Power Edge Set and Zero Forcing Set Remain Difficult in Cubic Graphs

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Abstract. This paper presents new complexity and non-approximation results concerning two color propagation problems, namely Power Edge Set and Zero Forcing Set. We focus on cubic graphs, exploiting their structural properties to improve and refine previous results. We also give hardness results for parameterized precolored versions of these problems, and a polynomial-time algorithm for Zero Forcing Set in proper interval graphs.

Keywords: Synchrophasor · Power Edge Set · Zero Forcing Set · Complexity

1 Introduction

Motivation. In power networks, synchrophasors are time-synchronized numbers that represent both the magnitude and phase angle of the sine waves on network links. A Phasor Measurement Unit (PMU) is an expensive measuring device used to continuously collect the voltage and phase angle of a single station and the electrical lines connected to it. The problem of minimizing the number of PMUs to place on a network for complete network monitoring is an important challenge for operators and has gained a considerable attention over the past decade [4, 7, 8, 12, 13, 15, 17, 19, 21, 22, 25]. The problem is known as Power Dominating Set [25] and we state it below. We model the network as a graph $G = (V, E)$ with $|V| =: n$ and $|E| =: m$. We denote the set of vertices and edges of G by $V(G)$ and $E(G)$, respectively. We let $N_G(v)$ denote the set of neighbors of $v \in V$ in G and $d_G(v) = |N_G(v)|$ its degree in G. Further, we let $N_G[v] = N_G(v) \cup \{v\}$ denote the *closed* neighborhood of v in G, and we let $G[W]$ denote the subgraph of G *induced* by vertices $W \subseteq V(G)$. The problem is described through *monitoring* of nodes of the network, corresponding to monitoring vertices $V(G)$ by PMUs, propagated using the following rules.

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RULE R_1*: A vertex v of G on which a PMU is placed will be called a *monitored* vertex, and all its neighbors vertices $N_G(v)$ automatically become monitored.
RULE R_2: if all but one neighbor of a monitored vertex are monitored, then this unmonitored vertex will become monitored as well.

Letting $\Gamma_P(G)$ denote the minimum number of PMUs to place on vertices to obtain a full monitoring of the network (using RULE R_2), the decision version of the problem is described as follows:

POWER DOMINATING SET (PDS)
Input: a graph $G = (V, E)$ and some $k \in \mathbb{N}$
Question: Is $\Gamma_P(G) \leq k$?

POWER DOMINATING SET is $\mathcal{NP}$-complete in general graphs [15]. A large amount of literature is devoted to this problem, describing a wide range of approaches, either exact such as integer linear programming [12] or branch-and-cut [21], or heuristic, such as greedy algorithms [17], approximations [4] or genetic algorithms [19]. The problem has also been shown to be polynomial-time solvable on grids [7], but $\mathcal{NP}$-complete in unit-disk graphs [22].

In this paper, we consider two variants of the problem, called POWER EDGE SET (PES) [23,24] and ZERO FORCING SET (ZFS) [3], which respectively consist in placing PMUs on the *links*, and reducing the monitoring range of a PMU placed on a node. This leads us to replace RULE R_1* in each of these problems as follows (RULE R_2 remains unchanged):

RULE R_1 (PES): two endpoints of an edge bearing a PMU are monitored.
RULE R_1 (ZFS): only the node bearing a PMU is monitored.

We let $pes(G)$ and $zfs(G)$ denote the minimum number of PMUs to place on the edges, resp. nodes, of G to entirely monitor G. Both PES and ZFS can be seen as a problem of color propagation with colors 0 (white) and 1 (black), respectively designating the states *not monitored* and *monitored* of a vertex of G. As input to PES or ZFS, we will consider a connected graph $G = (V, E)$. For each vertex $v \in V$, let $c(v)$ be the color assigned to v (we abbreviate $\bigcup_{v \in X} c(v) =: c(X)$). Before placing the PMUs, we have $c(V) = \{0\}$ and the aim is to obtain $c(V) = \{1\}$ using RULE R_1 and RULE R_2 while minimizing the number of PMUs. See Figs. 1 and 2 for detailed examples illustrating the differences between PES and ZFS.

POWER EDGE SET(PES)
Input: a graph G, some $k \in \mathcal{N}$
Question: Is $pes(G) \leq k$?

ZERO FORCING SET(ZFS)
Input: a graph G, some $k \in \mathcal{N}$
Question: Is $zfs(G) \leq k$?

