Chapter 12
Comments on an Orthogonal Family of Monogenic Functions on Spheroidal Domains

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Celebrating Wolfgang Sprößig 70th birthday

Abstract The problem of building an orthogonal basis for the space of square-integrable harmonic functions defined in a spheroidal (either oblate or prolate) domain leads to special functions, which provide an elegant analysis of a variety of physical problems. Many generalizations of these ideas in the context of Quaternionic Analysis possess a similar elegant mathematical structure. A brief descriptive review is given of these developments.

Keywords Quaternionic analysis · Spherical harmonics · Spheroidal harmonics · Monogenic functions

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12.1 Introduction

The origins behind the study of orthogonal bases of polynomials for the spaces of square-integrable harmonic functions defined in a prolate or oblate spheroid are to be found in [16]. The orthogonality was taken with respect to certain inner products, each of which lead to the discussion of a PDE by means of the kernel of the orthogonal system corresponding to that inner product. As regards treatises on the subject, we add the names of Laplace [28], Lamé [27], Heine [23], Liouville [34], Thomson and Tait [51], Hilbert [24], Niven [47], Klein [26], Lindemann [33], Stieltjes [49], Darwin [11], Ferrers [15], Féjer [14], Whittaker and Watson [52], among others, while more general aspects of their theory were given by Hobson

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[25], Szegö [50], Byerly [6], Sansone [48], Lebedev [31], and Dassios [12]. In this connection, recently in [17] the spheroidal harmonics were defined following [16], with a rescaling factor which permits including the unit ball as a limit of both the prolate and oblate cases, combined into a single one-parameter family.

Multi-dimensional extensions of the prolate spheroidal harmonics to the framework of Quaternionic Analysis were originally developed in [36] and subsequently in [37], which provided many of their properties and have subsequently attracted special attention. In [43] it was shown that the underlying prolate spheroidal monogenics play an important role in defining and studying the monogenic Szegö kernel function for prolate spheroids. In [46] the authors developed an orthogonal basis of oblate spheroidal monogenics and some recurrence formulae were found. It was shown that in the case of an oblate spheroid a basis can only be either orthogonal or Appell basis. Some aspects on generating monogenic functions that are orthogonal in a region outside a prolate spheroid were considered in [44]. Generalization of these results has been recently done in [38].

The object of the present note is twofold: to review the construction of a single one-parameter family of spheroidal harmonics with special emphasis on those orthogonal in the L_2-Hilbert space structure; and to construct an orthogonal basis of spheroidal monogenics, whose elements are parametrized by the shape of the corresponding spheroid. We observe that this analysis cannot be done with models in which the unit ball only is approximated as a degenerate case and requires a separate, yet completely analogous, treatment for prolate and oblate spheroids [16, 25]. The proofs of the main results are simplified, in accordance with developments of the theory later in date than the original proofs; other results are given in a form more general than that in which they were first discovered. The references given are to be regarded solely as indicating sources of information from which I have drawn, or where more detailed information on the various topics is to be found.

12.2 Background on Spheroidal Harmonics

We consider the family of coaxial spheroidal domains Ω_μ, scaled so that the major axis is of length 2:

$$\Omega_\mu = \{x \in \mathbb{R}^3 |\ x_0^2 + \frac{x_1^2 + x_2^2}{e^{2\nu}} < 1\}, \tag{12.2.1}$$

where $\nu \in \mathbb{R}$ and $\mu = (1 - e^{2\nu})^{\frac{1}{2}}$ will be useful in later formulas. This follows the notation in [17]. The equations relating the Cartesian coordinates of a point $x = (x_0, x_1, x_2)$ inside Ω_μ to *spheroidal coordinates* $(\eta, \vartheta, \varphi)$ are

$$x_0 = \mu \cosh\eta \cos\vartheta, \quad x_1 = \mu \sinh\eta \sin\vartheta \cos\varphi, \quad x_2 = \mu \sinh\eta \sin\vartheta \sin\varphi, \tag{12.2.2}$$

where in the case of the *prolate spheroid* ($\nu < 0$) the coordinates range over $\eta \in [0, \pi]$, $\vartheta \in [0, \operatorname{arctanh} e^{\nu}]$, $\varphi \in [0, 2\pi)$, and $0 < \mu < 1$ is the eccentricity, while for the *oblate spheroid* ($\nu > 0$) we have $\eta \in [0, \pi]$ and $\vartheta \in [0, \operatorname{arccoth} e^{\nu}]$, $\varphi \in [0, 2\pi)$ and μ is imaginary, $\mu/i > 0$. The spheroids reduce to the unit ball for $\nu = 0$, $\mu = 0$: $\Omega_0 = \{x \in \mathbb{R}^3 : |x|^2 < 1\}$.

In terms of the coordinates (12.2.2), *the spheroidal harmonics* are

$$U^{\pm}_{l,m}[\mu](x) := U_{l,m}[\mu](\eta, \vartheta)\, \Phi^{\pm}_m(\varphi), \tag{12.2.3}$$

where

$$U_{l,m}[\mu](\eta, \vartheta) = \alpha_{l,m}\, \mu^l\, P^m_l(\cos \vartheta)\, P^m_l(\cosh \eta) \tag{12.2.4}$$

for $\mu \neq 0$. Here P^m_l are the *associated Legendre functions of the first kind* (see [25, Ch. III]) of degree l and order m, and we write $\Phi^+_m(\varphi) = \cos(m\varphi)$, $\Phi^-_m(\varphi) = \sin(m\varphi)$, and

$$\alpha_{l,m} = \frac{(l-m)!}{(2l-1)!!} \tag{12.2.5}$$

with use of the symbol $n!! = \prod_{k=0}^{\lceil n/2\rceil - 1}(n - 2k)$ for the double factorial. To avoid repetition, we state once and for all that $U^-_{l,m}[\mu]$ is only defined for $m \geq 1$, i.e. $U^-_{l,0}[\mu]$ is expressly excluded from all statements of theorems.

