Dual Domination

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Abstract. Inspired by the feedback scenario, which characterizes online social networks, we introduce a novel domination problem, which we call *Dual Domination (DD)*. We assume that the nodes in the input network are partitioned into two categories: Positive nodes (V^+) and negative nodes (V^-). We are looking for a set $D \subseteq V^+$ that dominates the largest number of positive nodes while avoiding as many negative nodes as possible. In particular, we study the *Maximum Bounded Dual Domination* (MBDD) problem, where given a bound k, the problem is to find a set $D \subseteq V^+$, which maximizes the number of nodes dominated in V^+, dominating at most k nodes in V^-. We show that the MBDD problem is hard to approximate to a factor better than $(1 - 1/e)$. We give a polynomial time approximation algorithm with approximation guaranteed $(1 - e^{-1/\Delta})$, where Δ represents the maximum number of neighbors in V^+ of any node in V^-. Furthermore, we give an $O(|V|k^2)$ time algorithm to solve the problem on trees.

1 Introduction

Let $G = (V, E)$ be an undirected graph modeling a network. We denote by $N_G(v)$ and by $d_G(v) = |N_G(v)|$, respectively, the neighborhood and the degree of the node v in G. In general, for each $S \subseteq V$ we denote by $N_G(S) = \bigcup_{v \in S} N_G(v)$ the neighborhood of the nodes in S. In the rest of the paper we will omit the subscript G whenever the graph G is clear from the context.

A dominating set for $G = (V, E)$ is a subset of the nodes $D \subseteq V$ such that each $v \in V - D$ has at least one neighbor in D. The concept of domination in graphs and its many related problems have been widely studied (see [17] and references therein quoted). Inspired by some scenarios in social networking, which we shall briefly describe in Sect. 1.2, we introduce a new domination problem, which we call *Dual Domination (DD)*. We assume that the nodes in the input network are partitioned into two categories: Positive nodes (V^+) and negative nodes (V^-); i.e., $V = V^+ \cup V^-$. For any $D \subseteq V^+$, we denote by $\Gamma(D)$ (resp. $\Gamma^+(D)$ and $\Gamma^-(D)$) the set of nodes (resp. positive and negative nodes) dominated by D. That is,

$$\Gamma(D) = D \cup N(D), \qquad \Gamma^+(D) = \Gamma(D) \cap V^+ \quad \text{and} \quad \Gamma^-(D) = \Gamma(D) \cap V^-.$$

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For sake of simplicity we pose $\Gamma^+(v) = \Gamma^+(\{v\})$ and $\Gamma^-(v) = \Gamma^-(\{v\})$.

Formally the problem we study in this paper is the following.

Problem 1. (Maximum Bounded Dual Domination (MBDD)) Given a network $G = (V = (V^+ \cup V^-), E)$ and an integer $k \geq 0$, find a set $D \subseteq V^+$ such that $|\Gamma^-(D)| \leq k$, which maximizes $|\Gamma^+(D)|$.

1.1 Our Results

We first show hardness results on the approximability of the MBDD problem, then we give a polynomial time approximation algorithm with approximation guaranteed $(1 - e^{-1/\Delta})$, where Δ represents the maximum number of neighbors in V^+ of any node in V^-. The algorithm uses the fact that $|\Gamma^+(D)|$ is a submodular, nondecreasing set function and is inspired by [27] where an approximation algorithm for maximizing a submodular set function subject to a knapsack constraint has been presented. However, we stress that the constraint, to which a solution of our MBDD problem is subject, is not a knapsack constraint since in our problem two or more positive nodes might share a negative neighbor. In Sect. 4, we depict an $O(|V|k^2)$ time algorithm for the MBDD problem on trees, state some related Dual Domination problems, and give some open problems.

Due to space constraint, most of the proofs are omitted or only sketched.

1.2 The Online Social Networks Context

Online social networks have become an important media for the dissemination of opinions, beliefs, new ideas etc. The increasing popularity of such platforms, together with the availability of large amounts of contents and user profile/behaviour information, has contributed to the rise of viral marketing as an effective advertising strategy. The idea is to exploit the word-of-mouth effect in such a way that an initial set of influential users could influence their friends, friends of friends, and so on, generating a large influence cascade. The key problem is how to select an initial set of users (given a limited budget) so to maximize the influence within the network. This *influence maximization* (IM) problem has been extensively studied in recent years [5–10,16] and a number of approximation algorithms and scalable heuristics have been devised. However, the studies above only look at networks with positive relationships/activities (e.g., positive feedback or influence), where in real scenarios, social actor relationships/activities also include negative ones (e.g., adverse opinion, negative feedback or distrust relationships). For instance in Ebay, buyers and sellers develop trust and distrust relationship; in online review and news forums, such as Slashdot, users comment (positively or negatively) reviews and articles of each other [21].

Research has provided evidence that the benefits of a marketing campaign are not purely increasing in the number of people reached and the exposure to different groups can help or hurt adoption [2,18,19]. As an other example, in a social network composed by individuals with some social problem, people can have both positive and negative impact on each other. In order to implement an intervention programme, it becomes important to target a group of users which allow to reinforce a positive behavior through the network while minimizing the negative reactions (also to maximize the impact of future campaigns) [28]. A somehow similar finding applies to political campaigns where candidates want to reinforce positive messages without promoting resistance to persuasion [25]. Such empirical research suggests that marketing campaigns can suffer negative payoff due to the existence of subsets of the population that will react negatively to the message/product. Hence, the marketing campaign can suffer negative payoff. These can come in the form of harm to the firm's reputation in several ways, as for example through negative reviews on rating sites [3,11,12]. Recently, a variation of the influence maximization problem named *opinion maximization* (OM) has been proposed [4,22]. The goal of opinion maximization is to maximize the number of positive opinions while minimizing the number of negative opinions generated by the activated users during the cascading behavior. A first algorithmic study of an OM problem was done in [1], where the authors propose a theoretical model for the problem of seeding a cascade when there are benefits from reaching positively inclined customers and costs from reaching negatively inclined customers. Namely, the problem studied in [1] is: Given a graph G with node set $V = V^+ \cup V^-$ partitioned into positive and negative nodes, determine a subset of the nodes S that can trigger a cascade which maximize the difference between positive and negative payoff.

