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Swanhild Bernstein
Editor

Topics in Clifford Analysis

Special Volume in Honor of Wolfgang Sprößig

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This volume is dedicated to our colleague, friend and teacher

Wolfgang Sprößig

on the occasion of his 70th birthday.

He started with quaternionic analysis in the 1970s and was the PhD-advisor of several well-known researchers in the field of quaternionic and Clifford analysis.

W. Sprößig (left) at the 5th International Conference on Clifford Algebras and their Applications in Mathematical Physics, 1999, in Ixtapa, Zihuatanejo (Mexico).[1] *In the rows behind Wolfgang are some of his colleagues and students. Second row (from left to right): U. Kähler (University of Aveiro, PhD-student of K. Gürlebeck), S. Bernstein (TU Bergakademie Freiberg, PhD-student of W. Sprößig), K. Gürlebeck (Bauhaus-Universität Weimar, PhD-student and co-author of W. Sprößig. Last row in the background: S.-L. Eriksson (University of Helsinki)).*

[1] With courtesy of P. Cerejeiras.

Preface

This volume, compiled in honour of Wolfgang Sprößig's 70th birthday, gives an overview of modern quaternionic and Clifford analysis.

When Wolfgang Sprößig began his research in the field of quaternion analysis and elliptic partial differential equations, little was known from multidimensional function theory. Back then, function theory was conceived as a theory in $\mathbb{C}$ or $\mathbb{C}^n$ or as a theory of harmonic functions, but not in $\mathbb{R}^n$ as a refinement of harmonic function theory.

It turned out that Clifford algebras are the appropriate tool for refining the harmonic analysis and for describing a higher-dimensional analogue of the Cauchy-Riemann system, the so-called generalized Cauchy-Riemann system, and in particular the Dirac operator, which is actually related to the Dirac operator in physics. Today, many completely different topics and theories rely on Clifford analysis as a tool or as a principal research topic. The contributions to this volume exemplify various approaches of Clifford analysis and its application to partial differential equations, distributions, harmonic analysis and frameworks, monogenic polynomials, numerical methods, differential geometry, as well as discrete Clifford analysis.

I would like to thank all the colleagues who contributed to this volume—some of them being long-term friends of Wolfgang Sprößig, some of them being his students, some of them being his friends' disciples, but all of them being researchers in the field of Clifford analysis.

Freiberg, Germany
March 2019

Swanhild Bernstein

Laudation

Klaus Gürlebeck and Helmuth R. Malonek

This collection[1] contains 23 original contributions, submitted by 40 colleagues and friends on the occasion of the 70th birthday of Professor Wolfgang Sprößig. The main goal of this preface is to provide a—admittedly subjective and incomplete—review of the life and work of the celebrant, and to highlight his contribution to and his influence on the development of quaternionic and Clifford analysis, a scientific field with origins in the first half of the twentieth century. The actual importance of this field, and its vast potential for practical applications in particular, have been fully understood in the late 1970s and early 1980s—to a considerable degree due to the work of Wolfgang Sprößig.

As the authors of this preface know all too well, Professor Sprößig is not the kind of person to revel in the spotlight. Still, we do hope that this book will give him the opportunity to look back on his life path with the perspective that only the passage of time may grant and recognize some of his many achievements, reflected here through the work and the eyes of other authors. The picture that emerges is certainly dazzling.

Wolfgang Sprößig was born in the Saxon city of Chemnitz on Sunday, October 13th, 1946, to his loving parents Dora and Karl Sprößig. From 1953 until 1965, he attended high school, and in 1965 began his studies in mathematics at the University of Technology in Karl-Marx-Stadt (today: Chemnitz). In 1969, he successfully finished his studies with the (German) diploma degree. At that time, the University of Technology in Karl-Marx-Stadt was facing big challenges. New departments

[1]The authors thank Marius Mitrea for the linguistic improvement of this laudation.

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were established in order to meet the high demand caused by the industrial development in the region, and the number of students was rapidly increasing. In 1968, a reorganization of all the universities had taken place that granted more freedom to research. Also, increasing attention was being paid to the applicability of theoretical results in practice. At that time, the University of Technology in Karl-Marx-Stadt had attracted several young mathematicians with a "habilitation" who had studied at the universities of Leningrad (today: St. Petersburg) and Moscow, some of whom had also graduated there with a doctoral degree. As a result, the level of activity in pure and applied mathematics increased drastically. Among the new professors who joined this movement was Siegfried Prößdorf, who had graduated under the supervision of Solomon G. Mikhlin. In 1969, Wolfgang Sprößig began to work as a young assistant in Prößdorf's research group. He completed and defended his dissertation in 1974. The thesis is entitled *"Über die Regularisierung eines Systems zweidimensionaler singulärer Integralgleichungen, dessen Symbol endlich viele Nullstellen ganzzahliger Ordnung besitzt"* (English translation: "On the regularization of a system of two-dimensional singular integral equations whose symbol has finitely many zeros of integer order").

Having acquired a great deal of professional expertise, Wolfgang Sprößig began to direct his interests to the field of quaternionic analysis—which, as we all know, has remained a vibrant area of activity until today. He wrote five original papers that set the ground for his habilitation thesis *"Eine mehrdimensionale Operatorenrechnung über beschränkten Gebieten des Euklidischen Raumes und ihre Anwendung auf die Lösung von Gleichungen"* (English translation: "A multidimensional operator calculus on bounded domains of Euclidean space, and its application to solving equations"), published in 1979. These first five papers as well as the habilitation thesis were the beginning of an ambitious research program, which would take Wolfgang many decades to complete. But step by step, it eventually became a reality—a process that has stretched through the entire academic life of Wolfgang Sprößig. Envisioning something ahead of one's time is impressive; but having the ability and tenacity to make it real is truly momentous. Wolfgang Sprößig's original insight is now part of the universal mathematical landscape.

In the early stages, quaternionic analysis had belonged to the fringes of mathematics and was regarded as merely esoteric. Wolfgang Sprößig challenged these narrow perceptions and succeeded in changing the status quo of quaternionic analysis. With his calm yet persuasive demeanour, he made a compelling case by highlighting both the theoretical importance of this field as well as its potential for practical applicability.

Wolfgang Sprößig has an innate talent for explaining complex topics. This made him a perspicuous teacher, as already his first students could experience. His lecture on partial differential equations in 1977 was a remarkable example. In this lecture, he had incorporated some of his own original research results that were new at the time. This gave the audience a chance to acquaint themselves in a most effective manner with quaternion-valued functions, spatial Cauchy-Riemann equations, the generalized Cauchy integral, and last but not least the famous T-operator. This type of lectures was both a revelation and a challenge for the involved students, since it blurred the boundaries between doing research and teaching in a traditional way. While traditional scientists who upheld Humboldt's pioneering ideas only spoke about the union of teaching and research, Wolfgang Sprößig's students could witness this symbiosis from the very beginning. For example, the students got the chance to learn about the T-operator and applications of the associated operator calculus to boundary-value problems before these results had actually been published (1978, 1979)!

The sheer number of students who subsequently decided to pursue academic careers testifies to Wolfgang Sprößig's ability to educate and inspire young students by his way of lecturing. Wolfgang Sprößig's exuberant enthusiasm unabatedly continued throughout his entire professional life, and he has always put great effort into importing recent, original research into his lectures. The lecture notes of an international intensive course taking place in Coimbra in 2000, a book on function theory in the plane and in the space published in 2006, and the small text book *"Vector analysis"*[2] prove this side of our celebrated colleague. Overall, this set of activities perfectly embodies Wolfgang Sprößig's professional attitude, according to which there is nothing that one cannot still improve.

In 1980, Wolfgang Sprößig was recognized for his achievements by being appointed as Associate Professor ("Hochschuldozent") for Analysis at the University of Technology in Karl-Marx-Stadt. At that time, the first author of this contribution, Klaus Gürlebeck, had the opportunity to work directly under the tutelage of Professor Sprößig as a member of his research group until 1986. This turned out to be a unique learning experience that transcended the boundaries of mathematics. Klaus still vividly remembers that, on his first day of work, Professor Sprößig made unmistakably clear that failure was not an option and that great things were expected of all concerned people. This clear statement of determination and, of course, the competent scientific guidance provided by Professor Sprößig ensured that things indeed turned out that way. Wolfgang Sprößig has always understood how to get the best out of his students, with a gentle style of guidance that fostered trust and encouraged personal initiative. The pleasant and open-minded atmosphere in his research group has always been a great incentive for all those involved.

It is also worth mentioning that Wolfgang Sprößig's cooperation with SKET Magdeburg (a company in the heavy engineering industry) was intensified during this period. At that time, such cross-pollination between academia and industry had by no means been normal at universities. Still, Wolfgang Sprößig fully embraced the challenge of doing sophisticated applied mathematics and at the same time serving as the Scientific Director of this cooperation project for several years. Together with the Head of the Science Department, Prof. Hans Jäckel, he succeeded in forming a group of interested students and young staff members who tackled concrete problems arising in engineering in a systematic and sustainable way. Overall, this cooperation project lasted for almost 14 years, namely from 1977 to 1990.

[2]W. Sprößig, A. Fichtner: Vektoranalysis. 2004. 1st edition, 79 pages. Leipzig: Edition am Gutenbergplatz Leipzig (EAGLE).

This explains why also titles like *"Mathematical Foundations for Heat Treatment Technologies of Industrial Steels"*[3] show up in Wolfgang Sprößig's publication list.

Eventually, quaternionic analysis emerged as an important tool for solving such problems—and as a consequence, several generations of students had the distinct pleasure of dealing with the T-operator. As a result of these endeavours, several publications emerged with the goal of solving concrete boundary-value problems in mathematical physics, e.g., *"A Hypercomplex Method of Calculating Stress in Three-Dimensional Bodies"* by W. Sprößig and K. Gürlebeck.[4] This work essentially marks the beginning of several productive decades of collaboration between the two authors.

In 1986, Wolfgang Sprößig was appointed Chair of Analysis at the Bergakademie Freiberg—the world's first and most traditional "Montanuniversität", founded in 1765. Six years later, in 1992, he was appointed Professor of Complex Analysis at the TU Bergakademie Freiberg, where he served as Deputy Director of the Institute for Applied Mathematics until 1996. Later on, he took over the leadership of the Institute for Applied Mathematics until 2003. And finally, until his retirement in 2012, he was the Head of the Institute for Applied Analysis. In all these functions, Wolfgang Sprößig has worked tirelessly while maintaining a highly productive scientific life. In addition to about 90 scientific papers, he published 11 books, including six books on his most favourite topic, namely applications of quaternionic and Clifford analysis to boundary-value problems in mathematical physics:

- K. Gürlebeck, W. Sprößig: "Quaternionic Analysis and Elliptic Boundary Value Problems", Akademieverlag Berlin, Math. Research 56, 1989 and ISNM 89, Birkhäuser, Basel, 1990;
- K. Gürlebeck, W. Sprößig: "Quaternionic Calculus for Engineers and Physicists", John Wiley & Sons, Chichester, 1997;
- K. Gürlebeck, W. Sprößig: "Introduction in analytical and numerical methods in Clifford Algebras", Dep. de Matematica da Universidade de Coimbra, Textos de Matematica, Serie B, 2000;
- K. Gürlebeck, K. Habetha, W. Sprößig: "Funktionentheorie in der Ebene und im Raum", Birkhäuser, 2006 (in German; English translation: Function theory in the plane and in space);
- K. Gürlebeck, K. Habetha, W. Sprößig: "Holomorphic Functions in the Plane and n-dimensional Space", Birkhäuser, 2008;
- K. Gürlebeck, K. Habetha, W. Sprößig: "Application of Holomorphic Functions in Two and Higher Dimensions", Birkhäuser, Basel, 2016.

At the core of this body of work was the task of obtaining explicit representations for solutions to partial differential equations in general, and to boundary-value problems associated with such PDEs in particular. These representations can be used

[3]Wissenschaftliche Schriftenreihe (WSR) of the University of Technology in Karl-Marx-Stadt, 10/1982, in German.

[4]Suppl. Rend. Circ. Mat. Palermo, Series II, number 6, 1984, 271–284.

for establishing the uniqueness, regularity, and stability of solutions. In addition to these purely analytical questions, Wolfgang Sprößig was always interested in the numerical aspects revolving around the topic of boundary-value problems. A substantial part of the joint work with Klaus Gürlebeck was dedicated to this area (33 papers and 6 books). The basic idea was—and still is—to develop a unified theory for the analytical and numerical investigation of boundary-value problems.

As is obvious from his research output, Wolfgang Sprößig was particularly fascinated by the field of flow problems. Here, his works range from purely abstract results on the existence and regularity of solutions via the study of the spectra of relevant operators to very concrete models. In this context, one should also mention his entirely new research on weather models, which was presented only recently.

Wolfgang Sprößig has also maintained a highly visible profile at the international level. In addition to his academic and administrative tasks as coordinator in various international programs—such as the Socrates, Erasmus and Leonardo program in the European Union—, he has also been very active on the lecture tour for many years. His recent involvement in the International Master Course "Techno-Mathematics" at Hanoi University of Technology (2005–2010) deserves special mention.

Besides his scientific work, Wolfgang Sprößig has also been significantly involved in editor work for a number of scientific publications. For example, he is

- Member of the Editorial Boards of
  - Revista Scientifica
  - Advances in Applied Clifford Algebras
  - Complex Analysis and Operator Theory (CAOT)
- Advisory Editor of the book series "Frontiers in Mathematics"

And perhaps the most recognizable activity in this category: He is

- Managing Editor of the journal "Mathematical Methods in Applied Sciences" (Wiley).

Several international conference series are also very closely associated with Wolfgang's name. For many years, he was a member of the ICCA Advisory Board (International Conferences on Clifford Algebras and Applications) and Head of the Advisory Board from 2008 to 2014. Also, Wolfgang Sprößig has regularly been a co-organizer of various other meetings. In this context, the annual work for the ICNAAM sessions from 2004 to 2017 are particularly noteworthy.

This small overview of Professor Wolfgang Sprößig's work undoubtedly shows what influence he has had on the development of his area of research. He is an academic as one would imagine an academic to be. In the years since his retirement, there has been no sign of retreat, so we are looking forward to further cooperation. Dear Wolfgang, we wish you all the best for your seventies—and above all, health and a productive time.

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Part I
Clifford Analysis Theories

Chapter 1
Cauchy's Formula in Clifford Analysis: An Overview

Fred Brackx, Hennie De Schepper, Roman Lávička, and Vladimir Souček

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract The Clifford-Cauchy integral formula has proven to be a corner stone of the monogenic function theory, as is the case for the traditional Cauchy formula in the theory of holomorphic functions in the complex plane. In the recent years, several new branches of Clifford analysis have emerged. Similarly as hermitian Clifford analysis was introduced in Euclidean space $\mathbb{R}^{2n}$ of even dimension as a refinement of Euclidean Clifford analysis by the introduction of a complex structure on $\mathbb{R}^{2n}$, quaternionic Clifford analysis arose as a further refinement by the introduction of a so-called hypercomplex structure $\mathbb{Q}$, i.e. three complex structures ($\mathbb{I}$, $\mathbb{J}$, $\mathbb{K}$) which submit to the quaternionic multiplication rules, on Euclidean space $\mathbb{R}^{4p}$, the dimension now being a fourfold. Two, respectively four differential operators are constructed, leading to invariant systems under the action of the respective symmetry groups $\mathrm{U}(n)$ and $\mathrm{Sp}(p)$. Their simultaneous null solutions are respectively called hermitian and quaternionic monogenic functions. The basics of hermitian monogenicity have been studied in e.g. Brackx et al. (Compl Anal Oper Theory 1(3):341–365, 2007; Complex Var Elliptic Equ 52(10–11):1063–1079, 2007; Appl Clifford Algebras 18(3–4):451–487, 2008). Quaternionic monogenicity has been developed in, amongst others, Peña-Peña (Complex Anal Oper Theory 1:97–113, 2007), Eelbode (Complex Var Elliptic Equ 53(10):975–987, 2008), Damiano et al. (Adv Geom 11:169–189, 2011), and Brackx et al. (Adv Appl Clifford Alg 24(4):955–980, 2014; Ann Glob Anal Geom 46:409–430, 2014). In

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this contribution, we give an overview of the ways in which a Cauchy integral representation formula has been established within each of these frameworks.

Keywords Cauchy's formula · Monogenic functions

Mathematics Subject Classification (2010) Primary 30G35

1.1 Introduction

In the theory of holomorphic functions in the complex plane, i.e. the null solutions of the Cauchy–Riemann operator $\partial_{\overline{z}}$ (with $z = x + iy$), the Cauchy formula as well as the Cauchy transform play an important rôle; they both involve the so-called Cauchy kernel

$$E(z) = \frac{1}{2\pi i}\frac{1}{z}$$

which is the fundamental solution of the Cauchy–Riemann operator, i.e.

$$\partial_{\overline{z}} E(z) = \delta(z)$$

Let D be a bounded domain in $\mathbb{C}$ with (piecewise) smooth boundary ∂D. Then the Cauchy formula reproduces a holomorphic function f in the interior of D from its boundary values on ∂D as follows:

$$f(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{f(\xi)}{\xi - z}\, d\xi, \quad z \in \overset{\circ}{D}$$

while the Cauchy transform serves to generate a holomorphic function H in the interior of D from a given smooth function h on ∂D:

$$H(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{h(\xi)}{\xi - z}\, d\xi, \quad z \in \overset{\circ}{D}$$

The Cauchy formula has been extended to the case of several complex variables in two ways. Taking a holomorphic kernel and an integral over the boundary $\partial_0 D = \prod_{j=1}^n \partial D_j$ of a polydisk $D = \prod_{j=1}^n D_j$ in $\mathbb{C}^n$ leads to

$$f(z_1, \ldots, z_n) = \frac{1}{(2\pi i)^n}\int_{\partial_0 D} \frac{f(\xi_1, \ldots, \xi_n)}{(\xi_1 - z_1)\cdots(\xi_n - z_n)}\, d\xi_1 \wedge \cdots \wedge d\xi_n\ , \quad z_j \in \overset{\circ}{D}_j$$

while taking an integral over the (piecewise) smooth boundary ∂D of a bounded domain D in $\mathbb{C}^n$ in combination with the Martinelli–Bochner kernel, see e.g.

[16, 17], which is no longer holomorphic but still harmonic, results into

$$f(z) = \int_{\partial D} f(\xi)\, U(\xi, z)\ , \quad z \in \overset{\circ}{D} \tag{1.1}$$

with

$$U(\xi, z) = \frac{(n-1)!}{(2\pi i)^n} \sum_{j=1}^{n} (-1)^{j-1} \frac{\xi_j^c - z_j^c}{|\xi - z|^{2n}} \left[d\xi_j\right]$$

where

$$\left[d\xi_j\right] = d\xi_1^c \wedge \cdots \wedge d\xi_{j-1}^c \wedge d\xi_{j+1}^c \wedge \cdots \wedge d\xi_n^c \wedge d\xi_1 \wedge \cdots \wedge d\xi_n$$

and $\cdot^c$ denotes the complex conjugate. For some historical background on (1.1), which was obtained independently by Martinelli and Bochner, we refer to [16]. The formula reduces to the traditional Cauchy integral formula when $n = 1$; for $n > 1$, it establishes a connection between harmonic and complex analysis.

An alternative for generalizing Cauchy's integral formula to higher dimension is offered by Clifford analysis, the theory of monogenic functions, i.e. continuously differentiable functions defined in an open region of Euclidean space $\mathbb{R}^m$, taking their values in the Clifford algebra $\mathbb{R}_{0,m}$, or subspaces thereof, and vanishing under the action of the Dirac operator

$$\partial = \sum_{\alpha=1}^{m} e_\alpha\, \partial_{X_\alpha}$$

which corresponds to the Clifford vector variable

$$\underline{X} = \sum_{\alpha=1}^{m} e_\alpha\, X_\alpha$$

where $(e_\alpha)_{\alpha=1}^m$ is an orthonormal basis of $\mathbb{R}^m$, underlying the construction of the Clifford algebra, see e.g. [2, 12, 14, 15]. Monogenic functions are the natural higher dimensional counterparts of holomorphic functions in the complex plane. The Dirac operator factorizes the Laplacian: $\Delta_m = -\partial^2$, and is invariant under the action of the Spin(m)-group which doubly covers the SO(m)-group, whence this framework is usually referred to as Euclidean (or orthogonal) Clifford analysis. Standard references in this respect are [2, 12, 14, 15]. In this framework the Cauchy formula for a monogenic function f on a bounded domain D in $\mathbb{R}^m$ with smooth boundary ∂D can be written as

$$f(\underline{X}) = \int_{\partial D} E(\underline{Y} - \underline{X})\, d\sigma_{\underline{Y}}\, f(\underline{Y})\ , \quad \underline{X} \in \overset{\circ}{D} \tag{1.2}$$

where now the Cauchy kernel $E(\underline{X})$ in the integral over the boundary is the fundamental solution of the Dirac operator, given by

$$E(\underline{X}) = \frac{1}{a_m} \frac{\overline{\underline{X}}}{|\underline{X}|^m}$$

a_m being the area of the unit sphere S^{m-1} in $\mathbb{R}^m$, $\bar{\cdot}$ denoting the Clifford conjugation and $d\sigma$ being a Clifford algebra valued differential form of order $(m-1)$, explicitly given by

$$d\sigma_{\underline{X}} = \sum_{j=1}^{m} e_j \, (-1)^{j-1} \, \widehat{dX_j}$$

where the notation indicates that in the jth term the differential dX_j is omitted, i.e.

$$\widehat{dX_j} = dX_1 \wedge \ldots \wedge dX_{j-1} \wedge dX_{j+1} \wedge \ldots \wedge dX_n, \qquad j = 1, \ldots, m$$

This Clifford-Cauchy integral formula, which enables us to reproduce monogenic functions from their boundary values, has been a corner stone in the function theoretic development of Euclidean Clifford analysis. Similarly, monogenic functions in the interior of D are generated by the Clifford-Cauchy transform acting on a smooth function h on ∂D:

$$H(\underline{X}) = \int_{\partial D} E(\underline{Y} - \underline{X}) \, d\sigma_{\underline{Y}} \, h(\underline{Y}) \ , \quad \underline{X} \in \overset{\circ}{D}$$

This paper is devoted to giving an overview of attempts to establish Cauchy-like formulae in recent branches of Clifford analysis, i.e. hermitian and quaternionic Clifford analysis. The ingredients in any setting should thus be: a differential operator $\mathcal{D}$, a fundamental solution $\mathcal{K}$ of this differential operator, which will serve as a kernel for an integral transform which will reproduce or generate null solutions of the differential operator in the interior of a bounded domain by means of given function values on the boundary of that domain.

1.2 Hermitian Monogenicity

The first refinement of monogenicity is so-called hermitian monogenicity, for which the setting is fixed as follows: take the dimension to be even: $m = 2n$, rename the variables as

$$(X_1, \ldots, X_{2n}) = (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$$

and consider the standard complex structure $\mathbb{I}_{2n}$, i.e. the complex linear real SO$(2n)$-matrix

$$\mathbb{I}_{2n} = \mathrm{diag}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for which $\mathbb{I}_{2n}^2 = -E_{2n}$, where E_{2n} denotes the identity matrix. We then define the rotated vector variable and the corresponding rotated Dirac operator

$$\underline{X}_{\mathbb{I}} = \mathbb{I}_{2n}[\underline{X}] = \sum_{k=1}^{n}(-y_k e_{2k-1} + x_k e_{2k})$$

$$\partial_{\mathbb{I}} = \mathbb{I}_{2n}[\partial] = \sum_{k=1}^{n}(-\partial_{y_k} e_{2k-1} + \partial_{x_k} e_{2k})$$

A differentiable function F taking values in the complex Clifford algebra $\mathbb{C}_{2n}$ then is called hermitian monogenic in some region Ω of $\mathbb{R}^{2n}$, if and only if in that region F is a solution of the system

$$\partial F = 0 = \partial_{\mathbb{I}} F$$

However, one can also introduce hermitian monogenics by means of the projection operators $\pi^{\pm} = \pm\frac{1}{2}(\mathbf{1} \pm i\,\mathbb{I}_{2n})$, involving a complexification. They produce the Witt basis vectors

$$\mathfrak{f}_k = \pi^{-}[e_{2k-1}] = -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I}_{2n})[e_{2k-1}], \qquad k = 1, \ldots, n$$

$$\mathfrak{f}_k^{\dagger} = \pi^{+}[e_{2k-1}] = \frac{1}{2}(\mathbf{1} + i\,\mathbb{I}_{2n})[e_{2k-1}], \qquad k = 1, \ldots, n$$

submitting to the properties

$$\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = 0, \quad \mathfrak{f}_j^{\dagger}\mathfrak{f}_k^{\dagger} + \mathfrak{f}_k^{\dagger}\mathfrak{f}_j^{\dagger} = 0, \quad \mathfrak{f}_j\mathfrak{f}_k^{\dagger} + \mathfrak{f}_k^{\dagger}\mathfrak{f}_j = \delta jk, \qquad j, k = 1, \ldots, n$$

which imply their isotropy. By means of these Witt bases, we obtain hermitian vector variables

$$\underline{z} = -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I}_{2n})[\underline{X}] = \sum_{k=1}^{n}(x_k + iy_k)\mathfrak{f}_k = \sum_{k=1}^{n} z_k\mathfrak{f}_k,$$

$$\underline{z}^{\dagger} = \frac{1}{2}(\mathbf{1} + i\,\mathbb{I}_{2n})[\underline{X}] = \sum_{k=1}^{n}(x_k - iy_k)\mathfrak{f}_k = \sum_{k=1}^{n} \overline{z}_k\mathfrak{f}_k^{\dagger}$$

where we have introduced complex variables $(z_k, \overline{z}_k)$ in n respective complex planes. Correspondingly, the hermitian Dirac operators arise:

$$\partial_{\underline{z}}^{\dagger} = \frac{1}{2}\pi^{-}[\partial] = -\frac{1}{4}(\mathbf{1} - i\,\mathbb{I}_{2n})[\partial] = \sum_{k=1}^{n} \partial_{\overline{z}_k} \mathfrak{f}_k,$$

$$\partial_{\underline{z}} = \frac{1}{2}\pi^{+}[\partial] = \frac{1}{4}(\mathbf{1} + i\,\mathbb{I}_{2n})[\partial] = \sum_{k=1}^{n} \partial_{z_k} \mathfrak{f}_k^{\dagger}$$

It follows that for a function F on $\mathbb{R}^{2n} \cong \mathbb{C}^n$ the hermitian monogenic system is equivalent to the system

$$\partial_{\underline{z}} F = 0 = \partial_{\underline{z}}^{\dagger} F$$

which can be shown to be invariant under the action of the group $\mathrm{U}(n)$. The basics of hermitian monogenicity can be found in e.g. [3–5, 18]. For group theoretical aspects we also refer to [11, 13].

In the real approach to hermitian monogenicity we have the fundamental solutions

$$E(\underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{X}}}{|\underline{X}|^{2n}}, \quad E_{\mathbb{I}}(\underline{X}) = \frac{1}{a_{2n}} \frac{\overline{\underline{X}_{\mathbb{I}}}}{|\underline{X}_{\mathbb{I}}|^{2n}}$$

for the operators ∂ and $\partial_{\mathbb{I}}$ respectively, where now a_{2n} denotes the area of the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$. By projection they give rise to their hermitian counterparts, explicitly given by:

$$E(\underline{z}) = 2\pi^{-}[E(\underline{X})] = -E(\underline{X}) + i\,E_{\mathbb{I}}(\underline{X}) = \frac{2}{a_{2n}} \frac{\underline{z}}{|\underline{z}|^{2n}}$$

$$E^{\dagger}(\underline{z}) = 2\pi^{+}[E(\underline{X})] = E(\underline{X}) + i\,E_{\mathbb{I}}(\underline{X}) = \frac{2}{a_{2n}} \frac{\underline{z}^{\dagger}}{|\underline{z}|^{2n}}$$

However, the latter turn out to be no fundamental solutions for the hermitian Dirac operators. Introducing the particular circulant (2×2) matrices

$$\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^{\dagger})} = \begin{pmatrix} \partial_{\underline{z}} & \partial_{\underline{z}^{\dagger}} \\ \partial_{\underline{z}^{\dagger}} & \partial_{\underline{z}} \end{pmatrix}, \ \mathbf{E}(\underline{z}) = \begin{pmatrix} E(\underline{z}) & E^{\dagger}(\underline{z}) \\ E^{\dagger}(\underline{z}) & E(\underline{z}) \end{pmatrix}, \ \boldsymbol{\delta}(\underline{z}) = \begin{pmatrix} \delta(\underline{z}) & 0 \\ 0 & \delta(\underline{z}) \end{pmatrix}$$

it was obtained that

$$\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^{\dagger})} \mathbf{E}(\underline{z}) = \boldsymbol{\delta}(\underline{z})$$

whence the concept of a fundamental solution has to be reinterpreted for a matrix Dirac operator.

Consequently, this also turned out to be the case for hermitian monogenicity: a circulant matrix

$$\boldsymbol{G}_2^1 = \begin{pmatrix} g_1 & g_2 \\ g_2 & g_1 \end{pmatrix}$$

with continuously differentiable entries g_1 and g_2 defined in Ω and taking values in $\mathbb{C}_{2n}$ was then called hermitian monogenic if and only if it satisfies the system

$$\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)} \boldsymbol{G}_2^1 = \boldsymbol{O}$$

where $\boldsymbol{O}$ denotes the matrix with zero entries. An important feature of this definition of matricial hermitian monogenicity is that, for the case of a diagonal matrix

$$\boldsymbol{G}_0 == \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix}$$

i.e. when $g_1 = g$ and $g_2 = 0$, the hermitian monogenicity of $\boldsymbol{G}_0$ coincides with the hermitian monogenicity of g. This is however not the case for a full matrix $\boldsymbol{G}_2^1$ versus its entries g_1 and g_2.

Also observe that the matrix Dirac operator still factorizes the Laplacian, since

$$4\boldsymbol{\mathcal{D}}_{(\underline{z},\underline{z}^\dagger)} \boldsymbol{\mathcal{D}}^\dagger_{(\underline{z},\underline{z}^\dagger)} = \boldsymbol{\Delta}_{2n}$$

where $\boldsymbol{\Delta}_{2n}$ denotes a diagonal matrix with the Laplace operator in dimension $2n$ on the diagonal.

1.3 Cauchy Integral Formulae in the Hermitian Context

In the actual dimension and with the new notations, the classical Clifford-Cauchy formula now reads

$$f(\underline{X}) = \int_{\partial D} E(\underline{Y} - \underline{X})\, d\sigma_{\underline{Y}}\, f(\underline{Y}) \,, \quad \underline{X} \in \overset{\circ}{D}$$

where $E(\underline{X})$ was given in the previous section and the differential form $d\sigma$ of order $(2n-1)$ is explicitly given by

$$d\sigma_{\underline{X}} = \sum_{j=1}^{n} \left(e_{2j-1}\, \widehat{dx_j} - e_{2j} \widehat{dy_j} \right)$$

A formal Cauchy integral formula for hermitian monogenic circulant matrix functions was first obtained in [6]. We recall the different steps needed to arrive at it. Introducing the notations

$$\widehat{dz_j} = dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_{j-1} \wedge d\overline{z}_{j-1} \wedge d\overline{z}_j \wedge dz_{j+1} \wedge d\overline{z}_{j+1} \wedge \ldots \wedge dz_n \wedge d\overline{z}_n \tag{1.3}$$

$$\widehat{d\overline{z}_j} = dz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge dz_{j-1} \wedge d\overline{z}_{j-1} \wedge dz_j \wedge dz_{j+1} \wedge d\overline{z}_{j+1} \wedge \ldots \wedge dz_n \wedge d\overline{z}_n \tag{1.4}$$

it is easily obtained that

$$\widehat{dz_j} = 2^{n-1}(-i)^n \left[\widehat{dx_j} + i\widehat{dy_j}\right]$$
$$\widehat{d\overline{z}_j} = 2^{n-1}(-i)^n \left[\widehat{dx_j} - i\widehat{dy_j}\right]$$

Then the hermitian differential forms are defined as

$$d\sigma_{\underline{z}} = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \, \widehat{dz_j}, \qquad d\sigma_{\underline{z}^\dagger} = -\sum_{j=1}^{n} \mathfrak{f}_j \, \widehat{d\overline{z}_j}$$

which may also be obtained by projection:

$$d\sigma_{\underline{z}} = (-i)^n 2^{n-1} \pi^-[d\sigma_{\underline{X}}] = -\frac{1}{2}(-i)^n 2^{n-1} \left(d\sigma_{\underline{X}} - i\, d\sigma_{\underline{X}_{\mathbb{I}}}\right)$$
$$d\sigma_{\underline{z}^\dagger} = (-i)^n 2^{n-1} \pi^+[d\sigma_{\underline{X}}] = \frac{1}{2}(-i)^n 2^{n-1} \left(d\sigma_{\underline{X}} + i\, d\sigma_{\underline{X}_{\mathbb{I}}}\right)$$

Then, for a bounded domain $D \in \mathbb{R}^{2n}$ with smooth boundary ∂D and a full hermitian monogenic circulant matrix $\boldsymbol{G}_2^1$ the Cauchy formula reads as follows:

$$\boldsymbol{G}_2^1(\underline{X}) = \frac{1}{(-2i)^n} \int_{\partial D} \boldsymbol{E}(\underline{v} - \underline{z})\, \boldsymbol{d\Sigma}_{(\underline{v},\underline{v}^\dagger)}\, \boldsymbol{G}_2^1(\underline{Y}), \quad \underline{X} \in \overset{\circ}{D}$$

where $\underline{v}$ is the hermitian vector variable corresponding to $\underline{Y} \in \partial D$, $\underline{z}$ is the one corresponding to $\underline{X}$ in the interior of D and where the differential form matrix $\boldsymbol{d\Sigma}$ is given by

$$\boldsymbol{d\Sigma}_{(\underline{z},\underline{z}^\dagger)} = \begin{pmatrix} d\sigma_{\underline{z}} & d\sigma_{\underline{z}^\dagger} \\ d\sigma_{\underline{z}^\dagger} & d\sigma_{\underline{z}} \end{pmatrix}$$

The multiplicative constant appearing at the right hand side of the formula originates from the re-ordering of $2n$ real variables into n complex planes.

Taking for $\boldsymbol{G}_2^1$ the diagonal matrix $\boldsymbol{G}_0$, the above formula reduces to a genuine Cauchy formula for the hermitian monogenic function g, which explicitly reads

$$g(\underline{X}) = \frac{1}{(-2i)^n} \int_{\partial D} \left[E(\underline{v} - \underline{z}) d\sigma_{\underline{v}} + E^\dagger(\underline{v} - \underline{z}) d\sigma_{\underline{v}^\dagger} \right] g(\underline{Y}) \tag{1.5}$$

together with the additional integral identity

$$\int_{\partial D} \left[E(\underline{v} - \underline{z}) d\sigma_{\underline{v}^\dagger} + E^\dagger(\underline{v} - \underline{z}) d\sigma_{\underline{v}} \right] g(\underline{Y}) = 0 \tag{1.6}$$

which thus should be fulfilled for every hermitian monogenic function g on D.

We will now further comment on these obtained results, which we will derive directly from the classical Clifford-Cauchy formula. To this end we will consider functions taking values in complex spinor space

$$\mathbb{S} = \mathbb{C}_{2n}\, I \cong \mathbb{C}_n\, I$$

which is realized here by means of the primitive idempotent $I = I_1 \dots I_n$, with

$$I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger, \qquad j = 1, \dots, n$$

In [4], it has been shown that $\mathbb{S}$, considered as a $\mathrm{U}(n)$-module, decomposes as

$$\mathbb{S} = \bigoplus_{r=0}^{n} \mathbb{S}^r = \bigoplus_{r=0}^{n} (\mathbb{C}\Lambda_n^\dagger)^{(r)} I \tag{1.7}$$

into the $\mathrm{U}(n)$-invariant and irreducible subspaces

$$\mathbb{S}^r = (\mathbb{C}\Lambda_n^\dagger)^{(r)} I, \qquad j = 0, \dots, n$$

consisting of r-vectors from $\mathbb{C}\Lambda_n^\dagger$ multiplied by the idempotent I, where $\mathbb{C}\Lambda_n^\dagger$ is the Grassmann algebra generated by the Witt basis elements $\{\mathfrak{f}_1^\dagger, \dots, \mathfrak{f}_n^\dagger\}$. The spaces $\mathbb{S}^r$ are also called the "homogeneous parts" of spinor space. Consequently, any spinor valued function g decomposes as

$$g = \sum_{r=0}^{n} g^{(r)}, \quad g^{(r)} : \mathbb{C}^n \longrightarrow \mathbb{S}^r, \qquad r = 0, \dots, n$$

in its so-called homogeneous components. It is worth noticing that the action of the hermitian Dirac operators on a function F^r taking values in a fixed part $\mathbb{S}^r$ will have the following effect:

$$\partial_{\underline{z}} F^r : \mathbb{C}^n \longrightarrow \mathbb{S}^{r+1}$$
$$\partial_{\underline{z}}^\dagger F^r : \mathbb{C}^n \longrightarrow \mathbb{S}^{r-1}$$

whence for such a function, the notions of monogenicity and hermitian monogenicity are equivalent. Indeed, seen the fact that

$$\partial = 2(\partial_{\underline{z}} - \partial_{\underline{z}^\dagger})$$

hermitian monogenicity clearly implies monogenicity for any function. If moreover the function g takes values in the homogeneous part $\mathbb{S}^r$, then we have seen above that $\partial_{\underline{z}} g$ will be $\mathbb{S}^{r+1}$ valued, while $\partial_{\underline{z}^\dagger} g$ will be mS^{r-1} valued, whence $\partial g = 0$ will force both terms to be separately zero. A similar decomposition, followed by an analysis of the values may now be applied to the classical Clifford-Cauchy formula (1.2). Indeed, since all building blocks of the hermitian framework were obtained by projection (up to constants), we may conversely decompose

$$E(\underline{X}) = \frac{1}{2}\left(E^\dagger(\underline{z}) - E(\underline{z})\right) \qquad \text{and} \qquad d\sigma_{\underline{X}} = \frac{i^n}{2^{n-1}}\left(d\sigma_{\underline{z}^\dagger} - d\sigma_{\underline{z}}\right)$$

Substituting this into (1.2) yields

$$g(\underline{X}) = \frac{1}{(-2i)^n}\int_{\partial D}\left(E^\dagger(\underline{v}-\underline{z}) - E(\underline{v}-\underline{z})\right)\left(d\sigma_{\underline{v}^\dagger} - d\sigma_{\underline{v}}\right) g(\underline{Y})$$

or still

$$g(\underline{X}) = \frac{1}{(-2i)^n}\left[\int_{\partial D}\left(E^\dagger(\underline{v}-\underline{z})d\sigma_{\underline{v}^\dagger} + E(\underline{v}-\underline{z})d\sigma_{\underline{v}}\right) g(\underline{Y}) \right.$$
$$\left. + \int_{\partial D}\left(E^\dagger(\underline{v}-\underline{z})d\sigma_{\underline{v}} + E(\underline{v}-\underline{z})d\sigma_{\underline{v}^\dagger}\right) g(\underline{Y})\right]$$

Seen the definitions of $E(\underline{z})$, $E^\dagger(\underline{z})$, $d\sigma_{\underline{z}}$ and $d\sigma_{\underline{z}^\dagger}$, we will have

$$\left(E^\dagger(\underline{v}-\underline{z})d\sigma_{\underline{v}^\dagger} + E(\underline{v}-\underline{z})d\sigma_{\underline{v}}\right) g(\underline{Y}) : \mathbb{C}^n \longrightarrow \mathbb{S}^r$$

while

$$E^\dagger(\underline{v}-\underline{z})d\sigma_{\underline{v}} g(\underline{Y}) : \mathbb{C}^n \longrightarrow \mathbb{S}^{r+2} \qquad \text{and} \qquad E(\underline{v}-\underline{z})d\sigma_{\underline{v}^\dagger} g(\underline{Y}) : \mathbb{C}^n \longrightarrow \mathbb{S}^{r-2}$$

We thus directly obtain (1.5), while (1.6) can be replaced by the even stronger result

$$\int_{\partial D} E(\underline{v} - \underline{z}) d\sigma_{\underline{v}^\dagger}\, g(\underline{Y}) = 0 = \int_{\partial D} E^\dagger(\underline{v} - \underline{z}) d\sigma_{\underline{v}}\, g(\underline{Y})$$

since both terms take values in different homogeneous parts.

This conclusion may be directly generalized for any spinor valued function g; it suffices to decompose such a function into its homogeneous parts and invoke the fact that g is hermitian monogenic if and only if all its homogeneous parts $g^{(r)}$ are. We may thus write the above results separately for each component $g^{(r)}$ and simply add them.

Remark 1.1 As mentioned above, in complex analysis, an alternative way of generalizing the Cauchy formula to higher dimension is by means of the Martinelli–Bochner kernel, see e.g. [16, 17], a kernel which is not holomorphic but still harmonic, in this way establishing a connection between harmonic and holomorphic functions. The above hermitian Cauchy formula reduces to the Martinelli-Bochner formula when the considered functions take their values in the nth homogeneous part of complex spinor space, where hermitian monogenicity coincides with holomorphicity in the variables $z_1, \ldots, z_n$, and thus establishes a connection between (hermitian) Clifford analysis and complex analysis in several variables.

If we want to consider the Cauchy transform in this framework, as a generator for hermitian monogenic functions, then the integral

$$\int_{\partial D} \boldsymbol{E}(\underline{v} - \underline{z})\, \boldsymbol{d\Sigma}_{(\underline{v},\underline{v}^\dagger)}\, \boldsymbol{H}_0(\underline{\Xi}), \quad \text{for } \underline{X} \in \overset{\circ}{D}$$

for a diagonal matrix $\boldsymbol{h}_0$ having a smooth spinor valued entry h defined on ∂D, should yield a hermitian monogenic diagonal matrix H_0. This will however only be the case if h fulfills the additional conditions

$$\int_{\partial D} E(\underline{v} - \underline{z}) d\sigma_{\underline{v}^\dagger}\, h = 0 = \int_{\partial D} E^\dagger(\underline{v} - \underline{z}) d\sigma_{\underline{v}}\, h$$

1.4 Quaternionic Monogenicity

A further refinement of hermitian monogenicity is obtained by taking the dimension to be a fourfold: $m = 2n = 4p$, renumbering the variables as

$$(X_1, \ldots, X_{4p}) = (x_1, y_1, x_2, y_2, \ldots, x_{2p}, y_{2p})$$

and considering the hypercomplex structure $\mathbb{Q} = (\mathbb{I}_{4p}, \mathbb{J}_{4p}, \mathbb{K}_{4p})$ on $\mathbb{R}^{4p}$. This hypercomplex structure arises by introducing, next to the complex structure $\mathbb{I}_{4p}$, a second one, $\mathbb{J}_{4p}$, given by

$$\mathbb{J}_{4p} = \text{diag}\begin{pmatrix} & & 1 & \\ & & & -1 \\ -1 & & & \\ & 1 & & \end{pmatrix}$$

Clearly $\mathbb{J}_{4p} \in \text{SO}(4p)$, with $\mathbb{J}_{4p}^2 = -E_{4p}$, and it anti-commutes with $\mathbb{I}_{4p}$. A third SO$(4p)$-matrix

$$\mathbb{K}_{4p} = \mathbb{I}_{4p}\,\mathbb{J}_{4p} = -\mathbb{J}_{4p}\,\mathbb{I}_{4p}$$

then arises, for which $\mathbb{K}_{4p}^2 = -E_{4p}$ and which anti-commutes with both $\mathbb{I}_{4p}$ and $\mathbb{J}_{4p}$. Note that the representation of vectors is assumed to be by rows and the action of matrices on vectors thus is given by right multiplication, whence the above relation between the matrices $\mathbb{K}$, $\mathbb{I}$ and $\mathbb{J}$ in fact signifies that $\mathbb{K} = \mathbb{J} \circ \mathbb{I}$.