It was shown in [17] that with the scale factor (12.2.5), the $U^{\pm}_{l,m}[\mu]$ are polynomials in the variables x_0, x_1, x_2, which are normalized so that the limiting case $\mu \to 0$ gives the classical *solid spherical harmonics* [45, 48],

$$U^{\pm}_{l,m}[0](x) = |x|^l P^m_l\left(\frac{x_0}{|x|}\right) \Phi^{\pm}_m(\varphi), \tag{12.2.6}$$

where we employ spherical coordinates $x_0 = \rho \cos\theta$, $x_1 = \rho \sin\theta \cos\varphi$, and $x_2 = \rho \sin\theta \sin\varphi$.

Moreover, in [16] it was shown that while the $U^{\pm}_{l,m}[\mu]$ are orthogonal in the Dirichlet norm on Ω_μ, the closely related functions, which we will call the *Garabedian spheroidal harmonics*,

$$V^{\pm}_{l,m}[\mu](x) = \frac{\partial}{\partial x_0} U^{\pm}_{l+1,m}[\mu](x) \tag{12.2.7}$$

form an orthogonal basis for $L_2(\Omega_\mu) \cap \operatorname{Har}(\Omega_\mu)$, the set of harmonic functions in $L_2(\Omega_\mu)$. This property makes the $V^{\pm}_{l,m}[\mu]$ of greater interest for many considerations.

In accordance with the notation already employed, we shall use $V^{\pm}_{l,m}[\mu] = V_{l,m}[\mu]\Phi^{\pm}_m$ when the factors $\Phi^{\pm}_m$ are not of interest. It will be convenient, before proceeding, to investigate the algebraical forms of the $V_{l,m}[\mu]$. We will assume that $\nu < 0$, because the case $\nu > 0$ is similar. From differentiating (12.2.2),

$$\frac{\partial}{\partial x_0} = \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)}\left(\cos\vartheta \sinh\eta \frac{\partial}{\partial\eta} - \sin\vartheta \cosh\eta \frac{\partial}{\partial\vartheta}\right),$$

from which the definition (12.2.7) gives

$$\frac{(\cosh^2\eta - \cos^2\vartheta)}{\alpha_{l+1,m}\,\mu^l} V_{l,m}[\mu] = \cos\vartheta \sinh^2\eta P^m_{l+1}(\cos\vartheta)(P^m_{l+1})'(\cosh\eta) + \sin^2\vartheta \cosh\eta P^m_{l+1}(\cosh\eta)(P^m_{l+1})'(\cos\vartheta). \tag{12.2.8}$$

There are many well-known recurrence relations for the associated Legendre functions (see for example [25, Ch. III]). The relation

$$(1-t^2)(P^m_{l+1})'(t) = (l+m+1)P^m_l(t) - (l+1)tP^m_{l+1}(t) \tag{12.2.9}$$

yields that (12.2.8) is equal to $(l+m+1)$ times

$$\cosh\eta\, P^m_l(\cos\vartheta)P^m_{l+1}(\cosh\eta) - \cos\vartheta\, P^m_{l+1}(\cos\vartheta)P^m_l(\cosh\eta).$$

It follows, then, that

$$V_{l,m}[\mu] = \frac{\alpha_{l+1,m}(l+m+1)\mu^l}{(\cosh^2\eta - \cos^2\vartheta)}\left[\cosh\eta\, P^m_l(\cos\vartheta)P^m_{l+1}(\cosh\eta) - \cos\vartheta\, P^m_l(\cosh\eta)P^m_{l+1}(\cos\vartheta)\right], \tag{12.2.10}$$

with the initial values

$$V_{l,l}[\mu] = (2l+1)U_{l,l}[\mu],$$

$$V_{l+1,l}[\mu] = 2(l+1)U_{l+1,l}[\mu].$$

In order to avoid the difficulties usually attendant on manipulations like those of the formulas (12.2.10), it will here be convenient to prove very simple recurrence relations for the functions $V_{l,m}[\mu]$. The following will be key in the proof of Theorem 12.3.1 and it is based on the results of [36].

Proposition 12.2.1 *For each $l \geq 2$, the functions $V_{l,m}[\mu]$ satisfy the recurrence relations*

$$V_{l,m}[\mu] = (l+m+1)U_{l,m}[\mu] + \frac{\mu^2(l+m+1)(l+m)}{(2l+1)(2l-1)} V_{l-2,m}[\mu]. \tag{12.2.11}$$

Proof Equation (12.2.10) together with the further relation

$$(l-m+1)P_{l+1}^m(t) = (2l+1)tP_l^m(t) - (l+m)P_{l-1}^m(t) \tag{12.2.12}$$

show that

$$\begin{aligned} V_{l+1,m}[\mu] = (l+m+1)U_{l,m}[\mu] \\ + \frac{\alpha_{l,m}\,\mu^l(l+m+1)(l+m)}{(\cosh^2\eta - \cos^2\vartheta)(2l+1)}[\cos\vartheta\; P_{l-1}^m(\cos\vartheta)P_l^m(\cosh\eta) \\ - \cosh\eta\; P_l^m(\cos\vartheta)P_{l-1}^m(\cosh\eta)], \end{aligned}$$

with

$$\alpha_{l,m} = \frac{2l+1}{l-m+1}\,\alpha_{l+1,m}.$$

Using again (12.2.12), we obtain

$$\begin{aligned} V_{l+1,m}[\mu] &= (l+m+1)U_{l,m}[\mu] \\ &+ \frac{\alpha_{l-1,m}\,\mu^l(l+m+1)(l+m)(l+m-1)}{(\cosh^2\eta - \cos^2\vartheta)(2l-1)(2l+1)} \\ &\times [\cosh\eta\; P_{l-2}^m(\cos\vartheta)P_{l-1}^m(\cosh\eta) \\ &- \cos\vartheta\; P_{l-1}^m(\cos\vartheta)P_{l-2}^m(\cosh\eta)]. \end{aligned}$$

The result now follows. □

Since the basic harmonics $U_{l,m}^{\pm}[\mu]$ of [16] are polynomials of degree l, it is clear that the operations of rescaling by $1/\mu$ or i/μ and multiplying by μ^l implied in (12.2.4) assure that the $V_{l,m}^{\pm}[\mu]$ are polynomials in μ. By Eq. (12.2.11) it is clear that $-\mu$ produces the same results as μ, so the only powers of μ which appear are even.

