv \in E(G)$, the sets $F_u(= \{(u, l_i)\}_{l_i \in L})$ and $F_v(= \{(v, l_i)\}_{l_i \in L})$ induce a perfect matching in G^L. Thus G^L is a $L-$lift of G.

Now we pick a shortest cycle C_0 in G^L. Its projection in G is also a cycle C. We claim that for every edge $uv \in C$ there are at least two edges in C_0 between F_u and F_v. Start at point (u, l_i) of C_0. Let the next vertex in C_0 be (v, l_j); so l_i and l_j differ only at the index of uv in π. Now to reach (u, l_i) we need to flip the value at the index of uv in π. This happens only if we traverse uv once again. So $|C_0| \geq 2|C|$. Hence girth of G^L is at least twice the girth of G.

To construct the oriented graph with arbitrarily high cop number and girth, do the following. Take a K_{2c+1} and go on applying the above-mentioned lift construction repeatedly until the girth is at least g. The resulting graph $G_{g,c}$ is Eulerian as degree of v in G is even and is the same as the degree of (v, l_i) in $G_{g,c}$. Make the Eulerian circuit a directed circuit by assigning orientations to the edges. This results in a strongly connected oriented graph with girth at least g. Observe that the out-degree of each vertex is c. Thus its normal cop number is at least $c + 1$ as we know that a strongly connected oriented graph with girth at least 5 have normal cop number $c_n(\overrightarrow{G}) \geq \delta^+(\overrightarrow{G}) + 1$, where $\delta^+(\overrightarrow{G})$ is the minimum out-degree of $\overrightarrow{G}$ [21].   □

Darlington et al. [6] characterized cop-win oriented paths and trees in the normal cop model. We are also going to do so for some other families of oriented graphs.

A *transitive-triangle-free* oriented graph is an oriented graph with no transitive triangles. The following theorem characterizes cop-win transitive-triangle-free oriented graphs, a superclass of triangle-free oriented graphs.

Proposition 3. *A transitive-triangle-free oriented graph $\overrightarrow{G}$ is cop-win if and only if it is a directed acyclic graph with one source.*

Proof. Observe that any directed acyclic graph with one source is cop-win and that every cop-win oriented graph has exactly one source. So it suffices to prove that if a transitive triangle-free oriented graph $\overrightarrow{G}$ is cop-win, then it is a directed acyclic graph.

Suppose $\overrightarrow{G}$ has a directed cycle[3] $\overrightarrow{C}$ on at least 3 vertices. We will now give a strategy for the robber $\mathcal{R}$ to escape. Note that the cop c must be placed at the source initially, as otherwise Player 2 places $\mathcal{R}$ on the source and wins. $\mathcal{R}$ initially places himself at some safe vertex of $\overrightarrow{C}$. Such a vertex exists, as any vertex in $\overrightarrow{G}$ cannot dominate two consecutive vertices in $\overrightarrow{C}$, else a transitive triangle is created. $\mathcal{R}$ moves to the next vertex in $\overrightarrow{C}$ whenever $\mathcal{R}$ lies in the out-neighbour of c. Whenever c attacks $\mathcal{R}$, the robber moves to the next vertex in $\overrightarrow{C}$ and evades the attack. Since $\overrightarrow{C}$ is a directed cycle, c cannot capture $\mathcal{R}$. This contradicts that $\overrightarrow{G}$ is a cop-win graph; hence the result. □

As bipartite graphs are triangle-free, we have the following corollary.

Corollary 1. *An oriented bipartite graph is cop-win if and only if it is a directed acyclic graph with one source.*

Next, we characterize cop-win oriented outerplanar graphs.

Proposition 4. *An oriented outerplanar graph $\overrightarrow{G}$ is cop-win if and only if it is a directed acyclic graph with one source.*

Proof. The 'if' part is obvious.

