# Chapter 14
# Connections in Euclidean and Non-commutative Geometry

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*To the 70th birthday of professor Wolfgang Sprössig*

**Abstract** In this paper, we trace the development of concepts of differential geometry such as a first order differential calculus, an algebra of differential forms and a connection from Euclidean geometry to noncommutative geometry. We begin with basic structures of differential geometry in $n$-dimensional Euclidean space such as vector field, differential form, connection and then, having explained a general idea of non-commutative geometry, we show how these notions can be developed in the assumption that the algebra of smooth functions on Euclidean space is replaced by its non-commutative analog and the differential graded algebra of differential forms is replaced by a $q$-differential graded algebra, where $q$ is a primitive $N$th root of unity and a differential $d$ of this algebra satisfies the equation $d^N = 0$.

**Keywords** Vector fields · Differential forms · Connections · Differential graded algebra · $q$-Differential graded algebra · Non-commutative differential calculus · Quantum hyperplane

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## 14.1 Introduction

A theory of connections is very important part of modern differential geometry. The discovery of interconnection between this theory and the gauge field theories has given a powerful impetus to development of the entire theory. In the present paper we track the development of a concept of connection from the simplest case of the canonical connection in the $n$-dimensional Euclidean space $\mathbb{R}^n$ to its generalizations in non-commutative geometry. The present paper is written on the basis of lectures delivered to the doctoral level students of Division of Applied Mathematics of University of Mälardalen within the framework of NordPlus Higher Education Program 2017.

In the first subsections of Sect. 14.2 we describe the structures of elementary differential geometry of $n$-dimensional Euclidean space $\mathbb{R}^n$ such as vector fields and differential 1-forms. We lay particular stress on algebraic aspects of these structures and bring a reader to an idea of algebraic structure which is called a first order differential calculus over an algebra. Then we explain a basic idea of non-commutative geometry, where the commutative algebra of smooth functions $C^\infty(\mathbb{R})^n$ on $\mathbb{R}^n$ is replaced by a non-commutative algebra. We show that in order to construct the differential 1-forms on a non-commutative space we should have a coordinate first order differential calculus with right partial derivatives over a non-commutative algebra. As an example of a coordinate first order differential calculus with right partial derivatives we consider the differential calculus on the quantum hyperplane. In Sect. 14.3 we describe the algebra of differential forms with exterior differential in the $n$-dimensional Euclidean space $\mathbb{R}^n$ and its generalization in an approach of non-commutative geometry, which is called a higher order differential calculus over an algebra. Particularly we explain the notion of the universal differential graded algebra. In Sect. 14.4 we describe the canonical connection in $n$-dimensional Euclidean space and derive its Cartan's structure equations. In Sect. 14.5 we develop a generalization of the theory of connection on modules with the help of the concept of $q$-differential graded algebra, where $q$ is a primitive $N$th root of unity.

## 14.2 Vector Fields, Differential 1-Forms in $\mathbb{R}^n$ and Non-commutative First Order Differential Calculus

In this section we describe the Lie algebra of smooth vector fields in $n$-dimensional Euclidean space $\mathbb{R}^n$ and an approach of non-commutative geometry to a first order differential calculus in a non-commutative space. In what follows we will use the Einstein summation convention over repeated subscript and superscript.

### 14.2.1 Vector Fields in the n-Dimensional Euclidean Space $\mathbb{R}^n$

Consider the $n$-dimensional space $\mathbb{R}^n$. This space has the structure of $n$-dimensional vector space with the component-wise addition of two vectors and the component-wise multiplication by real numbers. If we consider an element $(v^1, v^2, \ldots, v^n)$ of $\mathbb{R}^n$ as a vector then it will be denoted by $\vec{v} = (v^1, v^2, \ldots, v^n)$. For any two vectors $\vec{v} = (v^1, v^2, \ldots, v^n), \vec{w} = (w^1, w^2, \ldots, w^n)$ we have the inner product $< \vec{v}, \vec{w} >= \sum_i v^i\, w^i$, which determines the Euclidean structure of $\mathbb{R}^n$. If we do not use the vector space structure of $\mathbb{R}^n$ then an element $(p^1, p^2, \ldots, p^n)$ will be called a point of $\mathbb{R}^n$ and denoted by $p = (p^1, p^2, \ldots, p^n)$. The coordinate functions of $\mathbb{R}^n$ will be denoted by $x^1, x^2, \ldots, x^n$ and by definition $x^i(p) = p^i$. The canonical basis for the Euclidean vector space $\mathbb{R}^n$ will be denoted by $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where the $i$th component of $\vec{e}_i$ is 1 and the others are zeros.

Let $U \subset \mathbb{R}^n$ be an open subset. A real-valued function $f : U \to \mathbb{R}$ is called a smooth function if it has continuous partial derivative of any order. The set of all smooth functions on the $n$-dimensional Euclidean space $\mathbb{R}^n$ will be denoted by $C^\infty(\mathbb{R}^n)$. The set $C^\infty(\mathbb{R}^n)$ endowed with the pointwise addition of smooth functions and the multiplication of a smooth function by a real number is the infinite dimensional vector space. We remind that a vector space $\mathscr{A}$ is said to be a *unital associative algebra* if $\mathscr{A}$ is equipped with a product $a \cdot b$, where $a, b \in \mathscr{A}$, such that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity), and this product has the identity element $e$ satisfying $a \cdot e = e \cdot a = a$. If, in addition to associativity, the product $a \cdot b$ of any two elements is commutative, i.e. $a \cdot b = b \cdot a$, a unital associative algebra $\mathscr{A}$ is called *commutative*. The vector space $C^\infty(\mathbb{R}^n)$ of smooth functions endowed with the product $fg$ of two smooth functions $f, g$, which is defined by $(fg)(p) = f(p)\, g(p)$, is the commutative unital associative algebra, where the identity element is the function, whose value at any point of the space is 1. If we ignore the multiplication of smooth functions by scalars (real numbers) then $C^\infty(\mathbb{R}^n)$ is the *commutative unital associative ring*.

A *tangent vector* to the $n$-dimensional Euclidean space $\mathbb{R}^n$ at a point $p \in \mathbb{R}^n$ is a pair $(p; \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n$, which will be denoted by $\vec{v}_p$, i.e. $\vec{v}_p = (p; \vec{v})$. A tangent space of all tangent vectors to $\mathbb{R}^n$ at a point $p$ will be denoted by $\mathrm{T}_p\mathbb{R}^n$. The vector space and Euclidean structure of $\mathbb{R}^n$ can be extended to any tangent space $\mathrm{T}_p\mathbb{R}^n$ in the natural way

$$\vec{v}_p + \vec{w}_p = (p; \vec{v} + \vec{w}), \quad a\vec{v}_p = (p; a\vec{v}), \quad < \vec{v}_p, \vec{w}_p >=< \vec{v}, \vec{w} >, \quad a \in \mathbb{R}.$$

Then any tangent vector $\vec{v}_p = (p; \vec{v}) = (p; v^1, v^2, \ldots, v^n)$ can be expressed as $\vec{v}_p = v^i\, \vec{e}_{p,i}$, where $\vec{e}_{p,i} = (p; \vec{e}_i)$. The disjoint union of tangent spaces

$$\mathrm{T}\mathbb{R}^n = \cup_{p\in\mathbb{R}^n} \mathrm{T}_p\mathbb{R}^n, \tag{14.2.1}$$

will be referred to as the *tangent bundle over the $n$-dimensional Euclidean space* $\mathbb{R}^n$. The projection $\pi : \mathrm{T}\mathbb{R}^n \to \mathbb{R}^n$ is defined by $\pi(\vec{v}_p) = p$. A *section* of the tangent bundle $\mathrm{T}\mathbb{R}^n$ is a smooth mapping $X : \mathbb{R}^n \to \mathrm{T}\mathbb{R}^n$ such that $\pi \circ X = \mathrm{id}_{\mathbb{R}^n}$. A smooth section $X$ of the tangent bundle $\mathrm{T}\mathbb{R}^n$ is called a *vector field* in the Euclidean space $\mathbb{R}^n$. Obviously any vector field $X$ is uniquely determined by $n$ smooth functions $X^1, X^2, \ldots, X^n$ such that

$$X : p \in \mathbb{R}^n \mapsto X_p = (p; X^1(p), X^2(p), \ldots, X^n(p)) \in \mathrm{T}_p\mathbb{R}^n.$$

The functions $X^1, X^2, \ldots, X^n$ will be called the components of a vector field $X$. The vector space structure of a tangent space $\mathrm{T}_p\mathbb{R}^n$ induces the vector space structure in the set $\mathfrak{D}$ of all vector fields.

Let $\mathscr{M}$ be an Abelian group and $\mathscr{A}$ be a unital associative ring. We remind that $\mathscr{M}$ together with a mapping $(a, u) \in \mathscr{A} \times \mathscr{M} \mapsto a \cdot u \in \mathscr{M}$ satisfying

$$(a+b)\cdot u = a\cdot u + b\cdot u, a\cdot(u+v) = a\cdot u + a\cdot v, (ab)\cdot u = a\cdot(b\cdot u), e\cdot u = u,$$

where $a, b \in \mathscr{A}$, $u, v \in \mathscr{M}$ and $e$ is the identity element of $\mathscr{A}$, is called a left $\mathscr{A}$-module. A notion of right $\mathscr{A}$-module is defined in a similar manner. $\mathscr{A}$*-bimodule* is an Abelian group $\mathscr{M}$, which is both left and right $\mathscr{A}$-module and $(a\cdot u)\cdot b = a\cdot(u\cdot b)$. One can extend the notion of a module (left, right or bimodule) over a ring to a notion of a module over a unital associative algebra assuming that in this case $\mathscr{A}$, $\mathscr{M}$ are vector spaces and scalars commute with everything. A left $\mathscr{A}$-module $\mathscr{M}$ is referred to as *finitely generated* if there is a set $\{u_1, u_2, \ldots, u_n\}$ of elements of $\mathscr{M}$ such that any element $u$ of $\mathscr{M}$ can be written as $u = a_1\, u_1 + a_2\, u_2 + \ldots + a_n\, u_n$, where $a_1, a_2, \ldots, a_n \in \mathscr{A}$. If $\{u_1, u_2, \ldots, u_n\}$ are linearly independent (over $\mathscr{A}$) then a finitely generated left $\mathscr{A}$-module $\mathscr{M}$ is called a *free module with a basis*. If a ring $\mathscr{A}$ has an invariant basis number then the cardinality of any basis for free left $\mathscr{A}$-module is called the *rank of a free module*.

It turns out that the concept of a module is applicable to the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ and the vector space of vector fields $\mathfrak{D}$, and plays an important role in constructing noncommutative generalizations of vector fields. Indeed given a smooth function $f$ and a vector field $X$ one can define the product $fX$ (left multiplication of vector fields by smooth functions) as the vector field $fX : p \mapsto f(p)\, X_p$. It is easy to check that this product defines the structure of left $C^\infty(\mathbb{R}^n)$-module in $\mathfrak{D}$. Analogously one can define a structure of right $C^\infty(\mathbb{R}^n)$-module in $\mathfrak{D}$, and then from $f(p)\, X_p = X_p\, f(p)$ (numbers commute with vectors) it follows that in the case of the Euclidean space $\mathbb{R}^n$ functions commute with vector fields $fX = Xf$.

Let us define the vector fields $E_i, i = 1, 2, \ldots, n$ by the formula $E_i(p) = \vec{e}_{p,i}$, where $p \in \mathbb{R}^n$. Making use of the left multiplication of vector fields by smooth functions one can express any vector field $X$, whose components are functions $X^1, X^2, \ldots, X^n$, in the form $X = X^i\, E_i$. Evidently the vector fields $E_i, i = 1, 2, \ldots, n$ are linearly independent (over the ring of smooth functions). The ring of smooth functions $C^\infty(\mathbb{R}^n)$ is commutative, hence it has an invariant basis number, and consequently the formula $X = X^i\, E_i$ shows that $\mathfrak{D}$ is the free left $C^\infty(\mathbb{R}^n)$-module of rank $n$. We can consider the basis $\{E_1, E_2, \ldots, E_n\}$ for the free left

$C^\infty(\mathbb{R}^n)$-module $\mathfrak{D}$ as the *frame field*, i.e. as the mapping which attaches to each point $p$ of the Euclidean space the frame $\{\vec{e}_{p,i}\}_{i=1}^n$ of tangent space $\mathrm{T}_p\mathbb{R}^n$. We will denote this frame field by $E$ and call it the *canonical frame field* for the tangent bundle $\mathrm{T}\mathbb{R}^n$. Obviously the canonical frame field $E$ is orthonormal, i.e. $< E_i, E_j >= \delta_{ij}$.

### *14.2.2 Vector Field as the Directional Derivative*

Let $f$ be a smooth function and $X$ be a vector field in $\mathbb{R}^n$. A vector field $X$ at a point $p$ is the tangent vector $X_p = (p; \vec{v}) \in \mathrm{T}_p\mathbb{R}^n$. How we can measure a rate of change of a function $f$ at a point $p$ in the direction of tangent vector $X_p$? For this purpose we can use the *directional derivative of a function*. Evidently there is a parametrized curve $\alpha : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an open interval, such that $\alpha(0) = p$ (curve passes through a point $p$), and $\vec{\alpha}'(0) = X_p$ (the tangent vector to a curve at a point $p$ is $X_p$). For instance one can take the straight line $\alpha(t) = p + t\,\vec{v}$. Then the directional derivative $Xf$ at a point $p$ is defined by

$$Xf\,(p) = \frac{d}{dt}(f \circ \alpha(t))|_{t=0}. \tag{14.2.2}$$

The directional derivative of a function $f$ determines a new smooth function $Xf$, whose value at any point of the Euclidean space is defined by (14.2.2). Hence a vector field $X$ induces the directional derivative and can be considered as a linear mapping $X : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, which satisfies the Leibniz rule $X(fg) = (Xf)\,g + f\,(Xg)$. An approach to a vector field $X$ as the directional derivative is very useful because it makes clear an algebraic nature of a vector field. We remind that a linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ of an algebra $\mathscr{A}$ is said to be a *derivation* if it satisfies $\delta(ab) = \delta(a)\,b + a\,\delta(b)$, where $a, b \in \mathscr{A}$. Thus a vector field considered as a directional derivative of a function is a derivation of the algebra $C^\infty(\mathbb{R}^n)$. It can be proved that the vector space of all derivations of the algebra $C^\infty(\mathbb{R}^n)$ coincides with the vector space of vector fields $\mathfrak{D}$, but generally if we consider the algebra of $N$th order differentiable functions this is not the case.