In this regard, from (12.2.11) we note that for spherical harmonics,

$$\frac{\partial}{\partial x_0}U_{l+1,m}^{\pm}[0](x) = (l+m+1)U_{l,m}^{\pm}[0](x), \tag{12.2.13}$$

whereas $V_{l,m}^{\pm}[\mu]$ is not so simply related to $U_{l,m}^{\pm}[\mu]$ for $\mu \neq 0$, as was proved in [36]:

Theorem 12.2.2 *Let $l \geq 0$, $0 \leq m \leq l$. The non-vanishing coefficients $v_{l,m,k}$ in the relation*

$$V_{l,m}^{\pm}[\mu] = \sum_{k=0}^{\left[\frac{l-m}{2}\right]} v_{l,m,k}\,\mu^{2k}U_{l-2k,m}^{\pm}[\mu] \tag{12.2.14}$$

are given by

$$v_{l,m,k} = \frac{(l+m+1)!(2l+1-4k)!!}{(l+m-2k)!(2l+1)!!}. \tag{12.2.15}$$

Proof Suppose inductively that the formula of theorem is true when l is replaced by $l' < l$. Then

$$V^{\pm}_{l,m}[\mu] = (l+m+1)U^{\pm}_{l,m}[\mu] + \frac{(l+m+1)(l+m)}{(2l+1)(2l-1)} \sum_{k=0}^{\left[\frac{l-2-m}{2}\right]} v_{l-2,m,k}\, \mu^{2(k+1)} U^{\pm}_{l-2(k+1),m}[\mu].$$

Since by (12.2.15)

$$v_{l,m,0} = l+m+1,$$
$$v_{l,m,k+1} = \frac{(l+m+1)(l+m)}{(2l+1)(2l-1)}\, v_{l-2,m,k},$$

we find that the stated formula is also true, completing the proof. □

An important result of [16] regarding the orthogonailty of the $V^{\pm}_{l,m}[\mu]$ in the L_2-Hilbert space can be restated as follows.

Theorem 12.2.3 *For a fixed* μ*, the functions* $V^{\pm}_{l,m}[\mu]$ $(l \geq 0)$ *form a complete orthogonal family in the closed subspace* $L_2(\Omega_\mu) \cap \mathrm{Har}(\Omega_\mu)$ *of* $L_2(\Omega_\mu)$ *with the norms*

$$\| V^{\pm}_{l,m}[\mu] \|^2_{L_2(\Omega_\mu)} = 2\pi(1+\delta_{0,m})\mu^{2l+3}\gamma_{l,m} I_{l,m}(\mu), \tag{12.2.16}$$

where $I_{l,m}(\mu)$ *is defined by*

$$I_{l_1,m}(\mu) := \int_1^{\frac{1}{\mu}} P^m_{l_1}(t) P^m_{l_1+2}(t) dt, \tag{12.2.17}$$

and

$$\gamma_{l,m} = \frac{(l+m+1)(l+2-m)!(l+m+1)!}{(2l+1)!!(2l+3)!!}. \tag{12.2.18}$$

For the limiting case, $\mu = 0$*,*

$$\| V^{\pm}_{l,m}[0] \|^2_{L_2(\Omega_0)} = \frac{2\pi(1+\delta_{0,m})(l+m+1)(l+m+1)!}{(2l+1)(2l+3)(l-m)!}. \tag{12.2.19}$$

12.3 An Orthogonal Basis of Spheroidal Monogenics

The standard bases for spheroidal harmonics have their counterparts for the corresponding spaces of monogenic functions taking values in $\mathbb{R}^3$. These monogenic polynomials are defined by regarding $\mathbb{R}^3$ as the subset of the *quaternions* $\mathbb{H} := \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\}$ for which $x_3 = 0$. Although this subspace is not closed under the quaternionic multiplication (which is defined, as usual, so that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$), it is possible to carry out a great deal of the analysis analogous to that of complex numbers [13, 21, 35, 39, 40, 42].

Consider the Cauchy-Riemann (or Fueter) operators

$$\overline{\partial} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}, \quad \partial = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2}. \tag{12.3.1}$$

Define the set of *monogenic functions*

$$\mathcal{M}(\Omega_\mu) := \left\{\mathbf{f} = [\mathbf{f}]_0 + [\mathbf{f}]_1\mathbf{i} + [\mathbf{f}]_2\mathbf{j} \in C^1(\Omega_\mu) : \overline{\partial}\mathbf{f} = 0\right\}.$$

Monogenic functions are harmonic, but not vice-versa. The *hypercomplex derivative* is simply denoted by $(1/2)\partial\mathbf{f}$ [18].

A basis of polynomials spanning the square-integrable solutions of $\overline{\partial}\mathbf{f} = 0$ was given in [36] (cf. [37]) for prolate spheroids and another in [46] for oblate spheroids, via explicit formulas. Note that the latter prolate and oblate spheroidal monogenics can be obtained as a special case of the present theory by appropriate interpretation. In the following, we consider the prolate and oblate cases of spheroids simultaneously.

In analogy to (12.2.7) the *basic monogenic spheroidal polynomials* are constructed as

$$\mathbf{X}^{\pm}_{l,m}[\mu] = \partial U^{\pm}_{l+1,m}[\mu]. \tag{12.3.2}$$

It was noted in [39] that Sc $\mathbf{X}^{\pm}_{l,m}[0]$ is equal to $V^{\pm}_{l,m}[0] = (l+m+1)U^{\pm}_{l,m}[0]$, and an explicit expression for the vector part was written out, which was later generalized from the sphere to the spheroid in [36].

