Chapter 11

Nonlinear Dynamic Simulation of Turbomachinery Components and Systems

The following chapters deal with the nonlinear transient simulation of turbomachinery systems. Power generation steam and gas turbine engines, combined cycle systems, aero gas turbine engines ranging from single spool engines to multi-spool high pressure core engines with an afterburner for supersonic flights, rocket propulsion systems and compression systems for transport of natural gas with a network of pipeline systems are a few examples of systems that heavily involve turbomachinery components.

Considering a power generation gas turbine engine as a turbomachinery system that is designed for steady state operation, its behavior during routine startups, shot downs, and operational load changes significantly deviates from the steady state design point. Aero gas turbine engines have to cover a relatively broad operational envelope that includes takeoff, low and high altitude operation conditions, as well as landing. During these operations, the components are in a continuous dynamic interaction with each other, where the aero-thermodynamic as well as the mechanical load conditions undergo temporal changes. As an example, the acceleration/deceleration process causes a dynamic mismatch between the turbine and compressor power resulting in temporal change of the shaft speed.

In the above cases, the turbomachinery systems are subjected to the operating modes that are specific to the system operation. Besides these foreseeable events, there are unforeseeable operation scenarios that are not accounted for when designing the system. System failures, such as, blade loss during a routine operation, loss of cooling mass flow through the cooled turbine blades, adverse

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operation conditions that force the compressor component to surge, and failure of the control system, are a few examples of adverse operational conditions. In all of these operations, the system experiences adverse changes in total fluid and the thermodynamic process leading to greater aerodynamic, thermal and mechanical stress conditions.

The trend in the development of gas turbine technology during the past decades shows a continuous increase in efficiency, performance, and specific load capacities. This trend is inherently associated with increased aerodynamic, thermal, and mechanical stresses. Under this circumstance, each component operates in the vicinity of its aerodynamic, thermal and mechanical stress limits. Adverse operational conditions that cause a component to operate beyond its limits can cause structural damages as a result of increased aerodynamic, thermal, and structural stresses. To prevent this, the total response of the system, including aerodynamic, thermal, and mechanical responses must be known in the stage of design and development of new turbomachinery systems.

This chapter describes the physical basis for the non-linear dynamic simulation of gas turbine components and systems. A brief explanation of the numerical method for solution is followed by detailed dynamic simulation of several components described in the following chapters.

11.1 Theoretical Background

Dynamic behavior of turbomachinery components and systems can generally be described by conservation laws of fluid mechanics and thermodynamics ([1], [2], [3], [4]). The fluid dynamic process that takes place within an inertial system is, as a rule, unsteady. The steady state, a special case, always originates from an unsteady condition during which the temporal changes in the process parameters have largely come to a standstill. Considering this fact, the conservation laws of fluid dynamics and thermodynamics must be rearranged such that temporal changes of thermo-fluid dynamic quantities are expressed in terms of spatial changes. A summary of relevant equations are found in [5]. For an unsteady flow, the conservation of mass derived in [5], Chapter 3, Equation (3.4), is called upon:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \boldsymbol{V}). \tag{11.1}$$

Equation (11.1) expresses the *temporal change* of the density in terms of *spatial change* of the specific mass flow. Neglecting the gravitational force, the Cauchy equation of motion in [5] Equation (3.22) was derived as:

$$\frac{\partial \boldsymbol{V}}{\partial t} + \boldsymbol{V} \cdot \nabla \boldsymbol{V} = \frac{1}{\rho} \nabla \cdot \boldsymbol{\Pi}. \tag{11.2}$$

We rearrange Equation (11.2) to include the density in the temporal and spatial derivatives by implementing the continuity equation [5], Equation (3.4)

$$\frac{\partial (\rho \boldsymbol{V})}{\partial t} + \nabla \cdot (\rho \boldsymbol{V} \boldsymbol{V}) = \nabla \cdot \boldsymbol{\Pi} \tag{11.3}$$

where the stress tensor $\boldsymbol{\Pi}$ can be decomposed into the pressure and shear stress tensor as follows:

$$\boldsymbol{\Pi} = \boldsymbol{I}p + \boldsymbol{T} \tag{11.4}$$

with Ip as the normal stress and T as the shear stress tensor. Inserting Equation (11.4) into Equation (11.3) results in

$$\frac{\partial(\rho \boldsymbol{V})}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}\boldsymbol{V}) = -\nabla p + \nabla \cdot \boldsymbol{T}. \tag{11.5}$$

Equation (11.5) directly relates the temporal changes of the specific mass flow to the spatial changes of the velocity, pressure, and shear stresses. For the complete description of the unsteady flow process, the total energy equation that includes mechanical and thermal energy balance is needed. Referring to [5], Equation (3.58) and neglecting the gravitational contribution, we find for mechanical energy that:

$$\rho \frac{D}{Dt}\left(\frac{V^2}{2}\right) = -\boldsymbol{V} \cdot \nabla p + \nabla \cdot (\boldsymbol{T} \cdot V) - \boldsymbol{T} : \boldsymbol{D}. \tag{11.6}$$