It can be shown that if we consider a vector field $X$ in the Euclidean space $\mathbb{R}^n$ as the derivation of the algebra $C^\infty(\mathbb{R}^n)$ then we can identify a vector field $X$ with the first order differential operator

$$X = X^i \frac{\partial}{\partial x^i}. \tag{14.2.3}$$

This formula is equivalent to $X = X^i\,E_i$, because the constant vector field $E_i$, considered as the directional derivative, can be identified with partial derivative $\frac{\partial}{\partial x^i}$. Consequently the vector fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ form the basis for the free left $C^\infty(\mathbb{R}^n)$-module $\mathfrak{D}$ of vector fields. From this it follows that at any point $p$ of the

$n$-dimensional Euclidean space there is the canonical basis $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$ for the tangent space $\mathrm{T}_p\mathbb{R}^n$.

Next algebraic structure, which plays an important role in the theory of vector fields, is a Lie algebra. We remind that a vector space $\mathfrak{g}$ is said to be a *Lie algebra* if it is equipped with a *Lie bracket* $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which for any $x, y, z \in \mathfrak{g}$ satisfies

- $[x, y] = -[y, x]$ (*skew-symmetry*),
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (*Jacobi identity*).

Consider the *commutator* of two vector fields $[X, Y] = X \circ Y - Y \circ X$. It is easy to verify that because of the symmetry (Schwarz's theorem) of second order partial derivatives the commutator of two vector fields is the vector field. Indeed if $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ then

$$[X, Y] = (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i})\frac{\partial}{\partial x^j}. \tag{14.2.4}$$

Clearly the commutator is skew-symmetric, and it can be checked by straightforward calculation that it satisfies the Jacobi identity. Hence the commutator of two vector fields is the Lie bracket, and it determines the structure of Lie algebra in $\mathfrak{D}$. It should be pointed out that this Lie algebra is infinite-dimensional.

### 14.2.3 Differential 1-Forms in the Euclidean Space $\mathbb{R}^n$

A calculus of differential forms is dual to the calculus of vector fields. Let $\mathrm{T}^*_p\mathbb{R}^n$ be the dual or *cotangent space* of the tangent space $\mathrm{T}_p\mathbb{R}^n$ at a point $p$. We remind that in the case of finite dimensional vector spaces the dual space $V^*$ of a vector space $V$ is the vector space of all linear $\mathbb{R}$-valued functions on $V$. We will call the elements of dual space *covectors*. We write an element of the cotangent space $\mathrm{T}^*_p\mathbb{R}^n$ at a point $p$ as the pair $(p; \phi)$, where $p$ is a point of Euclidean space and $\phi : \mathbb{R}^n \to \mathbb{R}$ is a linear function, and define $(p; \phi)(\vec{v}_p) = \phi(\vec{v})$, where $\vec{v}_p = (p; \vec{v}) \in \mathrm{T}_p\mathbb{R}^n$. Consider the disjoint union

$$\mathrm{T}^*\mathbb{R}^n = \cup_{p\in\mathbb{R}^n}\mathrm{T}^*_p\mathbb{R}^n,$$

which will be referred to as the *cotangent bundle over the Euclidean space* $\mathbb{R}^n$. Define the projection $\tilde{\pi} : \mathrm{T}^*\mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{\pi}(p; \phi) = p$. Then a differential form of degree 1 or 1-form $\omega$ is a smooth section of the cotangent bundle $\omega : \mathbb{R}^n \to \mathrm{T}^*\mathbb{R}^n$, i.e. it satisfies $\tilde{\pi} \circ \omega = \mathrm{id}_{\mathbb{R}^n}$. Hence a differential 1-form is a smooth mapping $\omega : p \mapsto \omega_p \in \mathrm{T}^*_p\mathbb{R}^n$.

The vector space structure of $\mathrm{T}^*_p\mathbb{R}^n$ induces the vector space structure in the set of all 1-forms, and this vector space will be denoted by $\Omega^1(\mathbb{R}^n)$. The infinite-dimensional vector space of 1-forms can be considered as a bimodule over the algebra of smooth functions. Indeed given a smooth function $f$ and a 1-form $\omega$ one can define the product $f \cdot \omega$ (left multiplication of 1-forms by functions) as the 1-form such that $(f \cdot \omega)_p = f(p)\,\omega_p$, where $p$ is a point of $\mathbb{R}^n$. Since real numbers commute with covectors the right-hand side of this formula can be written as $\omega_p\, f(p)$, which means that we can define the product $\omega \cdot f$ (right multiplication of 1-forms by functions) by simply setting it equal to $f \cdot \omega$. These two products $f \cdot \omega$ and $\omega \cdot f$ determine the $C^\infty(\mathbb{R}^n)$-bimodule structure of $\Omega^1(\mathbb{R}^n)$. Next we define the value of differential 1-form $\omega$ on a vector field $X$ as the function $f = \omega(X)$, whose value at a point $p$ is defined by $f(p) = \omega_p(X_p)$. Evidently for any functions $g, h$ and any vector fields $X, Y$ it holds $\omega(gX + hY) = f\,\omega(X) + g\,\omega(Y)$. This shows that a differential 1-form $\omega$ determines the homomorphism $\omega : \mathfrak{D} \to C^\infty(\mathbb{R}^n)$ of $C^\infty(\mathbb{R}^n)$-bimodules. Two differential forms $\omega_1, \omega_2$ are equal $\omega_1 \equiv \omega_2$ iff for any vector field $X$ it holds $\omega_1(X) = \omega_2(X)$. Hence a 1-form $\omega$ is uniquely determined if we show how to compute its value on any vector field $X$ (this dependence on a vector field should be $C^\infty(\mathbb{R}^n)$-linear). We can apply this way of constructing differential 1-forms to functions. Indeed given a function $f \in C^\infty(\mathbb{R}^n)$ we can define the differential 1-form $df$ by means of the formula

$$df(X) = Xf, \tag{14.2.5}$$

where $X$ is a vector field. Hence any smooth function $f$ induces the differential 1-form $df$, i.e. we have the linear mapping $f \in C^\infty(\mathbb{R}^n) \to df \in \Omega^1(\mathbb{R}^n)$. Because a vector field $X$ is the derivation of the algebra of smooth functions, it holds

$$d(fg)(X) = X(fg) = (Xf)\,g + f\,(Xg) = df(X)\,g + f\,dg(X),$$

or, omitting a vector field $X$ and making use of $C^\infty(\mathbb{R}^n)$-bimodule structure of $\Omega^1(\mathbb{R}^n)$, we can write

$$d(fg) = df \cdot g + f \cdot dg. \tag{14.2.6}$$

We see that the formula $df(X) = Xf$ defines the linear mapping $d : C^\infty(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ from the algebra to bimodule over this algebra, which satisfies (14.2.6).

Now our aim is to find an expression for a 1-form $\omega$ in the coordinates $x^i$ of the $n$-dimensional Euclidean space $\mathbb{R}^n$. For this purpose we remind that if $V^*$ is the dual space of a finite dimensional vector space $V$, $\vec{e}_i$ is a basis for $V$ then the elements $e^i$ of the dual space $V^*$ defined by $e^i(\vec{e}_j) = \delta^i_j$ form the basis for $V^*$, which is called the dual basis of $\vec{e}_i$. Any element (covector) $\phi$ of the dual space $V^*$ can be expressed in terms of dual basis as $\phi = \phi_i\, e^i$, where $\phi_i$ are real numbers. We also remind that $x^i$ are regarded as the coordinate functions on the Euclidean space $\mathbb{R}^n$. Hence each coordinate function $x^i$ induces the 1-form $dx^i$, and, according to the definition, we have

$$dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

This shows that at any point $p$ the covectors $dx^1_p, dx^2_p, \dots, dx^n_p$ form the basis for the cotangent space $\mathrm{T}^*_p\mathbb{R}^n$, which is dual to the canonical basis $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \dots, \frac{\partial}{\partial x^n}|_p$. From this it is easy to conclude that any 1-form $\omega$ can be expressed as follows $\omega = \omega_i\, dx^i$, where $\omega_i$ are smooth functions. This also shows that the 1-forms $dx^i$ form the basis for the bimodule of 1-forms $\Omega^1(\mathbb{R}^n)$ over the algebra $C^\infty(\mathbb{R}^n)$. Hence $\Omega^1(\mathbb{R}^n)$ is free left (or right) $C^\infty(\mathbb{R}^n)$-module of rank $n$.

### *14.2.4 Non-commutative First Order Differential Calculus*

Now we can draw some conclusions from the previous considerations. The important conclusion is that the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ is the basic structure for the calculus of vector fields and differential 1-forms. Indeed we see that a vector field can be identified with the derivation of this algebra and differential 1-forms $\Omega^1(\mathbb{R}^n)$ can be considered as elements of the bimodule over this algebra. This observation underlies an approach used in non-commutative geometry. The algebra $C^\infty(\mathbb{R}^n)$ is commutative, but we can consider a non-commutative algebra, which by its properties should be close, in a sense, to $C^\infty(\mathbb{R}^n)$. This non-commutative algebra will mimic an algebra of functions on our space, and, making use of this algebra, we can then develop structures of differential geometry such as a calculus of vector fields, differential forms and so on. Peculiar property of this approach to geometry is that we do not need a notion of a point of our space because the only thing which we use to develop a differential geometry is an algebra of functions. It is worth to mention that our main goal is the algebraic aspect of noncommutative geometry approach, that is, we ignore the topological questions of functional spaces.

Let $\mathscr{A}$ be a unital associative algebra, which is not necessarily commutative. In order to be able to model the algebraic aspect of the calculus of differential forms developed in the previous subsection, we should have a bimodule over this algebra. We will denote this $\mathscr{A}$-bimodule by $\mathscr{M}$. Now we can give a general definition of a *first order differential calculus over a unital associative algebra* [14]. A triple $(\mathscr{A}, d, \mathscr{M})$, where $\mathscr{A}$ is a unital associative algebra, $\mathscr{M}$ is an $\mathscr{A}$-bimodule and $d : \mathscr{A} \to \mathscr{M}$ is a linear mapping, is said to be a first order differential calculus over an algebra $\mathscr{A}$ if $d$ satisfies the Leibniz rule $d(ab) = da \cdot b + a \cdot db$, where $a, b \in \mathscr{A}$. If $\mathscr{A}$ is a commutative (non-commutative) algebra then a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ is called a commutative (non-commutative) first order differential calculus. Particularly the triple $(C^\infty(\mathbb{R}^n), d, \Omega^1(\mathbb{R}^n))$, where $d$ is defined by (14.2.5), is the commutative first order differential calculus over the algebra of smooth functions in the $n$-dimensional Euclidean space $\mathbb{R}^n$.

Given a unital associative algebra $\mathscr{A}$ one can construct a universal first order differential calculus. Indeed the tensor product $\mathscr{A}^{\otimes 2} = \mathscr{A} \otimes \mathscr{A}$ is the $\mathscr{A}$-bimodule, where the left and right multiplications by elements of $\mathscr{A}$ are defined by

$$a \cdot (b \otimes c) = (ab) \otimes c, \quad (b \otimes c) \cdot a = b \otimes (ca).$$

For any $a \in \mathscr{A}$ define the linear mapping $d : \mathscr{A} \to \mathscr{A}^{\otimes 2}$ by $da = e \otimes a - a \otimes e$, where $e$ is the identity element of $\mathscr{A}$. Applying this mapping to product of two elements

$$\begin{aligned} d(ab) &= e \otimes (ab) - (ab) \otimes e = e \otimes (ab) - a \otimes b + a \otimes b - (ab) \otimes e \\ &= (e \otimes a - a \otimes e)\, b + a\, (e \otimes b - b \otimes e) = da \cdot b + a \cdot db, \end{aligned}$$

we see that $d$ satisfies the Leibniz rule and thus $(\mathscr{A}, d, \mathscr{A}^{\otimes 2})$ is the first order differential calculus, which is referred to as the *universal first order differential calculus* over $\mathscr{A}$ [7].