Using (12.2.10) and further properties of the Legendre functions, we can verify that

$$V_{l,-1}[\mu] = \begin{cases} -\frac{1}{(l+1)(l+2)}V_{l,1}[\mu] & l = 1, 2, \ldots, \\ 0 & l = 0. \end{cases} \tag{12.3.3}$$

These functions will appear in the representation (12.3.4) for the case of zero-order monogenic polynomials (see Theorem 12.3.1 below). Similar results can be found in [36].

Theorem 12.3.1 *For each $l \geq 0$ and $0 \leq m \leq l+1$, the basic spheroidal monogenic polynomials (12.3.2) are equal to*

$$\mathbf{X}^{\pm}_{l,m}[\mu] = V^{\pm}_{l,m}[\mu] + \frac{\mathbf{i}}{2}\left[(l+m+1)V^{\pm}_{l,m-1}[\mu] - \frac{1}{l+m+2}V^{\pm}_{l,m+1}[\mu]\right]$$
$$\mp \frac{\mathbf{j}}{2}\left[(l+m+1)V^{\mp}_{l,m-1}[\mu] + \frac{1}{l+m+2}V^{\mp}_{l,m+1}[\mu]\right], \qquad (12.3.4)$$

where the harmonic polynomials $V^{\pm}_{l,m}[\mu]$ are defined by (12.2.7). Moreover, the set $\{\mathbf{X}^{\pm}_{l,m}[\mu] : l \geq 0,\ 0 \leq m \leq l+1\}$ is orthogonal in the sense of the scalar product defined by

$$\langle \mathbf{f}, \mathbf{g}\rangle_{[\mu]} = \mathrm{Sc} \iiint_{\Omega_\mu} \bar{\mathbf{f}}\mathbf{g}\, dx. \qquad (12.3.5)$$

Their norms are given by

$$\|\mathbf{X}^{\pm}_{l,m}[\mu]\|^2_{L_2(\Omega_\mu)} = \frac{\pi\, \mu^{2l+3}}{(l+2)(l+m+2)(2l+1)!!(2l+3)!!}$$
$$\big[(l+2)(l+m)(l+m+1)(l-m+3)!(l+m+2)!I_{l,m-1}$$
$$+ 2\delta_{0,m}(l+m+2)(l+1)!(l+2)!I_{l,1}$$
$$+ (l+2)(l-m+1)!(l+m+2)!\big(I_{l,m+1}$$
$$+ 2(l-m+2)(l+m+1)(1+\delta_{0,m})I_{l,m}\big)\big],$$

where $I_{l,m}(\mu)$ is defined by (12.2.17). For the limiting case, $\mu = 0$,

$$\|\mathbf{X}^{\pm}_{l,m}[0]\|^2_{L_2(\Omega_0)} = \frac{2\pi(1+\delta_{0,m})(l+1)(l+1+m)!}{(2l+3)(l+1-m)!}.$$

Proof The full operator (12.3.1) in spheroidal coordinates (12.2.2) is

$$\partial = \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)}\left(\cos\vartheta \sinh\eta \frac{\partial}{\partial\eta} - \sin\vartheta \cosh\eta \frac{\partial}{\partial\vartheta}\right)$$
$$- \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)}(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j})\left(\sin\vartheta\cosh\eta\frac{\partial}{\partial\eta} + \cos\vartheta\sinh\eta\frac{\partial}{\partial\vartheta}\right)$$
$$- \frac{1}{\mu\sin\vartheta\sinh\eta}(-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j})\frac{\partial}{\partial\varphi}.$$

The first line of this expression applied to $U^{\pm}_{l+1,m}[\mu]$ produces the scalar part of $\mathbf{X}^{\pm}_{l,m}[\mu]$ in (12.3.4) and was calculated in [36]. For the non-scalar part, we use the relation (12.2.9) to obtain

$$\begin{aligned}
&\frac{2}{\mu^{l+1}\alpha_{l+1,m}\Phi^{\pm}_m}\left(\cos\vartheta\,\sinh\eta\,\frac{\partial}{\partial\vartheta}+\sin\vartheta\,\cosh\eta\,\frac{\partial}{\partial\eta}\right)U^{\pm}_{l+1,m}[\mu]\\
&=(l+m+1)(l-m+2)\Big[\sin\vartheta\,\cosh\eta\,P^m_{l+1}(\cos\vartheta)\,P^{m-1}_{l+1}(\cosh\eta)\\
&\quad-\cos\vartheta\,\sinh\eta\,P^{m-1}_{l+1}(\cos\vartheta)\,P^m_{l+1}(\cosh\eta)\Big]\\
&\quad+\sin\vartheta\,\cosh\eta\,P^m_{l+1}(\cos\vartheta)\,P^{m+1}_{l+1}(\cosh\eta)\\
&\quad+\cos\vartheta\,\sinh\eta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^m_{l+1}(\cosh\eta).
\end{aligned}$$

Next, we use the relation

$$\sqrt{1-t^2}\,P^m_{l+1}(t)=(l-m)t\,P^{m-1}_{l+1}(t)-(l+m)P^{m-1}_l(t)$$

(valid for $|t|<1$, and replacing $1-t^2$ with t^2-1 for $|t|>1$) produces

$$\begin{aligned}
-\frac{(\cosh^2\eta-\cos^2\vartheta)}{\mu^l\alpha_{l+1,m-1}}V_{l,m-1}[\mu]&=\sin\vartheta\,\cosh\eta\,P^m_{l+1}(\cos\vartheta)\,P^{m-1}_{l+1}(\cosh\eta)\\
&\quad-\cos\vartheta\,\sinh\eta\,P^m_{l+1}(\cosh\eta)\,P^{m-1}_{l+1}(\cos\vartheta).
\end{aligned}$$

