How to Morph a Tree on a Small Grid

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Abstract. In this paper we study planar morphs between straight-line planar grid drawings of trees. A morph consists of a sequence of morphing steps, where in a morphing step vertices move along straight-line trajectories at constant speed. We show how to construct planar morphs that simultaneously achieve a reduced number of morphing steps and a polynomially-bounded resolution. We assume that both the initial and final drawings lie on the grid and we ensure that each morphing step produces a grid drawing; further, we consider both upward drawings of rooted trees and drawings of arbitrary trees.

1 Introduction

The problem of morphing combinatorial structures is a consolidated research topic with important applications in several areas of Computer Science such as Computational Geometry, Computer Graphics, Modeling, and Animation. The structures of interest typically are drawings of graphs; a *morph* between two drawings Γ_0 and Γ_1 of the same graph G is defined as a continuously changing family of drawings $\{\Gamma_t\}$ of G indexed by time $t \in [0, 1]$, such that the drawing at time $t = 0$ is Γ_0 and the drawing at time $t = 1$ is Γ_1. A morph is usually required to preserve a certain drawing standard and pursues certain qualities.

The *drawing standard* is the set of the geometric properties that are maintained at any time during the morph. For example, if both Γ_0 and Γ_1 are planar drawings, then the drawing standard might require that all the drawings of the morph are planar. Other properties that might be required to be preserved are the convexity of the faces, or the fact that the edges are straight-line segments, etc.

Regarding the *qualities* of the morph, the research up to now mainly focused on limiting the number of *morphing steps*, where in a morphing step vertices

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move along straight-line trajectories at constant speed. A morph $\mathcal{M}$ can then be described as a sequence of drawings $\mathcal{M} = \langle \Gamma_0 = \Delta_0, \Delta_1, \dots, \Delta_k = \Gamma_1 \rangle$ where the morph $\langle \Delta_{i-1}, \Delta_i \rangle$, for $i = 1, \dots, k$, is a morphing step. Following the pioneeristic works of Cairns and Thomassen [8,13], most of the literature focused on the straight-line planar drawing standard. A sequence of recent results in [1–5] proved that a linear number of morphing steps suffices, and is sometimes necessary, to construct a morph between any two straight-line planar drawings of a graph.

Although the results mentioned in the previous paragraph establish strong theoretical foundations for the topic of morphing graph drawings, they produce morphs that are not appealing from a visualization perspective. Namely, such algorithms produce drawings that have poor *resolution*, i.e., they may have an exponential ratio of the distances between the farthest and closest pairs of geometric objects (points representing vertices or segments representing edges), even if the same ratio is polynomially bounded in the initial and final drawings. Indeed, most of the above cited papers mention the problem of constructing morphs with bounded resolution as the main challenge in this research area.

The only paper we are aware of where the resolution problem has been successfully addressed is the one by Barrera-Cruz et al. [6], who showed how to construct a morph with polynomially-bounded resolution between two *Schnyder drawings* Γ_0 and Γ_1 of the same planar triangulation. The model they use in order to ensure a bound on the resolution requires that $\Gamma_0 = \Delta_0, \Delta_1, \dots, \Delta_k = \Gamma_1$ are *grid drawings*, i.e., vertices have integer coordinates, and the resolution is measured by comparing the area of Γ_0 and Γ_1 with the area of the Δ_i's. We remark that morphs between planar orthogonal drawings of maximum-degree-4 planar graphs, like those in [7,12], inherently have polynomial resolution.

In this paper we show how to construct morphs of tree drawings that simultaneously achieve a reduced number of morphing steps and a polynomially-bounded resolution. Adopting the setting of [6], we assume that Γ_0 and Γ_1 are grid drawings and we ensure that each morphing step produces a grid drawing.

We present three algorithms. The first two algorithms construct morphs between any two strictly-upward straight-line planar grid drawings Γ_0 and Γ_1 of n-node rooted trees; *strictly-upward* drawings are such that each node lies above its children. Both algorithms construct morphs in which each intermediate grid drawing has linear width and height, where the input size is measured by n and by the width and the height of Γ_0 and Γ_1. The first algorithm employs $\Theta(n)$ morphing steps. The second algorithm employs $\Theta(1)$ morphing steps, however it only applies to binary trees. The third algorithm allows us to achieve our main result, namely that for any two straight-line planar grid drawings Γ_0 and Γ_1 of an n-node tree, there is a planar morph with $\Theta(n)$ morphing steps between Γ_0 and Γ_1 such that each intermediate grid drawing has polynomial area, where the input size is again measured by n and by the width and the height of Γ_0 and Γ_1.

The first algorithm uses recursion; namely, it eliminates a leaf in the tree, it recursively morphs the drawings of the remaining tree and it then reintroduces the removed leaf in suitable positions during the morph. The second algorithm morphs the given drawings by independently changing their x- and y-coordinates; this technique is reminiscent of a recent paper by Da Lozzo et al. [10]. Finally, the third algorithm scales the given drawings up in order to make room for a bottom-up modification of each drawing into a "canonical" drawing of the tree.

Missing proofs can be found in the full version of the paper.

2 Preliminaries

In this section we introduce some definitions and preliminaries; see also [11].

Trees. The node and edge sets of a tree T are denoted by $V(T)$ and $E(T)$, respectively. The *degree* $\deg(v)$ of a node v of T is the number of its neighbors. In an *ordered* tree, a counter-clockwise order of the edges incident to each node is specified.