For proving the 'only if' part, first note that a graph cannot be cop-win if it has no source or at least two sources. Thus suppose that there exists an oriented outerplanar cop-win graph $\overrightarrow{G}$ containing a directed cycle $\overrightarrow{C}$ with exactly one source v. The cop c must be initially placed on the source v.

Note that at most two vertices of $\overrightarrow{C}$ can have a path made up of vertices from outside $\overrightarrow{C}$ connecting v in order to avoid a K_4-minor. So there is at least one safe vertex u in $\overrightarrow{C}$ such that any directed path connecting v to u must go through some vertex of $\overrightarrow{C}$ other than u. Thus if the robber $\mathcal{R}$ places itself on u and does not move until c comes on a vertex of $\overrightarrow{C}$, it cannot be captured.

If c is on a vertex of $\overrightarrow{C}$ and starts moving towards $\mathcal{R}$ following the direction of the arcs of $\overrightarrow{C}$, then $\mathcal{R}$ also moves forward and evades c.

Thus c must go out of $\overrightarrow{C}$ in order to try and capture $\mathcal{R}$. The moment c goes out to some vertex w outside $\overrightarrow{C}$, $\mathcal{R}$ either is on a safe vertex or it can move to a safe vertex on $\overrightarrow{C}$ in its next move as w can be adjacent to at most two vertices of $\overrightarrow{C}$ in order to avoid a K_4-minor.

This brings us to a situation similar to the initial situation. Thus, the robber will always evade the cop, a contradiction. □

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[3] We use the term *directed cycle* instead of *oriented cycle* as it is commonly used.

4   Strong Cop Model

The strong cop number of an oriented graph is upper bounded by cop number in classical version of the game on the underlying undirected graph. We begin this section by finding strong cop number of planar graphs, outerplanar graphs, and series-parallel graphs. But first, we find a lower bound of the strong cop number of a specific oriented graph.

Construction: Given an undirected graph G on vertex set $\{v_1, \ldots, v_n\}$, we form an oriented graph $\overrightarrow{H}$ from G by replacing each edge $v_i v_j$ in G by a directed 4-cycle $v_i u_{ij} v_j u_{ji} v_i$. We have the following lemma relating strong cop number of $\overrightarrow{H}$ with cop number of G. Here $c(G)$ is the cop number of the undirected graph G.

Lemma 1. $c_s(\overrightarrow{H}) \geq c(G)$.

Proof. Each vertex $v_i \in V(G)$ corresponds to the set $N^+[v_i]$ in $V(\overrightarrow{H})$. Note the sets $N^+[v_i]$ partition $V(\overrightarrow{H})$. For each $v \in N^+[v_i]$ in $\overrightarrow{H}$, define its image in G as $I(v) = v_i$. We know that $c_s(\overrightarrow{H})$ cops have a strategy to capture the robber $\mathcal{R}$ in $\overrightarrow{H}$. We will show that $c_s(\overrightarrow{H})$ cops have a winning strategy in G.

We use the winning strategy of $c_s(\overrightarrow{H})$ cops in $\overrightarrow{H}$ to obtain a winning strategy in G. As the game is played in G, we also (sort of) play it in $\overrightarrow{H}$ by following $\mathcal{R}$'s move in G. The move of the cops in $\overrightarrow{H}$ following the winning strategy is translated to G using the images. This procedure is done as follows.

Initially in G, place the cops and then $\mathcal{R}$ is placed. In $\overrightarrow{H}$, place the cops and $\mathcal{R}$ at the vertices with same labels as in the occupied vertices in G. The cops in G pass their first move and then $\mathcal{R}$ moves or passes its move. For each move of $\mathcal{R}$ in G (say from v_i to v_j), we play two turns in $\overrightarrow{H}$: in the first turn $\mathcal{R}$ moves from v_i to u_{ij} and then to v_j in the second turn. In each of these two turns in $\overrightarrow{H}$, the cops move following their winning strategy. After two turns in $\overrightarrow{H}$, the cops in G move to the images of cops in $\overrightarrow{H}$ (this is always possible). Following the winning strategy, when $\mathcal{R}$ is captured in $\overrightarrow{H}$, $\mathcal{R}$ is also captured in G.   □

As a result of Lemma 1, we find the strong cop number of oriented planar graphs and then form oriented graphs with arbitrarily high strong cop number.