Equation (11.6) exhibits the mechanical energy balance in differential form. The first term on the right-hand side represents the mechanical energy contribution due to the pressure gradient. The second term is the contribution of the shear stress work. The third term represents the production of the irreversible mechanical energy due to the shear stress. It dissipates as heat and increases the internal energy of the system. Before rearranging Equation (11.6) and performing the material differentiation, we revert to the thermal energy balance, [5], Equation (3.66),

$$\rho \frac{Du}{Dt} = -\nabla \cdot \dot{\boldsymbol{q}} - p \nabla \cdot \boldsymbol{V} + \boldsymbol{T} : \boldsymbol{D}. \tag{11.7}$$

The combination of the mechanical and thermal energy balances, Equations (11.6) and (11.7), results in:

$$\rho \frac{D}{Dt}\left(u + \frac{V^2}{2}\right) = -\nabla \cdot \dot{\boldsymbol{q}} + \nabla \cdot (\boldsymbol{\Pi} \cdot \boldsymbol{V}). \tag{11.8}$$

Since all components of a turbomachinery system are considered open systems, it is appropriate to use enthalpy h rather than internal energy u. The state properties h and u can be expressed as a function of other state properties such as T, v, p etc.

$$u = u(T, \nu),\ du = \left(\frac{\partial u}{\partial T}\right)_{\nu} dT + \left(\frac{\partial u}{\partial \nu}\right)_{T} d\nu \tag{11.9}$$

$$h = h(T, p),\ dh = \left(\frac{\partial h}{\partial T}\right)_{p} dT + \left(\frac{\partial h}{\partial p}\right)_{T} dp \tag{11.10}$$

where T and h are the absolute static temperature and enthalpy. With the definitions:

$$c_\nu = \left(\frac{\partial u}{\partial T}\right)_\nu, \; c_p = \left(\frac{\partial h}{\partial T}\right)_p \tag{11.11}$$

and the application of the first law, the following relation between c_p and c_ν is established:

$$c_p = c_\nu + \left[\left(\frac{\partial u}{\partial \nu}\right)_T + p\right]\left(\frac{\partial \nu}{\partial T}\right)_p. \tag{11.12}$$

For the open cycle gas turbines and also jet engines with moderate pressure ratios, the working fluids air and combustion gases behave practically like ideal gases whose internal energy is only a function of temperature and not volume. This circumstance considerably simplifies the interconnection of different thermodynamic properties if the equation of state of ideal gases is considered:

$$p\nu = RT. \tag{11.13}$$

Introducing the Gibbs's enthalpy function:

$$h = u + p\nu \tag{11.14}$$

the differentiation gives:

$$dh = \frac{\kappa}{\kappa - 1} d(pv) \text{ and } dh = \kappa du \tag{11.15}$$

with

$$c_p - c_\nu = R \text{ and } \frac{c_p}{c_\nu} = \kappa. \tag{11.16}$$

As a consequence, for ideal gases, the result of this operation leads to:

$$\left(\frac{\partial \nu}{\partial T}\right)_p = \frac{R}{p}. \tag{11.17}$$

Thus, the state properties, u and h, as well as the specific heat capacities c_p, and c_ν and their ratio κ are solely functions of temperature. This is also valid for combustion gases with approximately ideal behavior. For combustion gases, there is a parametric dependency of the above stated properties and the fuel-air ratio. After this preparation, Equation (11.7) can be written in terms of h and p:

$$\rho \frac{Dh}{Dt} = -\nabla \cdot \dot{\boldsymbol{q}} + \frac{\boldsymbol{D}p}{\boldsymbol{D}t} + \boldsymbol{T} : \boldsymbol{D} \tag{11.18}$$

with Equation (11.15) and considering the continuity equation (11.54), the substantial change of the static pressure is:

$$\frac{Dp}{Dt} = \frac{\kappa - 1}{\kappa} \rho \frac{Dh}{Dt} - p \nabla \cdot \boldsymbol{V}. \tag{11.19}$$

Introducing Equation (11.19) into Equation (11.18):

$$\frac{\rho}{\kappa}\frac{Dh}{Dt} = -\nabla\dot{\boldsymbol{q}} - p\nabla\cdot\boldsymbol{V} + \boldsymbol{T}:\boldsymbol{D}. \tag{11.20}$$