Previous work. Assigning a minimum number of PMUs to monitor the whole network is known to be $\mathcal{NP}$-hard in general for both PES and ZFS. For the former, some complexity results and a lower bound on approximation of $1.12 - \epsilon$ with $\epsilon > 0$ have been shown by Toubaline et al. [23], who also present a linear-time algorithm on trees by reduction to PATH COVER. Poirion et al. [20] propose

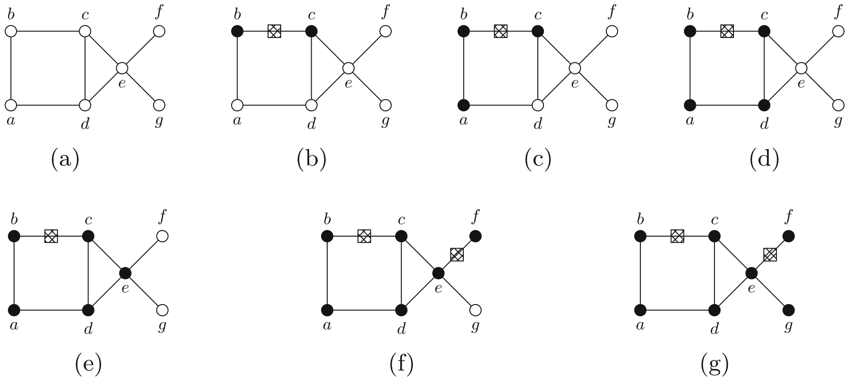


Fig. 1. PMU propagation on PES problem: before any PMU placement, all vertices are white (a). A PMU on $\{b, c\}$ induces $c(b) = c(c) = 1$ (black) by Rule R_1 (b). By applying Rule R_2 on b, we obtain $c(a) = 1$ (c). Then Rule R_2 on a induces $c(d) = 1$ (d), and Rule R_2 on c or d induces $c(e) = 1$ (e). A second PMU is required to obtain a complete coloring. Placing a PMU on $\{e, f\}$ gives us $c(f) = 1$ by Rule R_1 (f). Finally, Rule R_2 on e induces $c(g) = 1$ (g). The set of edges where PMUs have been placed is $S = \{bc, ef\}$, giving (b, c, a, d, e, f, g) as a valid order for G.

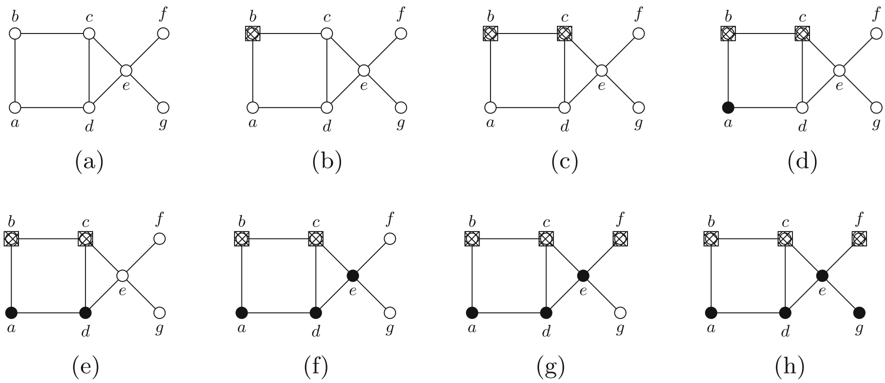


Fig. 2. PMU propagation on ZFS problem: before any PMU placement, all vertices are white (a). Placing one PMU on $\{b\}$ allows to monitor it. (b). Placing a second PMU on $\{c\}$ allows to monitor it (c), and now we can apply Rule R_2 on b, to obtain $c(a) = 1$ (d). Then Rule R_2 on a induces $c(d) = 1$ (e), and Rule R_2 on c or d induces $c(e) = 1$ (f). A third PMU is required to obtain a complete coloring. For example, placing a PMU on f (g) allows to apply Rule R_2 on e to obtain $c(g) = 1$ (h).

a linear program with binary variables indexed by the necessary iterations using propagation rules. Recently, inapproximability results have been proposed on planar or bipartite graphs [5]. In this work, we develop hardness results on complexity and approximation for special cases of Power Edge Set and Zero Forcing Set.

Preliminaries. In the following, we will consider a total order σ of vertices of a graph G as a sequence $(v_1, v_2, \ldots)$ such that v_i occurs before v_j in the sequence if and only if $v_i <_\sigma v_j$.