Corollary 2. *The strong cop number of the family of oriented planar graphs is three.*

Proof. Recall that the strong cop number of an oriented graph is upper bounded by cop number in classical version of the game on the underlying undirected graph. The cop number of planar graphs is three [1]. Apply the construction used in Lemma 1 to a planar graph with cop number 3 to get an oriented planar graph with $c_s \geq 3$ (the construction used in Lemma 1 maintains planarity).   □

Corollary 3. *For every $k \in \mathbb{N}$, there exists an oriented graph $\overrightarrow{H}$ such that $c_s(\overrightarrow{H}) \geq k$.*

Proof. Apply the construction used in Lemma 1 to a graph with cop number at least k (whose existence is given in [1]).   □

Next, we find strong cop numbers of the family of oriented outerplanar and series-parallel graphs.

Theorem 2. *The strong cop number of the family of oriented outerplanar graphs is two.*

Proof. The cop number of outerplanar graphs in the classical game on undirected graphs is two [5]. Hence it suffices to construct an oriented graph which is not strong cop-win.

Consider an outerplanar graph on 47 vertices formed by joining 2 copies of the following biconnected outerplanar graph at a vertex, say v_0. Take a cycle C_1 (see Fig. 1) on $v_0, v_1, \ldots, v_{23}$ arranged in counterclockwise manner with arcs $v_i v_{i-1}$ and $v_{2i} v_{2i+2}$ (under modulo 24). Let the other copy C_2 of the cycle be on vertices $v_0 = u_0, u_1, \ldots, u_{23}$ with arcs $u_i u_{i-1}$ and $u_{2i} u_{2i+2}$ (under modulo 24). The arcs of the form $v_i v_{i-1}$ are called *cycle arcs* and the arcs of the form $v_{2i} v_{2i+2}$ are called *chord arcs*. The vertices v_i and u_j with even indices are called *even vertices* and with odd indices are called *odd vertices*.
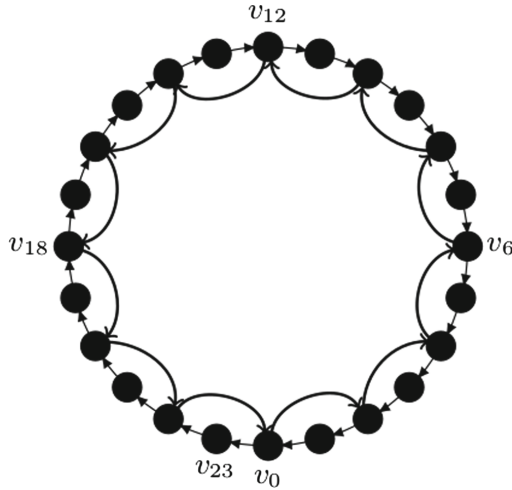

Fig. 1. The biconnected outerplanar graph C_1.

If the cop c is placed at v_0, then robber $\mathcal{R}$ enters at v_4; else $\mathcal{R}$ starts at v_4 or u_4 depending on whether c starts at C_2 or C_1 respectively. In the latter case (that is c is not placed at v_0), $\mathcal{R}$ passes his moves until c is at v_0 (in order to catch $\mathcal{R}$, c has to go through v_0). Once c reaches v_0, $\mathcal{R}$ passes his move once more; reducing this case to the former case. Hence, without loss of generality, assume that c and $\mathcal{R}$ start at v_0 and v_4 respectively.