The combination of thermal energy equation (11.20) and mechanical energy equation (11.6) leads to:

$$\frac{\partial H}{\partial t} = -kV\cdot\nabla H(-\kappa-1)\left(\frac{1}{\rho}\nabla\cdot(\rho\boldsymbol{V})(H+K) + \frac{\boldsymbol{V}\cdot\partial(\rho\boldsymbol{V})}{\rho\partial t}\right) + \left(-\frac{\kappa\nabla\cdot\dot{\boldsymbol{q}}}{\rho} + \frac{\kappa}{\rho}\nabla\cdot(\boldsymbol{V}\cdot\boldsymbol{T})\right) \tag{11.21}$$

with $H = h + V^2/2$ as the total enthalpy and $K = V^2/2$ the kinetic energy. Equation (11.21) can also be obtained by introducing the relationship between the pressure and enthalpy into Equation (11.21). The total enthalpy can be expressed in terms of total temperature:

$$c_p\frac{\partial T_0}{\partial t} = -kV\cdot\nabla(c_pT_0) - (\kappa-1)\left(\frac{1}{\rho}\nabla\cdot(\rho V)(c_pT_0+K) + \frac{V\cdot\partial(\rho V)}{\rho\partial t}\right) + \left(-\frac{\kappa\nabla\cdot\dot{q}}{\rho} + \frac{\kappa}{\rho}\nabla\cdot(V\cdot T)\right). \tag{11.22}$$

Finally, the total energy equation in terms of total pressure can be established by inserting Equations (11.14) and (11.16) into Equation (11.8):

$$\frac{\partial P}{\partial t} = -k\nabla\cdot(\boldsymbol{V}P) + (\kappa-1)[-\nabla\cdot(\dot{\boldsymbol{q}}) + \nabla\cdot(\boldsymbol{V}\cdot\boldsymbol{T})] - (\kappa-2)\left[\frac{\partial(\rho K)}{\partial t} + \nabla\cdot(\rho K\boldsymbol{V})\right] + \rho g\cdot\boldsymbol{V} \tag{11.23}$$

where $P = p + \rho V^2/2$ is the total pressure. Equations (11.21), (11.22) and (11.23) express the same physical principles, namely the law of conservation of energy in two different forms. In physical terms, they are fully equivalent and mathematically convertible to each other. As shown in the following chapters, one or the other of these equations is called upon in conjunction with the other laws to deal with various dynamic problems. For example, in dealing with an unsteady exchange of energy and impulse, it is useful to apply the differential equation for total temperature. If the principal goal of a problem is the determination of the unsteady changes in pressure, the total pressure differential equation should be used. A summary of the working equations is given in Table 11.1.

Table 11.1: Summary of thermo-fluid dynamic equations.

Equations in terms of substantial derivatives $\boldsymbol{D}/\boldsymbol{D}t$	Eq. No.
Continuity $\frac{\boldsymbol{D}\rho}{\boldsymbol{D}t} = -\rho\nabla\cdot\boldsymbol{V}$	(11.1)
Motion $\rho\frac{\boldsymbol{D}\boldsymbol{V}}{\boldsymbol{D}t} = \nabla\cdot\mathit{\Pi} + gz$	(11.2)
Mechanical Engineering $\rho\frac{\boldsymbol{D}}{\boldsymbol{D}t}\left(\frac{\boldsymbol{V}^2}{2}\right) = -\boldsymbol{V}\cdot\nabla p + \nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{V}) = \boldsymbol{T}:\boldsymbol{D} + \rho\boldsymbol{V}\cdot g$	(11.6)

Table : Summary of thermo-fluid dynamic equations (continued).

Equations in terms of substantial derivatives $\boldsymbol{D}/\boldsymbol{D}t$, $\partial/\partial t$	Eq. No.
Equation of thermal energy in terms of u $\rho\frac{\boldsymbol{D}u}{\boldsymbol{D}t} = -\nabla\cdot\dot{\boldsymbol{q}} - p\nabla\cdot\boldsymbol{V} + \boldsymbol{T}:\boldsymbol{D}$	(11.7)
Equation of thermal energy in terms of h for ideal gas $\rho\frac{\boldsymbol{D}h}{\boldsymbol{D}t} = -\nabla\cdot\dot{\boldsymbol{q}} + \frac{\boldsymbol{D}p}{\boldsymbol{D}t} + \boldsymbol{T}:\boldsymbol{D}$	(11.18)
Equation of total enthalpy $\rho\frac{\boldsymbol{D}H}{\boldsymbol{D}t} = \rho\frac{\boldsymbol{D}}{\boldsymbol{D}t}(h + \frac{\boldsymbol{V}^2}{2}) = \frac{\partial p}{\partial t} + -\nabla\cdot\dot{\boldsymbol{q}} + \nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{V}) + \rho\boldsymbol{V}\cdot g$	(11.21)
Equation of thermal energy in terms of c_v and T for ideal gas $\rho c_\nu\frac{\boldsymbol{D}T}{\boldsymbol{D}t} = -\nabla\cdot\dot{\boldsymbol{q}} - p\nabla\cdot\boldsymbol{V} + \boldsymbol{T}:\boldsymbol{D}$	(11.7)
Equation of thermal energy in terms of c_p and T for ideal gas $\rho c_p\frac{\boldsymbol{D}\boldsymbol{T}}{\boldsymbol{D}t} = -\nabla\cdot\dot{\boldsymbol{q}} + \frac{\boldsymbol{D}p}{\boldsymbol{D}t} + \boldsymbol{T}:\boldsymbol{D}$	(11.7)
Equation of continuity $\frac{\partial\rho}{\partial t} = -\nabla\cdot(\rho\boldsymbol{V})$	(11.1)
Equation of motion in terms of total stress tensor $\frac{\partial(\rho\boldsymbol{V})}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}) = \nabla\cdot\Pi$	(11.2)
Equation of motion, stress tensor decomposed $\frac{\partial(\rho\boldsymbol{V})}{\partial t} + \nabla\cdot(\rho\boldsymbol{V}\boldsymbol{V}) = -\nabla p + \nabla\cdot\boldsymbol{T}$	(11.5)
Equation of mechanical energy including ρ $\frac{\partial(\rho K)}{\partial t} = -\nabla\cdot(\rho K\boldsymbol{V}) - \boldsymbol{V}\cdot\nabla p + \nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{V}) + \rho\boldsymbol{V}\cdot g$	(11.6)
Equation of thermal energy in terms of u for ideal gas $\frac{\partial(\rho u)}{\partial t} = -\nabla\cdot(\rho u\boldsymbol{V}) - \nabla\dot{\boldsymbol{q}} - p\nabla\cdot\boldsymbol{V} + \boldsymbol{T}:\nabla\boldsymbol{V}$	(11.7) Rearranged