A first order differential calculus over an algebra is very general algebraic concept and in order to make it more close to differential 1-forms in the $n$-dimensional Euclidean space we can use the fact that $\Omega^1(\mathbb{R}^n)$ is the free left (or right) $C^\infty(\mathbb{R}^n)$-module of rank $n$ and the set $\{dx^i\}_{i=1}^n$ of 1-forms can be taken as the basis for this module. Let $(\mathscr{A}, d, \mathscr{M})$ be a non-commutative first order differential calculus over an algebra $\mathscr{A}$. Assume that the right $\mathscr{A}$-module $\mathscr{M}$ is free module of rank $n$ and $\xi^i, i = 1, 2, \ldots, n$ is a basis for this module. In analogy with the differential calculus in the Euclidean space $\mathbb{R}^n$ one can define the right partial derivatives $\partial_i : \mathscr{A} \to \mathscr{A}$ by $da = \xi^i\, \partial_i a$. In this case a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ is called a first order differential calculus *with right partial derivatives* (r.p.d.). Now the left $\mathscr{A}$-module structure of $\mathscr{M}$ induces the mappings $H^i_j : a \in \mathscr{A} \mapsto H^i_j(a) \in \mathscr{A}$ defined by the formula $a\, \xi^i = \xi^j\, H^i_j(a)$, where $a$ is an element of algebra $\mathscr{A}$. It can be proved [4] that the right partial derivatives $\partial_i$ satisfy the *twisted Leibniz rule*

$$\partial_i(ab) = (\partial_i a)\, b + H_i^j(a)\, (\partial_j b), \;\; a, b \in \mathscr{A}. \tag{14.2.7}$$

We can compose the $n$th order matrix $H(a) = (H_i^j(a))$ over $\mathscr{A}$ by positioning the element $H_i^j(a)$ at the intersection of $j$th column and $i$th row. It can be proved that $H(ab) = H(a)\, H(b)$, which shows that $H$ is the homomorphism from an algebra $\mathscr{A}$ to the algebra of $n$th order matrices $\mathrm{Mat}_n(\mathscr{A})$ over $\mathscr{A}$.

Next assume that a unital associative algebra $\mathscr{A}$ is generated by variables $x^i, i = 1, 2, \ldots, n$ which obey the relations $f_\alpha(x^1, x^2, \ldots, x^n) = 0, \alpha = 1, 2, \ldots, m$, where each $f_\alpha(x^1, x^2, \ldots, x^n)$ is the finite polynomial of variables $x^1, x^2, \ldots, x^n$ and $dx^i = \xi^i$. In this case a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ with r.p.d. is called a *coordinate first order differential calculus* [4]. Clearly in this case the generators $x^1, x^2, \ldots, x^n$ can be viewed as analogs of coordinate functions.

### *14.2.5 Two Dimensional Quantum Space*

A well known example of first order non-commutative differential calculus can be constructed in the case of the quantum hyperplane [13]. As it was mentioned before, in non-commutative geometry approach one constructs and studies various structures of differential geometry in a non-commutative space by means of algebra of functions on this space. The algebra of functions on the *quantum hyperplane* is the algebra of finite polynomials over $\mathbb{C}$ generated by variables $x^1, x^2, \ldots, x^n$, which obey relations

$$x^i x^j = q\, x^j x^i, \tag{14.2.8}$$

where $q$ is a non-zero complex number. The generators of the algebra $x^1, x^2, \ldots, x^n$ can be considered as the coordinate functions on the quantum hyperplane. Particularly if $n = 2$ then the algebra of functions generated by $x, y$, which obey the relations

$$xy = q\, yx, \tag{14.2.9}$$

will be referred to as the algebra of functions on the *quantum plane* and denoted by $\mathfrak{C}_q$. Our aim in this subsection is to construct a first order differential calculus over the algebra of functions on the quantum plane. It is useful to write the relation (14.2.9) in the form $r(x, y) = 0$, where $r(x, y) = xy - q\, yx$.

According to the notion of first order differential calculus over an algebra explained in the previous subsection, we have to construct a $\mathfrak{C}_q$-bimodule $\mathfrak{M}_q$ together with a differential $d : \mathfrak{C}_q \to \mathfrak{M}_q$, which satisfies the Leibniz rule. For this purpose we consider the right $\mathfrak{C}_q$-module $\mathfrak{M}_q$ freely generated by $\xi, \eta$. We define the $\mathfrak{C}_q$-bimodule structure of $\mathfrak{M}_q$ by putting

$$x\xi = \xi\, H^1_1(x) + \eta\, H^1_2(x),\;\; x\eta = \xi\, H^2_1(x) + \eta\, H^2_2(x), \tag{14.2.10}$$

$$y\xi = \xi\, H^1_1(y) + \eta\, H^1_2(y),\;\; y\eta = \xi\, H^2_1(y) + \eta\, H^2_2(y), \tag{14.2.11}$$

where

$$H : x \mapsto \begin{pmatrix} H^1_1(x) & H^2_1(x) \\ H^1_2(x) & H^2_2(x) \end{pmatrix}, \quad H : y \mapsto \begin{pmatrix} H^1_1(y) & H^2_1(y) \\ H^1_2(y) & H^2_2(y) \end{pmatrix}, \tag{14.2.12}$$

is a homomorphism from the algebra of functions $\mathfrak{C}_q$ to the algebra of $2\times 2$-matrices over $\mathfrak{C}_q$ defined on the generators. Hence for any two functions $f, g \in \mathfrak{C}_q$ it holds $H(fg) = H(f)H(g)$. Now we can define a differential $d : \mathfrak{C}_q \to \mathfrak{M}_q$. Since differential is a linear mapping and it satisfies the Leibniz rule, it suffices to define it on the generators $x, y$. In order to have a coordinate first order differential calculus with r.p.d. we put $dx = \xi, dy = \eta$. As it is shown in the previous subsection the differential $d$ induces the right partial derivatives

$$df = dx\, \partial_x f + dy\, \partial_y f, \quad f \in \mathfrak{C}_q, \tag{14.2.13}$$

which satisfy the twisted Leibniz rule (14.2.7)

$$\partial_x(fg) = (\partial_x f)\, g + H_1^1(f)\, \partial_x g + H_1^2(f)\, \partial_y g, \tag{14.2.14}$$

$$\partial_y(fg) = (\partial_y f)\, g + H_2^1(f)\, \partial_x g + H_2^2(f)\, \partial_y g. \tag{14.2.15}$$

Thus our first order differential calculus is coordinate differential calculus with r.p.d.

Since we defined the differential $d$ by $dx = \xi, dy = \eta$, the relations (14.2.10) and (14.2.11) can be considered as commutation relations between coordinate functions $x, y$ and their differentials $dx, dy$. It is worth to mention that two matrices $H(x), H(y)$ completely determine the coordinate first order differential calculus with r.p.d. over the algebra of functions on the quantum plane. Hence the matrices (14.2.12) can be considered as parameters of a possible differential calculus. Obviously these matrices should be compatible with the defining relation of quantum plane $r(x, y) = xy - q\, yx = 0$. Hence the matrices $H(x), H(y)$ have to satisfy the following conditions

$$\partial_x r(x, y) = 0,\ \ \partial_y r(x, y) = 0,\ \ H(r(x, y)) = 0. \tag{14.2.16}$$

We find

$$\partial_x(xy - q\, yx) = y + H_1^2(x) - q\, H_1^1(y),\ \ \partial_y(xy - q\, yx) = H_2^2(x) - qx - q\, H_2^1(y).$$

Hence the conditions $\partial_x r(x, y) = 0,\ \partial_y r(x, y) = 0$ imply

$$H_1^2(x) = qH_1^1(y) - y,\ \ \ H_2^2(x) = qH_2^1(y) + qx, \tag{14.2.17}$$

which can be written as

$$H^2(x) = qH^1(y) + \begin{pmatrix} -y \\ qx \end{pmatrix}, \tag{14.2.18}$$

where $H^i(x)(H^i(y))$ is the $i$th column of the matrix $H(x)$ $(H(y))$.

From $H(r(x, y)) = 0$ it follows

$$H^2(y)\, H_2^i(x) = q^{-1} H(x)\, H^i(y) - H^1(y)\, H_1^i(x),\ \ i = 1, 2. \tag{14.2.19}$$

The simplest case is when the matrices $H(x), H(y)$ depend linearly on the coordinates $x, y$. Hence we assume

$$H(x) = Ax + By,\ \ H(y) = Cx + Dy,$$

where $A, B, C, D$ are complex matrices. It follows from (14.2.18) that these matrices must satisfy

$$A^2 = q\,C^1 + \begin{pmatrix} 0 \\ q \end{pmatrix}, \quad B^2 = q\,D^1 + \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{14.2.20}$$

From (14.2.19) it follows that

$$C^2 A_2^i - q^{-1} A C^i + C^1 A_1^i = 0, \tag{14.2.21}$$

$$D^2 A_2^i - A D^i + D^1 A_1^i + q C^2 B_2^i - q^{-1} B C^i + q C^1 B_1^i = 0, \tag{14.2.22}$$

$$D^2 B_2^i - q^{-1} B D^i + D^1 B_1^i = 0. \tag{14.2.23}$$

Since $H(y) = Cx + Dy$ it is natural to seek a solution on the assumption $C = 0$. Then the first condition in (14.2.20) immediately gives

$$A^2 = \begin{pmatrix} 0 \\ q \end{pmatrix}.$$

Thus $A_1^2 = 0$, $A_2^2 = q$. Now the condition (14.2.21) is identically satisfied and the condition (14.2.22) takes the form

$$D^2 A_2^i - A D^i + D^1 A_1^i = 0. \tag{14.2.24}$$

The second natural assumption is that the matrices $A, D$ are diagonal, i.e.

$$A = \begin{pmatrix} q^2 & 0 \\ 0 & q \end{pmatrix}, \quad D = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are different from zero. Then from the second formula in (14.2.20) we get $B_1^2 = q\gamma_1 - 1$, $B_2^2 = 0$. It is easy to verify that now the condition (14.2.24) is identically satisfied, while Eq. (14.2.24) gives

$$(1 - q^{-1})\gamma_1 B_1^1 = 0, \;\; (\gamma_2 - q^{-1}\gamma_1) B_2^1 = 0, \;\; (\gamma_1 - q^{-1}\gamma_2) B_1^2 = 0. \tag{14.2.25}$$

Since we assume $\gamma_1 \neq 0$, it follows from the first relation that $B_1^1 = 0$. The second and third relations have a symmetric form and we can solve them either by putting $\gamma_2 - q^{-1}\gamma_1 = 0$, $B_1^2 = 0$ or $\gamma_1 - q^{-1}\gamma_2 = 0$, $B_2^1 = 0$ (other choices lead either to restriction of $q$, which is unacceptable, or to $B = 0$, which makes the whole construction very indeterminate). In order to be more specific, we take $\gamma_1 - q^{-1}\gamma_2 = 0$, $B_2^1 = 0$ and fix $\gamma_1 = q$. Then $\gamma_2 = q^2$, and we finally obtain the well known first order coordinate differential calculus with r.p.d. on the quantum plane

$$x\,dx = q^2 dx\,x, \quad x\,dy = (q^2 - 1)\,dx\,y + q\,dy\,x, \tag{14.2.26}$$

$$y\,dx = q\,dx\,y, \quad y\,dy = q^2\,dy\,y. \tag{14.2.27}$$

## 14.3 Algebra of Differential Forms and Differential Graded Algebra

In this section we describe the higher order differential forms in the $n$-dimensional Euclidean space. We lay particular stress on an algebraic structure of differential forms and bring a reader to idea of a notion of differential graded algebra. We analyze in details the structure of differential graded algebra by pointing out that it contains the first order differential calculus. We explain the notion of universal differential graded algebra over a first order differential calculus.

### *14.3.1 Algebra of Differential Forms in* $\mathbb{R}^n$

In the previous section we showed how one can construct the calculus of differential 1-forms in the $n$-dimensional Euclidean space and its possible generalizations within the framework of noncommutative geometry. In order to continue this construction to higher degree differentials forms in $\mathbb{R}^n$ we attach to each point $p$ of the Euclidean space a vector space of totally skew-symmetric multilinear real-valued $k$-forms $\wedge^k(\mathrm{T}_p^*\mathbb{R}^n)$. The disjoint union $\wedge^k(\mathrm{T}^*\mathbb{R}^n) = \cup_p \wedge^k (\mathrm{T}_p^*\mathbb{R}^n)$ is referred to as the *vector bundle of exterior k-forms* over the Euclidean space $\mathbb{R}^n$. An element of this bundle can be written in the form $(p;\varphi)$, where $p$ is a point of the Euclidean space and $\varphi$ is a totally skew-symmetric multilinear $k$-form on the vector space $\mathbb{R}^n$, that is

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \ (k \text{ times}) \to \mathbb{R},$$

which for any permutation $\sigma = (i_1, i_2, \ldots, i_k)$ of integers $(1, 2, \ldots, k)$ satisfies

$$\varphi(\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_k}) = (-1)^{|\sigma|}\,\varphi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k),$$

where $|\sigma|$ is the parity of a permutation. We consider the pair $\varphi_p = (p;\varphi) \in \wedge^k(\mathrm{T}_p^*\mathbb{R}^n)$ as the $k$-form on the tangent space $T_p\mathbb{R}^n$, where

$$\varphi_p(\vec{v}_{p;1}, \vec{v}_{p;2}, \ldots, \vec{v}_{p;k}) = \varphi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k), \quad \vec{v}_{p;i} = (p;\vec{v}_i).$$

The projection $\pi_{(k)} : \wedge^k(\mathrm{T}^*\mathbb{R}^n) \to \mathbb{R}^n$ is defined in the natural way $\pi_{(k)}\,\varphi_p = p$. A smooth section $\theta : \mathbb{R}^n \to \wedge^k(\mathrm{T}^*\mathbb{R}^n)$ of the vector bundle of exterior $k$-forms is referred to as a differential $k$-form. The vector space of all differential $k$-forms will be denoted by $\Omega^k(\mathbb{R}^n)$ and the degree of a differential $k$-form $\theta$ will be denoted by $|\theta|$, i.e. $|\theta| = k$. Similar to differential 1-forms we can define the products $f\theta, \theta f$, where $f$ is a smooth function, by means of point-wise multiplication and since scalars commute with vectors we have $f\theta = \theta f$. Hence the vector space $\Omega^k(\mathbb{R}^n)$ can be regarded as the bimodule over the algebra

$C^\infty(\mathbb{R}^n)$. The value of a differential $k$-form on vector fields $X_1, X_2, \ldots, X_n$ is the function $f = \theta(X_1, X_2, \ldots, X_n)$, whose value at a point $p$ is defined by $f(p) = \theta_p((X_1)_p, (X_2)_p, \ldots, (X_k)_p)$.