Next to the vector variable

$$\underline{X} = \sum_{k=1}^{n} (x_k e_{2k-1} + y_k e_{2k} + x_{k+1} e_{2k+1} + y_{k+1} e_{2k+2})$$

we now introduce the rotated variables

$$\underline{X}_{\mathbb{I}} = \sum_{k=1}^{n} (-y_k e_{2k-1} + x_k e_{2k} - y_{k+1} e_{2k+1} + x_{k+1} e_{2k+2})$$

$$\underline{X}_{\mathbb{J}} = \sum_{k=1}^{n} (-x_{k+1} e_{2k-1} + y_{k+1} e_{2k} + x_k e_{2k+1} - y_k e_{2k+2})$$

$$\underline{X}_{\mathbb{K}} = \sum_{k=1}^{n} (y_{k+1} e_{2k-1} + x_{k+1} e_{2k} - y_k e_{2k+1} - x_k e_{2k+2})$$

and we introduce the concept of quaternionic monogenicity by means of the Dirac operator

$$\partial = \sum_{k=1}^{n} (\partial_{x_k} e_{2k-1} + \partial_{y_k} e_{2k} + \partial_{x_{k+1}} e_{2k+1} + \partial_{y_{k+1}} e_{2k+2})$$

and the additional rotated Dirac operators

$$\partial_{\mathbb{I}} = \mathbb{I}_{4p}[\partial] = \sum_{k=1}^{n}(-\partial_{y_k} e_{2k-1} + \partial_{x_k} e_{2k} - \partial_{y_{k+1}} e_{2k+1} + \partial_{x_{k+1}} e_{2k+2})$$

$$\partial_{\mathbb{J}} = \mathbb{J}_{4p}[\partial] = \sum_{k=1}^{n}(-\partial_{x_{k+1}} e_{2k-1} + \partial_{y_{k+1}} e_{2k} + \partial_{x_k} e_{2k+1} - \partial_{y_k} e_{2k+2})$$

$$\partial_{\mathbb{K}} = \mathbb{K}_{4p}[\partial] = \sum_{k=1}^{n}(\partial_{y_{k+1}} e_{2k-1} + \partial_{x_{k+1}} e_{2k} - \partial_{y_k} e_{2k+1} - \partial_{x_k} e_{2k+2})$$

A differentiable function $F : \mathbb{R}^{4p} \longrightarrow \mathbb{S}$ is called quaternionic monogenic in some region Ω of $\mathbb{R}^{4p}$, if and only if in that region F is a solution of the system

$$\partial F = \partial_{\mathbb{I}} F = \partial_{\mathbb{J}} F = \partial_{\mathbb{K}} F = 0$$

Also here an alternative characterization is possible by means of complexification. In the actual dimension the hermitian vector variables read

$$\underline{z} = -\frac{1}{2}(\mathbf{1} - i\,\mathbb{I}_{4p})[\underline{X}] = \sum_{j=1}^{p}(z_{2j-1}\mathfrak{f}_{2j-1}^{\dagger} + z_{2j}\mathfrak{f}_{2j}^{\dagger})$$

$$\underline{z}^{\dagger} = \frac{1}{2}(\mathbf{1} + i\,\mathbb{I}_{4p})[\underline{X}] = \sum_{j=1}^{p}(\overline{z}_{2j-1}\mathfrak{f}_{2j-1} + \overline{z}_{2j}\mathfrak{f}_{2j})$$

and their images under the action of $\mathbb{J}_{4p}$ turn out to be

$$\underline{z}^{J} = \mathbb{J}[\underline{z}] = -\frac{1}{2}(\mathbb{J}_{4p} - i\,\mathbb{K}_{4p})[\underline{X}] = \sum_{j=1}^{p}(z_{2j}\mathfrak{f}_{2j-1}^{\dagger} - z_{2j-1}\mathfrak{f}_{2j}^{\dagger})$$

$$\underline{z}^{\dagger J} = \mathbb{J}[\underline{z}^{\dagger}] = \frac{1}{2}(\mathbb{J}_{4p} + i\,\mathbb{K}_{4p})[\underline{X}] = \sum_{j=1}^{p}(\overline{z}_{2j}\mathfrak{f}_{2j-1} - \overline{z}_{2j-1}\mathfrak{f}_{2j})$$

The corresponding quaternionic Dirac operators are

$$\partial_{\underline{z}} = \frac{1}{4}(\mathbf{1} + i\,\mathbb{I}_{2n})[\partial] = \sum_{j=1}^{p}(\partial_{z_{2j-1}}\mathfrak{f}_{2j-1}^{\dagger} + \partial_{z_{2j}}\mathfrak{f}_{2j}^{\dagger}),$$

$$\partial_{\underline{z}}^{\dagger} = -\frac{1}{4}(\mathbf{1} - i\,\mathbb{I}_{2n})[\partial] = \sum_{j=1}^{p}(\partial_{\overline{z}_{2j-1}}\mathfrak{f}_{2j-1} + \partial_{\overline{z}_{2j}}\mathfrak{f}_{2j})$$

$$\partial_{\underline{z}}^{J} = \frac{1}{4}(\mathbb{J}_{4p} + i\,\mathbb{K}_{4p})[\partial] = \mathbb{J}_{4p}[\partial_{\underline{z}}] = \sum_{j=1}^{p}(\partial_{z_{2j}}\mathfrak{f}_{2j-1} - \partial_{z_{2j-1}}\mathfrak{f}_{2j}),$$

$$\partial_{\underline{z}}^{\dagger J} = -\frac{1}{4}(\mathbb{J}_{4p} - i\,\mathbb{K}_{4p})[\partial] = \mathbb{J}_{4p}[\partial_{\underline{z}}^{\dagger}] = \sum_{j=1}^{p}(\partial_{\overline{z}_{2j}}\mathfrak{f}_{2j-1}^{\dagger} - \partial_{\overline{z}_{2j-1}}\mathfrak{f}_{2j}^{\dagger})$$

For a function F on $\mathbb{R}^{4p} \cong \mathbb{C}^{2n}$ the quaternionic system then is easily seen to be equivalent to

$$\partial_{\underline{z}} F = \partial_{\underline{z}}^{\dagger} F = \partial_{\underline{z}}^{J} F = \partial_{\underline{z}}^{\dagger J} F = 0$$

which can be shown to be invariant under the action of the symplectic group action of $\mathrm{Sp}(p)$. The basics of the quaternionic monogenic function theory were developed in [7, 8]. For group theoretical aspects we refer to [9, 10].

In the real approach to quaternionic monogenicity we have the fundamental solutions

$$E(\underline{X}) = \frac{1}{a_{4p}}\,\frac{\overline{\underline{X}}}{|\underline{X}|^{4p}}, \qquad E_{\mathbb{I}}(\underline{X}) = \frac{1}{a_{4p}}\,\frac{\overline{\underline{X}}_{\mathbb{I}}}{|\underline{X}_{\mathbb{I}}|^{4p}},$$

$$E_{\mathbb{J}}(\underline{X}) = \frac{1}{a_{4p}}\,\frac{\overline{\underline{X}}_{\mathbb{J}}}{|\underline{X}_{\mathbb{J}}|^{4p}}, \qquad E_{\mathbb{K}}(\underline{X}) = \frac{1}{a_{4p}}\,\frac{\overline{\underline{X}}_{\mathbb{K}}}{|\underline{X}_{\mathbb{K}}|^{4p}}$$

for the operators ∂, $\partial_{\mathbb{I}}$, $\partial_{\mathbb{J}}$ and $\partial_{\mathbb{K}}$, respectively, where now a_{4p} denotes the area of the unit sphere S^{4p-1} in $\mathbb{R}^{4p}$. By similar projections/decompositions as above they give rise to their quaternionic counterparts, explicitly given by:

$$E(\underline{z}) = -E(\underline{X}) + i\,E_{\mathbb{I}}(\underline{X}) = \frac{2}{a_{4p}}\,\frac{\underline{z}}{|\underline{z}|^{4p}}$$

$$E^{\dagger}(\underline{z}) = E(\underline{X}) + i\,E_{\mathbb{I}}(\underline{X}) = \frac{2}{a_{4p}}\,\frac{\underline{z}^{\dagger}}{|\underline{z}|^{4p}}$$

$$E^{J}(\underline{z}) = -E_{\mathbb{J}}(\underline{X}) + i\,E_{\mathbb{K}}(\underline{X}) = \frac{2}{a_{4p}}\,\frac{\underline{z}^{J}}{|\underline{z}|^{4p}}$$

$$E^{\dagger J}(\underline{z}) = E_{\mathbb{J}}(\underline{X}) + i\,E_{\mathbb{K}}(\underline{X}) = \frac{2}{a_{4p}}\,\frac{\underline{z}^{\dagger J}}{|\underline{z}|^{4p}}$$

However, as could be expected, the latter are no fundamental solutions for the quaternionic Dirac operators, whence again, a circulant matrix approach has to be followed. Looking closer at the explicit computations shows that

$$\partial_{\underline{z}} E(\underline{z}) = \frac{1}{2p}\beta\delta(\underline{z}) + \frac{2}{a_{4p}}\beta \mathrm{Fp}\frac{1}{|\underline{z}|^{4p}} - \frac{2}{a_{4p}}(2p)\mathrm{Fp}\frac{\underline{z}^{\dagger}\underline{z}}{|\underline{z}|^{4p+2}}$$

$$\partial_{\underline{z}}^{\dagger} E^{\dagger}(\underline{z}) = \frac{1}{2p}(2p-\beta)\delta(\underline{z}) + \frac{2}{a_{4p}}(2p-\beta)\mathrm{Fp}\frac{1}{|\underline{z}|^{4p}} - \frac{2}{a_{4p}}(2p)\mathrm{Fp}\frac{\underline{z}\underline{z}^{\dagger}}{|\underline{z}|^{4p+2}}$$

where β is a Clifford constant and Fp stands for the finite parts distribution. Similarly, we also have

$$\partial_{\underline{z}}^{J} E^{J}(\underline{z}) = \frac{1}{2p}(2p-\beta)\delta(\underline{z}) + \frac{2}{a_{4p}}(2p-\beta)\mathrm{Fp}\frac{1}{|\underline{z}|^{4p}} - \frac{2}{a_{4p}}(2p)\mathrm{Fp}\frac{\underline{z}^{\dagger J}\underline{z}^{J}}{|\underline{z}|^{4p+2}}$$

$$\partial_{\underline{z}}^{\dagger J} E^{\dagger J}(\underline{z}) = \frac{1}{2p}\beta\delta(\underline{z}) + \frac{2}{a_{4p}}\beta \mathrm{Fp}\frac{1}{|\underline{z}|^{4p}} - \frac{2}{a_{4p}}(2p)\mathrm{Fp}\frac{\underline{z}^{J}\underline{z}^{\dagger J}}{|\underline{z}|^{4p+2}}$$

Introducing the operator matrix

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \partial_{\underline{z}} & \partial_{\underline{z}}^{\dagger} & \partial_{\underline{z}}^{J} & \partial_{\underline{z}}^{\dagger J} \\ \partial_{\underline{z}}^{\dagger J} & \partial_{\underline{z}} & \partial_{\underline{z}}^{\dagger} & \partial_{\underline{z}}^{J} \\ \partial_{\underline{z}}^{J} & \partial_{\underline{z}}^{\dagger J} & \partial_{\underline{z}} & \partial_{\underline{z}}^{\dagger} \\ \partial_{\underline{z}}^{\dagger} & \partial_{\underline{z}}^{J} & \partial_{\underline{z}}^{\dagger J} & \partial_{\underline{z}} \end{pmatrix},$$

as well as the matrices

$$\mathbf{E}(\underline{z}) = \begin{pmatrix} E & E^{\dagger} & E^{J} & E^{\dagger J} \\ E^{\dagger J} & E & E^{\dagger} & E^{J} \\ E^{J} & E^{\dagger J} & E & E^{\dagger} \\ E^{\dagger} & E^{J} & E^{\dagger J} & E \end{pmatrix} \text{ and } \boldsymbol{\delta}(\underline{z}) = \begin{pmatrix} \delta(\underline{z}) & 0 & 0 & 0 \\ 0 & \delta(\underline{z}) & 0 & 0 \\ 0 & 0 & \delta(\underline{z}) & 0 \\ 0 & 0 & 0 & \delta(\underline{z}) \end{pmatrix}$$

it thus is easily obtained that

$$\boldsymbol{\mathcal{D}}\boldsymbol{E}^{\mathrm{T}}(\underline{z}) = 2\boldsymbol{\delta}(\underline{z})$$

whence a matrix fundamental solution has been found. Also this matrix Dirac operator still factorizes the Laplacian, in the sense that $2\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^{\dagger} = \boldsymbol{\Delta}_{4p}$. Notice that taking the transpose of the matrix $\mathbf{E}$ was not needed in the hermitian case, since a circulant 2×2 matrix always is symmetric. A similar (yet slightly different) strategy was developed in [1].

However, another approach is possible as well, since the actions $\partial_{\underline{z}} E^\dagger(\underline{z})$, $\partial_{\underline{z}}^\dagger E(\underline{z})$, $\partial_{\underline{z}}^J E^{\dagger J}(\underline{z})$ and $\partial_{\underline{z}}^{\dagger J} E^J(\underline{z})$ all equal zero, meaning that we can also consider

$$\boldsymbol{\mathcal{D}} = \begin{pmatrix} \partial_{\underline{z}} & \partial_{\underline{z}}^\dagger & 0 & 0 \\ \partial_{\underline{z}}^\dagger & \partial_{\underline{z}} & 0 & 0 \\ 0 & 0 & \partial_{\underline{z}}^J & \partial_{\underline{z}}^{\dagger J} \\ 0 & 0 & \partial_{\underline{z}}^{\dagger J} & \partial_{\underline{z}}^J \end{pmatrix}, \text{ and } \mathbf{E} = \begin{pmatrix} E & E^\dagger & 0 & 0 \\ E^\dagger & E & 0 & 0 \\ 0 & 0 & E^J & E^{\dagger J} \\ 0 & 0 & E^{\dagger J} & E^J \end{pmatrix} \tag{1.8}$$

for which it holds that $4\boldsymbol{\mathcal{D}}\boldsymbol{\mathcal{D}}^\dagger = \boldsymbol{\Delta}$ and

$$\boldsymbol{\mathcal{D}} \boldsymbol{E}(\underline{z}) = \boldsymbol{\delta}(\underline{z}) \tag{1.9}$$

In the next section we will see which of both possibilities is best suited for establishing a Cauchy-type formula in the quaternionic Clifford setting.

1.5 Cauchy Integral Formulae in the Quaternionic Context

In order to make a deliberate choice between both approaches, we will first have a look at the underlying group symmetry of quaternionic monogenic functions. To this end we will again consider functions taking values in complex spinor space, which now is given by

$$\mathbb{S} = \mathbb{C}_{4p}\, I \cong \mathbb{C}_{2p}\, I$$

and realized by means of the primitive idempotent $I = I_1 \dots I_{2p}$, with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger$, $j = 1, \dots, 2p$. We already know that, as a $\mathrm{U}(n)$-module, it decomposes into homogeneous parts as

$$\mathbb{S} = \bigoplus_{r=0}^{2p} \mathbb{S}^r = \bigoplus_{r=0}^{2p} (\mathbb{C}\Lambda_{2p}^\dagger)^{(r)} I$$

An important observation is that the spaces $\mathbb{S}^r$ are invariant and irreducible $\mathrm{U}(n)$ modules, but they are reducible under the action of the fundamental symmetry group $\mathrm{Sp}(p)$.

It still holds that a spinor valued function g is hermitian monogenic if and only if all its homogeneous parts $g^{(r)}$ are; however for a fixed component $g^{(r)}$ quaternionic monogenicity is not equivalent to monogenicity. Yet we have the following result (see [8]).

Proposition 1.1 *For a function $g^{(r)}$ defined on (a domain in) $\mathbb{R}^{4p} \cong \mathbb{C}^{2n}$ and taking values in $\mathbb{S}^r$, $r \in \{1, \ldots, 2p\}$, it holds that $g^{(r)}$ is quaternionic monogenic if and only if it is simultaneously ∂ and ∂^J monogenic.*

This result shows that the second attempt (1.8)–(1.9) is the right one to pursue in view of establishing a Cauchy formula, since the structure of the involved matrices reflects the importance of ∂ and ∂^J monogenicity in this setting. We thus introduce the concept of matricial quaternionic monogenicity: a block diagonal matrix

$$\boldsymbol{G} = \begin{pmatrix} g_1 & g_2 & 0 & 0 \\ g_2 & g_1 & 0 & 0 \\ 0 & 0 & g_3 & g_4 \\ 0 & 0 & g_4 & g_3 \end{pmatrix}$$

with continuously differentiable entries g_1, g_2, g_3, g_4 defined on (a domain in) $\mathbb{R}^{4p} \cong \mathbb{C}^{2p}$ and taking values in $\mathbb{C}_{4p}$ is called quaternionic monogenic if and only if it satisfies the system

$$\boldsymbol{\mathcal{D}G} = \boldsymbol{O}$$

In the case of a diagonal matrix $\boldsymbol{G}_0$ with $g_1 = g = g_3$ and $g_2 = 0 = g_4$, the quaternionic monogenicity of $\boldsymbol{G}_0$ coincides with the quaternionic monogenicity of g, which is not the case in general.

Next we define, in a similar way, the differential form matrix

$$\boldsymbol{d\Sigma} = \begin{pmatrix} d\sigma_{\underline{z}} & d\sigma_{\underline{z}^\dagger} & 0 & 0 \\ d\sigma_{\underline{z}^\dagger} & d\sigma_{\underline{z}} & 0 & 0 \\ 0 & 0 & d\sigma_{\underline{z}^J} & d\sigma_{\underline{z}^{\dagger J}} \\ 0 & 0 & d\sigma_{\underline{z}^{\dagger J}} & d\sigma_{\underline{z}^J} \end{pmatrix}$$

where, as above, we have introduced

$$d\sigma_{\underline{z}} = \sum_{j=1}^{2p} \left(\mathfrak{f}_{2j-1}^\dagger \widehat{dz_{2j-1}} + \mathfrak{f}_{2j}^\dagger \widehat{dz_{2j}} \right)$$

$$d\sigma_{\underline{z}^\dagger} = -\sum_{j=1}^{2p} \left(\mathfrak{f}_{2j-1} \widehat{d\overline{z}_{2j-1}} + \mathfrak{f}_{2j} \widehat{d\overline{z}_{2j}} \right)$$

where the notations $\widehat{dz_k}$ and $\widehat{d\overline{z}_k}$ keep their original definition, see (1.3)–(1.4), whence

$$d\sigma_{\underline{z}} = (-i)^{2p} 2^{2p-1} \pi^- [d\sigma_{\underline{X}}] = -\frac{1}{2}(-i)^{2p} 2^{2p-1} \left(d\sigma_{\underline{X}} - i\, d\sigma_{\underline{X}_{\mathbb{I}}} \right)$$

$$d\sigma_{\underline{z}^\dagger} = (-i)^{2p} 2^{p-1} \pi^+ [d\sigma_{\underline{X}}] = \frac{1}{2}(-i)^{2p} 2^{2p-1} \left(d\sigma_{\underline{X}} + i\, d\sigma_{\underline{X}_{\mathbb{I}}} \right)$$

Similarly we have defined

$$d\sigma_{\underline{z}^J} = \sum_{j=1}^{2p} \left(\mathfrak{f}_{2j-1}\, \widehat{dz_{2j}} - \mathfrak{f}_{2j}\, \widehat{dz_{2j-1}} \right)$$

$$d\sigma_{\underline{z}^{\dagger J}} = -\sum_{j=1}^{2p} \left(\mathfrak{f}^{\dagger}_{2j-1}\, \widehat{d\overline{z}_{2j}} - \mathfrak{f}^{\dagger}_{2j}\, \widehat{d\overline{z}_{2j-1}} \right)$$

or, expressed in the original real variables

$$d\sigma_{\underline{z}} = J[d\sigma_{\underline{z}}] = -\frac{1}{2}(-i)^{2p}2^{2p-1}\left(d\sigma_{\underline{X}_{\mathbb{J}}} - i\, d\sigma_{\underline{X}_{\mathbb{K}}} \right)$$

$$d\sigma_{\underline{z}^{\dagger}} = J[d\sigma_{\underline{z}^{\dagger}}] = \frac{1}{2}(-i)^{2p}2^{2p-1}\left(d\sigma_{\underline{X}_{\mathbb{J}}} + i\, d\sigma_{\underline{X}_{\mathbb{K}}} \right)$$

The resulting Cauchy formula then reads, for a bounded domain $D \in \mathbb{R}^{4p}$ with smooth boundary ∂D and a full quaternionic monogenic matrix $\boldsymbol{G}$:

$$\boldsymbol{G}(\underline{X}) = \frac{1}{(-2i)^{2p}} \int_{\partial D} \boldsymbol{E}(\underline{v} - \underline{z})\, \boldsymbol{d\Sigma}_{(\underline{v},\underline{v}^{\dagger})}\, \boldsymbol{G}(\underline{Y}), \quad \underline{X} \in \overset{\circ}{D}$$

where $\underline{v}$ is the hermitian vector variable corresponding to $\underline{Y} \in \partial D$, and $\underline{z}$ is the one corresponding to $\underline{X}$ in the interior of D. Again, the multiplicative constant appearing at the right hand side originates from the re-ordering of $4p$ real variables into $2p$ complex planes.

Taking for $\boldsymbol{G}$ the diagonal matrix $\boldsymbol{G}_0$, the above formula reduces to a genuine Cauchy formula for the hermitian monogenic function g, which splits into two reproducing formulae, given by

$$g(\underline{X}) = \frac{1}{(-2i)^n} \int_{\partial D} \left[E(\underline{v} - \underline{z}) d\sigma_{\underline{v}} + E^{\dagger}(\underline{v} - \underline{z}) d\sigma_{\underline{v}^{\dagger}} \right] g(\underline{Y})$$

$$g(\underline{X}) = \frac{1}{(-2i)^n} \int_{\partial D} \left[E^J(\underline{v} - \underline{z}) d\sigma_{\underline{v}^J} + E^{\dagger^J}(\underline{v} - \underline{z}) d\sigma_{\underline{v}^{J\dagger}} \right] g(\underline{Y})$$

stemming from the ∂ and the ∂_J part, and two additional integral identities

$$\int_{\partial D} \left[E(\underline{v} - \underline{z}) d\sigma_{\underline{v}^{\dagger}} + E^{\dagger}(\underline{v} - \underline{z}) d\sigma_{\underline{v}} \right] g(\underline{Y}) = 0$$

$$\int_{\partial D} \left[E^J(\underline{v} - \underline{z}) d\sigma_{\underline{v}^{J\dagger}} + E^{\dagger^J}(\underline{v} - \underline{z}) d\sigma_{\underline{v}^J} \right] g(\underline{Y}) = 0$$

which thus should be fulfilled for every quaternionic monogenic function g on D. The same formulae can be obtained by, as has been done explicitly for the hermitian case, splitting a spinor valued function in its homogeneous components, writing down the Cauchy formulae for ∂ and ∂_J monogenicity for each part, while invoking the structural decompositions for all building blocks involved and the subsequent splitting of the values, and finally adding all obtained results.

1.6 Future Work and Ideas

Interesting results were obtained in the hermitian framework by restricting the values of the considered functions to the different homogenous parts of spinor space, which are suggested by the $\mathrm{U}(n)$ symmetry. For quaternionic monogenics the underlying $\mathrm{Sp}(p)$ invariance has not yet been fully exploited, since the homogeneous parts of spinor space are reducible under $\mathrm{Sp}(p)$ and split further into so-called symplectic cells, see e.g. [7, 8]. This splitting is caused by the action of the multiplication operators

$$P = \mathfrak{f}_2\mathfrak{f}_1 + \mathfrak{f}_4\mathfrak{f}_3 + \ldots + \mathfrak{f}_{2p}\mathfrak{f}_{2p-1}, \qquad Q = \mathfrak{f}_1^\dagger\mathfrak{f}_2^\dagger + \mathfrak{f}_3^\dagger\mathfrak{f}_4^\dagger + \ldots + \mathfrak{f}_{2p-1}^\dagger\mathfrak{f}_{2p}^\dagger$$

for which we define, for $r = 0, \ldots, p$, the kernel spaces

$$\mathbb{S}_r^r = \mathrm{Ker}\, P|_{\mathbb{S}^r}, \qquad \mathbb{S}_r^{2p-r} = \mathrm{Ker}\, Q|_{\mathbb{S}^{2p-r}}$$

and for $k = 0, \ldots, p-r$, the subspaces obtained by iterative action of Q on the kernel of P and vice versa:

$$\mathbb{S}_r^{r+2k} = Q^k\, \mathbb{S}_r^r, \qquad \mathbb{S}_r^{2p-r-2k} = P^k\, \mathbb{S}_r^{2p-r}$$

It was shown that, for all $r = 0, \ldots, p$,

$$\mathbb{S}^r = \bigoplus_{j=0}^{\lfloor \frac{r}{2} \rfloor} \mathbb{S}_{r-2j}^r, \qquad \mathbb{S}^{2p-r} = \bigoplus_{j=0}^{\lfloor \frac{r}{2} \rfloor} \mathbb{S}_{r-2j}^{2p-r}$$

and each of the symplectic cells $\mathbb{S}_s^r$ in the above decompositions is an irreducible $\mathrm{Sp}(p)$-representation, whence we can now decompose a function $F : \mathbb{R}^{4p} \longrightarrow \mathbb{S}$ into components taking values in these symplectic cells:

$$F = \sum_{r=0}^{n} F^r = \sum_{r=0}^{n} \sum_{s} F_s^r, \qquad F_s^r : \mathbb{R}^{4p} \longrightarrow \mathbb{S}_s^r$$

The quaternionic monogenicity of F then is shown to be equivalent with the quaternionic monogenicity of each of its components F_s^r, creating an opportunity to refine even further the results obtained above.

Moreover, in [10] it was shown, that from a group theoretical point of view, the definition of quaternionic monogenicity is not the best possible one. For instance, spaces $\mathcal{Q}_{a,b}^{r,s}$ of quaternionic monogenic bi-homogeneous polynomials with values in a symplectic cell still remain reducible under the action of the group $\mathrm{Sp}(p)$, an unfortunate situation. This has lead to the definition of so-called $\mathfrak{osp}(4|2)$ monogenicity in [9, 10], where a function, apart from being quaternionic monogenic, is requested to be in the kernel of the multiplication operator P and of the Euler like scalar differential operator

$$\mathcal{E} = \sum_{k=1}^{p} z_{2k-1}\, \partial_{\overline{z}_{2k}} - z_{2k}\, \partial_{\overline{z}_{2k-1}}$$

which arises when computing the anti-commutators of all operators in the odd part of the involved Lie (super)algebra.

It remains to establish suitable integral formulae for the reproduction of such functions from their boundary values.

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Chapter 2
Quaternionic Hyperbolic Function Theory

Sirkka-Liisa Eriksson and Heikki Orelma

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract We are studying hyperbolic function theory in the skew-field of quaternions. This theory is connected to k-hyperbolic harmonic functions that are harmonic with respect to the hyperbolic Riemannian metric

$$ds_k^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^k}$$

in the upper half space $\mathbb{R}_+^4 = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : x_3 > 0\}$. In the case $k = 2$, the metric is the hyperbolic metric of the Poincaré upper half-space. Hempfling and Leutwiler started to study this case and noticed that the quaternionic power function x^m $(m \in \mathbb{Z})$, is a conjugate gradient of a 2-hyperbolic harmonic function. They researched polynomial solutions. We find fundamental k-hyperbolic harmonic functions depending only on the hyperbolic distance and x_3. Using these functions we are able to verify a Cauchy type integral formula. Earlier these results have been verified for quaternionic functions depending only on reduced variables (x_0, x_1, x_2). Our functions are depending on four variables.

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2.1 Introduction

We study hyperbolic function theory in the skew-field of quaternions, denoted by $\mathbb{H}$. This theory was initiated by Hempfling and Leutwiler in [14]. They studied quaternion valued twice continuous differentiable functions $f(x)$ defined in the full space $\mathbb{R}^4$ satisfying the following modified Cauchy-Riemann system

$$x_3\left(\frac{\partial f_0}{\partial x_0}-\frac{\partial f_1}{\partial x_1}-\frac{\partial f_2}{\partial x_2}-\frac{\partial f_3}{\partial x_3}\right)+2f_3=0,$$
$$\frac{\partial f_0}{\partial x_i}=-\frac{\partial f_i}{\partial x_0}\ \text{ for all }\ i=1,2,3,$$
$$\frac{\partial f_i}{\partial x_j}=\frac{\partial f_j}{\partial x_i}\ \text{ for all }\ i,j=1,2,3.$$

In [16] Leutwiler noticed that the power function x^m, where $m\in\mathbb{Z}$, calculated using quaternions, is a conjugate gradient of a hyperbolic harmonic function h which satisfies the equation

$$\Delta_2 h = x_3^2\Delta h - 2x_3\frac{\partial h}{\partial x_3}=0$$

where as usual

$$\Delta h=\frac{\partial^2 h}{\partial x_0^2}+\frac{\partial^2 h}{\partial x_1^2}+\frac{\partial^2 h}{\partial x_2^2}+\frac{\partial^2 h}{\partial x_3^2}.$$

The operator Δ_2 is the hyperbolic Laplace-Beltrami operator with respect to the Poincaré hyperbolic metric

$$ds^2=\frac{dx_0^2+dx_1^2+dx_2^2+dx_3^2}{x_3^2}.$$

Leutwiler and the first author in [11] studied the total Clifford algebra valued functions, called hypermonogenic functions. Their Cauchy-type formula was proved in [3, 9] and the key ideas are the relations between k and $-k$-

hypermonogenic functions, introduced in [10]. An introduction to the theory is given in [17] and in more recent papers [6] and [7].

In this paper, we verify the Cauchy type theorems for quaternionic valued functions called k-hyperregular. Our Cauchy type theorems are not directly following from the theory of quaternionic valued hypermonogenic functions, which are depending only on three variables. Our functions are depending on four variables and k is an arbitrary real coefficient. However, it is possible to deduce some results from the theory of paravector valued k-hypermonogenic funcions (see [5]) which domain of the definition is an open subset of $\mathbb{R}^4$ and the values are in the Clifford algebra $\mathcal{C}\ell_{0,3}$. These methods are rather complicated in case of quaternions and we prefer the direct methods.

2.2 Preliminaries

The space of quaternions $\mathbb{H}$ is four dimensional associative division algebra over reals with an identity $\mathbf{1}$ and generated by the elements $\mathbf{1}$, e_1, e_2 and e_3 satisfying the relations

$$e_3 = e_1 e_2$$

and

$$e_i e_j + e_j e_i = -2\delta_{ij}\mathbf{1},$$

where δ_{ij} is the usual Kronecker delta. The elements $\alpha\mathbf{1}$ and α may be identified.

We denote the coefficients of the components of a quaternion x with respect to the base $\{1, e_0, e_1, e_2\}$ by x_0, x_1, x_2 and x_3, that is

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3$$

where x_0, x_1, x_2 and x_3 are real numbers. The spaces $\mathbb{R}^4$ and $\mathbb{H}$ may be identified as vector spaces.

We denote the upper half space by

$$\mathbb{H}_+ = \{x \mid x_i \in \mathbb{R}, i = 0, 1, 2, 3 \text{ and } x_3 > 0\}$$

and the lower half space by

$$\mathbb{H}_- = \{x \mid x_i \in \mathbb{R}\ i = 0, 1, 2, 3 \text{ and } x_3 < 0\}.$$

The hyperbolic distance $d_h(x,a)$ between the points x and a in $\mathbb{H}_+$ may be computed from the formula $d_h(x,a) = \operatorname{arcosh} \lambda(\mathrm{x}, \mathrm{a})$, where

$$\begin{aligned}\lambda(x,a) &= \frac{(x_0-a_0)^2+(x_1-a_1)^2+(x_2-a_2)^2+x_3^2+a_3^2}{2x_3a_3} \\ &= \frac{\|x-a\|^2+\|x-a^*\|^2}{4x_3a_3} \\ &= \frac{\|x-a\|^2}{2x_3a_3}+1 = \frac{\|x-a^*\|^2}{2x_3a_3}-1,\end{aligned}$$

$a^* = a_0 + a_1e_1 + a_2e_2 - a_3e_3$ and the distance

$$\|x-a\| = \sqrt{(x_0-a_0)^2+(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2}$$

is the usual Euclidean distance (see the proof for example in [17]). Similarly, we may compute the hyperbolic distance between the points x and a in $\mathbb{H}_-$. Notice that if both x and a belong to $\mathbb{H}_+$ or in $\mathbb{H}_-$ then

$$d_h(x,a) = d_h\left(x^*, a^*\right).$$

We recall the following simple calculation rules

$$\|x-a\|^2 = 2x_3a_3\left(\lambda(x,a)-1\right), \tag{2.2.1}$$

$$\left\|x-a^*\right\|^2 = 2x_3a_3\left(\lambda(x,a)+1\right), \tag{2.2.2}$$

$$\frac{\|x-a\|^2}{\|x-a^*\|^2} = \frac{\lambda(x,a)-1}{\lambda(x,a)+1} = \tanh^2\left(\frac{d_h(x,a)}{2}\right). \tag{2.2.3}$$

We remind that hyperbolic balls are also Euclidean balls with a shifted center given by the next result.

Proposition 2.2.1 *The hyperbolic ball $B_h(a, r_h)$ with the hyperbolic center a in $\mathbb{H}_+$ and the radius r_h is the same as the Euclidean ball with the Euclidean center*

$$c_a(r_h) = a_0 + a_1e_1 + a_2e_2 + a_3 \cosh r_h e_3$$

and the Euclidean radius $r_e = a_3 \sinh r_h$. Conversely, if $b = (b_0, b_1, b_2, b_3)$ is a point in $\mathbb{H}_+$ and $r_e < b_3$ then the Euclidean ball $B_e(b, r_e)$ is the same as the hyperbolic ball with the hyperbolic radius

$$r_h = \operatorname{artanh}\left(\frac{r_e}{b_3}\right)$$

and the hyperbolic center

$$a = \left(b_0, b_1, b_2, \frac{b_3}{\cosh r_h} \right).$$

Corollary 2.2.2 *The hyperbolic metric in $\mathbb{H}_+$ (resp. in $\mathbb{H}_-$) is equivalent with the Euclidean metric in $\mathbb{H}_+$ (resp. in $\mathbb{H}_-$), that is they generate the same topology.*

We may extend the hyperbolic topology to the whole space. Indeed, if $U \subset \mathbb{H}$ and the set $U \cap \{x \in \mathbb{H} \mid x_3 = 0\}$ is non-empty then we call the set U open if it is open with respect to usual Euclidean topology. The inner product $\langle x, y \rangle$ in $\mathbb{H}$ is defined by

$$\langle x, y \rangle = \sum_{i=0}^{3} x_i y_i$$

similarly as in the Euclidean space $\mathbb{R}^4$.

The elements

$$x = x_0 + x_1 e_1 + x_2 e_2$$

are called *reduced quaternions* if x_0, $x_1$and x_2 are real numbers. The set of reduced quaternions is identified with $\mathbb{R}^3$.

We recall that the *prime involution* in $\mathbb{H}$ is the mapping $x \to x'$ defined by

$$x' = x_0 - x_1 e_1 - x_2 e_2 + x_3 e_3.$$

Similarly, the *reversion* in $\mathbb{H}$ is the mapping $x \to x^*$ defined by

$$x^* = x_0 + x_1 e_1 + x_2 e_2 - x_3 e_3.$$

The *conjugation* in $\mathbb{H}$ is the mapping $x \to \overline{x}$ defined by $\overline{x} = \left(x'\right)^* = (x^*)'$, that is

$$\overline{x} = x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3.$$

These involutions satisfy the following product rules

$$(xy)' = x'y',$$
$$(xy)^* = y^* x^*$$

and

$$\overline{xy} = \overline{y}\,\overline{x}$$

for all $x, y \in \mathbb{H}$.

The prime involution may be characterized also as

$$xe_3 = e_3x'$$

for all quaternions x.

The real part of a quaternion x is defined by

$$\mathrm{Re}\, x = x_0$$

and the vector part by

$$\mathrm{Vec}\, x = x_1e_1 + x_2e_2 + x_3e_3.$$

We recall the product rule

$$xy = -\langle x, y\rangle + x \times y$$

if $\mathrm{Re}\, x = \mathrm{Re}\, y = 0$, where $\times$ is the usual cross product in $\mathbb{R}^3$.

We define the mappings $S : \mathbb{H} \to \mathbb{R}^3$ and $T : \mathbb{H} \to \mathbb{R}$ by

$$Sa = a_0 + a_1e_1 + a_2e_2$$

and

$$Ta = a_3$$

for $a = a_0 + a_1e_1 + a_2e_2 + a_3e_3 \in \mathbb{H}$. Using the reversion, we compute the formulas

$$Sa = \frac{1}{2}\left(a + a^*\right), \tag{2.2.4}$$

$$Ta = -\frac{1}{2}\left(a - a^*\right)e_3. \tag{2.2.5}$$

We recall the identities

$$ab + ba = 2a\mathrm{Re}\, b + 2b\mathrm{Re}\, a - 2\langle a, b\rangle \tag{2.2.6}$$

and

$$\frac{1}{2}\left(a\overline{b}c + c\overline{b}a\right) = \langle b, c\rangle\, a - [a, b, c] \tag{2.2.7}$$

valid for all quaternions a, b and c. The term $[a, b, c]$ is called a *triple product* and is defined by

$$[a, b, c] = \langle a, c\rangle b - \langle a, b\rangle c.$$

If a, b and c are quaternions with $\operatorname{Re} a = \operatorname{Re} b = \operatorname{Re} c = 0$, then (cf. [13])

$$[a, b, c] = a \times (b \times c).$$

2.3 Hyperregular Functions

We use the following hyperbolic modifications H_k^l and H_k^r of the Cauchy-Riemann operators

$$H_k^l f(x) = D_l f(x) + k\frac{f_3}{x_3}, \quad \overline{H}_k^l f(x) = \overline{D}_l f(x) - k\frac{f_3}{x_3},$$
$$H_k^r f(x) = D_r f(x) + k\frac{f_3}{x_3}, \quad \overline{H}_k^r f(x) = \overline{D}_r f(x) - k\frac{f_3}{x_3},$$

where the parameter $k \in \mathbb{R}$ and the generalized Cauchy-Riemann operators are defined by

$$D_l f = \sum_{i=0}^{3} e_i \frac{\partial f}{\partial x_i}, \qquad \overline{D}_l f = \sum_{i=0}^{3} \overline{e_i} \frac{\partial f}{\partial x_i},$$
$$D_r f = \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} e_i, \qquad \overline{D}_r f = \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} \overline{e_i}.$$

We also abbreviate $D_l f$ by Df and H_k^l by H_k.

Definition 2.3.1 Let $\Omega \subset \mathbb{H}$ be open. A function $f : \Omega \to \mathbb{H}$ is called *k-hyperregular*, if $f \in \mathcal{C}^1(\Omega)$ and

$$H_k^l f(x) = H_k^r f(x) = 0.$$

for any $x \in \Omega \setminus \{x_3 = 0\}$.

We may simply compute the components of the operators H_k^l and H_k^r as follows.

Lemma 2.3.2 *Let $\Omega \subset \mathbb{H}$ be open. If a function $f : \Omega \to \mathbb{H}$ is differentiable then the coordinate functions of H_k^l and H_k^r are given by*

$$\begin{aligned}
\left(H_k^l f\right)_0 &= \tfrac{\partial f_0}{\partial x_0} - \tfrac{\partial f_1}{\partial x_1} - \tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_3}{\partial x_3} + k\tfrac{f_3}{x_3}, & \left(H_k^r f\right)_0 &= \left(H_k^l f\right)_0, \\
\left(H_k^l f\right)_1 &= \tfrac{\partial f_0}{\partial x_1} + \tfrac{\partial f_1}{\partial x_0} - \tfrac{\partial f_2}{\partial x_3} + \tfrac{\partial f_3}{\partial x_2}, & \left(H_k^r f\right)_1 &= \tfrac{\partial f_0}{\partial x_1} + \tfrac{\partial f_1}{\partial x_0} + \tfrac{\partial f_2}{\partial x_3} - \tfrac{\partial f_3}{\partial x_2}, \\
\left(H_k^l f\right)_2 &= \tfrac{\partial f_0}{\partial x_2} + \tfrac{\partial f_2}{\partial x_0} + \tfrac{\partial f_1}{\partial x_3} - \tfrac{\partial f_3}{\partial x_1}, & \left(H_k^r f\right)_2 &= \tfrac{\partial f_0}{\partial x_2} + \tfrac{\partial f_2}{\partial x_0} - \tfrac{\partial f_1}{\partial x_3} + \tfrac{\partial f_3}{\partial x_1}, \\
\left(H_k^l f\right)_3 &= \tfrac{\partial f_0}{\partial x_3} + \tfrac{\partial f_3}{\partial x_0} - \tfrac{\partial f_1}{\partial x_2} + \tfrac{\partial f_2}{\partial x_1}, & \left(H_k^r f\right)_3 &= \tfrac{\partial f_0}{\partial x_3} + \tfrac{\partial f_3}{\partial x_0} + \tfrac{\partial f_1}{\partial x_2} - \tfrac{\partial f_2}{\partial x_1},
\end{aligned}$$

where $(\cdot)_j$ denotes the real coefficient of the element e_j for each $j = 0, 1, 2, 3$.

We obtain immediately the following result.

Proposition 2.3.3 *Let $\Omega \subset \mathbb{H}$ be open and a function $f : \Omega \to \mathbb{H}$ continuously differentiable. A function f is k-hyperregular in Ω if and only if*

$$\begin{aligned}
\tfrac{\partial f_0}{\partial x_0} - \tfrac{\partial f_1}{\partial x_1} - \tfrac{\partial f_2}{\partial x_2} - \tfrac{\partial f_3}{\partial x_3} + k\tfrac{f_3}{x_3} &= 0, \text{ if } x_3 \neq 0, \\
\tfrac{\partial f_0}{\partial x_i} &= -\tfrac{\partial f_i}{\partial x_0} \text{ for all } i = 1, 2, 3, \\
\tfrac{\partial f_i}{\partial x_j} &= \tfrac{\partial f_j}{\partial x_i} \text{ for all } i, j = 1, 2, 3.
\end{aligned}$$

Our operators are connected to the hyperbolic metric via the hyperbolic Laplace operator as follows.

Proposition 2.3.4 *Let $f : \Omega \to \mathbb{H}$ be twice continuously differentiable. Then*

$$\begin{aligned}
H_k^l \overline{H}_k^l f =& \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3}{x_3^2} e_3 + \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right) \\
&+ \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right) e_2 \\
=& \overline{H}_k^l H_k^l f
\end{aligned}$$

and

$$\begin{aligned}
H_k^r \overline{H}_k^r f =& \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3 e_3}{x_3^2} + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) \\
&+ \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right) e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right) e_2 \\
=& \overline{H}_k^r H_k^r f.
\end{aligned}$$

Proof We just compute

$$D_l \overline{H_k^l} f = D_l \overline{D}_l f - k\frac{Df_3}{x_3} + \frac{kf_3 e_3}{x_3^2}$$
$$= \Delta f - k\frac{\frac{\partial f_3}{\partial x_0} + \frac{\partial f_3}{\partial x_1}e_1 + \frac{\partial f_3}{\partial x_2}e_2 + \frac{\partial f_3}{\partial x_3}e_3}{x_3} + \frac{kf_3 e_3}{x_3^2}$$

and

$$\left(\overline{H}_k^l f\right)_3 = \left(\overline{D}_l f\right)_3 = -\frac{\partial f_0}{\partial x_3} + \frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0}.$$

Hence we obtain

$$H_k^l \overline{H}_k^l f = \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{kf_3}{x_3^2}e_3 + \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_2} - \frac{\partial f_2}{\partial x_1}\right)$$
$$+ \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right)e_2.$$

Similarly, we compute

$$D_r \overline{H_k^r} f = D_r \overline{D}_r f - k\frac{D_r f_3}{x_3} + \frac{kf_3 e_3}{x_3^2}$$
$$= \Delta f - k\frac{\frac{\partial f_3}{\partial x_0} + \frac{\partial f_3}{\partial x_1}e_1 + \frac{\partial f_3}{\partial x_2}e_2 + \frac{\partial f_3}{\partial x_3}e_3}{x_3} + \frac{kf_3 e_3}{x_3^2}$$

and

$$\left(\overline{H}_k^r f\right)_3 = \left(\overline{D}_r f\right)_3 = -\frac{\partial f_0}{\partial x_3} - \frac{\partial f_1}{\partial x_2} + \frac{\partial f_2}{\partial x_1} + \frac{\partial f_3}{\partial x_0}.$$

Hence we have

$$H_k^r \overline{H}_k^r f = \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{kf_3 e_3}{x_3^2} + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right)$$
$$+ \frac{k}{x_3}\left(\frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1}\right)e_1 + \frac{k}{x_3}\left(\frac{\partial f_2}{\partial x_3} - \frac{\partial f_3}{\partial x_2}\right)e_2.$$

Moreover, we easily deduce that $\overline{H}_k^l H_k^l f = H_k^l \overline{H}_k^l f$ and $\overline{H}_k^r H_k^r f = H_k^r \overline{H}_k^r f$. □

We immediately obtain two corollaries.

Corollary 2.3.5 *If $f : \Omega \to \mathbb{H}$ is twice continuously differentiable and $k \neq 0$ then*

$$H_k^l \overline{H}_k^l f = H_k^r \overline{H}_k^r f = \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3 e_3}{x_3^2}$$

if and only if $\frac{\partial f_i}{\partial x_j} = \frac{\partial f_j}{\partial x_i}$ for all $i, j = 1, 2, 3$.

Corollary 2.3.6 *If $f : \Omega \to \mathbb{R}$ is real valued and twice continuously differentiable then*

$$x_3^k H_k^l \overline{H}_k^l f = x_3^k H_k^r \overline{H}_k^r f = \Delta_k f,$$

where the operator

$$\Delta_k = x_3^k \left(\Delta - \frac{k}{x_3}\frac{\partial}{\partial x_3} \right)$$

is the Laplace-Beltrami operator (see [18]) with respect to the Riemannian metric

$$ds_k^2 = \frac{dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2}{x_3^k}. \tag{2.3.1}$$

Differentiating the first equation of Proposition 2.3.3 with respect to x_i and applying the rest of the equations of Proposition 2.3.3 we obtain the following result.

Proposition 2.3.7 *Let $\Omega \subset \mathbb{H}$ be open and a function $f : \Omega \to \mathbb{H}$ twice continuously differentiable. If f is k-hyperregular then*

$$x_3^k H_k^l \overline{H}_k^l f = x_3^k H_k^r \overline{H}_k^r f = \Delta_k f + x_3^{k-2} k f_3 e_3 = 0.$$

The previous results motivate the following definition.

Definition 2.3.8 Let $\Omega \subset \mathbb{H}$ be open. A twice continuously differentiable function $f : \Omega \to \mathbb{H}$ is called *k-hyperbolic*, if

$$\Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3} + \frac{k f_3 e_3}{x_3^2} = 0.$$

There exists a characterization of k-hyperregular functions in terms of k-hyperbolic functions.

Theorem 2.3.9 *Let $\Omega \subset \mathbb{H}$ be open. A twice continuously differentiable hyperbolic harmonic function $f : \Omega \to \mathbb{H}$ is k-hyperregular if and only if the functions f and $xf + fx$ are k-hyperbolic and $H_k^l f = H_k^r f$.*

Proof In order to abbreviate notations, we denote $g = xf + fx$. Using the standard formulas $\Delta(xf) = x\Delta f + 2D_l f$ and $\Delta(fx) = (\Delta f)x + 2D_r f$ we obtain by virtue of Proposition 2.3.7, that

$$
\begin{aligned}
x_3^2\Delta g - kx_3\frac{\partial g}{\partial x_3} + kg_3e_3 =& x_3^2 x H_k^l\overline{H}_k^l f + x_3^2\left(H_k^l\overline{H}_k^l f\right)x + 2x_3^2 H_k^l f + 2x_3^2 H_k^r f \\
& - 4kx_3f_3 - kx_3\left(e_3 f + fe_3\right) + 2k\left(x_0f_3 + x_3f_0\right)e_3 \\
& - 2kf_3\left(x_0e_3 - x_3\right) \\
=& x_3^2 x H_k^l\overline{H}_k^l f + x_3^2\left(H_k^l\overline{H}_k^l f\right)x \\
& + 2x_3^2 H_k^l f + 2x_3^2 H_k^r f.
\end{aligned}
$$

If f is k-hyperregular then

$$
x_3^2 H_k^l\overline{H}_k^l f = x_3^2\Delta f - kx_3\frac{\partial f}{\partial x_3} + kf_3e_3 = 0
$$

and $H_k^l f = H_k^r f = 0$ which implies that g is k-hyperbolic. Conversely, if g and f are k-hyperbolic and $H_k^l f = H_k^r f$ then

$$
H_k^l f + H_k^r f = 0.
$$

Hence f is k-hyperregular. □

Real valued k-hyperbolic functions are especially important, since they produce k-hyperregular functions.

Theorem 2.3.10 *Let Ω be an open subset of $\mathbb{H}$. If h is real valued k-hyperbolic on Ω then the function $f = \overline{D}h$ is k-hyperregular on Ω. Conversely, if f is k-hyperregular on Ω, there exists locally a real valued k-hyperbolic function h satisfying $f = \overline{D}h$.*

Proof Let h be real k-hyperbolic on Ω and denote $f = \overline{D}h$. Applying Proposition 2.3.6 we obtain

$$
H_k^l f = H_k^l\overline{H}_k^l h = \Delta h - \frac{k}{x_3}\frac{\partial h}{\partial x_3} = 0 = H_k^r\overline{H}_k^r h = H_k^r f.
$$

Hence f is k-hyperregular. The converse statement is verified similarly as in [11]. □

We use the following transformation property proved in [4] and [15].