Furthermore, using the expression

$$(1-t^2)^{1/2}\,P^m_{l+1}(t)=\frac{1}{2l+3}\,(P^{m+1}_{l+2}(t)-P^{m+1}_l(t)),$$

and its counterpart for $|t|>1$, and then applying (12.2.12), we arrive at

$$\begin{aligned}
&\cosh\eta\,\sin\vartheta\,P^m_{l+1}(\cos\vartheta)\,P^{m+1}_{l+1}(\cosh\eta)+\sinh\eta\,\cos\vartheta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^m_{l+1}(\cosh\eta)\\
&=\frac{(\cosh^2\eta-\cos^2\vartheta)}{(l+1-m)(l+2+m)\mu^l\alpha_{l+1,m+1}}\,V_{l,m+1}[\mu].
\end{aligned}$$

With these calculations at hand, we have

$$\begin{aligned}
&-\frac{1}{\mu(\cosh^2\eta-\cos^2\vartheta)}\left(\sin\vartheta\,\cosh\eta\frac{\partial}{\partial\eta}+\cos\vartheta\,\sinh\eta\frac{\partial}{\partial\vartheta}\right)U^{\pm}_{l+1,m}[\mu]\\
&=\frac{(l+1+m)}{2}V_{l,m-1}[\mu]\Phi^{\pm}_m-\frac{1}{2(l+2+m)}V_{l,m+1}[\mu]\Phi^{\pm}_m.
\end{aligned}$$

Similarly, one can prove that

$$\begin{aligned}
&\frac{1}{\sin\vartheta\,\sinh\eta}\frac{\partial}{\partial\varphi}U^{\pm}_{l+1,m}[\mu]\\
&= \mp\,\frac{m\,\mu^{l+1}\alpha_{l+1,m}}{\cosh^2\eta-\cos^2\vartheta}\Phi^{\mp}_{m}\\
&\quad\times\left[\frac{\sinh\eta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^{m}_{l+1}(\cosh\eta)}{\sin\vartheta}+\frac{\sin\vartheta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^{m}_{l+1}(\cosh\eta)}{\sinh\eta}\right]\\
&= \pm\,\frac{\mu}{2}\left[\frac{1}{l+2+m}V_{l,m+1}[\mu]+(l+1+m)V_{l,m-1}[\mu]\right]\Phi^{\mp}_{m}.
\end{aligned}$$

Combining these three formulas one straightforward obtains the desired expressions for $(\partial/\partial x_1)U^{\pm}_{l+1,m}[\mu]$ and $(\partial/\partial x_2)U^{\pm}_{l+1,m}[\mu]$.

In the sequel, we will denote by $[\mathbf{f}]_i$ $(i=0,1,2)$ the components of a function $\mathbf{f}\colon\Omega_\mu\to\mathbb{R}^3$. By definition of the integral (12.3.5) it follows that

$$\begin{aligned}
&\langle\mathbf{X}^{\pm}_{l_1,m_1}[\mu],\mathbf{X}^{\pm}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)}\\
&=\iiint_{\Omega_\mu}\Big([\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_0[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_0+[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_1\\
&\quad+[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_2\Big)\,dx.
\end{aligned}$$

By Eqs. (12.3.3) and (12.3.4), and Theorem 12.2.3 we have

$$\iiint_{\Omega_\mu}[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_0[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_0\,dx=\|V^{\pm}_{l_1,m_1}[\mu]\|^2_{L_2(\Omega_\mu)}\,\delta_{l_1,l_2}\delta_{l_1,l_2}\tag{12.3.6}$$

and

$$\begin{aligned}
&\iiint_{\Omega_\mu}\Big([\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_1+[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_2\Big)\,dx\\
&=\frac{\pi p_1(l_2+m_1+1)\delta_{m_1,m_2}}{2}\int_0^{\operatorname{arctanh}e^{\nu}}\int_0^{\pi}V_{l_1,m_1-1}[\mu]V_{l_2,m_1-1}[\mu]\,dR\\
&\quad\pm\frac{\pi}{(l_1+2)(l_2+2)}\delta_{m_1,0}\int_0^{\operatorname{arctanh}e^{\nu}}\int_0^{\pi}V_{l_1,1}[\mu]V_{l_2,1}[\mu]\,dR\\
&\quad+\frac{\pi}{2p_1(l_2+m_1+1)}\delta_{m_1,m_2}\int_0^{\operatorname{arctanh}e^{\nu}}\int_0^{\pi}V_{l_1,m_1+1}[\mu]V_{l_2,m_1+1}[\mu]\,dR,
\end{aligned}$$

where $dR=\mu^3(\cosh^2\eta-\cos^2\vartheta)\sin\vartheta\,\sinh\eta d\vartheta d\eta$.

Using Proposition 12.2.1, and applying again the orthogonality of Theorem 12.2.3, we are left with

$$\iiint_{\Omega_\mu} \left([\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_1 + [\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_2\right) dx$$
$$= \frac{\pi \mu^{2l_1+3}}{(l_1+2)(2l_1+1)!!(2l_1+3)!!}$$
$$\times \Big[(l_1+2)(l_1+m_1+1)!$$
$$\big((l_1+m_1)(l_1+m_1+1)(l_1-m_1+3)!I_{l_1,m_1-1}$$
$$+ (l_1-m_1+1)!I_{l_1,m_1+1}\big) + 2(l_1+1)!(l_1+2)!I_{l_1,1}\delta_{0,m}\Big]\delta_{m_1,m_2}\delta_{l_1,l_2} \tag{12.3.7}$$

with $I_{l,m}$ defined in (12.2.17). Combining (12.3.6) and (12.3.7), we conclude that

$$\langle \mathbf{X}^{+}_{l_1,m_1}[\mu], \mathbf{X}^{+}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)} = 0$$

when $l_1 \neq l_2$ or $m_1 \neq m_2$. Similarly, $\langle \mathbf{X}^{-}_{l_1,m_1}[\mu], \mathbf{X}^{-}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)} = 0$ when $l_1 \neq l_2$ or $m_1 \neq m_2$.