Let $\omega$ be a differential $k$-form and $\theta$ be a differential $l$-form. The wedge product $\omega \wedge \theta$ of two differential forms $\omega, \theta$ is the differential $(k+l)$-form, which is defined by

$$\omega \wedge \theta(X_1, X_2, \ldots, X_{k+l}) = \sum_\sigma (-1)^{|\sigma|} \omega(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) \times \theta(X_{j_1}, X_{j_2}, \ldots, X_{j_l}), \quad (14.3.1)$$

where $\sigma = (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l)$ is a permutation of integers $(1, 2, \ldots, k+l)$ such that $i_1 < i_2 < \ldots < i_k$, $j_1 < j_2 < \ldots < j_l$ and sum is taken over all such permutations. It can be proved that the wedge product of differential forms has the following properties:

(i) $\omega \wedge \theta = (-1)^{|\omega||\theta|} \theta \wedge \omega$,
(ii) $(\omega \wedge \theta) \wedge \chi = \omega \wedge (\theta \wedge \chi)$, i.e. the wedge product of differential forms is associative.

It is useful to add the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ to the sequence $\Omega^k(\mathbb{R}^n)$, $k = 1, 2, \ldots$, of the vector spaces of differential forms by assigning degree zero to functions. Hence we identify the vector space of differential 0-forms with $C^\infty(\mathbb{R}^n)$, i.e. $\Omega^0(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$. In order to complete the construction of algebra of differential forms we introduce the direct sum of vector spaces $\Omega(\mathbb{R}^n) = \oplus_i \Omega^i(\mathbb{R}^n)$. Evidently $\Omega(\mathbb{R}^n)$ is closed under the wedge product of differential forms and hence it is the associative unital algebra, which is called the algebra of differential forms in the $n$-dimensional Euclidean space. By unital we mean that the constant function **1**, whose value at any point is one, can be taken as the identity element of the algebra of differential forms.

Now we remind the notion of a graded algebra. A unital associative algebra $\mathscr{A}$ is called a *graded algebra* if $\mathscr{A} = \oplus_{k\in\mathbb{Z}} \mathscr{A}^k$ and for any elements $u \in \mathscr{A}^i$, $v \in \mathscr{A}^j$ it holds $u \cdot v \in \mathscr{A}^{i+j}$. If $u \in \mathscr{A}^i$ then $u$ is an element of degree $i$ and we will denote its degree by $|u|$. An element of graded algebra, which has the certain degree, is called homogeneous. A graded algebra is said to be a graded commutative if for any two homogeneous elements $u, v \in \mathscr{A}$ it holds $u \cdot v = (-1)^{|u||v|} v \cdot u$. It is useful to introduce the *graded commutator* $[u, v]$ of two homogeneous elements $u, v$ by $[u, v] = u \cdot v - (-1)^{|u||v|} v \cdot u$. Then the condition of graded commutativity can be given in the form $[u, v] = 0$.

Making use of the notion of graded algebra, we can say that the algebra of differential forms is the graded algebra because for any two homogeneous forms $\omega, \theta$ it holds $|\omega \wedge \theta| = |\omega| + |\theta|$. Moreover, because of the first property of the wedge product, the algebra of differential forms is graded commutative.

A differential 1-form $\omega$ can be expressed in terms of coordinate functions $x^i$ of the $n$-dimensional Euclidean space as $\omega = \omega_i \, dx^i$, where the coefficients $\omega_i$ are the smooth functions and the 1-forms $dx^i$ form the basis for the bimodule $\Omega^1(\mathbb{R}^n)$. It follows from the properties of the wedge product that $dx^i \wedge dx^j = -dx^j \wedge dx^i$ or, equivalently, $dx^i \wedge dx^i = 0$. It is easy to show that differential 2-forms $dx^i \wedge dx^j$, where $i < j$, form the basis for the bimodule of 2-forms $\Omega^2(\mathbb{R}^n)$ and any differential 2-form $\theta$ can be written as follows

$$\theta = \frac{1}{2}\,\theta_{ij}dx^i \wedge dx^j,$$

where indices $i, j$ run independently from 1 to $n$ and the functions $\theta_{ij}$ satisfy $\theta_{ij} = -\theta_{ji}$. Analogously any differential $k$-form can be written as follows

$$\theta = \frac{1}{k!}\theta_{i_1 i_2 \ldots i_k}\, dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_k}, \tag{14.3.2}$$

where the functions $\theta_{i_1 i_2 \ldots i_k}$ are totally skew-symmetric under permutations of subscripts. From the expression for a differential $k$-form (14.3.2) and the property $dx^i \wedge dx^i = 0$ it follows that the highest degree of non-trivial differential form in the $n$-dimensional Euclidean space is $n$. Hence $\Omega(\mathbb{R}^n) = \oplus_{k=0}^n \Omega^k(\mathbb{R}^n)$.

Finally we would like to point out that at any fixed point $p$ of the Euclidean space the wedge product of differential forms induces the wedge products of covectors $dx^1|_p, dx^2|_p, \ldots, dx^n|_p$, which are subjected to the commutation relations

$$dx^i|_p \wedge dx^j|_p = -dx^j|_p \wedge dx^i|_p.$$

These relations show that $dx^i|_p$ are the generators of *Grassmann algebra* $\wedge(\mathrm{T}^*_p\mathbb{R}^n) = \oplus_k \wedge^k (\mathrm{T}^*_p\mathbb{R}^n)$, which is called the *exterior algebra of the cotangent space* $\mathrm{T}^*_p\mathbb{R}^n$.

The exterior differential $d$ of the algebra of differential forms is defined as follows:

(i) if $f$ is a smooth function then the exterior differential $df$ is the 1-form defined for any vector field $X$ by $df(X) = Xf$,
(ii) for any differential $k$-form $\theta$ the exterior differential $d\theta$ is the differential $(k+1)$-form defined by the formula

$$\begin{aligned} d\theta(X_1, X_2, .., X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1}X_i\theta(X_1, X_2, .., \hat{X}_i, .., X_{k+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\theta([X_i, X_j], X_1, X_2, .., \hat{X}_i, .., \hat{X}_j, .., X_{k+1}), \end{aligned}$$

where hat over $X_i$ means that this vector field is omitted.

It can be proved that the exterior differential has the following properties:

1. the exterior differential has the degree 1, i.e. $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$,
2. for any homogeneous forms $\omega, \theta$ it holds $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{|\omega|}\omega \wedge d\theta$, and this property is referred to as the *graded Leibniz rule*,
3. $d^2 = 0$.

It should be mentioned that the exterior differential $d$ is uniquely determined by the above properties. The last property is very important and it is the key property for a concept of *de Rham cohomology*.

### *14.3.2 Non-commutative Higher Order Differential Calculus*

In the previous subsection it was shown that the triple $(C^\infty(\mathbb{R}^n), d, \Omega^1(\mathbb{R}^n))$ is the commutative first order differential calculus in the Euclidean space $\mathbb{R}^n$ and this calculus was included as the subalgebra of the algebra of differential forms $\Omega(\mathbb{R}^n)$ by assigning degree zero to smooths functions, i.e. $\Omega^0(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$. Hence we can look at the algebra of differential forms with exterior differential $d$ as the extension of the first order differential calculus $(C^\infty(\mathbb{R}^n), d, \Omega^1(\mathbb{R}^n))$ to higher degree differentials forms, which satisfies the listed above properties of exterior differential.

A general approach to this kind of extensions of first order differential calculus is provided by the notion of differential graded algebra. A *differential graded algebra* (DGA) $\mathscr{G}$ is a graded algebra $\mathscr{G} = \oplus_k \mathscr{G}^k$ endowed with a linear mapping $d : \mathscr{G}^k \to \mathscr{G}^{k+1}$ of degree 1, which satisfies the graded Leibniz rule $d(uv) = du\, v + (-1)^{|u|}u\, dv$, where $u, v \in \mathscr{G}$, $|u|$ is the degree of $u$, and $d^2 = 0$. Hence the algebra of differential forms in the Euclidean space $\mathbb{R}^n$ is the commutative differential graded algebra.

Firstly it follows from the definition of a DGA that the subspace of elements of degree zero $\mathscr{G}^0$ is the *subalgebra* of $\mathscr{G}$. Indeed for $u, v \in \mathscr{G}^0$ we have $|uv| = |u| + |v| = 0$, which means that the product of two degree zero elements $uv$ is the element of degree zero, hence $uv \in \mathscr{G}^0$. Secondly any subspace $\mathscr{G}^k$ of elements of degree $k \geq 0$ is the $\mathscr{G}^0$-bimodule. Indeed if we multiply an element $w \in \mathscr{G}^k$ of degree $k$ by an element $u$ of degree zero either from the left or from the right then the degree of products is $|w|+|u| = |w| = k$, and thus the products are the elements of $\mathscr{G}^k$. Consequently the multiplication by elements of degree zero determines the mappings $\mathscr{G}^0 \times \mathscr{G}^k \to \mathscr{G}^k$, $\mathscr{G}^k \times \mathscr{G}^0 \to \mathscr{G}^k$, and it is easy to verify that all axioms of bimodule are fulfilled. Thirdly the triple $(\mathscr{G}^0, d, \mathscr{G}^1)$ is the first order differential calculus over the algebra of elements of degree zero $\mathscr{G}^0$, because the graded Leibniz rule in the case of zero degree elements reduces to ordinary Leibniz rule. This suggests the following definition: If $(\mathscr{A}, d, \mathscr{M})$ is a first order differential calculus and $\mathscr{G}$ is a DGA with differential $d'$ such that $\mathscr{G}^0 \equiv \mathscr{A}, \mathscr{G}^1 \equiv \mathscr{M}$ and $d'$ coincides with $d$, when restricted to $\mathscr{G}^0$, then a DGA $\mathscr{G}$ will be referred to as a *higher order differential calculus over a first order differential calculus* $(\mathscr{A}, d, \mathscr{M})$.

Assume $(\mathscr{A}, d, \mathscr{M})$ is a first order differential calculus and $\mathscr{G}$ is a higher order differential calculus over $\mathscr{A}$. A first order differential calculus, where $\mathscr{A}$-bimodule $\mathscr{M}$ does not contain unnecessary elements, is of most interest in a theory of higher order calculus over an algebra. Hence we are most interested in the case, where $\mathscr{M}$ is generated by elements of $\mathscr{A}$ and their differentials, i.e. $\mathscr{M} = \mathscr{A}\, d\mathscr{A}\, \mathscr{A}$. But it immediately follows from the Leibniz rule that $\mathscr{M} = \mathscr{A}\, d\mathscr{A}\, \mathscr{A} = d\mathscr{A}\, \mathscr{A}$. Indeed we have $a\, db = d(ab) - da\, b$, which implies $a\, db\, c = (d(ab) - da\, b)\, c = d(ab)\, c - da\, (bc)$. It can be proved [12] that if $(\mathscr{A}, d, \mathscr{M})$ is a first order differential calculus, where $\mathscr{M} = d\mathscr{A}\, \mathscr{A}$, then there exists a DGA $\mathscr{G}$ generated by $\mathscr{G}^0 = \mathscr{A}$ such that its differential coincides with $d$, when restricted to $\mathscr{A}$. This DGA is usually referred to as the *universal differential graded algebra* of $(\mathscr{A}, d, \mathscr{M})$.

The structure of the universal differential graded algebra of a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ with right partial derivatives, where $\mathscr{A}$ is generated by a set of variables $x^i, i \in I$ (with relations $f_\alpha = 0$) and the right $\mathscr{A}$-module is freely generated by $\omega^k, k \in K$, is of interest because it is similar to algebra of differential forms in the Euclidean space $\mathbb{R}^n$. Hence we have for the generators $x^i$ of $\mathscr{A}$ and for the $\mathscr{A}$-bimodule structure of $\mathscr{M}$ the following relations

$$f_\alpha(x^i) = 0, \tag{14.3.3}$$

$$x^i \omega^k = \omega^l\, H_l^{ik}, \tag{14.3.4}$$

where $f_\alpha(x^i)$ are finite polynomials and $H_l^{ik} = H_l^k(x^i)$. Now let $\bar{\mathscr{G}}$ be the algebra generated by variables $x^i, \omega^k$, which obey relations (14.3.3), (14.3.4). In order to consider a case more general than a coordinate calculus we assume $dx^i = \omega^k g_k^i$, where $\omega^k \in d\mathscr{A}$, i.e. $d\omega^k = 0$ ($\omega^k$ are closed "differential 1-forms"). Extending a differential $d$ to elements of $\mathscr{M}$ by the graded Leibniz rule and differentiating (14.3.4) and $dx^i = \omega^k g_k^i$ we get

$$\omega^l \omega^m \big(H_m^k(g_l^i) + \partial_m H_l^{ik}\big) = 0, \quad \omega^k \omega^m\, \partial_m g_k^i = 0. \tag{14.3.5}$$

Now consider the algebra $\mathscr{G}$ generated by $x^i, \omega^k$, which are subjected to the relations (14.3.3), (14.3.4), (14.3.5). This algebra is endowed with differential $d : \mathscr{A} \to \mathscr{M}$. From (14.3.4) it follows that any element of $\mathscr{G}$ can be expressed as follows

$$\omega^{k_1} \omega^{k_2} \dots \omega^{k_n} h_{k_1 k_2 \dots k_n}, \quad h_{k_1 k_2 \dots k_n} \in \mathscr{A}. \tag{14.3.6}$$

This implies that $\mathscr{G}$ is the graded algebra with the degree of a homogeneous element determined by the number of $\omega^k$ in the expression (14.3.6). Now it can be proved [4] that if we extend a differential $d : \mathscr{A} \to \mathscr{M}$ to the algebra $\mathscr{G}$ by means of the formula

$$d(\omega^{k_1} \omega^{k_2} \dots \omega^{k_n} h_{k_1 k_2 \dots k_n}) = (-1)^n \omega^{k_1} \omega^{k_2} \dots \omega^{k_n} \omega^k\, \partial_k h_{k_1 k_2 \dots k_n}, \tag{14.3.7}$$

then $\mathscr{G}$ is the universal differential graded algebra over coordinate first order differential calculus $(\mathscr{A}, d, \mathscr{M})$.