Lemma 2.3.11 *Let Ω be an open set contained in $\mathbb{H}_+$ or in $\mathbb{H}_-$. A function $f : \Omega \to \mathbb{R}$ is k-hyperbolic harmonic if and only if the function $g(x) = x_3^{\frac{2-k}{2}} f(x)$ satisfies the equation*

$$\Delta_2 Sg + \frac{1}{4}\left(9 - (k+1)^2\right) Sg = 0. \tag{2.3.2}$$

2.4 Cauchy Type Integral Formulas

We first recall the quaternionic version of the Stokes theorem verified for example in [13] as follows. If Ω is an open subset of $\mathbb{H}$, K a 3-chain satisfying $\overline{K} \subset \Omega$ and $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$, then

$$\int_{\partial K} g\nu f \, d\sigma = \int_K (D_r g f + g D_l f)\, dm \tag{2.4.1}$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ is the outer normal, $d\sigma$ the surface element and dm is the usual Lebesgue volume element in $\mathbb{R}^4$ identified with $\mathbb{H}$ as a vector space.

The T-part and S-part play a strong role in our operator H_k. We have therefore two versions of the Stokes theorem. The first version deals with T-parts and the second one with S-parts.

Theorem 2.4.1 *Let Ω be an open subset of $\mathbb{H} \setminus \{x_3 = 0\}$ and K a 3-chain satisfying $\overline{K} \subset \Omega$. If $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$, then*

$$\int_{\partial K} g\nu f \, d\sigma = \int_K \left(\left(H_{-k}^r g\right) f + g H_k^l f + \frac{k}{x_3}\left((g_3) Sf - Sg f_3\right)\right) dm$$

and therefore

$$T\left(\int_{\partial K} g\nu f \, d\sigma\right) = \int_K T\left(\left(H_{-k}^r g\right) f + g H_k^l f\right) dm$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ is the outer normal, $d\sigma$ the surface element and dm is the usual Lebesgue volume element in $\mathbb{R}^4$.

Proof Since $D_r g = H_{-k}^r g + k\frac{g_3}{x_3}$ and $D_l f = H_k^l f - k\frac{f_3}{x_3}$ we deduce using (2.4.1) that

$$\begin{aligned}\int_{\partial K} (g d\sigma f) &= \int_K \left(\left(H_{-k}^r g\right) f + g H_k^l f + \frac{k}{x_3}\left((g_3) f - g f_3\right)\right) dm \\ &= \int_K \left(\left(H_{-k}^r g\right) f + g H_k^l f + \frac{k}{x_3}\left((g_3) Sf - Sg f_3\right)\right) dm,\end{aligned}$$

completing the proof. □

We may also prove

Theorem 2.4.2 *Let* Ω *be an open subset of* $\mathbb{H}^4 \setminus \{x_3 = 0\}$ *and* K *a* 3*-chain satisfying* $\overline{K} \subset \Omega$*. If* $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$*, then*

$$\int_{\partial K} f \nu g d\sigma = \int_K \left(\left(H_k^r f \right) g + f H_{-k}^l g + \frac{k}{x_3} \left((g_3) Sf - Sg f_3 \right) \right) dm$$

and therefore

$$T \left(\int_{\partial K} f \nu g d\sigma \right) = \int_K T \left(\left(H_k^r f \right) g + f H_{-k}^l g \right) dm,$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ *is the outer normal,* $d\sigma$ *the surface element and* dm *is the usual Lebesgue volume element in* $\mathbb{R}^4$*.*

Proof Since $D_l g = H_{-k}^l g + k \frac{g_3}{x_3}$ and $D_r f = H_k^r f - k \frac{f_3}{x_3}$ we deduce using (2.4.1) that

$$\begin{aligned} \int_{\partial K} (g \nu f) \, d\sigma &= \int_K \left(\left(H^r f \right) g + f H_{-k}^l g + \frac{k}{x_3} \left(f g_3 - f_3 g \right) \right) dm \\ &= \int_K \left(\left(H^r f \right) f + g H_{-k}^l g + \frac{k}{x_3} \left((g_3) Sf - Sg f_3 \right) \right) dm, \end{aligned}$$

completing the proof. □

Combining previous results we conclude the following results.

Theorem 2.4.3 *Let* Ω *be an open subset of* $\mathbb{R}^4 \setminus \{x_3 = 0\}$ *and* K *a* 3*-chain satisfying* $\overline{K} \subset \Omega$*. If* $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$*, then*

$$\int_{\partial K} T \left(g \nu f + f \nu g \right) d\sigma = \int_K T \left(H_{-k}^r g f + g H_k^l f + H_k^r f g + f H_{-k}^l g \right) dm,$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ *is the outer normal,* $d\sigma$ *the surface element and* dm *is the usual Lebesgue volume element in* $\mathbb{R}^4$*.*

Theorem 2.4.4 *Let* Ω *be an open subset of* $\mathbb{R}^4 \setminus \{x_3 = 0\}$ *and* K *a* 3*-chain satisfying* $\overline{K} \subset \Omega$*. If* $f, g \in \mathcal{C}^1(\Omega, \mathbb{H})$*, then*

$$\int_{\partial K} S \left(g \nu f + f \nu g \right) \frac{d\sigma}{x_3^k} = \int_K S \left(H_k^r g f + g H_k^l f + H_k^r f g + f H_k^l g \right) \frac{dm}{x_3^k},$$

where $\nu = \nu_0 + \nu_1 e_1 + \nu_2 e_2 + \nu_3 e_3$ *is the outer normal,* $d\sigma$ *the surface element and* dm *is the usual Lebesgue volume element in* $\mathbb{R}^4$*.*

Proof Applying (2.4.1), we deduce

$$\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} = \int_K \left(D_r g f + g D_l f - k\frac{g e_3 f}{x_3} \right) \frac{dm}{x_3^k}.$$

Since $H_k^r g = D_r g + \frac{kg_3}{x_3}$ and $H_k^l f = D_l g + \frac{kf_3}{x_3}$, we infer

$$\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} = \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_3 f + g f_3 + g e_3 f}{x_3} \right) \frac{dm}{x_3^k}.$$

Using the formula $ge_3 f = ge_3 Sf - gf_3$, we obtain

$$\begin{aligned}\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} &= \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_3 f + g e_3 S f}{x_3} \right) \frac{dm}{x_3^k} \\ &= \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_3 f_3 e_3 + S g e_3 S f}{x_3} \right) \frac{dm}{x_3^k}.\end{aligned}$$

If we compute the coordinates of Sge_3Sf, we have

$$\begin{aligned}\int_{\partial K} g\nu f \frac{d\sigma}{x_3^k} =& \int_K \left(H_k^r g f + g H_k^l f - k\frac{g_0 f_0 + g_1 f_1 + g_2 f_2 + g_3 f_3}{x_3} e_3 \right) \frac{dm}{x_3^k} \\ &- \int_K k\frac{g_1 f_2 - g_2 f_1 + (g_2 f_0 - g_0 f_2)\, e_1 + (g_0 f_1 - g_1 f_0)\, e_2}{x_3^{k+1}} dm.\end{aligned}$$

If we interchange the roles of f and g, we infer

$$\begin{aligned}\int_{\partial K} f\nu g \frac{d\sigma}{x_3^k} =& \int_K \left(H_k^r f g + f H_k^l g - k\frac{g_0 f_0 + g_1 f_1 + g_1 f_1 + g_3 f_3}{x_3} e_3 \right) \frac{dm}{x_3^k} \\ &- \int_K k\frac{f_1 g_2 - f_2 g_1 + (f_2 g_0 - f_0 g_2)\, e_1 + (f_0 g_1 - f_1 g_0)\, e_2}{x_3^{k+1}} dm\end{aligned}$$

Hence

$$\begin{aligned}\int_{\partial K} (g\nu f + f\nu g) \frac{d\sigma}{x_3^k} =& \int_K \left(H_k^r g f + g H_k^l f + H_k^r f g + f H_k^l g \right) \frac{dm}{x_3^k} \\ &- 2ke_3 \int_K \frac{g_0 f_0 + g_1 f_1 + g_1 f_1 + g_3 f_3}{x_3} \frac{dm}{x_3^k}\end{aligned}$$

and therefore

$$\int_{\partial K} S\left(g\nu f + f\nu g\right)\frac{d\sigma}{x_3^k} = \int_K S\left(H_k^r gf + gH_k^l f + H_k^r fg + fH_k^l g\right)\frac{dm}{x_3^k}.$$

□

The hyperbolic Laplace operator of functions depending on λ is computed in [4] as follows.

Lemma 2.4.5 *Let x and y be points in the upper half space. If f is twice continuously differentiable depending only on $\lambda = \lambda(x, y)$, then*

$$\Delta_h f(x) = \left(\lambda^2 - 1\right)\frac{\partial^2 f}{\partial \lambda^2} + 4\lambda \frac{\partial f}{\partial \lambda}.$$

We recall the definition of the associated Legendre function of the second kind

$$Q_\nu^\mu(\lambda) = C\left(\lambda^2 - 1\right)^{\frac{\mu}{2}} \lambda^{-\nu-\mu-1}\, {}_2F_1\left(\frac{\nu+\mu+2}{2}, \frac{\mu+\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{\lambda^2}\right)$$

where

$$C = -\frac{\sqrt{\pi}\,\Gamma(\nu+\mu+1)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}.$$

and the hypergeometric function is defined by

$${}_2F_1(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m}\frac{x^m}{m!},$$

converging in the usual sense at least for x satisfying $|x| < 1$. Associated Legendre functions satisfies the differential equation (see [19])

$$(\lambda^2 - 1)u''(\lambda) + 2\lambda u'(\lambda) - \left(\nu(\nu+1) - \frac{\mu^2}{1-\lambda^2}\right)u(\lambda) = 0. \qquad (2.4.2)$$

We are looking for solutions of the equation

$$\Delta_h f(\lambda) + \gamma f(\lambda) = 0$$

in the form

$$f(\lambda) = \left(\lambda^2 - 1\right)^\alpha g(\lambda).$$

We just compute that

$$\left(\lambda^2-1\right)g''(\lambda)+(4\alpha+4)\lambda g'(\lambda)+\left(4\alpha^2+6\alpha+\gamma+\frac{2\alpha(2+2\alpha)}{\lambda^2-1}\right)g(\lambda)=0.$$

In order to compute the solutions using Legendre functions, we compare this equation with (2.4.2) and first we set $4\alpha+4=2$ and therefore $\alpha=-\frac{1}{2}$. Then we have the equation

$$\left(\lambda^2-1\right)g''(\lambda)+2\lambda g'(\lambda)+\left(-2+\gamma-\frac{1}{1-\lambda^2}\right)g(\lambda)=0$$

and again comparing with (2.4.2), we obtain equations

$$\nu(\nu+1)=2-\gamma,$$
$$\mu^2=\frac{(n-1)^2}{4}.$$

Hence $\mu=\pm 1$ and $\nu=\frac{\sqrt{9-4\gamma}-1}{2}$. Setting $-\gamma=\frac{1}{4}\left((k+1)^2-9\right)$, we obtain

$$\nu=\frac{\pm|k+1|-1}{2}.$$

Consequently, we found a solution $\left(\lambda^2-1\right)^{-\frac{1}{2}}Q^1_{\frac{|k+1|-1}{2}}(\lambda)$. Note that $Q^1_{\frac{|k+1|-1}{2}}(\lambda)$ is well defined since $\lambda>1$ and $\frac{|k+1|-1}{2}>-1$.

Denote $\nu=\frac{|k+1|-1}{2}$. Applying [19, S.2.9-4.] and the definition of $Q^1_\nu(\lambda)$, we obtain

$$\begin{aligned}Q^1_\nu(\lambda)&=-\frac{\nu+1}{2^{\nu+1}}\frac{\int_0^\pi(\lambda+\cos\alpha)^{-\nu}\sin^{2\nu+1}\alpha\,d\alpha}{\left(\lambda^2-1\right)^{\frac{1}{2}}}\\&=-\frac{\sqrt{\pi}\Gamma(\nu+2)\lambda^{-\nu}\,{}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\lambda^2}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)\left(\lambda^2-1\right)}.\end{aligned}$$

We recall that the volume measure of the Riemannian metric ds_k defined in (2.3.1) is

$$dm_k=y_3^{-2k}dm$$

where dm is the usual Lebesgue measure. Its surface element is defined by $d\sigma_{(k)} = y_3^{-\frac{3k}{2}} d\sigma$. The outer normal in $\partial B_h(x, R_h)$ is denoted by n_e and the outer normal derivative is defined by $\frac{\partial u}{\partial n^k} = y_3^{\frac{k}{2}} \frac{\partial u}{\partial n_e}$.

We prove that the function

$$F_k(x, y) = -\frac{x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}} Q_\nu^1(\cosh d_h(x, y))}{\omega_3 \sinh d_h(x, y)}$$

is the fundamental k-hyperbolic harmonic function at the point x (symmetrically y), that is $-\Delta_k F_k = \delta_x$ in the distributional sense with respect to the volume measure of the Riemannian metric ds_k and $\omega_3 = 2\pi^2$ is the Euclidean surface area of the unit ball in $\mathbb{H}$. We also remind that the fundamental k-harmonic function is unique up to the k-hyperbolic harmonic function.

We first verify the following crucial result.

Lemma 2.4.6 *Let x be a point in the upper half space and denote $\nu = \frac{|k+1|-1}{2}$. The function*

$$g_k(d_h(x, y)) = \frac{\nu+1}{2^{\nu+1}} \int_0^\pi (\cosh d_h(x, y) + \cos\alpha)^{-\nu} \sin^{2\nu+1}\alpha \, d\alpha$$
$$= \frac{\sqrt{\pi}\,\Gamma(\nu+2)\, \lambda^{-\nu}\, {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; \frac{1}{\cosh^2 d_h(x,y)}\right)}{2^{\nu+1}\Gamma\left(\nu+\frac{3}{2}\right)}$$

is positive and continuous for any $y \in \mathbb{H}_+$ and

$$g_k(0) = 1.$$

Proof Applying properties of hypergeometric functions (see for example [2]) and the Gamma function, we infer that

$${}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{2\nu+3}{2}; 1\right) = \frac{\Gamma\left(\nu+\frac{3}{2}\right)\Gamma(1)}{\Gamma\left(\frac{\nu+3}{2}\right)\Gamma\left(\frac{\nu+2}{2}\right)} = \frac{\Gamma\left(\nu+\frac{3}{2}\right)2^{\nu+1}}{\sqrt{\pi}\,\Gamma(\nu+2)}.$$

Hence $g_k(0) = 1$. □

Next we prove that $F_k(x, y)$ is integrable in the hyperbolic ball $B_h(a, R_h)$ with respect to the Riemannian volume measure dm_k.

Lemma 2.4.7 *The function $F_k(x, y)$ is integrable in the hyperbolic ball $B_h(x, R_h)$ with respect to the volume measure dm_k in the hyperbolic ball $B_h(x, R_h)$ and*

$$\int_{B_h(x,R_h)} F_k(d_h(y, x))\, dm_k(y) \leq 2^{-\frac{3k+4}{2}} M e^{\frac{|3k+2|}{2}} x_3^{-k} \sinh^2 R_h,$$

where $M = \max_{y \in \overline{B_h(x,R_h)}} (g_k(y, x)) \geq 1$.

Proof Using Proposition 2.2.1 we infer that the hyperbolic ball $B_h(x, R_h)$ is an Euclidean ball with the Euclidean center $c_x(R_h) = x_0 + x_1e_1 + x_2e_2 + x_2 \cosh R_h$ and the Euclidean radius $R_e = x_3 \sinh R_h$. Hence we deduce

$$\frac{g_k(d_h(x, y))}{x_3^2 \sinh^2 d_h(y, x)} = \frac{g_k(d_h(x, y))}{\|y - c_x(R_h)\|^2}$$

and in $B_h(x, R_h)$

$$2x_3e^{-R_h} = x_3(\cosh R_h - \sinh R_h) \leq y_3 \leq x_3(\cosh R_h + \sinh R_h) = 2x_3e^{R_h}$$

for all $y \in B_h(x, R_h)$. Since $g_k(d_h(x, y))$ is a continuous function, it attains its maximum in the closure of the ball $B_h(x, R_h)$. Since

$$\begin{aligned}\int_{B_h(x,R_h)} x_3^{-2} \sinh^{-2} d_h(y, x)\, dm(y) &= \int_{B_e(c_x(R_h), x_3 \sinh R_h)} \frac{dm(y)}{\|y - c_x(R_h)\|^2} \\ &= \int_0^{x_3 \sinh R_h} r \int_{\partial B_h(c_x(r_h),1)} dS dr \\ &= \frac{\omega_3 x_3^2 \sinh^2 R_h}{2}\end{aligned}$$

we conclude

$$\int_{B_h(x,R_h)} F_k(y, x)\, dm_k(y) \leq 2^{-\frac{3k+4}{2}} M e^{\frac{|3k+2|}{2}} x_3^{-k} \sinh^2 R_h.$$

□

We also need the result

Lemma 2.4.8 *Let $\Omega \subset \mathbb{H}_+$ be open and $\overline{B_h(x, R_h)} \subset \Omega$. Let u be a continuous real valued function in Ω. Then*

$$\lim_{R_h \to 0} \int_{\partial B_h(x,R_h)} u \frac{\partial F_k(x, y)}{\partial n^k} d\sigma_{(k)}(y) = -u(x).$$

Proof Applying Proposition 2.2.1 we obtain that the outer normal at $y \in \partial B_h(x, R_h)$ is

$$n_e = (n_0, n_1, n_2, n_3) = \frac{(y_0 - x_0, y_1 - x_1, y_2 - x_2, y_3 - x_3 \cosh R_h)}{x_3 \sinh R_h}$$

In order to abbreviate the notations, we denote briefly $r_h = d_h(y, x)$. We compute the outer normal derivative by

$$\begin{aligned} \frac{\partial F_k(x, y)}{\partial n^k} =& y_3^{\frac{k}{2}} \frac{\partial F_k(x, y)}{\partial n_e} = y_3^{\frac{k}{2}} \langle n_e, \operatorname{grad} F_k(x, y)\rangle \\ =& y_3^{k-1} x_3^{\frac{k-2}{2}} \frac{\partial \frac{g_k(r_h)}{\sinh^2 r_h}}{\partial r_h} \sum_{i=0}^{3} n_i \frac{\partial r_h}{\partial y_i} \\ &+ \frac{k-2}{2} y_3^{\frac{k-2}{2}} n_3 F_k(x, y). \end{aligned}$$

Since $r_h = \operatorname{arcos} \lambda(y, x)$ we deduce

$$\frac{\partial r_h}{\partial y_i} = \frac{\partial \operatorname{arccos} \lambda(y, x)}{\partial y_i} = \frac{y_i - x_i - x_3(\cosh r_h - 1)\delta_{i3}}{y_3 x_3 \sinh r_h}$$

and therefore the identity

$$\sum_{i=0}^{3} n_i \frac{\partial r_h}{\partial y_i} = \frac{1}{y_3}$$

holds. Hence we compute further

$$\begin{aligned} \frac{\partial F_k(x, y)}{\partial n^k} =& \frac{y_3^{k-2} x_3^{\frac{k-2}{2}}}{\omega_3 \sinh^2 r_h} \frac{\partial g_k(r_h)}{\partial r_h} + \frac{k-2}{2\omega_3} y_3^{k-2} n_3 F_k(x, y) \\ &- \frac{y_3^{k-2} x_3^{\frac{k-2}{2}} g_k(r_h) \cosh r_h}{\omega_3 \sinh^3 r_h}. \end{aligned}$$

Since $B_h(x, R_h) = B(c_x(R_h), x_3 \sin R_h)$ for

$$c_x(R_h) = x_0 + x_1 e_1 + x_2 e_2 + x_2 \cosh R_h$$

we infer that

$$\lim_{R_h \to 0} \frac{x_3^{\frac{k-4}{2}}}{\omega_3 x_3^3 \sinh^3 R_h} \int_{\partial B_h(x, R_h)} \sinh R_h\, y_3^{k-2} \frac{\partial g_k}{\partial r_h}(R_h)\, d\sigma_{(k)} = 0.$$

Similarly, we compute that

$$\lim_{R_h \to 0} \frac{(k-2)\, x_3^{\frac{k-6}{2}}}{2\omega_3 x_3^3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} y_3^{k-2} \left(y_3 - x_3 \cosh R_h \right) g_k \left(R_h \right) d\sigma_{(k)} = 0.$$

Finally, manipulating the last integral, we obtain

$$\begin{aligned} &\lim_{R_h \to 0} - \frac{g_k \left(R_h \right) \cosh R_h}{\omega_3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} y_3^{k-2} x_3^{\frac{k-2}{2}} \, d\sigma_{(k)} \\ &= \lim_{r_h \to 0} - \frac{x_3^{\frac{k+4}{2}} \cosh R_h g_k \left(R_h \right)}{\omega_3 x_3^3 \sinh^3 R_h} \int_{\partial B_h(x,R_h)} y_3^{-\frac{k+4}{2}} \, d\sigma \\ &= -u \left(x \right), \end{aligned}$$

completing the proof. □

Theorem 2.4.9 *Let $\Omega \subset \mathbb{H}_+$ be open and $B_h(a,\rho)$ a hyperbolic ball with a center a and the hyperbolic radius ρ satisfying $\overline{B_h(a,\rho)} \subset \Omega$. If u is a twice continuously differentiable functions in Ω and $x \in B_h(a,\rho)$ then*

$$\begin{aligned} u\left(x \right) = &\int_{\partial B_h(a,\rho)} \left(F_k\left(y,x \right) \frac{\partial u\left(y \right)}{\partial n^k} - u\left(y \right) \frac{\partial F_k\left(y,x \right)}{\partial n^k} \right) d\sigma_{(k)}\left(y \right) \\ &- \int_{B_h(a,\rho)} \Delta_k u\left(y \right) F_k\left(y,x \right) dm_k\left(y \right), \end{aligned}$$

where $dm_k = y_3^{-2k} dx$, $d\sigma_{(k)} = y_n^{-\frac{3k}{2}} d\sigma$ and the outer normal $\frac{\partial u}{\partial n^k} = y_3^{\frac{k}{2}} \frac{\partial u}{\partial n_e}$.

Proof Denote $B_h(a,\rho) = B$ and pick a hyperbolic ball such that $\overline{B_h(x,R_h)} \subset B$. Denote $R = B \backslash \overline{B_h(x,R_h)}$. Since F_k is k-hyperbolic harmonic in R, we may apply the Green's formula

$$\int_R \left(u \Delta_k v - v \Delta_k u \right) dm_k = \int_{\partial R} \left(u \frac{\partial v}{\partial n^k} - v \frac{\partial u}{\partial n^k} \right) d\sigma_{(k)}$$

of the Laplace-Beltrami operator

$$\Delta_k = x_3^k \left(\Delta - \frac{k}{x_3} \frac{\partial}{\partial x_3} \right)$$

with respect to the Riemannian metric ds_k^2 (see [1]) and obtain

$$\int_R F_k(y,x)\,\Delta_k u dx_k = \int_{\partial B}\left(F_k(y,x)\frac{\partial u}{\partial n^k} - u\frac{\partial F_k(y,x)}{\partial n^k}\right) d\sigma\,(k)$$
$$- \int_{\partial B_h(x,R_h)}(F_k(y,x)\frac{\partial u}{\partial n^k} - u\frac{\partial F_k(y,x)}{\partial n^k})d\sigma_{(k)}.$$

Since $\frac{\partial u}{\partial n^k}$ and $y_3^{-\frac{2k+2}{2}} x_3^{\frac{k-2}{2}} g_k(d_h(x,y))$ are bounded we obtain

$$\int_{\partial B_h(x,R_h)} |F_k(y,x)\frac{\partial u}{\partial n^k}|d\sigma_{(k)}(y) \leq \frac{M}{\sinh^2 R}\int_{\partial B_h(x,R_h)} d\sigma = M\sinh R_h$$

and therefore

$$\lim_{R_h\to 0}\int_{\partial B_h(x,R_h)} |F_k(y,x)\frac{\partial u}{\partial n^k}|d\sigma_{(k)}(y) = 0.$$

Moreover, since $F_k(x,y)$ is integrable and u is bounded on $\overline{B}$ we infer

$$\int_{B_h(a,\rho)} \Delta_k u(y)\,F_k(y,x)\,dm_k = \lim_{R_h\to 0}\int_{R_h} F_k(y,x)\,\Delta_k u\;dm_k.$$

Then applying the previous result we conclude the result. □

Using the standard methods, we deduce that

$$\phi(x) = -\int \Delta_k\phi(y)\,F_k(y,x)\,dm_k$$

for all $\phi \in \mathcal{C}_0^\infty(\mathbb{H}_+)$. Hence we have reached our main result.

Theorem 2.4.10 *Let x and y be points in the upper half space and denote $\nu = \frac{|k+1|-1}{2}$. The fundamental k-hyperbolic harmonic function is*

$$F_k(x,y) = -\frac{x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}} Q_\nu^1(\lambda(x,y))}{2^{\nu+1}\omega_3\left(\lambda(x,y)^2-1\right)^{\frac{1}{2}}}$$
$$= \frac{(\nu+1)\,x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}} \int_0^\pi(\lambda(x,y)+\cos\alpha)^{-\nu}\sin^{2\nu+1}\alpha\,d\alpha}{2^{\nu+1}\omega_3\left(\lambda(x,y)^2-1\right)}$$
$$= \frac{\sqrt{\pi}\,\Gamma(\nu+2)\,x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}-1}\lambda^{-\nu}\;{}_2F_1\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{2\nu+3}{2};\frac{1}{\lambda^2}\right)}{2^{\nu+1}\omega_3\Gamma\left(\nu+\frac{3}{2}\right)\left(\lambda(x,y)^2-1\right)}.$$

Corollary 2.4.11 *Let x and y be points in the upper half-space $\mathbb{H}_+$. Then*

$$F_k\left(x, y\right) = x_3^{k+1} y_3^{k+1} F_{-k-2}\left(x, y\right).$$

The previous result follows also from the correspondence principle of Weinstein (see [20]).

Lemma 2.4.12 *If we denote*

$$K_k\left(f\right) = \Delta f - \frac{k}{x_3}\frac{\partial f}{\partial x_3}$$

then

$$K_k\left(f\right) = x_3^{k+1} K_{-k-2}\left(x_3^{-k-1} f\right).$$

A kind of fundamental k-hyperbolic harmonic function has also been computed by GowriSankaram and Singman in [12] using more technical deductions. In order to compare the results, we first verify the following lemma.

Lemma 2.4.13 *Let $\lambda > 1$ and $\nu > -1$. Denoting $\nu + 1 = \beta$, then*

$$\int_0^\pi (\lambda - \cos\alpha)^{-\beta} \sin^{2\beta-1}\alpha \, d\alpha = 2^\beta Q_\nu^0(\lambda)$$

and therefore

$$\begin{aligned}\left(\lambda^2 - 1\right)^{-\frac{1}{2}} Q_\nu^1(\lambda) &= -\beta 2^{-\beta} \int_0^\pi (\lambda - \cos\alpha)^{-\beta-1} \sin^{2\beta-1}\alpha \, d\alpha \\ &= A \int_0^\pi \left(\|x-y\|^2 + 2x_3 y_3 \left(1 - \cos\alpha\right)\right)^{-\beta-1} \sin^{2\beta-1}\alpha \, d\alpha.\end{aligned}$$

where $A = -2\beta x_2^{\beta+1} y_3^{\beta+1}$.

Proof Applying [19, S.2.9-4.] and using complex numbers in computations, we obtain

$$\begin{aligned}Q_\nu^0(\lambda) = e^{i(\beta)\pi} Q_\nu^0(-\lambda) &= e^{i(\beta)\pi} 2^{-(\beta)} \int_0^\pi (-\lambda + \cos\alpha)^{-\beta} \sin^{2\beta-1}\alpha \, d\alpha \\ &= 2^{-(\beta)} \int_0^\pi (\lambda - \cos\alpha)^{-\beta} \sin^{2\beta-1}\alpha \, d\alpha\end{aligned}$$

Recalling the known formula

$$Q_\nu^1\left(\lambda\right) = \left(\lambda^2 - 1\right)^{\frac{1}{2}} \frac{d}{d\lambda} Q_\nu^0(\lambda)$$

we obtain the first equality. The second one follows from it when we substitute $\lambda = \frac{\|x-y\|^2+2x_3y_3}{2x_3y_3}$. □

Theorem 2.4.14 *Let x and y be points in the upper half space and denote $\nu = \frac{|k+1|-1}{2}$. The fundamental k-hyperbolic harmonic function is*

$$\omega_3 F_k(x, y) = \frac{(\nu+1)\, x_3^{\frac{k-2}{2}} y^{\frac{k-2}{2}} \int_0^\pi (\lambda - \cos\alpha)^{-\nu-2} \sin^{2\nu+1}\alpha\, d\alpha}{2^{\nu+1}}$$
$$= B \int_0^\pi \left(\|x-y\|^2 + 2x_3y_3(1-\cos\alpha)\right)^{-\nu-2} \sin^{2\nu+1}\alpha\, d\alpha$$

where

$$B = 2(\nu+1)\, x_3^{\frac{k-2}{2}+\nu+2} y_3^{\frac{k-2}{2}+\nu+2}.$$

Moreover, if $k \le -1$ then

$$\omega_3 F_k(x, y) = -k \int_0^\pi \left(\|x-y\|^2 + 2x_3y_3(1-\cos\alpha)\right)^{\frac{k-2}{2}} \sin^{-k-1}\alpha\, d\alpha,$$

and if $k \ge -1$ then

$$\frac{\omega_3 F_k(x, y)}{k+2} = x_3^{k+1} y_3^{k+1} \int_0^\pi \left(\|x-y\|^2 + 2x_3y_3(1-\cos\alpha)\right)^{-\frac{k+4}{2}} \sin^{k+1}\alpha\, d\alpha.$$

We may compute the following special cases.

1. Let $k = 0$. Then

$$F_0(x, y) = \frac{1}{2\omega_3 x_3 y_3}\left(\frac{1}{\lambda-1} - \frac{1}{\lambda+1}\right)$$
$$\frac{1}{\omega_3}\left(\frac{1}{\|x-y\|^2} - \frac{1}{\|x-y^*\|^2}\right)$$

2. Let $k = -2$. Then

$$F_{-2}(x, y) = \frac{1}{2\omega_3 x_3^2 y_3^2} \int_0^\pi (\cosh d_h(x, y) - \cos\alpha)^{-2} \sin\alpha d\alpha$$
$$= \frac{1}{2\omega_3 x_3^2 y_3^2}\left(\frac{1}{\lambda-1} - \frac{1}{\lambda+1}\right)$$

$$= \frac{1}{\omega_3 x_3^2 y_3^2 \left(\lambda^2 - 1\right)}$$

$$= \frac{1}{2\omega_3 x_3 y_3}\left(\frac{1}{\|x-y\|^2} - \frac{1}{|x-y^*|^2}\right)$$

$$= \frac{4}{\omega_3 \|x-y\|^2 \|x-y^*\|^2}.$$

3. Let $k = 2$, then

$$\begin{aligned}
2\omega_3^{-1} F_2(x, y) &= \int_0^\pi (\cosh d_h(x, y) - \cos\alpha)^{-3} \sin^3 \alpha d\alpha \\
&= \left[-2^{-1} (\cosh d_h(x, y) - \cos\alpha)^{-2} \sin^2\alpha\right]_0^\pi \\
&\quad + \int_0^\pi (\cosh d_h(x, y) - \cos\alpha)^{-2} \sin\alpha \cos\alpha d\alpha \\
&= -\left[(\cosh d_h(x, y) - \cos\alpha)^{-1} \cos\alpha\right]_0^\pi \\
&\quad - \int_0^\pi (\cosh d_h(x, y) - \cos\alpha)^{-1} \sin\alpha d\alpha \\
&= \frac{1}{\lambda - 1} + \frac{1}{\lambda + 1} - (\log(\lambda + 1) - \log(\lambda - 1)) \\
&= \frac{2\lambda}{\lambda^2 - 1} - \log(\lambda + 1) + \log(\lambda - 1).
\end{aligned}$$

Comparing this function with the kernel function computed in [8], we obtain

$$\begin{aligned}
-\int_{\frac{\|a-x\|}{\|x-a^*\|}}^1 \frac{\left(1 - s^2\right)^2}{s^3} ds &= -\int_{\frac{\|a-x\|}{\|x-a^*\|}}^1 \left(s^{-3} - 2s^{-1} + s\right) ds \\
&= \frac{|x - a^*|^2}{2\|a-x\|^2} + 2\log\frac{\|a-x\|}{\|x-a^*\|} - \frac{1}{2}\frac{\|a-x\|^2}{\|x-a^*\|^2}.
\end{aligned}$$

Applying the properties (2.2.1) and (2.2.2), we infer that

$$-\frac{1}{4}\int_{\frac{\|a-x\|}{\|x-a^*\|}}^1 \frac{\left(1 - s^2\right)^2}{s^3} ds = \frac{\lambda}{\lambda^2 - 1} - \frac{\log(\lambda + 1)}{2} + \frac{\log(\lambda - 1)}{2}$$

In order to compute the kernel function for k-hyperregular functions, we need the following lemma (see [8]).

Lemma 2.4.15 *If* $a \in \mathbb{R}^{n+1}_+$ *and* $c_a(d_h(x,a)) = a_0 + a_1e_1 + a_2e_2 + a_3 \cosh d_h(x,a)\, e_3$ *then*

$$\overline{D}^x \lambda(x,a) = \frac{\overline{x - c_a(d_h(x,a))}}{x_3 a_3}.$$

Theorem 2.4.16 *Denote* $r_h = d_h(x,y)$, $t = \frac{k-2}{2}$, $\nu = \frac{|k+1|-1}{2}$ *and define as earlier*

$$g_k(d_h(x,y)) = \frac{\nu+1}{2^{\nu+1}} \int_0^{\pi} (\cosh d_h(x,y) + \cos\alpha)^{-\nu} \sin^{2\nu+1}\alpha\, d\alpha.$$

The k*-hyperregular kernel is the function*

$$\begin{aligned} h_k(x,y) &= \frac{1}{2}\overline{D}^x (F_k(x,y)) \\ &= r(x,y) w_k(x,y)\, p(x,y) \\ &= r(x,y) p(x,y)\, v_k(x,y) \end{aligned}$$

where $r(x,y) = \frac{1}{2} x_3^{\frac{k-2}{2}} y_3^{\frac{k+4}{2}}$,

$$w_k(x,y) = -te_3 g_k(r_h)\frac{x - Py}{y_3} + \sinh r_h g_k'(r_h) - (t+2)\, g_k(r_h) \cosh r_h,$$

$$v_k(x,y) = -t\frac{x - Py}{y_3} e_3 g_k(r_h) + \sinh r_h g_k'(r_h) - (t+2)\, g_k(r_h) \cosh r_h,$$

and

$$p(x,y) = \frac{\left(x - c_y(r_h)\right)^{-1}}{x_3 \|x - c_y(r_h)\|^2}$$

is 2-hyperregular with respect to x.

Proof The function $F_k(x,y)$ is k-hyperbolic and therefore the function $h_k = \overline{D}^x F_k(x,y)$ is k-hyperregular outside y and y^*. Denoting $t = \frac{k-2}{2}$ and $\lambda(x,y) = \cosh r_h$, we compute as follows

$$\frac{2h_k(x,y)}{x_3^{\frac{k-2}{2}} y_3^{\frac{k-2}{2}}} = -\frac{te_3 g(r_h)}{x_3 \sinh^2 r_h} + \left(\frac{\sinh r_h g'(r_h) - 2g(r_h)\cosh r_h}{\sinh^3 r_h}\right)\overline{D}^x r_h.$$

Applying [8] we obtain

$$\overline{D}^x r_h = \frac{\overline{x - c_y(r_h)}}{x_3 y_3 \sinh r_h}$$

and

$$\frac{x_3 \overline{D}^x r_h}{y_3^3 \sinh^3 r_h} = \frac{\overline{x - c_y(r_h)}}{\|x - c_y(r_h)\|^4} = \frac{\left(x - c_y(r_h)\right)^{-1}}{\|x - c_y(r_h)\|^2}.$$

Since

$$\begin{aligned}\frac{x - c_y(r_h)}{x_3 y_3} \frac{\left(x - c_y(r_h)\right)^{-1}}{\|x - c_y(r_h)\|^2} &= \frac{1}{x_3 y_3 \|x - c_y(r_h)\|^2} \\ &= \frac{1}{x_3 y_3^3 \sinh^2 r_h}.\end{aligned}$$

Hence we obtain

$$\frac{h_k(x,y)}{y_3^{t+3} x_3^t} = w_k(x,y) \frac{\left(x - c_y(r_h)\right)^{-1}}{x_3 \|x - c_y(r_h)\|^2},$$

where

$$w_k(x,y) = -t e_3 g_k(r_h) \frac{x - Py}{y_3} + \sinh r_h g_k'(r_h) - (t+2) g_k(r_h) \cosh r_h.$$

Similarly we prove the other equation. □

Using the similar deductions as in [3] we may prove the formula for S and T-parts.

Theorem 2.4.17 *Let Ω and be an open subsets of $\mathbb{H}_+$ (or $\mathbb{H}_-$). Assume that K is an open subset of Ω and $\overline{K} \subset \Omega$ is a compact set with the smooth boundary whose outer unit normal field is denoted by ν. If f is k-hyperregular in Ω and $a \in K$, then*

$$\begin{aligned}Sf(a) &= -\frac{1}{2} \int_{\partial K} S\left(h_k(y,a)\nu f + f \nu h_k(y,a)\right) \frac{d\sigma}{y_3^k} \\ &= \frac{1}{2} \int_{\partial K} S\left[h_k(y,a), \overline{\nu}, f\right] \frac{d\sigma}{y_3^k} - \frac{1}{2} \int_{\partial K} S h_k(y,a) \langle \overline{\nu}, f \rangle \frac{d\sigma}{y_3^k}.\end{aligned}$$

Proof Let $a \in K$. Denote $R = K \setminus B_h(a, r_h)$ and

$$A = \int_{\partial K} S\left(h_k(y,a)\nu f(y) + f(y) \nu h_k(y,a)\right) \frac{d\sigma}{y_3^k}.$$

Then we obtain

$$0 = \int_{\partial R} S\left(h_k\left(y, a\right) \nu f\left(y\right) + f\left(y\right) \nu h_k\left(y, a\right)\right) \frac{d\sigma}{y_3^k}$$
$$= A - \int_{\partial B_h(a, r_h)} S\left(h_k\left(y, a\right) \nu\left(y\right) f\left(y\right) + f\left(y\right) \nu\left(y\right) h_k\left(y, a\right)\right) \frac{d\sigma}{y_3^k}.$$

By virtue of Proposition 2.2.1, we deduce that

$$\nu\left(y\right) = \frac{y - c_a\left(r_h\right)}{\left\| y - c_a\left(r_h\right)\right\|}.$$

Hence we obtain

$$A = -\lim_{r_h} \frac{a_3^{\frac{k-4}{2}}}{2\omega_3 \left\| a - c_a\left(r_h\right)\right\|^3} \int_{\partial B_h(a, r_h)} S\left(w_k\left(y, a\right) f + f v_k\left(y, a\right)\right) \frac{d\sigma}{y_3^{\frac{k-4}{2}}}$$
$$= -f\left(a\right).$$

The last formula follows from (2.2.7) and the definition of the triple product. □

Similarly we may verify the result for the T-part. The main difference is that we use the surface measure $d\sigma$, not $y_3^{-k} d\sigma$.

Theorem 2.4.18 *Let Ω be an open subsets of $\mathbb{H}_+$ (or $\mathbb{H}_-$). Assume that K is an open subset of Ω and $\overline{K} \subset \Omega$ is a compact set with the smooth boundary whose outer unit normal field at y is denoted by ν. If f is k-hyperregular in Ω and $a \in K$,*

$$Tf\left(a\right) = -\frac{a_3^k}{2} \int_{\partial K} T\left(h_{-k}\left(y, a\right) \nu f + f \nu\, h_{-k}\left(y, a\right)\right) d\sigma$$
$$= \frac{a_3^k}{2} \left(\int_{\partial K} T[h_{-k}\left(y, a\right), \overline{\nu}, f] d\sigma - \int_{\partial K} T h_{-k}\left(y, a\right) \langle \overline{\nu}, f\rangle\, d\sigma \right).$$

2.5 Conclusion

Our main results produce integral formulas for the T- and S-parts of k-hyperregular functions. An interesting problem is to research integral operators produced by these formulas. However, these results requires much computations and therefore they are left to the consequent publications.

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Chapter 3
Slice Regularity and Harmonicity on Clifford Algebras

Alessandro Perotti

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract We present some new relations between the Cauchy-Riemann operator on the real Clifford algebra $\mathbb{R}_n$ of signature $(0, n)$ and slice-regular functions on $\mathbb{R}_n$. The class of slice-regular functions, which comprises all polynomials with coefficients on one side, is the base of a recent function theory in several hypercomplex settings, including quaternions and Clifford algebras. In this paper we present formulas, relating the Cauchy-Riemann operator, the spherical Dirac operator, the differential operator characterizing slice regularity, and the spherical derivative of a slice function. The computation of the Laplacian of the spherical derivative of a slice regular function gives a result which implies, in particular, the Fueter-Sce Theorem. In the two four-dimensional cases represented by the paravectors of $\mathbb{R}_3$ and by the space of quaternions, these results are related to zonal harmonics on the three-dimensional sphere and to the Poisson kernel of the unit ball of $\mathbb{R}^4$.

Keywords Slice-regular functions · Dirac operator · Clifford analysis · Quaternions

Mathematics Subject Classification (2010) Primary 30G35; Secondary 32A30, 33C55, 31A30, 15A66

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3.1 Introduction and Preliminaries

Let $\mathbb{R}_n$ denote the real Clifford algebra of signature $(0, n)$, with basis vectors $e_1, \ldots, e_n$. Consider the Dirac operator

$$\mathcal{D} = e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n}$$

and the Cauchy-Riemann operators

$$\overline{\partial} = \frac{\partial}{\partial x_0} + e_1 \frac{\partial}{\partial x_1} + \cdots + e_n \frac{\partial}{\partial x_n} \quad \text{and} \quad \partial = \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - \cdots - e_n \frac{\partial}{\partial x_n}$$

on $\mathbb{R}_n$. The operator $\overline{\partial}$ factorizes the Laplacian operator

$$\partial\overline{\partial} = \overline{\partial}\partial = \Delta_{n+1} = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}$$

of the paravector space

$$V = \{x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{R}_n \mid x_0, \ldots, x_n \in \mathbb{R}\} \simeq \mathbb{R}^{n+1}.$$

This property is one of the most attractive aspects of Clifford analysis, the well-developed function theory based on Dirac and Cauchy-Riemann operators (see [4, 8, 19] and the vast bibliography therein).

In this paper we prove some new relations between the Cauchy-Riemann operator, the spherical Dirac operator, the Laplacian operator and the class of *slice-regular* functions on a Clifford algebra. Slice-regular functions constitute a recent but rapidly expanding function theory in several hypercomplex settings, including quaternions and real Clifford algebras (see [5, 10–12, 16, 17]). This class of functions was introduced by Gentili and Struppa [10] in 2006–2007 for functions of one quaternionic variable. Let $\mathbb{H}$ denote the skew field of quaternions, with basic elements i, j, k. For each quaternion J in the sphere of imaginary units

$$\mathbb{S}_{\mathbb{H}} = \{J \in \mathbb{H} \mid J^2 = -1\} = \{x_1 i + x_2 j + x_3 k \in \mathbb{H} \mid x_1^2 + x_2^2 + x_3^2 = 1\},$$

let $\mathbb{C}_J = \langle 1, J \rangle \simeq \mathbb{C}$ be the subalgebra generated by J. Then we have the "slice" decomposition

$$\mathbb{H} = \bigcup_{J \in \mathbb{S}_{\mathbb{H}}} \mathbb{C}_J, \quad \text{with } \mathbb{C}_J \cap \mathbb{C}_K = \mathbb{R} \quad \text{for every } J, K \in \mathbb{S}_{\mathbb{H}},\ J \neq \pm K.$$

A differentiable function $f : \Omega \subseteq \mathbb{H} \to \mathbb{H}$ is called *(left) slice-regular* on Ω if, for each $J \in \mathbb{S}_{\mathbb{H}}$, the restriction

$$f_{|\Omega\cap\mathbb{C}_J} : \Omega \cap \mathbb{C}_J \to \mathbb{H}$$

is holomorphic with respect to the complex structure defined by left multiplication by J. For example, polynomials $f(x) = \sum_m x^m a_m$ with quaternionic coefficients on the right are slice-regular on $\mathbb{H}$. More generally, convergent power series are slice-regular on an open ball centered at the origin. Observe that nonconstant polynomials do not belong to the kernel of the Cauchy-Riemann-Fueter operator

$$\overline{\partial}_{CRF} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}.$$

Here x_0, x_1, x_2, x_3 denote the real components of a quaternion $x = x_0 + x_1 i + x_2 j + x_3 k$.

Let $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ be the algebra of complex quaternions, with elements $w = a + \imath b$, $a, b \in \mathbb{H}$, $\imath^2 = -1$. Every quaternionic polynomial $f(x) = \sum_m x^m a_m$ lifts to a unique polynomial function $F : \mathbb{C} \to \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ which makes the following diagram commutative for all $J \in \mathbb{S}_{\mathbb{H}}$:

$$\begin{array}{ccc} \mathbb{C} \simeq \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{\ F\ } & \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \\ \Phi_J \downarrow & & \downarrow \Phi_J \\ \mathbb{H} & \xrightarrow{\ f\ } & \mathbb{H} \end{array}$$

where $\Phi_J : \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{H}$ is defined by $\Phi_J(a + \imath b) := a + Jb$. The lifted polynomial is simply $F(z) = \sum_m z^m a_m$, with variable $z = \alpha + \imath\beta \in \mathbb{C}$. This lifting property is equivalent to the following fact: for each $z = \alpha + \imath\beta \in \mathbb{C}$, the restriction of f to the 2-sphere $\alpha + \mathbb{S}_{\mathbb{H}}\beta = \cup_{J \in \mathbb{S}_{\mathbb{H}}} \Phi_J(z)$, is a quaternionic left-affine function with respect to $J \in \mathbb{S}_{\mathbb{H}}$, namely of the form $J \mapsto a + Jb$ $(a, b \in \mathbb{H})$.

In this lifting, the usual product of polynomials with coefficients in $\mathbb{H}$ on one fixed side (the one obtained by imposing commutativity of the indeterminate with the coefficients when two polynomials are multiplied together) corresponds to the pointwise product in the algebra $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$.

More generally, if a quaternionic function f (not necessarily a polynomial) has a holomorphic lifting F then f is called *(left) slice-regular*.