1.3 Related Domination Problems

Domination in graphs, and its several variants, is a widely studied problem in graph theory [17]. The variation of the domination problem which we study in this paper is related, but not equivalent, to the concepts of signed and minus domination introduced in [14,15]. For instance, in signed domination the sign of the nodes is not part of the input; namely, given an input graph $G = (V, E)$ one looks for a function of the form $f : V \to \{-1, 1\}$ such that, $\sum_{u \in N(v) \cup \{v\}} f(u) \geq 1$ for all $v \in V$.

Another recently studied related problem is *domination with required and forbidden nodes* [13]: Given a graph G and two disjoint sets $R, F \subset V$, construct dominating set D of G such that no forbidden node is in D and every required node of R is in D, that is $F \cap D = \emptyset$ and $R \subseteq D$.

2 Hardness Results

Theorem 1. *The MBDD problem is such that:*
(i) There is no polynomial time approximation algorithm with any constant factor better than $(1 - 1/e)$ *unless P=NP.*
(ii) There is no polynomial time approximation algorithm providing an $n^{-1/polyloglog\ n}$*-approximation unless the exponential time hypothesis is false.*

Proof. (Sketch.) We are going to show that both the k-MaxVD problem [24] and the Densest k-subgraph (DkS) problem [23] are reducible (preserving the approximation factor) in polynomial time, to the MBDD problem. The results (i) and (ii) will then follow from [24] and [23] respectively. For space reasons, the reduction from the DkS problem is omitted.

The k-MaxVD problem is one of the optimization versions of the well known Dominating Set problem [17] and it is defined as follows.

Problem 2. $k-$MAXIMUM VERTEX DOMINATION (k-MaxVD): Given a network $G = (V, E)$ and an integer $k \geq 0$, find a set $D \subseteq V$ with $|D| \leq k$, which maximizes the cardinality of the dominated nodes $|\Gamma(D)|$.

Consider an instance of the k-MaxVD problem, consisting of a graph $G = (V, E)$ having $n = |V|$ nodes and a bound k. Let $V = \{v_1, v_2, \ldots, v_n\}$, we build a graph $G' = (V' = (V^+ \cup V^-), E')$ as follows:

Replace each v_i by a gadget G'_i having two nodes v_i^+ and v_i^-. The node v_i^+ plays the role of v_i in G and is also connected to v_i^-. Formally,

$$V' = V^+ \cup V^- \text{ where } V^+ = \{v_i^+ | 1 \leq i \leq n\} \text{ and } V^- = \{v_i^- | 1 \leq i \leq n\}$$

$$E' = \{(v_i^+, v_j^+) | (v_i, v_j) \in E\} \cup \{(v_i^+, v_i^-) | 1 \leq i \leq n\}.$$

Notice that G corresponds to the subgraph of G' induced by V^+. We prove that:

Given an integer t, there exists a set D of nodes in G of size at most k such that $|\Gamma(D)| \geq t$ iff there exists a set $D' \subseteq V^+$ such that $|\Gamma^-(D')| \leq k$ and $|\Gamma^+(D')| \geq t$ in G'.

Assume that there exists a dominating set $D \subseteq V$ in G such that $|D| \leq k$ and $|\Gamma(D)| \geq t$. Then let $D' = \{v_i^+ \in V^+ | v_i \in D\}$, since G is isomorphic to the subgraph of G' induced by V^+, we have that D' dominates the corresponding of all the nodes in $\Gamma(D)$. Hence, $|\Gamma^+(D')| = |\Gamma(D)| \geq t$. Moreover, by construction, in G' each positive node has exactly one connection with a negative one. Hence, $|\Gamma^-(D')| = |D'| = |D| \leq k$.

On the other hand, assume that there exists a set $D' \subseteq V^+$ in G' such that $|\Gamma^-(D')| \leq k$ and $|\Gamma^+(D')| \geq t$. Then, by using exactly the same argument above, the reader can easily see that the set $D = \{v_i \in V | v_i^+ \in D'\}$ satisfies $|D| \leq k$ and $|\Gamma(D)| \geq t$ and this completes the proof. □

3 An Approximation Algorithm for MBDD

Theorem 2. *Let $G = (V, E)$ be any graph with $V = V^+ \cup V^-$. There exists a polynomial time approximation algorithm for the MBDD problem on G with approximation factor $1 - e^{-1/\Delta}$, where $\Delta = max_{v \in V^-} |\Gamma^+(v)|$ is the maximum degree[1] of any negative node in V^-.*

[1] We can assume that no edge exists between two nodes in V^-, since such edges are irrelevant for our problem.

In order to prove Theorem 2, we distinguish two cases on the value of Δ.

If $\Delta = 1$ then two nodes in V^+ cannot share a neighbor in V^-. As a consequence, if we define the weight of a node in V^+ as the number of its neighbors in V^-, then the problem reduces to select a set of nodes in V^+ so that the union of their neighborhood sets in V^+ has maximum size and the sum of their weights is at most k. This is the Budgeted maximum coverage problem and its approximation factor is $(1 - 1/e)$ [20].