## 14.4 Connection in Euclidean Space

A concept of connection arises when we consider the problem of *parallel translation of a vector* in the $n$-dimensional Euclidean space $\mathbb{R}^n$. Assume $\alpha : I \to \mathbb{R}^n, I \subset \mathbb{R}$ is a parametrized curve which passes through a point $p = \alpha(0)$. Let $\vec{v}_p = (p; \vec{v}) \in \mathrm{T}_p\mathbb{R}^n$ be a tangent vector at a point $p \in \mathbb{R}^n$ of the space and our goal is to move this vector in parallel way along a curve to some other point of a curve $q = \alpha(t_0), t_0 \in I$. Because we know what does it mean the parallelism in the Euclidean space $\mathbb{R}^n$ the solution is easy. In order to extend a tangent vector $\vec{v}_p$ in parallel way along a curve $\alpha$ we construct the constant vector field $V(\alpha(t)) = (\alpha(t); \vec{v})$ along $\alpha$. But this problem becomes less trivial and leads to interesting geometric structure if we consider this problem of parallel translation of a vector in curvilinear coordinates. Let us assume that $U$ is an open subset of the Euclidean space $\mathbb{R}^n$ and $x'^1, x'^2, \ldots, x'^n$ are curvilinear coordinates determined in $U$. We also assume that these curvilinear coordinates can be expressed in terms of the Cartesian coordinates $x^1, x^2, \ldots, x^n$ by means of smooth functions, i.e. $x'^i = x'^i(x^1, x^2, \ldots, x^n)$, and vice versa the Cartesian coordinates can be expressed in terms of curvilinear coordinates by means of smooth functions $x^i = x^i(x'^1, x'^2, \ldots, x'^n)$. We also assume that the coordinate lines of curvilinear coordinates are orthogonal and hence we can construct the orthonormal frame field $E' = \{E'_1, E'_2, \ldots, E'_n\}$ by means of the vector fields $\frac{\partial}{\partial x'^i}$ (normalizing them if necessary). This orthonormal frame can be expressed in terms of the canonical frame field $E = \{E_1, E_2, \ldots, E_n\}$ as follows $E'_i = g^j_i\, E_j$, where the matrix $G = (g^j_i)$ depends on a point of $U$ and for any $G \in \mathrm{SO}(n)$, i.e. $G\,G^T = I$, $\mathrm{Det}\, G = 1$ and $I$ is the unit matrix.

Now we can write the constant vector field $V(\alpha(t))$ as follows

$$V(\alpha(t)) = V^i(\alpha(t))\, E'_i(\alpha(t)),$$

and our aim is to find unknown functions $V^i(\alpha(t))$. Differentiating both sides with respect to $t$ we get zero at the left-hand side because $V(\alpha(t))$ is the constant vector field. The right-hand side can be written as follows

$$\frac{d}{dt}\big(V^i(\alpha(t))\, E'_i(\alpha(t))\big) = \frac{d}{dt}\big(V^i(\alpha(t))\big)\, E'_i(\alpha(t)) + V^i(\alpha(t))\, \frac{d}{dt}\big(E'_i(\alpha(t))\big).$$

Making use of the definition of directional derivative of a function we can interpret the coefficients $\frac{d}{dt}\big(V^i(\alpha(t))\big)$ in the first sum as the directional derivatives of functions $V^i(\alpha(t)$ in the direction of the tangent vector field $X(\alpha(t)) = (\alpha(t); \vec{\alpha}'(t))$ along a curve, i.e. we can write them as $XV^i$. The derivatives in the second sum

$\frac{d}{dt}\big(E_i'(\alpha(t))\big)$ can be regarded as analogs of directional derivatives for vector fields, and this suggests us to introduce a new derivative for vector fields, which is called a covariant derivative.

Let $X, Y$ be two vector fields, $p \in \mathbb{R}^n$ be a point, $\alpha : I \to \mathbb{R}^n$ be a curve such that $\alpha(0) = p, \vec{\alpha}'(0) = X_p$. The *covariant derivative of a vector field $Y$* with respect to a vector field $X$ at a point $p$ is the tangent vector $(D_X Y)_p \in \mathrm{T}_p\mathbb{R}^n$, which is defined by

$$(D_X Y)_p = \frac{d}{dt}\big(Y|_{\alpha(t)}\big)|_{t=0}.$$

Hence the covariant derivative determines the vector field $p \mapsto (D_X Y)_p$, which will be denoted by $D_X Y$. From this definition it follows that in any curvilinear coordinate system $x'^i$ and at any point $p$ we have

$$(D_X Y)^i_p = \frac{\partial Y^i}{\partial x'^j}|_p \frac{dx'^j}{dt}|_{t=0} = X^j(p)\frac{\partial Y^i}{\partial x'^j}|_p = (XY^i)(p). \tag{14.4.1}$$

Hence $D_X Y$ is the vector field which in curvilinear coordinates $x'^i$ can be written as follows

$$D_X Y = (XY^i)\,\frac{\partial}{\partial x'^i}. \tag{14.4.2}$$

From (14.4.2) it follows that covariant derivative has the following properties:

(i) $D_{X_1+X_2}Y = D_{X_1}Y + D_{X_2}Y,\ D_{fX}Y = f\,D_X Y$;
(ii) $D_X(Y_1 + Y_2) = D_X Y_1 + D_X Y_2,\ D_X(fY) = (Xf)\,Y + f\,D_X Y$;
(iii) $D_X Y - D_Y X = [X, Y]$;
iv) $X < Y, Z >=< D_X Y, Z > + < Y, D_X Z >$.

The property (i) shows that the covariant derivative $D_X Y$ depends linearly on a vector field $X$, and this clearly suggests that we can describe the structure of covariant derivative in the terms of differential 1-forms. Indeed the formula (14.4.2) can be written in the form

$$D_X Y = dY^i(X)\,\frac{\partial}{\partial x'^i}, \tag{14.4.3}$$

where $dY^i$ is the exterior differential of the function $Y^i$. Now our aim is to get rid of a vector field $X$ in the above formula and to use differential 1-forms. For this purpose we consider the right-hand side of (14.4.3) as a vector field valued differential 1-form. In order to assign a geometric meaning to these words we attach to each point $p$ of the space $\mathbb{R}^n$ the tensor product $\wedge^k(\mathrm{T}^*_p\mathbb{R}^n) \otimes \mathrm{T}_p\mathbb{R}^n$ and consider the vector bundle $\wedge^k(\mathrm{T}^*\mathbb{R}^n) \otimes \mathrm{T}\mathbb{R}^n = \cup_p \wedge^k (\mathrm{T}^*_p\mathbb{R}^n) \otimes \mathrm{T}_p\mathbb{R}^n$ with obvious projection. A smooth section of this bundle is referred to as a *vector field valued differential k-form* in the Euclidean space $\mathbb{R}^n$. The vector space of vector field valued differential

$k$-forms will be denoted by $\Omega^k(\mathbb{R}^n, \mathfrak{D})$. It is worth to point out that this vector space can be considered as the left $C^\infty(\mathbb{R}^n)$-module, i.e. we can multiply vector field valued forms by functions from the left. For instance any vector field valued 1-form $\omega$ can be written in the form

$$\omega = \omega^i_j \, dx'^j \otimes \frac{\partial}{\partial x'^i} = \omega^i \otimes \frac{\partial}{\partial x'^i} = dx'^j \otimes X_j,$$

where $\omega^i_j$ are smooth functions, $\omega^i = \omega^i_j \, dx'^j$ are $\mathbb{R}$-valued differential 1-forms and $X_j = \omega^i_j \frac{\partial}{\partial x'^i}$ are vector fields. If $X$ is a vector field and a vector field valued differential 1-form is written as $\omega = \omega^i \otimes \frac{\partial}{\partial x'^i}$ then its value on a vector field $X$ is the vector field defined by

$$\omega(X) = \omega^i(X) \frac{\partial}{\partial x'^i}.$$

Now making use of vector field valued differential forms we can omit a vector field $X$ in the formula (14.4.3) and write it in the equivalent form

$$DY = dY^i \otimes \frac{\partial}{\partial x'^i}. \tag{14.4.4}$$

Clearly for any vector field $X$ we have $DY(X) = D_X Y$. Thus starting with the covariant derivative $D_X Y$ we constructed the mapping $Y \mapsto DY$, which assigns to any vector field the vector field valued differential 1-form. What are the properties of this mapping? Now the property (i) of the covariant derivative is obvious, because $DY$ is the differential 1-form. The property (ii) of covariant derivative shows that $D : \mathfrak{D} \to \Omega^1(\mathbb{R}^n, \mathfrak{D})$ is the linear mapping of vector spaces. The second part of this properties gives

$$D(fY) = df \otimes Y + f DY. \tag{14.4.5}$$

In order to write the property (iv) in the terms of $D$ we must extend the scalar product of vector fields to vector field valued differential form and we can do this by means of the formula

$$< \omega \otimes X, \theta \otimes Y > = < X, Y > \omega \wedge \theta.$$

Particularly the scalar product of vector field valued 1-form $\omega \otimes X$ and a vector field $Y$ is the 1-form $< \omega \otimes X, Y > = < X, Y > \omega$. Now the property (iv) implies

$$d < Y, Z > = < DY, Z > + < Y, DZ >, \tag{14.4.6}$$

and this property is usually referred to as the condition of consistency of the covariant derivative with the metric (inner product) of the Euclidean space $\mathbb{R}^n$. Till now we used the frame field $\{\frac{\partial}{\partial x'^i}\}_{i=1}^n$ for the module of vector fields and the basis $\{dx'^i\}_{i=1}^n$ for the module of differential forms to obtain formulae for the covariant derivatives. Now our aim is to study the structure of covariant derivative with the help of the frame field $E' = \{E'_i\}_{i=1}^n$. Let $\{\theta^i\}_{i=1}^n$ be the dual basis, where $\theta^i$ are differential 1-forms, which satisfy $\theta^i(E'_j) = \delta^i_j$. We can easily find the expression for these differential forms in the terms of Cartesian coordinates $dx^i$. Indeed if we denote $\theta^i = \theta^i_j\, dx^j$ then

$$\delta^i_k = \theta^i(E'_k) = \theta^i(g^j_k\, E_j) = \theta^i_m\, g^j_k\, dx^m(E_j) = \theta^i_m\, g^m_k.$$

Thus $\theta^i_m = (G^{-1})^i_m$ and $\theta^i = (G^{-1})^i_m\, dx^m$.

Given a vector field $X$ we can write it in the frame field $E'$ induced by curvilinear coordinates as follows $X = X^i\, E'_i$. Making use of the properties of covariant derivative we find

$$DX = D(X^i\, E'_i) = dX^i \otimes E'_i + X^i\, DE'_i. \tag{14.4.7}$$

The covariant derivative $DE'_i$ is the vector field valued 1-form and hence it can be expanded as $\omega^j_i \otimes E'_j$, where $\omega^j_i$ are the differential 1-forms. The matrix $\omega = (\omega^j_i)$, whose elements are differential 1-forms, is referred to as the *matrix of connection*. Thus if we fix a frame field (a basis for the module of vector fields) then the covariant derivative induces the matrix of connection, which depends on a choice of a frame field. Before we compute the matrix of connection, we can derive its very important property from the consistency with the Euclidean metric (14.4.6). For two vector fields $E'_i, E'_j$ of the frame field the consistency condition (14.4.6) takes on the form

$$d < E'_i, E'_j >=< DE'_i, E'_j > + < E'_i, DE'_j > .$$

Taking into account that $< E'_i, E'_j >= \delta_{ij}$ and substituting $DE'_i = \omega^k_i \otimes E'_k$, we obtain $\omega^i_j + \omega^j_i = 0$. Thus the matrix of connection is the skew-symmetric matrix $\omega + \omega^T = 0$. If we analyze the origin of this property of the matrix of connection we can see that the reason lies in the orthogonality of the attitude matrix $G = (g^i_j)$, which determines the transformation of the canonical frame field $E$ into the orthonormal frame field $E'$, induced by curvilinear coordinates. We remind that the Lie algebra so$(n)$ of the special orthogonal group SO$(n)$ is the vector space of skew-symmetric matrices, i.e.

$$\mathrm{so}(n) = \{h \in \mathrm{Mat}_n(\mathbb{R}) : h + h^T = 0\}.$$

Consequently we conclude that if we use the matrix group SO$(n)$ for a transition from one frame field to another, or, by other words, we consider the action of the matrix group SO$(n)$ on the set of orthonormal frames of the tangent space $\mathrm{T}_p\mathbb{R}^n$ at any point $p \in U$ then the matrix of connection is so$(n)$-valued differential 1-form, i.e. Lie algebra valued 1-form.