A *rooted tree* T is a tree with one distinguished node, which is called *root* and is denoted by $r(T)$. For any node $u \in V(T)$ with $u \neq r(T)$, the *parent* $p(u)$ of u is the neighbor of u in the unique path from u to $r(T)$. For any node $u \in V(T)$ with $u \neq r(T)$, the *children* of u are the neighbors of u different from $p(u)$; the *children* of $r(T)$ are all its neighbors. The nodes that have children are called *internal*; a non-internal node is a *leaf*. For any node $u \in V(T)$ with $u \neq r(T)$, the *subtree* T_u of T rooted at u is defined as follows: remove from T the edge $(u, p(u))$, thus separating T in two trees; the one containing u is the subtree of T rooted at u. If each node of T has at most two children, then T is a *binary tree*.

An *ordered rooted tree* is a tree that is rooted and ordered. In an ordered rooted tree T, for each node $u \in V(T)$, a *left-to-right* (linear) order $u_1, \dots, u_k$ of the children of u is specified. If T is binary then the first (second) child in the left-to-right order of the children of any node u is the *left* (*right*) *child* of u, and the subtree rooted at the left (right) child of u is the *left* (*right*) *subtree* of u.

Tree Drawings. In a *straight-line drawing* Γ of a tree T each node u is represented by a point of the plane (whose coordinates are denoted by $x_\Gamma(u)$ and $y_\Gamma(u)$) and each edge is represented by a straight-line segment between its end-points. All the drawings considered in this paper are straight-line, even when not specified. In a *planar* drawing no two edges intersect except, possibly, at common end-points. For a rooted tree T, a *strictly-upward* drawing Γ is such that each edge $(u, p(u)) \in E(T)$ is represented by a curve monotonically increasing in the y-direction from u to $p(u)$; if Γ is a straight-line drawing, this is equivalent to requiring that $y_\Gamma(u) < y_\Gamma(p(u))$. For an ordered tree T, an *order-preserving* drawing Γ is such that, for each node $u \in V(T)$, the counter-clockwise order of the edges incident to u in Γ is the same as the order associated with u in T.

The *bounding box* of a drawing Γ is the smallest axis-parallel rectangle enclosing Γ. In a *grid* drawing Γ each node has integer coordinates; then the *width* and the *height* of Γ, denoted by $w(\Gamma)$ and $h(\Gamma)$, respectively, are the number of grid columns and rows intersecting the bounding box of Γ, while the *area* of Γ is its width times its height. For a node v in a drawing Γ, an *ℓ-box centered at* v is the convex hull of the square whose corners are $(x_\Gamma(v) \pm \frac{\ell}{2}, y_\Gamma(v) \pm \frac{\ell}{2})$.

Morphs. A morph is *planar* if all its intermediate drawings are planar. A morph between two strictly-upward drawings of a rooted tree is *upward* if all its intermediate drawings are strictly-upward. A morph is *linear* if each node moves along a straight-line trajectory at constant speed. Whenever the linear morph between two straight-line planar drawings Γ_0 and Γ_1 of a graph G is not planar, one is usually interested in the construction of a piecewise-linear morph with small complexity between Γ_0 and Γ_1. This is formalized by defining a *morph* between Γ_0 and Γ_1 as a sequence $\langle \Gamma_0 = \Delta_0, \Delta_1, \dots, \Delta_k = \Gamma_1 \rangle$ of drawings of G such that the linear morph $\langle \Delta_{i-1}, \Delta_i \rangle$ is planar, for $i = 1, \dots, k$; each linear morph $\langle \Delta_{i-1}, \Delta_i \rangle$ is called a *morphing step* or simply a *step*.

The *width* $w(\mathcal{M})$ of a morph $\mathcal{M} = \langle \Delta_0, \Delta_1, \dots, \Delta_k \rangle$, where Δ_i is a grid drawing, for $i = 0, 1, \dots, k$, is equal to $\max\{w(\Delta_0), w(\Delta_1), \dots, w(\Delta_k)\}$. The *height* $h(\mathcal{M})$ of $\mathcal{M}$ is defined analogously. The *area* of a morph $\mathcal{M}$ is defined as $w(\mathcal{M}) \times h(\mathcal{M})$.

The algorithms we design in this paper receive in input two order-preserving straight-line planar grid drawings Γ_0 and Γ_1 of an ordered tree and construct morphs $\langle \Gamma_0 = \Delta_0, \Delta_1, \dots, \Delta_k = \Gamma_1 \rangle$ with few steps and small area.

Remark 1. A necessary and sufficient condition for the existence of a planar morph between two straight-line planar drawings Γ_0 and Γ_1 of a tree T is that they are "topologically-equivalent", i.e., the counter-clockwise order of the edges incident to each node $u \in V(T)$ is the same in Γ_0 and Γ_1. In order to better exploit standard terminology about tree drawings, we ensure that Γ_0 and Γ_1 are topologically-equivalent by assuming that T is ordered and that Γ_0 and Γ_1 are order-preserving drawings; hence, dealing with ordered trees and with order-preserving drawings is not a loss of generality.

Remark 2. The width and height of the morphs we construct are expressed not only in terms of the number of nodes of the input tree T, but also in terms of the width and height of the input drawings Γ_0 and Γ_1 of T; this is necessary, given that $\max\{w(\Gamma_0), w(\Gamma_1)\}$ and $\max\{h(\Gamma_0), h(\Gamma_1)\}$ are obvious lower bounds for the width and height of any morph between Γ_0 and Γ_1, respectively.

Remark 3. The morphs $\langle \Delta_0, \Delta_1, \dots, \Delta_k \rangle$ we construct in this paper are such that $\Delta_0, \Delta_1, \dots, \Delta_k$ are *grid* drawings, even when not explicitly specified.

3 Upward Planar Morphs of Rooted-Tree Drawings

In this section we study small-area morphs between order-preserving strictly-upward straight-line planar grid drawings of rooted ordered trees.