Algorithm 1. The Dual Domination algorithm: $\mathtt{DUAL}(G, k)$

Input: A graph $G = (V^+ \cup V^-, E)$ (with $\Delta \geq 2$) and a positive integer k.

$P = \emptyset$
forall the $u \in V^+$ **do**
  if $|\Gamma^-(u)| \leq k$ **then** $P =$ the largest set between P and $\{v \in V^+ \mid \Gamma^-(v) \subseteq \Gamma^-(u)\}$
  forall the $v \in V^+ - \{u\}$ **do**
    if $|\Gamma^-(\{u, v\})| \leq k$ **then** $P =$ the largest set between P and $\mathtt{DD}(G, \{u, v\}, k)$;
return P

Algorithm 2. $\mathtt{DD}(G, U, k)$

Input: A graph $G = (V^+ \cup V^-, E)$, a set $U \subseteq V^+$ with $|U| = 2$, a positive integer k.

Set $I = V^+$, $S = U$ and $P = \{w \in V^+ \mid \Gamma^-(w) \subseteq \Gamma^-(S)\}$
while $(I - P \neq \emptyset)$ **do**
  forall the $u \in I - P$ **do** $P_u = \{w \in I - P \mid \Gamma^-(w) \subseteq \Gamma^-(S \cup \{u\})\}$
  $v = \arg\max_{u \in I-P} \frac{|\Gamma^+(P \cup P_u) - \Gamma^+(P)|}{|\Gamma^-(S \cup \{u\}) - \Gamma^-(S)|}$
  if $|\Gamma^-(S \cup \{v\})| \leq k$ **then** $\{S = S \cup \{v\};\ P = P \cup P_v\}$ **else** $I = I - \{v\}$
return P

The rest of this section is devoted to prove Theorem 2 in the case $\Delta \geq 2$.

The proposed Algorithm $\mathtt{DUAL}(G, k)$ first computes all the feasible solutions of cardinality one, by simply enumerating all nodes in V^+. In order to consider feasible solutions with cardinality two or more, it exploits Algorithm $\mathtt{DD}(G, U, k)$ to greedily enlarge each feasible solution of cardinality two. It is worth noticing that the algorithm $\mathtt{DD}(G, U, k)$ is executed for each couple of nodes in G. This fact will be exploited to obtain the desired approximation factor.

Algorithm $\mathtt{DD}(G, U, k)$, starting from a partial solution $S = U$ of cardinality 2, greedily adds nodes to such a solution, until no feasible node is available: For each node u not in the current solution, the algorithm measures

- the *cost* of u (how many new nodes of V^- are dominated by adding u) and
- the *profit* of u (how many more nodes of V^+ we can dominate by adding u, as well as all the other nodes which become "*cost-free*" because of u, that is, all their neighbors in V^- are already neighbors of the current solution augmented by u).

The algorithm then selects the node, say v, that provides the best *profit/cost* ratio; if the current solution augmented by v is feasible (i.e., $|\Gamma^-(S \cup \{v\})| \leq k$) then v is added to it, otherwise v is definitively discarded (because v will make any solution, that includes the current solution, infeasible).

Define the padding set of $X \subseteq V^+$ as

$$P_X = \{v \in V^+ \mid \Gamma^-(v) \subseteq \Gamma^-(X)\} \tag{1}$$

Notice $\Gamma^-(X) = \Gamma^-(P_X)$. Also, define *the padding set of u with respect to a ground set I and a set X* as

$$P_u(I, X) = \{v \in I - P_X \mid \Gamma^-(v) \subseteq \Gamma^-(\{u\} \cup X)\}. \tag{2}$$

Starting from any set $U \subset V^+$ consisting of two nodes such that $|\Gamma^-(U)| < k$, Algorithm 2 greedily augments U while preserving the constraint. For a given $U = \{w_1, w_2\}$, the algorithm starts with $S = U$ and a padding set $P = P_S$, fixes the initial ground set I to V^+, and iteratively adds feasible nodes to the solution.

At each iteration, Algorithm 2 maintains the relation $P = P_S$ between the sets S and P. For each node $u \in I - P$, the algorithm identifies the set $P_u \subseteq I - P$ whose neighbors in V^- are dominated when we add u to the current set S, namely the algorithm sets

$$P_u = P_u(I, S).$$

The node to be added to the solution is chosen as to maximize the ratio of the number of positive nodes that will be dominated thanks to the contribution of u to the cost of u (i.e., the increment on the number of negative nodes dominated by $\{u\} \cup S$.) Once a node v has been selected:

- If $S \cup \{v\}$ is not feasible (i.e., $|\Gamma^-(S \cup \{v\})| > k$), then v is removed from the ground set I.
- If $|\Gamma^-(S \cup \{v\})| \leq k$, then the algorithm augments S by v and consequently P by P_v thus maintaining the equality $P = P_S$.

The algorithm ends when $I = P = P_S$.

In the following, we analyze Algorithm 2 and derive the desired approximation factor. Let OPT be an optimal solution to the MBDD problem on G. Let u_1, u_2 be respectively, the two nodes in OPT that dominate the maximum number of positive neighbors, namely

$$u_1 = \underset{u \in \mathtt{OPT}}{\arg\max} |\Gamma^+(P_{\{u\}})| \quad \text{and} \quad u_2 = \underset{u \in \mathtt{OPT} - P_{\{u_1\}}}{\arg\max} \left|\Gamma^+(P_u(V^+, \{u_1\})\right|. \tag{3}$$

Recalling that Algorithm 2 is executed for each pair of nodes in G, from now on we focus on the execution of Algorithm 2 on input $U = \{u_1, u_2\}$. Hence, Algorithm 2 initially (at at line 1) sets $S = \{u_1, u_2\}$ and $P = P_{\{u_1,u_2\}}$.