Definition 1 (valid order). *Let $G = (V, E)$ be a graph, let $S \subseteq E$ (resp. $S \subseteq V$), and let σ be a total order of V, such that for each $v \in V(G)$, there is an edge incident to v in S (resp. $v \in S$) or there is a vertex $u \in N_G(v)$ which verifies $N_G[u] \leq_\sigma v$. Then, $<_\sigma$ is called* valid *for S.*

Given a graph $G = (V, E)$, any set $S \subseteq V$ (or $S \subseteq E$) such that repeated application of Rule R_1 (ZFS) (or Rule R_1 (PES)) and Rule R_2 leads to G being completely monitored is called a *zero forcing set* (or *power edge set*). Using Definition 1, we can formally define the propagation process in G. For instance, in Fig. 1, a valid order for $S = \{bc, ef\}$ is (b, c, a, d, e, f, g).

Observation 1. *Let $G = (V, E)$ be a graph and let $S \subseteq E$ (resp. $S \subseteq V$). Then, S is a power edge set (res. a zero forcing set) if and only if there is a valid order σ on G, with respect to S.*

Note that, for a graph $G = (V, E)$, any set $S \subseteq E$ is a power edge set if and only if $\bigcup_{e \in S} e$ is a zero forcing set for G. It is therefore a natural and unambiguous to also call such an edge set *zero forcing set.*

Finally, we call a vertex v *propagating to* $x \in N_G(v)$ if $c(x) = 0$ and for all $y \in N_G[v] \setminus \{x\}$, we have $c(y) = 1$. Note that each maximal clique of G can contain at most one propagating vertex.

Lemma 1. *Let $G = (V, E)$ be a graph, let S be a zero forcing set of G, and let $\mathcal{C} := \{C_1, \ldots, C_c\}$ be a set of maximal cliques in G covering E. Then $|V \setminus S| \leq c$.*

Proof. Let σ be a valid order for S. We show that each C_i contains at most one edge uv such that $v \notin S$ and $N_G[u] \leq_\sigma v$. Since $\mathcal{C}$ covers E, this implies $|V \setminus S| \leq |\mathcal{C}| = c$. Let $C \in \mathcal{C}$ and let C contain an edge uv such that $N_G[u] \leq_\sigma v$ and $v \notin S$. Then, $C \subseteq N_G[u]$, implying $C \leq_\sigma v$. Thus, v is the last vertex of C with respect to σ and this vertex is *unique.* □

Contribution. The next section is devoted to the NP-completeness for cubic graph for Power Edge Set. We show that Power Edge Set and Zero Forcing Set are W[2]-hard parameterized by the size of the solution in Sect. 3. Section 4 is mainly dedicated to inapproximability and we show that there is an $\frac{n}{2}$-approximation for Power Edge Set. In the last section, we propose a linear polynomial-time algorithm on proper interval graph for Zero Forcing Set.

2 Computational Results

Most results presented in this section rely on reductions from graph problems using gadgets for vertices or edges of the original instance. We model the propagation process using the notion of valid order with respect to the solution set, whatever the nature of it: set of edges for PES, of vertices for ZFS.

We present new lower bounds for POWER EDGE SET that hold even in the very restricted case that G is cubic (*i.e.* all vertices in G have degree three). Previous results show that the problem is $\mathcal{NP}$-complete even if G is a subgraph of the grid with bounded degree at most three [5]. In this paper, we show the problem remains $\mathcal{NP}$-complete if G is cubic and planar. The proof is done by reduction from VERTEX COVER (see below) on 3-regular, planar graphs, which is $\mathcal{NP}$-complete [11] but admits a PTAS [1], and a 3/2-approximation [2].

3-REGULAR PLANAR VERTEX COVER (3RPVC).
Input: a 3-regular planar graph $G = (V, E)$, some $k \in \mathcal{N}$.
Question: Is there a size-k set $S \subseteq V$ covering E, *i.e.* $\forall_{e \in E}\ e \cap S \neq \varnothing$?

Construction 1. *For a given cubic planar graph $G = (V, E)$ with n vertices, we construct a graph G' as follows:*

– *For each $v \in V$, construct H_v (see Fig. 3).*
– *If x is adjacent to y in G, we add exactly one of the edge between x_0, x_1 or x_2 and y_0, y_1 or y_2 to connect H_x and H_y*

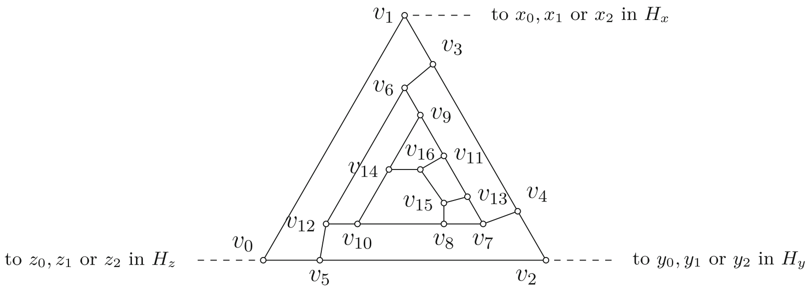


Fig. 3. The gadget H_v for a vertex v.