In the rest of this proof, we show that if c (at v_0) tries to capture $\mathcal{R}$ (at v_4), then $\mathcal{R}$ reaches the *initial configuration* (c at v_0 and $\mathcal{R}$ at v_4) or its *equivalent configuration* (c at v_0 and $\mathcal{R}$ at u_4) without being captured. Precisely, we show that if c pursues $\mathcal{R}$, then $\mathcal{R}$ reaches v_0 two turns before c. So after two more turns $\mathcal{R}$ can be at u_4 or v_4 depending on whether c is in C_1 or C_2 respectively; and then passes its moves until c is at v_0. So $\mathcal{R}$ evades capture indefinitely; thereby proving that the graph constructed above is not strong cop-win.

To simplify our presentation, we use the following notations. Let variable $X = \{c, \mathcal{R}\}$; variables U, V denote two adjacent vertices; and symbol $*$ denote $\circlearrowright$ for clockwise or $\circlearrowleft$ for counter-clockwise. Read $X(U * V)$ as "X moves from U to V in $*$ sense". Read $X(*)$ as "X moves in $*$ sense to an adjacent vertex". Let d_c denote the distance between c and $\mathcal{R}$ at the given instant in the underlying undirected graph. Note that $\mathcal{R}(*)$ results in a fixed final position, where as $c(*)$ results in two possible final positions.

$\mathcal{R}$ moves according to the following rules. All operations are performed under modulo 24.

R0: At any turn, if c passes its move then $\mathcal{R}$ passes its move.
R1: For $i = 1$ to 7, $\mathcal{R}(v_{2i} \circlearrowright v_{2i+2})$ only if $c(v_{2i-4} \circlearrowleft v_{2i-2})$ or $c(v_{2i-3} \circlearrowleft v_{2i-2})$; else it passes its move.
R2: For $i = 8$ to 11, $\mathcal{R}(v_{2i} \circlearrowright v_{2i+2})$ irrespective of c's move.
R3: If $\mathcal{R}$ is at an even vertex v_{2i}, for $i \leq 7$, and $\mathcal{C}(\circlearrowleft)$, then
   - $\mathcal{R}(\circlearrowright)$, if d_c increases to at least 4.
   - $\mathcal{R}$ passes its move, if d_c increases but remains less than 4.
   - $\mathcal{R}(\circlearrowleft)$, if d_c decreases.
R4: If $\mathcal{R}$ is at an odd vertex v_{2i+1}, for $i < 7$, then $\mathcal{R}(\circlearrowright)$ irrespective of c's move.

We claim that $\mathcal{R}$ reaches v_0 at least two turns before c. Once $\mathcal{R}$ is at v_{14} and $c(v_{10} \circlearrowleft v_{12})$, then $\mathcal{R}$ keeps on moving counter-clockwise and reaches v_0 at least two turns before c. However if $c(\circlearrowleft)$ and if d_c increases to at least 4, then $\mathcal{R}(\circlearrowright)$, else if $d_c < 4$ then $\mathcal{R}$ passes its move. The restriction $d_c \geq 4$ ensures that if c moves counter-clockwise then $\mathcal{R}$ can safely move clockwise to the next even vertex. For subsequent steps, if $c(\circlearrowleft)$ and $\mathcal{R}$ is on an even vertex v_{2i}, for $i = 1$ to 7, then $\mathcal{R}(\circlearrowright)$, provided the restrictions in *R3* are met. In any intermediate step if $c(\circlearrowleft)$, then $\mathcal{R}(\circlearrowright)$ if it is at an odd vertex; else $\mathcal{R}(\circlearrowleft)$ or $\mathcal{R}$ passes its move depending on whether c attacks it or not. In such a case $\mathcal{R}$ always stays at least two moves away from c and hence evades capture.

The only way left for c to capture $\mathcal{R}$ is if c continues moving counter-clockwise along the chord arcs and then tries to capture $\mathcal{R}$ which now moves counter-clockwise along the cycle arcs. However in such a case also it is easy to see that $\mathcal{R}$ reaches v_0 at least two moves before c. Hence, either $\mathcal{R}$ evades capture indefinitely or reaches the initial or its equivalent configuration; which implies that $\mathcal{R}$ is never captured. Hence the constructed graph is not strong cop-win. $\square$

It is known that the cop number of series-parallel graphs in the classical game on undirected graphs is two [24]. Since outerplanar graphs are also series-parallel graphs, we have the following corollary.