We denote by S^i the partial set S after exactly i nodes are added to it. Namely,

$$S^0 = \{u_1, u_2\}$$

and for $i \geq 1$, we define S^i as the solution consisting of the initial set $\{u_1, u_2\}$ and the i nodes $v_1, v_2, \ldots, v_i$ (added at line 5 of the algorithm), i.e.,

$$S^i = \{u_1, u_2\} \cup \{v_1, v_2, \ldots, v_i\}.$$

We also set

$$P^0 = P_{S^0} = P_{\{u_1,u_2\}} \text{ and } P^i = P_{S^i} = P_{\{u_1,u_2\}\cup\{v_1,v_2,\ldots,v_i\}} \quad \text{for each } i \geq 1$$

and denote by I^i the ground set I at the end of the iteration in which v_i (cfr. line 5) is added to have S^i. Moreover, for each $i \geq 0$, and $u \in I^i - P^i$, we set

$$c_{i,u} = |\Gamma^-(S^i \cup \{u\}) - \Gamma^-(S^i)|$$

the increment in number of dominated nodes in V^- with respect to $\Gamma^-(S^i)$. Furthermore, recalling that the set P_u at line 3 is $P_u(I^i, S^i)$, we denote by

$$\theta_{i+1} = \max_{u \in I^i - P^i} \frac{|\Gamma^+(P^i \cup P_u(I^i, S^i))| - |\Gamma^+(P^i)|}{c_{i,u}}.$$

Hence, the node v selected at line 4 of Algorithm 2 satisfies the equality

$$|\Gamma^+(P^i \cup P_v(I^i, S^i))| - |\Gamma^+(P^i)| = c_{i,v}\theta_{i+1} \tag{4}$$

while for any other node $u \in (I^i - S^i) - \{v\}$ it holds

$$|\Gamma^+(P^i \cup P_u(I^i, S^i))| - |\Gamma^+(P^i)| \leq c_{i,u}\theta_{i+1}. \tag{5}$$

In the following we use c_u instead of $c_{i,u}$ whenever the index i is clear from the context. Furthermore, we use P_u instead of $P_u(I, S)$ whenever the ground set I and the set S are clear from the context.

We assume that the solution provided by the Algorithm 2 is not the optimal solution OPT. Let $S^t = \{u_1, u_2\} \cup \{v_1, \ldots, v_t\}$, for some $t \geq 0$, be the partial set constructed by Algorithm 2 when, for the first time, the node v selected at line 4 satisfies both the following conditions

1. $v \in$ OPT;
2. v is discarded, i.e. v is removed from the ground set because $|\Gamma^-(S^t \cup \{v\})| > k$.

We notice that,

– it is possible that other nodes have been previously discarded by the algorithm but these nodes do not belong to OPT.

– t is well defined. Indeed, since the solution provided by the Algorithm 2 differs from OPT, there exists at least one node $v \in \texttt{OPT}$, which is discarded by the Algorithm 2.

Let $I' \subseteq I^t$ denote the ground set when v is selected and let

$$\theta = (|\Gamma^+(P^t \cup P_v(I', S^t))| - |\Gamma^+(P^t)|)/c_v \tag{6}$$

Then for any other node $u \in (I' - S^t) - \{v\}$ it holds

$$|\Gamma^+(P^t \cup P_u(I', S^t))| - |\Gamma^+(P^t)| \leq c_u\theta. \tag{7}$$

We prove now some claims, relating S^t, the discarded node v, and the optimal solution OPT, that will be useful to prove the desired approximation ratio.

Claim 1

$$|\Gamma^-(P^0)| + \sum_{i=1}^{t} c_{v_i} + c_v > k. \tag{8}$$

Proof. It suffices to notice that $S^t \cup \{v\}$ dominates more than k nodes in V^-.

□

Claim 2. For any $i = 0, \ldots, t$, it holds

$$\sum_{u \in \texttt{OPT} - P^i} c_u \leq (k - |\Gamma^-(P^0)|)\Delta \tag{9}$$

Proof. Since Δ is the maximum degree of any node in V^-, a node $x \in \Gamma^-(\texttt{OPT}) - \Gamma^-(P^i)$ can have at most Δ neighbors in $\texttt{OPT} - P^i$, hence we have[2]

$$\begin{aligned}
\sum_{u \in \texttt{OPT} - P^i} c_u &= \sum_{u \in \texttt{OPT} - P^i} |\Gamma^-(S^i \cup \{u\}) - \Gamma^-(S^i)| \\
&\leq |\Gamma^-(\texttt{OPT}) - \Gamma^-(P^i)|\Delta \\
&\leq |\Gamma^-(\texttt{OPT}) - \Gamma^-(P^0)|\Delta \qquad \text{since } P^0 \subseteq P^i \\
&= (|\Gamma^-(\texttt{OPT})| - |\Gamma^-(P^0)|)\Delta \qquad \text{since } P^0 \subseteq \texttt{OPT} \\
&\leq (k - |\Gamma^-(P^0)|)\Delta \qquad \text{since } |\Gamma^-(\texttt{OPT})| \leq k.
\end{aligned}$$

□

Given a set $A \subseteq V^+$ such that $P^0 \subseteq A$, we define the function

$$g(A) = |\Gamma^+(A)| - |\Gamma^+(P^0)|.$$

In order to obtain the desired bound on the approximation factor of Algorithm 2, we first prove some preliminary results regarding the function $g(\cdot)$ which will be exploited to derive a lower bound on the ratio $g(P^t \cup P_v)/g(\texttt{OPT})$.