Table : Summary of thermo-fluid dynamic equations (continued).

Equations in terms of local derivatives $\partial/\partial t$	Eq. No.
Equation of thermal energy in terms of h for ideal gas $\frac{\partial(\rho h)}{\partial t} = -\nabla\cdot(\rho h\boldsymbol{V}) - \nabla\dot{\boldsymbol{q}} + \frac{\boldsymbol{D}p}{\boldsymbol{D}t} + \boldsymbol{T}:\nabla\boldsymbol{V}$	(11.7) Rearranged
Equation of thermal energy in terms of c_v and T $\frac{\partial(\rho c_v \boldsymbol{T})}{\partial \boldsymbol{T}} = -\nabla\cdot(\rho u\boldsymbol{V}) - \nabla\dot{\boldsymbol{q}} - p\nabla\cdot\boldsymbol{V} + \boldsymbol{T}:\nabla\boldsymbol{V}$	(11.7) Rearranged
Equation of thermal energy in terms of static h for ideal gases $\frac{\partial(\rho c_p \boldsymbol{T})}{\partial t} = -\nabla\cdot(\rho h\boldsymbol{V}) - \nabla\dot{\boldsymbol{q}} + \frac{\boldsymbol{D}p}{\boldsymbol{D}t} + \boldsymbol{T}:\nabla\boldsymbol{V}$	(11.7) Rearranged
Energy equation in terms of total enthalpy $\frac{\partial H}{\partial t} = -k\boldsymbol{V}\cdot\nabla H - (\kappa-1)\left(\frac{1}{\rho}\nabla\cdot(\rho\boldsymbol{V})(H+K) + \frac{\boldsymbol{V}\cdot\partial(\rho\boldsymbol{V})}{\rho\partial t}\right) + \left(-\frac{\kappa\nabla\cdot q}{\rho} + \frac{\kappa}{\rho}\nabla\cdot(\boldsymbol{V}\cdot\boldsymbol{T})\right)$	(11.21)
Energy equation in terms of total temperature $c_p\frac{\partial T_0}{\partial t} = -k\boldsymbol{V}\cdot\nabla(c_pT_0) - (\kappa-1)\left(\frac{1}{\rho}\nabla\cdot(\rho\boldsymbol{V})(c_pT_0+K) + \frac{\boldsymbol{V}\cdot\partial(\rho\boldsymbol{V})}{\rho\partial t}\right) - \left(\frac{\kappa\nabla\cdot\dot{\boldsymbol{q}}}{\rho} - \frac{\kappa}{\rho}\nabla\cdot(\boldsymbol{V}\cdot\boldsymbol{T})\right)$	(11.22)
Energy equation in terms of total pressure $\frac{\partial P}{\partial \boldsymbol{T}} = -k\nabla\cdot(\boldsymbol{V}P) - (\kappa-1)[\nabla\cdot\dot{\boldsymbol{q}} + \nabla\cdot(\boldsymbol{V}\cdot\boldsymbol{T})] - (\kappa-2)\left[\frac{\partial(\rho K)}{\partial t} + \nabla\cdot(\rho K\boldsymbol{V})\right]$	(11.23)

11.2 Preparation for Numerical Treatment

The thermo-fluid dynamic equations discussed above constitute the theoretical basis describing the dynamic process that takes place within a turbomachinery component during a transient operation. A four-dimensional time space treatment that involves the Navier solution is, at least for the time being, out of reach. For simulation of dynamic behavior of a turbomachine that consists of many components, it is not primarily important to calculate the unsteady three-dimensional flow processes in great detail. However, it is critical to accurately predict the response of each individual component as a result of dynamic operation conditions. A one-dimensional time dependent calculation procedure provides sufficiently accurate results. In the following, the conservation equations are presented in index notation. For the one-dimensional time dependent treatment, the basic equations are prepared first by setting the index $i = 1$ in Equations (11.24)-(11.27).