Corollary 4. *The strong cop number of oriented series-parallel graphs is two.*

As mentioned earlier all the oriented graphs whose underlying graphs are cop-win graphs in the classical (undirected graph) version are strong cop-win. Next, we find some families of oriented graphs which are strong cop-win but whose underlying undirected graphs are not cop-win in the classical version. We begin with a specific class of outerplanar graphs.

We need the following definitions. For a plane graph G (i.e. the planar embedding of G), its *dual graph* has vertices that represent faces of G and edges represent the adjacency of faces in G separated by an edge. The *weak dual* of G is the induced subgraph of the dual graph whose vertices correspond to the bounded faces of G.

Theorem 3. *Oriented outerplanar graphs whose weak dual is a collection of paths are strong cop-win.*

Proof. Let G be an outerplanar graph on n vertices, whose weak dual is a collection of paths and $\overrightarrow{G}$ denote the oriented outerplanar graph on G. We call the edges in the outer face of G as *cycle edges*. For a cycle C in $\overrightarrow{G}$, *image* $I_C(\mathcal{R})$ of the robber $\mathcal{R}$, is the set of vertices in C that are closest to $\mathcal{R}$.

First we claim that $|I_C(\mathcal{R})| \leq 2$. Suppose $|I_C(\mathcal{R})| > 2$; then let $u_1, u_2, u_3 \in I_C(\mathcal{R})$ be three vertices arranged in a cyclic order in C. Since distance from $\mathcal{R}$ to u_1, u_2, u_3 are same, the paths from the robber to u_i does not contain u_j, for $i \neq j$ and $i, j \leq 3$. So u_2 does not lie in the outer face; a contradiction.

Furthermore, if $|I_C(\mathcal{R})| = \{u_1, u_2\}$, then u_1 and u_2 are adjacent, else the internal vertices on a $u_1 u_2$ path does not lie in the outer face. If $\mathcal{R}$ is in the cycle then $I_C(\mathcal{R})$ contains the vertex occupied by $\mathcal{R}$. So if $|I_C(\mathcal{R})| = 2$, then $\mathcal{R}$ does not lie in the cycle C.

Now we prove the theorem by induction on the order of G. The base case is easy to verify. Now assume every outerplanar graph of order less than n, whose weak dual is a collection of paths is strong cop-win.

Now consider an outerplanar graph $\overrightarrow{G}$ of order n, whose weak dual is a collection of paths. Select a cycle C in $\overrightarrow{G}$ and place the cop c in some vertex of C. After $\mathcal{R}$ is placed in $\overrightarrow{G}$, we find $I_C(\mathcal{R})$ and capture it in subsequent moves. If $|I_C(\mathcal{R})| = 2$, then we capture any one of them. This can always be done in a cycle that is not directed. If C is directed, c moves against the orientations. So a vertex in $I_C(\mathcal{R})$ can be captured by c. If $\mathcal{R}$ is in the cycle then it is captured. If $|I_C(\mathcal{R})| = 2$, then $\mathcal{R}$ is not in the cycle when some vertex of $I_C(\mathcal{R})$ is captured by c. Once c captures a vertex in $I_C(\mathcal{R})$, the robber cannot enter C. So $\mathcal{R}$ is now trapped in one component of $\overrightarrow{G}$ obtained after deleting the cycle edges of C. By our inductive hypothesis, $\mathcal{R}$ can be captured in this component. Hence oriented outerplanar graphs whose weak dual is a collection of paths are cop-win in the strong cop model. □

Our next class of strong cop-win graphs are oriented grids.