This approach to slice regularity can be pursued on an ample class of real algebras. Here we recall the basic definitions and refer to [12, 16] for details and other references. Let A be a real alternative algebra with unity e. The real multiples of e in A are identified with the real numbers. Assume that A is a *-algebra, i.e. it is equipped with a linear antiinvolution $x \mapsto x^c$, such that $(xy)^c = y^c x^c$ for all $x, y \in A$ and $x^c = x$ for x real. Let $t(x) := x + x^c \in A$ be the *trace* of x and

$n(x) := xx^c \in A$ the *norm* of x. Let

$$\mathbb{S}_A := \{J \in A \mid t(x) = 0,\ n(x) = 1\}$$

be the "sphere" of the imaginary units of A compatible with the *-algebra structure of A. Assuming $\mathbb{S}_A \neq \emptyset$, one can consider the *quadratic cone* of A, defined as the subset of A

$$\mathcal{Q}_A := \bigcup_{J \in \mathbb{S}_A} \mathbb{C}_J$$

where $\mathbb{C}_J = \langle 1, J \rangle$ is the complex "slice" of A generated as a subalgebra by $J \in \mathbb{S}_A$. It holds $\mathbb{C}_J \cap \mathbb{C}_K = \mathbb{R}$ for each $J, K \in \mathbb{S}_A$, $J \neq \pm K$. The quadratic cone is a real cone invariant with respect to translations along the real axis.

Observe that t and n are real-valued on $\mathcal{Q}_A$ and that $\mathcal{Q}_A = A$ if and only if $A \simeq \mathbb{C}, \mathbb{H}, \mathbb{O}$ (where $\mathbb{O}$ is the algebra of octonions).

The remark made above about quaternionic polynomials suggests a way to define polynomials with coefficients in A or more generally A-valued functions on the quadratic cone of A. Let $J \in \mathbb{S}_A$ and let $\Phi_J : A \otimes_{\mathbb{R}} \mathbb{C} \to A$ defined by $\Phi_J(a + \imath b) := a + Jb$ for any $a, b \in A$. By imposing commutativity of diagrams

$$\begin{array}{ccc} \mathbb{C} \simeq \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} & \xrightarrow{F} & A \otimes_{\mathbb{R}} \mathbb{C} \\ {\scriptstyle \Phi_J} \downarrow & & \downarrow {\scriptstyle \Phi_J} \\ \mathcal{Q}_A & \xrightarrow{f} & A \end{array} \tag{3.1.1}$$

for all $J \in \mathbb{S}_A$, we get the class of *slice functions* on A. This is the class of functions which are compatible with the slice character of the quadratic cone.

More precisely, let $D \subseteq \mathbb{C}$ be a set that is invariant with respect to complex conjugation. In $A \otimes_{\mathbb{R}} \mathbb{C}$ consider the complex conjugation mapping $w = a + \imath b$ to $\overline{w} = a - \imath b$ $(a, b \in A)$. If a function $F : D \to A \otimes_{\mathbb{R}} \mathbb{C}$ satisfies $F(\overline{z}) = \overline{F(z)}$ for every $z \in D$, then F is called a *stem function* on D. Let Ω_D be the *circular* subset of the quadratic cone defined by

$$\Omega_D = \bigcup_{J \in \mathbb{S}_A} \Phi_J(D).$$

The stem function $F = F_1 + \imath F_2 : D \to A \otimes_{\mathbb{R}} \mathbb{C}$ induces the *(left) slice function* $f = \mathcal{I}(F) : \Omega_D \to A$ in the following way: if $x = \alpha + J\beta = \Phi_J(z) \in \Omega_D \cap \mathbb{C}_J$, then

$$f(x) = F_1(z) + JF_2(z),$$

where $z = \alpha + \imath\beta$. The slice function $f = \mathcal{I}(F)$ is called *(left) slice-regular* if F is holomorphic. The function $f = \mathcal{I}(F)$ is called *slice-preserving* if F_1 and F_2 are real-valued (this is the case already considered by Fueter [9] for quaternionic functions and by Gürlebeck and Sprössig (cf. [18, 19]) for *radially holomorphic* functions on Clifford algebras). In this case, the condition $f(x^c) = f(x)^c$ holds for each $x \in \Omega_D$.

When A is the algebra of real quaternions and the domain D intersects the real axis, this definition of slice regularity is equivalent to the one proposed by Gentili and Struppa [10].

In this paper we are mainly interested in the case where A is the real 2^n-dimensional Clifford algebra $\mathbb{R}_n$ with signature $(0, n)$. Let $e_1, \dots, e_n$ be the basic units of $\mathbb{R}_n$, satisfying $e_ie_j + e_je_i = -2\delta_{ij}$ for each i, j, and let e_K denote the basis elements $e_K = e_{i_1} \cdots e_{i_k}$, with $e_\emptyset = 1$ and $K = (i_1, \dots, i_k)$ an increasing multi-index of length k, with $0 \le k \le n$. Every element $x \in \mathbb{R}_n$ can be written as $x = \sum_K x_K e_K$, with $x_K \in \mathbb{R}$. We will identify paravectors, i.e. elements $x \in \mathbb{R}_n$ of the form $x = x_0 + x_1e_1 + \cdots + x_ne_n$, with elements of the Euclidean space $\mathbb{R}^{n+1}$.

The Clifford conjugation $x \mapsto x^c$ is the unique antiinvolution of $\mathbb{R}_n$ such that $e_i^c = -e_i$ for $i = 1, \dots, n$. If $x = x_0 + x_1e_1 + \cdots + x_ne_n \in \mathbb{R}^{n+1}$ is a paravector, then $x^c = x_0 - x_1e_1 - \cdots - x_ne_n$. Therefore $t(x) = x + x^c = 2x_0$ and $n(x) = xx^c = |x|^2$, the squared Euclidean norm. The same formulas for t and n hold on the entire quadratic cone of $\mathbb{R}_n$, that in this case (cf. [12, 13]) can be defined as

$$\mathcal{Q}_{\mathbb{R}_n} = \{x \in \mathbb{R}_n \mid t(x) \in \mathbb{R},\ n(x) \in \mathbb{R}\}.$$

The quadratic cone of $\mathbb{R}_n$ is a real algebraic set which contains the paravector space $\mathbb{R}^{n+1}$ as a proper (if $n > 2$) subset. For example, $\mathcal{Q}_{\mathbb{R}_1} = \mathbb{R}_1 \simeq \mathbb{C}$, $\mathcal{Q}_{\mathbb{R}_2} = \mathbb{R}_2 \simeq \mathbb{H}$, while

$$\mathcal{Q}_{\mathbb{R}_3} = \{x \in \mathbb{R}_3 \mid x_{123} = x_1x_{23} - x_2x_{13} + x_3x_{12} = 0\}$$

is a real algebraic set of dimension 6.

Each $x \in \mathcal{Q}_{\mathbb{R}_n}$ can be written as $x = \mathrm{Re}(x) + \mathrm{Im}(x)$, with $\mathrm{Re}(x) = \frac{x+x^c}{2}$, $\mathrm{Im}(x) = \frac{x-x^c}{2} = \beta J$, where $\beta = |\mathrm{Im}(x)|$ and $J \in \mathbb{S}_{\mathbb{R}_n}$ (the “sphere” of imaginary units in $\mathbb{R}_n$ compatible with the Clifford conjugation). The choice of J is unique if $x \notin \mathbb{R}$.

The class of slice-regular functions on $\mathbb{R}_n$, defined as explained before by means of holomorphic stem functions, is an extension of the class of *slice-monogenic* functions introduced by Colombo, Sabadini and Struppa in 2009 [5]. More precisely, let $\mathbb{S}^{n-1} = \{x_1e_1 + \cdots + x_ne_n \in \mathbb{R}_n \mid x_1^2 + \cdots + x_n^2 = 1\}$, a subset of $\mathbb{S}_{\mathbb{R}_n}$. A function $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is slice-monogenic if, for every $J \in \mathbb{S}^{n-1}$, the restriction $f_{|\Omega \cap \mathbb{C}_J} : \Omega \cap \mathbb{C}_J \to \mathbb{R}_n$ is holomorphic with respect to the complex structure L_J defined by $L_J(v) = Jv$. When the domain Ω intersects the real axis, the definition of slice regularity on $\mathbb{R}_n$ is equivalent to the one of slice monogenicity, in the sense that the restriction to the paravector space of a $\mathbb{R}_n$-valued slice-regular function is a slice-monogenic function.

In the next sections we will introduce a differential operator $\overline{\vartheta}$ characterizing slice regularity [7, 15] and the notion of *spherical derivative* of a slice function [12]. Then we will prove a formula, relating the operator $\overline{\vartheta}$, the spherical derivative, the spherical Dirac operator and the Cauchy-Riemann operator on $\mathbb{R}_n$. The computation of the Laplacian of the spherical derivative of a slice regular function gives a result which implies, in particular, the Fueter-Sce Theorem for monogenic functions (i.e. the functions belonging to the kernel of the Cauchy-Riemann operator $\overline{\partial}$). We recall that Fueter's Theorem [9], generalized by Sce [23], Qian [22] and Sommen [24] on Clifford algebras and octonions, in our language states that applying to a slice-preserving slice-regular function the Laplacian operator of $\mathbb{R}^4$ (in the quaternionic case) or the iterated Laplacian operator $\Delta_{n+1}^{(n-1)/2}$ of $\mathbb{R}^{n+1}$ (in the Clifford algebra case with n odd), one obtains a function in the kernel, respectively, of the Cauchy-Riemann-Fueter operator or of the Cauchy-Riemann operator. This result was extended in [6] to the whole classes of quaternionic slice-regular functions and of slice-monogenic functions defined by means of stem functions.

These results take a particularly neat form in the two four-dimensional cases represented by paravectors in the Clifford algebra $\mathbb{R}_3$ and by the space $\mathbb{H}$ of quaternions. As we will show in Sects. 3.5 and 3.6, in these cases there appear unexpected relations between slice-regular functions, the zonal harmonics on the three-dimensional sphere and the Poisson kernel of the unit ball.

As an application, a stronger version of Liouville's Theorem for entire slice-regular functions is given. See [5, 10, 11] for the generalization of the complex Liouville's Theorem to quaternionic functions and slice monogenic functions. The formulas obtained in the present paper have found application also to four dimensional Jensen formulas for quaternionic slice-regular functions [2, 21].

The present work can be considered also a continuation of [20], where other relations, of a different nature, between the two function theories, the one of monogenic functions and the one of slice-regular functions, were presented.

3.2 The Slice Derivatives and the Operator $\overline{\vartheta}$

The commutative diagrams (3.1.1) suggest a natural definition of the *slice derivatives* $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial x^c}$ of a slice functions f. They are the slice functions induced, respectively, by the derivatives $\frac{\partial F}{\partial z}$ and $\frac{\partial F}{\partial \overline{z}}$:

$$\frac{\partial f}{\partial x} = \mathcal{I}\left(\frac{\partial F}{\partial z}\right) \quad \text{and} \quad \frac{\partial f}{\partial x^c} = \mathcal{I}\left(\frac{\partial F}{\partial \overline{z}}\right).$$

With this notation a slice function is slice-regular if and only if $\frac{\partial f}{\partial x^c} = 0$ and if this is the case also the slice derivative $\frac{\partial f}{\partial x}$ is slice-regular.

For each alternative *-algebra A there exists [15] a global differential operator $\overline{\vartheta}$ which characterizes slice-regular functions among the class of slice functions. If Ω_D is a circular domain in the quadratic cone of A, the operator

$$\overline{\vartheta} : \mathcal{C}^1(\Omega_D \setminus \mathbb{R}, A) \to \mathcal{C}^0(\Omega_D \setminus \mathbb{R}, A)$$

is defined on the class $\mathcal{C}^1(\Omega_D \setminus \mathbb{R}, A)$ of A-valued functions of class $\mathcal{C}^1$ on $\Omega_D \setminus \mathbb{R}$. In particular, when A is the Clifford algebra $\mathbb{R}_n$, the operator $\overline{\vartheta}$ has the following expression

$$\overline{\vartheta} = \frac{\partial}{\partial x_0} + \frac{\mathrm{Im}(x)}{|\mathrm{Im}(x)|^2} \sum_{|K| \equiv 1,2 \, (\mathrm{mod}\, 4)} x_K \frac{\partial}{\partial x_K}. \tag{3.2.1}$$

When the operator $\overline{\vartheta}$ is applied to a slice function f, it yields two times the slice derivative $\frac{\partial f}{\partial x^c}$. Let ϑ be the conjugated operator of $\overline{\vartheta}$. Then $\vartheta f = 2\frac{\partial f}{\partial x}$ for each slice function f.

Theorem 3.2.1 ([15]) *If $f \in \mathcal{C}^1(\Omega_D)$ is a slice function, then f is slice-regular if and only if $\overline{\vartheta} f = 0$ on $\Omega_D \setminus \mathbb{R}$. If $\Omega_D \cap \mathbb{R} \neq \emptyset$ and $f \in \mathcal{C}^1(\Omega_D)$ (not a priori a slice function), then f is slice-regular if and only if $\overline{\vartheta} f = 0$.*

As seen in the introduction, the paravector space $\mathbb{R}^{n+1}$ is a subspace of the quadratic cone of $\mathbb{R}_n$, proper if $n > 2$. The action of $\overline{\vartheta}$ on functions defined on $\Omega_D \cap \mathbb{R}^{n+1}$ (corresponding to the terms with $|K| = 1$ in the summation (3.2.1)) coincides, up to a factor 2, with the action of the *radial Cauchy-Riemann operator* $\overline{\partial}_{rad}$ (cf. e.g. [18]). An equivalent operator defined on the paravector space $\mathbb{R}^{n+1}$ of $\mathbb{R}_n$ was given in [7].

In the case $n = 2$, the algebra $\mathbb{R}_2 \simeq \mathbb{H}$ is four-dimensional and coincides with the quadratic cone (while paravectors form a three-dimensional subspace). When applied on functions defined on open subsets of the full algebra $\mathbb{H}$, the operator $\overline{\vartheta}$ contains also the term with $|K| = 2$ in (3.2.1).

3.3 Spherical Operators

Let $f = \mathcal{I}(F)$ be a slice function on Ω_D, induced by the stem function $F = F_1 + \imath F_2$, with $F_1, F_2 : D \subseteq \mathbb{C} \to \mathbb{R}_n$. We recall some definitions from [12]:

Definition 3.3.1 The function $f_s^\circ : \Omega_D \to \mathbb{R}_n$, called *spherical value* of f, and the function $f_s' : \Omega_D \setminus \mathbb{R} \to \mathbb{R}_n$, called *spherical derivative* of f, are defined as

$$f_s^\circ(x) := \frac{1}{2}(f(x) + f(x^c)) \quad \text{and} \quad f_s'(x) := \frac{1}{2}\,\mathrm{Im}(x)^{-1}(f(x) - f(x^c)).$$

If $x = \alpha + \beta J \in \Omega_D$, $z = \alpha + \imath\beta \in D$, then $f_s^\circ(x) = F_1(z)$ and $f_s'(x) = \beta^{-1}F_2(z)$. Therefore f_s° and f_s' are slice functions, constant on every set $\mathbb{S}_x = \alpha + \beta\,\mathbb{S}_{\mathbb{R}_n}$. They are slice-regular only if f is locally constant. Moreover, the formula

$$f(x) = f_s^\circ(x) + \mathrm{Im}(x)\,f_s'(x) \tag{3.3.1}$$

holds for each $x \in \Omega_D \setminus \mathbb{R}$. If $F \in \mathcal{C}^1$, the formula holds also for $x \in \Omega_D \cap \mathbb{R}$. In particular, if f is slice-regular, f_s' extends with the values of the slice derivative $\frac{\partial f}{\partial x}$ on the real points $x \in \Omega_D \cap \mathbb{R}$.

Since the paravector space $\mathbb{R}^{n+1}$ is contained in the quadratic cone $\mathcal{Q}_{\mathbb{R}_n}$, we can consider the restriction of a slice function on domains of the form $\Omega = \Omega_D \cap \mathbb{R}^{n+1}$ in $\mathbb{R}^{n+1}$. Thanks to the representation formula (see e.g. [12, Prop. 6]), this restriction uniquely determines the slice function. We will therefore use the same symbol to denote the restriction.

To simplify notation, in the following we will denote the partial derivatives $\frac{\partial}{\partial x_i}$ also with the symbol ∂_i $(i = 0, \dots, n)$.

For any i, j with $1 \le i < j \le n$, let $L_{ij} = x_i \partial_j - x_j \partial_i$ be the angular momentum operators and $\Gamma = -\sum_{i<j} e_{ij} L_{ij}$ the *spherical Dirac operator* on $\mathbb{R}_n$ (see e.g. [18, §2.1.5], [4, §8.7] or [25]). The operators L_{ij} are tangential differential operators for the spheres $\mathbb{S}_x \cap \mathbb{R}^{n+1} = \alpha + \beta\,\mathbb{S}^{n-1}$. The spherical Dirac operator Γ factorizes the Laplace-Beltrami operator $\Delta_{LB} = \sum_{i<j} L_{ij}^2$ on $\mathbb{S}^{n-1}$ since $\Delta_{LB} = \Gamma(-\Gamma + n - 2)$. We show that the function obtained applying the operator Γ to a slice function f is equal, up to a multiple of $\mathrm{Im}(x)$, to the spherical derivative f_s'.

Proposition 3.3.2 *Let $\Omega = \Omega_D \cap \mathbb{R}^{n+1}$ be an open subset of $\mathbb{R}^{n+1}$. For each slice function $f : \Omega \to \mathbb{R}_n$ of class $\mathcal{C}^1(\Omega)$, the following formulas hold on $\Omega \setminus \mathbb{R}$:*

(a) $\Gamma f = (n-1)\,\mathrm{Im}(x)\,f_s'$.
(b) $\overline{\partial} f - \overline{\vartheta} f = (1-n)\,f_s'$.

Proof Let f be a slice function. Since the functions f_s° and f_s' are constant on the spheres $\mathbb{S}_x$, every L_{ij} vanishes on them. Using formula (3.3.1) and Leibniz rule we get

$$L_{ij} f = L_{ij}(f_s^\circ(x) + \mathrm{Im}(x) f_s'(x)) = L_{ij}(\mathrm{Im}(x))\,f_s'.$$

A direct computation gives

$$\Gamma x = -\sum_{i<j} e_{ij} L_{ij}(\mathrm{Im}(x)) = -\sum_{i<j} e_{ij}(x_i e_j - x_j e_i) = (n-1)\,\mathrm{Im}(x).$$

It follows that

$$\Gamma f = -\sum_{i<j} e_{ij} L_{ij} f = -\sum_{i<j} e_{ij} L_{ij}(\mathrm{Im}(x))\,f_s' = (n-1)\,\mathrm{Im}(x)\,f_s'$$

and point (a) is proved. To prove (b), we can use the decomposition of the Cauchy-Riemann operator given in [25]:

$$\overline{\partial} = \partial_0 + \omega \ell_\omega + \frac{1}{|\operatorname{Im}(x)|} L$$

where $\omega = \frac{\operatorname{Im}(x)}{|\operatorname{Im}(x)|}$, $\ell_\omega = \frac{1}{|\operatorname{Im}(x)|} \sum_{i=1}^n x_i \partial_i$ and $L = \omega \Gamma$. Since f depends only on paravector variables, $\overline{\vartheta} f$ coincides with the radial part $(\partial_0 + \omega \ell_\omega) f$ of $\overline{\partial} f$. Then $\overline{\partial} f - \overline{\vartheta} f = \frac{1}{|\operatorname{Im}(x)|} L f = \frac{1}{|\operatorname{Im}(x)|} \omega \Gamma f = \omega^2 (n-1) f'_s = (1-n) f'_s$. □

Corollary 3.3.3 *Let $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ be a slice function of class $\mathcal{C}^1(\Omega)$. Then*

(a) *f is slice-regular if and only if $\overline{\partial} f = (1-n) f'_s$.*
(b) *Let $n > 1$. Then f is slice-regular and monogenic (i.e. it belongs to the kernel of $\overline{\partial}$) if and only if f is (locally) constant.*
(c) *$\partial f - \vartheta f = (n-1) f'_s$ and $\vartheta f'_s = \partial f'_s$.*

Proof The first statement is immediate from point (b) of Proposition 3.3.2 and Theorem 3.2.1. If $\overline{\partial} f = \overline{\vartheta} f = 0$, then $f'_s \equiv 0$. This means that the component F_2 of the inducing stem function F vanishes identically. From the holomorphicity of F it follows that F_1 is locally constant, and then also f is locally constant. The first statement in (c) is a consequence of point (b) of Proposition 3.3.2, taking account of the equalities

$$\partial + \overline{\partial} = 2\partial_0 = \vartheta + \overline{\vartheta}.$$

The last statement comes from the property $(f'_s)'_s = 0$, which holds for every slice function f. □

Statement (b) of the previous Corollary shows that when $n > 1$ the two function theories, the one of monogenic functions and the one of slice-regular functions, are really skew. This is in contrast with the classical case ($n = 1$), when $\overline{\partial} f = \overline{\vartheta} f$ and the two theories coincide.

3.4 The Laplacian of Slice Functions

Let $f = \mathcal{I}(F)$ be a slice function on Ω_D, with $F = F_1 + \imath F_2$ a stem function with real analytic components F_1, F_2. Since $F(\overline{z}) = \overline{F(z)}$ for every $z = \alpha + \imath \beta$, the functions $F_1, F_2 : D \subseteq \mathbb{C} \to \mathbb{R}_n$ are, respectively, even and odd functions with respect to the variable β. Therefore there exist G_1 and G_2, again real analytic, such that

$$F_1(\alpha, \beta) = G_1(\alpha, \beta^2), \quad F_2(\alpha, \beta) = \beta G_2(\alpha, \beta^2).$$

If $x = \alpha + \beta J$, $z = \alpha + \imath\beta$, then

$$f_s^\circ(x) = G_1(\alpha, \beta^2) = G_1(\mathrm{Re}(x), |\,\mathrm{Im}(x)|^2), \tag{3.4.1}$$

$$f_s'(x) = G_2(\alpha, \beta^2) = G_2(\mathrm{Re}(x), |\,\mathrm{Im}(x)|^2). \tag{3.4.2}$$

The functions G_1 and G_2 are useful in the computation of the Laplacian of the spherical derivative and of the spherical value of a slice regular function. Let $\Omega = \Omega_D \cap \mathbb{R}^{n+1}$.

Observe that if f is slice regular, then F_1 and F_2 have harmonic real components with respect to the two-dimensional Laplacian Δ_2 of the plane. For $j = 1, 2$, let $\partial_1 G_j(u, v)$ stand for the partial derivative $\frac{\partial G_j}{\partial u}(u, v)$ and $\partial_2 G_j(u, v)$ for the partial derivative $\frac{\partial G_j}{\partial v}(u, v)$.

Theorem 3.4.1 *Let $\Omega = \Omega_D \cap \mathbb{R}^{n+1}$ be an open subset of $\mathbb{R}^{n+1}$. Let $f = \mathcal{I}(F) : \Omega \to \mathbb{R}_n$ be (the restriction of) a slice-regular function. Let $f_s'(x) = G_2(\mathrm{Re}(x), |\,\mathrm{Im}(x)|^2)$ as in (3.4.2) and let Δ_{n+1} be the Laplacian operator on $\mathbb{R}^{n+1}$. Then it holds:*

(a)

$$\Delta_{n+1} f_s'(x) = 2(n-3)\,\partial_2 G_2(\mathrm{Re}(x), |\,\mathrm{Im}(x)|^2).$$

(b) *For each $k = 1, 2, \ldots, \left[\frac{n-1}{2}\right]$,*

$$\Delta_{n+1}^k f_s'(x) = 2^k (n-3)(n-5)\cdots(n-2k-1)\,\partial_2^k G_2(\mathrm{Re}(x), |\,\mathrm{Im}(x)|^2).$$

(c)

$$\Delta_{n+1} f_s'(x) = \frac{n-3}{|\,\mathrm{Im}(x)|^2}\left(\frac{\partial f_s^\circ}{\partial x_0}(x) - f_s'(x)\right).$$

More precisely, the formulas in (a) and (b) hold if and only if F_2 has harmonic components on D.

Proof Let $x_0 = \mathrm{Re}(x)$, $r = |\,\mathrm{Im}(x)|$. By direct computation, from (3.4.2) we get

$$\Delta_2 F_2(\alpha, \beta) = \left(\partial_1^2 + 4\beta^2\,\partial_2^2 + 6\,\partial_2\right) G_2(\alpha, \beta^2) \tag{3.4.3}$$

and

$$\Delta_{n+1} G_2(x_0, r^2) = \frac{\partial^2 G_2}{\partial x_0^2}(x_0, r^2) + \sum_{i=1}^n \frac{\partial^2 G_2}{\partial x_i^2}(x_0, r^2)$$
$$= \left(\partial_1^2 + 4r^2\partial_2^2 + 2n\,\partial_2\right) G_2(x_0, r^2).$$

Therefore $\Delta_2 F_2 = 0$ on D if and only if $\Delta_{n+1} f'_s(x) = \Delta_{n+1} G_2(x_0, r^2) = (2n - 6)\, \partial_2 G_2(x_0, r^2)$. This proves (a). To obtain (b) we use induction on k, starting from the case $k = 1$ given by (a). For every k with $1 < k \leq \left[\frac{n-1}{2}\right] - 1$, if $\Delta_2 F_2(\alpha, \beta) = 0$ it holds

$$\begin{aligned}\Delta_{n+1} \partial_2^k G_2(x_0, r^2) &= \left(\partial_1^2 \partial_2^k + 4r^2 \partial_2^{k+2} + 2n\, \partial_2^{k+1}\right) G_2(x_0, r^2)\\ &= \left(\partial_2^k \left(\partial_1^2 + 4r^2 \partial_2^2 + 2n\, \partial_2\right) - 4k\, \partial_2^{k+1}\right) G_2(x_0, r^2)\\ &= \left(\partial_2^k\, (-6\, \partial_2 + 2n\, \partial_2) - 4k \partial_2^{k+1}\right) G_2(x_0, r^2)\\ &= 2(n - 2k - 3)\, \partial_2^{k+1} G_2(x_0, r^2).\end{aligned}$$

By the induction hypothesis

$$\begin{aligned}\Delta_{n+1}^{k+1} f'_s(x) &= 2^k (n-3)(n-5) \cdots (n-2k-1)\, \Delta_{n+1} \partial_2^k G_2(x_0, r^2)\\ &= 2^k (n-3)(n-5) \cdots (n-2k-1)\, 2(n-2k-3)\, \partial_2^{k+1} G_2(x_0, r^2)\end{aligned}$$

and (b) is proved. Statement (c) follows from the holomorphicity of F. Since $\partial_\alpha F_1(\alpha, \beta) = \partial_\beta F_2(\alpha, \beta) = \partial_\beta(\beta G_2(\alpha, \beta^2)) = G_2(\alpha, \beta^2) + 2\beta^2 \partial_2 G_2(\alpha, \beta^2)$, it holds, for $r \neq 0$,

$$\partial_2 G_2(x_0, r^2) = \frac{1}{2r^2} \left(\partial_1 F_1(x_0, r^2) - G_2(x_0, r^2)\right) = \frac{1}{2r^2} \left(\frac{\partial f_s^\circ}{\partial x_0}(x) - f'_s(x)\right).$$

Together with (a), this proves (c). □

These results take a particularly attractive form when the paravectors space is four-dimensional, i.e. $n = 3$.

Corollary 3.4.2 *Let $f : \Omega \subseteq \mathbb{R}^4 \to \mathbb{R}_3$ be (the restriction of) a slice-regular function. Then it holds:*

(a) *The spherical derivative f'_s is harmonic on Ω, i.e. its eight real components are harmonic functions.*
(b) *The following generalization of Fueter-Sce Theorem for the Clifford algebra $\mathbb{R}_3$ holds:*

$$\overline{\partial} \Delta_4 f = \Delta_4 \overline{\partial} f = -2 \Delta_4 f'_s = 0.$$

(c) $\Delta_4^2 f = 0$, *i.e. every slice regular function on $\mathbb{R}_3$ is biharmonic on Ω.*
(d) $\Delta_4 f = -4 \dfrac{\partial f'_s}{\partial x}$. *Therefore also $\dfrac{\partial f'_s}{\partial x}$ is harmonic on Ω.*

Proof The first statement is immediate from point (a) of Theorem 3.4.1. Point (b) is a consequence of (a) and of Corollary 3.3.3. Statement (c) follows from (b) and the factorization $\partial\overline{\partial} = \Delta_4$. Since $2\frac{\partial f}{\partial x} = \vartheta f$ (cf. [15]) for any slice function, from Corollary 3.3.3 it follows that

$$4\frac{\partial f_s'}{\partial x} = 2\vartheta(f_s') = \partial(2f_s') = -\partial\overline{\partial} f = -\Delta_4 f.$$

This proves (d). □

Remark 3.4.3 If f is only *slice-harmonic* on Ω_D, i.e. $f = \mathcal{I}(F)$ is induced by a harmonic stem function F on $D \subseteq \mathbb{C}$, then F_2 has harmonic real components and therefore f_s' is still harmonic.

Examples

(1) If $f = x^3$, a slice-regular function, then

$$\overline{\partial} f = -2f_s' = -2\left(3x_0^2 - x_1^2 - x_2^2 - x_3^2\right)$$

is harmonic on $\mathbb{R}^4$ and $\Delta_4 f = -4\,(3x_0 + x_1e_1 + x_2e_2 + x_3e_3)$ is monogenic.

(2) If $f = (x^c)^3$, a slice-harmonic function, then

$$f_s' = -3x_0^2 + x_1^2 + x_2^2 + x_3^2$$

is harmonic on $\mathbb{R}^4$ while $\Delta_4 f = 4\,(-3x_0 + x_1e_1 + x_2e_2 + x_3e_3)$ is not monogenic.

(3) Let $f = x\left(1 - \frac{\operatorname{Im}(x)}{|\operatorname{Im}(x)|}e_1\right)$. The function f is slice regular on $\mathcal{Q}_{\mathbb{R}_3} \setminus \mathbb{R}$, a set that contains $\mathbb{R}^4 \setminus \mathbb{R}$. Then $f_s' = 1 - \frac{x_0}{|\operatorname{Im}(x)|}e_1$ is harmonic on $\mathbb{R}^4 \setminus \mathbb{R}$. The Laplacian

$$\Delta_4 f = \frac{2}{|\operatorname{Im}(x)|^3}\left(-x_0x_1 + (x_1^2 + x_2^2 + x_3^2)e_1 - x_0x_2e_{12} - x_0x_3e_{13}\right)$$

is monogenic on $\mathbb{R}^4 \setminus \mathbb{R}$.

Now consider the higher dimensional case, with $n > 3$ odd.

Corollary 3.4.4 *Let $n > 3$ odd. If $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is (the restriction of) a slice-regular function, then*

(a) $(\Delta_{n+1})^{\frac{n-3}{2}} f_s'$ *is harmonic on* Ω.

(b) *The following generalization of Fueter-Sce Theorem for $\mathbb{R}_n$ holds:*

$$\overline{\partial}(\Delta_{n+1})^{\frac{n-1}{2}} f = (\Delta_{n+1})^{\frac{n-1}{2}}\overline{\partial} f = (1-n)\,(\Delta_{n+1})^{\frac{n-1}{2}} f_s' = 0.$$

(c) $(\Delta_{n+1})^{\frac{n+1}{2}} f = 0$, *i.e. every slice regular function on $\mathbb{R}_n$ is polyharmonic.*

Proof The proof follows the same lines as above, using point (b) of Theorem 3.4.1 instead of (a). □

The harmonicity properties of slice-regular functions imply a stronger form of the Liouville's Theorem for entire slice regular functions. See [5] for the generalization of the classical result to slice monogenic functions. In the next Corollary we give also a new proof of this last result (at least in the case of n odd).

Corollary 3.4.5 *Let $n \geq 3$ be odd. Let $f \in \mathcal{SR}(\mathcal{Q}_{\mathbb{R}_n})$ be an entire slice regular function. If f is bounded on $\mathbb{R}^{n+1}$, then f is constant. If the spherical derivative f'_s is bounded (equivalently, if $\overline{\partial} f$ is bounded) on $\mathbb{R}^{n+1}$, then f is a left-affine function, of the form $f(x) = a + xb$ $(a, b \in \mathbb{R}_n)$.*

Proof Let $f = \mathcal{I}(F)$ be induced by the holomorphic stem function $F = F_1 + \imath F_2$, with $F_1, F_2 : \mathbb{C} \to \mathbb{R}_n$. If f is bounded, then the real components of f are polyharmonic and bounded on $\mathbb{R}^{n+1}$. Then f is constant from the Liouville's Theorem for polyharmonic functions.

If f'_s is bounded, the real components of its continuous extension to $\mathbb{R}^{n+1}$ are polyharmonic and bounded and then constant. Therefore $F_2(\alpha, \beta) = \beta b$, with $b \in \mathbb{R}_n$. By the Cauchy-Riemann equations, it follows that $F_1(\alpha, \beta) = a + \alpha b$, with $a \in \mathbb{R}_n$. Therefore $f(x) = (a + x_0 b) + \mathrm{Im}(x) b = a + xb$. □

As regards the spherical value of a slice-regular function, we can still compute its Laplacian. In general, even in the four-dimensional case, the spherical value is not a harmonic function, nonetheless in the even-dimensional case it is always polyharmonic.

Theorem 3.4.6 *Let $f = \mathcal{I}(F) : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ be (the restriction of) a slice-regular function. Let $f^\circ_s(x) = G_1(\mathrm{Re}(x), |\mathrm{Im}(x)|^2)$ as in (3.4.1). It holds:*

(a)

$$\Delta_{n+1} f^\circ_s(x) = (\Delta_{n+1} f)^\circ_s(x) = 2(n-1)\, \partial_2 G_1(\mathrm{Re}(x), |\mathrm{Im}(x)|^2).$$

(b) *For each $k = 1, 2, \ldots, \left[\frac{n-1}{2}\right]$*

$$\Delta^k_{n+1} f^\circ_s(x) = 2^k (n-1)(n-3)\cdots(n-2k+1)\, \partial^k_2 G_1(\mathrm{Re}(x), |\mathrm{Im}(x)|^2).$$

(c)

$$\Delta_{n+1} f^\circ_s(x) = (1-n) \frac{\partial f'_s}{\partial x_0}(x).$$

(d) *When $n = 3$, $\Delta^2_4 f^\circ_s = 0$. In general, if n is odd, $(\Delta_{n+1})^{\frac{n+1}{2}} f^\circ_s = 0$.*

Proof Let $x_0 = \mathrm{Re}(x)$, $r = |\mathrm{Im}(x)|$. By direct computation, from (3.4.1) we get

$$\Delta_2 F_1(\alpha, \beta) = \left(\partial^2_1 + 4\beta^2\, \partial^2_2 + 2\, \partial_2\right) G_1(\alpha, \beta^2). \tag{3.4.4}$$

The proofs of (a) and (b) follows the same lines as the corresponding proofs of Theorem 3.4.1, using (3.4.4) in place of (3.4.3). We prove (c): since $\partial_\alpha F_2(\alpha,\beta) = -\partial_\beta F_1(\alpha,\beta) = -2\beta\partial_2 G_1(\alpha,\beta^2)$, it holds, for $r \neq 0$,

$$\partial_2 G_1(x_0, r^2) = -\frac{1}{2r} r\, \partial_1 G_2(x_0, r^2) = -\frac{1}{2}\frac{\partial f_s'}{\partial x_0}(x).$$

Together with (a), this proves (c). Finally, (d) is immediate from (b). □

Again, points (a) and (b) remain valid if f is slice-harmonic.

3.5 The Four-Dimensional Case: Zonal Harmonics and the Poisson Kernel

Thanks to Corollary 3.4.2, for any polynomial $f(x) = \sum_{m=0}^{d} x^m a_m$ with coefficients in $\mathbb{R}_3$, the spherical derivative

$$f_s'(x) = \sum_{m=0}^{d} (x^m)_s' a_m$$

is a harmonic polynomial on $\mathbb{R}^4$. In particular, for every $m \in \mathbb{N}$ the spherical derivative $(x^m)_s' = -\frac{1}{2}\overline{\partial}(x^m)$ of a Clifford power x^m is a homogeneous harmonic polynomial of degree $m-1$ in the variables x_0, x_1, x_2, x_3, with real coefficients. Observe that $(x^m)_s'$ can be written as

$$\begin{aligned}(x^m)_s' &= \frac{\operatorname{Im}(x)^{-1}}{2}\left(x^m - (x^c)^m\right) = (x - x^c)^{-1}\left(x^m - (x^c)^m\right)\\ &= \sum_{k=0}^{m-1} x^{m-k-1}(x^c)^k = \sum_{\nu=0}^{\left[\frac{m-2}{2}\right]} t(x^{m-1-2\nu})n(x)^\nu + n(x)^{\frac{m-1}{2}}\end{aligned}$$

(where the last term is present only if m is odd).

Let $\mathbb{B}$ be the open unit ball in $\mathbb{R}^4$. Let $\mathcal{Z}_m(x,a)$ denote the four-dimensional *(solid) zonal harmonic* of degree m with pole $a \in \partial\mathbb{B}$ (see e.g. [3, Ch. 5]). From the uniqueness properties of zonal harmonics and their invariance with respect to four-dimensional rotations, we get the following result.

Proposition 3.5.1 *The spherical derivatives of Clifford powers x^m coincide on $\mathbb{R}^4$, up to a constant, with the zonal harmonics with pole $1 \in \partial\mathbb{B}$. More precisely, for every $m \geq 1$ and every $a \in \partial\mathbb{B}$, it holds:*

(a) $\mathcal{Z}_{m-1}(x,1) = m(x^m)_s'$. *Therefore* $\overline{\partial}(x^m) = -\frac{2}{m}\mathcal{Z}_{m-1}(x,1)$.

(b) $\mathcal{Z}_{m-1}(x,a) = \mathcal{Z}_{m-1}(xa^c,1) = m(x^m)'_{s\,|x=xa^c}$
$= m\sum_{k=0}^{m-1}(xa^c)^{m-k-1}(ax^c)^k$.

(c) $(x^{-m})'_s = -\mathcal{K}\left[(x^m)'_s\right]$, *where* $\mathcal{K}$ *is the Kelvin transform in* $\mathbb{R}^4$. *The functions* $(x^{-m})'_s$ *are harmonic on* $\mathbb{R}^4 \setminus \{0\}$.

(d) *The restriction of* $(x^m)'_s$ *to the unit sphere* $\partial\mathbb{B}$ *is equal to the Gegenbauer polynomial* $C^{(1)}_{m-1}(\mathrm{Re}(x))$.

Proof Let $\alpha, \beta \in \mathbb{R}$ with $\beta > 0$ and $\alpha^2 + \beta^2 = 1$. The spherical derivatives $(x^m)'_s$ are constant on every "parallel" $\mathbb{S}_x = \alpha + \beta\mathbb{S}^2$ in $\partial\mathbb{B}$ orthogonal to the real axis. From [3, Theorem 5.37] it follows that $(x^m)'_s$ is a constant multiple of $\mathcal{Z}_{m-1}(x,1)$. To determine the constant, it is sufficient to compute $(x^m)'_s$ at $x = 1$. On real points the spherical derivative coincides with the slice derivative $\frac{\partial f}{\partial x}$, and then $(x^m)'_{s\,|x=1} = \frac{\partial x^m}{\partial x}_{|x=1} = mx^{m-1}{}_{|x=1} = m$. Since $\mathcal{Z}_{m-1}(1,1) = m^2$, (a) is proved. Point (b) is a consequence of the rotational properties of zonal harmonics (cf. [3, 5.27]). Statement (c) follows from direct computation:

$$(x^{-m})'_s = (x - x^c)^{-1}\left(\frac{(x^c)^m}{|x|^{2m}} - \frac{x^m}{|x|^{2m}}\right)$$

and then

$$\begin{aligned}\mathcal{K}\left[(x^{-m})'_s\right] &= |x|^{-2}\left(\frac{x}{|x|^2} - \frac{x^c}{|x|^2}\right)^{-1}\left(\left(\frac{x^c}{|x|^2}\right)^m - \left(\frac{x}{|x|^2}\right)^m\right)\left|\frac{x^c}{|x|^2}\right|^{-2m}\\ &= (x - x^c)^{-1}((x^c)^m - x^m) = -(x^m)'_s.\end{aligned}$$

Since $\mathcal{K}[\mathcal{K}[f]] = f$, this proves (c). Statement (d) follows from (a) and a well-known property of zonal harmonics. Note that $C^{(1)}_{m-1}(1) = m$ for each $m \geq 1$. □

Corollary 3.5.2 *The Clifford powers* x^m *of the paravector variable of* $\mathbb{R}_3$ *can be expressed in terms of the four-dimensional zonal harmonics: for each* $m \geq 1$,

$$\begin{aligned}x^m &= \frac{1}{m+1}\mathcal{Z}_m(x,1) + \frac{(x - 2\,\mathrm{Re}(x))}{m}\mathcal{Z}_{m-1}(x,1)\\ &= \frac{1}{m+1}\mathcal{Z}_m(x,1) - x^c\frac{1}{m}\mathcal{Z}_{m-1}(x,1).\end{aligned}$$

Proof Let $x_0 = \mathrm{Re}(x)$. Applying the Leibniz rule for the spherical derivative (cf. [12, §5]), we get

$$(x^{m+1})'_s = (x \cdot x^m)'_s = (x)'_s(x^m)^\circ_s + (x)^\circ_s(x^m)'_s = (x^m)^\circ_s + x_0(x^m)'_s.$$

Therefore

$$(x^m)^\circ_s = (x^{m+1})'_s - x_0(x^m)'_s = \frac{1}{m+1}\mathcal{Z}_m(x,1) - x_0\frac{1}{m}\mathcal{Z}_{m-1}(x,1)$$

and

$$x^m = \frac{1}{m+1}\mathcal{Z}_m(x,1) - x_0\frac{1}{m}\mathcal{Z}_{m-1}(x,1) + \mathrm{Im}(x)\frac{1}{m}\mathcal{Z}_{m-1}(x,1).$$

□

Let $\mathcal{P}(x,a) = \frac{1-|x|^2}{|x-a|^4}$ be the Poisson kernel for the unit ball $\mathbb{B}$ in $\mathbb{R}^4$ ($x \in \mathbb{B}$, $a \in \partial\mathbb{B}$). This harmonic kernel is related with the slice-regular function induced by a famous holomorphic function. Let $f_K(x) = (1-x)^{-2}x$ be the *Cliffordian Koebe function*. It is the slice-preserving slice-regular function induced by the classical Koebe function $F_K(z) = (1-z)^{-2}z$.

Corollary 3.5.3 *The Cliffordian Koebe function $f_K(x) = (1-x)^{-2}x$ is slice-regular on $\mathcal{Q}_{\mathbb{R}_3} \setminus \{1\} \supset \mathbb{B}$ and has the following properties. For every $x \in \mathbb{B}$,*

$$(f_K)'_s(x) = \mathcal{P}(x,1) = \frac{1-|x|^2}{|x-1|^4}.$$

For every $a \in \partial\mathbb{B}$ and $x \in \mathbb{B}$,

$$(f_K)'_s(xa^c) = \mathcal{P}(x,a) = \frac{1-|x|^2}{|x-a|^4}.$$

Proof The formulas can be checked directly or by means of the relation of $(x^m)'_s$ with zonal harmonics proved in Proposition 3.5.1. The power series $\sum_{m=0}^{\infty}(m+1)z^{m+1}$ converges to $F_K(z)$ on the complex unit disc. This implies the expansion $f_K(x) = \sum_{m=0}^{\infty}(m+1)x^{m+1}$ on $\mathbb{B}$. Therefore, for every $x \in \mathbb{B}$, it holds

$$(f_K)'_s(x) = \sum_{m=0}^{\infty}(m+1)(x^{m+1})'_s = \sum_{m=0}^{\infty}\mathcal{Z}_m(x,1) = \mathcal{P}(x,1),$$

where the last equality follows from the zonal harmonic expansion $\mathcal{P}(x,a) = \sum_{m=0}^{\infty}\mathcal{Z}_m(x,a)$ (cf. [3, Theorem 5.33]). The last statement is a consequence of point (b) of Proposition 3.5.1. □

3.6 The Quaternionic Case

When $n = 2$, the Clifford algebra $\mathbb{R}_2$ is isomorphic to the field $\mathbb{H}$ of quaternions. In this case the paravector space has dimension three and then Corollary 3.4.4 and its consequences are not applicable. However, similar results still hold since the computations made in Proposition 3.3.2 on the paravector space can be repeated anytime there is a real subspace of the quadratic cone containing the real axis. The simplest example of this setting is given by the quaternions, where the quadratic cone coincides with the whole algebra: $\mathcal{Q}_{\mathbb{H}} = \mathbb{H}$.

By means of the identifications $e_1 = i, e_2 = j, e_{12} = ij = k$, in the coordinates (x_0, x_1, x_2, x_3) of $x = x_0 + x_1 i + x_2 j + x_3 k \in \mathbb{H}$, the differential operator $\overline{\vartheta}$ takes the form [15]

$$\overline{\vartheta} = \frac{\partial}{\partial x_0} + \frac{\operatorname{Im}(x)}{|\operatorname{Im}(x)|^2} \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i}.$$

For every slice function $f : \Omega \subseteq \mathbb{R}^3 \to \mathbb{R}_2$, of class $\mathcal{C}^1$ on a domain Ω in the three-dimensional space of (quaternionic) paravectors, Proposition 3.3.2 gives

$$\overline{\partial} f - \overline{\vartheta} f = -f_s'. \tag{3.6.1}$$

If we consider the whole quaternion algebra, we must instead use the Cauchy-Riemann-Fueter operator

$$\overline{\partial}_{CRF} = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} + j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3} = \overline{\partial} + k\frac{\partial}{\partial x_3}.$$

Let ϑ and ∂_{CRF} be the conjugated differential operators:

$$\vartheta = \frac{\partial}{\partial x_0} - \frac{\operatorname{Im}(x)}{|\operatorname{Im}(x)|^2} \sum_{i=1}^{3} x_i \frac{\partial}{\partial x_i} \quad \text{and} \quad \partial_{CRF} = \frac{\partial}{\partial x_0} - i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} - k\frac{\partial}{\partial x_3}.$$

The Cauchy-Riemann-Fueter operator $\overline{\partial}_{CRF}$ factorizes the Laplacian operator of $\mathbb{R}^4$:

$$\partial_{CRF}\overline{\partial}_{CRF} = \overline{\partial}_{CRF}\partial_{CRF} = \Delta_4.$$

For any i, j with $1 \le i < j \le 3$, let $L_{ij} = x_i \partial_j - x_j \partial_i$ and let

$$\Gamma = -iL_{23} + jL_{13} - kL_{12}$$

be the *quaternionic spherical Dirac operator* on $\operatorname{Im}(\mathbb{H})$. The operators L_{ij} are tangential differential operators for the spheres $\mathbb{S}_x = \alpha + \beta\,\mathbb{S}^2$. For the Cauchy-Riemann-Fueter operator the analogous of Proposition 3.3.2 is the following result. Compare point (b) with formula (3.6.1).

Proposition 3.6.1 *Let $\Omega = \Omega_D$ be an open circular domain in $\mathbb{H}$. For every slice function $f : \Omega \to \mathbb{H}$, of class $\mathcal{C}^1(\Omega)$, the following formulas hold on $\Omega \setminus \mathbb{R}$:*

(a) $\Gamma f = 2\,\mathrm{Im}(x)\,f'_s$.
(b) $\overline{\partial}_{CRF} f - \overline{\vartheta} f = -2 f'_s$.