11.3 One-Dimensional Approximation

The thermo-fluid dynamic equations discussed above constitute the theoretical basis describing the dynamic process that takes place within a turbomachinery component during a transient operation. A four-dimensional time space treatment that involves the Navier solution is, at least for the time being, out of reach. For simulation of the dynamic behavior of a turbomachine that consists of many components, it is not primarily important to calculate the unsteady three-dimensional flow processes in great detail. However, it is critical to accurately predict the response of each individual component as a result of dynamic operation conditions. A one-dimensional time dependent calculation procedure provides sufficiently accurate results. For this purpose, firstly, the basic equations are prepared for the one-dimensional treatment.

11.3.1 Time Dependent Equation of Continuity

In Cartesian coordinate system the continuity equation (11.1) is:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_i}(\rho V_i). \tag{11.24}$$

Equation (11.24), after setting $\rho V_1 = \dot{m}/S$, becomes:

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x_1}\left(\frac{\dot{m}}{S}\right). \tag{11.25}$$

with $x_1 \equiv x$ as the length in streamwise direction and $S = S(x)$ the cross sectional area of the component under investigation. Equation (11.25) expresses the fact that the temporal change of the density is determined from the spatial change of the specific mass flow within a component. The partial differential Equation (11.25) can be approximated as an ordinary differential equation by

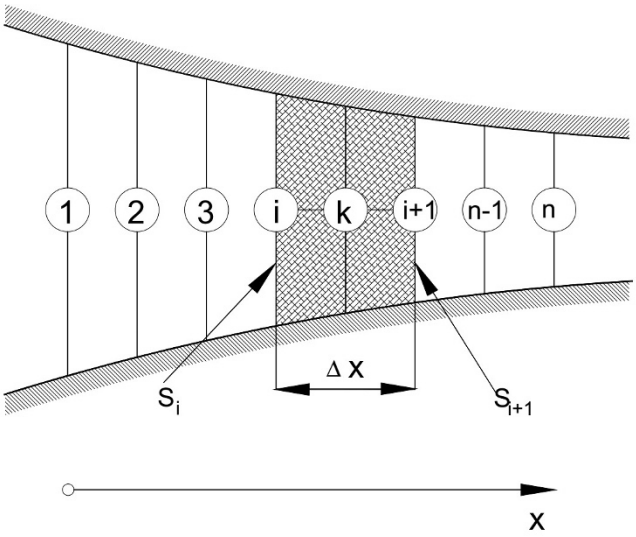

Figure 11.1: Discretization of an arbitrary flow path with variable cross section $S = S(x)$.

means of conversion into a difference equation. The ordinary differential equation can then be solved numerically with the prescribed initial and boundary conditions. For this purpose, the flow field is equidistantly divided into a number of discrete zones with prescribed length, ΔX, inlet and exit cross sections and S_i and S_{i+1} as Figure 11.1 shows. The quantities pertaining to the inlet and exit cross sections represent averages over the inlet and exit heights. For the annular nozzles and diffusers the quantities are thought of as averages taken in circumferential as well as radial direction. Using the nomenclature in Figure 11.1, Equation (11.25) is approximated as:

$$\frac{\partial \rho_k}{\partial t} = -\frac{1}{\Delta x}\left(\frac{\dot{m}_{i+1}}{S_{i+1}} - \frac{\dot{m}_i}{S_i}\right) \tag{11.26}$$

with $\dot{m}_i$ and $\dot{m}_{i+1}$ as the mass flows at stations i and $i+1$ with the corresponding cross sections. For a constant cross section, Equation (11.26) reduces to:

$$\frac{\partial \rho_k}{\partial t} = -\frac{1}{\Delta x S}(\dot{m}_{i+1} - \dot{m}_i) = -\frac{1}{\Delta V}(\dot{m}_{i+1} - \dot{m}_i) \tag{11.27}$$

with $\Delta V = \Delta x S$ as the volume of the element k enclosed between the surfaces i and $i+1$. The index k refers to the position at $\Delta x/2$, Figure 11.1.

11.3.2 Time Dependent Equation of Motion

The index notation of the momentum equation (11.5) is:

$$\frac{\partial(\rho V_i)}{\partial t} = -\frac{\partial}{\partial x_j}(\rho V_i V_j) - \frac{\partial p}{\partial x_i} + \frac{\partial T_{ij}}{\partial x_j}. \tag{11.28}$$

In the divergence of the shear stress tensor in Equation (11.28), $\nabla \cdot T = e_i \partial T_{ij}/\partial x_j$ represents the shear force acting on the surface of the component. For a one-dimensional flow, the only non-zero term is $\partial \tau_{21}/\partial x_2$. It can be related to the wall shear stress τ_w which is a function of the friction coefficient c_f.