Our first result shows that such morphs can always be constructed consisting of a linear number of steps. This is obtained via an inductive algorithm which is described in the following. Let T be an n-node rooted ordered tree. The *rightmost path* of T is the maximal path $(s_0, \dots, s_m)$ such that $s_0 = r(T)$ and s_i is the rightmost child of s_{i-1}, for $i = 1, \dots, m$. Note that s_m is a leaf, which

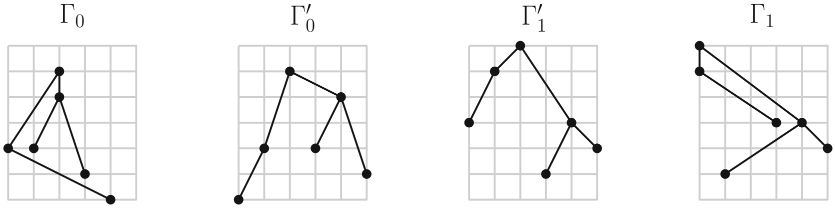


Fig. 1. The 3-step morph $\langle \Gamma_0, \Gamma'_0, \Gamma'_1, \Gamma_1 \rangle$.

is called the *rightmost leaf* $l_{\overrightarrow{T}}$ of T. For a straight-line grid drawing Γ, denote by ℓ_Γ the rightmost vertical line intersecting Γ; note that ℓ_Γ is a grid column.

Let Γ_0 and Γ_1 be two order-preserving strictly-upward straight-line planar grid drawings of T. We inductively construct a morph $\mathcal{M}$ from Γ_0 to Γ_1 as follows.

In the base case $n = 1$; then $\mathcal{M}$ is the linear morph $\langle \Gamma_0, \Gamma_1 \rangle$.

In the inductive case $n > 1$. Let $l = l_{\overrightarrow{T}}$ be the rightmost leaf of T. Let $\pi = p(l)$ be the parent of l. Let T' be the $(n-1)$-node tree obtained from T by removing the node l and the edge (π, l). Let Γ'_0 and Γ'_1 be the drawings of T' obtained from Γ_0 and Γ_1, respectively, by removing the node l and the edge (π, l). Inductively compute a k-step upward planar morph $\mathcal{M}' = \langle \Gamma'_0 = \Delta'_1, \Delta'_2, \dots, \Delta'_k = \Gamma'_1 \rangle$.

We now construct a morph $\mathcal{M} = \langle \Gamma_0, \Delta_1, \Delta_2, \dots, \Delta_k, \Gamma_1 \rangle$. For each $i = 2, 3, \dots, k-1$, we define Δ_i as the drawing obtained from Δ'_i by placing l one unit below π and one unit to the right of $\ell_{\Delta'_i}$. Further, we define Δ_1 (Δ_k) as the drawing obtained from Δ'_1 (resp. from Δ'_k) by placing l one unit below π and one unit to the right of ℓ_{Γ_0} (resp. ℓ_{Γ_1}). Note that the point at which l is placed in Δ_1 (in Δ_k) is one unit to the right of $\ell_{\Delta'_1}$ (resp. $\ell_{\Delta'_k}$), similarly as in $\Delta_2, \Delta_3, \dots, \Delta_{k-1}$, except if l is to the right of every other node of Γ_0 (of Γ_1); in that case l might be several units to the right of $\ell_{\Delta'_1}$ (resp. $\ell_{\Delta'_k}$). This completes the construction of $\mathcal{M}$. We get the following.

Theorem 1. *Let T be an n-node rooted ordered tree, and let Γ_0 and Γ_1 be two order-preserving strictly-upward straight-line planar grid drawings of T. There exists a $(2n-1)$-step upward planar morph $\mathcal{M}$ from Γ_0 to Γ_1 with $h(\mathcal{M}) = \max\{h(\Gamma_0), h(\Gamma_1)\}$ and $w(\mathcal{M}) = \max\{w(\Gamma_0), w(\Gamma_1)\} + n - 1$.*

In view of Theorem 1, it is natural to ask whether a sub-linear number of steps suffices to construct a small-area morph between any two order-preserving strictly-upward straight-line planar grid drawings of a rooted ordered tree. In the following we prove that this is indeed the case for binary trees, for which just three morphing steps are sufficient.

Our algorithm borrows ideas from a recent paper by Da Lozzo *et al.* [10], which deals with upward planar morphs of *upward plane graphs*.

Consider any two order-preserving strictly-upward straight-line planar grid drawings Γ_0 and Γ_1 of an n-node rooted ordered binary tree T. We define two order-preserving strictly-upward straight-line planar grid drawings Γ'_0 and Γ'_1 of T such that the 3-step morph $\langle \Gamma_0, \Gamma'_0, \Gamma'_1, \Gamma_1 \rangle$ is upward and planar.

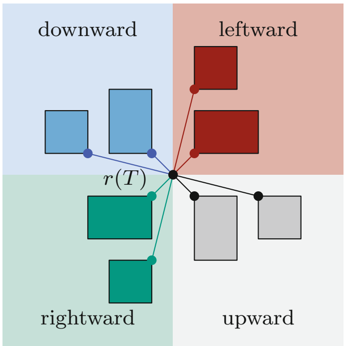


Fig. 2. Four canonical drawings of a tree T (each shown in a differently colored quadrant).