The matrix of connection depends on a choice of a frame field. Let us find how the matrix of connection transforms when we pass from one frame field to another. Let $\{E'_i\}, \{E''_i\}$ be two orthonormal frame fields and an orthogonal matrix $G = (g_i^j) \in \mathrm{SO}(n)$ be a transition matrix from $\{E'_i\}$ to $\{E''_i\}$, i.e. $E''_i = g_i^j\, E'_j$. We will write this symbolically as $E'' = G \cdot E'$. It is worth to mention that if we consider the previous formula at a fixed point $p$, i.e. $(E''_i)_p = g_i^j(p)\,(E'_j)_p$ (symbolically $E''_p = G(p)\cdot E'_p$), then it determines the action of the orthogonal group $\mathrm{SO}(n)$ on the set of all orthonormal frames of the tangent space $\mathrm{T}_p\mathbb{R}^n$. This suggests us to attach to each point $p$ of the Euclidean space the set $\mathscr{F}_p$ of all orthonormal frames for $\mathrm{T}_p\mathbb{R}^n$ and to consider the disjoint union $\mathscr{F}(U) = \cup_p \mathscr{F}_p$. We will refer to $\mathscr{F}(U)$ as the *bundle of orthonormal frames over an open subset* $U$ of the Euclidean space $\mathbb{R}^n$, and to $\mathscr{F}_p$ as the *fiber* of this bundle at a point $p$. The *projection* $\pi : \mathscr{F}(U) \to U$ is defined in the obvious way and any orthonormal frame field is a smooth section of the bundle $\mathscr{F}(U)$. The special orthogonal group $\mathrm{SO}(n)$ acts on the bundle of orthonormal frames from the left as it is shown above, i.e. $E_p \to G\cdot E_p$, and we will denote this *left action* by $L : (G, E_p) \mapsto G \cdot E_p$, i.e. $L : \mathrm{SO}(n) \times \mathscr{F}(U) \to \mathscr{F}(U)$. This action is

(i) *transitive*, i.e. for any two $E'_p, E''_p \in \mathscr{F}_p$ there exists $G \in \mathrm{SO}(n)$ such that $E''_p = G \cdot E'_p$,
(ii) *effective*, i.e. $G \cdot E'_p = E'_p$ implies $G = I$.

Now let $E', E''$ be two orthonormal frame fields, i.e. two sections of the bundle of orthonormal frames, and $G = (g_i^j) : U \to \mathrm{SO}(n)$ be the $\mathrm{SO}(n)$-valued function such that $E'' = G\cdot E'$. Following the terminology used in a gauge field theory we can call this transformation (from one frame field to another) the *gauge transformation of first kind*. Hence $E''_i = g_i^j\, E'_j$, where $g_i^j$ depend smoothly on a point $x \in U$. Let $\tilde{\omega}, \omega$ be the matrices of connection in a frame fields $E'', E'$ respectively. Then $DE''_i = \tilde{\omega}_i^k \otimes E''_k$, $DE'_i = \omega_i^k \otimes E'_k$. On the one hand $DE''_i = \tilde{\omega}_i^k \otimes (g_k^m\, E'_m) = (g_k^m\, \tilde{\omega}_i^k) \otimes E'_m$. On the other hand

$$DE''_i = dg_i^m \otimes E'_m + g_i^j\, DE'_j = (dg_i^m + \omega_j^m\, g_i^j) \otimes E'_m,$$

and we get

$$g_k^m \tilde{\omega}_i^k = dg_i^m + \omega_j^m\, g_i^j,$$

or, written in the matrix form

$$\tilde{\omega} = G^{-1}\omega\, G + G^{-1} dG. \tag{14.4.8}$$

We derived the transformation rule of the matrix of connection and this is usually called in a gauge field theory the *gauge transformation of second kind*.

Particularly in the case of $E' = G \cdot E$, where $E$ is the canonical frame field, let us denote the connection matrix in the frame field $E'$ by $\omega$ and the connection matrix in the canonical frame field $E$ by $\omega_0$. Since the canonical frame field $E$ consists of constant vector fields, the gauge transformation (14.4.8) takes the form $\omega = G^{-1}d\,G$, because in the case of the canonical frame field $DE_i = 0$ ($E_i$ is the constant vector field) and hence $\omega_0 = 0$. In a gauge field theory the connection $\omega = G^{-1}d\,G$ is referred to as the *pure gauge*. It is easy to show that $\omega = G^{-1}d\,G$ is the so($n$)-valued differential 1-form. Indeed we have $G^{-1}\,G = I$ and, differentiating both sides, we obtain

$$dG^{-1}\,G + G^{-1}dG = 0. \tag{14.4.9}$$

But the first term can be written $dG^{-1}\,G = (dG)^T(G^{-1})^T = (G^{-1}dG)^T = \omega^T$ and we conclude $\omega + \omega^T = 0$.

We remind that the dual 1-forms $\theta^i$ for $E'_i$ are $\theta^i = (G^{-1})^i_j\,dx^j$. Differentiating and making use of (14.4.9) written in the form $dG^{-1} = -G^{-1}dG\,G^{-1}$, we get

$$d\theta^i = d(G^{-1})^i_j \wedge dx^j = -(G^{-1}dG\,G^{-1})^i_j \wedge dx^j = -(G^{-1}dG)^i_k \wedge ((G^{-1})^k_j \wedge dx^j),$$

or

$$d\theta^i = -\omega^i_k \wedge \theta^k. \tag{14.4.10}$$

Equation (14.4.10) is called the *first Cartan's structure equation*. Analogously computing the exterior differential of the matrix of connection $d\omega$, we obtain

$$d\omega = d(G^{-1}dG) = dG^{-1} \wedge dG = -(G^{-1}dG) \wedge (G^{-1}dG),$$

or

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j \quad \Leftrightarrow \quad d\omega = -\omega \wedge \omega. \tag{14.4.11}$$

Equation (14.4.11) is called the *second Cartan's structure equation*. Remind that the matrix of connection $\omega$ can be considered as the so($n$)-valued differential 1-form. Let $\mathfrak{f}_\alpha$, where $\alpha = 1, 2, \ldots, \frac{n(n-1)}{2}$, be a basis for the Lie algebra so($n$). Then $\omega = \omega^\alpha\,\mathfrak{f}_\alpha$ or $\omega^i_j = \omega^\alpha(\mathfrak{f})^i_j$, where $\omega^\alpha$ is the differential 1-form. Define

$$[\omega, \omega] = \omega^\alpha \wedge \omega^\beta\,[\mathfrak{f}_\alpha, \mathfrak{f}_\beta]. \tag{14.4.12}$$

From this it follows

$$[\omega, \omega]^i_j = \omega^\alpha \wedge \omega^\beta\left((\mathfrak{f}_\alpha)^i_k(\mathfrak{f}_\beta)^k_j - (\mathfrak{f}_\beta)^i_k(\mathfrak{f}_\alpha)^k_j\right) = 2\,\omega^i_k \wedge \omega^k_j.$$

Now the second Cartan's structure equation can be written in the matrix form as follows

$$d\omega = -\frac{1}{2}[\omega, \omega]. \tag{14.4.13}$$

## 14.5 $q$-Differential Graded Algebra and $N$-Connection

In this section we describe a generalization of the notion of connection which arises in the framework of non-commutative geometry. First of all we would like to remind a reader that the notion of a connection in the Euclidean space $\mathbb{R}^n$, described in the previous section, can be extended to a vector bundle over a smooth manifold. A smooth $n$-dimensional manifold $M$ is a Hausdorff topological space, which is locally homeomorphic to open subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$ (this is called a local chart), and the smooth structure of $M$ is determined by the condition that the transition functions of any two local charts must be smooth. A vector bundle $\mathfrak{V}$ over a manifold $M$ is a triple $(\mathfrak{V}, \pi, M)$, where $\mathfrak{V}$ is a $(n+r)$-dimensional manifold, $\pi : \mathfrak{V} \to M$ is a differentiable map, which is called a *projection*, such that for any point $x$ of a manifold $M$ the fiber $\pi^{-1}(x)$ is an $r$-dimensional vector space. Additionally it is required that locally a vector bundle is trivial, i.e. for any point $x \in M$ there exists its neighborhood $U \subset M$ such that $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^r$. A *section* of a vector bundle $\mathfrak{V}$ is a differentiable map $s : M \to \mathfrak{V}$ such that $\pi \circ s = \mathrm{id}_M$. Let $\mathfrak{C} = C^\infty(M)$ be the algebra of smooth functions on $M$, $\Omega(M) = \oplus_i \Omega^i(M)$ be the algebra of differential forms on $M$, $\mathscr{E}(\mathfrak{V})$ be the vector space of smooth sections of a vector bundle $\mathfrak{V}$. This vector space of sections $\mathscr{E}(\mathfrak{V})$ and the algebra of differential forms $\Omega(M)$ can be endowed with the structure of module over the algebra of functions $\mathfrak{C}$ by means of pointwise multiplication. Now we can extend the notion of a vector field valued differential form to a notion of vector bundle valued differential form by considering the tensor product $\Omega(\mathfrak{V}) = \Omega(M) \otimes_{\mathfrak{C}} \mathscr{E}(\mathfrak{V})$. It is important here that the first factor in this tensor product is the DGA. Now in accordance with the formula (14.4.5) we can define a connection in a vector bundle $\mathfrak{V}$ as a linear mapping $D : \mathscr{E}(\mathfrak{V}) \to \Omega^1(\mathfrak{V})$, which assigns to each section of a vector bundle the vector bundle valued 1-form and satisfies

$$D(f \cdot s) = df \otimes s + f \cdot Ds.$$

Hence we see that important ingredient in the structure of connection is the DGA of differential forms $\Omega(M)$. In this section we describe a generalization of the notion of connection which can be constructed if instead of a DGA we consider a more general structure, which is called a $q$-differential graded algebra ($q$-DGA), where $q$ is a primitive $N$th root of unity.

### *14.5.1 q-Differential Graded Algebra*

A basic algebraic structure used in the theory of connections on modules is a DGA. Therefore if we consider a generalization of a DGA, where the basic property of differential $d^2 = 0$ is given in a more general form $d^N = 0,\ N \geq 2$ and the graded Leibniz rule is replaced by the graded $q$-Leibniz rule, where $q$ is a primitive $N$th root of unity, we can develop a generalization of the theory of connections on modules.

A notion of $q$-differential graded algebra was introduced in [5] and studied in the series of papers [1, 6, 9, 11]. Let $N \geq 2$, $q$ be a primitive $N$th root of unity and $\mathscr{G}_q = \oplus_k {\mathscr{G}_q}^k$ be an associative unital $\mathbb{Z}_N$-graded algebra over a field of complex numbers. An algebra $\mathscr{G}_q$ is said to be a *q-differential graded algebra* ($q$-DGA) if it is endowed with a linear mapping $d$ of degree one, satisfying the graded $q$-Leibniz rule

$$d(u\,v) = d(u)\,v + q^k u\,d(v), \tag{14.5.1}$$

where $u \in {\mathscr{G}_q}^k$, $v \in \mathscr{G}_q$, and the $N$-nilpotency condition

$$d^N = 0. \tag{14.5.2}$$

A concept of $q$-DGA is related to a monoidal structure introduced in [11] for a category of $N$-complexes. It is proved in [8] that the monoids of the category of $N$-complexes can be determined as the $q$-DGA. In agreement with the terminology developed in [5] we shall call $d$ the $N$-differential of $q$-DGA $\mathscr{G}_q$.

Clearly in the case $N = 2$ and $q = -1$ we get a notion of DGA, which allows us to consider a concept of a $q$-DGA as a generalization of a DGA.

Let $\mathscr{G}_q$ be a $q$-DGA and $\mathfrak{A}$ be an unital associative algebra over the field of complex numbers. The subspace $\mathscr{G}_q^0 \subset \mathscr{G}_q$ of elements of degree zero is the subalgebra of an algebra $\mathscr{G}_q$. Obviously the triple $(\mathfrak{A}, d, \mathscr{G}_q^1)$ is the first order differential calculus over the the algebra $\mathfrak{A}$ provided that $\mathfrak{A} = \mathscr{G}_q^0$. The triple $(\mathfrak{A}, d, \mathscr{G}_q)$ is said to be an $N$-differential calculus over the algebra $\mathfrak{A}$. Every subspace $\mathscr{G}_q^k$ can be viewed as the bimodule over the algebra $\mathscr{G}_q^0$ if we determine the structure of a bimodule with the mappings $\mathscr{G}_q^0 \times \mathscr{G}_q^k \to \mathscr{G}_q^k$ and $\mathscr{G}_q^k \times \mathscr{G}_q^0 \to \mathscr{G}_q^k$ defined by $(u, w) \mapsto uw$ and $(w, v) \mapsto wv$, where $u, v \in \mathscr{G}_q^0$ and $w \in \mathscr{G}_q^k$. Hence we have the following sequence of bimodules over the algebra $\mathscr{G}_q^0$

$$\ldots \xrightarrow{d} \mathscr{G}_q^{k-1} \xrightarrow{d} \mathscr{G}_q^k \xrightarrow{d} \mathscr{G}_q^{k+1} \xrightarrow{d} \ldots \tag{14.5.3}$$

The sequence (14.5.3) can be considered as a cochain $N$-complex of modules or simply $N$-complex with $N$-differential $d$ [6]. The generalized cohomologies of this

$N$-complex are defined by the formula $H^k_m(\mathscr{G}_q) = Z^k_m(\mathscr{G}_q)/B^k_m(\mathscr{G}_q)$, where

$$Z^k_m(\mathscr{G}_q) = \{u \in \mathscr{G}^k_q : d^m u = 0\} \subset \mathscr{G}^k_q,$$
$$B^k_m(\mathscr{G}_q) = \{u \in \mathscr{G}^k_q : \exists\, v \in \mathscr{G}^{k+m-N}_q, u = d^{N-m} v\} \subset Z^k_m(\mathscr{G}_q).$$

Given a $q$-DGA $\mathscr{G}_q$ one can associate to it the generalized homologies $H_m(\mathscr{G}_q) = \oplus_{k\in\mathbb{Z}_N} H^k_m(\mathscr{G}_q)$ of the corresponding $N$-complex (14.5.3).