$$\tau_w = c_f \frac{\rho}{2} V^2. \tag{11.29}$$

In the near of the wall, the change of the shear stress can be approximated as the difference between the wall shear stress τ_w and the shear stress at the edge of the boundary layer, which can be set as $\tau_e \approx 0$

$$\left(\frac{\partial \tau_{12}}{\partial x_2}\right)_{x_2=0} = \frac{\tau_e - \tau_w}{\Delta x_2} = -\frac{\tau_w}{\Delta x_2}. \tag{11.30}$$

The distance in Δx_2 can be replaced by a characteristic length such as the hydraulic diameter D_h. Expressing the wall shear stress in Equation (11.30) by the skin friction coefficient

$$\left(\frac{\partial \tau_{12}}{\partial x_2}\right)_{x_2=0} = -c_f \frac{\rho}{D_h} \frac{V^2}{2} = -c_f \frac{\dot{m}^2}{2 D_h \rho S^2} \tag{11.31}$$

and inserting Equation (11.31) into the one-dimensional version of Equation (11.28), we obtain

$$\frac{\partial \dot{m}}{\partial t} = -\frac{\partial}{\partial x_1}(\dot{m} V_1 + pS) + (\dot{m} V_1 + pS)\frac{1}{S}\frac{\partial S}{\partial x_1} - c_f \frac{\dot{m}^2}{2 D_h \rho S}. \tag{11.32}$$

Equation (11.32) relates the temporal change of the mass flow to the spatial change of the velocity, pressure and shear stress momentums. As we will see in the following sections, mass flow transients can be accurately determined using Equation (11.32). Using the nomenclature from Figure 11.1, we approximate (11.32) as:

$$\begin{aligned}\frac{\partial \dot{m}_k}{\partial t} =& -\frac{1}{\Delta x}(\dot{m}_{i+1} V_{i+1} - \dot{m}_i V_i + p_{i+1} S_{i+1} - p_i S_i) \\ &+ \left(\frac{\dot{m}_k V_k + P_k S_k}{S_k}\right)\left(\frac{S_{i+1} - S_i}{\Delta x}\right) - c_f \frac{m_k^2}{2 D_{h_k} \rho_k S_k}.\end{aligned} \tag{11.33}$$

For a constant cross section, Equation (11.33) is modified as:

$$\frac{\partial \dot{m}_k}{\partial t} = -\frac{1}{\Delta x}[\dot{m}_{i+1} V_{i+1} - \dot{m}_i V_i + (p_{i+1} - p_i) S] - c_f \frac{m_k^2}{2 D_{h_k} \rho_k S_k}. \tag{11.34}$$

11.3.3 Time Dependent Equation of Total Energy

The energy equation in Equation (11.21), in terms of total enthalpy, is written in index notation

$$\frac{\partial H}{\partial t} = -kV_i\frac{\partial H}{\partial x_i} - \frac{\kappa-1}{\rho}\left[(H+K)\frac{\partial(\rho V_i)}{\partial x_i} + \frac{V_i\cdot\partial(\rho V_i)}{\partial t}\right] - \frac{\kappa}{\rho}\left[\frac{\partial \dot{q}_i}{\partial x_i} - \frac{\partial(V_j T_{ij})}{\partial x_i}\right] \tag{11.35}$$

expressing the total enthalpy, Equation (11.35), in terms of total temperature results in:

$$\frac{\partial(c_pT_0)}{\partial t} = -kV_i\frac{\partial(c_pT_0)}{\partial x_i} - \frac{\kappa-1}{\rho}\left[(c_pT_0+K)\frac{\partial(\rho V_i)}{\partial x_i} + \frac{V_i\cdot\partial(\rho V_i)}{\partial t}\right] - \frac{\kappa}{\rho}\left[\frac{\partial \dot{q}_i}{\partial x_i} - \frac{\partial(V_j T_{ij})}{\partial x_i}\right]. \tag{11.36}$$

For calculating the total pressure, the equation of total energy is written in terms of total pressure already derived as Equation (11.23), which is presented for the Cartesian coordinate system as:

$$\frac{\partial P}{\partial T} = -\kappa\frac{\partial}{\partial x_i}(PV_i) - (\kappa-1)\left(\frac{\partial \dot{q}_i}{\partial x_i} - \frac{\partial}{\partial x_i}(V_jT_{ij})\right) - (\kappa-2)\left(\frac{\partial(\rho K V_i)}{\partial x_i} + \frac{\partial(\rho K)}{\partial t}\right). \tag{11.37}$$

Before treating the energy equation, the shear stress work needs to be evaluated:

$$\nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{V}) = \delta_{ij}\delta_{km}\frac{\partial(\tau_{jk}V_m)}{\partial x_i} = \frac{\partial(\tau_{ij}V_j)}{\partial x_i}. \tag{11.38}$$

For a two-dimensional flow, Equation (11.38) gives

$$\nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{V}) = \frac{\partial(\tau_{ij}V_j)}{\partial x_i} = \frac{\partial(\tau_{11}V_1+\tau_{12}V_2)}{\partial x_1} + \frac{\partial(\tau_{21}V_1+\tau_{22}V_2)}{\partial x_2}. \tag{11.39}$$

Assuming a one-dimensional flow with $V_2 = 0$, the contribution of the shear stress work (11.39) is reduced to

$$\nabla\cdot(\boldsymbol{T}\cdot\boldsymbol{V}) = \frac{\partial(\tau_{11}V_1)}{\partial x_1} \approx \frac{(\tau_{11\ \text{inlet}}V_{\text{inlet}} - \tau_{11\ \text{exit}}V_{1\ \text{exit}})}{\Delta x_1}. \tag{11.40}$$

The differences in τ_{11} at the inlet and exit of the component under simulation stem from velocity deformation at the inlet and exit. Its contribution, however, compared to the enthalpy terms in the energy equation, is negligibly small.