Using once more the orthogonality of the system $\{\Phi^{\pm}_m\}$ on $[0, 2\pi]$, we conclude that

$$\langle \mathbf{X}^{\pm}_{l_1,m_1}[\mu], \mathbf{X}^{\mp}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)} = 0$$

when the indices do not coincide. The calculation of the norms comes from taking $l_1 = l_2$ and $m_1 = m_2$ in (12.3.7) and adding the expression (12.2.16). By the symmetric form taken by $\mathbf{X}^{\pm}_{l,m}[\mu]$ in (12.3.4), we know that when $m \neq 0$,

$$\|\mathbf{X}^{+}_{l,m}[\mu]\|_{L_2(\Omega_\mu)} = \|\mathbf{X}^{-}_{l,m}[\mu]\|_{L_2(\Omega_\mu)}.$$

The limiting case, $\mu = 0$, follows with the use of Eq. (12.2.19). □

The solid spherical monogenics $\mathbf{X}^{\pm}_{l,m}[0]$ are embedded generically in this one-parameter family of spheroidal monogenics. In contrast, in treatments such as [16, 25, 36, 37, 44, 46], the spheroidal monogenics degenerate as the eccentricity of the spheroid decreases.

For a general orientation, the reader is urged to read some of the existing works where the spherical monogenics emerged [7, 9, 10]. It is worth mentioning that at the time of the publications [8–10] a closed-form representation corresponding to the $\mathbf{X}^{\pm}_{l,m}[0]$ in terms of the basic solid spherical harmonics (12.2.6), originally stated in [39], were not at disposal for the investigation of some basic properties of these functions. They played a fundamental role in [19, 20, 22, 39, 40] (cf. [35]) in the study of higher-dimensional counterparts of the well-known Bohr theorem, Borel-Carathéodory's theorem and Hadamard real part theorems on the majorant of a Taylor's series, as well as Bloch's theorem, in the context of Quaternionic Analysis, where they were investigated in detail. In a different context, orthogonal Appell bases of monogenic polynomials were constructed in [1], [4] and [41] (cf. [2, 3]) using a basis of quaternionic-valued spherical monogenics orthogonal with respect to the quaternionic inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{[0]} = \iiint_{\Omega_0} \bar{\mathbf{f}} \mathbf{g} \, dx.$$

These bases were rediscovered in [29] (cf. [5, 30]) using a different algebraic approach based on *Gelfand-Tsetlin schemes*.

In [32] it is shown that the dimension of the space $\mathcal{M}^{(l)}$ of homogeneous monogenic polynomials of degree l in x_0, x_1, x_2 is $2l + 3$ (this does not depend on the domain). Since the polynomials $\mathbf{X}^{\pm}_{l,m}[\mu]$ are not homogeneous, we consider the space

$$\mathcal{M}^{(l)}_* = \bigcup_{0 \le k \le l} \mathcal{M}^{(l)}$$

of monogenic polynomials of degree l, a class which is not altered by adding monogenic polynomials of lower degree. Thus

$$\dim \mathcal{M}^{(l)}_* = \sum_{k=0}^{l} (2k + 3) = (l + 3)(l + 1). \tag{12.3.8}$$

Consider the collections of $2k + 3$ polynomials

$$\mathbf{B}_k[\mu] := \{\mathbf{X}^{+}_{k,m}[\mu],\ 0 \le m \le k + 1\} \cup \{\mathbf{X}^{-}_{k,m}[\mu],\ 1 \le m \le k + 1\}.$$

By Theorem 12.3.1 and (12.3.8), the union

$$\bigcup_{0 \le k \le l} \mathbf{B}_k[\mu] \tag{12.3.9}$$

is an orthogonal basis for $\mathcal{M}^{(l)}_*$. In addition, $\mathcal{M}^{(l)}_*$ is dense in $L_2(\Omega_\mu) \cap \mathcal{M}(\Omega_\mu)$. Therefore the following result, which will be of use in the further discussion, can now be established:

Proposition 12.3.2 *For a fixed μ, the function set (12.3.9) forms an orthogonal basis of $L_2(\Omega_\mu) \cap \mathcal{M}(\Omega_\mu)$.*

Furthermore, it would be useful in practice if the foregoing orthogonal basis (12.3.9) has the *Appell property* also. It was shown in [46] that there does not exist an orthogonal Appell basis in the case of spaces of solid oblate spheroidal monogenics. We shall proceed in such a manner that we compute the hypercomplex derivative of a spheroidal monogenic of degree l and show, as expected, that the obtained polynomial is not a member of the family with degree $l-1$ like in cases of Appell bases [4, 7, 8, 10]. We find that the hypercomplex derivative of a basic spheroidal monogenic is a combination of $[(l-m)/2]+1$ spheroidal monogenics of lower degrees. Basically, it can be represented by all polynomials of degree at most $l-1$.

Theorem 12.3.3 *For a fixed μ, the hypercomplex derivative of $\mathbf{X}^{\pm}_{l,m}[\mu]$ has the form:*

$$(\frac{1}{2}\partial)\mathbf{X}^{\pm}_{l,m}[\mu] = \sum_{k=0}^{[\frac{l-m}{2}]} v_{l,m,k}\,\mu^{2k}\mathbf{X}^{\pm}_{l-1-2k,m}[\mu], \tag{12.3.10}$$

where the constants $v_{l,m,k}$ are given by (12.2.15).

Proof Since $\partial/\partial x_0$ is a linear operator, we find, by Theorem 12.2.2, the relation:

$$\frac{\partial}{\partial x_0}V^{\pm}_{l,m}[\mu] = \sum_{k=0}^{[\frac{l-m}{2}]} v_{l,m,k}\,\mu^{2k}V^{\pm}_{l-1-2k,m}[\mu].$$

The rest of the proof is straightforward. □

An advantage of Eq. (12.3.10) is that it furnishes a concise expression for the hypercomplex derivatives of the basic monogenic spheroidal polynomials by means of which many of their properties may be easily investigated.

The next proposition shows that there are two *hyperholomorphic constants* among the basic spheroidal monogenic polynomials, i.e., functions whose hypercomplex derivative is identically zero.