Theorem 4. *Oriented grids are strong cop-win.*

Outline of the Proof. Fix a $m \times n$ grid with points $\{(i,j)|0 \leq i \leq m-1, 0 \leq j \leq n-1\}$. The cop c starts at $(0,0)$. If the robber $\mathcal{R}$ is at (x_r, y_r), then define the vertices $(x_r \pm 1, y_r \pm 1)$ (when they exist) as *guard positions*. The reader can check that c can *guard* $\mathcal{R}$ at $(x_r - 1, y_r - 1)$ (see [7, Step 1 in Thm. 1]). So $\mathcal{R}$ can only move either up or to the right if the orientations allow (else $\mathcal{R}$ is caught). Note that once it reaches $(m-1, n-1)$, it gets captured. If $\mathcal{R}$ stays stagnant at a vertex, then c can force it to move. Although the guard position is lost, after a few steps c can regain the guard position (or capture $\mathcal{R}$). So the Y-coordinate of $\mathcal{R}$ gradually increases. Eventually $\mathcal{R}$ ends up at $(m-1, n-1)$ or gets captured. □

5   Weak Cop Model

A vertex u in a directed graph is said to be a *corner vertex*, if there exists a vertex v such that $N^+[u] \cup N^-(u) \subseteq N^+[v]$ where $N^\alpha[v] = N^\alpha(v) \cup \{v\}$ for each $\alpha \in \{+,-\}$. We also say that v *dominates* u.

Now we characterize all cop-win directed graphs in this model, which is adapted from the cop-win characterization of undirected graphs (whose proof follows from a couple of lemmas).

Theorem 5. *A directed graph is cop-win in the weak cop model if and only if by successively removing corner vertices, it can be reduced to a single vertex.*

Lemma 2. *If a directed graph has no corner vertex, then it is not weak cop-win.*

Proof. Let $\overrightarrow{G}$ have no corner vertex. The robber $\mathcal{R}$ starts from a vertex that is not an out-neighbour of the cop c. The robber does not move unless c attacks it. Whenever $\mathcal{R}$ is under attack, it can move to a vertex that in not an out-neighbour of c (as there are no corner vertices in $\overrightarrow{G}$). Hence $\mathcal{R}$ never gets caught.   □

Lemma 3. *A directed graph $\overrightarrow{G}$ with a corner u is weak cop-win if and only if $\overrightarrow{H} = \overrightarrow{G} \setminus \{u\}$ is weak cop-win.*

Proof. Let vertex v dominate u in G. Suppose $\overrightarrow{H}$ is cop-win. Define the image $I_\mathcal{R}$ of the robber $\mathcal{R}$ as follows: $I_\mathcal{R}(u) = v$ and $I_\mathcal{R}(x) = x$ for all $x \in V(\overrightarrow{H})$. So $I_\mathcal{R}$ is restricted to $\overrightarrow{H}$ and it can be captured by the cop c. If $\mathcal{R}$ is not on u, then it is captured. If $\mathcal{R}$ is on u, then c is on v and will capture $\mathcal{R}$ in its next move.

Suppose, on the other hand, $\overrightarrow{H}$ is not weak cop-win. Define the image $I_\mathcal{C}$ of the cop c as follows: $I_\mathcal{C}(u) = v$ and $I_\mathcal{C}(x) = x$ for all $x \in V(\overrightarrow{H})$. So $I_\mathcal{C}$ is restricted to $\overrightarrow{H}$ and $\mathcal{R}$ has a winning strategy against $I_\mathcal{C}$. If c is not on u, then $\mathcal{R}$ follows its winning strategy and does not get captured in c's next move. If c is on u, then $\mathcal{R}$ follows its winning strategy assuming c is on $I_\mathcal{C}(u) = v$. Since $\mathcal{R}$ has a winning strategy against c if c were at v instead, $\mathcal{R}$ does not get captured in c's next move (as v dominates u). So $\mathcal{R}$ evades capture; hence $\overrightarrow{G}$ is not weak cop-win.   □

Finally, we are ready to prove Theorem 5.