Thus, the one-dimensional approximation of total energy equation (11.35) in terms of total enthalpy reads:

$$\frac{\partial H}{\partial t} = -\frac{\kappa \dot{m}}{\rho S}\frac{\partial H}{\partial x_1} - \frac{\kappa - 1}{\rho}\left[(H+K)\frac{\partial}{\partial x_1}\left(\frac{\dot{m}}{S}\right) + \frac{1}{2\rho S^2}\frac{\partial \dot{m}^2}{\partial t}\right] - \frac{\kappa}{\rho}\frac{\partial}{\partial}. \quad (11.41)$$

For a steady state case without changes of specific mass $\dot{m}/S$, Equation (11.41) leads to:

$$\frac{\partial H}{\partial x_1} = -\frac{S}{\dot{m}}\frac{\partial \dot{q}_1}{\partial x_i}. \quad (11.42)$$

Assuming a constant cross section and mass flow, Equation (11.42) gives

$$\frac{\partial H}{\partial x_1} = -\frac{\partial}{\partial x_1}\left(\frac{S\dot{q}_i}{\dot{m}}\right). \quad (11.43)$$

Integrating (11.43) in a streamwise direction results in:

$$H_{\text{out}} - H_{\text{in}} = -\left(\frac{S}{\dot{m}}\right)\Delta\dot{q} \quad (11.44)$$

for Equation (11.44) to be compatible with the energy equation, a modified version of Equation (3.27) is presented.

$$H_{\text{Out}} - H_{\text{In}} = q + w_{\text{Shaft}}. \quad (11.45)$$

Equating Equations (11.44) and (11.45) in the absence of a specific shaft power, the following relation between the heat flux vector and the heat added or rejected from the element must hold:

$$q = -\left(\frac{S}{\dot{m}}\right)\Delta\dot{q}. \quad (11.46)$$

From Equation (11.46), it immediately follows that

$$\Delta\dot{q} = -\frac{q\dot{m}}{S} = -\frac{\dot{Q}}{S} \quad (11.47)$$

where $\dot{Q}$ is the thermal energy flow added to or rejected from the component. In the presence of shaft power, the specific heat in Equation (11.47) may be replaced by the sum of the specific heat and specific shaft power:

$$\Delta\dot{q} = -\frac{\dot{m}q + \dot{m}l_m}{S} = -\left(\frac{\dot{Q}+L}{S}\right). \quad (11.48)$$

Equation (11.48) in differential form in terms of $\dot{Q}$ and L is

$$\frac{\partial \dot{q}}{\partial x} = -\frac{\partial}{\partial x}\left(\frac{\dot{m}q + \dot{m}l_m}{S}\right) = -\frac{\partial}{\partial x}\left(\frac{\dot{Q}+L}{S}\right). \quad (11.49)$$

With Equation (11.49), we find:

$$\frac{\partial H}{\partial t} = -\frac{\kappa \dot{m}}{\rho S}\frac{\partial H}{\partial x_1} - \frac{\kappa - 1}{\rho}\left[(H+K)\frac{\partial}{\partial x_1}\left(\frac{\dot{m}}{S}\right) + \frac{1}{2\rho S^2}\frac{\partial \dot{m}^2}{\partial t}\right] - \frac{\kappa}{\rho}\frac{\partial}{\partial x}\left(\frac{\dot{Q}+L}{S}\right). \tag{11.50}$$

Using the nomenclature in Figure 11.1, Equation (11.50) is written as:

$$\frac{\partial H}{\partial t} = -\kappa_k \frac{\dot{m}_k}{\rho_k S_k}\left(\frac{H_{i+1}-H_i}{\Delta x}\right) - \left(\frac{\kappa-1}{\rho}\right)_k\left[\left(\frac{H_k+K_k}{\Delta x}\right)\left(\frac{\dot{m}_{i+1}}{S_{i+1}} - \frac{\dot{m}_i}{S_i}\right) + \frac{\dot{m}_k}{\rho_k S_k^2}\frac{\partial \dot{m}_{i+1}}{\partial t}\right] - \frac{\kappa_k}{\rho_k}\left(\frac{\Delta\dot{Q}+\Delta L}{\Delta V}\right). \tag{11.51}$$