The graph G' is clearly cubic and planar and Construction 1 is applied in polynomial time. The construction is linear in n and k.

Lemma 2. *The gadget H_v needs at least one PMU to be fully colored: if x_1, y_2 and z_0 are propagating respectively to v_1, v_2 and v_0, then one PMU is sufficient; otherwise two PMUs are needed to fully color H_v.*

Proof. First, if x_1, y_2 and z_0 are propagating respectively to v_1, v_2 and v_0, then, after application of RULE R_2, $c(v_0) = c(v_1) = c(v_2) = 1$. Thus this is the beginning of a valid order: $(v_0, v_1, v_2, v_3, v_5, v_4, v_6, v_7, v_{12}, v_9, v_{10})$. There is no more possible propagation, it is necessary to assign a new PMU. If we place it on the edge $v_{14}v_{16}$, the remainder of a valid order for H_v is: $(v_{14}, v_{16}, v_{11}, v_8, v_{13}, v_{15}, v_{16})$.

Second, we show that H_v may be colored by two PMUs in every case. If PMUs are assigned to the edges $v_{11}v_{13}$ and $v_{15}v_{16}$, we the following order is valid: $(v_{11}, v_{13}, v_{15}, v_{16}, v_7, v_8, v_9, v_{14}, v_4, v_6, v_{10}, v_3, v_{12}, v_3, v_{12}, v_1, v_2, v_5, v_0)$.

Third, we show that even if x_1 and z_0 are propagating to respectively v_1 and v_0, and y_2 is not, we need two PMUs to color H_v. The beginning of the propagation is given by the following order: (v_0, v_1, v_3, v_5). There is no more possible propagation, therefore we have to put one more PMU. As more than two uncolored vertices remain, so we have to initiate propagation with this PMU. So the potential edges are v_6v_{12}, v_4v_2, v_6v_9 or $v_{10}v_{12}$ (other edges won't start a propagation, and we need to color more than two vertices). By exhaustive search, we find that it is impossible to color H_v with only one PMU on any one of these edges. We use the same kind of argument if x_1 and y_2 or y_2 and z_0 propagate. □

Theorem 1. POWER EDGE SET *remains $\mathcal{NP}$-complete on planar cubic graphs.*

Proof. Let G' be the graph obtained by using Construction 1 on $G = (V, E)$, a cubic planar graph. We show that G has a size-k vertex cover iff POWER EDGE SET has a solution of size $n + k$ on G'. Clearly, POWER EDGE SET is in $\mathcal{NP}$.

"⇒": With a size-k vertex cover S for G, we build a power edge set S' for G':

$$S' := \bigcup_{v \in S} \{v_{11}v_{16}, v_{13}v_{15}\} \cup \bigcup_{v \in V \setminus S} \{v_{14}v_{16}\}$$

Then, $|S'| = n + k$ and, by Lemma 2, all vertices of G' are colored by S'.

"⇐": Suppose that G' is colored with $n + k$ PMUs. By Lemma 2, there is at least one PMU on each gadget. Further, if a gadget H_v is colored with a single PMU, then every H_x with $x \in N_G(v)$ is colored with two PMUs inside (by Lemma 2). Then, $\{v \mid H_v \text{ admits two PMUs}\}$ is a vertex cover for G. □

3 Parameterized Hardness

In what follows, we introduce parameterized versions of our problems and recall the notion of parameterized reduction. Using known results for DOMINATING SET, we deduce hardness results for POWER EDGE SET and ZERO FORCING SET. First, we recall the parameterized DOMINATING SET problem. We obtain hardness results for a restricted version of our problems, when a precoloring exists on a particular set of vertices.

Dominating Set (DS)
Input: a graph $G = (V, E)$, some $k \in \mathbb{N}^*$
Question: Does G have a size-k dominating set?
Parameter: k

Precolored Zero Forcing Set/Precolored Power Edge Set
Input: a graph $G = (V, E)$, a set $B \subseteq V$, some $c : V \to \{0, 1\}$ with $c^{-1}(1) = B$, and an integer k
Question: Is there a set $S' \subseteq V$ (resp. $S' \subseteq E$) of size k such that $B \cup S'$ (resp. $B \cup \bigcup_{e \in S'} e$) is a zero forcing set for G?
Parameter: k

We prove the hardness using a parameterized reduction from Dominating Set. First, we introduce a gadget which allows to propagate a coloration, but only in one direction. It is called "check-valve".