For $i = 0, 1$, we define Γ_i' recursively as follows; see Fig. 1. Let $x_{\Gamma_i'}(r(T)) = 0$ and let $y_{\Gamma_i'}(r(T)) = y_{\Gamma_i}(r(T))$. If the left subtree L of $r(T)$ is non-empty, then recursively construct a drawing of it. Let x_M be the maximum x-coordinate of a node in the constructed drawing of L; horizontally translate such a drawing by subtracting $x_M + 1$ from the x-coordinate of every node in L, so that the maximum x-coordinate of any node in L is now -1. Symmetrically, if the right subtree R of $r(T)$ is non-empty, then recursively construct a drawing of it. Let x_m be the minimum x-coordinate of a node in the constructed drawing of R; horizontally translate such a drawing by subtracting $x_m - 1$ from the x-coordinate of every node in R, so that the minimum x-coordinate of any node in R is now 1.

Theorem 2. *Let T be an n-node rooted ordered binary tree, and let Γ_0 and Γ_1 be two order-preserving strictly-upward straight-line planar grid drawings of T. There exists a 3-step upward planar morph $\mathcal{M}$ from Γ_0 to Γ_1 with $h(\mathcal{M}) = \max\{h(\Gamma_0), h(\Gamma_1)\}$ and $w(\mathcal{M}) = \max\{w(\Gamma_0), w(\Gamma_1), n\}$.*

The algorithm presented before Theorem 2 can be easily generalized to rooted ordered trees with unbounded degree. Thus, there exists a 3-step upward planar morph between any two order-preserving strictly-upward straight-line planar grid drawings of an n-node rooted ordered tree. However, the generalized version of the algorithm does not guarantee polynomial bounds on the width of the morph.

4 Planar Morphs of Tree Drawings

In this section we show how to construct small-area morphs between straight-line planar grid drawings of trees. In particular, we prove the following result.

Theorem 3. *Let T be an n-node ordered tree and let Γ_0 and Γ_1 be two order-preserving straight-line planar grid drawings of T. There exists an $O(n)$-step planar morph $\mathcal{M}$ from Γ_0 to Γ_1 with $h(\mathcal{M}) \in O(D^3 n \cdot H)$ and $w(\mathcal{M}) \in O(D^3 n \cdot W)$, where $H = \max\{h(\Gamma_0), h(\Gamma_1)\}$, $W = \max\{w(\Gamma_0), w(\Gamma_1)\}$, and $D = \max\{H, W\}$.*

The rest of this section is devoted to the proof of Theorem 3. We are going to use the following definition (see Fig. 2).

Definition 1. *An upward canonical drawing of a rooted ordered tree T is an order-preserving strictly-upward straight-line planar grid drawing Γ of T satisfying the following properties:*

- *if $|V(T)| = 1$, then Γ is a grid point in the plane, representing $r(T)$;*
- *otherwise, let $\Gamma_1, \ldots, \Gamma_k$ be upward canonical drawings of the subtrees $T_1, \ldots, T_k$ of $r(T)$ (in their left-to-right order), respectively; then Γ is such that:*
  - *$r(T)$ is one unit to the left and one unit above the top-left corner of the bounding box of Γ_1;*
  - *the top sides of the bounding boxes of $\Gamma_1, \ldots, \Gamma_k$ have the same y-coordinate; and*
  - *the right side of the bounding box of Γ_i is one unit to the left of the left side of the bounding box of Γ_{i+1}, for $i = 1, \ldots, k-1$.*

By counter-clockwise rotating an upward canonical drawing of T by $\frac{\pi}{2}$, π, and $\frac{3\pi}{2}$ radians, we obtain a *leftward*, a *downward*, and a *rightward canonical drawing* of T, respectively. A *canonical drawing* of T is an upward, leftward, downward, or rightward canonical drawing of T. In an upward, leftward, downward, or rightward canonical drawing Γ of T, $r(T)$ is placed at the top-left, bottom-left, bottom-right, and top-right corner of the bounding box of Γ, respectively.

Remark 4. If T has n nodes, then a canonical drawing of T lies in the $2n$-box centered at $r(T)$.

The following lemma allows us to morph one canonical drawing into another in a constant number of morphing steps.

Lemma 1 (Pinwheel). *Let Γ and Γ' be two canonical drawings of a rooted ordered tree T, where $r(T)$ is at the same point in Γ and Γ'. If Γ and Γ' are upward and leftward, or leftward and downward, or downward and rightward, or rightward and upward, then the morph $\langle \Gamma, \Gamma' \rangle$ is planar and lies in the interior of the right, top, left, or bottom half of the $2n$-box centered at $r(T)$, respectively.*

We now describe the proof of Theorem 3. Let T be an n-node ordered tree and let Γ_0 and Γ_1 be two order-preserving straight-line planar grid drawings of T. In order to compute a morph $\mathcal{M}$ from Γ_0 to Γ_1, we root T at any leaf $r(T)$. Since T is ordered, this determines a left-to-right order of the children of each node.

We construct three morphs: a morph $\mathcal{M}^0$ from Γ_0 to a canonical drawing Γ_0^* of T, a morph $\mathcal{M}^1$ from Γ_1 to a canonical drawing Γ_1^* of T, and a morph $\mathcal{M}^{0,1}$ from Γ_0^* to Γ_1^*. Then $\mathcal{M}$ is composed of $\mathcal{M}^0$, of $\mathcal{M}^{0,1}$, and of the reverse of $\mathcal{M}^1$. The morph $\mathcal{M}^{0,1}$ consists of $O(1)$ steps and can be constructed by applying Lemma 1. We describe how to construct $\mathcal{M}^0$; the construction of $\mathcal{M}^1$ is analogous.