Next we give the statement of theorem which allows us to construct various $N$-complexes. Let $\mathscr{G} = \oplus_{k\in\mathbb{Z}_N}\mathscr{G}^k$ be an associative unital $\mathbb{Z}_N$-graded algebra over the field of complex numbers and $e$ be the identity element of this algebra. The graded subspace $\mathscr{Z}(\mathscr{G}) \subset \mathscr{G}$ generated by homogeneous elements $u \in \mathscr{G}^k$, which for any $v \in \mathscr{G}^l$ satisfy $uv = (-1)^{kl} vu$, is called a *graded center* of an algebra $\mathscr{G}$.

Let us generalize the notions of graded commutator and graded derivation of a graded algebra with the help of $q$-deformations. In general $q$ may be any complex number different from one but for the structures we construct we need $q$ to be a primitive $N$th root of unity. *The graded $q$-commutator* $[\ ,\ ]_q : \mathscr{G}^k \otimes \mathscr{G}^l \to \mathscr{G}^{k+l}$ is defined by

$$[u, v]_q = uv - q^{kl} vu,$$

where $u \in \mathscr{G}^k, \mathscr{G}^l$ are homogeneous elements and $q$ is a primitive $N$th root of unity. *A graded $q$-derivation of degree $m$* of a graded algebra $\mathscr{G}$ is a linear mapping $\delta : \mathscr{G} \to \mathscr{G}$ of degree $m$ with respect to the graded structure of $\mathscr{G}$, i.e. $\delta : \mathscr{G}^k \to \mathscr{G}^{k+m}$ satisfying the graded $q$-Leibniz rule

$$\delta(u\, v) = \delta(u)\, v + q^{ml} u\, \delta(v),$$

where $u \in \mathscr{G}^l$.

The following theorem [2] can be used to construct the structure of a $q$-DGA for a certain class of graded associative unital algebra.

**Theorem 14.5.1** *If there exists an element $v \in \mathscr{G}^1$ of degree one which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{G})$, where $N \geq 2$, then an algebra $\mathscr{G}$ equipped with the linear mapping $d : \mathscr{G} \to \mathscr{G}$ defined by the formula $d(u) = [v, u]_q$, $u \in \mathscr{G}$ is the $q$-DGA and $d$ is its $N$-differential.*

## *14.5.2 Connection on Module*

In this section we propose a notion of $N$-connection, which can be viewed as a generalization of a concept of connection on modules. In our generalization we use an algebraic approach based on the concept of $q$-DGA to define a notion of $N$-connection and show that in the case of $N = 2$ we get the algebraic analog of a classical connection. A theory of connection on modules can be found in an review [7]. We study the structure of an $N$-connection, define its curvature and prove the Bianchi identity [1, 2]. We begin this section by recalling the notion of connection on modules given in [7] and called $\Omega$-connection. Suppose that $\mathfrak{A}$ is an unital associative algebra over the field of complex numbers and $\mathcal{E}$ is a left module over $\mathfrak{A}$. Let $\Omega$ be a DGA with differential $d$, such that $\Omega^0 = \mathfrak{A}$, it means that the triple $(\mathfrak{A}, d, \Omega^1)$ is the first oder differential calculus over $\mathfrak{A}$. Since an subspace of elements of grading one can be viewed as a $(\mathfrak{A}, \mathfrak{A})$-bimodule, the tensor product $\Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ clearly has the structure of left $\mathfrak{A}$-module.

A linear map $\nabla : \mathcal{E} \to \Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ is called an $\Omega$-*connection* if it satisfies

$$\nabla(us) = du \otimes_{\mathfrak{A}} s + u\nabla(s)$$

for any $u \in \mathfrak{A}$ and $s \in \mathcal{E}$. Similarly to the case of connections on vector bundles, this map has a natural extension $\nabla : \Omega \otimes_{\mathfrak{A}} \mathcal{E} \to \Omega \otimes_{\mathfrak{A}} \mathcal{E}$ by setting

$$\nabla(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + (-1)^p \omega\nabla(s),$$

where $\omega \in \Omega^p$ and $s \in \mathcal{E}$.

We will generalize a notion of $\Omega$-connection taking $q$-DGA $\Omega_q$ instead of DGA $\Omega$. Let $\mathfrak{A}$ be an unital associative algebra over a field of complex numbers, $\Omega_q$ is a $q$-DGA with $N$-differential $d$ and $\mathfrak{A} = \Omega_q^0$. Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. Considering algebra $\Omega_q$ as the $(\mathfrak{A}, \mathfrak{A})$-bimodule we take the tensor product of left $\mathfrak{A}$-modules $\Omega_q \otimes_{\mathfrak{A}} \mathcal{E}$ which has the structure of left $\mathfrak{A}$-module. To minimize the notation, we denote this left $\mathfrak{A}$-module by $\mathfrak{F}$. Taking into account that an algebra $\Omega_q$ can be viewed as the direct sum of $(\mathfrak{A}, \mathfrak{A})$-bimodules $\Omega_q^k$ we can split the left $\mathfrak{A}$-module $\mathfrak{F}$ into the direct sum of the left $\mathfrak{A}$-modules $\mathfrak{F}^k = \Omega_q^k \otimes_{\mathfrak{A}} \mathcal{E}$, i.e. $\mathfrak{F} = \oplus_k \mathfrak{F}^k$, which means that $\mathfrak{F}$ inherits the graded structure of algebra $\Omega_q$, and $\mathfrak{F}$ is the graded left $\mathfrak{A}$-module. It is worth noting that the left $\mathfrak{A}$-submodule $\mathfrak{F}^0 = \mathfrak{A} \otimes_{\mathfrak{A}} \mathcal{E}$ of elements of grading zero is isomorphic to a left $\mathfrak{A}$-module $\mathcal{E}$, where isomorphism $\varphi : \mathcal{E} \to \mathfrak{F}^0$ can be defined for any $s \in \mathcal{E}$ by $\varphi(s) = e \otimes_{\mathfrak{A}} s$, where $e$ is the identity element of algebra $\mathfrak{A}$. Since a graded $q$-DGA $\Omega_q$ can be viewed as the $(\Omega_q, \Omega_q)$-bimodule, the left $\mathfrak{A}$-module $\mathfrak{F}$ can be also considered as the left $\Omega_q$-module and we will use this structure to describe a concept of $N$-connection. Let us mention that multiplication by elements of $\Omega^k$, where $k \neq 0$, does not preserve the graded structure of the left $\Omega_q$-module $\mathfrak{F}$.

The tensor product $\mathfrak{F} = \Omega_q \times \mathcal{E}$ as the tensor product of two vector spaces has also the structure of the vector space over $\mathbb{C}$. Obviously $\mathfrak{F}$ has a graded structure, i.e. $\mathfrak{F} = \oplus_k \mathfrak{F}^k$, where $\mathfrak{F}^k = \Omega_q^k \otimes_{\mathbb{C}} \mathcal{E}$. Due to the structure of vector space of $\mathfrak{F}$ we can introduce the notion of linear operator on $\mathfrak{F}$. We denote the vector space of linear operators on $\mathfrak{F}$ by $\mathrm{Lin}(\mathfrak{F})$. The structure of the graded vector space of $\mathfrak{F}$ induces the structure of a graded vector space on $\mathrm{Lin}(\mathfrak{F})$, and we shall denote the subspace of homogeneous linear operators of degree $k$ by $\mathrm{Lin}^k(\mathfrak{F})$.

An *N-connection* on the left $\Omega_q$-module $\mathfrak{F}$ is a linear operator $\nabla_q : \mathfrak{F} \to \mathfrak{F}$ of degree one satisfying the condition

$$\nabla_q(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + q^{|\omega|}\, \omega\, \nabla_q(s), \tag{14.5.4}$$

where $\omega \in \Omega^i_q$, $s \in \mathcal{E}$, and $|\omega|$ is the grading of the homogeneous element of algebra $\Omega_q$.

It is worth to mention that if $N = 2$ then $q = -1$, and in this particular case we get the algebraic analog of a classical connection. A connection on vector bundle can be viewed as a linear map on a left module of sections of vector bundle, taking values a algebra of differential 1-forms with values in this vector bundle, which clearly has a structure of a left module over an algebra of smooth functions on a base manifold. Therefore a concept of a $N$-connection can be viewed as a generalization of a classical connection.

We use the following proposition proved in [1] to define the curvature of $N$-connection.

**Proposition 14.5.2** *The $N$-th power of any $N$-connection $\nabla_q$ is the endomorphism of degree $N$ of the left $\Omega_q$-module $\mathfrak{F}$.*

The endomorphism $F = \nabla_q^N$ of degree $N$ of the left $\Omega_q$-module $\mathfrak{F}$ is said to be the *curvature of an $N$-connection* $\nabla_q$.

Let us show that the curvature of an $N$-connection satisfies Bianchi identity. We proceed to show that the graded vector space $\mathrm{Lin}(\mathfrak{F})$ has a structure of graded algebra. To this end, we take the product $A \circ B$ of two linear operators $A, B$ of the vector space $\mathfrak{F}$ as an algebra multiplication. If $A : \mathfrak{F} \to \mathfrak{F}$ is a homogeneous linear operator than we can extend it to the linear operator $L_A : \mathrm{Lin}(\mathfrak{F}) \to \mathrm{Lin}(\mathfrak{F})$ on the whole graded algebra of linear operators $\mathrm{Lin}(\mathfrak{F})$ by means of the graded $q$-commutator: $L_A(B) = [A, B]_q = A \circ B - q^{|A||B|} B \circ A$, where $B$ is a homogeneous linear operator. It makes allowable to extend an $N$-connection $\nabla_q$ to the linear operator on the vector space $\mathrm{Lin}(\mathfrak{F})$

$$\nabla_q(A) = [\nabla_q, A]_q = \nabla_q \circ A - q^{|A|}\, A \circ \nabla_q, \tag{14.5.5}$$

where $A$ is a homogeneous linear operator. $N$-connection $\nabla_q$ is the linear operator of degree one on the vector space $\mathrm{Lin}(\mathfrak{F})$, i.e. $\nabla_q : \mathrm{Lin}^k(\mathfrak{F}) \to \mathrm{Lin}^{k+1}(\mathfrak{F})$, and $\nabla_q$ satisfies the graded $q$-Leibniz rule with respect to the algebra structure of $\mathrm{Lin}(\mathfrak{F})$. Consequently the curvature $F$ of an $N$-connection can be viewed as the linear operator of degree $N$ on the vector space $\mathfrak{F}$, i.e. $F \in \mathrm{Lin}^N(\mathfrak{F})$. Therefore one can act on $F$ by $N$-connection $\nabla_q$, and it holds that for any $N$-connection $\nabla_q$ the curvature $F$ of this connection satisfies the Bianchi identity

$$\nabla_q(F) = 0. \tag{14.5.6}$$

### 14.5.3 Local Structure of N-Connection

Connection on the vector bundle of finite rank over a finite dimensional smooth manifold can be studied locally by choosing a local trivialization of the vector bundle and this leads to the basis for the module of sections of this vector bundle.

In order to construct an algebraic analog of the local structure of an $N$-connection $\nabla_q$ we assume $\mathcal{E}$ to be a finitely generated free left $\mathfrak{A}$-module. Let $\mathfrak{e} = \{\mathfrak{e}_\mu\}_{\mu=1}^r$ be a basis for a left module $\mathcal{E}$. This basis induces the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, where $\mathfrak{f}_\mu = e \otimes_{\mathfrak{A}} \mathfrak{e}_\mu$, for the left $\mathfrak{A}$-module $\mathfrak{F}^0 \cong \mathcal{E}$. For any $\xi \in \mathfrak{F}^0$ we have $\xi = \xi^\mu \mathfrak{f}_\mu$. Taking into account that $\mathfrak{F}^0 \subset \mathfrak{F}$ and $\mathfrak{F}$ is the left $\Omega_q$-module we can multiply the elements of the basis $\mathfrak{f}$ by elements of an $q$-DGA $\Omega_q$. It is easy to see that if $\omega \in \Omega_q^k$ then for any $\mu$ we have $\omega\mathfrak{f}_\mu \in \mathfrak{F}^k$. Consequently we can express any element of the $\mathfrak{F}^k$ as a linear combination of $\mathfrak{f}_\mu$ with coefficients from $\Omega_q^k$. Indeed let $\omega \otimes_{\mathfrak{A}} s$ be an element of $\mathfrak{F}^k = \Omega^k \otimes_{\mathfrak{A}} \mathcal{E}$. Then

$$\begin{aligned}\omega \otimes_{\mathfrak{A}} s &= (\omega\, e) \otimes_{\mathfrak{A}} (s^\mu \mathfrak{e}_\mu) = (\omega\, e\, s^\mu) \otimes_{\mathfrak{A}} \mathfrak{e}_\mu \\ &= (\omega s^\mu\, e) \otimes_{\mathfrak{A}} \mathfrak{e}_\mu = \omega s^\mu\, (e \otimes_{\mathfrak{A}} \mathfrak{e}_\mu) = \omega^\mu \mathfrak{f}_\mu,\end{aligned}$$

where $\omega^\mu = \omega s^\mu \in \Omega_q^k$.

Let $\mathfrak{F}^0$ be a finitely generated free module with a basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, and $s = s^\mu \mathfrak{f}_\mu \in \mathfrak{F}^0$, where $s^\mu \in \mathfrak{A}$. Since $N$-connection $\nabla_q$ is a linear operator of degree one, it follows that $\nabla_q(s) \in \mathfrak{F}^1$, and making use of $q$-Leibniz rule we can express the element $\nabla_q(s)$ as follows: $\nabla_q(s) = \nabla_q(s^\mu \mathfrak{f}_\mu)$

Denote by $\mathfrak{M}_r(\Omega_q)$ be the vector space of square matrices of order $r$ whose entries are the elements of an $q$-DGA $\Omega_q$. If each entry of a matrix $\Theta = (\theta_\mu^\nu)$ is an element of a homogeneous subspace $\Omega_q^k$, i.e. $\theta_\mu^\nu \in \Omega_q^k$ then $\Theta$ will be refereed to as a homogeneous matrix of degree $k$ and we shall denote the vector space of such matrices by $\mathfrak{M}_r^k(\Omega_q)$. Obviously $\mathfrak{M}_r(\Omega_q) = \oplus_k \mathfrak{M}_r^k(\Omega_q)$. The vector space $\mathfrak{M}_r(\Omega_q)$ of $r \times r$-matrices becomes the associative unital graded algebra if we define the product of two matrices $\Theta = (\theta_\mu^\nu)$, $\Theta' = (\theta'^\nu_\mu) \in \mathfrak{M}_r(\Omega_q)$ by $(\Theta\,\Theta')_\mu^\nu = \theta_\mu^\sigma\, \theta'^\nu_\sigma$.