Proof The proof of (a) is the same as the one given for point (a) of Proposition 3.3.2. To prove (b), set $r = |\,\mathrm{Im}(x)|$, $\omega = \frac{\mathrm{Im}(x)}{|\,\mathrm{Im}(x)|}$, $\ell_\omega = \frac{1}{|\,\mathrm{Im}(x)|}\sum_{i=1}^{3} x_i\partial_i$ and $L = \omega\Gamma$. It holds

$$r\omega\,(i\partial_i + j\partial_2 + k\partial_3) = \mathrm{Im}(x)\,(i\partial_i + j\partial_2 + k\partial_3) = -\sum_{i=1}^{3} x_i\partial_i - \Gamma.$$

Therefore

$$\overline{\partial}_{CRF} = \partial_0 + (i\partial_i + j\partial_2 + k\partial_3) = \partial_0 + \frac{\omega}{r}\left(\sum_{i=1}^{3} x_i\partial_i + \Gamma\right) = \partial_0 + \omega\ell_\omega + \frac{1}{r}L.$$

Since $\overline{\vartheta} f$ coincides with the radial part $(\partial_0 + \omega\ell_\omega) f$ of $\overline{\partial}_{CRF} f$, we get $\overline{\partial}_{CRF} f - \overline{\vartheta} f = \frac{1}{|\,\mathrm{Im}(x)|} L f = \frac{1}{|\,\mathrm{Im}(x)|}\omega\Gamma f = 2\omega^2 f'_s = -2 f'_s$ and (b) is proved. □

Corollary 3.6.2 *Let $\Omega = \Omega_D$ be an open circular domain in $\mathbb{H}$. Let $f : \Omega \to \mathbb{H}$ be a slice function of class $\mathcal{C}^1(\Omega)$. Then*

(a) *f is slice-regular if and only if $\overline{\partial}_{CRF} f = -2 f'_s$.*
(b) *f is slice-regular and Fueter-regular (i.e. it belongs to the kernel of $\overline{\partial}_{CRF}$) if and only if f is (locally) constant.*
(c) *$\partial_{CRF} f - \vartheta f = 2 f'_s$ and $\vartheta f'_s = \partial_{CRF} f'_s$.*

Proof The proofs of (a) and (b) are the same as the ones given in Corollary 3.3.3. From $\overline{\partial}_{CRF} + \partial_{CRF} = 2\partial_0 = \vartheta + \overline{\vartheta}$ and point (b) of Proposition 3.6.1, it follows the first statement in (c). The last statement comes from the property $(f'_s)'_s = 0$, which holds for every slice function f. □

Theorem 3.6.3 *Let $\Omega = \Omega_D$ be an open circular domain in $\mathbb{H}$. If $f : \Omega \to \mathbb{H}$ is slice-regular, then it holds:*

(a) *The spherical derivative f'_s is harmonic on Ω (i.e. its four real components are harmonic).*
(b) *The following generalization of Fueter's Theorem holds:*

$$\overline{\partial}_{CRF}\Delta_4 f = \Delta_4\overline{\partial}_{CRF} f = -2\Delta_4 f'_s = 0.$$

As a consequence, $\Delta_4^2 f = 0$: every quaternionic slice-regular function is biharmonic.

(c) *$\Delta_4 f = -4\,\dfrac{\partial f'_s}{\partial x}$. In particular, $\dfrac{\partial f'_s}{\partial x}$ is harmonic on Ω.*

Proof We proceed as in Sect. 3.4. Let $f = \mathcal{I}(F)$ and let G_2 be the real analytic function on D such that $F_2(\alpha, \beta) = \beta G_2(\alpha, \beta^2)$. If $x = \alpha + \beta J$, $z = \alpha + \imath\beta$, then

$$f_s'(x) = G_2(\alpha, \beta^2) = G_2(\mathrm{Re}(x), |\mathrm{Im}(x)|^2).$$

Since

$$\Delta_2 F_2(\alpha, \beta) = \left(\partial_1^2 + 4\beta^2\, \partial_2^2 + 6\, \partial_2\right) G_2(\alpha, \beta^2) \tag{3.6.2}$$

and

$$\begin{aligned}\Delta_4 G_2(x_0, r^2) &= \frac{\partial^2 G_2}{\partial x_0^2}(x_0, r^2) + \sum_{i=1}^{3} \frac{\partial^2 G_2}{\partial x_i^2}(x_0, r^2)\\ &= \left(\partial_1^2 + 4r^2 \partial_2^2 + 6\, \partial_2\right) G_2(x_0, r^2),\end{aligned}$$

$\Delta_2 F_2 = 0$ on D if and only if $\Delta_4 f_s'(x) = \Delta_4 G_2(x_0, r^2) = 0$ on Ω. This proves (a). Point (b) is immediate from (a) and Proposition 3.6.1. The last statement can be proved as point (d) of Corollary 3.4.2. □

Remark 3.6.4 The spherical value f_s° of a slice-regular function is biharmonic. This can be proved as in Theorem 3.4.6.

Remark 3.6.5 As in the case of $\mathbb{R}_3$, if f is *slice-harmonic* on $\Omega = \Omega_D$, i.e. $f = \mathcal{I}(F)$ is induced by a harmonic stem function F on $D \subseteq \mathbb{C}$, then F_2 has harmonic real components and therefore f_s' is still harmonic.

Remark 3.6.6 As it was proved in [1], the harmonic functions $f_s'(y)$ and $\frac{\partial f_s'}{\partial x}(y)$ are, respectively, the first and the second coefficients of the *spherical expansion* at y of a slice-regular function f (see [14, 26]).

The link existing between Clifford powers and zonal harmonics (Proposition 3.5.1 and its corollaries) has a quaternionic counterpart: the spherical derivatives of the quaternionic powers x^m coincide on $\mathbb{R}^4$, up to a constant, with the four-dimensional zonal harmonics with pole $1 \in \partial\mathbb{B}$. We do not repeat the proofs given in Sect. 3.5.

Corollary 3.6.7 *For every $m \geq 1$ and every $a \in \partial\mathbb{B}$, it holds:*

(a) $\mathcal{Z}_{m-1}(x, 1) = m(x^m)_s'$. *Therefore* $\overline{\partial}_{CRF}(x^m) = -\frac{2}{m}\mathcal{Z}_{m-1}(x, 1)$.
(b) $\mathcal{Z}_{m-1}(x, a) = \mathcal{Z}_{m-1}(x\overline{a}, 1) = m(x^m)'_{s\,|x=x\overline{a}} = m\sum_{k=0}^{m-1}(x\overline{a})^{m-k-1}(a\overline{x})^k$.
(c) $(x^{-m})_s' = -\mathcal{K}\left[(x^m)_s'\right]$, *where* $\mathcal{K}$ *is the Kelvin transform in* $\mathbb{R}^4$. *The functions* $(x^{-m})_s'$ *are harmonic on* $\mathbb{R}^4 \setminus \{0\}$.
(d) $x^m = \frac{1}{m+1}\mathcal{Z}_m(x, 1) - \overline{x}\frac{1}{m}\mathcal{Z}_{m-1}(x, 1)$.
(e) *The restriction of* $(x^m)_s'$ *to the unit sphere* $\partial\mathbb{B}$ *is equal to the Gegenbauer polynomial* $C_{m-1}^{(1)}(\mathrm{Re}(x))$.

(f) *The* quaternionic Koebe function $f_K(x) = (1-x)^{-2}x$ *is slice-regular on* $\mathbb{H}\setminus\{1\}$ *and it holds*

$$(f_K)'_s(x) = \mathcal{P}(x, 1), \quad (f_K)'_s(x\overline{a}) = \mathcal{P}(x, a).$$

for every $x \in \mathbb{B}$, $a \in \partial\mathbb{B}$, *where* $\mathcal{P}(x, a)$ *is the Poisson kernel of* $\mathbb{B}$.

□

The harmonicity properties of slice-regular functions imply also in the quaternionic case a stronger form of the Liouville's Theorem for entire slice regular functions. See [10, 11] for the generalization of the classical result to quaternionic functions. In the next Corollary we give also a new proof of this last result.

Corollary 3.6.8 *Let* $f \in \mathcal{SR}(\mathbb{H})$ *be an entire slice regular function. If* f *is bounded, then* f *is constant. If the spherical derivative* f'_s *is bounded (equivalently, if* $\overline{\partial}_{CRF} f$ *is bounded), then* f *is a quaternionic left-affine function, of the form* $f(x) = a + xb$ $(a, b \in \mathbb{H})$.

Proof We can repeat the same arguments of the proof of Corollary 3.4.5, using the harmonicity of f'_s and the biharmonicity of f. □

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Chapter 4
Some Notions of Subharmonicity over the Quaternions

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Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract This work introduces several notions of subharmonicity for real-valued functions of one quaternionic variable. These notions are related to the theory of slice regular quaternionic functions introduced by Gentili and Struppa in 2006. The interesting properties of these new classes of functions are studied and applied to construct the analogs of Green's functions.

Keywords Subharmonic functions · Green functions · Quaternions

Mathematics Subject Classification (2010) 31C05, 30G35

4.1 Introduction

Let $\mathbb{H} = \mathbb{R} + i\mathbb{R} + j\mathbb{R} + k\mathbb{R}$ denote the real algebra of quaternions and let

$$\mathbb{S} := \{q \in \mathbb{H} : q^2 = -1\} = \{\alpha i + \beta j + \gamma k : \alpha^2 + \beta^2 + \gamma^2 = 1\}$$

denote the 2-sphere of quaternionic imaginary units. For each $I \in \mathbb{S}$, the subalgebra $L_I = \mathbb{R} + I\mathbb{R}$ generated by 1 and I is isomorphic to $\mathbb{C}$. In recent years, this elementary fact has been the basis for the introduction of a theory of quaternionic functions.

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Definition 4.1 ([5]) Let f be a quaternion-valued function defined on a domain Ω. For each $I \in \mathbb{S}$, let $\Omega_I = \Omega \cap L_I$ and let $f_I = f_{|\Omega_I}$ be the restriction of f to Ω_I. The restriction f_I is called *holomorphic* if it has continuous partial derivatives and

$$\frac{1}{2}\left(\frac{\partial}{\partial x} + I\frac{\partial}{\partial y}\right) f_I(x + yI) \equiv 0. \tag{4.1}$$

The function f is called *(slice) regular* if, for all $I \in \mathbb{S}$, f_I is holomorphic.

The study of regular quaternionic functions has then grown into a full theory, described in the monograph [6]. It resembles the theory of holomorphic complex functions, but in a many-sided way that reflects the richness of the non-commutative setting.

In the present work, we consider several notions of subharmonicity related to the class of regular quaternionic functions. This study is distinct from the one performed in [10]. Indeed, that work studied the relation between regularity and real harmonicity; moreover, it introduced the notion of *slice harmonic function*: a quaternion-valued (or Clifford-valued) slice functions induced by a harmonic stem function. The present work searches instead for new notions of subharmonicity for real-valued functions of a quaternionic variable, compatible with composition with regular functions.

The first attempt is $\mathbb{J}$*-plurisubharmonicity*. However, this property is quite restrictive, besides being preserved by composition with a regular function f only if f is *slice preserving*, that is, if $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$.

For this reason, the alternative notions of *weakly subharmonic* and *strongly subharmonic* function are introduced. Composition with regular functions turns out to map strongly subharmonic functions into weakly subharmonic ones. Moreover, composition with slice preserving regular functions is proven to preserve weak subharmonicity.

These new notions of subharmonicity turn out to have many nice properties that recall the complex and pluricomplex cases, including mean-value properties and versions of the maximum modulus principle.

These results are finally applied to construct quaternionic analogs of Green's functions, which reveal many peculiarities due to the non-commutative setting.

An appendix comprises the classic properties of subharmonic and plurisubharmonic functions used for our new constructions.

4.2 Prerequisites

Let us recall a few properties of the algebra of quaternions $\mathbb{H}$, on which we consider the standard Euclidean metric and topology.

- For each $I \in \mathbb{S}$, the couple $1, I$ can be completed to a (positively oriented) orthonormal basis $1, I, J, K$ by choosing $J \in \mathbb{S}$ with $I \perp J$ and setting $K = IJ$.

- The coordinates of any $q \in \mathbb{H}$ with respect to such a basis can be recovered as

$$x_0(q) = \frac{1}{4}(q - IqI - JqJ - KqK)$$

$$x_1(q) = \frac{1}{4I}(q - IqI + JqJ + KqK)$$

$$x_2(q) = \frac{1}{4J}(q + IqI - JqJ + KqK)$$

$$x_3(q) = \frac{1}{4K}(q + IqI + JqJ - KqK).$$

- Mapping each $v \in T_{q_0}\mathbb{H} \cong \mathbb{H}$ to Iv for all $q_0 \in \mathbb{H}$ defines an (orthogonal) complex structure on $\mathbb{H}$, called *constant*. A biholomorphism between $(\mathbb{H}, I)$ and $(L_I^2, I) \cong (\mathbb{C}^2, i)$ can be constructed by mapping each q to $(z_1(q), z_2(q))$, where

$$z_1(q) = x_0(q) + Ix_1(q) = (q - IqI)\frac{1}{2},$$

$$z_2(q) = x_2(q) + Ix_3(q) = (q + IqI)\frac{1}{2J},$$

are such that $z_1(q) + z_2(q)J = q$. Both z_1 and z_2 depend on the choice of I; z_2 also depends on J, but only up to a multiplicative constant $c \in L_I$.

For every domain Ω and every function $f : \Omega \to \mathbb{H}$, let us denote by $f = f_1 + f_2 J$ the corresponding decomposition with f_1, f_2 ranging in L_I. Furthermore $\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2 : C^1(\Omega, L_I) \to C^0(\Omega, L_I)$ will denote the corresponding complex derivatives. In other words,

$$\partial_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} - I\frac{\partial}{\partial x_1}\right)$$

$$\bar{\partial}_1 = \frac{1}{2}\left(\frac{\partial}{\partial x_0} + I\frac{\partial}{\partial x_1}\right)$$

$$\partial_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} - I\frac{\partial}{\partial x_3}\right)$$

$$\bar{\partial}_2 = \frac{1}{2}\left(\frac{\partial}{\partial x_2} + I\frac{\partial}{\partial x_3}\right).$$

We notice that these derivatives commute with each other, and that $\partial_1, \bar{\partial}_1$ depend only on I, while $\partial_2, \bar{\partial}_2$ depend on both I and J.

The definition of regular function (Definition 4.1) amounts to requiring that the restriction to Ω_I be holomorphic from (Ω_I, I) to $(\mathbb{H}, I)$ for all $I \in \mathbb{S}$. Curiously, if the domain is carefully chosen then a stronger property holds.

Definition 4.2 Let Ω be a domain in $\mathbb{H}$. Ω is a *slice domain* if it intersects the real axis $\mathbb{R}$ and if, for all $I \in \mathbb{S}$, the intersection Ω_I with the complex plane L_I is connected. Moreover, Ω is termed *symmetric* if it is axially symmetric with respect to the real axis $\mathbb{R}$.

If we denote by $\partial_c f$ the *slice derivative*

$$\partial_c f(x+Iy) = \frac{1}{2}\left(\frac{\partial}{\partial x} - I\frac{\partial}{\partial y}\right) f_I(x+yI)$$

introduced in [5] and by $\partial_s f$ the *spherical derivative*

$$\partial_s f(q) = (q-\bar{q})^{-1}\left(f(q) - f(\bar{q})\right)$$

introduced in [8], then the aforementioned property can be stated as follows.

Theorem 4.3 ([12]) *Let Ω be a symmetric slice domain, let $f : \Omega \to \mathbb{H}$ be a regular function and let $q_0 \in \Omega$. Chosen $I, J \in \mathbb{S}$ so that $q_0 \in L_I$ and $I \perp J$, let $z_1, z_2, \bar{z}_1, \bar{z}_2$ be the induced coordinates and let $\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2$ be the corresponding derivations. Then*

$$\left.\begin{pmatrix} \bar{\partial}_1 f_1 & \bar{\partial}_2 f_1 \\ \bar{\partial}_1 f_2 & \bar{\partial}_2 f_2 \end{pmatrix}\right|_{q_0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.2}$$

Furthermore, if $q_0 \notin \mathbb{R}$ then

$$\left.\begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{pmatrix}\right|_{q_0} = \begin{pmatrix} \partial_c f_1(q_0) & -\overline{\partial_s f_2(q_0)} \\ \partial_c f_2(q_0) & \overline{\partial_s f_1(q_0)} \end{pmatrix}. \tag{4.3}$$

If, on the contrary, $q_0 \in \mathbb{R}$ then

$$\left.\begin{pmatrix} \partial_1 f_1 & \partial_2 f_1 \\ \partial_1 f_2 & \partial_2 f_2 \end{pmatrix}\right|_{q_0} = \begin{pmatrix} \partial_c f_1(q_0) & -\overline{\partial_c f_2(q_0)} \\ \partial_c f_2(q_0) & \overline{\partial_c f_1(q_0)} \end{pmatrix}. \tag{4.4}$$

We point out that we have not proven that f is holomorphic with respect to the constant structure I: with respect to the basis $1, I, J, IJ$, equality (4.2) only holds at those points q_0 that lie in L_I. In fact, regularity is related to a different notion of holomorphy, which involves non-constant orthogonal complex structures. Let us recall the notations $Re(q) = x_0(q)$, $Im(q) = q - Re(q)$ for $q \in \mathbb{H}$ and let us set

$$\mathbb{J}_{q_0} v := \frac{Im(q_0)}{|Im(q_0)|} v \qquad \forall\, v \in T_{q_0}(\mathbb{H} \setminus \mathbb{R}) \cong \mathbb{H}.$$

Then $\pm\mathbb{J}$ are orthogonal complex structures on

$$\mathbb{H} \setminus \mathbb{R} = \bigcup_{I \in \mathbb{S}} (\mathbb{R} + I\mathbb{R}^+)$$

and they are induced by the natural identification with the complex manifold $\mathbb{CP}^1 \times (\mathbb{R} + i\mathbb{R}^+)$.

Theorem 4.4 ([7]) *Let Ω be a symmetric slice domain, and let $f : \Omega \to \mathbb{H}$ be an injective regular function. Then the real differential of f is invertible at each $q \in \Omega$ and the push-forward of $\mathbb{J}$ via f, that is,*

$$\mathbb{J}^f_{f(q)} v = \frac{Im(q)}{|Im(q)|} v \qquad \forall\, v \in T_{f(q)} f(\Omega \setminus \mathbb{R}) \cong \mathbb{H}\,, \tag{4.5}$$

is an orthogonal complex structure on $f(\Omega \setminus \mathbb{R})$.

In the hypotheses of the previous theorem, f is (obviously) a holomorphic map from $(\Omega \setminus \mathbb{R}, \mathbb{J})$ to $\big(f(\Omega \setminus \mathbb{R}), \mathbb{J}^f\big)$. Furthermore, there is a special class of regular functions such that $\mathbb{J}^f = \mathbb{J}$.

Remark 4.5 Let $f : \Omega \to \mathbb{H}$ be a *slice preserving* regular function, namely a regular function such that $f(\Omega_I) \subseteq L_I$ for all $I \in \mathbb{S}$. Then $f(\Omega \setminus \mathbb{R}) = f(\Omega) \setminus \mathbb{R}$ and f is a holomorphic map from $(\Omega \setminus \mathbb{R}, \mathbb{J})$ to $(f(\Omega) \setminus \mathbb{R}, \mathbb{J})$.

4.3 Quaternionic Notions of Subharmonicity

Let Ω be a domain in $\mathbb{H}$ and let

$$\mathsf{us}(\Omega) = \{u : \Omega \to [-\infty, +\infty),\, u \text{ upper semicontinuous},\, u \not\equiv -\infty\}.$$

For $u \in \mathsf{us}(\Omega)$, we aim at defining some notion of subharmonicity that behaves well when we compose u with a regular function. Remark 4.5 encourages us to consider $\mathbb{J}$-*plurisubharmonic* and $\mathbb{J}$-*harmonic* functions, i.e., functions that are pluri(sub)harmonic with respect to the complex structure $\mathbb{J}$. However, the notion of $\mathbb{J}$-plurisubharmonicity on a symmetric slice domain Ω is induced by plurisubharmonicity in $\mathbb{CP}^1 \times D_\Omega$ with

$$D_\Omega = \{x + iy \in \mathbb{R} + i\mathbb{R}^+ : x + y\mathbb{S} \subset \Omega\},$$

which amounts to constancy in the first variable and subharmonicity in the second variable. We conclude:

Proposition 4.6 *Let Ω be a symmetric domain in $\mathbb{H}$. A function $u \in \mathsf{us}(\Omega)$ is $\mathbb{J}$-pluri(sub)harmonic in $\Omega \setminus \mathbb{R}$ if, and only if, there exists a (sub)harmonic function*

$\upsilon : D_\Omega \to \mathbb{R}$ such that $u(x+Iy) = \upsilon(x+iy)$ for all $I \in \mathbb{S}$ and for all $x+iy \in D_\Omega$. In particular, a function $u \in C^2(\Omega, \mathbb{R})$ is $\mathbb{J}$-plurisubharmonic in $\Omega \setminus \mathbb{R}$ if, and only if, $u(x+Iy)$ does not depend on I and

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x+Iy) \geq 0; \tag{4.6}$$

it is $\mathbb{J}$-harmonic if, and only if, equality holds at all points.

Clearly, if we complete $1, I$ to a basis $1, I, J, IJ$ with $I \perp J$, consider the induced coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$ and let $\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2$ be the corresponding derivations, then inequality (4.6) is equivalent to $\bar{\partial}_1 \partial_1 u_I \geq 0$.

In order to get a richer class of functions, we need to suitably weaken the notion of subharmonicity considered. We are thus encouraged to give the following definition.

Definition 4.7 Let Ω be a domain in $\mathbb{H}$ and let $u \in \mathsf{us}(\Omega)$. We call u *weakly subharmonic* if for all $I \in \mathbb{S}$ the restriction $u_I = u_{|\Omega_I}$ is subharmonic (after the natural identification between L_I and $\mathbb{C}$). We say that u is *weakly harmonic* if, for all $I \in \mathbb{S}$, u_I is harmonic.

Remark 4.8 A function $u \in C^2(\Omega, \mathbb{R})$ is weakly subharmonic if, and only if, inequality (4.6) holds for all $I \in \mathbb{S}$; u is weakly harmonic if, and only if, equality holds at all points.

By construction:

Proposition 4.9 *Let Ω be a symmetric domain in $\mathbb{H}$ and let $u \in \mathsf{us}(\Omega)$. If u is $\mathbb{J}$-pluri(sub)harmonic in $\Omega \setminus \mathbb{R}$ then it is weakly (sub)harmonic in $\Omega \setminus \mathbb{R}$. If, moreover, u is continuous at all points of $\Omega \cap \mathbb{R}$ then u is weakly (sub)harmonic in Ω.*

The converse implication is not true, as shown by the next example. Here, $\langle \cdot, \cdot \rangle$ denotes the Euclidean scalar product on $Im(\mathbb{H}) \cong \mathbb{R}^3$.

Example 4.10 All real affine functions $u : \mathbb{H} \to \mathbb{R}$ are weakly harmonic, including the coordinates x_0, x_1, x_2, x_3 with respect to any basis $1, I, J, IJ$ with $I, J \in \mathbb{S}$, $I \perp J$. On the other hand, x_1, x_2, x_3 are not $\mathbb{J}$-plurisubharmonic in $\mathbb{H} \setminus \mathbb{R}$, as $x_1(x+Iy) = y\langle I, i\rangle$, $x_2(x+Iy) = y\langle I, j\rangle$, $x_3(x+Iy) = y\langle I, k\rangle$ are not constant in I.

Actually, a stronger property holds for real affine functions $u : \mathbb{H} \to \mathbb{R}$: they are pluriharmonic with respect to any constant orthogonal complex structure. This motivates the next definition.

Definition 4.11 Let Ω be a domain in $\mathbb{H}$ and let $u \in \mathsf{us}(\Omega)$. We say that u is *strongly (sub)harmonic* if it is pluri(sub)harmonic with respect to every constant orthogonal complex structure on Ω.

Remark 4.12 A function $u \in C^2(\Omega, \mathbb{R})$ is strongly subharmonic if, for all $I, J \in \mathbb{S}$ with $I \perp J$,

$$H_{I,J}(u) = \begin{pmatrix} \bar{\partial}_1 \partial_1 u & \bar{\partial}_1 \partial_2 u \\ \bar{\partial}_2 \partial_1 u & \bar{\partial}_2 \partial_2 u \end{pmatrix} \tag{4.7}$$

is a positive semidefinite matrix at each $q \in \Omega$. The function u is strongly harmonic if for all $I, J \in \mathbb{S}$ with $I \perp J$ the matrix $H_{I,J}(u)$ has constant rank 0.

Clearly, if the matrix $H_{I,J}(u)$ is positive semidefinite then its $(1, 1)$-entry $\bar{\partial}_1 \partial_1 u$ is non-negative. Similarly, if $H_{I,J}(u)$ has constant rank 0 then $\bar{\partial}_1 \partial_1 u \equiv 0$. This leads to the next result, which, however, is not only true for $u \in C^2(\Omega, \mathbb{R})$ but also for $u \in \mathsf{us}(\Omega)$.

Proposition 4.13 *Let $u \in \mathsf{us}(\Omega)$. If u is strongly (sub)harmonic then it is weakly (sub)harmonic.*

Proof Let $u \in \mathsf{us}(\Omega)$ be strongly (sub)harmonic, let $I \in \mathbb{S}$ and let us pr ove that u_I is (sub)harmonic. By construction, u is pluri(sub)harmonic with respect to the constant orthogonal complex structure I. Moreover, the inclusion map $incl : \Omega_I \to \Omega$ is a holomorphic map from (Ω_I, I) to (Ω, I). As a consequence, $u_I = u \circ incl$ is (sub)harmonic, as desired. □

Example 4.10 shows that strong (sub)harmonicity does not imply $\mathbb{J}$-pluri(sub)harmonicity. The converse implication does not hold, either:

Example 4.14 The function $u(q) = Re(q^2)$ is $\mathbb{J}$-pluriharmonic in $\mathbb{H} \setminus \mathbb{R}$, as $u(x + Iy) = x^2 - y^2$ does not depend on I and $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(x + Iy) \equiv 0$. On the other hand, u is not strongly subharmonic. Actually, it is not plurisubharmonic with respect to any constant orthogonal complex structure I: after choosing $J \in \mathbb{S}$ with $J \perp I$, we compute

$$u(z_1 + z_2 J) = Re\left(z_1^2 - z_2\bar{z}_2 + (z_1 z_2 + z_2\bar{z}_1)J\right) = \frac{z_1^2 + \bar{z}_1^2}{2} - z_2\bar{z}_2$$

for all $z_1, z_2 \in L_I$, so that $H_{I,J}(u) \equiv \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$.

We have proven the following implications (none of which can be reversed):

plurisubharmonic
w.r.t. all OCS's in $\Omega \setminus \mathbb{R}$

↙ ↘

$\mathbb{J}$-plurisubharmonic in $\Omega \setminus \mathbb{R}$ | strongly subharmonic in $\Omega \setminus \mathbb{R}$

↘ ↙

weakly subharmonic in $\Omega \setminus \mathbb{R}$

A similar scheme can be drawn for the quaternionic notions of harmonicity. We show with a further example that a strongly subharmonic function is not necessarily strongly *harmonic* when it is weakly harmonic, or even $\mathbb{J}$-pluriharmonic.

Example 4.15 Consider the function $u : \mathbb{H} \to \mathbb{R}$ with $u(q) := \log|q|$ for all $q \in \mathbb{H} \setminus \{0\}$ and $u(0) := -\infty$. u is $\mathbb{J}$-pluriharmonic in $\mathbb{H} \setminus \mathbb{R}$, as $u(x+Iy) = \frac{1}{2}\log(x^2 + y^2)$. As a consequence, u is also weakly harmonic. On the other hand, for any choice of $I, J \in \mathbb{S}$, the fact that $u(z_1 + z_2 J) = \frac{1}{2}\log(z_1\bar{z}_1 + z_2\bar{z}_2)$ implies that

$$
\begin{aligned}
H_{I,J}(u)_{|z_1+z_2J} &= \frac{1}{2(z_1\bar{z}_1 + z_2\bar{z}_2)^2}\begin{pmatrix} z_2\bar{z}_2 & -z_1\bar{z}_2 \\ -z_2\bar{z}_1 & z_1\bar{z}_1 \end{pmatrix} \\
&= \frac{1}{2(|z_1|^2 + |z_2|^2)^2}\begin{pmatrix} |z_2|^2 & -z_1\bar{z}_2 \\ -z_2\bar{z}_1 & |z_1|^2 \end{pmatrix}.
\end{aligned}
$$

Hence, u is strongly subharmonic but it is not strongly harmonic.

Let us review a few classical constructions in our new environment.

Remark 4.16 On a given domain Ω, let us denote by $\mathsf{wsh}(\Omega)$ the set of weakly subharmonic functions, by $\mathsf{ssh}(\Omega)$ that of strongly subharmonic functions, and by $\mathsf{psh}_{\mathbb{J}}(\Omega)$ that of $\mathbb{J}$-plurisubharmonic functions on Ω (if Ω equals a symmetric domain minus $\mathbb{R}$). If S is any of these sets then:

1. S is a convex cone;
2. for all $u \in S$, if φ is a real-valued C^2 function on a neighborhood of $u(\mathbb{R})$ and if φ is increasing and convex then $\varphi \circ u : \Omega \to \mathbb{R}$ also belongs to S;
3. for all $u_1, u_2 \in S$, the function $u(q) = \max\{u_1(q), u_2(q)\}$ belongs to S;
4. if $\{u_\alpha\}_{\alpha\in A}$ (with $A \neq \emptyset$) is a family in S, locally bounded from above, and if $u(q) = \sup_{\alpha\in A} u_\alpha(q)$ for all $q \in \Omega$ then the upper semicontinuous regularization u^* belongs to S.

Example 4.17 For any $\alpha > 0$, the function $u : \mathbb{H} \to \mathbb{R}\ q \mapsto |q|^\alpha$ is strongly subharmonic in $\mathbb{H}$ and it is $\mathbb{J}$-plurisubharmonic in $\mathbb{H} \setminus \mathbb{R}$.

Example 4.18 The functions $Re^2(q) = x_0^2(q)$ and $|Im(q)|^2 = x_1^2(q) + x_2^2(q) + x_3^2(q)$ are strongly subharmonic in $\mathbb{H}$. (They are also $\mathbb{J}$-plurisubharmonic in $\mathbb{H} \setminus \mathbb{R}$, as $Re^2(x+Iy) = x^2$ and $|Im(x+Iy)|^2 = y^2$.)

We conclude this section showing that any given subharmonic function on an axially symmetric planar domain extends to a weakly subharmonic function on the corresponding symmetric domain of $\mathbb{H}$.

Remark 4.19 If we start with a domain $D \subseteq \mathbb{C}$ that is symmetric with respect to the real axis and a (sub)harmonic function υ on D, we may define a weakly (sub)harmonic function u on the symmetric domain $\Omega = \bigcup_{x+iy\in D} x + y\mathbb{S}$ by setting

$$u(x+Iy) := \frac{1+\langle I, i\rangle}{2}\upsilon(x+iy) + \frac{1-\langle I, i\rangle}{2}\upsilon(x-iy)$$

for all $x \in \mathbb{R}, I \in \mathbb{S}, y > 0$ such that $x + Iy \in \Omega$ and $u(x) := v(x)$ for all $x \in \Omega \cap \mathbb{R}$.

4.4 Composition with Regular Functions

We now want to understand the behavior of the different notions of subharmonicity we introduced, under composition with regular functions. For $\mathbb{J}$-pluri(sub)harmonicity, Remark 4.5 immediately implies:

Proposition 4.20 *Let Ω be a symmetric domain in $\mathbb{H}$ and let $u \in \mathsf{us}(\Omega)$. u is $\mathbb{J}$-pluri(sub)harmonic in $\Omega \setminus \mathbb{R}$ if, and only if, for every symmetric domain Ω' in $\mathbb{H}$ and every slice preserving regular function $f : \Omega' \setminus \mathbb{R} \to \Omega \setminus \mathbb{R}$, the composition $u \circ f$ is $\mathbb{J}$-pluri(sub)harmonic in $\Omega' \setminus \mathbb{R}$.*

It is essential to restrict to slice preserving regular functions. If we compose u with an injective regular function f then the only sufficient condition we know in order for $u \circ f$ to be $\mathbb{J}$-pluri(sub)harmonic is, that u be $\mathbb{J}^f$-pluri(sub)harmonic (see Theorem 4.4).

For weak (sub)harmonicity, we can prove the next result.

Theorem 4.21 *Let $u \in \mathsf{us}(\Omega)$. u is weakly (sub)harmonic in Ω if, and only if, for every symmetric domain Ω' in $\mathbb{H}$ and every slice preserving regular function $f : \Omega' \to \Omega$, the composition $u \circ f$ is weakly (sub)harmonic in Ω'.*

Proof If $u \circ f$ is weakly (sub)harmonic for all slice preserving regular f then in particular $u = u \circ id$ is weakly (sub)harmonic.

Conversely, let $u \in \mathsf{us}(\Omega)$ be weakly (sub)harmonic, let $f : \Omega' \to \Omega$ be a slice preserving regular function and let us prove that $u \circ f$ is weakly (sub)harmonic. For each $I \in \mathbb{S}$, u_I is (sub)harmonic in Ω_I and the restriction f_I is a holomorphic map from (Ω'_I, I) to (Ω_I, I). As a consequence, $(u \circ f)_I = u_I \circ f_I$ is (sub)harmonic in Ω'_I. □

As for strong harmonicity and subharmonicity, they are preserved under composition with quaternionic affine transformations, as the latter are holomorphic with respect to any constant structure $I \in \mathbb{S}$:

Remark 4.22 If u is a strongly (sub)harmonic function on a domain $\Omega \subseteq \mathbb{H}$ then, for any $a, b \in \mathbb{H}$ with $b \neq 0$, the function $v(q) = u(a + qb)$ is strongly (sub)harmonic in $\Omega b^{-1} - a$.

However, strong harmonicity and subharmonicity are not preserved by composition with other regular functions, not even slice preserving regular functions (see Example 4.14). For this reason, we address the study of their composition with regular functions by direct computation, starting with the C^2 case.

Lemma 4.23 *Let $I, J \in \mathbb{S}$ with $I \perp J$ and let us consider the associated $\partial_1, \partial_2, \bar{\partial}_1, \bar{\partial}_2$. Let Ω be a domain in $\mathbb{H}$ and let $u \in C^2(\Omega, \mathbb{R})$. For every symmetric* *slice domain* Ω' *and for every regular function* $f : \Omega' \to \Omega$, *we have*

$$\bar{\partial}_1 \partial_1 (u \circ f)_{|_{q_0}} = (\overline{\partial_1 f_1}, \overline{\partial_1 f_2})|_{q_0} \cdot H_{I,J}(u)_{|_{f(q_0)}} \cdot \left. \begin{pmatrix} \partial_1 f_1 \\ \partial_1 f_2 \end{pmatrix} \right|_{q_0} \tag{4.8}$$

at each $q_0 \in \Omega'_I$. *If, moreover,* f *is slice preserving then*

$$\bar{\partial}_1 \partial_1 (u \circ f)_{|_{q_0}} = |\partial_1 f_1|^2_{|_{q_0}} \cdot \bar{\partial}_1 \partial_1 u_{|_{f(q_0)}} \tag{4.9}$$

at each $q_0 \in \Omega'_I$. *The same is true if there exists a constant* $c \in \mathbb{H}$ *such that* $f_I + c$ *maps* Ω'_I *to* L_I.

Proof We compute:

$$\partial_1 (u \circ f) = (\partial_1 u) \circ f \cdot \partial_1 f_1 + (\partial_2 u) \circ f \cdot \partial_1 f_2 + (\bar{\partial}_1 u) \circ f \cdot \partial_1 \bar{f}_1 + (\bar{\partial}_2 u) \circ f \cdot \partial_1 \bar{f}_2$$

and

$$\begin{aligned} \bar{\partial}_1 \partial_1 (u \circ f) = & \bar{\partial}_1((\partial_1 u) \circ f) \cdot \partial_1 f_1 + (\partial_1 u) \circ f \cdot \bar{\partial}_1 \partial_1 f_1 + \\ & + \bar{\partial}_1((\partial_2 u) \circ f) \cdot \partial_1 f_2 + (\partial_2 u) \circ f \cdot \bar{\partial}_1 \partial_1 f_2 + \\ & + \bar{\partial}_1((\bar{\partial}_1 u) \circ f) \cdot \partial_1 \bar{f}_1 + (\bar{\partial}_1 u) \circ f \cdot \bar{\partial}_1 \partial_1 \bar{f}_1 + \\ & + \bar{\partial}_1((\bar{\partial}_2 u) \circ f) \cdot \partial_1 \bar{f}_2 + (\bar{\partial}_2 u) \circ f \cdot \bar{\partial}_1 \partial_1 \bar{f}_2 \,. \end{aligned}$$

If we evaluate the previous expression at a point $q \in L_I$, equality (4.2) guarantees the vanishing of all terms but the first and the third. Hence,

$$\bar{\partial}_1 \partial_1 (u \circ f)_{|_q} = \bar{\partial}_1((\partial_1 u) \circ f)_{|_q} \cdot \partial_1 f_{1|_q} + \bar{\partial}_1((\partial_2 u) \circ f)_{|_q} \cdot \partial_1 f_{2|_q}$$

where

$$\begin{aligned} \bar{\partial}_1((\partial_1 u) \circ f)_{|_q} = & \partial_1 \partial_1 u_{|_{f(q)}} \cdot \bar{\partial}_1 f_{1|_q} + \partial_2 \partial_1 u_{|_{f(q)}} \cdot \bar{\partial}_1 f_{2|_q} + \\ & + \bar{\partial}_1 \partial_1 u_{|_{f(q)}} \cdot \bar{\partial}_1 \bar{f}_{1|_q} + \bar{\partial}_2 \partial_1 u_{|_{f(q)}} \cdot \bar{\partial}_1 \bar{f}_{2|_q} = \\ = & \bar{\partial}_1 \partial_1 u_{|_{f(q)}} \cdot \overline{\partial_1 f_1}_{|_q} + \bar{\partial}_2 \partial_1 u_{|_{f(q)}} \cdot \overline{\partial_1 f_2}_{|_q} \end{aligned}$$

and

$$\begin{aligned} \bar{\partial}_1((\partial_2 u) \circ f)_{|_q} = & \partial_1 \partial_2 u_{|_{f(q)}} \cdot \bar{\partial}_1 f_{1|_q} + \partial_2 \partial_2 u_{|_{f(q)}} \cdot \bar{\partial}_1 f_{2|_q} + \\ & + \bar{\partial}_1 \partial_2 u_{|_{f(q)}} \cdot \bar{\partial}_1 \bar{f}_{1|_q} + \bar{\partial}_2 \partial_2 u_{|_{f(q)}} \cdot \bar{\partial}_1 \bar{f}_{2|_q} = \\ = & \bar{\partial}_1 \partial_2 u_{|_{f(q)}} \cdot \overline{\partial_1 f_1}_{|_q} + \bar{\partial}_2 \partial_2 u_{|_{f(q)}} \cdot \overline{\partial_1 f_2}_{|_q} \,. \end{aligned}$$

Thus,

$$\bar{\partial}_1\partial_1(u\circ f)_{|_q} = \bar{\partial}_1\partial_1 u_{|f(q)} \cdot |\partial_1 f_1|^2_{|_q} + \bar{\partial}_2\partial_1 u_{|f(q)} \cdot \overline{\partial_1 f_2}_{|_q} \cdot \partial_1 f_{1|_q} + \\ + \bar{\partial}_1\partial_2 u_{|f(q)} \cdot \overline{\partial_1 f_1}_{|_q} \partial_1 f_{2|_q} + \bar{\partial}_2\partial_2 u_{|f(q)} \cdot |\partial_1 f_2|^2_{|_q};$$

that is,

$$\bar{\partial}_1\partial_1(u\circ f)_{|_q} = (\overline{\partial_1 f_1}, \overline{\partial_1 f_2})|_q \cdot \begin{pmatrix} \bar{\partial}_1\partial_1 u & \bar{\partial}_1\partial_2 u \\ \bar{\partial}_2\partial_1 u & \bar{\partial}_2\partial_2 u \end{pmatrix}\Bigg|_{f(q)} \cdot \begin{pmatrix} \partial_1 f_1 \\ \partial_1 f_2 \end{pmatrix}\Bigg|_q .$$

Finally, if there exists $c = c_1 + c_2 J \in \mathbb{H}$ such that $f_I + c$ maps Ω'_I to L_I, then $f_2 \equiv -c_2$ in Ω'_I so that $\partial_1 f_2$ vanishes identically in Ω'_I and

$$\bar{\partial}_1\partial_1(u\circ f)_{|_q} = |\partial_1 f_1|^2_{|_q} \cdot \bar{\partial}_1\partial_1 u_{|f(q)},$$

as desired. □

We are now ready to study the composition of strongly (sub)harmonic C^2 functions with regular functions.

Theorem 4.24 *Let $u \in C^2(\Omega, \mathbb{R})$. u is strongly (sub)harmonic if, and only if, for every symmetric slice domain Ω' and for every regular function $f : \Omega' \to \Omega$, the composition $u \circ f$ is weakly (sub)harmonic.*

Proof If $u : \Omega \to \mathbb{R}$ is strongly subharmonic then, for all $I, J \in \mathbb{S}$ (with $I \perp J$), the matrix $H_{I,J}(u)$ is positive semidefinite. For every regular function $f : \Omega' \to \Omega$ and for all $I \in \mathbb{S}$, Lemma 4.23 implies $\bar{\partial}_1\partial_1(u\circ f)_{|_{z_1}} \geq 0$ at each $z_1 \in \Omega_I$. Hence, $u \circ f$ is weakly subharmonic.

Conversely, if $u \circ f$ is weakly subharmonic for every regular function $f : \Omega' \to \Omega$ then we can prove that u is strongly subharmonic in the following way. Let us fix $I, J \in \mathbb{S}$, $p \in \Omega$ (with $I \perp J$) and prove that $H_{I,J}(u)$ is positive semidefinite at p, i.e., that

$$(\bar{v}_1, \bar{v}_2) \cdot H_{I,J}(u)_{|_p} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \geq 0$$

for arbitrary $v_1, v_2 \in L_I$. Let us set $v := v_1 + v_2 J$ and $f(q) := qv + p$ for $q \in B(0, R)$ (with $R > 0$ small enough to guarantee the inclusion of $f(B(0, R)) = B(p, |v|R)$ into Ω). By direct computation, $\partial_c f \equiv v$. Formula (4.3) yields the equalities $\partial_1 f_{1|_0} = v_1$, $\partial_1 f_{2|_0} = v_2$. Taking into account Lemma 4.23 and the fact that $f(0) = p$, we conclude that

$$(\bar{v}_1, \bar{v}_2) \cdot H_{I,J}(u)_{|_p} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \bar{\partial}_1\partial_1(u\circ f)_{|_0}.$$

Since $u \circ f$ is weakly subharmonic, $\bar{\partial}_1 \partial_1 (u \circ f)_{|0} \geq 0$ and we have proven the desired inequality.

Analogous reasonings characterize strong *harmonicity*. □

The previous result allows us to construct a large class of examples of weakly subharmonic functions.

Example 4.25 For any regular function $f : \Omega \to \mathbb{H}$ on a symmetric slice domain Ω, the components of f with respect to any basis $1, I, J, IJ$ with $I, J \in \mathbb{S}, I \perp J$ are weakly harmonic. Furthermore, for all $\alpha > 0$ the functions $\log|f|, |f|^\alpha, Re^2 f, |Imf|^2$ are weakly subharmonic.

4.5 Mean-Value Property and Consequences

We can characterize weak and strong (sub)harmonicity of $u \in \mathsf{US}(\Omega)$ in terms of mean-value properties. For each $I \in \mathbb{S}, a \in \Omega, b \in \mathbb{H} \setminus \{0\}$ such that Ω includes the disc $\Gamma_{I,a,b} := \{a + \lambda b : \lambda \in L_I, |\lambda| \leq 1\}$, we will use the notation

$$l_I(u; a, b) := \frac{1}{2\pi} \int_0^{2\pi} u(a + e^{I\vartheta} b) d\vartheta. \tag{4.10}$$

Proposition 4.26 *Let Ω be a domain in $\mathbb{H}$ and let $u \in \mathsf{US}(\Omega)$. u is weakly subharmonic if, and only if, the inequality*

$$u(a) \leq l_I(u; a, b) \tag{4.11}$$

holds for all $I \in \mathbb{S}, a \in \Omega_I, b \in L_I \setminus \{0\}$ such that $\Gamma_{I,a,b} \subset \Omega_I$. u is weakly harmonic if, and only if, (u does not take the value $-\infty$ and) equality always holds in formula (4.11)*.*

Proof Fix any $I \in \mathbb{S}$. By the mean-value characterization of subharmonic functions (see Theorem 4.51), u_I is subharmonic in Ω_I if, and only if, $u(a) \leq l_I(u; a, b)$ for all $a \in \Omega_I, b \in L_I \setminus \{0\}$ such that $\Gamma_{I,a,b} \subset \Omega_I$. The corresponding equalities characterize harmonicity because u_I is harmonic if, and only if, u_I and $-u_I$ are both subharmonic (see Corollary 4.53). □

Proposition 4.27 *Let Ω be a domain in $\mathbb{H}$ and let $u \in \mathsf{US}(\Omega)$. u is strongly subharmonic if, and only if, the inequality*

$$u(a) \leq l_I(u; a, b) \tag{4.12}$$

holds for all $I \in \mathbb{S}, a \in \Omega, b \in \mathbb{H} \setminus \{0\}$ such that $\Gamma_{I,a,b} \subset \Omega$. u is strongly harmonic if, and only if, (u does not take the value $-\infty$ and) equality always holds in formula (4.12)*.*

Proof For each $I \in \mathbb{S}$, let us apply the mean-value characterization of plurisubharmonic functions (see Theorem 4.56) to establish whether u is I-plurisubharmonic. This happens if, and only if, $u(a) \leq l_I(u; a, b)$ for all $a \in \Omega$, $b \in \mathbb{H} \setminus \{0\}$ such that $\Gamma_{I,a,b} \subset \Omega$. The corresponding equalities characterize I-pluriharmonicity. □

As an application of the previous results, we can extend Theorem 4.24 to all $u \in \mathsf{us}(\Omega)$.