Let $T[0]$ be the tree T together with a labeling of each of the k internal nodes of T as `unvisited` and of each leaf as `visited`. We perform a bottom-up visit of

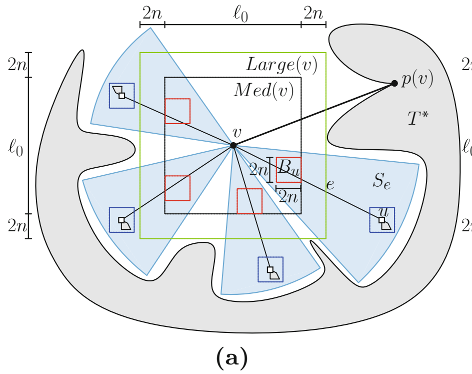


Fig. 3. (a) A partially-canonical drawing Δ_{i-1} of tree $T[i-1]$; the subtree T^* lies in the gray region, `visited` and `unvisited` nodes are represented as squares and circles, respectively. (b) Drawing Δ' of the morph $\langle \Delta_{i-1}, \Delta' \rangle$ of Claim 3.1.

T, labeling one-by-one the internal nodes of T as `visited`. We label a node v as `visited` only after all of its children have been labeled as `visited`. We denote by $T[i]$ the tree T once i of its internal nodes have been labeled as `visited`.

Let $D_0 = \max\{w(\Gamma_0), h(\Gamma_0)\}$. Let Γ be a drawing of T and let v be a node of T. We denote by $Large(v)$, $Med(v)$, and $Small(v)$ the $(\ell_0 + 4n)$-box, the ℓ_0-box, and the $2n$-box centered at v in Γ, respectively, where $\ell_0 = k_0 D_0^2 n$ for some constant $k_0 > 1$ to be determined later. We have the following definition.

Definition 2. *An order-preserving straight-line planar grid drawing Γ of T is a partially-canonical drawing of $T[i]$ if it satisfies the following properties (Fig. 3a):*

- *(a) for each* ***`visited`*** *node u of T, the drawing Γ_u of T_u in Γ is upward canonical or downward canonical; further, if $u \neq r(T)$, then Γ_u is upward canonical, if $y_\Gamma(u) \leq y_\Gamma(p(u))$, or downward canonical, if $y_\Gamma(u) > y_\Gamma(p(u))$;*
- *(b) for each edge $e = (v, u)$ of T, where v is the parent of u and v is* ***`unvisited`****, there exists a sector S_e of a circumference centered at v such that:*
- *(b.i) S_e encloses $Small(u)$;*
- *(b.ii) S_e contains no node with the exception of v and of, possibly, the nodes of T_u, and no edge with the exception of (u, v) and of, possibly, the edges of T_u;*
- *(b.iii) the intersection between S_e and $Med(v)$ contains a $2n$-box B_u whose corners have integer coordinates and whose center c_u is such that $y_\Gamma(c_u) \leq y_\Gamma(v)$ if and only if $y_\Gamma(u) \leq y_\Gamma(v)$; and*
- *(b.iv) for any edge $e' \neq e$ incident to v, the sectors S_e and $S_{e'}$ are internally disjoint;*
- *(c) for any two* ***`unvisited`*** *nodes v and w, it holds $Large(v) \cap Large(w) = \emptyset$; and*

(d) for each **unvisited** *node v of T, $Large(v)$ contains no node different from v, and any edge e or any sector S_e intersecting $Large(v)$ is such that e is incident to v.*

Note that, by Property (a), a partially-canonical drawing of $T[k]$ is a canonical drawing.

The algorithm to construct $\mathcal{M}^0$ is as follows. First, we scale Γ_0 up by a factor in $O(D_0^3 n)$ so that the resulting drawing Δ_0 is a partially-canonical drawing of $T[0]$ (see Lemma 2). Clearly, the morph $\mathcal{M}_0 = \langle \Gamma_0, \Delta_0 \rangle$ is planar, $w(\mathcal{M}_0) = w(\Delta_0)$, and $h(\mathcal{M}_0) = h(\Delta_0)$.

For $i = 1, \dots, k$, let v_i be the node that is labeled as `visited` at the i-th step of the bottom-up visit of T. Starting from a partially-canonical drawing Δ_{i-1} of $T[i-1]$, we construct a partially-canonical drawing Δ_i of $T[i]$ and a morph $\mathcal{M}_{i-1,i}$ from Δ_{i-1} to Δ_i with $O(\deg(v_i))$ steps, with $w(\mathcal{M}_{i-1,i}) = w(\Delta_{i-1})$ and $h(\mathcal{M}_{i-1,i}) = h(\Delta_{i-1})$ (see Lemma 3).

Composing $\mathcal{M}_0, \mathcal{M}_{0,1}, \mathcal{M}_{1,2}, \dots, \mathcal{M}_{k-1,k}$ yields the desired morph $\mathcal{M}^0$ from Γ_0 to a canonical drawing $\Delta_k = \Gamma_0^*$ of T. The morph has $\sum_i deg(v_i) \in O(n)$ steps (by Lemma 3). Further, $w(\mathcal{M}^0) = w(\Delta_0)$ and $h(\mathcal{M}^0) = h(\Delta_0)$ (by Lemma 3), hence $w(\mathcal{M}^0) \in O(D_0^3 n \cdot w(\Gamma_0))$ and $h(\mathcal{M}^0) \in O(D_0^3 n \cdot h(\Gamma_0))$ (by Lemma 2).