Proof of Theorem 5. Lemma 3 implies that upon removing the corner vertices, the weak cop-win property of the graph remains the same. Now remove all possible corner vertices successively in the directed graph. If we end up with a single vertex, then it is weak cop-win. Otherwise we end up with some other graph that has no corner vertices, Lemma 2 implies that it is not weak cop-win.   □

References

1. Aigner, M., Fromme, M.: A game of cops and robbers. Discret. Appl. Math. **8**, 1–12 (1984)
2. Amit, A., Linial, N.: Random graph coverings I: general theory and graph connectivity. Combinatorica, **22**(1), 1–18 (2002)
3. Alspach, B.: Sweeping and searching in graphs: a brief survey. Matematiche **59**, 5–37 (2006)
4. Bonato, A., Nowakowski, R.: The Game of Cops and Robbers on Graphs. American Mathematical Society, Providence (2011)
5. Clarke, N.E.: Constrained cops and robber. Ph.D. thesis, Dalhousie University (2002)
6. Darlington, E., Gibbons, C., Guy, K., Hauswald, J.: Cops and robbers on oriented graphs. Rose-Hulman Undergrad. Math. J. **17**(1), Article 14 (2016)
7. Das, S., Gahlawat, H.: Variations of cops and robbers game on grids. In: Panda, B.S., Goswami, P.P. (eds.) CALDAM 2018. LNCS, vol. 10743, pp. 249–259. Springer, Cham (2018). https://doi.org/10.1007/978-3-319-74180-2_21
8. Fomin, F., Fraigniaud, P., Nisse, N., Thilikos, D.M.: Report on GRASTA 2014, pp. 14 (hal-01084230) (2014)
9. Frieze, A., Krivelevich, M., Loh, P.: Variations on cops and robbers. J. Graph Theory **69**, 383–402 (2012)
10. Goldstein, A.S., Reingold, E.M.: The complexity of pursuit on a graph. Theoret. Comput. Sci. **143**, 93–112 (1995)
11. Hahn, G., MacGillivray, G.: A note on k-cop, l-robber games on graphs. Discret. Math. **306**, 2492–2497 (2006). Creation and Recreation: A Tribute to the Memory of Claude Berge
12. Hamidoune, Y.O.: On a pursuit game on Cayley digraphs. Eur. J. Comb. **8**, 289–295 (1987)
13. Hosseini, S.A., Mohar, B.: Game of cops and robbers in oriented quotients of the integer grid. Discret. Math. **341**, 439–450 (2018)
14. Hosseini, S.A.: Game of Cops and Robbers on Eulerian Digraphs. Ph.D. thesis, Simon Fraser University (2018)
15. Isaza, A., Lu, J., Bulitko, V., Greiner, R.: A cover-based approach to multi-agent moving target pursuit. In: Proceedings of The 4th Conference on Artificial Intelligence and Interactive Digital Entertainment (2008)
16. Isler, V., Kannan, S., Khanna, S.: Randomized pursuit-evasion with local visibility. SIAM J. Discret. Math. **1**, 26–41 (2006)
17. Khatri, D., et al.: A study of cops and robbers in oriented graphs. ArXiv e-prints, arXiv:1811.06155 (2019)
18. Kilbane, J.: Notes on Graphs of Large Girth

19. Kinnersley, W.B.: Cops and robbers is EXPTIME-complete. J. Comb. Theory Ser. B **111**, 201–220 (2015)
20. Kinnersley, W.B.: Bounds on the length of a game of cops and robbers. Discret. Math. **341**, 2508–2518 (2018)
21. Loh, P., Oh, S.: Cops and robbers on planar directed graphs. ArXiv e-prints, arXiv:1507.01023 (2015)
22. Nowakowski, R., Winkler, P.: Vertex-to-vertex pursuit in a graph. Discret. Math. **43**, 253–259 (1983)
23. Seymour, P.D., Thomas, R.: Graph searching and a min-max theorem for tree-width. J. Comb. Theory Ser. B **58**, 22–33 (1993)
24. Theis, D.O.: The cops and robber game on series-parallel graphs. ArXiv e-prints, arXiv:0712.2908 (2008)