If $\Theta, \Theta' \in \mathfrak{M}_r(\Omega_q)$ are homogeneous matrices then we define the graded $q$-commutator by $[\Theta, \Theta']_q = \Theta\,\Theta' - q^{|\Theta||\Theta'|}\Theta'\,\Theta$. We extend the $N$-differential $d$ of an $q$-DGA $\Omega_q$ to the algebra $\mathfrak{M}_r(\Omega_q)$ as follows $d\Theta = d(\theta_\mu^\nu) = (d\theta_\mu^\nu)$.

Since any element of a left $\mathfrak{A}$-module $\mathfrak{F}^1$ can be expressed in terms of the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$ with coefficients from $\Omega_q^1$, we have

$$\nabla_q(\mathfrak{f}_\mu) = \theta_\mu^\nu\, \mathfrak{f}_\nu, \tag{14.5.7}$$

where $\theta_\mu^\nu \in \Omega_q^1$. An $r \times r$-matrix $\Theta = (\theta_\mu^\nu)$, whose entries $\theta_\mu^\nu$ are the elements of $\Omega_q^1$ i.e. $\Theta \in \mathrm{Mat}_r^1(\Omega_q)$, is said to be a *matrix of an $N$-connection* $\nabla_q$ with respect to the basis $\mathfrak{f}$ of the left $\mathfrak{A}$-module $\mathfrak{F}^0$. Using the definition of $N$-connection we obtain

$$\nabla_q(s) = (ds^\mu + s^\nu \theta^\mu_\nu)\, \mathfrak{f}_\mu. \tag{14.5.8}$$

Let $\mathfrak{f}' = \{\mathfrak{f}'_\mu\}^r_{\mu=1}$ be another basis for the left $\mathfrak{A}$-module $\mathfrak{F}^0$ with the same number of elements (this will always be the case if $\mathfrak{A}$ is a division algebra or if $\mathfrak{A}$ is commutative). Then $\mathfrak{f}'_\mu = g^\nu_\mu \mathfrak{f}_\nu$, where $G = (g^\nu_\mu) \in \mathrm{Mat}^0_r(\Omega_q)$ is a transition matrix from the basis $\mathfrak{f}$ to the basis $\mathfrak{f}'$. It is well known [10] that in the case of finitely generated free module transition matrix is an invertible matrix. If we denote by $\theta'^\mu_\nu$ the coefficients of $\nabla_q$ with respect to a basis $\mathfrak{f}'$ and $\tilde{g}^\mu_\nu$ are the entries of the inverse matrix $G^{-1}$ then

$$\theta'^\mu_\nu = dg^\sigma_\nu\, \tilde{g}^\mu_\sigma + g^\sigma_\nu\, \theta^\tau_\sigma\, \tilde{g}^\mu_\tau,$$

and this clearly shows that the components of $\nabla_q$ with respect to different bases of module $\mathfrak{F}^0$ are related by the gauge transformation.

Our next aim is to express the components of the curvature $F$ of a $N$-connection $\nabla_q$ in the terms of the entries of the matrix $\Theta$ of an $N$-connection $\nabla_q$. Computation in successive steps allows us to introduce polynomials $\psi^{l,\mu}_\nu \in \Omega^l_q$ on the entries of the matrix of $N$-connection and their differentials. We have

$$\begin{aligned}\nabla_q(s) &= (ds^\mu + s^\nu \theta^\mu_\nu)\, \mathfrak{f}_\mu,\\ \psi^{1,\mu}_\nu &:= \theta^\mu_\nu,\end{aligned}$$

$$\begin{aligned}\nabla^2_q(s) &= (d^2 s^\mu + [2]_q ds^\nu \theta^\mu_\nu + s^\nu(d\theta^\mu_\nu + q\theta^\sigma_\nu \theta^\mu_\sigma))\mathfrak{f}_\mu,\\ \psi^{2,\mu}_\nu &:= d\theta^\mu_\nu + q\, \theta^\sigma_\nu \theta^\mu_\sigma,\end{aligned} \tag{14.5.9}$$

$$\begin{aligned}\nabla^3_q(s) &= \Big(d^3 s^\mu + [3]_q d^2 s^\nu \theta^\mu_\nu + [3]_q ds^\nu (d\theta^\mu_\nu + q\theta^\sigma_\nu \theta^\mu_\sigma)\\ &\quad + s^\nu (d^2\theta^\mu_\nu + (q+q^2) d\theta^\sigma_\nu \theta^\mu_\sigma + q^2 \theta^\sigma_\nu d\theta^\mu_\sigma + q^3 \theta^\tau_\nu \theta^\sigma_\tau \theta^\mu_\sigma)\Big)\mathfrak{f}_\mu,\\ \psi^{(3,k)\mu}_\nu &:= d^2\theta^\mu_\nu + (q+q^2)\, d\theta^\sigma_\nu \theta^\mu_\sigma + q^2\, \theta^\sigma_\nu\, d\theta^\mu_\sigma + q^3\, \theta^\tau_\nu \theta^\sigma_\tau \theta^\mu_\sigma\end{aligned} \tag{14.5.10}$$

Therefore, the $k$th power of $N$-connection $\nabla_q$ has the following form

$$\begin{aligned}\nabla^k_q(s) &= \sum_{l=0}^{k} \begin{bmatrix} k \\ l \end{bmatrix}_q d^{k-l} s^\mu\ \psi^{l,\nu}_\mu\ \mathfrak{f}_\nu\\ &= (d^k s^\mu\ \psi^{0,\nu}_\mu + [k]_q\, d^{k-1} s^\mu\ \psi^{1,\nu}_\mu + \ldots + s^\mu\ \psi^{k,\nu}_\mu)\, \mathfrak{f}_\nu,\end{aligned} \tag{14.5.11}$$

We can calculate the polynomials $\psi_\mu^{l,\nu}$ by means of the following recursion formula

$$\psi_\mu^{l,\nu} = d\psi_\mu^{l-1,\nu} + q^{l-1}\,\psi_\mu^{l-1,\sigma}\,\theta_\sigma^\nu, \tag{14.5.12}$$

or in the matrix form

$$\Psi^l = d\Psi^{l-1} + q^{l-1}\,\Psi^{l-1}\,\Theta, \tag{14.5.13}$$

We begin with the polynomial $\psi_\mu^{0,\nu} = \delta_\mu^\nu\, e \in \mathfrak{A}$, and $e$ is the identity element of $\mathfrak{A} \subset \Omega_q$. From (14.5.11) it follows that if $k = N$ then the first term $d^N \xi^\mu\ \psi_\mu^{(0,N)\nu}$ in this expansion vanishes because of the $N$-nilpotency of the $N$-differential $d$, and the next terms corresponding to the $l$ values from 1 to $N-1$ also vanish because of the property of $q$-binomial coefficients. Hence if $k = N$ then the formula (14.5.11) takes on the form

$$\nabla_q^N(s) = s^\mu\,\psi_\mu^{(N,N)\nu}\,\mathfrak{f}_\nu. \tag{14.5.14}$$

In order to simplify the notations and assuming that $N$ is fixed we shall denote $\psi_\mu^\nu = \psi_\mu^{(N,N)\nu}$.

An $(r \times r)$-matrix $\Psi = (\psi_\mu^\nu)$, whose entries are the elements of degree $N$ of a graded $q$-differential algebra $\Omega_q$, is said to be the *curvature matrix* of a $N$-connection $\nabla_q$.

Obviously $\Psi \in \mathfrak{M}_r^N(\Omega_q)$. In new notations the formula (14.5.14) can be written as follows $\nabla_q^N(s) = s^\mu\,\psi_\mu^\nu\,\mathfrak{f}_\nu$, and it shows that $\nabla_q^N$ is the endomorphism of degree $N$ of the left $\Omega_q$-module $\mathfrak{F}$.

Let us consider the expressions for curvature in the case when $N = 2$. If $N = 2$ then $q = -1$, and a graded $q$-differential algebra $\Omega_q$ is a graded differential algebra with differential $d$ satisfying $d^2 = 0$. This is a classical case, and if we assume that $\Omega_q$ is the algebra of differential forms on a smooth manifold $M$ with exterior differential $d$ and exterior multiplication $\wedge$, $\mathcal{E}$ is the module of smooth sections of a vector bundle $E$ over $M$, $\nabla_q$ is a connection on $E$, $\mathfrak{e}$ is a local frame of a vector bundle $E$ then $\Theta$ is the matrix of 1-forms of a connection $\nabla_q$ and we have for the components of curvature $\psi_\mu^\nu = d\theta_\mu^\nu - \theta_\mu^\sigma\theta_\sigma^\nu$. In this case $\Omega_q$ is super-commutative algebra and we can put the expressions for components of curvature into the form $\psi_\mu^\nu = d\theta_\mu^\nu + \theta_\sigma^\nu\theta_\mu^\sigma$. or by means of matrices $\Psi = d\Theta + \Theta\cdot\Theta$ in which we recognize the classical expression for the curvature.

From the previous section it follows that the curvature of a $N$-connection satisfies the Bianchi identity. If $\theta_\nu^\mu$, $\psi_\nu^\mu$ are the components of an $N$-connection $\nabla_q$ and its curvature $F$ with respect to a basis $\mathfrak{f}$ for the module $\mathfrak{F}$ then the Bianchi identity takes on the form

$$d\psi_\nu^\mu = \theta_\mu^\sigma\psi_\sigma^\nu - \psi_\mu^\sigma\theta_\sigma^\nu.$$

Let us consider now the structure of $N$-connection forms and their curvature. We apply the algebra of polynomials $\mathfrak{P}[\mathfrak{d}, a]$ over $\mathbb{C}$, constructed in the paper [3] to study the structure of $N$-curvature. Let $\Omega_q$ be an $q$-DGA We will call an element of degree one $\Theta \in \Omega_q^1$ an *$N$-connection form* in a graded $q$-differential algebra $\Omega_q$. The linear operator of degree one $\nabla_q = d + \Theta$ will be referred to as a *covariant $N$-differential* induced by a $N$-connection form $\Theta$.

We remind that $d$ is an $N$-differential which means that $d^k \neq 0$ for $1 \leq k \leq N-1$ and if we successively apply it to an $N$-connection form $\Theta$ we get the sequence of elements $\Theta, d\Theta, d^2\Theta, \ldots, d^{N-1}\Theta$, where $d^k\Theta \in \Omega_q^{k+1}$. Let us denote

$$\Theta_1 = \Theta$$
$$\Theta_2 = d\Theta$$
$$\vdots$$
$$\Theta_N = d^{N-1}\Theta.$$

We denote by $\Omega_q[\Theta]$ the graded subalgebra of $\Omega_q$ generated by elements $\Theta_1, \Theta_2, \ldots, \Theta_N$. For any integer $k = 1, 2, \ldots, N$ we define the polynomial $F_k \in \Omega_q[\Theta]$ by the formula $F_k = \nabla_q^{k-1}(\Theta)$. Evidently the subalgebra $\Omega_q[\Theta]$ is isomorphic to the $q$-DGA $\mathfrak{P}_q[a]$ of [3] if we identify $\Theta_k \to a_k$. Then the polynomials $F_k$ are identified with the polynomials $f_k$ and we can apply all formulae proved in the case of $\mathfrak{P}_q[a]$ to study the structure of $\Omega_q[\Theta]$.

It follows from [3] that for any integer $1 \leq k \leq N$ the $k$ th power of the covariant $N$-differential $\nabla_q$ can be expanded as follows

$$(\nabla_q)^k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q F_{(i)}\, d^{k-1} = d^k + [k]_q F_1 d^{k-1} + \ldots + [n]_q F_{k-1} d + F_k,$$

where $F_k = (\nabla_q)^{k-1}(\Theta)$. Particularly if $k = N$ then the $N$th power of the covariant $N$-differential $\nabla_q$ is the operator of multiplication by the element $F_N$ of grading zero. It makes possible to define the curvature of an $N$-connection form $\Theta$ : the *$N$-curvature form* of an $N$-connection form $\Theta$ is the element of grading zero $F_N \in \mathfrak{A}$.

We get the explicit power expansion formula for $N$-curvature form of an $N$-connection

$$F_k = \sum_{\sigma \in \Upsilon_k} \begin{bmatrix} k_2 - 1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3 - 1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k - 1 \\ k_{r-1} \end{bmatrix}_q \Theta_{i_1}\Theta_{i_2}\ldots\Theta_{i_r},$$

where $\Upsilon_k$ is the set of all compositions of an integer $1 \leqslant k \leqslant N$, $\sigma = (i_1, i_2, \ldots, i_r)$ is composition of an integer $k$ in the form of a sequence of strictly positive integers, where $i_1 + i_2 + \ldots + i_r = N$, and

$$\begin{aligned} k_1 &= i_1, \\ k_2 &= i_1 + i_2, \\ k_3 &= i_1 + i_2 + i_3, \\ &\ldots \\ k_{r-1} &= i_1 + i_2 + \ldots + i_{r-1}. \end{aligned}$$

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