Definition 2 (Check-valve). *A* check-valve $C_{x,y}$ *from x to y is a graph $G = (V, E)$, with $V = \{x, y, x_1, x_2, x_3, x_4\}$ and $E = \{xx_1,\ xx_2,\ x_1x_3,\ x_1x_4,\ x_2x_4,\ x_3y,\ x_4y\}$, with a coloring function $c : V \to \mathbb{N}$, such that $c(x) = c(x_1) = 1$ and all other vertices are colored by 0. A check-valve $C_{x,y}$ is illustrated on Fig. 4.*

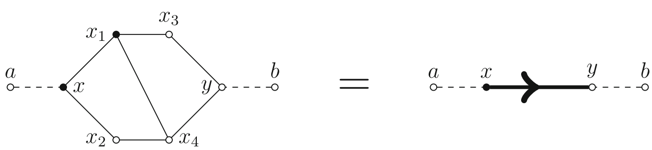


Fig. 4. The check-valve $C_{x,y}$

Observation 2. *Let $C_{x,y}$ be a check-valve inserted between two vertices a and b, depicted by Fig. 4. Then:*

1. *If $c(a) = 1$ then $c(b) = 1$ after exhaustive application of* Rule R_2.
2. *If $c(b) = 1$, and $c(a) = 0$, then $c(a)$ is still 0 after exhaustive application of* Rule R_2, *and it is necessary to add a PMU in order to have $c(a) = 1$.*

Construction 2. *Let xy be a edge such that $c(x) = 1$ and $c(y) = 0$, we construct the gadget C_{xy}: we add vertices x_1, x_2, x_3 and x_4 and we add edges xx_1, x_1x_2, x_3y, yx_4, x_4x_2 et x_2x. Notice that xy is deleted.*

Construction 3. *For given $G = (V, E)$, construct $G' = (V', E')$ as follows:*

1. *For all $x \in V$, build J_x depicted in Fig. 5, containing a* core graph $(\{E_x, R_x, V_x,\ \ x_1, x_2, x_3, x_4\},\ \ \{E_x x_3, E_x x_4, E_x V_x, x_1 x_3, x_2 x_4, R_x x_1, R_x x_2, R_x V_x\})$ *with precolored vertices V_x, x_3 and x_4, and outgoing check-valves: $d_G(x)$ many $C_{x^1_{v_i}, x^2_{v_i}}$ connected to E_x, and n many $C_{x^1_{s_i}, x^2_{s_i}}$ connected to R_x.*

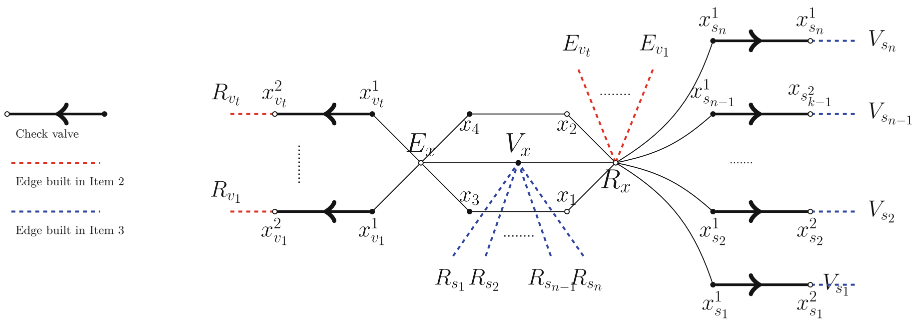


Fig. 5. The gadget J_x for a vertex x. Note that for sake of clarity, some external vertices have been duplicated. Indeed, $\{R_{v_1}, \ldots R_{v_t}\}$, where $v_1, \ldots, v_t$ are neighbors of x, is included in $\{R_{s_1}, \ldots, R_{s_n}\}$.

2. For all $v_i \in N(x)$, add edges $x^2_{v_i} R_{v_i}$ with $x^2_{v_i} \in J_x$ and $R_{v_i} \in J_{v_i}$.
3. For all $s_i \in V$, add edges $x^2_{s_i} V_{s_1}$ with $x^2_{s_i} \in J_x$ and $V_{s_1} \in J_{s_i}$.