Proposition 12.3.4 *For a fixed μ, $\mathbf{X}^{\pm}_{l,l+1}[\mu]$ are hyperholomorphic constants.*

Proof The proof is a consequence of Theorem 12.3.3. □

It can be further shown that $\mathbf{X}^{\pm}_{l,l+1}[\mu] = \mathbf{X}^{\pm}_{l,l+1}[0]$; that is, the hyperholomorphic constants $\mathbf{X}^{\pm}_{l,l+1}[\mu]$ do not depend on the parameter μ.

The hypercomplex derivatives of the prescribed monogenic polynomials in its extended signification being thus computed, no difficulties can arise in restricting it to a particular limiting case. In fact, when $\mu = 0$, we have readily [7, 10]:

$$(\frac{1}{2}\partial)\mathbf{X}^{\pm}_{l,m}[0] = (l+m+1)\mathbf{X}^{\pm}_{l-1,m}[0]. \tag{12.3.11}$$

The reader might find without any additional work that, using (12.3.11) and setting for each $l \geq 0, 0 \leq m \leq l+1$,

$$\mathbf{Y}^{\pm}_{l,m} := \frac{l!(m+1)!}{(l+m+1)!}\mathbf{X}^{\pm}_{l,m}[0], \tag{12.3.12}$$

the equality follows:

$$(\frac{1}{2}\partial)\mathbf{Y}^{\pm}_{l,m} = l\,\mathbf{Y}^{\pm}_{l-1,m}. \tag{12.3.13}$$

Thus the application of the hypercomplex derivative to $\mathbf{Y}^{\pm}_{l,m}$ results again in a real multiple of the similar function one degree lower [41]. The special normalization (12.3.13) is called *Appell property*. In [8] it is proved that the solid spherical monogenics (12.3.12) form, indeed, an *orthogonal Appell basis* for $\mathcal{M}^{(l)}(\Omega_0)$, $l \geq 0$. In [1] and [3], fundamental recursion formulas were obtained for the elements of the prescribed Appell basis.

We turn now to show that the Appell property holds for a part of the $\mathbf{X}^{\pm}_{l,m}[\mu]$ (providing the prescribed normalization (12.3.12)).

Corollary 12.3.5 *Let μ be fixed. For $l-m=0,1$, the hypercomplex derivatives of $\mathbf{X}^{\pm}_{l,m}[\mu]$ follow the rule*

$$(\frac{1}{2}\partial)\mathbf{X}^{\pm}_{l,m}[\mu] = (l+1+m)\mathbf{X}^{\pm}_{l-1,m}[\mu].$$

Proof It is an immediate consequence of Theorem 12.3.3. □

One of our leading results is that the three-dimensional spherical monogenics considered, e.g., in [4, 8, 10] are embedded in the prescribed one-parameter family of internal spheroidal monogenics. Hence, the latter can be naturally seen as an extention of the former functions to arbitrarily spheroidal domains. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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References

1. S. Bock, Über funktionentheoretische Methoden in der räumlichen Elastizitätstheorie, Ph.D. thesis, Bauhaus-University Weimar (2009)
2. S. Bock, On a three dimensional analogue to the holomorphic z-powers: Laurent series expansions. Complex Var. Elliptic Equ. **57**(12), 1271–1287 (2012)
3. S. Bock, On a three dimensional analogue to the holomorphic z-powers: power series and recurrence formulae. Complex Var. Elliptic Equ. **57**(12), 1349–1370 (2012)

4. S. Bock, K. Gürlebeck, On a generalized Appell system and monogenic power series. Math. Methods Appl. Sci. **33**(4), 394–411 (2010)
5. S. Bock, K. Gürlebeck, R. Lávička, V. Souček, The Gel'fand-Tsetlin bases for spherical monogenics in dimension 3. Rev. Mat. Iberoam. **28**(4), 1165–1192 (2012)
6. W.E. Byerly, *An Elementary Treatise on Fourier's Series, and Spherical, Cylindrical, and Ellipsoidal Harmonics, with Applications to Problems in Mathematical Physics* (Dover, New York, 1959), pp. 251–258
7. I. Cação, Constructive approximation by monogenic polynomials, Ph.D. dissertation, Universidade de Aveiro (2004)
8. I. Cação, Complete orthonormal sets of polynomial solutions of the Riesz and Moisil-Teodorescu systems in $\mathbb{R}^3$. Numer. Algorithms **55**(2–3), 191–203 (2010)
9. I. Cação, K. Gürlebeck, S. Bock, Complete orthonormal systems of spherical monogenics – a constructive approach, in *Methods of Complex and Clifford Analysis*, ed. by L.H. Son, et al. (SAS International Publications, Delhi, 2005), pp. 241–260
10. I. Cação, K. Gürlebeck, B. Bock, On derivatives of spherical monogenics. Complex Var. Elliptic Equ. **51**(8–11), 847–869 (2006)
11. G. Darwin, Ellipsoidal harmonic analysis. Philos. Trans. R. Soc. A **197**, 461–557 (1902)
12. G. Dassios, *Ellipsoidal Harmonics, Theory and Applications* (Cambridge University Press, Cambridge, 2012)
13. R. Delanghe, On homogeneous polynomial solutions of the Riesz system and their harmonic potentials. Complex Var. Elliptic Equ. **52**(10–11), 1047–1062 (2007)
14. L. Fejér, Über die Laplacesche Reihe. Math. Ann. **67**, 76–109 (1909)
15. N.M. Ferrers, On the potentials of ellipsoids, ellipsoidal shells, elliptic laminae, and elliptic rings, of variable densities. Q. J. Pure Appl. Math. **14**, 1–22 (1897)
16. P. Garabedian, Orthogonal harmonic polynomials. Pacific J. Math. **3**(3), 585–603 (1953)
17. R. García, A.J. Morais, R.M. Porter, Contragenic functions on spheroidal domains. Math. Methods Appl. Sci. (2017). https://doi.org/10.1002/mma.4759
18. K. Gürlebeck, H. Malonek, A hypercomplex derivative of monogenic functions in $\mathbb{R}^{n+1}$ and its applications. Complex Var. Elliptic Equ. **39**(3), 199–228 (1999)
19. K. Gürlebeck, J. Morais, Bohr type theorems for monogenic power series. Comput. Methods Funct. Theory **9**(2), 633–651 (2009)
20. K. Gürlebeck, J. Morais, On Bohr's phenomenon in the context of quaternionic analysis and related problems, in *Algebraic Structures in Partial Differential Equations Related to Complex and Clifford Analysis*, ed. by L.H. Son, T. Wolfgang (University of Education Press, Ho Chi Minh City, 2010), pp. 9–24
21. K. Gürlebeck, K. Habetha, W. Sprössig, *Holomorphic Functions in the Plane and n-Dimensional Space*. Contemporary Mathematics, vol. 212 (Birkhäuser Verlag, Basel, 2008), pp. 95–107
22. K. Gürlebeck, J. Morais, P. Cerejeiras, Borel-Carathéodory type theorem for monogenic functions. Complex Anal. Oper. Theory **3**(1), 99–112 (2009)
23. E. Heine, *Handbuch der Kugelfunktionen* (Verlag G. Reimer, Berlin, 1878)
24. D. Hilbert, Eigenschaften spezieller binärer Formen, insbesondere der Kugelfunktionen Drck Leupold (1885), p. 30
25. E. Hobson, *The Theory of Spherical and Ellipsoidal Harmonics* (Cambridge Universtiy Press, Cambridge, 1931)
26. F. Klein, Über Lamé'sche Functionen. Math. Ann. **18**, 237–246 (1881)
27. G. Lamé, *Mémoire sur l'équilibre des températures dans un ellipsoïde à trois axes inégaux*. J. Math. Pure Appl. **4**, 126–163 (1839)
28. P.S. Laplace, *Théorie des Attractions des Sphéroïdes et de la Figure des Planètes,* vol. III (Mechanique Celeste, Paris, 1785)
29. R. Lávička, Canonical bases for $sl(2, \mathbb{C})$-modules of spherical monogenics in dimension 3, Arch. Math. (Brno) **46**(5), 339–349 (2010)
30. R. Lávička, Complete orthogonal appell systems for spherical monogenics. Complex Anal. Oper. Theory **6**(2), 477–489 (2012)