[2] (Notice that $S^0 \subseteq \texttt{OPT}$ and since OPT is an optimal solution we have $P_{\texttt{OPT}} \subseteq \texttt{OPT}$ and then $P^0 \subseteq \texttt{OPT}$.)

Claim 3. For $i = 0, \dots, t$, it holds $g(P^i) = \sum_{\ell=1}^{i} c_{v_\ell}\theta_\ell$.

Proof. Recalling that for $i = 0, \dots, t$, we have $P^i = P^0 \cup P_{v_1} \cup \cdots \cup P_{v_i}$, we get

$$\begin{aligned} g(P^i) &= |\Gamma^+(P^i)| - |\Gamma^+(P^0)| = |\Gamma^+(P^0 \cup P_{v_1} \cup \cdots \cup P_{v_i})| - |\Gamma^+(P^0)| \\ &= \sum_{\ell=1}^{i} (|\Gamma^+(P^0 \cup P_{v_1} \cup \cdots \cup P_{v_\ell})| - |\Gamma^+(P^0 \cup P_{v_1} \cup \cdots \cup P_{v_{\ell-1}})|) \\ &= \sum_{\ell=1}^{i} |\Gamma^+(P^{\ell-1} \cup P_{v_\ell})| - |\Gamma^+(P^{\ell-1})| = \sum_{\ell=1}^{i} c_{v_\ell}\theta_\ell \qquad \text{by (4).} \end{aligned}$$

□

Claim 4. Let $P_v = P_v(I', S^t)$, then $g(P^t \cup P_v) = \sum_{\ell=1}^{t} c_{v_\ell}\theta_\ell + c_v\theta$.

Proof. We have that $P^t = P^0 \cup P_{v_1} \cup \cdots \cup P_{v_t}$ and v was selected when the value of S in Algorithm 2 was S^t. We can then apply Claim 3 and (6) to get the claim.

□

Claim 5

$$g(\mathtt{OPT}) \le \min_{0 \le i \le t} g_i \text{ where } g_i = \begin{cases} \sum_{\ell=1}^{i} c_{v_\ell}\theta_\ell + \theta_{i+1}(k - |\Gamma^-(P^0)|)\Delta & \text{if } 0 \le i \le t-1, \\ \sum_{\ell=1}^{t} c_{v_\ell}\theta_\ell + \theta(k - |\Gamma^-(P^0)|)\Delta & \text{if } i = t. \end{cases} \tag{10}$$

Proof. Fix any $i = 0, \cdots, t$. We notice that the set function g is *non-decreasing*, indeed $g(A) \le g(A')$ for all $A \subseteq A'$. Moreover, recalling that a set function $f : 2^X \to \mathbb{R}^+$ on the ground set X is *submodular* iff $f(A) + f(A') \ge f(A \cup A') + f(A \cap A')$, for all $A, A' \subseteq X$, it is easy to see that $g(A)$ is also a submodular function on the ground set of the subsets of V^+ that contain P^0. Considering that $P^0 \subseteq \mathtt{OPT}$, we can apply to g a result in [26] and we have that

$$g(\mathtt{OPT}) \le g(P^i) + \sum_{u \in \mathtt{OPT} - P^i} (g(P^i \cup \{u\}) - g(P^i)) \tag{11}$$

Hence, for any $i = 0, \cdots, t-1$ we get

$$\begin{aligned} g(\mathtt{OPT}) &\le g(P^i) + \sum_{u \in \mathtt{OPT} - P^i} (g(P^i \cup \{u\}) - g(P^i)) && \text{by (11)} \\ &= g(P^i) + \sum_{u \in \mathtt{OPT} - P^i} (|\Gamma^+(P^i \cup \{u\})| - |\Gamma^+(P^i)|) && \text{by the definition of } g(\cdot) \\ &\le g(P^i) + \sum_{u \in \mathtt{OPT} - P^i} (|\Gamma^+(P^i \cup P_u)| - |\Gamma^+(P^i)|) && \text{since } \{u\} \subseteq P_u \\ &\le g(P^i) + \sum_{u \in \mathtt{OPT} - P^i} c_u\theta_{i+1} && \text{by (5), since } u \in \mathtt{OPT} - P^i \subseteq I^i - P^i \\ &\le g(P^i) + \theta_{i+1}(k - |\Gamma^-(P^0)|)\Delta && \text{by (9)} \\ &= \sum_{\ell=1}^{i} c_{v_\ell}\theta_\ell + \theta_{i+1}(k - |\Gamma^-(P^0)|)\Delta && \text{by Claim 3.} \end{aligned}$$

and, following the above reasoning for $i = t$ we get

$$\begin{aligned} g(\texttt{OPT}) &\le g(P^t) + \sum_{u \in \texttt{OPT} - P^t} (|\Gamma^+(P^t \cup P_u)| - |\Gamma^+(P^t)|) \\ &\le g(P^t) + \sum_{u \in \texttt{OPT} - P^t} c_u \theta \qquad \text{by (7), since } u \in \texttt{OPT} - P^t \subseteq I' - P^t \\ &= \sum_{\ell=1}^{t} c_{v_\ell} \theta_\ell + \theta(k - |\Gamma^-(P^0)|)\Delta \qquad \text{by Claim 3 and (9)} . \end{aligned}$$

□

Lemma 1.