Lemma 3. *For all $x \in V$, if $c(E_x) = 1$ then, after exhaustive application of* Rule R_2*, $c(J_x) = 1$ and $c(R_v) = 1$ for all $v \in N(x)$.*

Proof. If $V = \{s_1, \ldots, s_n\}$ and $N(x) = \{v_1, \ldots, v_t\}$, then the following sequence is a valid order: $(E_x, x_1, x_2, R_x, x^2_{s_1}, \ldots, x^2_{s_n}, x^2_{v_1}, \ldots, x^2_{v_t}, R_{v_1}, \ldots R_{v_t})$. ☐

Lemma 4. *Let $c(R_x) = 1$ for all $x \in V$. Then, after exhaustive application of* Rule R_2*, G' becomes fully colored.*

Proof. Clearly, all vertices in $N(V_x) \setminus \{E_x\}$ are colored by R_x for all $x \in V$. Then, E_x is colored by V_x. By Lemma 3, $c(E_x) = 1$ leads to J_x being fully colored. As $V' = \bigcup_{z \in V} V(J_z)$, G' becomes fully colored. ☐

Theorem 2. Precolored Zero Forcing Set *and* Precolored Power Edge Set *are $W[2]$-hard wrt. the solution size k.*

Proof. Let $G = (V, E)$ be a graph and let G' the product of Construction 3 on G. We show that G has a size-k dominating set if and only if G' has a size-k zero forcing set (power edge set).

"⇒": Let S be a size-k dominating set for G. A size-k zero forcing set S' for G' is obtained as follows: for all $x \in S$, we place a PMU on E_x, (resp. $E_x x_4$). By Lemmas 3 and 4, G' is fully colored after applying Rule R_2 exhaustively.

"⇐": Let S' be a zero forcing set of size k for G'. Let S be the set of vertices $x \in V(G)$ such that J_x has at least one vertex, resp. one edge, in S' (for each $x, y \in V$, if there is a PMU on the edge $E_x R_y$ or $R_x V_y$ it counts as an edge of J_x). Suppose that S is not a dominating set for G. So, there is some $y \in V$ such that no $u \in V(J_y)$ is in S' and no $v \in V(J_x)$ is in S' for any $x \in N(y)$. (for PES, there is some $y \in V$ such that no $u_1 u_2 \in E(J_y)$ is in S' and no $v_1 v_2 \in E(J_x)$ is in S' for any $x \in N(y)$). Since J_y is fully colored, this coloration comes from a vertex (resp. an edge) outside of J_y. Four cases have to be considered:

Case 1: There is some $v_i \in N(y)$ such that $c(y^2_{v_i}) = 1$ for $y^2_{v_i} \in J_y$ and this coloration comes from $R_{v_i} \in J_{v_i}$. By Observation 2, we have $c(E_y) = 1$ only if at least one PMU is assigned on the check-valve.

Case 2: There is some $s_i \in V$ such that $c(y^2_{s_i}) = 1, \in J_y$, and this coloration comes from $V_{s_i} \in J_{s_i}$. By Observation 2, for R_y to be colored, at least one PMU has to be assigned to the check-valve.

Case 3: V_y be a propagator. But then, S' is not zero forcing since $c(E_y) = 0$ and $c(R_y) = 0$ and they are in $N(y)$.

Case 4: There is some $v_i \in N(y)$ such that a coloration happens on $R_y \in J_y$ from $E_{v_i} \in J_{v_i}$. Then, either there is some $t \in J_{v_i} \cap S$ and so S is a dominating set, or no PMU is assigned on J_{v_i}, but we already know that E_{v_i} cannot be colored (see Case 1). Consequently, if $c(E_{v_i}) = 1$ then $c(w) = 1$ for some $w \in J_{v_i}$ contradicting S not being a dominating set.

Thus S is a dominating set of G. Further, Construction 3 can be carried out in polynomial time and $|S| = |S'|$, yielding the desired result. □

4 Non-approximation

In this section, we will show that the reductions presented in the proofs of Theorems 1 and 2 are L-reductions.

But above all, it is clear it exists a $\frac{n}{2}$-approximation; it is sufficient to put one PMU incident to each vertex (at most n), and the lower bound for optimal solution is at least two PMUs (in cubic graph) so we obtain a $\frac{n}{2}$-approximation.

Theorem 3. Power Edge Set *is* $\frac{n}{2}$*-approximable*

For the first, by construction, we have $OPT(I') = OPT(I) + n$. Let S be a solution to I, suppose that $n > 3|S|$. By the pigeon hole principle, there is a vertex which cover at least four edges, which is impossible because the degree of each vertex is three, so $n \leq 3|S|$. Thus $OPT(I') \leq 4OPT(I)$.

Moreover, by construction, we have

$$val(g(S') \leq val(S') - n \leq val(S') - OPT(I') + OPT(I)$$

Thus, we construct an L-reduction with $\alpha_1 = 4$ and $\alpha_2 = 1$.