Theorem 4.28 *Let $u \in \mathsf{us}(\Omega)$. u is strongly (sub)harmonic if, and only if, for every regular function $f : \Omega' \to \Omega$ the composition $u \circ f$ is weakly (sub)harmonic.*

Proof Let us suppose the composition $u \circ f$ with any regular function $f : \Omega' \to \Omega$ to be weakly subharmonic and let us prove that u is strongly subharmonic. By Proposition 4.27, it suffices to prove that, for any $I \in \mathbb{S}$, $a \in \Omega$, $b \in \mathbb{H} \setminus \{0\}$ such that $\Gamma_{I,a,b} \subset \Omega$, the inequality

$$u(a) \leq l_I(u; a, b)$$

holds. If we set $f(q) := a + qb$, then $f(0) = a$ and f maps the disc $\Gamma_{I,0,1}$ into the disc $\Gamma_{I,a,b}$. Thus, it suffices to prove that

$$u(f(0)) \leq l_I(u \circ f; 0, 1).$$

But this inequality is true by Proposition 4.26, since $u \circ f$ is weakly subharmonic in a domain Ω' such that $\Gamma_{I,0,1} \subset \Omega'_I$. Analogous considerations can be made for the harmonic case.

Conversely, let $u \in \mathsf{us}(\Omega)$ be strongly (sub)harmonic, let $f : \Omega' \to \Omega$ be a regular function and let us prove that $u \circ f$ is weakly (sub)harmonic. For each $I \in \mathbb{S}$, u is I-pluri(sub)harmonic and the restriction f_I is a holomorphic map from (Ω'_I, I) to (Ω, I). As a consequence, $(u \circ f)_I = u \circ f_I$ is (sub)harmonic in Ω'_I, as desired. □

A form of maximum modulus principle holds for weakly or strongly plurisubharmonic functions.

Proposition 4.29 *Let Ω be a domain in $\mathbb{H}$ and suppose $u \in \mathsf{us}(\Omega)$ to be weakly subharmonic. If u has a local maximum point $p \in \Omega_I$ then u_I is constant in the connected component of Ω_I that includes p. If, moreover, u is strongly subharmonic, then u is constant in Ω.*

Proof In our hypotheses, u_I is a subharmonic function with a local maximum point $p \in \Omega_I$. Thus, u_I is constant in the connected component of Ω_I that includes p by the maximum modulus principle for subharmonic functions (see Theorem 4.52).

If, moreover, u is strongly subharmonic then it is I-plurisubharmonic. Since we assumed Ω to be connected, u is constant in Ω by the maximum modulus principle for plurisubharmonic functions (see Theorem 4.58). □

Let us now consider maximality.

Definition 4.30 Let S be a class of real-valued functions on an open set D and let υ be an element of S. Suppose that, for any relatively compact subset G of D and for all $\nu \in S$ with $\nu \leq \upsilon$ in ∂G, the inequality $\nu \leq \upsilon$ holds throughout G. In this situation, we say that υ is *maximal* in S (or among the elements of S).

Bearing in mind that harmonic functions of one complex variable are the maximal elements of the class of plurisubharmonic functions (see Remark 4.59), we can characterize weak harmonicity as follows.

Remark 4.31 Let Ω be a domain in $\mathbb{H}$ and let $u \in \mathsf{wsh}(\Omega)$. u is weakly harmonic if, and only if, for all $I \in \mathbb{S}$, the restriction u_I is maximal among subharmonic functions on Ω_I. As a consequence, if u is weakly harmonic then u is maximal in $\mathsf{wsh}(\Omega)$.

Let us now consider strongly subharmonic functions. Since they are plurisubharmonic with respect to all constant structures, we can make the following observation.

Remark 4.32 Let Ω be a domain in $\mathbb{H}$ and let $u \in \mathsf{ssh}(\Omega)$. If u is strongly harmonic then it is maximal in $\mathsf{ssh}(\Omega)$. Furthermore, if $u \in C^2(\Omega)$ then u is maximal in $\mathsf{ssh}(\Omega)$ if and only if $\det H_{I,J}(u) \equiv 0$ for all $I, J \in \mathbb{S}$ with $I \perp J$.

It is easy to exhibit a maximal element of $\mathsf{ssh}(\Omega)$ that is not strongly harmonic.

Example 4.33 The function $u(q) = \log |q|$ is a strongly subharmonic function on $\mathbb{H}$. The explicit computations in Example 4.15 show that u is maximal but not strongly harmonic.

4.6 Approximation

An approximation result holds for strongly subharmonic functions. For all $\varepsilon > 0$, let

$$\Omega_\varepsilon := \begin{cases} \{q \in \Omega : \operatorname{dist}(q, \partial\Omega) > \varepsilon\} & \text{if } \Omega \neq \mathbb{H} \\ \mathbb{H} & \text{if } \Omega = \mathbb{H} \end{cases}$$

and let $u * \chi_\varepsilon$ denote the convolution of u with the standard smoothing kernels χ_ε of $\mathbb{H} \cong \mathbb{R}^4$ (see Definition 4.54).

Proposition 4.34 *Let* $u \in \mathsf{ssh}(\Omega)$. *If* $\varepsilon > 0$ *is such that* Ω_ε *is not empty, then* $u * \chi_\varepsilon \in C^\infty \cap \mathsf{ssh}(\Omega_\varepsilon)$. *Moreover,* $u * \chi_\varepsilon$ *monotonically decreases with decreasing* ε *and*

$$\lim_{\varepsilon \to 0^+} u * \chi_\varepsilon(q) = u(q) \tag{4.13}$$

for each $q \in \Omega$.

Proof Fix any $I \in \mathbb{S}$, so that u is I-plurisubharmonic. The fact that $u * \chi_\varepsilon \in C^\infty \cap \mathsf{ssh}(\Omega_\varepsilon)$, as well as our second statement, follow from the corresponding classic properties of plurisubharmonic functions (see Theorem 4.57). □

On the other hand, convolution with the standard smoothing kernel χ_ε does not preserve weak subharmonicity. This can be shown with an example, which uses the notation $l_I(u; a, b)$ established in formula (4.10). As customary, $l_I(u; \cdot, b)$ denotes the function $a \mapsto l_I(u; a, b)$.

Example 4.35 The function $u(q) = Re(q^2)$ is in $\mathsf{wsh}(\mathbb{H})$, but $u*\chi_\varepsilon$ does not belong to $\mathsf{wsh}(\mathbb{H})$. Indeed, we saw that for each orthonormal basis $1, I, J, IJ$ of $\mathbb{H}$ we have $H_{I,J}(u) \equiv \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$. Hence, $-u$ is strongly subharmonic and the same is true for $-u * \chi_\varepsilon$ by the previous proposition. In particular, $-u * \chi_\varepsilon \in \mathsf{wsh}(\mathbb{H})$ so that $u * \chi_\varepsilon$ can only be in $\mathsf{wsh}(\mathbb{H})$ if it is weakly *harmonic*. This amounts to requiring that for each $I \in \mathbb{S}$, $a \in \Omega_I$, $b \in L_I \setminus \{0\}$ such that $\Gamma_{I,a,b} \subset \Omega$ the equality

$$u * \chi_\varepsilon(a) = l_I(u * \chi_\varepsilon; a, b) = l_I(u; \cdot, b) * \chi_\varepsilon(a),$$

holds. But this happens if, and only if, $u(a - q) = l_I(u; a - q, b)$ for all q in the support $\overline{B(0, \varepsilon)}$ of χ_ε. This cannot be true, since (if z_1, z_2 denote the complex variables with respect to the orthonormal basis $1, I, J, IJ$) the function $-u$ is strictly subharmonic in z_2.

4.7 Green's Functions

We now consider the analogs of Green's functions in the context of weakly and strongly subharmonic functions.

Definition 4.36 Let Ω be a domain in $\mathbb{H}$, let $q_0 \in \Omega$, and set

$$\mathsf{wsh}_{q_0}(\Omega) := \left\{ u \in \mathsf{wsh}(\Omega) : u < 0, \limsup_{q \to q_0} \left| u(q) - \log|q - q_0| \right| < \infty \right\}$$

$$\mathsf{ssh}_{q_0}(\Omega) := \mathsf{wsh}_{q_0}(\Omega) \cap \mathsf{ssh}(\Omega)\,.$$

For all $q \in \Omega$, let us define

$$w(q) := \begin{cases} -\infty & \text{if } \mathsf{wsh}_{q_0}(\Omega) = \emptyset \\ \sup\{u(q) : u \in \mathsf{wsh}_{q_0}(\Omega)\} & \text{otherwise} \end{cases}$$

$$s(q) := \begin{cases} -\infty & \text{if } \mathsf{ssh}_{q_0}(\Omega) = \emptyset \\ \sup\{u(q) : u \in \mathsf{ssh}_{q_0}(\Omega)\} & \text{otherwise} \end{cases}$$

The *Green function* of Ω with logarithmic pole at q_0, denoted $g_{q_0}^{\Omega}$, is the upper semicontinuous regularization w^* of w. The *strongly subharmonic Green function* of Ω with logarithmic pole at q_0, denoted $G_{q_0}^{\Omega}$, is the upper semicontinuous regularization s^* of s.

Remark 4.37 By construction, $G_{q_0}^{\Omega}(q) \le g_{q_0}^{\Omega}(q)$ for all $q \in \Omega$. Moreover, any inclusion $\Omega' \subseteq \Omega$ implies $g_{q_0}^{\Omega}(q) \le g_{q_0}^{\Omega'}(q)$ and $G_{q_0}^{\Omega}(q) \le G_{q_0}^{\Omega'}(q)$.

Let us construct a basic example. We will use the notations $\mathbb{B} := B(0, 1)$, where

$$B(q_0, R) := \{q \in \mathbb{H} : |q - q_0| < R\}$$

for all $q_0 \in \mathbb{H}$, $R > 0$, and $\mathbb{B}_I := \mathbb{B} \cap L_I$.

Example 4.38 We can easily prove that

$$G_0^{\mathbb{B}}(q) = g_0^{\mathbb{B}}(q) = \log|q|$$

for all $q \in \mathbb{B}$. Indeed, $q \mapsto \log|q|$ is clearly an element of $\mathsf{ssh}_0(\mathbb{B}) \subseteq \mathsf{wsh}_0(\mathbb{B})$. Furthermore, for each $u \in \mathsf{wsh}_0(\mathbb{B})$, the inequality $u(q) \le \log|q|$ holds throughout $\mathbb{B}$. Indeed, for all $I \in \mathbb{S}$ it holds $u_I(z) \le \log|z|$ for all $z \in \mathbb{B}_I$ because $z \mapsto \log|z|$ is the (complex) Green function of the disc $\mathbb{B}_I$.

Further examples can be derived by means of the next results.

Lemma 4.39 *Let f be any affine transformation of $\mathbb{H}$, let Ω be a domain in $\mathbb{H}$ and fix $q_0 \in \Omega$. Then*

$$G_{f(q_0)}^{f(\Omega)}(f(q)) = G_{q_0}^{\Omega}(q)$$

for all $q \in \Omega$.

Proof By repeated applications of Remark 4.22, we conclude that

$$\mathsf{ssh}_{q_0}(\Omega) = \{u \circ f : u \in \mathsf{ssh}_{f(q_0)}(f(\Omega))\}.$$

Thanks to this equality, the statement immediately follows from Definition 4.36. □

Lemma 4.40 *Let Ω be a symmetric slice domain in $\mathbb{H}$, fix $q_0 \in \Omega$ and take a regular function $f : \Omega \to \mathbb{H}$. Then*

$$G_{f(q_0)}^{f(\Omega)}(f(q)) \le g_{q_0}^{\Omega}(q)$$

for all $q \in \Omega$. If, moreover, f is slice preserving then

$$g_{f(q_0)}^{f(\Omega)}(f(q)) \le g_{q_0}^{\Omega}(q)$$

for all $q \in \Omega$. If, additionally, f admits a regular inverse $f^{-1} : f(\Omega) \to \Omega$, then the last inequality becomes an equality at all $q \in \Omega$.

Proof By Theorem 4.28,

$$\mathsf{wsh}_{q_0}(\Omega) \supseteq \{u \circ f : u \in \mathsf{ssh}_{f(q_0)}(f(\Omega))\}.$$

If f is a slice preserving regular function then, by Theorem 4.21,

$$\mathsf{wsh}_{q_0}(\Omega) \supseteq \{u \circ f : u \in \mathsf{wsh}_{f(q_0)}(f(\Omega))\}.$$

The last inclusion is actually an equality if f admits a regular inverse $f^{-1} : f(\Omega) \to \Omega$, which is necessarily slice preserving. The three statements now follow from Definition 4.36. □

In the last statement, we assumed Ω to be a symmetric slice domain for the sake of simplicity. The result could, however, be extended to all slice domains.

Lemmas 4.39 and 4.40, along with the preceding example, yield what follows.

Example 4.41 For each $x_0 \in \mathbb{R}$ and each $R > 0$, the equalities

$$\log \frac{|q - x_0|}{R} = G_{x_0}^{B(x_0,R)}(q) = g_{x_0}^{B(x_0,R)}(q)$$

hold for all $q \in B(x_0, R)$. We point out that this function is strongly subharmonic in $B(x_0, R)$ and weakly harmonic in $B(x_0, R) \setminus \{x_0\}$.

Example 4.42 For each $q_0 \in \mathbb{H}$ and each $R > 0$ it holds

$$\log \frac{|q - q_0|}{R} = G_{q_0}^{B(q_0,R)}(q) \leq g_{q_0}^{B(q_0,R)}(q)$$

for all $q \in B = B(q_0, R)$. We point out that $G_{q_0}^B$, though strongly and weakly subharmonic, is not weakly *harmonic* if $q_0 \notin \mathbb{R}$. Indeed, if we fix an orthonormal basis $1, I, J, IJ$ and write $q_0 = q_1 + q_2 J$ with $q_1, q_2 \in L_I$, then by Lemma 4.23

$$\left(\bar{\partial}_1 \partial_1 G_{q_0}^B\right)_{|z} = \frac{1}{R^2} \left(\bar{\partial}_1 \partial_1 G_0^{\mathbb{B}}\right)_{|\frac{z-q_0}{R}} = \frac{|q_2|^2}{2(|z - q_1|^2 + |q_2|^2)^2}$$

for $z \in L_I$. This expression only vanishes when $q_2 = 0$, that is, when $q_0 \in L_I$.

We are now in a position to make the next remarks.

Remark 4.43 Let Ω be a bounded domain in $\mathbb{H}$ and let $q_0 \in \Omega$. For all $r, R > 0$ such that $B(q_0, r) \subseteq \Omega \subseteq B(q_0, R)$, we have that

$$\log \frac{|q - q_0|}{R} \leq G_{q_0}^{\Omega}(q) \leq \log \frac{|q - q_0|}{r}.$$

As a consequence, $G_{q_0}^{\Omega}$ is not identically equal to $-\infty$, it belongs to $\mathsf{ssh}_{q_0}(\Omega)$ and it coincides with the supremum s appearing in Definition 4.36.

Remark 4.44 Let Ω be a bounded domain in $\mathbb{H}$ and let $q_0 \in \Omega$. For all $R > 0$ such that $\Omega \subseteq B(q_0, R)$, we have that

$$\log \frac{|q - q_0|}{R} \leq g_{q_0}^{\Omega}(q)\,,$$

whence $g_{q_0}^{\Omega}$ is not identically equal to $-\infty$. If, moreover, $q_0 = x_0 \in \mathbb{R}$ then for all $r > 0$ such that $B(x_0, r) \subseteq \Omega$ we have that

$$g_{x_0}^{\Omega}(q) \leq \log \frac{|q - x_0|}{r}.$$

In this case, $g_{x_0}^{\Omega}$ belongs to $\mathsf{wsh}_{x_0}(\Omega)$ and it coincides with the supremum w appearing in Definition 4.36.

When Ω admits a well-behaved exhaustion function, we can prove a few further properties.

Theorem 4.45 *Let Ω be a bounded domain in $\mathbb{H}$. Suppose there exists $\rho \in C^0(\Omega, (-\infty, 0)) \cap \mathsf{ssh}(\Omega)$ such that $\{q \in \Omega : \rho(q) < c\} \subset\subset \Omega$ for all $c < 0$. Then for all $q_0 \in \Omega$ and for all $p \in \partial\Omega$*

$$\lim_{q \to p} G_{q_0}^{\Omega}(q) = 0.$$

Moreover, $G_{q_0}^{\Omega}$ is continuous in $\Omega \setminus \{q_0\}$.

Proof Let $B(q_0, r) \subseteq \Omega \subseteq B(q_0, R)$ and let $C > 0$ be such that $C\rho < \log \frac{r}{R}$ in $\overline{B(q_0, r)}$. If we set

$$u(q) = \begin{cases} \log \frac{|q-q_0|}{R} & q \in \overline{B(q_0, r)} \\ \max\left\{C\rho(q), \log \frac{|q-q_0|}{R}\right\} & q \in \Omega \setminus B(q_0, r) \end{cases}$$

then $u \in \mathsf{ssh}_{q_0}(\Omega)$. Thus, $u \leq G_{q_0}^{\Omega} \leq 0$, where $\lim_{q \to p} u(q) = 0$ for all $p \in \partial\Omega$. This proves the first statement.

To prove the second statement, we only need to prove the lower semicontinuity of $G_{q_0}^{\Omega}$. Let us choose $\lambda \in (0, 1)$ such that $\rho < -\lambda$ in $\overline{B(q_0, \lambda)}$. For any $\varepsilon \in (0, \lambda)$ such that $\log \frac{\varepsilon}{R} > (1 - \varepsilon) \log \varepsilon^2$ (whence $\log \frac{\varepsilon}{R} > \varepsilon - \frac{1}{\varepsilon}$), let us set

$$\alpha(q) = (1 - \varepsilon) \log (\varepsilon|q - q_0|) - \varepsilon$$

on $\overline{B(q_0, \varepsilon)}$.

By Proposition 4.34, for each sufficiently small $\delta > 0$, the convolution $G_{q_0}^{\Omega} * \chi_\delta$ is an element of $C^\infty \cap \mathsf{ssh}(\Omega_\delta)$. If we choose $\eta \in (0, \varepsilon)$ so that $(1-\varepsilon)\log(\varepsilon\eta) > \log\frac{\eta}{R}$ then we may choose $\delta = \delta_\varepsilon > 0$ so that:

- $(1-\varepsilon)\log(\varepsilon\eta) > G_{q_0}^{\Omega} * \chi_\delta$ in $\partial B(q_0, \eta)$,
- Ω_δ includes $\rho^{-1}([-\infty, -\varepsilon^3])$, and
- $G_{q_0}^{\Omega} * \chi_\delta < 0$ in $\rho^{-1}(-\varepsilon^3)$.

We may then set

$$\beta(q) := G_{q_0}^{\Omega} * \chi_{\delta_\varepsilon}(q) - \varepsilon$$

for all $q \in \Omega_{\delta_\varepsilon}$ and

$$\gamma(q) := \varepsilon^{-2}\rho(q)$$

for all $q \in \Omega$. By construction, α, β, γ can be patched together in a continuous strongly subharmonic function defined on Ω, namely

$$u_\varepsilon := \begin{cases} \alpha & \text{in } \overline{B(q_0, \eta)} \\ \max\{\alpha, \beta\} & \text{in } \overline{B(q_0, \varepsilon)} \setminus B(q_0, \eta) \\ \beta & \text{in } \rho^{-1}([-\infty, -\varepsilon]) \setminus B(q_0, \varepsilon) \\ \max\{\beta, \gamma\} & \text{in } \rho^{-1}([-\varepsilon, -\varepsilon^3]) \\ \gamma & \text{in } \Omega \setminus \rho^{-1}([-\infty, -\varepsilon^3)) \end{cases}$$

We remark that $\rho^{-1}([-\infty, -\varepsilon]) \setminus B(q_0, \varepsilon)$ increases as $\varepsilon \to 0$ and that

$$\bigcup_{\varepsilon \in (0,\lambda)} \left(\rho^{-1}([-\infty, -\varepsilon]) \setminus B(q_0, \varepsilon)\right) = \Omega \setminus \{q_0\}\,.$$

Thus, for all $q \in \Omega \setminus \{q_0\}$,

$$\lim_{\varepsilon \to 0} u_\varepsilon(q) = \lim_{\varepsilon \to 0} G_{q_0}^{\Omega} * \chi_{\delta_\varepsilon}(q) - \varepsilon = G_{q_0}^{\Omega}(q)\,.$$

Moreover, for each $\varepsilon \in (0, \lambda)$ it holds $\frac{u_\varepsilon}{1-\varepsilon} \in \mathsf{ssh}_{q_0}(\Omega)$, whence $\frac{u_\varepsilon}{1-\varepsilon} \leq G_{q_0}^{\Omega}$ in Ω. Thus,

$$G_{q_0}^{\Omega}(q) = \sup_{\varepsilon \in (0,\lambda)} \frac{u_\varepsilon(q)}{1-\varepsilon}\,,$$

whence the lower semicontinuity of $G_{q_0}^{\Omega}$ immediately follows. □

Similarly:

Proposition 4.46 *Let Ω be a bounded domain in $\mathbb{H}$. Suppose there exists $\rho \in C^0(\Omega, (-\infty, 0)) \cap \mathsf{wsh}(\Omega)$ such that $\{q \in \Omega : \rho(q) < c\} \subset\subset \Omega$ for all $c < 0$. Then for all $q_0 \in \Omega$ and for all $p \in \partial\Omega$*

$$\lim_{q \to p} g_{q_0}^{\Omega}(q) = 0.$$

Proof Let $B(q_0, r) \subseteq \Omega \subseteq B(q_0, R)$ and let $C > 0$ be such that $C\rho < \log \frac{r}{R}$ in $\overline{B(q_0, r)}$. If we set

$$u(q) = \begin{cases} \log \frac{|q-q_0|}{R} & q \in \overline{B(q_0, r)} \\ \max \left\{ C\rho(q), \log \frac{|q-q_0|}{R} \right\} & q \in \Omega \setminus B(q_0, r) \end{cases}$$

then $u \in \mathsf{wsh}_{q_0}(\Omega)$. Thus, $u \leq g_{q_0}^{\Omega} \leq 0$, where $\lim_{q \to p} u(q) = 0$ for all $p \in \partial\Omega$. □

For a special class of domains Ω and points q_0, the Green function $g_{q_0}^{\Omega}$ can be easily determined, as follows.

Theorem 4.47 *Let Ω be a bounded symmetric slice domain and let $x_0 \in \Omega \cap \mathbb{R}$. Consider the slice $\Omega_i = \Omega \cap \mathbb{C}$ of the domain and the (complex) Green function of Ω_i with logarithmic pole at x_0, which we will denote as $\gamma_{x_0}^{\Omega_i}$. If we set*

$$u(x + Iy) := \gamma_{x_0}^{\Omega_i}(x + iy)$$

for all $x, y \in \mathbb{R}$ and $I \in \mathbb{S}$ such that $x + Iy \in \Omega$, then:

- *u is a well-defined function on Ω;*
- *u is $\mathbb{J}$-plurisubharmonic in $\Omega \setminus \mathbb{R}$ and it belongs to $\mathsf{wsh}_{x_0}(\Omega)$;*
- *u coincides with $g_{x_0}^{\Omega}$.*

Proof The slice Ω_i of the domain is a bounded domain in $\mathbb{C}$. Hence, the function $\gamma_{x_0}^{\Omega_i}$ is a negative plurisubharmonic function on Ω_i with a logarithmic pole at x_0 (see Proposition 4.60). Moreover, since Ω_i is symmetric with respect to the real axis, it holds $\gamma_{x_0}^{\Omega_i}(x + iy) = \gamma_{x_0}^{\Omega_i}(x - iy)$ for all $x + iy \in \Omega_i$. It follows at once that u is well-defined, that it belongs to $\mathsf{wsh}_{x_0}(\Omega)$, and that it is $\mathbb{J}$-plurisubharmonic in $\Omega \setminus \mathbb{R}$.

Moreover, let us fix any other $v \in \mathsf{wsh}_0(\mathbb{B})$: we can prove that $v(q) \leq u(q)$ for all $q \in \Omega$, as follows. For each $I \in \mathbb{S}$, the inequality $v_I \leq u_I$ holds in Ω_I because (up to identifying L_I with $\mathbb{C}$) the function u_I is the (complex) Green function of Ω_I with logarithmic pole at x_0 and v_I is a negative subharmonic function on Ω_I with a logarithmic pole at x_0. As a consequence, u coincides with $g_{x_0}^{\Omega}$. □

4.7.1 A Significant Example

An interesting example to consider is that of the unit ball $\mathbb{B}$ with a pole other than 0. It is natural to address it by means of the *classical Möbius transformations* of $\mathbb{B}$, namely the conformal transformations $v^{-1}M_{q_0}u$, where u, v are constants in $\partial\mathbb{B}$, $q_0 \in \mathbb{B}$ and M_{q_0} is the transformation of $\mathbb{B}$ defined as

$$M_{q_0}(q) := (1 - q\bar{q}_0)^{-1}(q - q_0)\,.$$

The transformation M_{q_0} has inverse $M_{q_0}^{-1} = M_{-q_0}$. It is regular if, and only if, $q_0 = x_0 \in \mathbb{R}$; in this case, it is also slice preserving. For more details, see [2, 11].

Our first observation can be derived from either Lemma 4.40 or Theorem 4.47.

Example 4.48 For each $x_0 \in \mathbb{B} \cap \mathbb{R}$, we have

$$G_{x_0}^{\mathbb{B}}(q) \leq g_{x_0}^{\mathbb{B}}(q) = \log \frac{|q - x_0|}{|1 - qx_0|}$$

for all $q \in \mathbb{B}$.

The same techniques do not work when the logarithmic pole q_0 is not real, as a consequence of the fact that M_{q_0} is not a regular function. Nevertheless, we can make the following observation.

Example 4.49 Let us fix $q_0 \in \mathbb{B} \setminus \mathbb{R}$. We will prove that

$$\log \frac{|q - q_0|}{|1 - q\bar{q}_0|} \leq g_{q_0}^{\mathbb{B}}(q)$$

by showing that $u(q) = \log \frac{|q-q_0|}{|1-q\bar{q}_0|}$ is weakly subharmonic. We will also prove that: (a) the restriction u_I to $\mathbb{B}_I \setminus \{q_0\}$ is harmonic if, and only if, $q_0 \in \mathbb{B}_I$; (b) the function u is not strongly subharmonic.

- We first observe that

$$u(q) = \log|q - q_0| - \log|q - \tilde{q}_0| - \log|\bar{q}_0|\,,$$

with $\tilde{q}_0 := \bar{q}_0^{-1} = q_0|q_0|^{-2}$. With respect to any orthonormal basis $1, I, J, IJ$ and to the associated coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$, if we split $q_0, \tilde{q}_0$ as $q_0 = q_1 + q_2J$, $\tilde{q}_0 = \tilde{q}_1 + \tilde{q}_2J$ with $q_1, q_2, \tilde{q}_1, \tilde{q}_2 \in \mathbb{B}_I$ then

$$H_{I,J}(u)_{|_q} = \frac{1}{2|q - q_0|^4}\begin{pmatrix} |z_2 - q_2|^2 & -(z_1 - q_1)(\bar{z}_2 - \bar{q}_2) \\ -(z_2 - q_2)(\bar{z}_1 - \bar{q}_1) & |z_1 - q_1|^2 \end{pmatrix} +$$
$$- \frac{1}{2\,|q - \tilde{q}_0|^4}\begin{pmatrix} |z_2 - \tilde{q}_2|^2 & -(z_1 - q_1)(\bar{z}_2 - \overline{\tilde{q}_2}) \\ -(z_2 - \tilde{q}_2)(\bar{z}_1 - \overline{\tilde{q}_1}) & |z_1 - \tilde{q}_1|^2 \end{pmatrix}.$$

- For all $z \in \mathbb{B}_I$ (that is, $z_1 = z, z_2 = 0$) it holds

$$\left(\bar{\partial}_1 \partial_1 u\right)_{|z} = \frac{|q_2|^2}{2(|z-q_1|^2+|q_2|^2)^2} - \frac{|q_2|^2|q_0|^4}{2\left(\left|z\,|q_0|^2-q_1\right|^2+|q_2|^2\right)^2}.$$

If $q_0 \in \mathbb{B}_I$ then $q_2 = 0$ and $\left(\bar{\partial}_1 \partial_1 u\right)$ vanishes identically in $\mathbb{B}_I$. Otherwise, $\left(\bar{\partial}_1 \partial_1 u\right)_{|z} > 0$ for all $z \in \mathbb{B}_I$ because

$$\begin{aligned}
&\left|z\,|q_0|^2-q_1\right|^2+|q_2|^2-|q_0|^2(|z-q_1|^2+|q_2|^2)\\
&=|z|^2|q_0|^4+|q_0|^2-|q_0|^2|z|^2-|q_0|^4\\
&=|q_0|^2(1-|q_0|^2)(1-|z|^2)>0\,.
\end{aligned}$$

- In general, $H_{I,J}(u)$ is not positive semidefinite. To see this, let us choose a basis $1, I, J, IJ$ so that $q_1 \neq 0 \neq q_2$ and let us choose $z_1 = 0, z_2 = q_2$. We get

$$\begin{aligned}
H_{I,J}(u)_{|q_2 J} = &\frac{1}{2|q_1|^4}\begin{pmatrix} 0 & 0 \\ 0 & |q_1|^2 \end{pmatrix} + \\
&-\frac{1}{2\left(|\tilde{q}_1|^2+|q_2-\tilde{q}_2|^2\right)^2}\begin{pmatrix} |q_2-\tilde{q}_2|^2 & q_1(\bar{q}_2-\overline{\tilde{q}_2}) \\ (q_2-\tilde{q}_2)\overline{\tilde{q}_1} & |\tilde{q}_1|^2 \end{pmatrix},
\end{aligned}$$

where $q_2 - \tilde{q}_2 \neq 0$ by construction.

We would now like to consider a different approach, through regular transformations of $\mathbb{B}$. Indeed, the work [11] proved the following facts.

- The only regular bijections $\mathbb{B} \to \mathbb{B}$ are the so-called *regular Möbius transformations* of $\mathbb{B}$, namely the transformations $\mathcal{M}_{q_0} * u = \mathcal{M}_{q_0} u$ with $u \in \partial\mathbb{B}, q_0 \in \mathbb{B}$ and

$$\mathcal{M}_{q_0}(q) := (1-q\bar{q}_0)^{-*} * (q-q_0)\,.$$

Here, the symbol $*$ denotes the multiplicative operation among regular functions and f^{-*} is the inverse of f with respect to this multiplicative operation.
- For all $q \in \mathbb{B}$, it holds

$$\mathcal{M}_{q_0}(q) = M_{q_0}(T_{q_0}(q))\,,$$

where $T_{q_0} : \mathbb{B} \to \mathbb{B}$ is defined as $T_{q_0}(q) = (1-qq_0)^{-1}q(1-qq_0)$ and has inverse $T_{q_0}^{-1}(q) = T_{\bar{q}_0}(q)$.
- $\mathcal{M}_{q_0}$ is slice preserving if, and only if, $q_0 = x_0 \in \mathbb{R}$ (in which case, $T_{q_0} = id_{\mathbb{B}}$ and $\mathcal{M}_{x_0} = M_{x_0}$).

If we fix $q_0 \in \mathbb{B} \setminus \mathbb{R}$ then, by Lemma 4.40,

$$\log\left|\mathcal{M}_{q_0}(q)\right| \leq g^{\mathbb{B}}_{q_0}(q) \tag{4.14}$$

for all $q \in \mathbb{B}$. The work [3] proved the quaternionic Schwarz-Pick Lemma and, in particular, the inequality

$$\log|f(q)| \leq \log\left|\mathcal{M}_{q_0}(q)\right|$$

valid for all regular $f : \mathbb{B} \to \mathbb{B}$ with $f(q_0) = 0$. It is therefore natural to ask ourselves whether an equality may hold in (4.14). However, this is not the case: as a consequence of the next result, inequality (4.14) is strict at all q not belonging to the same slice $\mathbb{B}_I$ as q_0. As a byproduct, we conclude that the set

$$\{\log|f| : f : \mathbb{B} \to \mathbb{H} \text{ regular}, f(q_0) = 0\}$$

is not a dense subset of $\mathsf{wsh}_{q_0}(\mathbb{B})$.

Theorem 4.50 *If $q_0 \in \mathbb{B}_I$, then*

$$|M_{q_0}(T_{q_0}(q))| = |\mathcal{M}_{q_0}(q)| \leq |M_{q_0}(q)| \tag{4.15}$$

for all $q \in \mathbb{B}$. Equality holds if, and only if, $q \in \mathbb{B}_I$.

Proof Inequality (4.15) is equivalent to

$$|T_{q_0}(q) - q_0|\,|1 - q\bar{q}_0| \leq |q - q_0|\,|1 - T_{q_0}(q)\bar{q}_0|\,.$$

Since $|T_{q_0}(q)| = |q|$, the last inequality is equivalent to

$$\begin{aligned}
0 \leq & \left(|q|^2 - 2\langle q, q_0\rangle + |q_0|^2\right)\left(1 - 2\langle T_{q_0}(q), q_0\rangle + |q|^2|q_0|^2\right) + \\
& - \left(|q|^2 - 2\langle T_{q_0}(q), q_0\rangle + |q_0|^2\right)\left(1 - 2\langle q, q_0\rangle + |q|^2|q_0|^2\right) \\
= & \,2\left(-|q|^2 - |q_0|^2 + 1 + |q|^2|q_0|^2\right)\langle T_{q_0}(q), q_0\rangle + \\
& 2\left(-1 - |q|^2|q_0|^2 + |q|^2 + |q_0|^2\right)\langle q, q_0\rangle \\
= & \,2(1 - |q_0|^2)(1 - |q|^2))\langle T_{q_0}(q) - q, q_0\rangle\,.
\end{aligned}$$

Thus, inequality (4.15) holds for $q \in \mathbb{B}$ if, and only if, $0 \leq \langle T_{q_0}(q) - q, q_0\rangle$. This is equivalent to the non-negativity of the real part of $(T_{q_0}(q) - q)\bar{q}_0$

$= ((1 - qq_0)^{-1} q(1 - qq_0) - q)\bar{q}_0$ or, equivalently, of the real part of

$$\begin{aligned}
&\left(\overline{(1 - qq_0)} q(1 - qq_0) - |1 - qq_0|^2 q\right) \bar{q}_0 \\
&= \overline{(1 - qq_0)}(q - q^2 q_0 - q + qq_0 q)\bar{q}_0 \\
&= (1 - \bar{q}_0 \bar{q})(qq_0 q\bar{q}_0 - q^2 |q_0|^2) \\
&= qq_0 q\bar{q}_0 - q^2 |q_0|^2 + |q|^2 |q_0|^2 (\bar{q}_0 q - q\bar{q}_0) \\
&= (|q_0|^2 - \bar{q}_0^2)(|z_2|^2 - z_1 z_2 J) + (\bar{q}_0 - q_0) z_2 J \,,
\end{aligned}$$

where the last equality can be obtained by direct computation after splitting q as $q = z_1 + z_2 J$, with $z_1, z_2 \in \mathbb{B}_I$ and $J \perp I$. If $q_0 = x_0 + I y_0$ then the real part of the last expression equals $(x_0^2 + y_0^2 - x_0^2 + y_0^2)|z_2|^2 = 2y_0^2 |z_2|^2$, which is clearly non-negative. Moreover, it vanishes if, and only if, $z_2 = 0$, i.e., $q \in \mathbb{B}_I$. □

An example wherein inequality (4.15) holds and is strict had been constructed in [4]. That construction was used to prove that regular Möbius transformations are not isometries for the Poincaré distance of $\mathbb{B}$, defined as

$$\delta_{\mathbb{B}}(q, q_0) := \frac{1}{2} \log \left(\frac{1 + |M_{q_0}(q)|}{1 - |M_{q_0}(q)|} \right)$$

for all $q, q_0 \in \mathbb{B}$. The subsequent work [1] proved that, for each $q_0 \in \mathbb{B} \setminus \mathbb{R}$, there exists no Riemannian metric on $\mathbb{B}$ having $\mathcal{M}_{q_0}$ as an isometry.

Our new inequality (4.15) is equivalent to

$$\delta_{\mathbb{B}}(T_{q_0}(q), q_0) \leq \delta_{\mathbb{B}}(q, q_0).$$

In other words, we have proven the following property of the transformation T_{q_0} of $\mathbb{B}$: while all points $q \in \mathbb{B}_I$ are fixed, all points $q \in \mathbb{B} \setminus \mathbb{B}_I$ are attracted to q_0 with respect to the Poincaré distance.

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Appendix

For the reader's convenience, we include in this appendix some classical results and definitions, which are used in the present work. We begin with some theorems concerning subharmonic functions of m real variables, along with some instrumental definitions. Let λ denote the Lebesgue measure on $\mathbb{R}^m$ and σ denote the surface area measure.

Theorem 4.51 ([9, Theorem 2.4.1 (iii)]) *Let D be an open subset of $\mathbb{R}^m$ and let $\upsilon : D \to [-\infty, +\infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of D. The function υ is subharmonic in D if, and only if, for any Euclidean ball $B(a, R)$ such that $\overline{B(a, R)} \subset D$, it holds*

$$\upsilon(a) \leq \mathbf{L}(\upsilon; a, R),$$

where

$$\mathbf{L}(\upsilon; a, R) := \frac{1}{s_m R^{m-1}} \int_{\partial B(a,R)} u(x) d\sigma(x), \quad s_m := \sigma(\partial B(0, 1)).$$

Theorem 4.52 ([9, Theorem 2.4.2]) *Let D be a bounded connected open subset of $\mathbb{R}^m$ and let $\upsilon : D \to [-\infty, +\infty)$ be subharmonic in D. Then either υ is constant or, for each $x \in D$,*

$$\upsilon(x) < \sup_{z \in \partial D} \left\{ \limsup_{D \ni y \to z} \upsilon(y) \right\}.$$

Corollary 4.53 ([9, Corollary 2.4.3]) *u is harmonic if, and only if, u and $-u$ are both subharmonic.*

Definition 4.54 ([9, §2.5]) Define $h : \mathbb{R} \to \mathbb{R}$ by the formula

$$h(t) := \begin{cases} \exp(-1/t) & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{cases}$$

and define $\chi : \mathbb{R}^m \to \mathbb{R}$ by the formula

$$\chi(x) := \frac{1}{c} h(1 - \|x\|^2), \quad c := \int_{B(0,1)} h(1 - \|x\|^2) d\lambda(x).$$

The *standard smoothing kernels* $\chi_\varepsilon : \mathbb{R}^m \to \mathbb{R}$ are defined, for all $\varepsilon > 0$, by the formula

$$\chi_\varepsilon(x) := \frac{1}{\varepsilon^m} \chi\left(\frac{x}{\varepsilon}\right).$$

Given a function υ on a open subset D of $\mathbb{R}^m$, the *convolution*

$$\upsilon * \chi_\varepsilon(x) = \chi_\varepsilon * \upsilon(x) := \int_{\mathbb{R}^m} \chi_\varepsilon(x-y)\upsilon(y)d\lambda(y)$$

is well-defined on

$$D_\varepsilon := \begin{cases} \{x \in D : \operatorname{dist}(x, \partial D) > \varepsilon\} & \text{if } D \neq \mathbb{R}^m \\ \mathbb{C}^n & \text{if } D = \mathbb{R}^m \end{cases}$$

Theorem 4.55 ([9, Theorem 2.5.5]) *Let D be an open subset of $\mathbb{R}^m$ and let $\upsilon : D \to [-\infty, +\infty)$ be subharmonic. For all $\varepsilon > 0$ such that D_ε is not empty, $\upsilon * \chi_\varepsilon$ is C^∞ and subharmonic in D_ε. Moreover, $\upsilon * \chi_\varepsilon$ monotonically decreases with decreasing ε and*

$$\lim_{\varepsilon \to 0^+} \upsilon * \chi_\varepsilon(x) = \upsilon(x) \tag{4.16}$$

for each $x \in D$.

We now recall some properties of plurisubharmonic functions of n complex variables.

Theorem 4.56 ([9, Theorem 2.9.1]) *Let D be an open subset of $\mathbb{C}^n$ and let $\upsilon : D \to [-\infty, +\infty)$ be an upper semicontinuous function which is not identically $-\infty$ on any connected component of D. υ is plurisubharmonic in D if, and only if, for any $a \in D, b \in \mathbb{C}^n$ such that $\{a + \lambda b : \lambda \in \mathbb{C}, |\lambda| \leq 1\} \subset D$, it holds*

$$\upsilon(a) \leq l(\upsilon; a, b)\,,$$

where

$$l(\upsilon; a, b) := \frac{1}{2\pi} \int_0^{2\pi} \upsilon(a + e^{it}b)dt.$$

Moreover, plurisubharmonicity is a local property.

Theorem 4.57 ([9, Theorem 2.9.2]) *Let D be an open subset of $\mathbb{C}^n$ and let $\upsilon : D \to [-\infty, +\infty)$ be plurisubharmonic. For all $\varepsilon > 0$ such that D_ε is not empty, $\upsilon * \chi_\varepsilon$ is C^∞ and plurisubharmonic in D_ε. Moreover, $\upsilon * \chi_\varepsilon$ monotonically decreases with decreasing ε and*

$$\lim_{\varepsilon \to 0^+} \upsilon * \chi_\varepsilon(z) = \upsilon(z) \tag{4.17}$$

for each $z \in D$.

Theorem 4.58 ([9, Corollary 2.9.9]) *Let D be a bounded connected open subset of $\mathbb{C}^n$ and let υ be a plurisubharmonic function on D. Then either υ is constant or, for each $z \in D$,*

$$\upsilon(z) < \sup_{w \in \partial D} \left\{ \limsup_{D \ni y \to w} \upsilon(y) \right\}.$$

Remark 4.59 ([9, §3.1]) Let D be an open subset of $\mathbb{C}$ and let υ be a plurisubharmonic function on D. The function υ is harmonic if, and only if, it is maximal among plurisubharmonic functions on D.

Proposition 4.60 ([9, Proposition 6.1.1 (iv)]) *Let D be a bounded domain in $\mathbb{C}^n$, let $w \in D$ and let γ_w^D denote the pluricomplex Green function of D with pole at w. Then γ_w^D is a negative plurisubharmonic function with a logarithmic pole at w.*

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Part II
Applications to Elliptic Partial Differential Equations

Chapter 5
A Fatou Theorem and Poisson's Integral Representation Formula for Elliptic Systems in the Upper Half-Space

Juan José Marín, José María Martell, Dorina Mitrea, Irina Mitrea, and Marius Mitrea

Dedicated with great pleasure to Wolfgang Sprössig on the occasion of his 70th birthday for his many contributions to partial differential equations

Abstract Let L be a second-order, homogeneous, constant (complex) coefficient elliptic system in $\mathbb{R}^n$. The goal of this article is to prove a Fatou-type result, regarding the a.e. existence of the nontangential boundary limits of any null-solution u of L in the upper half-space, whose nontangential maximal function satisfies an integrability condition with respect to the weighted Lebesgue measure $(1+|x'|^{n-1})^{-1}dx'$ in $\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$. This is the best result of its kind in the literature. In addition, we establish a naturally accompanying integral representation formula involving the Agmon-Douglis-Nirenberg Poisson kernel for the system L. Finally, we use this machinery to derive well-posedness results for the Dirichlet boundary value problem for L in $\mathbb{R}^n_+$ formulated in a manner which allows for the simultaneous treatment of a variety of function spaces.

Keywords Fatou type theorem · Poisson integral representation formula · Nontangential maximal function · Second-order elliptic system · Dirichlet problem · Hardy-Littlewood maximal operator · Green function

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5.1 Introduction

Let $n \in \mathbb{N}$ with $n \geq 2$ denote the dimension of the Euclidean ambient space. Fix an integer $M \in \mathbb{N}$ and consider the second-order, homogeneous, $M \times M$ system, with constant complex coefficients in $\mathbb{R}^n$, written (with the usual convention of summation over repeated indices in place) as

$$Lu := \Big(a_{rs}^{\alpha\beta}\partial_r\partial_s u_\beta\Big)_{1\leq\alpha\leq M}, \tag{5.1.1}$$

when acting on vector-valued distributions $u = (u_\beta)_{1\leq\beta\leq M}$ in an open subset of $\mathbb{R}^n$. Throughout, we shall assume that L is `elliptic` in the sense that there exists a real number $c > 0$ such that the following Legendre-Hadamard condition is satisfied:

$$\begin{gathered}\mathrm{Re}\,\big[a_{rs}^{\alpha\beta}\xi_r\xi_s\overline{\eta_\alpha}\eta_\beta\,\big] \geq c|\xi|^2|\eta|^2 \quad\text{for every}\\ \xi = (\xi_r)_{1\leq r\leq n} \in \mathbb{R}^n \quad\text{and}\quad \eta = (\eta_\alpha)_{1\leq\alpha\leq M} \in \mathbb{C}^M.\end{gathered} \tag{5.1.2}$$

Examples to keep in mind are the Laplacian and the Lamé system.

As is known from the classical work of Agmon et al. in [1, 2], every operator L as in (5.1.1), (5.1.2) has a Poisson kernel, denoted by P^L (an object whose properties mirror the most basic characteristics of the classical harmonic Poisson kernel). For details, see Theorem 5.2.3 below.

The main goal of this paper is to establish a Fatou-type theorem and a naturally accompanying Poisson integral representation formula for null-solutions of an elliptic system L, as above, in the upper half-space

$$\mathbb{R}^n_+ := \big\{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R} : x_n > 0\big\}. \tag{5.1.3}$$

Among other things, this is going to yield versatile well-posedness results for the Dirichlet problem in $\mathbb{R}^n_+$ for such systems. Prior to formulating the main result, some comments on the notation used are in order. Given a function u defined in $\mathbb{R}^n_+$, by $\mathcal{N}_\kappa u$ we shall denote the nontangential maximal function of u with aperture κ; see (5.2.2) for a precise definition. Next, by $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ we denote the (κ-)nontangential limit of the given function u on the boundary of the upper half-space (canonically identified with $\mathbb{R}^{n-1}$), as defined in (5.2.3). Finally, given any $d \in \mathbb{N}$, the Lebesgue measure in $\mathbb{R}^d$ will be denoted by $\mathscr{L}^d$.

Theorem 5.1.1 (A Fatou-Type Theorem and Poisson's Integral Formula) *Let L be an $M \times M$ system with constant complex coefficients as in (5.1.1) and (5.1.2), and fix some aperture parameter $\kappa > 0$. Then*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \\ \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x') \, \frac{dx'}{1+|x'|^{n-1}} < \infty, \end{cases} \tag{5.1.4}$$

implies that

$$\begin{cases} u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ exists at } \mathscr{L}^{n-1}\text{-a.e. point in } \ \mathbb{R}^{n-1}, \\ u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \ \text{ belongs to } \ \Big[L^1\Big(\mathbb{R}^{n-1}\,,\, \dfrac{dx'}{1+|x'|^{n-1}}\Big)\Big]^M, \\ u(x',t) = \Big(P^L_t * \Big(u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}\Big)\Big)(x') \ \text{ for each } \ (x',t) \in \mathbb{R}^n_+, \end{cases} \tag{5.1.5}$$

where $P^L = \big(P^L_{\beta\alpha}\big)_{1\leq\beta,\alpha\leq M}$ is the Agmon-Douglis-Nirenberg Poisson kernel for the system L in $\mathbb{R}^n_+$ and $P^L_t(x') := t^{1-n} P^L(x'/t)$ for each $x' \in \mathbb{R}^{n-1}$ and $t > 0$.