$$\frac{g(P^t \cup P_v)}{g(\texttt{OPT})} > 1 - e^{-1/\Delta} \tag{12}$$

Proof. We need some definitions. Define

$$B_0 = 0, \qquad B_i = \sum_{\ell=1}^{i} c_{v_\ell} \quad \text{for } i = 1, \cdots, t, \qquad B_{t+1} = \sum_{\ell=1}^{t} c_{v_\ell} + c_v.$$

By (8) we have

$$\beta = k - |\Gamma^-(P^0)| < B_{t+1} \tag{13}$$

Furthermore, for $i = 0, \cdots, t$ define

$$\rho_j = \begin{cases} \theta_i & \text{if } j = B_{i-1} + 1, \cdots, B_i \\ \theta & \text{if } j = B_t + 1, \cdots, B_{t+1} \end{cases} \tag{14}$$

Hence, for $i = 1, \cdots, t$, we have

$$\sum_{j=1}^{B_i} \rho_j = \sum_{\ell=1}^{i} c_{v_\ell} \theta_\ell \quad \text{and} \quad \sum_{j=1}^{B_{t+1}} \rho_j = \sum_{\ell=1}^{t} c_{v_\ell} \theta_\ell + c_v \theta. \tag{15}$$

We use now the above definitions to bound $g(\texttt{OPT})$ and $g(P^t \cup P_v)$. By (10)

$$\begin{aligned} g(\texttt{OPT}) &\le \min\left\{ \min_{0 \le i \le t-1} \left\{ \sum_{\ell=1}^{i} c_{v_\ell} \theta_\ell + \theta_{i+1} \beta \Delta \right\}, \sum_{\ell=1}^{t} c_{v_\ell} \theta_\ell + \theta \beta \Delta \right\} \\ &\qquad \text{by the definition of } \beta \text{ in (13)} \\ &= \min_{0 \le i \le t} \left\{ \sum_{j=1}^{B_i} \rho_j + \rho_{B_{i+1}} \beta \Delta \right\} \qquad \text{by (15) and (14)} \\ &= \min_{1 \le s \le B_{t+1}} \left\{ \sum_{j=1}^{s-1} \rho_j + \rho_s \beta \Delta \right\} \qquad \text{by the definition of } B_{t+1}. \end{aligned}$$

By Claim 4 and (15) we have $g(P^t \cup P_v) = \sum_{\ell=1}^{t} c_{v_\ell}\theta_\ell + c_v\theta = \sum_{j=1}^{B_{t+1}} \rho_j$. Hence,

$$\frac{g(P^t \cup P_v)}{g(\mathtt{OPT})} \geq \frac{\sum_{j=1}^{B_{t+1}} \rho_j}{\min_{s=1,\cdots,B_{t+1}} \left\{\sum_{j=1}^{s-1} \rho_j + \rho_s \beta \Delta\right\}} \tag{16}$$

In order to bound the right end side of (16), we use the following fact.

Fact 1 **([26]).** *If a and b are arbitrary positive integers, ρ_j for $j = 1, \cdots, a$ are arbitrary non negative reals and $\rho_0 > 0$*

$$\frac{\sum_{j=1}^{a} \rho_j}{\min_{s=1,\cdots,a} \left\{\sum_{j=1}^{s-1} \rho_j + b\rho_s\right\}} > 1 - e^{-a/b}$$

Hence, we get $\frac{g(P^t \cup P_w)}{g(\mathtt{OPT})} \geq 1 - e^{-B_{t+1}/(\beta\Delta)} > 1 - e^{-1/\Delta}$ where the last inequality holds since $B_{t+1} > \beta$ by (13). □

We show now that the bound $1 - e^{-1/\Delta}$ also holds for $|\Gamma^+(P^t)|/|\Gamma^+(\mathtt{OPT})|$. Recalling that we are considering Algorithm 2 with input $U = \{u_1, u_2\}$, where u_1 and u_2 are the nodes defined in (3), we are able to prove the following claim.

Claim 6. $|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)| \leq |\Gamma^+(P^0)|/2$.

Proof. Recalling that the set $P^t = P^0 \cup P_{v_1} \cup \cdots \cup P_{v_t}$ is the union of disjoint sets, and that $|\Gamma^+(\cdot)|$ is a submodular set function, we can write

$$\begin{aligned}|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)| &= |\Gamma^+(P^0 \cup P_{v_1} \cup \cdots \cup P_{v_t} \cup P_v)| - |\Gamma^+(P^0 \cup P_{v_1} \cup \cdots \cup P_{v_t})| \\ &\leq |\Gamma^+(P_v)| - |\Gamma^+(\emptyset)| = |\Gamma^+(P_v)|\end{aligned}$$

Furthermore, recalling that $P_v = P_v(I', S^t) = \{u \in I' - P_{S^t} \mid \Gamma^-(u) \subseteq \Gamma^-(\{v\} \cup S^t)\}$ and that $P_{\{v\}} = \{u \in V^+ \mid \Gamma^-(u) \subseteq \Gamma^-(v)\}$ we have $P_v \subseteq P_{\{v\}}$. Since $|\Gamma^+(\cdot)|$ is not decreasing then $|\Gamma^+(P_v)| \leq |\Gamma^+(P_{\{v\}})|$. From this and using the definition of u_1 in (3) we have

$$|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)| \leq |\Gamma^+(P_{\{v\}})| \leq |\Gamma^+(P_{\{u_1\}})|. \tag{17}$$