Lemma 2. *There is an integer $B_0 \in O(D_0^3 n)$ such that the drawing Δ_0 obtained by scaling the drawing Γ_0 of T up by B_0 is a partially-canonical drawing of $T[0]$.*

Lemma 3. *For any $i \in \{1, \dots, k\}$, let Δ_{i-1} be a partially-canonical drawing of $T[i-1]$. There exists a partially-canonical drawing Δ_i of $T[i]$ and an $O(\deg(v_i))$-step planar morph $\mathcal{M}_{i-1,i}$ from Δ_{i-1} to Δ_i such that $w(\mathcal{M}_{i-1,i}) \leq w(\Delta_0) + \ell_0 + 4n$ and $h(\mathcal{M}_{i-1,i}) \leq h(\Delta_0) + \ell_0 + 4n$.*

The rest of the section is devoted to the proof of Lemma 3. We denote by T^* the tree obtained by removing from T the nodes of T_{v_i} and their incident edges. Let Δ_i be the straight-line drawing of T obtained from Δ_{i-1} by redrawing T_{v_i} so that it is upward canonical, if $y_{\Delta_{i-1}}(v_i) \leq y_{\Delta_{i-1}}(p(v_i))$, or downward canonical, otherwise, while keeping the placement of v_i and of every node of T^* unchanged.

Lemma 4. *The drawing Δ_i is a partially-canonical drawing of $T[i]$.*

We show how to construct a morph $\mathcal{M}_{i-1,i}$ from Δ_{i-1} to Δ_i satisfying the properties of the statement of the lemma. This is done in several stages as follows.

First, consider the drawing Δ' of T obtained as described next; refer to Fig. 3b. Initialize $\Delta' = \Delta_{i-1}$. Then, for each child u of v_i, translate the drawing of T_u so that u is at the center of a $2n$-box B_u that lies in the intersection between S_e and $Med(v_i)$, whose corners have integer coordinates, and whose center c_u is such that $y_{\Delta_{i-1}}(c_u) \leq y_{\Delta_{i-1}}(v_i)$ if and only if $y_{\Delta_{i-1}}(u) \leq y_{\Delta_{i-1}}(v_i)$; such a box exists by Property (b.iii) of Δ_{i-1}. Also, redraw the edge (v_i, u) as a straight-line segment in Δ'.

Claim 3.1 *The morph $\langle \Delta_{i-1}, \Delta' \rangle$ is planar.*

Second, we show how to move the subtrees rooted at the children of v_i in the interior of $Large(v_i)$, so that they land in the position they have in Δ_i. The way we deal with such subtrees depends on their placement with respect to v_i and to the drawing of edge $(v_i, p(v_i))$. We consider the case in which $y(p(v_i)) \geq y(v_i)$ and $x(p(v_i)) \geq x(v_i)$; the other cases can be treated similarly. In particular, we distinguish four regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$, and $\mathcal{R}_4$ defined as follows; refer to Fig. 4. Let $h_{\rightarrow}(v)$ and $h_{\leftarrow}(v)$ be the horizontal rays originating at a node v and directed rightward and leftward, respectively. Further, let $h_{\uparrow}(v)$ be the horizontal ray originating at a node v and directed upward.

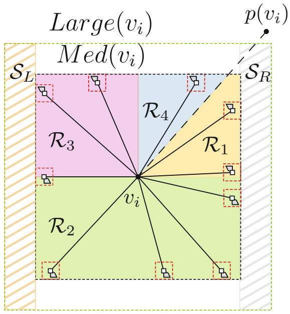
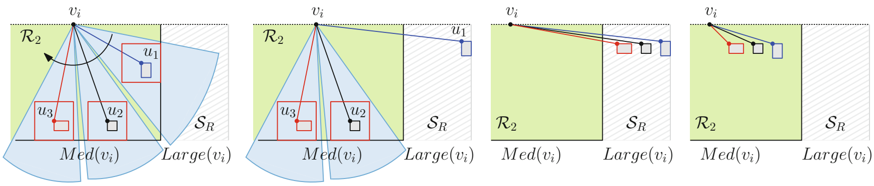


Fig. 4. Regions for v_i.



(a) Δ' **(b)** Ψ_1 **(c)** Ψ_3 **(d)** Δ^+

Fig. 5. Illustrations for Lemma 3, focused on the children of v_i that lie in $\mathcal{R}_2$.

Region $\mathcal{R}_1$ is defined as the intersection of $Med(v_i)$ with the wedge centered at v_i obtained by counter-clockwise rotating $h_{\rightarrow}(v_i)$ until it passes through $p(v_i)$; note that, if $(v_i, p(v_i))$ is a horizontal segment, then $\mathcal{R}_1 = \emptyset$.

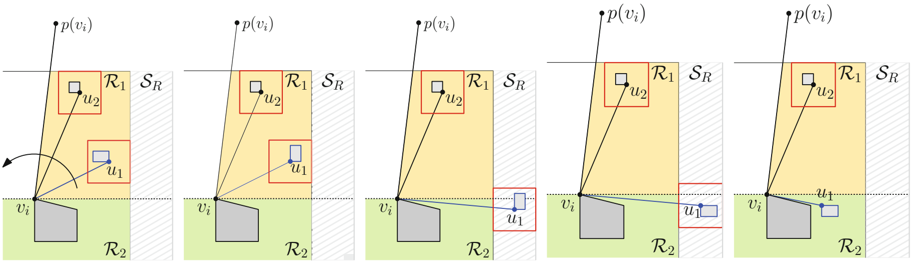
Region $\mathcal{R}_2$ is the rectangular region that is the lower half of $Med(v_i)$;

Region $\mathcal{R}_3$ is defined as the intersection of $Med(v_i)$ with the wedge centered at v_i obtained by clockwise rotating $h_{\leftarrow}(v_i)$ until it coincides with $h_{\uparrow}(v_i)$; and

Region $\mathcal{R}_4$ is defined as the intersection of $Med(v_i)$ with the wedge centered at v_i obtained by clockwise rotating $h_{\uparrow}(v_i)$ until it passes through $p(v_i)$; note that, if $(v_i, p(v_i))$ is a vertical segment, then $\mathcal{R}_4 = \emptyset$.