Assuming $\mathcal{P} \neq \mathcal{NP}$, Vertex Cover is hard to approximate to a factor 1.36 [6] and [9], thus yielding the desired result:

$$\begin{aligned} |S'| &\geq |g(S')| + OPT(I') - OPT(I) \\ &\geq 1.36OPT(I) + OPT(I') - OPT(I) \\ &\geq 1.09OPT(I') \end{aligned}$$

□

Corollary 1. *Under* $\mathcal{P} \neq \mathcal{NP}$, Power Edge Set *on cubic graph cannot be approximated to within a factor better than* 1.09*.*

Assuming, Vertex Cover is hard to approximate to a factor $2-\epsilon$ [16] and [9], thus yielding the desired result:

$$
\begin{aligned}
|S'| &\geq |g(S')| + OPT(I') - OPT(I) \\
&\geq 2-\epsilon\ OPT(I) + OPT(I') - OPT(I) \\
&\geq 5/4 - \epsilon\ OPT(I')
\end{aligned}
$$

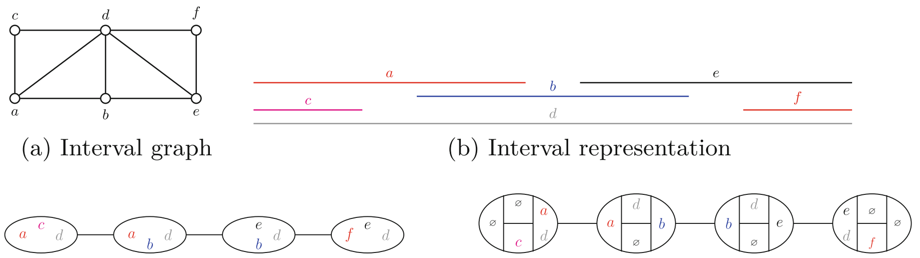
□

Corollary 2. *Under* $\mathcal{UGC}$*,* Power Edge Set *on cubic graph cannot be approximated to within a factor better than* 5/4*.*

Previous results mainly show that Power Edge Set do not admit a PTAS algorithm, even on cubic graphs.

For the second, we got $OPT(I) = OPT(I')$, so clearly it is a S-reduction. Dominating Set do not admit a polynomial time approximation algorithm with ratio $O(log\ n)$ ([18]), so Precolored Power Edge Set and Precolored Zero Forcing Set do not too.

Corollary 3. *Under* $NP \neq DTIME(n^{polylogn})$*,* Precolored Power Edge Set *and* Precolored Zero Forcing Set *do not admit a polynomial time approximation algorithm with ratio* $O(log\ n)$*.*



(a) Interval graph (b) Interval representation

(c) Perfect path decomposition (d) Bags partitions

Fig. 6. An interval graph (a), with its interval representation (b), a perfect path decomposition of this graph (c) and its bag partition according to Definition 4 (d).

5 ZFS on Proper Interval Graphs

Preliminaries. A graph G is an *interval graph* if it is the intersection graph of a family of intervals on the real line. Each interval is represented by a vertex of G and an intersection between two intervals is represented by an edge between the corresponding vertices (see Fig. 6). G is called *proper interval* if it has an interval representation in which no interval is properly contained in another.

In the following, we use perfect path decompositions to solve Power Edge Set on proper interval graphs.

Definition 3. *A* path decomposition $\mathcal{D}$ *of a graph* $G = (V, E)$ *is a sequence* $(X_i)_{i=1...\ell}$ *of subsets of* V *(called* bags*), verifying the following properties:*

(a) for each $xy \in E$*, there is some* X_i *with* $x, y \in X_i$ *(each edge is in a bag),*
(b) for $i \leq j \leq k$*,* $X_i \cap X_k \subseteq X_j$ *(bags containing any* $v \in V$ *are consecutive).*

$\mathcal{D}$ *is called* perfect *if the number of bags and their sizes are minimal under (a) and (b). The* pathwidth *of* $\mathcal{D}$ *is the size of the largest* X_i *minus one.*

Lemma 5. *If* G *is connected, then* $X_i \cap X_{i-1} \neq \varnothing$ *for all* $i > 1$*.*

Proof. Towards a contradiction, assume that $X_i \cap X_{i-1} = \varnothing$. Then, by Definition 3(b), $A := \bigcup_{1 \leq k \leq i-1} X_k$ and $B := \bigcup_{i \leq l} X_l$ are disjoint. Since G is connected, there is an edge xy between A and B, but no bag contains both x and y, contradicting Definition 3(a). □

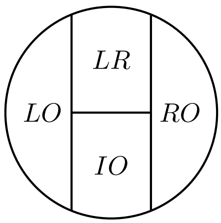
Lemma 6. *Let* G *be an interval graph. A perfect path decomposition* $\mathcal{D}$ *of* G *can be computed in linear time and*
each bag of $\mathcal{D}$ *is a maximal clique in* G*.*

Proof. Being an interval graph, G admits a linear order of its maximal cliques such that, for each vertex v, all maximal cliques containing v are consecutive [10] and this order can be computed in $O(n + m)$ time [14]. Such a "clique path" naturally corresponds to a perfect path decomposition and we know that vertices of each bags induce maximal cliques. In a clique path, the size and the number of bags are minimal. □

Now we can present our algorithm, using previous results:

The Algorithm. In the following, G is a connected proper interval graph and $\mathcal{D} = (X_1, ..., X_\ell)$ is a perfect path decomposition of G. We show that it is possible to apply Rule R_2 once per maximal clique X_i in interval graphs. The central concept is a partition of the bags of $\mathcal{D}$ into four sets.