We now derive a further bound on $|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)|$. To this aim, we notice that $P^0 = P_{\{u_1\}} \cup P_{u_2}(V^+, \{u_1\})$ is the union of disjoint sets and that $P_v \subseteq P_v(V^+, \{u_1\}))$; using this and the definition of u_2 in (3), we have

$$\begin{aligned}&|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)| = \\ &\leq |\Gamma^+(P_{\{u_1\}} \cup P_v)| - |\Gamma^+(P_{\{u_1\}})| \leq |\Gamma^+(P_{\{u_1\}} \cup P_v(V^+, \{u_1\}))| - |\Gamma^+(P_{\{u_1\}})| \\ &\leq |\Gamma^+(P_{\{u_1\}} \cup P_{u_2}(V^+, \{u_1\}))| - |\Gamma^+(P_{\{u_1\}})| = |\Gamma^+(P^0)| - |\Gamma^+(P_{\{u_1\}})| \end{aligned} \tag{18}$$

The claim follows by summing up (17) and (18). □

We are now ready to conclude the proof of Theorem 2. We have

$$
\begin{aligned}
|\Gamma^+(P^t)| &= |\Gamma^+(P^0)| + g(P^t) \\
&= |\Gamma^+(P^0)| + g(P^t \cup P_v) - (g(P^t \cup P_v) + g(P^t)) \\
&= |\Gamma^+(P^0)| + g(P^t \cup P_v) - (|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)|) \\
&\geq |\Gamma^+(P^0)| + (1-e^{-1/\Delta})g(\texttt{OPT}) - (|\Gamma^+(P^t \cup P_v)| - |\Gamma^+(P^t)|) \text{ by Lemma 1} \\
&\geq |\Gamma^+(P^0)| + (1-e^{-1/\Delta})g(\texttt{OPT}) - |\Gamma^+(P^0)|/2 \qquad \text{by Claim 6} \\
&= |\Gamma^+(P^0)|/2 + (1-e^{-1/\Delta})|\Gamma^+(\texttt{OPT})| - (1-e^{-1/\Delta})|\Gamma^+(P^0)| \text{ by def. of } g(\cdot) \\
&= |\Gamma^+(P^0)|(1/2 - (1-e^{-1/\Delta})) + (1-e^{-1/\Delta})|\Gamma^+(\texttt{OPT})| \\
&\geq (1 - e^{-1/\Delta})|\Gamma^+(\texttt{OPT})| \qquad \text{since } \Delta \geq 2.
\end{aligned}
$$

Hence, after the first iteration in which the algorithm eliminates (at line 5) an element of the optimal solution OPT, it holds that $\frac{|\Gamma^+(P^t)|}{|\Gamma^+(\texttt{OPT})|} \geq 1 - e^{-1/\Delta}$.

Noticing that subsequent iterations of Algorithm 2 can only improve the ratio, we can conclude that Theorem 2 holds.

4 Concluding Remarks: Extensions and Open Problems

In this section we summarize some additional results and problems related to the MBDD problem. Namely, we consider the following Problems 3 and 4.

Problem 3. (Maximum Dual Domination (MDD)) Given a network $G = (V = (V^+ \cup V^-), E)$, find $D \subseteq V^+$ which maximizes $|\Gamma^+(D)| - |\Gamma^-(D)|$.

Problem 4. (Minimum Negative Dual Domination (mNDD)) Given $G = (V = (V^+ \cup V^-), E)$, find $D \subseteq V^+$ which dominates all positive nodes ($\Gamma^+(D) = V^+$) and minimizes the number of dominated negative nodes $|\Gamma^-(D)|$.

First of all, we mention that the MBDD problem is at least as hard as solving any of the Problems 3 and 4. Indeed any optimal strategy OPT that solves the MBDD problem can be used to solve with an extra polynomial time both the Problems 3 and 4. Indeed for the Problem 3 it is sufficient to run the OPT strategy for any budget $i = 1, \ldots, |V^-|$ and than choose the value that maximizes the difference $|\Gamma^+(S_i)| - |\Gamma^-(S_i)|$, where S_i denotes the output of the OPT strategy with budget i. Similarly for the Problem 4 it is sufficient to run the OPT strategy increasing the value of the budget until $\Gamma^+(S_i) = V^+$.

4.1 Trees

The MBDD problem, defined in Sect. 1, can be solved in polynomial time when the graph G is a tree. Let $T = (V = (V^+ \cup V^-), E)$ be a tree network and k be an integer that represents our budget. Without loss of generality, we can root the tree at a node $r \in V^+$. The idea is then that, considering a node v and one of its children u, there are three possibilities: v dominates u (i.e., $v \in S$),

v is dominated by u (i.e., $v \notin S, u \in S$); they do not dominate each other (i.e., $u, v \notin S$). Taking this into account, one can design a dynamic programming algorithm that traverses the input tree T bottom–up, in such a way that each node is considered after all of its children have been processed.

Such an algorithm can be easily adapted to deal with Problems 3 and 4. Summarizing, we have the following results, whose proof is omitted.

Theorem 3. *The MBDD, mNDD, and MDD problems are solvable in linear time on trees.*

4.2 Hardness

By the same construction of the graph G' as in the proof of Theorem 1, it is possible to show that:

- There exists a dominating set D in G of size at most k iff there exists a set $D' \subseteq V^+$ such that $|\Gamma^+(D')| - |\Gamma^-(D')| \geq n - k$ in G'.
- There exists a dominating set D in G of size at most k iff there exists a set $D' \subseteq V^+$ such that $\Gamma^+(D') = V^+$ and $|\Gamma^-(D')| \leq k$.

Hence, DS is reducible in polynomial time to both Problems 3 and 4 and the following result holds.

Theorem 4. *The MDD problem is NP-hard.*

For the mNDD problem, noticing that the above reduction is gap preserving, we have the following result.

Theorem 5. *There is no polynomial time approximation algorithm with any constant factor better than* $\log |V|$ *for the mNDD problem unless P=NP.*

4.3 Open Problem

From the above, we have that it is a natural question to ask if a logarithmic approximation algorithm can be devised for the $mNDD$ problem.