Note that $Med(v_i) = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4$.

We define two more regions (see Fig. 4), which will be exploited as "buffers" that allow us to rotate subtrees via Lemma 1 without introducing crossings. Let $\mathcal{S}_L$ and $\mathcal{S}_R$ be the rectangular regions in Δ' containing all the points in $Large(v_i) - Med(v_i)$ to the left of the left side of $Med(v_i)$ and to the right of the right side of $Med(v_i)$, respectively. Observe that, since Δ_{i-1} satisfies Property (d) of a partially-canonical drawing and by the construction of Δ', the region $\mathcal{S}_L$ is empty, while the region $\mathcal{S}_R$ may only be traversed by the edge $(v_i, p(v_i))$.



(a) $\xi_0^4 = \Delta^+$ **(b)** ξ_1^1 **(c)** ξ_1^2 **(d)** ξ_1^3 **(e)** ξ_1^4

Fig. 6. Illustrations for Lemma 3, focused on the children of v_i that lie in $\mathcal{R}_1$.

We start by dealing with the children u_j of v_i that lie in the interior of $\mathcal{R}_2$; refer to Fig. 5. Consider the edges (v_i, u_j) in the order $(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_m)$ in which such edges are encountered while clockwise rotating $h_{\rightarrow}(v_i)$; see Fig. 5a. Let Ψ_1 be the drawing obtained from Δ' by translating the drawing of the tree T_{u_1} so that u_1 lies in the interior of $\mathcal{S}_R$ and one unit below v_i and so that the right side of the bounding box of the drawing of T_{u_1} lies upon the right side of $Large(v_i)$, and by redrawing the edge (v_i, u_1) as a straight-line segment.

Claim 3.2 *The morph* $\langle \Delta', \Psi_1 \rangle$ *is planar.*

For $j = 2, \ldots, m$, let Ψ_j be the drawing obtained from Ψ_{j-1} by translating the drawing of the tree T_{u_j} so that u_j lies in the interior of $\mathcal{S}_R$ and one unit below v_i and so that the right side of the bounding box of the drawing of T_{u_j} lies one unit to the left of u_{j-1}, and by redrawing the edge (v_i, u_j) as a straight-line segment.

Claim 3.3 *For* $j = 2, \ldots, m$, *the morph* $\langle \Psi_{j-1}, \Psi_j \rangle$ *is planar.*

Let Δ^+ be the drawing obtained from Ψ_m by horizontally translating T_{u_j} so that u_j lands at its final position in Δ_i, and by redrawing the edge (v_i, u_j) as a straight-line segment, for $j = 1, 2, \ldots, m$; see Fig. 5c and d.

Claim 3.4 *The morph* $\langle \Psi_m, \Delta^+ \rangle$ *is planar.*

Next, we deal with the children u_j of v_i that lie in the interior of $\mathcal{R}_1$. Consider the edges (v_i, u_j) in the order $(v_i, u_1), (v_i, u_2), \ldots, (v_i, u_\ell)$ in which such edges are encountered while counter-clockwise rotating $h_{\rightarrow}(v_i)$ around v_i; refer to Fig. 6. We are going to move the subtrees rooted at the children of v_i in $\mathcal{R}_1$, one by one in the order $T_{u_1}, T_{u_2}, \ldots, T_{u_\ell}$, so that they land in the position that they have in Δ_i. Such a movement consists of four linear morphs. First, we rotate the drawing of T_{u_j} so that it becomes leftward canonical (see Fig. 6b). Second, we translate the drawing of T_{u_j} so that u_j lies in the interior of S_R and one unit below v_i (see Fig. 6c). Third, we rotate the drawing of T_{u_j} so that it becomes upward canonical (see Fig. 6d). Finally, we horizontally translate the drawing of T_{u_j} to its final position in Δ_i (see Fig. 6e).

We now provide the details of the above four linear morphs. For $j = 1, \ldots, \ell$, let ξ^4_{j-1} be a drawing of T with the following properties, where $\xi^4_0 = \Delta^+$ (refer to Fig. 6a and e): (P1) the drawing of T^* is the same as in Δ_i; (P2) v_i lies at the same point as in Δ_i; (P3) the drawing of the subtrees of the children of v_i belonging to $\mathcal{R}_2$ (in Δ'), and the drawing of the subtrees $T_{u_1}, T_{u_2}, \ldots, T_{u_{j-1}}$ is the same as in Δ_i; (P4) the drawing of the subtrees $T_{u_j}, T_{u_{j+1}}, \ldots, T_{u_\ell}$ is the same as in Δ^+; and (P5) the drawing of the subtrees rooted at the children of v_i that lie in the interior of $\mathcal{R}_3$ and $\mathcal{R}_4$ is the same as in Δ^+.

For $j = 1, \ldots, \ell$, we construct a drawing ξ^1_j from ξ^4_{j-1} by rotating tree T_{u_j} so that it is leftward canonical in ξ^1_j, and by leaving the position of the nodes not in T_{u_j} unaltered. This rotation can be accomplished via a linear morph $\langle \xi^4_{j-1}, \xi^1_j \rangle$ by Lemma 1; see Fig. 6b.

Claim 3.5 *For $j = 1, \ldots, \ell$, the morph $\langle \xi^4_{j-1}, \xi^1_j \rangle$ is planar.*

For $j = 1, \ldots, \ell$, let ξ^2_j be the drawing obtained from ξ^1_j by translating the drawing of T_{u_j} so that $Small(u_j)$ lies in the interior of $\mathcal{S}_R$ and so that u_j lies one unit below v_i, and by redrawing the edge (v_i, u_j) as a straight-line segment; see Fig. 6c.