Definition 4 (Bag partition, see Fig. 6**).** *Let* X_i *be a bag in a perfect path decomposition of an interval graph.*



- *IO (Inside Only) is the set* $X_i \setminus (X_{i-1} \cup X_{i+1})$*.*
- *LO (Left Only) is the set* $X_i \cap X_{i-1} \setminus X_{i+1}$*.*
- *RO (Right Only) is the set* $X_i \cap X_{i+1} \setminus X_{i-1}$*.*
- *LR (Left Right), contains all remaining vertices of* X_i*.*

Note that $RO(X_i)$ and $RO(X_j)$ are disjoint for $i \neq j$. Further, since G is proper interval, $RO(X_i) \neq \varnothing$ for all $i < \ell$. Our algorithm will simply choose any vertex of $RO(X_i) \cup IO(X_i)$ for all i. This can clearly be done in linear time and we show that it is correct and optimal.

Lemma 7. *Let* G *be a connected interval graph and let* $\mathcal{D} = (X_1, \ldots, X_\ell)$ *be a perfect path decomposition of* G*. Let* $\overline{S}$ *be a set intersecting each* $RO(X_i)$ *for all* $1 \leq i < \ell$ *in exactly one vertex and intersecting* $IO(X_\ell)$ *in exactly one vertex. Then,* $S := V \setminus \overline{S}$ *is an optimal zero forcing set for* G*.*

Proof. For each i, let x_i be the i^{th} vertex of $\overline{S}$, that is, $\overline{S} \cap (RO(X_i) \cup IO(X_i)) = \{x_i\}$ for each $X_i \in \mathcal{D}$. We show that the order σ consisting of S in any order followed by $(x_1, \ldots, x_\ell)$ is valid for S. To this end, let $1 \leq j < \ell$. Note that $IO(X_j) \cup LO(X_j) = ((X_j \setminus X_{j+1}) \cap X_{j-1}) \cup ((X_j \setminus X_{j+1}) \setminus X_{j+1}) = X_j \setminus X_{j+1}$. Thus, there is some $u \in IO(X_j) \cup LO(X_j)$ as otherwise, $X_j \subseteq X_{j+1}$ contradicting $\mathcal{D}$ being perfect. Towards a contradiction, assume $N_G[u] \not\leq_\sigma x_j$, that is, there is some $v \in N_G[u]$ with $x_j <_\sigma v$. By construction of σ, there is a $k > j$ such that $v = x_k$. By construction of $\overline{S}$, we have $x_k \in IO(X_k) \cup RO(X_k)$, implying $x_k \notin X_{k-1}$ by definition of RO and IO. Further, since ux_k is an edge of G, there is a bag X_i containing both u and x_k and, since $u \in IO(X_j) \cup LO(X_j)$ we know $i \leq j$. But then, x_k occurs in X_i, not in X_{k-1} but again in X_k, contradicting $\mathcal{D}$ being perfect. It remains to treat x_ℓ, but since x_ℓ is the last vertex of σ, $N_G[u] \leq_\sigma x_\ell$ for all $u \in N_G(x_\ell)$.

Finally, optimality of S is implied by Lemma 1 as $|\overline{S}| = |\mathcal{D}|$. □

Theorem 4. Zero Forcing Set *is solvable in* $O(n+m)$ *time on proper interval graphs.*

Proof. We know that our algorithm is exact, compute the path decomposition PD can done in linear time (Lemma 6) and partitioning its vertices is easy. So there is a polynomial time algorithm for Zero Forcing Set in proper interval graph. □

6 Conclusion and Perspectives

In this article, we investigated Power Edge Set and Zero Forcing Set from the point of view of computational complexity. We obtained a series of negative results, refining the previous hardness results and excluding certain exact algorithms. On the positive side, we give a linear-time algorithm in case the input is a proper interval graph and a naive approximation algorithm. There is a big gap between positive and negative result in approximation so further research will be focused on developing efficient polynomial-time approximation algorithms, as well as considering more special cases and structural parameterizations.

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