31. N. Lebedev, *Special Functions and their Applications* (Dover, New York, 1972), Chaps. 7, 8
32. H. Leutwiler, Quaternionic analysis in $\mathbb{R}^3$ versus its hyperbolic modification, in *Proceedings of the NATO Advanced Research Workshop held in Prague, October 30–November 3, 2000*, ed. by F. Brackx, J.S.R. Chisholm, V. Soucek, vol. 25 (Kluwer Academic Publishers, Dordrecht, 2001)
33. F. Lindemann, Entwicklung der Functionen einer complexen Variabeln nach Lamé'shen Functionen und nach Zugeordneten der Kugelfunctionen. Math. Ann. **19**, 323–386 (1882)
34. J. Liouville, Sur diverses questions d'analyse et de Physique mathématique concernant L'ellipsoïde. J. Math. Pures Appl. **11**, 217–236 (1846)
35. J. Morais, Approximation by homogeneous polynomial solutions of the Riesz system in $\mathbb{R}^3$, Ph.D. thesis, Bauhaus-Universität Weimar (2009)
36. J. Morais, A complete orthogonal system of spheroidal monogenics. J. Numer. Anal. Ind. Appl. Math. **6**(3–4), 105–119 (2011)
37. J. Morais, An orthogonal system of monogenic polynomials over prolate spheroids in $\mathbb{R}^3$. Math. Comput. Model. **57**, 425–434 (2013)
38. J. Morais, A quaternionic version of a standard theory related to spheroidal functions, Habilitation thesis. Preprint (2019)
39. J. Morais, K. Gürlebeck, Real-part estimates for solutions of the Riesz system in $\mathbb{R}^3$. Complex Var. Elliptic Equ. **57**(5), 505–522 (2012)
40. J. Morais, K. Gürlebeck, Bloch's theorem in the context of quaternion analysis. Comput. Methods Funct. Theory **12**(2), 541–558 (2012)
41. J. Morais, H.T. Le, Orthogonal appell systems of monogenic functions in the cylinder. Math. Methods Appl. Sci. **34**(12), 1472–1486 (2011)
42. J. Morais, K. Avetisyan, K. Gürlebeck, On Riesz systems of harmonic conjugates in $\mathbb{R}^3$. Math. Methods Appl. Sci. **36**(12), 1598–1614 (2013)
43. J. Morais, K.I. Kou, W. Sprössig, Generalized holomorphic Szegö kernel in 3D spheroids. Comput. Math. Appl. **65**, 576–588 (2013)
44. J. Morais, M.H. Nguyen, K.I. Kou, On 3D orthogonal prolate spheroidal monogenics. Math. Methods Appl. Sci. **39**(4), 635–648 (2016)
45. C. Müller, *Spherical Harmonics*. Lectures Notes in Mathematics, vol. 17 (Springer, Berlin, 1966)
46. H.M. Nguyen, K. Gürlebeck, J. Morais, S. Bock, On orthogonal monogenics in oblate spheroidal domains and recurrence formulae. Integral Transforms Spec. Funct. **25**(7), 513–527 (2014)
47. C. Niven, On the conduction of heat in ellipsoids of revolution. Philos. Trans. R. Soc. Lond. **171**, 117 (1880)
48. G. Sansone. *Orthogonal Functions*. Pure and Applied Mathematics, vol. IX (Interscience Publishers, New York, 1959)
49. T.J. Stieltjes, Sur certains polynômes qui vérifient une équation différentielle linéaire du second ordre et sur la théorie des fonctions de Lamé. Acta Math. **6**, 321–326 (1885)
50. G. Szegö, A problem concerning orthogonal polynomials. Trans. Am. Math. Soc. **37**, 196–206 (1935)
51. W. Thomson, P.G. Tait Oxford, *Treatise on Natural Philosophy* (Clarendon Press, Oxford, 1867)
52. E.T. Whittaker, G.N. Watson, *A Course in Modern Analysis* (Cambridge University Press, Cambridge, 1927)