Claim 3.6 *For $j = 1, \ldots, \ell$, the morph $\langle \xi^1_j, \xi^2_j \rangle$ is planar.*

For $j = 1, \ldots, \ell$, let ξ^3_j be the drawing obtained from ξ^2_j by rotating tree T_{u_j} so that it is upward canonical in ξ^3_j, and by leaving the position of the nodes not in T_{u_j} unaltered. This rotation can be accomplished via a linear morph $\langle \xi^2_j, \xi^3_j \rangle$, by Lemma 1; see Fig. 6d.

Claim 3.7 *For $j = 1, \ldots, \ell$, the morph $\langle \xi^2_j, \xi^3_j \rangle$ is planar.*

Finally, for $j = 1, \ldots, \ell$, let ξ^4_j be the drawing obtained from ξ^3_j by horizontally translating T_{u_j} so that u_j lies at its final position in Δ_i, and by leaving the position of the nodes not in T_{u_j} unaltered; see Fig. 6e.

Claim 3.8 *For $j = 1, \ldots, \ell$, the morph $\langle \xi^3_j, \xi^4_j \rangle$ is planar.*

Note that the drawing ξ^4_ℓ coincides with Δ_i, except for the drawing of the subtrees lying in the interior of $\mathcal{R}_3$ and $\mathcal{R}_4$.

Subtrees in $\mathcal{R}_3$ are treated symmetrically to the ones in $\mathcal{R}_1$. In particular, the subtrees of the children of v_i that lie in $\mathcal{R}_3$ are processed according to the clockwise order of the edges from v_i to their roots, while the role played by $\mathcal{S}_R$ is now assumed by $\mathcal{S}_L$.

The treatment of the subtrees in $\mathcal{R}_4$ is similar to the one of the subtrees in $\mathcal{R}_3$. However, when a subtree is considered, it is first horizontally translated in the interior of $\mathcal{R}_3$ and then processed according to the rules for such a region.

Altogether, we have described a morph $\mathcal{M}_{i-1,i}$ from the partially-canonical drawing Δ_{i-1} of $T[i-1]$ to Δ_i, which is a partially-canonical drawing of $T[i]$ by Lemma 4. Next, we argue about the properties of $\mathcal{M}_{i-1,i}$.

We deal with the area requirements of $\mathcal{M}_{i-1,i}$. Consider the drawing Δ_0 and place the boxes $Large(v)$ around the nodes v of T; the bounding box of the arrangement of such boxes has width $w(\Delta_0)+\ell_0+4n$ and height $h(\Delta_0)+\ell_0+4n$. We claim that the drawings of $\mathcal{M}_{i-1,i}$ lie inside such a bounding box. Assume this is true for Δ_{i-1} (this is indeed the case when $i=1$); all subsequent drawings of $\mathcal{M}_{i-1,i}$ coincide with Δ_{i-1}, except for the placement of the subtrees rooted at the children of v_i, which however lie inside $Large(v_i)$ in each of such drawings. Since v_i has the same position in Δ_i as in Δ_0 and since $Large(v_i)$ has width and height equal to $\ell_0 + 4n$, the claim follows.

Finally, we deal with the number of linear morphs composing $\mathcal{M}_{i-1,i}$. The morph $\mathcal{M}_{i-1,i}$ consists of the morph $\langle \Delta_{i-1}, \Delta' \rangle$, followed by the morphs needed to drive the subtrees rooted at the children of v_i to their final positions in Δ_i. Since the number of morphing steps needed to deal with each of such subtrees is constant, we conclude that $M_{i-1,i}$ consists of $O(\deg(v_i))$ linear morphing steps. This concludes the proof of Lemma 3.

5 Conclusions and Open Problems

We presented an algorithm that, given any two order-preserving straight-line planar grid drawings Γ_0 and Γ_1 of an n-node ordered tree T, constructs a morph $\langle \Gamma_0 = \Delta_0, \Delta_1, \ldots, \Delta_k = \Gamma_1 \rangle$ such that k is in $O(n)$ and such that the area of each intermediate drawing Δ_i is polynomial in n and in the area of Γ_0 and Γ_1. Better bounds can be achieved if T is rooted and Γ_0 and Γ_1 are also strictly-upward drawings, especially in the case in which T is a binary tree.

We make a remark about the generality of the model that we adopted. At a first glance, our assumption that Γ_0 and Γ_1 are grid drawings seems restrictive, and it seems more general to consider drawings that have bounded resolution. However, by using an observation from [9], one can argue that two morphing steps suffice to transform a drawing with resolution r in a grid drawing whose area is polynomial in r. Namely, it suffices to scale each input drawing so that the smallest distance between any pair of geometric objects (points representing vertices or segments representing edges) is 2; this is a single morphing step which does not change the resolution of the drawing, hence the largest distance between any pair of geometric objects is in $O(r)$. Then each node can be moved to the nearest grid point; this is another morphing step, which is ensured to be planar by the fact that each node moves by at most $\sqrt{2}/2$, hence this motion only brings any two geometric objects closer by $\sqrt{2}$, while their distance is at least 2. Thus, this results in a grid drawing on an $O(r) \times O(r)$ grid.

Several problems are left open. Is it possible to generalize our results to graph classes richer than trees? Is it possible to improve our area bounds for morphs of straight-line planar grid drawings of trees or even just of paths? Is there a trade-off between the number of steps and the area required by a morph? Are there other relevant tree drawing standards for which it makes sense to consider the morphing problem?

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