References

1. Abebe, R., Adamic, L., Kleinberg, J.: Mitigating overexposure in viral marketing. In: The Thirty-Second AAAI Conference on Artificial Intelligence (AAAI 2018), pp. 241–248 (2018)
2. Berger, J., Heath, C.: Where consumers diverge from others: identity signaling and product domains. J. Consum. Res. **34**(2), 121–134 (2007)
3. Byers, J.W., Mitzenmacher, M., Zervas, G.: The groupon effect on yelp ratings: a root cause analysis. In: Proceedings of 13th ACM Conference on Electronic Commerce, EC 2012, pp. 248–265 (2012)
4. Chen, Y., Li, H., Qu, Q.: Negative-aware influence maximization on social networks. In: Proceedings of AAAI Conference on Artificial Intelligence (2018)

5. Cordasco, G., Gargano, L., Rescigno, A.A., Vaccaro, U.: Optimizing spread of influence in social networks via partial incentives. In: Scheideler, C. (ed.) Structural Information and Communication Complexity. LNCS, vol. 9439, pp. 119–134. Springer, Cham (2015). https://doi.org/10.1007/978-3-319-25258-2_9
6. Cordasco, G., Gargano, L., Rescigno, A.A.: Influence propagation over large scale social networks. In: Proceedings of ASONAM 2015, pp. 1531–1538 (2015)
7. Cordasco, G., et al.: Whom to befriend to influence people. Theoret. Comput. Sci. (to appear)
8. Cordasco, G., Gargano, L., Mecchia, M., Rescigno, A.A., Vaccaro, U.: Discovering small target sets in social networks: a fast and effective algorithm. Algorithmica **80**(6), 1804–1833 (2018)
9. Cordasco, G., Gargano, L., Rescigno, A.A.: Active influence spreading in social networks. Theoret. Comput. Sci. **764**, 15–29 (2019)
10. Cordasco, G., Gargano, L., Rescigno, A.A., Vaccaro, U.: Evangelism in social networks: algorithms and complexity. Networks **71**(4), 346–357 (2018)
11. Cravens, K., Goad Oliver, E., Ramamoorti, S.: The reputation index: measuring and managing corporate reputation. Eur. Manag. J. **21**(2), 201–212 (2003)
12. Cretu, A.E., Brodie, R.J.: The influence of brand image and company reputation where manufacturers market to small firms: a customer value perspective. Ind. Mark. Manag. **36**(2), 230–240 (2007)
13. Delbot, F., Laforest, C., Phan, R.: Graphs with forbidden and required vertices. In: 17th Rencontres Francophones sur les Aspects Algorithmiques des Télécom (ALGOTEL 2015) (2015)
14. Dunbar, J.E., Hedetniemi, S.T., Henning, M.A., Slater, P.J.: Signed domination in graphs. In: Proceedings of the Seventh International Conference in Graph Theory, Combinatorics, Algorithms, and Applications, pp. 311–321 (1995)
15. Dunbar, J., Hedetniemi, S., Henning, M.A., McRae, A.: Minus domination in graphs. Discret. Math. **199**(1), 35–47 (1999)
16. Easley, D., Kleinberg, J.: Networks, Crowds, and Markets: Reasoning About a Highly Connected World. Cambridge University Press, New York (2010)
17. Haynes, T.W., Hedetniemi, S., Slater, P.: Fundamentals of Domination in Graphs (Pure and Applied Mathematics (Marcel Dekker)). CRC, Boca Raton (1998)
18. Hu, Y., Van den Bulte, C.: Nonmonotonic status effects in new product adoption. Mark. Sci. **33**(4), 509–533 (2014)
19. Joshi, Y.V., Reibstein, D.J., Zhang, Z.J.: Turf wars: product line strategies in competitive markets. Mark. Sci. **35**(1), 128–141 (2016)
20. Khuller, S., Moss, A., Naor, J.: The budgeted maximum coverage problem. Inf. Process. Lett. **70**(1), 39–45 (1999)
21. Li, Y., Chen, W., Wang, Y., Zhang, Z.-L.: Influence diffusion dynamics and influence maximization in social networks with friend and foe relationships. In: Proc. of WSDM 2013, pp. 657–666 (2013)
22. Liu, X., Kong, X., Yu, P.S.: Active opinion maximization in social networks. In: Proceedings of the 24th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining, (KDD 2018) (2018)
23. Manurangsi, P.: Almost-polynomial ratio ETH-hardness of approximating densest K-subgraph. In: Proceedings of the 49th ACM SIGACT Symposium on Theory of Computing (STOC 2017), pp. 954–961 (2017)
24. Miyano, E., Ono, H.: Maximum domination problem. In: Proceedings of the 17th Computing on The Australasian Theory Symposium (CATS 2011), vol. 119, pp. 55–62 (2011)

25. Neiheisel, J.R., Niebler, S.: On the limits of persuasion: campaign ads and the structure of voters' interpersonal discussion networks. Polit. Commun. **32**(3), 434–452 (2015)
26. Nemhauser, G.L., Wolsey, L.A., Fisher, M.L.: An analysis of approximations for maximizing submodular set functions-I. Math. Prog. **14**(1), 265–294 (1978)
27. Sviridenko, M.: A note on maximizing a submodular set function subject to a knapsack constraint. Oper. Res. Lett. **32**(1), 41–43 (2004)
28. Wakefield, M., Flay, B., Nichter, M., Giovino, G.: Effects of anti-smoking advertising on youth smoking: a review. J. Health Comm. **8**(3), 229–247 (2003)