In terms of total temperature, Equation (11.51) is rearranged as:

$$\frac{\partial c_p T_0}{\partial t} = -\kappa_k \frac{\dot{m}_k}{\rho_k S_k}\left(\frac{c_p T_{0_{i+1}} - c_p T_{0_i}}{\Delta x}\right) - \left(\frac{\kappa-1}{\rho}\right)_k\left[\left(\frac{c_p T_{0_k}+K_k}{\Delta x}\right)\left(\frac{\dot{m}_{i+1}}{S_{i+1}} - \frac{\dot{m}_i}{S_i}\right) + \frac{\dot{m}_k}{\rho_k S_k^2}\frac{\partial \dot{m}_{i+1}}{\partial t}\right] - \frac{\kappa_k}{\rho_k}\left(\frac{\Delta\dot{Q}+\Delta L}{\Delta V}\right). \tag{11.52}$$

In terms of total pressure, Equation (11.37) is:

$$\frac{\partial P}{\partial t} = -\kappa\frac{\partial}{\partial x_1}(PV_1) - (\kappa-1)\left(\frac{\partial \dot{q}_1}{\partial x_1} - \frac{\partial}{\partial x_1}(V_j T_{1j})\right) - (\kappa-2)\left(\frac{\partial(\rho K V_1)}{\partial x_1} + \frac{\partial(\rho K)}{\partial t}\right) \tag{11.53}$$

which is approximated as:

$$\frac{\partial P_k}{\partial t} = -\frac{\kappa_k}{\Delta x}\left(\frac{\dot{m}_{i+1}p_{i+1}}{\rho_{i+1}S_{i+1}} - \frac{\dot{m}_i p_i}{\rho_i S_i}\right) - (\kappa_k - 1)\left(\frac{\dot{m}_k q_k}{\Delta V} + c_{f_k}\frac{\dot{m}_{i+1}\dot{m}_1^2}{2D_{h_{i+1}}S_{i+1}\rho}\right) - (\kappa_k-2)\frac{\dot{m}_k}{\rho_k S_k^2}\left(\frac{1}{2}\frac{\dot{m}_k}{\rho_k}\frac{1}{\Delta x}\left(\frac{\dot{m}_{i+1}}{S_{i+1}} - \frac{\dot{m}_i}{S_i}\right) + \frac{\partial \dot{m}_{i+1}}{\partial t}\right) - \frac{(\kappa_k-2)}{2\Delta x}\left(\frac{\dot{m}_{i+1}^3}{\rho_{i+1}^2 S_{i+1}^3} - \frac{\dot{m}_i^3}{\rho_i^2 S_i^3}\right) \tag{11.54}$$

with:

$$\rho_k = \frac{1}{R}\left(\frac{p_{i+1}+p_i}{T_{i+1}+T_i}\right),\ c_{p_k} = \frac{H_{i+1}-H_i}{T_{i+1}-T_i},\ \kappa_k = \frac{c_{p_k}}{c_{p_k}-R} \tag{11.55}$$

11.4 Numerical Treatment

The above partial differential equations can be reduced to a system of ordinary differential equations by a one-dimensional approximation. The simulation of a complete gas turbine system is accomplished by combining individual components that have been modeled mathematically. The result is a system of ordinary differential equations that can be dealt with numerically. For weak transients, Runge-Kutta or Predictor-Corrector procedures may be used for the solution. When strong transient processes are simulated, the time constants of the differential equation system can differ significantly so that difficulties must be expected with stability and convergence with the integration methods. An implicit method avoids this problem. The system of ordinary differential equations generated in a mathematical simulation can be represented by:

$$\frac{dX}{dt} = G(X,t) \tag{11.56}$$

with X as the state vector sought. If the state vector X is known at the time t, it can be approximated as follows for the time $t+dt$ by the trapezoidal rule:

$$X_{t+\Delta t} = X_t + \frac{1}{2}\Delta t(G_{t+\Delta t} + G_t). \tag{11.57}$$

Because the vector X and the function G are known at the time t, i.e., X_t and G_t are known, Equation (11.57) can be expressed as:

$$X_{t+\Delta t} - X_t - \frac{1}{2}\Delta t(G_{t+\Delta t} + G_t) = F(X_{t+\Delta t}). \tag{11.58}$$

As a rule, the function F is non-linear. It can be used to determine X_{t+dt} by iteration when X_t is known. The iteration process is concluded for the time $t+dt$ when the convergence criterion

$$\frac{X_i^{(k+1)} - X_i^{(k)}}{X_i^{(k+1)}} < \epsilon \tag{11.59}$$

is fulfilled. If the maximum number of iterations, $k = k_{\max}$, is reached without fulfilling the convergence criterion, the time interval Δ_t is halved, and the process of iteration is repeated until the criterion of convergence is met. This integration process, based on the implicit one-step method described by Liniger and Willoughby [6] is reliable for the solution of stiff differential equations. The computer time required depends, first, on the number of components in the system and, second, on the nature of the transient processes. If the transients

are very strong, the computer time can be 10 times greater than the real time because of the halving of the time interval. For weak transients, this ratio is less than 1.

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