This refines [6, Theorem 6.1, p. 956]. We also wish to remark that even in the classical case when $L := \Delta$, the Laplacian in $\mathbb{R}^n$, Theorem 5.1.1 is more general (in the sense that it allows for a larger class of functions) than the existing results in the literature. Indeed, the latter typically assume an L^p integrability condition for the harmonic function which, in the range $1 < p < \infty$, implies our weighted L^1 integrability condition for the nontangential maximal function demanded in (5.1.4). In this vein see, e.g., [4, Theorems 4.8–4.9, pp. 174–175], [13, Corollary, p. 200], [14, Proposition 1, p. 119].

A remarkable feature of Theorem 5.1.1 is that while its statement is phrased exclusively in terms of the Agmon-Douglis-Nirenberg Poisson kernel P^L, its proof is actually carried out largely in terms of the Green function associated with the system L in the upper half-space. The latter entity has been studied at length in [7], and here we make ample use of the results established therein.

A special case of Theorem 5.1.1 worth singling out is as follows. Recall the Agmon-Douglis-Nirenberg kernel function

$$\begin{aligned} &K^L \in \textstyle\bigcap_{\varepsilon>0} \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(0,\varepsilon)\big)\big]^{M\times M}, \\ &K^L(x) := P^L_t(x') \ \text{ for all } \ x = (x',t) \in \mathbb{R}^n_+, \end{aligned} \tag{5.1.6}$$

associated with the elliptic system L as in Theorem 5.2.3. Fix some $t_o > 0$ and define

$$u(x) := K^L(x', t+t_o) = P^L_{t+t_o}(x') \ \text{ for all } \ x = (x',t) \in \mathbb{R}^n_+. \tag{5.1.7}$$

Then

$$u \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+}\big)\big]^{M\times M}, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+, \quad u\big|_{\partial\mathbb{R}^n_+} = P^L_{t_o} \ \text{ on } \ \mathbb{R}^{n-1}. \tag{5.1.8}$$

In addition, (5.2.12) ensures that there exists a finite constant $C_{t_o} > 0$ with the property that $|u(x)| \le C_{t_o}(1+|x|)^{1-n}$ for each $x \in \mathbb{R}^n_+$. For each fixed $\kappa > 0$ this readily entails

$$\big(\mathcal{N}_\kappa u\big)(x') \le \frac{C}{1+|x'|^{n-1}}, \qquad \forall\, x' \in \mathbb{R}^{n-1}. \tag{5.1.9}$$

This, in turn, guarantees that the finiteness condition demanded in (5.1.4) is presently satisfied. Having verified all hypotheses of Theorem 5.1.1, from the Poisson integral representation formula in the last line of (5.1.5) and (5.1.7)–(5.1.8) we conclude that

$$P^L_{t+t_o}(x') = u(x',t) = \big(P^L_t * P^L_{t_o}\big)(x') \ \text{ for all } \ (x',t) \in \mathbb{R}^n_+, \tag{5.1.10}$$

where the convolution between the two matrix-valued functions in (5.1.10) is understood in a natural fashion, taking into account the algebraic multiplication of matrices. Ultimately, this provides an elegant proof of the following result (first established in [6, Theorem 5.1] via a conceptually different argument):

the Agmon-Douglis-Nirenberg Poisson kernel P^L associated with any given elliptic system L as in Theorem 5.2.3 satisfies the semi-group property $P^L_{t_0+t_1} = P^L_{t_0} * P^L_{t_1}$ for all $t_0, t_1 > 0$. (5.1.11)

Here is another important corollary of Theorem 5.1.1, which refines [6, Theorem 3.2, p. 935].

Corollary 5.1.2 (A General Uniqueness Result) *Let L be an $M \times M$ system with constant complex coefficients as in* (5.1.1) *and* (5.1.2)*, and fix an aperture parameter $\kappa > 0$. Then*

$$\left.\begin{array}{l} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \text{ in } \ \mathbb{R}^n_+ \\[4pt] \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x')\,\frac{dx'}{1+|x'|^{n-1}} < +\infty \\[4pt] u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \ \text{ at } \mathscr{L}^{n-1}\text{-a.e. point on } \ \mathbb{R}^{n-1} \end{array}\right\} \Longrightarrow u = 0 \ \text{ in } \ \mathbb{R}^n_+. \tag{5.1.12}$$

Theorem 5.1.1 also interfaces tightly with the topic of boundary value problems. To elaborate on this aspect, we need more notation. Denote by $\mathbb{M}$ the collection of all (equivalence classes of) Lebesgue measurable functions $f : \mathbb{R}^{n-1} \to [-\infty, \infty]$

such that $|f| < \infty$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Also, call a subset $\mathbb{Y}$ of $\mathbb{M}$ a `function lattice` if the following properties hold:

(i) whenever $f, g \in \mathbb{M}$ satisfy $0 \leq f \leq g$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$ and $g \in \mathbb{Y}$ then necessarily $f \in \mathbb{Y}$;
(ii) $0 \leq f \in \mathbb{Y}$ implies $\lambda f \in \mathbb{Y}$ for every $\lambda \in (0, \infty)$;
(iii) $0 \leq f, g \in \mathbb{Y}$ implies $\max\{f, g\} \in \mathbb{Y}$.

In passing, note that, granted (i), one may replace (ii)–(iii) above by the condition: $0 \leq f, g \in \mathbb{Y}$ implies $f + g \in \mathbb{Y}$. As usual, we set $\log_+ t := \max\big\{0\,,\, \ln t\big\}$ for each $t \in (0, \infty)$. Also, the symbol $\mathcal{M}$ is reserved for the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$; see (5.2.6).

We are now in a position to discuss the following refinement of [6, Theorem 1.1, p. 915].

Corollary 5.1.3 (A Template for the Dirichlet Problem) *Let L be an $M \times M$ system with constant complex coefficients as in* (5.1.1) *and* (5.1.2)*, and fix an aperture parameter $\kappa > 0$. Also, assume that*

$$\mathbb{Y} \subseteq L^1\Big(\mathbb{R}^{n-1}\,,\, \frac{dx'}{1+|x'|^{n-1}}\Big), \quad \mathbb{Y} \textit{ is a function lattice}, \tag{5.1.13}$$

and that

$$\mathbb{X} \textit{ is a collection of } \mathbb{C}^M\textit{-valued measurable functions on } \mathbb{R}^{n-1} \textit{ satisfying } \mathcal{M}\mathbb{X} \subseteq \mathbb{Y}. \tag{5.1.14}$$

Then the $(\mathbb{X}, \mathbb{Y})$-Dirichlet boundary value problem for the system L in the upper half-space, formulated as

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \\ Lu = 0 \ \textit{ in } \ \mathbb{R}^n_+, \\ \mathcal{N}_\kappa u \in \mathbb{Y}, \\ u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathbb{X}, \end{cases} \tag{5.1.15}$$

has a unique solution. Moreover, the solution u of (5.1.15) *is given by*

$$u(x) = (P^L_t * f)(x') \quad \textit{for all} \quad x = (x', t) \in \mathbb{R}^{n-1} \times (0, \infty) = \mathbb{R}^n_+, \tag{5.1.16}$$

where P^L is the Poisson kernel for L in $\mathbb{R}^n_+$, and satisfies

$$\big(\mathcal{N}_\kappa u\big)(x') \leq C\, \mathcal{M} f(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{5.1.17}$$

for some constant $C \in (0, \infty)$ that depends only on L, n, and κ.

Corollary 5.1.3 contains as particular cases a multitude of well-posedness results for elliptic systems in the upper half-space. For example, one may take Muckenhoupt weighted Lebesgue spaces $\mathbb{X} := \big[L^p(\mathbb{R}^{n-1}, w\mathscr{L}^{n-1})\big]^M$ and $\mathbb{Y} := L^p(\mathbb{R}^{n-1}, w\mathscr{L}^{n-1})$ with $p \in (1,\infty)$ and $w \in A_p$, or Morrey spaces in $\mathbb{R}^{n-1}$; for more on this, as well as other examples, see [6].

Here we wish to identify the most inclusive setting in which Corollary 5.1.3 yields a well-posedness result. Specifically, in view of the assumptions made in (5.1.13)–(5.1.14) it is natural to consider the linear space

$$\begin{aligned}\mathscr{Z} &:= \Big\{f \in \big[L^1\big(\mathbb{R}^{n-1}\,,\,\tfrac{dx'}{1+|x'|^{n-1}}\big)\big]^M : \mathcal{M}f \in L^1\big(\mathbb{R}^{n-1}\,,\,\tfrac{dx'}{1+|x'|^{n-1}}\big)\Big\}\\ &= \Big\{f : \mathbb{R}^{n-1} \to \mathbb{C}^M : \text{ measurable and } \mathcal{M}f \in L^1\big(\mathbb{R}^{n-1}\,,\,\tfrac{dx'}{1+|x'|^{n-1}}\big)\Big\}\end{aligned} \tag{5.1.18}$$

(recall that $\mathcal{M}$ is the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$) equipped with the norm

$$\begin{aligned}\|f\|_{\mathscr{Z}} &:= \|f\|_{[L^1(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1}})]^M} + \|\mathcal{M}f\|_{L^1\big(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1}}\big)}\\ &\approx \|\mathcal{M}f\|_{L^1\big(\mathbb{R}^{n-1},\frac{dx'}{1+|x'|^{n-1}}\big)}, \qquad \forall\, f \in \mathscr{Z}.\end{aligned} \tag{5.1.19}$$

Then, Corollary 5.1.3 applied with $\mathbb{X} := \mathscr{Z}$ and $\mathbb{Y} := L^1\big(\mathbb{R}^{n-1}\,,\,\frac{dx'|}{1+|x'|^{n-1}}\big)$ yields the following result.

Corollary 5.1.4 (The Most Inclusive Well-Posedness Result) *Let L be an $M \times M$ system with constant complex coefficients as in* (5.1.1) *and* (5.1.2)*, and fix an aperture parameter $\kappa > 0$. Then the following boundary value problem is well-posed:*

$$\begin{cases} u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \quad Lu = 0 \ \ in \ \ \mathbb{R}^n_+,\\ \displaystyle\int_{\mathbb{R}^{n-1}} \big(\mathcal{N}_\kappa u\big)(x')\,\frac{dx'}{1+|x'|^{n-1}} < \infty,\\ u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+} = f \in \mathscr{Z}.\end{cases} \tag{5.1.20}$$

The relevance of the fact that (5.1.4) implies (5.1.5) in the context of all the aforementioned boundary value problems (cf. (5.1.15), (5.1.20)) is that the nontangential boundary trace $u\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ is guaranteed to exist by the other conditions imposed on the function u in the formulation of the said problems, and that the solution may be recovered from the boundary datum via convolution with the Poisson kernel canonically associated with the system L.

The type of boundary value problems treated here, in which the size of the solution is measured in terms of its nontangential maximal function and its trace is taken in a nontangential pointwise sense, has been dealt with in the particular case when $L = \Delta$, the Laplacian in $\mathbb{R}^n$, in a number of monographs, including [3, 4, 13, 14], and [15]. In all these works, the existence part makes use of the explicit form of the harmonic Poisson kernel, while the uniqueness relies on either the Maximum Principle, or the Schwarz reflection principle for harmonic functions. Neither of the latter techniques may be adapted successfully to prove uniqueness in the case of general systems treated here, and our approach is more in line with the work in [6] (which involves Green function estimates and a sharp version of the Divergence Theorem), with some significant refinements. A remarkable aspect is that our approach works for the entire class of elliptic systems L as in (5.1.1) and (5.1.2).

5.2 Preliminary Matters

Throughout, $\mathbb{N}$ stands for the collection of all strictly positive integers, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As such, for each $k \in \mathbb{N}$, we denote by $\mathbb{N}_0^k$ the collection of all multi-indices $\alpha = (\alpha_1, \dots, \alpha_k)$ with $\alpha_j \in \mathbb{N}_0$ for $1 \le j \le k$. Also, fix $n \in \mathbb{N}$ with $n \ge 2$. We shall work in the upper half-space $\mathbb{R}^n_+$, whose topological boundary $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$ will be frequently identified with the horizontal hyperplane $\mathbb{R}^{n-1}$ via $(x', 0) \equiv x'$. The origin in $\mathbb{R}^{n-1}$ is denoted by $0'$ and we let $B_{n-1}(x', r)$ stand for the $(n-1)$-dimensional Euclidean ball of radius r centered at $x' \in \mathbb{R}^{n-1}$. Having fixed $\kappa > 0$, for each boundary point $x' \in \partial\mathbb{R}^n_+$ introduce the conical nontangential approach region with vertex at x' as

$$\Gamma_\kappa(x') := \big\{ y = (y', t) \in \mathbb{R}^n_+ : |x' - y'| < \kappa\, t \big\}. \tag{5.2.1}$$

Given a vector-valued function $u : \mathbb{R}^n_+ \to \mathbb{C}^M$, the nontangential maximal function of u is defined by

$$\big(\mathcal{N}_\kappa u\big)(x') := \sup\big\{ |u(y)| : \; y \in \Gamma_\kappa(x') \big\}, \qquad x' \in \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}. \tag{5.2.2}$$

Whenever meaningful, we also define the nontangential trace of u as

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+}(x') := \lim_{\Gamma_\kappa(x') \ni y \to (x', 0)} u(y) \quad \text{for } x' \in \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}. \tag{5.2.3}$$

In the sequel, we shall need to consider a localized version of the nontangential maximal operator. Specifically, given any $E \subset \mathbb{R}^n_+$, for each $u : E \to \mathbb{C}^M$ we set

$$\big(\mathcal{N}^E_\kappa u\big)(x') := \sup\big\{ |u(y)| : \; y \in \Gamma_\kappa(x') \cap E \big\}, \qquad x' \in \partial\mathbb{R}^n_+ \equiv \mathbb{R}^{n-1}. \tag{5.2.4}$$

Hence, $\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa \widetilde{u}$ where $\widetilde{u}$ is the extension of u to $\mathbb{R}^n_+$ by zero outside E. In the scenario when u is originally defined in the entire upper half-space $\mathbb{R}^n_+$ we may therefore write

$$\mathcal{N}_\kappa^E u = \mathcal{N}_\kappa(\mathbf{1}_E u), \tag{5.2.5}$$

where $\mathbf{1}_E$ denotes the characteristic function of E.

The action of the Hardy-Littlewood maximal operator in $\mathbb{R}^{n-1}$ on any Lebesgue measurable function f defined in $\mathbb{R}^{n-1}$ is given by

$$\big(\mathcal{M}f\big)(x') := \sup_{r>0} ⨍_{B_{n-1}(x',r)} |f|\, d\mathscr{L}^{n-1}, \qquad \forall\, x' \in \mathbb{R}^{n-1}, \tag{5.2.6}$$

where the barred integral denotes mean average (for functions which are $\mathbb{C}^M$-valued the average is taken componentwise).

We next recall a useful weak compactness result from [6, Lemma 6.2, p. 956]. To state it, denote by $\mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$ the space of continuous functions in $\mathbb{R}^{n-1}$ vanishing at infinity.

Lemma 5.2.1 *Let $v : \mathbb{R}^{n-1} \to (0,\infty)$ be a Lebesgue measurable function and consider a sequence $\{f_j\}_{j\in\mathbb{N}}$ in the weighted Lebesgue space $L^1(\mathbb{R}^{n-1}, v\mathscr{L}^{n-1})$ such that*

$$F := \sup_{j\in\mathbb{N}} |f_j| \in L^1(\mathbb{R}^{n-1}, v\mathscr{L}^{n-1}). \tag{5.2.7}$$

Then there exists a subsequence $\big\{f_{j_k}\big\}_{k\in\mathbb{N}}$ of $\{f_j\}_{j\in\mathbb{N}}$ and a function $f \in L^1(\mathbb{R}^{n-1}, v\mathscr{L}^{n-1})$ with the property that

$$\int_{\mathbb{R}^{n-1}} f_{j_k}(x')\varphi(x')v(x')\,dx' \longrightarrow \int_{\mathbb{R}^{n-1}} f(x')\varphi(x')v(x')\,dx' \;\; \text{as} \;\; k\to\infty, \tag{5.2.8}$$

for every $\varphi \in \mathscr{C}_{\rm van}(\mathbb{R}^{n-1})$.

We next discuss the notion of Poisson kernel in $\mathbb{R}^n_+$ for an operator L as in (5.1.1) and (5.1.2).

Definition 5.2.2 Let L be an $M\times M$ system with constant complex coefficients as in (5.1.1) and (5.1.2). A `Poisson kernel` for L in $\mathbb{R}^n_+$ is a matrix-valued function

$$P^L = \big(P^L_{\alpha\beta}\big)_{1\le\alpha,\beta\le M} : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^{M\times M} \tag{5.2.9}$$

such that the following conditions hold:

(a) there exists $C \in (0, \infty)$ such that $|P^L(x')| \leq \dfrac{C}{(1+|x'|^2)^{\frac{n}{2}}}$ for each $x' \in \mathbb{R}^{n-1}$;

(b) the function P^L is Lebesgue measurable and $\displaystyle\int_{\mathbb{R}^{n-1}} P^L(x')\,dx' = I_{M\times M}$, the $M \times M$ identity matrix;

(c) if $K^L(x', t) := P^L_t(x') := t^{1-n} P^L(x'/t)$, for each $x' \in \mathbb{R}^{n-1}$ and $t \in (0, \infty)$, then the function $K^L = \big(K^L_{\alpha\beta}\big)_{1\leq\alpha,\beta\leq M}$ satisfies (in the sense of distributions)

$$LK^L_{\cdot\beta} = 0 \ \text{ in } \ \mathbb{R}^n_+ \ \text{ for each } \ \beta \in \{1, \dots, M\}, \tag{5.2.10}$$

where $K^L_{\cdot\beta} := \big(K^L_{\alpha\beta}\big)_{1\leq\alpha\leq M}$.

Poisson kernels for elliptic boundary value problems in a half-space have been studied extensively in [1, 2], [5, §10.3], [10–12]. Here we record a corollary of more general work done by Agmon et al. in [2].

Theorem 5.2.3 *Any $M \times M$ system L with constant complex coefficients as in (5.1.1) and (5.1.2) has a Poisson kernel P^L in the sense of Definition 5.2.2, which has the additional property that the function*

$$K^L(x', t) := P^L_t(x') \quad \textit{for all} \ \ (x', t) \in \mathbb{R}^n_+, \tag{5.2.11}$$

satisfies $K^L \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(0, \varepsilon)\big)\big]^{M\times M}$ for every $\varepsilon > 0$, and has the property that for each multi-index $\alpha \in \mathbb{N}^n_0$ there exists $C_\alpha \in (0, \infty)$ such that

$$\big|(\partial^\alpha K^L)(x)\big| \leq C_\alpha\, |x|^{1-n-|\alpha|}, \quad \textit{for every} \ \ x \in \overline{\mathbb{R}^n_+} \setminus \{0\}. \tag{5.2.12}$$

Here and elsewhere, the convolution between two functions, which are matrix-valued and vector-valued, respectively, takes into account the algebraic multiplication between a matrix and a vector in a natural fashion.

The next result we recall has been proved in [6, Theorem 3.1, p. 934].

Proposition 5.2.4 *Let L be an $M \times M$ system with constant complex coefficients as in (5.1.1)–(5.1.2), and recall the Poisson kernel P^L for L in $\mathbb{R}^n_+$ from Theorem 5.2.3. Also, fix some arbitrary aperture parameter $\kappa > 0$. Given a function*

$$f \in \Big[L^1\Big(\mathbb{R}^{n-1}, \frac{dx'}{1+|x'|^n}\Big)\Big]^M, \tag{5.2.13}$$

set

$$u(x', t) := (P^L_t * f)(x'), \qquad \forall\, (x', t) \in \mathbb{R}^n_+. \tag{5.2.14}$$

Then u is meaningfully defined via an absolutely convergent integral,

$$u \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M, \;\; Lu = 0 \textit{ in } \mathbb{R}^n_+, \;\; u\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = f \;\textit{ at } \mathscr{L}^{n-1}\textit{-a.e. point in } \mathbb{R}^{n-1} \tag{5.2.15}$$

(with the last identity valid in the set of Lebesgue points of f), and there exists a constant $C = C(n, L, \kappa) \in (0, \infty)$ with the property that

$$\big(\mathcal{N}_\kappa u\big)(x') \le C\big(\mathcal{M} f\big)(x'), \qquad \forall\, x' \in \mathbb{R}^{n-1}. \tag{5.2.16}$$

A key ingredient in the proof of our main result is understanding the nature of the Green function associated with a given elliptic system. While we elaborate on this topic in Theorem 5.2.6 below, we begin by providing a suitable definition for the said Green function (which, in particular, is going to ensure its uniqueness). To set the stage, denote by $\mathcal{D}'(\mathbb{R}^n_+)$ the space of distributions in $\mathbb{R}^n_+$.

Definition 5.2.5 Let L be an $M \times M$ system with constant complex coefficients as in (5.1.1)–(5.1.2). Call $G^L(\cdot,\cdot) : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \text{diag} \to \mathbb{C}^{M\times M}$ a `Green function` for L in $\mathbb{R}^n_+$ provided for each $y = (y', y_n) \in \mathbb{R}^n_+$ the following properties hold (for some aperture parameter $\kappa > 0$):

$$G^L(\cdot\,, y) \in \big[L^1_{\text{loc}}(\mathbb{R}^n_+)\big]^{M\times M}, \tag{5.2.17}$$

$$G^L(\cdot\,, y)\big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} = 0 \;\text{ at } \mathscr{L}^{n-1}\text{-a.e. point in } \;\mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+, \tag{5.2.18}$$

$$\int_{\mathbb{R}^{n-1}} \Big(\mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus\overline{B(y,y_n/2)}}\, G^L(\cdot\,, y)\Big)(x')\frac{dx'}{1+|x'|^{n-1}} < \infty, \tag{5.2.19}$$

$$L\big[G^L(\cdot\,, y)\big] = -\delta_y\, I_{M\times M} \;\text{ in }\; \big[\mathcal{D}'(\mathbb{R}^n_+)\big]^{M\times M}, \tag{5.2.20}$$

where the $M \times M$ system L acts in the "dot" variable on the columns of G.

The existence and basic properties of the Green function just defined are discussed in our next theorem (a proof of which may be found in [7]). Before stating it, we make two conventions regarding notation. First, we agree to abbreviate $\text{diag} := \{(x, x) : x \in \mathbb{R}^n_+\}$ for the diagonal in the Cartesian product $\mathbb{R}^n_+ \times \mathbb{R}^n_+$. Second, given a function $G(\cdot,\cdot)$ of two vector variables, $(x, y) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ \setminus \text{diag}$, for each $k \in \{1, \dots, n\}$ we agree to write $\partial_{X_k} G$ and $\partial_{Y_k} G$, respectively, for the partial derivative of G with respect to x_k, and y_k. This convention may be iterated, lending a natural meaning to $\partial^\alpha_X \partial^\beta_Y G$, for each pair of multi-indices $\alpha, \beta \in \mathbb{N}^n_0$. We are now ready to present the result alluded to above.

Theorem 5.2.6 *Assume that L is an $M \times M$ system with constant complex coefficients as in* (5.1.1) *and* (5.1.2)*. Then there exists a unique Green function*

$G^L(\cdot,\cdot)$ for L in $\mathbb{R}^n_+$, in the sense of Definition 5.2.5. Moreover, this Green function also satisfies the following additional properties:

1. *Given $\kappa > 0$, for each $y \in \mathbb{R}^n_+$ and each compact neighborhood K of y in $\mathbb{R}^n_+$ there exists a finite constant $C_y = C(n, L, \kappa, K, y) > 0$ such that for every $x' \in \mathbb{R}^{n-1}$ there holds*

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus K}\big(G^L(\cdot,y)\big)(x') \leq C_y\,\frac{1+\log_+|x'|}{1+|x'|^{n-1}}. \tag{5.2.21}$$

Moreover, for any multi-indices $\alpha, \beta \in \mathbb{N}_0^n$ such that $|\alpha|+|\beta| > 0$, there exists some constant $C_y = C(n, L, \kappa, \alpha, \beta, K, y) \in (0,\infty)$ such that

$$\mathcal{N}_\kappa^{\mathbb{R}^n_+\setminus K}\big((\partial_X^\alpha \partial_Y^\beta G^L)(\cdot,y)\big)(x') \leq \frac{C_y}{1+|x'|^{n-2+|\alpha|+|\beta|}}. \tag{5.2.22}$$

2. *For each fixed $y \in \mathbb{R}^n_+$, there holds*

$$G^L(\cdot\,,y) \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+}\setminus B(y,\varepsilon)\big)\big]^{M\times M} \quad \textit{for every} \quad \varepsilon > 0. \tag{5.2.23}$$

As a consequence of (5.2.23) and (5.2.18), for each fixed $y \in \mathbb{R}^n_+$ one has

$$G^L(\cdot,y)\Big|_{\partial\mathbb{R}^n_+} = 0 \quad \textit{everywhere on} \quad \mathbb{R}^{n-1}. \tag{5.2.24}$$

3. *For each $\alpha, \beta \in \mathbb{N}_0^n$ the function $\partial_X^\alpha \partial_Y^\beta G^L$ is translation invariant in the tangential variables, in the sense that*

$$\begin{gathered}\big(\partial_X^\alpha \partial_Y^\beta G^L\big)\big(x-(z',0),\,y-(z',0)\big) = \big(\partial_X^\alpha \partial_Y^\beta G^L\big)(x,y)\\ \textit{for each} \quad (x,y)\in\mathbb{R}^n_+\times\mathbb{R}^n_+\setminus\mathrm{diag} \quad \textit{and} \quad z'\in\mathbb{R}^{n-1},\end{gathered} \tag{5.2.25}$$

and is positive homogeneous, in the sense that

$$\begin{gathered}\big(\partial_X^\alpha \partial_Y^\beta G^L\big)(\lambda x,\lambda y) = \lambda^{2-n-|\alpha|-|\beta|}\big(\partial_X^\alpha \partial_Y^\beta G^L\big)(x,y)\\ \textit{for each} \quad x,y\in\mathbb{R}^n_+ \quad \textit{with} \quad x\neq y \quad \textit{and} \quad \lambda\in(0,\infty),\\ \textit{provided either } n\geq 3, \textit{ or } |\alpha|+|\beta|>0.\end{gathered} \tag{5.2.26}$$

4. *If $G^{L^\top}(\cdot,\cdot)$ denotes the (unique, by the first part of the statement) Green function for $L^\top$ (the transposed of L) in $\mathbb{R}^n_+$, then*

$$G^L(x,y) = \Big[G^{L^\top}(y,x)\Big]^\top, \qquad \forall\,(x,y)\in\mathbb{R}^n_+\times\mathbb{R}^n_+\setminus\mathrm{diag}. \tag{5.2.27}$$

Hence, as a consequence of (5.2.27)*,* (5.2.18)*, and* (5.2.23)*, for each fixed* $x \in \mathbb{R}^n_+$ *and* $\varepsilon > 0$*,*

$$G^L(x,\cdot) \in \big[\mathscr{C}^\infty\big(\overline{\mathbb{R}^n_+} \setminus B(x,\varepsilon)\big)\big]^{M\times M} \quad \text{and} \quad G^L(x,\cdot)\Big|_{\partial\mathbb{R}^n_+} = 0 \ \text{ on } \ \mathbb{R}^{n-1}. \tag{5.2.28}$$

5. *For any multi-indices* $\alpha, \beta \in \mathbb{N}^n_0$ *there exists a finite constant* $C_{\alpha\beta} > 0$ *such that*

$$\begin{gathered}\big|\big(\partial_X^\alpha \partial_Y^\beta G^L\big)(x,y)\big| \le C_{\alpha\beta}|x-y|^{2-n-|\alpha|-|\beta|},\\ \forall\,(x,y)\in\mathbb{R}^n_+\times\mathbb{R}^n_+\setminus\text{diag}, \textit{ if either } n\ge 3, \textit{ or } |\alpha|+|\beta|>0,\end{gathered} \tag{5.2.29}$$

and, corresponding to $|\alpha| = |\beta| = 0$ *and* $n = 2$*, there exists* $C \in (0,\infty)$ *such that*

$$\big|G^L(x,y)\big| \le C + C\big|\ln|x-\overline{y}|\big|, \quad \forall\,(x,y)\in\mathbb{R}^2_+\times\mathbb{R}^2_+\setminus\text{diag}, \tag{5.2.30}$$

where $\overline{y} := (y', -y_n) \in \mathbb{R}^n$ *is the reflexion of* $y = (y', y_n) \in \mathbb{R}^n_+$ *across the boundary of the upper half-space.*

6. *The Agmon-Douglis-Nirenberg Poisson kernel* $P^L = \big(P^L_{\gamma\alpha}\big)_{1\le\gamma,\alpha\le M}$ *for* L *in* $\mathbb{R}^n_+$ *from Theorem* 5.2.3 *is related to the Green function* G^L *for* L *in* $\mathbb{R}^n_+$ *according to the formula*

$$\begin{gathered}P^L_{\gamma\alpha}(z') = a^{\beta\alpha}_{nn}\big(\partial_{Y_n} G^L_{\gamma\beta}\big)\big((z',1),0\big), \quad \forall\, z'\in\mathbb{R}^{n-1},\\ \textit{for each } \ \alpha,\gamma\in\{1,\dots,M\}.\end{gathered} \tag{5.2.31}$$

We shall now record the following versatile version of interior estimates for second-order elliptic systems. A proof may be found in [8, Theorem 11.9, p. 364].

Theorem 5.2.7 *Consider a homogeneous, constant coefficient, second-order, system* L *satisfying the weak ellipticity condition* $\det[L(\xi)] \neq 0$ *for each* $\xi \in \mathbb{R}^n \setminus \{0\}$*. Then for each null-solution* u *of* L *in a ball* $B(x,R)$ *(where* $x \in \mathbb{R}^n$ *and* $R > 0$*),* $0 < p < \infty$*,* $\lambda \in (0,1)$*,* $\ell \in \mathbb{N}_0$*, and* $0 < r < R$*, one has*

$$\sup_{z\in B(x,\lambda r)} |\nabla^\ell u(z)| \le \frac{C}{r^\ell}\bigg(\fint_{B(x,r)} |u|^p\, d\mathscr{L}^n\bigg)^{1/p}, \tag{5.2.32}$$

where $C = C(L,p,\ell,\lambda,n) > 0$ *is a finite constant.*

We conclude by recording a suitable version of the Divergence Theorem recently obtained in [9]. To state it requires a few preliminaries which we dispense with first. We shall write $\mathcal{E}'(\mathbb{R}^n_+)$ for the subspace of $\mathcal{D}'(\mathbb{R}^n_+)$ consisting of those distributions which are compactly supported. Hence,

$$\mathcal{E}'(\mathbb{R}^n_+) \hookrightarrow \mathcal{D}'(\mathbb{R}^n_+) \quad \text{and} \quad L^1_{\text{loc}}(\mathbb{R}^n_+) \hookrightarrow \mathcal{D}'(\mathbb{R}^n_+). \tag{5.2.33}$$

For each compact set $K \subset \mathbb{R}^n_+$, define $\mathcal{E}'_K(\mathbb{R}^n_+) := \{u \in \mathcal{E}'(\mathbb{R}^n_+) : \operatorname{supp} u \subset K\}$ and consider

$$\mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+) := \Big\{u \in \mathcal{D}'(\mathbb{R}^n_+) : \exists\, v_1 \in \mathcal{E}'_K(\mathbb{R}^n_+) \text{ and } \exists\, v_2 \in L^1(\mathbb{R}^n_+) \\ \text{such that } u = v_1 + v_2 \text{ in } \mathcal{D}'(\mathbb{R}^n_+)\Big\}. \tag{5.2.34}$$

Also, introduce $\mathscr{C}^\infty_b(\mathbb{R}^n_+) := \mathscr{C}^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$ and let $\big(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\big)^*$ denote its algebraic dual. Moreover, we let ${}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\cdot\,,\,\cdot\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)}$ denote the natural duality pairing between these spaces. It is useful to observe that for every compact set $K \subset \mathbb{R}^n_+$ one has

$$\mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+) \subset \big(\mathscr{C}^\infty_b(\mathbb{R}^n_+)\big)^*. \tag{5.2.35}$$

Theorem 5.2.8 ([9]) *Assume that $K \subset \mathbb{R}^n_+$ is a compact set and that $\vec{F} \in \big[L^1_{\rm loc}(\mathbb{R}^n_+)\big]^n$ is a vector field satisfying the following conditions (for some aperture parameter $\kappa > 0$):*

(a) $\operatorname{div} \vec{F} \in \mathcal{E}'_K(\mathbb{R}^n_+) + L^1(\mathbb{R}^n_+)$*, where the divergence is taken in the sense of distributions;*

(b) the nontangential maximal function $\mathcal{N}_\kappa^{\mathbb{R}^n_+ \setminus K} \vec{F}$ *belongs to* $L^1(\mathbb{R}^{n-1})$*;*

(c) the nontangential boundary trace $\vec{F}\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}$ *exists (in $\mathbb{C}^n$) at $\mathscr{L}^{n-1}$-a.e. point in* $\mathbb{R}^{n-1}$.

Then, with $e_n := (0, \ldots, 0, 1) \in \mathbb{R}^n$ and "dot" denoting the standard inner product in $\mathbb{R}^n$,

$${}_{(\mathscr{C}^\infty_b(\mathbb{R}^n_+))^*}\big(\operatorname{div} \vec{F}, 1\big)_{\mathscr{C}^\infty_b(\mathbb{R}^n_+)} = -\int_{\mathbb{R}^{n-1}} e_n \cdot \big(\vec{F}\,\big|^{\kappa-\mathrm{n.t.}}_{\partial\mathbb{R}^n_+}\big)\, d\mathscr{L}^{n-1}. \tag{5.2.36}$$

5.3 Proofs of Main Results

We take on the task of presenting the proof of Theorem 5.1.1.

Proof of Theorem 5.1.1 Fix an arbitrary point $x^\star \in \mathbb{R}^n_+$ and bring in $G^{L^\top}(\cdot\,, x^\star)$, the Green function with pole at $x^\star$ for $L^\top$, the transposed of the operator L (cf. Definition 5.2.5 and Theorem 5.2.6 for details on this matter). For ease of notation, abbreviate

$$G(\cdot) := G^{L^\top}(\cdot\,, x^\star) \text{ in } \mathbb{R}^n_+ \setminus \{x^\star\}. \tag{5.3.1}$$

By design, this is a matrix-valued function, say $G = (G_{\alpha\gamma})_{1\le\alpha,\gamma\le M}$. We shall apply Theorem 5.2.8 to a suitably chosen vector field. To set the stage, consider the compact set

$$K_\star := \overline{B(x^\star, r)} \subset \mathbb{R}^n_+, \quad \text{where } r := \operatorname{dist}(x^\star, \partial\mathbb{R}^n_+) \cdot \frac{\kappa}{2\sqrt{4+\kappa^2}}. \tag{5.3.2}$$

For each $\varepsilon > 0$ consider the function $u^\varepsilon : \overline{\mathbb{R}^n_+} \to \mathbb{C}^M$ given by

$$u^\varepsilon(x) := u(x', x_n + \varepsilon) \text{ for all } x = (x', x_n) \in \overline{\mathbb{R}^n_+}. \tag{5.3.3}$$

Then

$$u^\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M, \quad Lu^\varepsilon = 0 \text{ in } \mathbb{R}^n_+, \text{ and } \mathcal{N}_\kappa u^\varepsilon \le \mathcal{N}_\kappa u \text{ on } \mathbb{R}^{n-1}. \tag{5.3.4}$$

Fix $\varepsilon > 0$ along with some $\beta \in \{1, \dots, M\}$ and, using the summation convention over repeated indices, define the vector field

$$\vec{F} := \Big(u^\varepsilon_\alpha\, a^{\gamma\alpha}_{kj}\, \partial_k G_{\gamma\beta} - G_{\alpha\beta}\, a^{\alpha\gamma}_{jk}\, \partial_k u^\varepsilon_\gamma\Big)_{1\le j\le n} \quad \text{at } \mathscr{L}^n\text{-a.e. point in } \mathbb{R}^n_+. \tag{5.3.5}$$

From (5.3.5), Theorem 5.2.6, and the fact that $u^\varepsilon \in \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+})\big]^M$ it follows that

$$\vec{F} \in \big[L^1_{\mathrm{loc}}(\mathbb{R}^n_+)\big]^n \cap \big[\mathscr{C}^\infty(\overline{\mathbb{R}^n_+} \setminus K_\star)\big]^n \tag{5.3.6}$$

and, on account of (5.2.24) (used for $L^\top$ in place of L), we have

$$\vec{F}\Big|_{\partial\mathbb{R}^n_+} = \Big(\big(u^\varepsilon_\alpha\big|_{\partial\mathbb{R}^n_+}\big) a^{\gamma\alpha}_{kj} \big(\partial_k G_{\gamma\beta}\big)\big|_{\partial\mathbb{R}^n_+}\Big)_{1\le j\le n}. \tag{5.3.7}$$

Next, in the sense of distributions in $\mathbb{R}^n_+$, we may compute

$$\begin{aligned} \operatorname{div} \vec{F} &= (\partial_j u^\varepsilon_\alpha)\, a^{\gamma\alpha}_{kj}\, (\partial_k G_{\gamma\beta}) + u^\varepsilon_\alpha\, a^{\gamma\alpha}_{kj}\, (\partial_j \partial_k G_{\gamma\beta}) \\ &\quad - (\partial_j G_{\alpha\beta})\, a^{\alpha\gamma}_{jk}\, (\partial_k u^\varepsilon_\gamma) - G_{\alpha\beta}\, a^{\alpha\gamma}_{jk}\, (\partial_j \partial_k u^\varepsilon_\gamma) \\ &=: I_1 + I_2 + I_3 + I_4, \end{aligned} \tag{5.3.8}$$

where the last equality defines the I_i's. Changing variables $j' = k$, $k' = j$, $\alpha' = \gamma$, and $\gamma' = \alpha$ in I_3 yields

$$I_3 = -(\partial_{k'} G_{\gamma'\beta})\, a^{\gamma'\alpha'}_{k'j'}\, (\partial_{j'} u^\varepsilon_{\alpha'}) = -I_1. \tag{5.3.9}$$

As regards I_4, we have

$$I_4 = -G_{\alpha\beta}\,(Lu^{\varepsilon})_{\alpha} = 0, \tag{5.3.10}$$

by (5.3.4). Finally,

$$\begin{aligned} I_2 &= u^{\varepsilon}_{\alpha}(L_{A^\top} G_{\cdot\beta})_{\alpha} = u^{\varepsilon}_{\alpha}(L^\top G_{\cdot\beta})_{\alpha} \\ &= -u^{\varepsilon}_{\alpha}\delta_{\alpha\beta}\delta_{x^\star} = -u^{\varepsilon}_{\beta}\,\delta_{x^\star} = -u^{\varepsilon}_{\beta}(x^\star)\,\delta_{x^\star}. \end{aligned} \tag{5.3.11}$$

Collectively, these equalities permit us to conclude that, in the sense of distributions in $\mathbb{R}^n_+$,

$$\operatorname{div}\vec{F} = -u^{\varepsilon}_{\beta}(x^\star)\,\delta_{x^\star} \in \mathcal{E}'(\mathbb{R}^n_+). \tag{5.3.12}$$

In particular,

$$\operatorname{div}\vec{F} \in \mathcal{D}'(\mathbb{R}^n_+) \text{ induces a continuous functional in } \big(\mathscr{C}^{\infty}_b(\mathbb{R}^n_+)\big)^*. \tag{5.3.13}$$

Moving on, fix $x' \in \mathbb{R}^{n-1} \equiv \partial\mathbb{R}^n_+$ and pick an arbitrary point

$$y = (y', y_n) \in \Gamma_{\kappa/2}(x') \setminus K_\star. \tag{5.3.14}$$

Choose a rectifiable path $\gamma : [0,1] \to \overline{\mathbb{R}^n_+}$ joining $(x', 0)$ with y in $\Gamma_{\kappa/2}(x') \setminus K_\star$ and whose length is $\le Cy_n$. Then, for some constant $C \in (0,\infty)$ independent of x' and y, we may estimate

$$\begin{aligned} |G(y)| &= |G(y) - G(x',0)| = \Big|\int_0^1 \frac{d}{dt}[G(\gamma(t))]\,dt\Big| \\ &= \Big|\int_0^1 (\nabla G)(\gamma(t))\cdot\gamma'(t)\,dt\Big| \le \Big(\sup_{\xi\in\gamma((0,1))}|(\nabla G)(\xi)|\Big)\int_0^1 |\gamma'(t)|\,dt \\ &\le Cy_n \cdot \mathcal{N}^{\mathbb{R}^n_+\setminus K_\star}_{\kappa/2}(\nabla G)(x'), \end{aligned} \tag{5.3.15}$$

using the fact that G vanishes on $\partial\mathbb{R}^n_+$, the Fundamental Theorem of Calculus, Chain Rule, and (5.2.4). Next, define

$$a := \frac{\kappa}{2(\kappa+1)} \in \big(0, \tfrac{1}{2}\big) \tag{5.3.16}$$

and write, using interior estimates (cf. Theorem 5.2.7) for the function u^ε,

$$|(\nabla u^\varepsilon)(y)| \leq \frac{C}{y_n} \fint_{B(y,a\cdot y_n)} |u^\varepsilon(z)|\,dz$$
$$\leq C y_n^{-1} \cdot \sup_{z\in\Gamma_\kappa(x')} |u^\varepsilon(z)| \leq C y_n^{-1} \cdot \big(\mathcal{N}_\kappa u^\varepsilon\big)(x'), \tag{5.3.17}$$

since having $z = (z', z_n) \in B(y, a \cdot y_n)$ entails

$$y_n \leq z_n + |z - y| < z_n + a \cdot y_n \Longrightarrow y_n < (1-a)^{-1} z_n, \tag{5.3.18}$$

which, bearing in mind that y is as in (5.3.14), permits us to conclude that

$$|z' - x'| \leq |z' - y'| + |y' - x'| \leq |z - y| + (\kappa/2)y_n < a \cdot y_n + (\kappa/2)y_n$$
$$= (\kappa/2 + a)y_n < \frac{\kappa/2 + a}{1-a} z_n = \kappa z_n, \quad \text{hence} \;\; z \in \Gamma_\kappa(x'). \tag{5.3.19}$$

Then combining (5.3.15) with (5.3.17) gives, on account of (5.2.22),

$$\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}\big(|G||\nabla u^\varepsilon|\big)(x') \leq C\big(\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}(\nabla G)\big)(x')\big(\mathcal{N}_\kappa u^\varepsilon\big)(x')$$
$$\leq C\big(\mathcal{N}_\kappa u\big)(x')\frac{1}{1+|x'|^{n-1}} \quad \text{at each point} \;\; x' \in \mathbb{R}^{n-1}. \tag{5.3.20}$$

Since we also have

$$\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}\big(|\nabla G||u^\varepsilon|\big)(x') \leq \big(\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star}(\nabla G)\big)(x')\big(\mathcal{N}_\kappa u^\varepsilon\big)(x')$$
$$\leq C\big(\mathcal{N}_\kappa u\big)(x')\frac{1}{1+|x'|^{n-1}} \quad \text{at each point} \;\; x' \in \mathbb{R}^{n-1}, \tag{5.3.21}$$

we conclude from (5.3.5), (5.3.20), (5.3.21), and the second line in (5.1.4) that

$$\mathcal{N}_{\kappa/2}^{\mathbb{R}^n_+\setminus K_\star} \vec{F} \in L^1(\mathbb{R}^{n-1}). \tag{5.3.22}$$

Having established (5.3.6), (5.3.7), (5.3.13), and (5.3.22), Theorem 5.2.8 applies. To write the Divergence Formula (5.2.36) in this case, express $x^\star$ as $(x', t) \in \mathbb{R}^{n-1} \times$

$(0, \infty)$. Then, in view of (5.3.12) and (5.3.7) we may write

$$
\begin{aligned}
u_\beta(x^\star + \varepsilon e_n) &= u_\beta^\varepsilon(x^\star) = -{}_{(\mathscr{C}_b^\infty(\mathbb{R}^n_+))^*}\big(\operatorname{div}\vec{F}\,,\ 1\big)_{\mathscr{C}_b^\infty(\mathbb{R}^n_+)} \\
&= \int_{\mathbb{R}^{n-1}} e_n \cdot \big(\vec{F}\,\big|_{\partial\mathbb{R}^n_+}\big)\, d\mathscr{L}^{n-1} \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon) a_{kn}^{\gamma\alpha}\big(\partial_k G_{\gamma\beta}\big)(y', 0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon) a_{nn}^{\gamma\alpha}\big(\partial_n G_{\gamma\beta}\big)(y', 0)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon) a_{nn}^{\gamma\alpha}(\partial_{X_n} G_{\gamma\beta}^{L^\top})\big((y', 0), x^\star\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon) a_{nn}^{\gamma\alpha}(\partial_{Y_n} G_{\beta\gamma}^{L})\big(x^\star, (y', 0)\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon) a_{nn}^{\gamma\alpha}(\partial_{Y_n} G_{\beta\gamma}^{L})\big((x'-y', t), 0\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon) t^{1-n} a_{nn}^{\gamma\alpha}(\partial_{Y_n} G_{\beta\gamma}^{L})\big(((x'-y')/t, 1), 0\big)\, dy' \\
&= \int_{\mathbb{R}^{n-1}} u_\alpha(y', \varepsilon)(P_{\beta\alpha}^{L})_t(x'-y')\, dy',
\end{aligned}
\tag{5.3.23}
$$

where the fifth equality uses the observation that $(\partial_k G)(y', 0) = 0$ whenever $k < n$ since G vanishes (in a smooth fashion) on $\mathbb{R}^{n-1} \times \{0\}$, the sixth equality is a consequence of (5.3.1), the seventh equality is implied by (5.2.27), the eighth equality makes use of (5.2.25) (bearing in mind that $x^\star = (x', t)$), the ninth equality is seen from (5.2.26), and the last equality comes from (5.2.31).

Since $\beta \in \{1, \dots, M\}$ and $x^\star = (x', t) \in \mathbb{R}^n_+$ have been arbitrarily chosen, the argument so far shows that

$$
u(x', t + \varepsilon) = \int_{\mathbb{R}^{n-1}} P_t^L(x' - y') f_\varepsilon(y')\, dy' \quad \text{for each} \quad x = (x', t) \in \mathbb{R}^n_+,
\tag{5.3.24}
$$

where we have abbreviated

$$
f_\varepsilon := u(\cdot, \varepsilon) : \mathbb{R}^{n-1} \longrightarrow \mathbb{C}^M \quad \text{for each} \quad \varepsilon > 0.
\tag{5.3.25}
$$

If we also consider the weight $v : \mathbb{R}^{n-1} \to (0,\infty)$ defined as $v(x') := (1 + |x'|^{n-1})^{-1}$ for each $x' \in \mathbb{R}^{n-1}$, then the last condition in (5.1.4) entails

$$\sup_{\varepsilon>0} |f_\varepsilon| \le \mathcal{N}_\kappa u \in L^1\big(\mathbb{R}^{n-1}, \, v\,\mathscr{L}^{n-1}\big). \tag{5.3.26}$$

Granted this, the weak-$*$ convergence result from Lemma 5.2.1 may be used for the sequence $\{f_\varepsilon\}_{\varepsilon>0} \subset L^1\big(\mathbb{R}^{n-1}, \, v\,\mathscr{L}^{n-1}\big)$ to conclude that there exists some $f \in L^1\big(\mathbb{R}^{n-1}, \, v\,\mathscr{L}^{n-1}\big)$ and some sequence $\{\varepsilon_j\}_{j\in\mathbb{N}} \subset (0,\infty)$ which converges to zero with the property that

$$\lim_{j\to\infty} \int_{\mathbb{R}^{n-1}} \varphi(y') f_{\varepsilon_j}(y') \frac{dy'}{1+|y'|^{n-1}} = \int_{\mathbb{R}^{n-1}} \varphi(y') f(y') \frac{dy'}{1+|y'|^{n-1}} \tag{5.3.27}$$

for every continuous function $\varphi \in \mathscr{C}_{\text{van}}(\mathbb{R}^{n-1})$. The fact that there exists a constant $C \in (0,\infty)$ for which

$$|P^L(z')| \le \frac{C}{(1+|z'|^2)^{n/2}} \quad \text{for each} \quad z' \in \mathbb{R}^{n-1} \tag{5.3.28}$$

(see item *(a)* of Definition 5.2.2) ensures for each fixed point $(x',t) \in \mathbb{R}^n_+$ the assignment

$$\begin{gathered} \mathbb{R}^{n-1} \ni y' \mapsto \varphi(y') := (1+|y'|^{n-1}) P^L_t(x'-y') \in \mathbb{C}^{M\times M} \\ \text{is a continuous function which vanishes at infinity.} \end{gathered} \tag{5.3.29}$$

At this stage, from (5.3.24) and (5.3.27) used for the function φ defined in (5.3.29) we obtain (bearing in mind that u is continuous in $\mathbb{R}^n_+$) that

$$u(x',t) = \int_{\mathbb{R}^{n-1}} P^L_t(x'-y') f(y')\, dy' \quad \text{for each} \quad x = (x',t) \in \mathbb{R}^n_+. \tag{5.3.30}$$

With this in hand, and since $L^1\big(\mathbb{R}^{n-1}, \, v\,\mathscr{L}^{n-1}\big) \subseteq L^1\Big(\mathbb{R}^{n-1}, \, \dfrac{dx'}{1+|x'|^n}\Big)$, we may invoke Proposition 5.2.4 to conclude that

$$u\Big|^{\kappa-\text{n.t.}}_{\partial\mathbb{R}^n_+} \text{ exists and equals } f \text{ at } \mathscr{L}^{n-1}\text{-a.e. point in } \mathbb{R}^{n-1}. \tag{5.3.31}$$

Once this has been established, all conclusions in (5.1.5) are implied by (5.3.30) and (5.3.31). □

We close by presenting the proof of Corollary 5.1.3.

Proof of Corollary 5.1.3 As a preamble, let us first show that

$$\mathbb{X} \subseteq \Big[L^1\Big(\mathbb{R}^{n-1}\,,\,\frac{dx'}{1+|x'|^n}\Big)\Big]^M. \tag{5.3.32}$$

To justify this, pick some arbitrary $f \in \mathbb{X}$. Then the inclusion in (5.1.14) gives that $\mathcal{M}f \in \mathbb{Y}$, hence $\mathcal{M}f$ is not identically $+\infty$. This implies that $f \in [L^1_{\rm loc}(\mathbb{R}^{n-1})]^M$ which, in concert with Lebesgue's Differentiation Theorem, implies that $|f| \leq \mathcal{M}f$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Since $\mathbb{Y}$ is a function lattice, it follows that $|f| \in \mathbb{Y}$. Thus, ultimately, (5.3.32) holds by virtue of the inclusion in (5.1.13).

To prove the existence of a solution for (5.1.15), given any $f \in \mathbb{X}$ define u as in (5.1.16). Note that (5.3.32) ensures that Proposition 5.2.4 is applicable. In turn, this guarantees that u is a well-defined null-solution of L belonging to $\big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$, satisfying the boundary condition $u\big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^n_+} = f$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$, as well as the pointwise estimate in (5.1.17). The latter property, together with the last conditions imposed in (5.1.14) and (5.1.13), guarantees $\mathcal{N}_\kappa u \in \mathbb{Y}$. Thus, u is indeed a solution for (5.1.15).

At this stage, there remains to establish that the boundary value problem (5.1.15) can have at most one solution. To this end, assume that both u_1 and u_2 solve (5.1.15) for the same datum $f \in \mathbb{X}$ and set $u := u_1 - u_2 \in \big[\mathscr{C}^\infty(\mathbb{R}^n_+)\big]^M$. Then $Lu = 0$ in $\mathbb{R}^n_+$ and $u\big|^{\kappa-{\rm n.t.}}_{\partial\mathbb{R}^n_+} = 0$ at $\mathscr{L}^{n-1}$-a.e. point in $\mathbb{R}^{n-1}$. Since we also have $\mathcal{N}_\kappa u_1, \mathcal{N}_\kappa u_2 \in \mathbb{Y}$, the pointwise estimate

$$0 \leq \mathcal{N}_\kappa u \leq \mathcal{N}_\kappa u_1 + \mathcal{N}_\kappa u_2 \leq 2\max\big\{\mathcal{N}_\kappa u_1\,,\,\mathcal{N}_\kappa u_2\big\} \ \text{ on } \ \mathbb{R}^{n-1} \tag{5.3.33}$$

forces $\mathcal{N}_\kappa u \in \mathbb{Y}$ by the properties of the function lattice $\mathbb{Y}$. Granted this, Corollary 5.1.2 applies (thanks to the first condition in (5.1.13)) and gives that $u \equiv 0$ in $\mathbb{R}^n_+$. Hence $u_1 = u_2$, as wanted. □

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Chapter 6
Hardy Spaces for the Three-Dimensional Vekua Equation

Briceyda B. Delgado and R. Michael Porter

To Wolfgang Sprößig, with affection and respect

Abstract Let Ω be a bounded Lipschitz domain in $\mathbb{R}^3$. We introduce the Vekua-Hardy spaces $H_f^p(\Omega)$ of solutions of the main Vekua equation $DW = (Df/f)\overline{W}$ where $1 < p < \infty$. Here W is quaternion-valued, D is the Moisil-Teodorescu operator, and the conductivity f is a bounded scalar function with bounded gradient. Using the Vekua-Hilbert transform $\mathcal{H}_f$ defined in previous work of the authors, we give some characterizations of $H_f^p(\Omega)$ analogous to those of the "classical" Hardy spaces of monogenic functions in $\mathbb{R}^3$. The main obstacle is the lack of several fundamental analogues of properties of solutions to the special case $DW = 0$ (monogenic, or hyperholomorphic functions), such as power series and the Cauchy integral formula.

Keywords Monogenic functions · Hardy spaces · Main Vekua equation · Vekua-Hardy spaces · Vekua-Hilbert transform · Conductivity equation

Mathematics Subject Classification (2010) Primary 30G20; Secondary 30H10, 35J25

6.1 Introduction

The study of Hardy spaces of holomorphic functions in planar domains with Lipschitz boundary was initiated in [24]. The analogous study for monogenic functions on n-dimensional Lipschitz domains began in [10, 11, 16]. Here we

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consider $\Omega \subseteq \mathbb{R}^3$, and generalizing the definition given in [31] for Hardy spaces of monogenic functions $H^p(\Omega)$, we introduce the *Vekua-Hardy space* $H_f^p(\Omega)$ of solutions to the main Vekua equation (6.2.1), with conductivity $f \in W^{1,\infty}(\Omega, \mathbb{R})$ and $1 < p < \infty$. The case $p = 2$ is of particular importance because the solution of (6.2.1) constructed through the Vekua-Hilbert transform $\mathcal{H}_f$, defined in [14], belongs to the corresponding Vekua-Hardy space $H_f^2(\Omega)$.

In the complex case, more precisely for the unit disk, there exist several classes of generalized Hardy spaces, for example, the Hardy space for general first-order elliptic systems [27], the Hardy space of generalized analytic functions [26] and the Hardy space of solutions to the conjugate Beltrami equation [3]. All these spaces preserve many properties inherited from analytic functions mainly due to the Similarity Principle [9, 28]. Since there is no n-dimensional Similarity Principle in the literature for $n > 2$, our results instead depend strongly on the intrinsic properties of the Teodorescu transform and the Cauchy operator in bounded domains in $\mathbb{R}^3$ as well as on the Vekua-Hilbert transform defined in [14].

In the half space $\mathbb{R}^m_+$, there are the Hardy spaces of solutions of generalized Riesz and Moisil-Teodorescu systems, which were characterized in [7]. Some aspects of (monogenic) Hardy and Bergman spaces in the unit ball in $\mathbb{R}^3$ whose functions take values in the reduced quaternions were considered in [32].

Another recent example [30] is the generalized Hardy space consisting of Clifford-valued solutions of perturbed Dirac operators in the exterior of uniformly rectifiable domains, radiating at infinity (domains with boundary Lipschitz are uniformly rectifiable [22]).

The structure of this paper is as follows. In Sect. 6.2 we summarize some known results concerning the main Vekua equation (6.2.1) in $\mathbb{R}^3$. In Sect. 6.3 we define the Vekua-Hardy space $H_f^p(\Omega)$ and the associated generalized Bergman space of L^p solutions of (6.2.1), and state a series of characterizations for them and for some Banach subspaces. To take advantage of the fact that solutions of the main Vekua equation satisfy a homogeneous div-curl system (6.2.3), the spaces $W^{p,\mathrm{div}}(\Omega, \mathbb{R}^3)$ and $W^{p,\mathrm{curl}}(\Omega, \mathbb{R}^3)$ are introduced with the aim of providing a natural weak characterization for the L^p-solutions of (6.2.1). Finally, in Sect. 6.4 we recall the definition of the Vekua-Hilbert transform (6.4.1) given in [14] and prove the main result of the paper (Theorem 6.4.4), which guarantees the existence of non-tangential limits and is a kind of Maximum Principle for a subspace of the Vekua-Hardy space $H_f^2(\Omega)$: for those functions in $H_f^2(\Omega)$ whose scalar part belongs to the Sobolev space $W^{1,2}(\Omega, \mathbb{R})$.

6.2 Notation and Background for the Main Vekua Equation

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded domain. We are interested in functions $W = W_0 + \vec{W} \colon \Omega \to \mathbb{H}$ with $W_0 = \mathrm{Sc}\, W$, $\vec{W} = \mathrm{Vec}\, W$, where $\mathbb{H}$ denotes the algebra of quaternions $x = x_0 + \sum_{i=1}^3 e_i x_i = \mathrm{Sc}\, x + \mathrm{Vec}\, x$ with $x_i \in \mathbb{R}$; the imaginary units

e_1, e_2, e_3 obey the standard laws of multiplication $e_i^2 = e_1e_2e_3 = -1$ for $i = 1, 2, 3$. Identifying the subspaces $\mathrm{Sc}\,\mathbb{H}$ and $\mathrm{Vec}\,\mathbb{H}$ with $\mathbb{R}$ and $\mathbb{R}^3$ respectively, we refer to elements $\vec{x} \in \Omega$. Some references for higher-dimensional function theory and its applications which will be used here are [6, 17, 20, 21, 29].

Let $D = \sum_{i=1}^{3} e_i \partial/\partial x_i$ be the Moisil-Teodorescu differential operator acting usually on the left on $\mathbb{H}$-valued functions defined in Ω, and consider a fixed nonvanishing scalar function $f \in W^{1,\infty}(\Omega, \mathbb{R})$ (the Sobolev space of bounded functions with bounded derivatives). Our object of interest is the *main Vekua equation* [28]

$$DW = \frac{Df}{f}\overline{W}, \tag{6.2.1}$$

whose theory appeared in [5, 36] for functions in $\mathbb{R}^2$. It plays an important role in the theory of pseudo-analytic functions (sometimes called generalized analytic functions). We write

$$\mathfrak{M}_f(\Omega) = \left\{W : \Omega \to \mathbb{H} \,\middle|\, \left(D - \frac{Df}{f}C_{\mathbb{H}}\right)W = 0\right\} \tag{6.2.2}$$

for the space of measurable weak solutions of (6.2.1), where $C_{\mathbb{H}}W = \mathrm{Sc}\,W - \mathrm{Vec}\,W$ is the quaternionic conjugate operator. For $f \equiv 1$, this reduces to $\mathfrak{M}_{f\equiv 1}(\Omega) = \mathfrak{M}(\Omega)$, the classical space of left-monogenic (or hyperholomorphic) functions, i.e. solutions of $DW = 0$. We will be interested in the subspace

$$\mathfrak{M}_f^p(\Omega) = \{W \in \mathfrak{M}_f(\Omega) : \|W\|_{L^p} < \infty\}$$

of L^p-solutions of (6.2.1). This is a nontrivial linear subspace over $\mathbb{R}$: in addition to the elementary solutions f and $(1/f)e_i$, $i = 1, 2, 3$, also note $(1/f)\,\mathrm{SI}(\Omega, \mathbb{R}^3) \cap L^p(\Omega, \mathbb{R}^3) \subseteq \mathfrak{M}_f^p(\Omega)$, where $\mathrm{SI}(\Omega, \mathbb{R}^3)$ is the collection of solenoidal-irrotational vector fields $\vec{W}$, that is, $D\vec{W} = \vec{W}D = 0$, or more explicitly

$$\mathrm{div}\,\vec{W} = 0, \quad \mathrm{curl}\,\vec{W} = 0,$$

a system equivalent to $\vec{W} \in \mathrm{SI}(\Omega, \mathbb{R}^3)$ as can be seen by rewriting DW in vector form

$$DW = -\,\mathrm{div}\,\vec{W} + \mathrm{grad}\,W_0 + \mathrm{curl}\,\vec{W}.$$

From this we also have [28, Th. 161] that W is solution of (6.2.1) if and only if W_0 and $\vec{W}$ satisfy the homogeneous div-curl system

$$\mathrm{div}(f\vec{W}) = 0, \quad \mathrm{curl}(f\vec{W}) = -f^2\nabla(W_0/f). \tag{6.2.3}$$

We will say that $\vec{W}$ is an f^2*-hyperconjugate* for W_0 when $W_0 + \vec{W}$ is a solution of (6.2.1). In [13, 14] two constructions for f^2-hyperconjugates were given. Both are direct applications of the solution of the "div-curl system"; in [13] this was done for star-shaped domains in $\mathbb{R}^3$ and in [14] for Lipschitz domains in $\mathbb{R}^3$ with connected complement. In Sect. 6.4 we will follow the second approach.

When we apply div, curl to the second equation of (6.2.3), we obtain

$$\nabla \cdot f^2 \, \nabla\left(\frac{W_0}{f}\right) = 0, \tag{6.2.4}$$

$$\operatorname{curl}\left(\frac{1}{f^2}\operatorname{curl}(f\vec{W})\right) = 0. \tag{6.2.5}$$

These are the so-called *conductivity equation* and the *double curl-type equation*. For brevity we will say that f^2 is a *conductivity* when f is a non-vanishing $\mathbb{R}$-valued function in the domain under consideration. The conductivity will be called *proper* when $\rho(f) = \sup(|f|, 1/|f|)$ is finite.

The *Cauchy kernel* $E(\vec{x}) = -\vec{x}/(4\pi|\vec{x}|^3)$ is a fundamental solution of D. The *Teodorescu transform*

$$T_\Omega[w](\vec{x}) = -\int_\Omega E(\vec{y} - \vec{x}) w(\vec{y}) \, d\vec{y}, \quad \vec{x} \in \mathbb{R}^3, \tag{6.2.6}$$

is a right inverse of D. Although T_Ω is not a left inverse of D, we have the *Borel-Pompeiu formula* [21, Th. 7.8]

$$T_\Omega[Dw](\vec{x}) + F_{\partial\Omega}[\operatorname{tr} w](\vec{x}) = w(\vec{x}), \quad \vec{x} \in \Omega, \tag{6.2.7}$$

where the *Cauchy operator*

$$F_{\partial\Omega}[\varphi](\vec{x}) = \int_{\partial\Omega} E(\vec{y} - \vec{x}) \eta(\vec{y}) \varphi(\vec{y}) \, ds_{\vec{y}}, \quad \vec{x} \in \mathbb{R}^3 \setminus \partial\Omega$$

(with η the outward pointing unit normal vector to $\partial\Omega$) satisfies $F_{\partial\Omega}[\varphi] \in \mathfrak{M}(\Omega)$.

We let $W^{1,p}(\Omega, \mathbb{H})$ denote the Sobolev space of functions in L^p with first derivatives in L^p; the space of their boundary values is $W^{1-1/p,p}(\partial\Omega, \mathbb{H})$. The latter space exists due to the Trace Theorem [19, Th. 1.5.1.10], valid for example in Lipschitz domains, and the trace operator

$$\operatorname{tr}\colon W^{1,p}(\Omega, \mathbb{H}) \to L^p(\partial\Omega, \mathbb{H}), \quad \operatorname{tr} u = u|_{\partial\Omega},$$

is a bounded linear operator.

Moreover, when $p = 2$, $\operatorname{tr}\colon W^{1,2}(\Omega, \mathbb{H}) \to H^{1/2}(\partial\Omega, \mathbb{H})$ is bounded; we use the common notation $H^{1/2}(\partial\Omega, \mathbb{H}) = W^{1-1/2,2}(\partial\Omega, \mathbb{H})$. The presence of the codomain $\mathbb{H}$ (or $\mathbb{R}$ or $\mathbb{R}^3$) will avoid confusion with the notation of the Hardy spaces $H^p(\Omega)$ given in the next section.

6.3 Hardy Spaces of Solutions of the Main Vekua Equation

For definitions we follow almost entirely the monographs [25, 31]. Similarly to the classical definition of Hardy spaces in domains in $\mathbb{C}$, several equivalent characterizations are known for the left-monogenic Hardy spaces in $\mathbb{R}^3$, which depend on basic facts such as the Maximum Principle and the Mean Value Property for monogenic functions as well as the regularity and boundedness on $L^p(\partial\Omega)$ of the Cauchy operator [8]. However, since such results are not all currently available for the Vekua equation, we give our main definition in terms of the non-tangential maximal function $\mathcal{N}_\alpha$. From now on Ω will be a bounded domain with Lipschitz boundary.

For a fixed $\alpha > 0$ and $\vec{x} \in \partial\Omega$, the region of non-tangential approach with vertex at $\vec{x}$ is given by

$$\Gamma_\alpha(\vec{x}) = \{\vec{y} \in \Omega\colon |\vec{x} - \vec{y}| \le (1+\alpha)\,\mathrm{dist}(\vec{y}, \partial\Omega)\}\,.$$

Then $\mathcal{N}_\alpha W\colon \partial\Omega \to [0, \infty]$ is given by

$$\mathcal{N}_\alpha W(\vec{x}) = \sup\{|W(\vec{y})|\colon \vec{y} \in \Gamma_\alpha(\vec{x})\}\,.$$

When measuring the growth of W, the choice of α is largely irrelevant because for $\alpha, \beta > 0$ we have estimates $C_1\|\mathcal{N}_\beta W\|_{L^p(\partial\Omega)} \le \|\mathcal{N}_\alpha W\|_{L^p(\partial\Omega)} \le C_2\|\mathcal{N}_\beta W\|_{L^p(\partial\Omega)}$ [22, Prop. 2.1.2]. See [17, 24, 25, 31] for the general theory of Hardy spaces of monogenic functions $H^p(\Omega)$, where one writes $\mathcal{N}$ for $\mathcal{N}_\alpha$ and

$$\|\mathcal{N} W\|_{L^p(\partial\Omega)} = \left(\int_{\partial\Omega} |\mathcal{N} W(\vec{x})|^p \, ds_{\vec{y}}\right)^{1/p} \tag{6.3.1}$$

as the definition of $\|W\|_{H^p}$.

Definition 6.3.1 Let $1 < p < \infty$ and let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. The *Vekua-Hardy space* $H^p_f(\Omega)$ consists of all functions W in $\mathfrak{M}_f(\Omega)$ whose non-tangential maximal function $\mathcal{N} W$ belongs to $L^p(\partial\Omega, \mathbb{R})$; that is,

$$\|W\|_{H^p_f} := \|\mathcal{N} W\|_{L^p(\partial\Omega)} < \infty.$$

Notice that by hypothesis $Df/f \in L^\infty(\Omega, \mathbb{R}^3)$. Also $H^p_f(\Omega)$ is a nontrivial linear subspace over $\mathbb{R}$, because it contains f as well as $(1/f)e_i$, $i = 1, 2, 3$.

We allow W to be a solution of (6.2.1) in the sense of distributions in Ω, so the div-curl system (6.2.3) is satisfied weakly. For the development of the Vekua-Hardy spaces $H_f^p(\Omega)$ we will need the following spaces linked to the operators div and curl appearing in many electromagnetism problems [12, 18]:

$$W^{p,\mathrm{div}}(\Omega,\mathbb{R}^3) = \{\vec{u} \in L^p(\Omega,\mathbb{R}^3)\colon\ \operatorname{div}\vec{u} \in L^p(\Omega,\mathbb{R})\},$$
$$W^{p,\mathrm{curl}}(\Omega,\mathbb{R}^3) = \{\vec{u} \in L^p(\Omega,\mathbb{R}^3)\colon\ \operatorname{curl}\vec{u} \in L^p(\Omega,\mathbb{R}^3)\},$$

which are Banach spaces with the norms

$$\|\vec{u}\|_{W^{p,\mathrm{div}}} = \|\vec{u}\|_{L^p} + \|\operatorname{div}\vec{u}\|_{L^p}, \quad \|\vec{u}\|_{W^{p,\mathrm{curl}}} = \|\vec{u}\|_{L^p} + \|\operatorname{curl}\vec{u}\|_{L^p}.$$

These are weaker than $\|\cdot\|_{W^{1,p}}$ because $W^{1,p}(\Omega,\mathbb{R}^3)$ is a proper subset of the intersection

$$W^{p,\,\mathrm{div\text{-}curl}}(\Omega,\mathbb{R}^3) = W^{p,\mathrm{div}}(\Omega,\mathbb{R}^3) \cap W^{p,\mathrm{curl}}(\Omega,\mathbb{R}^3).$$

The *normal trace operator* is defined in $W^{p,\mathrm{div}}(\Omega,\mathbb{R}^3)$ by

$$\gamma_{\mathbf{n}}\colon W^{p,\mathrm{div}}(\Omega,\mathbb{R}^3) \to (W^{1-1/q,q}(\partial\Omega,\mathbb{R}))^*$$
$$\gamma_{\mathbf{n}}(\vec{u}) = \vec{u}|_{\partial\Omega}\cdot\eta,$$

which by the Divergence Theorem is weakly defined for $\vec{u} \in W^{p,\mathrm{div}}(\Omega,\mathbb{R}^3)$ by

$$\langle\gamma_{\mathbf{n}}(\vec{u}), \operatorname{tr} v_0\rangle_{\partial\Omega} = \int_\Omega \vec{u}\cdot\nabla v_0\,d\vec{y} + \int_\Omega (\operatorname{div}\vec{u})v_0\,d\vec{y} \tag{6.3.2}$$

for all $v_0 \in W^{1,q}(\Omega,\mathbb{R})$, where the symbol $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing between $W^{1-1/q,q}(\partial\Omega)$ and its dual space $(W^{1-1/q,q}(\partial\Omega))^*$. Analogously, the *tangential trace operator* is defined in $W^{p,\mathrm{curl}}(\Omega,\mathbb{R}^3)$ by

$$\gamma_{\mathbf{t}}\colon W^{p,\mathrm{curl}}(\Omega,\mathbb{R}^3) \to (W^{1-1/q,q}(\partial\Omega,\mathbb{R}^3))^*$$
$$\gamma_{\mathbf{t}}(\vec{u}) = \vec{u}|_{\partial\Omega}\times\eta,$$
$$\langle\gamma_{\mathbf{t}}(\vec{u}), \operatorname{tr}\vec{v}\rangle_{\partial\Omega} = \int_\Omega \vec{u}\cdot\operatorname{curl}\vec{v}\,d\vec{y} - \int_\Omega \operatorname{curl}\vec{u}\cdot\vec{v}\,d\vec{y} \tag{6.3.3}$$

for all $\vec{u} \in W^{p,\mathrm{curl}}(\Omega,\mathbb{R}^3)$ and $\vec{v} \in W^{1,q}(\Omega,\mathbb{R}^3)$.

Note by (6.2.3) that when $W \in \mathfrak{M}^p_f(\Omega)$, we have $\operatorname{curl}\vec{W} \in L^p(\Omega, \mathbb{R}^3)$ if and only if $W_0 \in W^{1,p}(\Omega, \mathbb{R})$. Thus, let us define

$$\begin{aligned}\widehat{L}^p(\Omega, \mathbb{H}) =& W^{1,p}(\Omega, \mathbb{R}) + L^p(\Omega, \mathbb{R}^3)\\ =&\{W \in L^p(\Omega, \mathbb{H})\colon\ \operatorname{Sc} W \in W^{1,p}(\Omega, \mathbb{R})\},\end{aligned} \tag{6.3.4}$$

which is a Banach space with the norm $\|W\|_{\widehat{L}^p} = \|W_0\|_{W^{1,p}} + \|\vec{W}\|_{L^p}$. Now we show that solutions of the main Vekua equation in $\widehat{L}^p(\Omega, \mathbb{H})$ have a natural weak characterization.

Lemma 6.3.2 *Let $W = W_0 + \vec{W} \in \widehat{L}^p(\Omega, \mathbb{H})$. Then $W \in \mathfrak{M}^p_f(\Omega)$ if and only if for every $v = v_0 + \vec{v} \in W^{1,q}_0(\Omega, \mathbb{H})$ (the subspace of $W^{1,q}(\Omega, \mathbb{H})$ with trace zero),*

$$\begin{aligned}&\int_\Omega f\vec{W}\cdot\nabla v_0\,d\vec{y} = 0,\\ &\int_\Omega f\vec{W}\cdot\operatorname{curl}\vec{v}\,d\vec{y} = -\int_\Omega f^2\nabla(W_0/f)\cdot\vec{v}\,d\vec{y}.\end{aligned} \tag{6.3.5}$$

Proof Let $W \in \mathfrak{M}^p_f(\Omega)$. By (6.2.3), $f\vec{W} \in W^{p,\,\text{div-curl}}(\Omega, \mathbb{R}^3)$. Therefore we can use (6.3.2) and (6.3.3). Let $v = v_0 + \vec{v} \in W^{1,q}_0(\Omega, \mathbb{H})$. Then (again by the Divergence Theorem)

$$\begin{aligned}0 &= \langle\gamma_{\mathbf{n}}(f\vec{W}), \operatorname{tr} v_0\rangle_{\partial\Omega} = \int_\Omega f\vec{W}\cdot\nabla v_0\,d\vec{y},\\ 0 &= \langle\gamma_{\mathbf{t}}(f\vec{W}), \operatorname{tr}\vec{v}\rangle_{\partial\Omega} = \int_\Omega f\vec{W}\cdot\operatorname{curl}\vec{v}\,d\vec{y} + \int_\Omega f^2\nabla(W_0/f)\cdot\vec{v}\,d\vec{y}.\end{aligned}$$

For the converse, begin with the system (6.3.5), and use (6.3.2) and (6.3.3) to obtain

$$\begin{aligned}\int_\Omega \operatorname{div}(f\vec{W})v_0\,d\vec{y} &= 0,\\ \int_\Omega (\operatorname{curl}(f\vec{W}) + f^2\nabla(W_0/f))\cdot\vec{v}\,d\vec{y} &= 0,\end{aligned}$$

for $v \in W^{1,q}_0(\Omega, \mathbb{H})$, which says that (6.2.3) is weakly satisfied. □

To investigate further how the scalar part of a solution of (6.2.1) influences the vector part, we introduce the subspace

$$\widehat{\mathfrak{M}}^p_f(\Omega) = \mathfrak{M}^p_f(\Omega) \cap \widehat{L}^p(\Omega, \mathbb{H}). \tag{6.3.6}$$

Theorem 6.3.3 *Let Ω be a bounded Lipschitz domain and let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. The space $\widehat{\mathfrak{M}}^p_f(\Omega)$ is closed in $\widehat{L}^p(\Omega, \mathbb{H})$ for $1 < p < \infty$.*

Proof Let $\{W_n\} \subseteq \widehat{\mathfrak{M}}^p_f(\Omega)$ be a sequence such that $W_n \to W$ in $\widehat{L}^p(\Omega, \mathbb{H})$, so for all pairs of test functions $v_0 \in W^{1,q}_0(\Omega, \mathbb{R})$ and $\vec{v} \in W^{1,q}_0(\Omega, \mathbb{R}^3)$ we have

$$\int_\Omega f\vec{W}_n \cdot \nabla v_0\, d\vec{y} \longrightarrow \int_\Omega f\vec{W} \cdot \nabla v_0\, d\vec{y},$$

$$\int_\Omega f\vec{W}_n \cdot \operatorname{curl}\vec{v}\, d\vec{y} \longrightarrow \int_\Omega f\vec{W} \cdot \operatorname{curl}\vec{v}\, d\vec{y},$$

$$\int_\Omega f^2\nabla(W_{0,n}/f) \cdot \vec{v}\, d\vec{y} \longrightarrow \int_\Omega f^2\nabla(W_0/f) \cdot \vec{v}\, d\vec{y}, \tag{6.3.7}$$

as $n \to \infty$. By Lemma 6.3.2, $W_n = W_{n,0} + \vec{W}_n$ satisfy (6.3.5), so W does also. By Lemma 6.3.2 again, $W \in \widehat{\mathfrak{M}}^p_f(\Omega)$. □

If instead of the first limit of (6.3.7) we use the fact [4] that $\mathrm{Sol}(\Omega, \mathbb{R}^3) \cap L^p(\Omega, \mathbb{R}^3)$ is closed in $L^p(\Omega, \mathbb{R}^3)$, then we can see immediately that $\operatorname{div} f\vec{W} = 0$. When $f \equiv 1$, then $\mathfrak{M}^p(\Omega)$ and $H^p(\Omega)$ are the classical Bergman and Hardy spaces, respectively. It is well known that $H^p(\Omega) \subseteq \mathfrak{M}^p(\Omega)$, but we do not know whether or not $H^p_f(\Omega) \subseteq \mathfrak{M}^p_f(\Omega)$ for every proper conductivity $f \in W^{1,\infty}(\Omega, \mathbb{R})$ (cf. Proposition 6.4.5 below).

A key consequence of $DT_\Omega = I$ which will enable us to derive information about (6.2.1) from known results on monogenic functions is the easily verified fact that $I - T_\Omega(Df/f)C_{\mathbb{H}}$ transforms solutions of the main Vekua equation to left-monogenic functions:

$$I - T_\Omega(Df/f)C_{\mathbb{H}} \colon \mathfrak{M}^p_f(\Omega) \to \mathfrak{M}(\Omega) \cap L^p(\Omega). \tag{6.3.8}$$

In fact, D is a right inverse of T_Ω even in the weak sense, by Weyl's Lemma [15, 35], so when W is a weak solution of (6.2.1), $U = W - T_\Omega[(Df/f)\overline{W}]$ is monogenic. Conversely, when U is monogenic, W is a (strong) solution of (6.2.1). A similar relation holds for Hardy-Vekua spaces:

Lemma 6.3.4 *Let Ω be a bounded Lipschitz domain. Let $W \in W^{m-1,p}(\Omega, \mathbb{H})$, where $f \in W^{m,\infty}(\Omega, \mathbb{R})$ is a proper conductivity, $m \in \mathbb{Z}^+$, $3/m < p < \infty$ and $m \geq 1$. Then*

$$W \in H^p_f(\Omega) \Leftrightarrow U = W - T_\Omega\left[\frac{Df}{f}\overline{W}\right] \in H^p(\Omega). \tag{6.3.9}$$

Proof By the comments preceding this statement of the Lemma, it is suffices to show that $\mathcal{N}T_\Omega[(Df/f)\overline{W}] \in L^p(\partial\Omega, \mathbb{R})$, because this implies that $\mathcal{N}U$ and $\mathcal{N}W$ are simultaneously in or not in $L^p(\partial\Omega, \mathbb{R})$.

By the dimension requirement of the Sobolev Imbedding Theorem [1, Th. 4.12], the inclusion $W^{m,p}(\Omega) \subseteq C^0(\Omega)$ is continuous because we are working in $\mathbb{R}^3$ and $3/m < p < \infty$. Since $T_\Omega \colon W^{m-1,p}(\Omega) \to W^{m,p}(\Omega)$ is continuous [20, Th. 2.3.8], we have $T_\Omega \colon W^{m-1,p}(\Omega) \to C^0(\Omega)$ is continuous. Thus since $(Df/f)\overline{W} \in W^{m-1,p}(\Omega)$,

$$
\begin{aligned}
\mathcal{N}_\alpha T_\Omega[(Df/f)\overline{W}](\vec{x}) &= \sup\{|T_\Omega[(Df/f)\overline{W}](\vec{y})| \colon \vec{y} \in \Gamma_\alpha(\vec{x})\} \\
&\leq \sup\{|T_\Omega[(Df/f)\overline{W}](\vec{y})| \colon \vec{y} \in \Omega\} \\
&\leq \|T_\Omega\| \|(Df/f)\overline{W}\|_{W^{m-1,p}(\Omega)} \\
&\leq C_f \|T_\Omega\| \|W\|_{W^{m-1,p}(\Omega)},
\end{aligned}
\tag{6.3.10}
$$

where C_f only depends on $\|Df/f\|_{W^{m-1,\infty}(\Omega)}$. Thus $\mathcal{N} T_\Omega[(Df/f)\overline{W}]$ is bounded and in particular p-integrable. □

A known result in harmonic analysis [8, 31] is the continuity of the Cauchy operator $F_{\partial\Omega} \colon L^p(\partial\Omega, \mathbb{H}) \to H^p(\Omega)$; that is, for every $\varphi \in L^p(\partial\Omega)$,

$$
\|\mathcal{N} F_{\partial\Omega}[\varphi]\|_{L^p(\partial\Omega)} \lesssim \|F_{\partial\Omega}\| \|\varphi\|_{L^p(\partial\Omega)}, \tag{6.3.11}
$$

where $\lesssim$ means that the left side does not exceed a constant times the right side.

Proposition 6.3.5 *Let Ω be a bounded Lipschitz domain, $m \in \mathbb{Z}^+$, $3/m < p < \infty$, $m \geq 2$, and let $f \in W^{m,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Suppose $W \in \mathfrak{M}_f(\Omega)$. If $W \in W^{m-1,p}(\Omega, \mathbb{H})$, then W belongs to the Vekua-Hardy space $H^p_f(\Omega)$, and*

$$
\|W\|_{H^p_f(\Omega)} \lesssim \|W\|_{W^{m-1,p}(\Omega)}. \tag{6.3.12}
$$

Proof Since the Borel-Pompeiu formula (6.2.7) is valid in $W^{1,p}(\Omega, \mathbb{H})$ [20, Cor. 2.5.4], we have

$$
W - T_\Omega[(Df/f)\overline{W}] = F_{\partial\Omega}[\operatorname{tr} W],
$$

for every $W \in \mathfrak{M}_f(\Omega) \cap W^{m-1,p}(\Omega, \mathbb{H})$. Thus $F_{\partial\Omega}[\operatorname{tr} W] \in H^p(\Omega)$. By (6.3.10)–(6.3.11) and using the fact that $\operatorname{tr} \colon W^{1,p}(\Omega) \to L^p(\partial\Omega)$ is continuous, we have

$$
\begin{aligned}
\|W\|_{H^p_f(\Omega)} &= \|\mathcal{N} W\|_{L^p(\partial\Omega)} \\
&\leq \|\mathcal{N} T_\Omega[(Df/f)\overline{W}] + \mathcal{N} F_{\partial\Omega}[\operatorname{tr} W]\|_{L^p(\partial\Omega)} \\
&\lesssim C_f \|T_\Omega\| \|W\|_{W^{m-1,p}(\Omega)} + \|F_{\partial\Omega}\| \|\operatorname{tr} W\|_{L^p(\partial\Omega)} \\
&\leq C'_f \|W\|_{W^{m-1,p}(\Omega)} < \infty,
\end{aligned}
$$

where $C'_f = C_f \|T_\Omega\| + \|\operatorname{tr}\| \|F_{\partial\Omega}\|$. Therefore (6.3.12) holds and $W \in H^p_f(\Omega)$. □

In particular, for $m = 2$ and $f \in W^{2,\infty}(\Omega, \mathbb{R})$, we have in the range $3/2 < p < \infty$ that

$$W^{1,p}(\Omega, \mathbb{H}) \cap \mathfrak{M}_f(\Omega) \subseteq H_f^p(\Omega).$$

6.4 Boundary Vekua-Hardy Spaces

We will write $\mathrm{tr}_+ W(\vec{x})$ for the non-tangential limit of $W(\vec{y})$ as $\vec{y} \in \Omega$ tends to $\vec{x} \in \partial\Omega$ within $\Gamma_\alpha(\vec{x}) \subseteq \Omega$. When W is in the Sobolev space $W^{1,p}(\Omega, \mathbb{H})$ $(1 < p < \infty)$, where the trace is well defined, tr coincides with the non-tangential limit tr_+ almost everywhere in $\partial\Omega$. Again, the equivalences of [31, Th. 4.1] tell us that $W \in H^p(\Omega)$ if and only if W has non-tangential boundary limit $tr_+ W(\vec{x})$ at almost any point $\vec{x} \in \partial\Omega$.

Proposition 6.4.1 *Let Ω be a bounded Lipschitz domain, $3 < p < \infty$, and let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Let $W \in H_f^p(\Omega) \cap L^p(\Omega, \mathbb{H})$. Then the non-tangential limit* $\mathrm{tr}_+ W(\vec{x})$ *exists for almost every $\vec{x} \in \partial\Omega$.*

Proof By (6.3.9) and the result [31, Th. 4.1] for classical Hardy spaces, we know that the monogenic function $W - T_\Omega[(Df/f)\overline{W}] \in H^p(\Omega)$ has a non-tangential limit for almost every $\vec{x} \in \partial\Omega$. Since $T_\Omega[(Df/f)\overline{W}] \in W^{1,p}(\Omega, \mathbb{H})$, where $\mathrm{tr} = \mathrm{tr}_+$, W also has a non-tangential limit for almost all $\vec{x} \in \partial\Omega$. □

Since $F_{\partial\Omega}[\mathrm{tr}\, T_\Omega[(Df/f)\overline{W}]] = 0$, we have for every $W \in H_f^p(\Omega) \cap L^p(\Omega, \mathbb{H})$ with $3 < p < \infty$,

$$W = T_\Omega[(Df/f)\overline{W}] + F_{\partial\Omega}[\mathrm{tr}_+ W].$$

6.4.1 The Vekua-Hilbert Transform

From this point we only discuss the case $p = 2$. At the beginning of this section we recall results mainly from [14]. The *Hilbert transform* $\mathcal{H}\colon H^{1/2}(\partial\Omega, \mathbb{R}) \to H^{1/2}(\partial\Omega, \mathbb{R}^3)$ for monogenic functions was defined in [14] in the context of Sobolev spaces for bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^3$ with connected complement, following a construction given in [33, 34] for scalar boundary data in Lipschitz domains or in the unit ball in $\mathbb{R}^n$, respectively (see also [2] for a generalization for k-forms on n-dimensional Lipschitz domains). The definition is

$$\mathcal{H}[\varphi_0] = \overrightarrow{K}(I + K_0)^{-1}\varphi_0, \tag{6.4.1}$$

where $K_0 = \mathrm{Sc}\, S_{\partial\Omega}$, $\vec{K} = \mathrm{Vec}\, S_{\partial\Omega}$, and $S_{\partial\Omega}$ is the principal value three-dimensional *singular Cauchy integral operator*

$$S_{\partial\Omega}[\varphi](\vec{x}) = 2\ \mathrm{P.V.}\int_{\partial\Omega} E(\vec{y}-\vec{x})\eta(\vec{y})\varphi(\vec{y})ds_{\vec{y}}, \quad \vec{x} \in \partial\Omega, \tag{6.4.2}$$

applied here only to scalar-valued functions.

The definition of the *Vekua-Hilbert transform*

$$\mathcal{H}_f\colon H^{1/2}(\partial\Omega, \mathbb{R}) \to H^{1/2}(\partial\Omega, \mathbb{R}^3)$$

in [14] requires additionally the Teodorescu transform (6.2.6) as follows. Given $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$, let W_0 be the solution of the conductivity Eq. (6.2.4) satisfying the boundary condition $\mathrm{tr}\, W_0 = f\varphi_0$. By [23, Th. 4.1],

$$\|W_0/f\|_{W^{1,2}(\Omega)} \le C_{\Omega,\rho(f)}\|\varphi_0\|_{H^{1/2}(\partial\Omega)}, \tag{6.4.3}$$

where $C_{\Omega,\rho(f)}$ only depends on Ω and $\rho(f)$. Set $\alpha_0 + \vec{\alpha} = \mathrm{tr}\, T_\Omega\left[-f^2\nabla(W_0/f)\right]$. The Vekua-Hilbert transform is given by

$$\mathcal{H}_f[\varphi_0] = \vec{\alpha} - \mathcal{H}[\alpha_0]. \tag{6.4.4}$$

Full details will be found in [14]. Further, by the Trace Theorem, $\alpha = \alpha_0 + \vec{\alpha} \in H^{1/2}(\partial\Omega, \mathbb{H})$. The characteristic property of $\mathcal{H}_f$, justifying the name of this operator, is expressed in the following result.

Proposition 6.4.2 *[14, Th. 4.3, Prop. 4.8] Let Ω be a bounded Lipschitz domain and let $f \in W^{1,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Suppose that $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. Then the quaternionic function $f\varphi_0 + (1/f)\mathcal{H}_f[\varphi_0]$ is the trace of a solution of the main Vekua equation (6.2.1) and the vector part $\vec{W} \in W^{1,2}(\Omega, \mathbb{R}^3)$ of the extension satisfies*

$$\|f\vec{W}\|_{W^{1,2}(\Omega)} \le C'_{\Omega,\rho(f)}\|\varphi_0\|_{H^{1/2}(\partial\Omega)}, \tag{6.4.5}$$

where $C'_{\Omega,\rho(f)}$ depends on $C_{\Omega,\rho(f)}$ and $\|f\|_{W^{1,\infty}(\Omega)}$.

In the following result we relate the solutions of the main Vekua equation constructed through the Vekua-Hilbert transform $\mathcal{H}_f$ (6.4.4) to elements of the Vekua-Hardy space $H^2_f(\Omega)$:

Proposition 6.4.3 *Let Ω be a bounded Lipschitz domain and let f be a proper conductivity in $W^{2,\infty}(\Omega, \mathbb{R})$. Suppose that $\varphi_0 \in H^{1/2}(\partial\Omega, \mathbb{R})$. Let $W = W_0 + \vec{W} \in$*

$W^{1,2}(\Omega, \mathbb{H})$ *be the solution of* (6.2.1) *such that*

$$\operatorname{tr} W_0 = f\varphi_0,$$
$$\operatorname{tr} f\vec{W} = \mathcal{H}_f[\varphi_0].$$

Then $W \in H^2_f(\Omega)$ *and*

$$\|W\|_{H^2_f(\Omega)} \lesssim \|\varphi_0\|_{H^{1/2}(\partial\Omega)}. \tag{6.4.6}$$

Proof This is a direct consequence of Propositions 6.4.2 and 6.3.5 with $m = 2$ and $p = 2$. The inequality (6.4.6) comes from (6.3.12), from the estimate (6.4.5) for $\vec{f W}$ and from the regularity property of the solutions of the conductivity Eq. (6.4.3). □

A fact that will be useful in the proof of Theorems 6.4.4 and 6.4.6 is that *when* W *and* Z *are solutions of* (6.2.1) *with identical scalar part,* $f(W - Z) = f(\vec{W} - \vec{Z})$ *is left-monogenic*. This can be seen immediately from the equivalent div-curl system (6.2.3) satisfied by the solutions of the main Vekua equation.

Consider the following linear subspace of the Vekua-Hardy space:

$$\widehat{H}^2_f(\Omega) = \{W \in H^2_f(\Omega)\colon\ \operatorname{Sc} W \in W^{1,2}(\Omega, \mathbb{R})\}. \tag{6.4.7}$$

Theorem 6.4.4 *Let* Ω *be a bounded Lipschitz domain and let* $f \in W^{2,\infty}(\Omega, \mathbb{R})$ *be a proper conductivity. Let* $W \in \widehat{H}^2_f(\Omega)$. *Then the non-tangential limit* $\operatorname{tr}_+ W(\vec{x})$ *exists for almost every* $\vec{x} \in \partial\Omega$. *Moreover,*

$$\|W\|_{H^2_f(\Omega)} \lesssim \|\operatorname{tr} W_0\|_{H^{1/2}(\partial\Omega)} + \|\operatorname{tr}_+ \vec{W}\|_{L^2(\partial\Omega)}. \tag{6.4.8}$$

Proof Let $W = W_0 + \vec{W} \in \widehat{H}^2_f(\Omega)$. Then $f(\vec{W} - \vec{Z})$ is left-monogenic, where $Z = W_0 + \vec{Z} \in W^{1,2}(\Omega, \mathbb{H})$ is the solution of (6.2.1) constructed through the Vekua-Hilbert transform $\mathcal{H}_f$; that is, $\mathcal{H}_f[\varphi_0] = \operatorname{tr} f\vec{Z}$ where $\varphi_0 = \operatorname{tr}(W_0/f)$. By Proposition 6.4.3, $Z \in H^2_f(\Omega)$. Since f is a proper conductivity, $f(W - Z) \in H^2(\Omega)$, and using the basic equivalences for monogenic Hardy spaces [31, Th. 4.1], the trace $\operatorname{tr}_+ f(W - Z)$ exists. Therefore $\operatorname{tr}_+ W$ exists, so tr_+ is well defined on $\widehat{H}^2_f(\Omega)$.

Since $f(W - Z) = F_{\partial\Omega}[\operatorname{tr}_+ f(W - Z)]$, by (6.3.11) we have

$$\|\mathcal{N} f(W - Z)\|_{L^2(\partial\Omega)} \lesssim \|F_{\partial\Omega}\| \|\operatorname{tr}_+ f(\vec{W} - \vec{Z})\|_{L^2(\partial\Omega)}. \tag{6.4.9}$$

By (6.4.6), (6.4.9) and the boundedness of $\mathcal{H}_f$ [14, Th. 4.7], we deduce

$$\begin{aligned}\|W\|_{H^2_f(\Omega)} &\lesssim \|W - Z\|_{H^2_f(\Omega)} + \|Z\|_{H^2_f(\Omega)}\\ &\lesssim \|1/f\|_{L^\infty} \|\mathcal{N} f(\vec{W} - \vec{Z})\|_{L^2(\partial\Omega)} + \|\varphi_0\|_{H^{1/2}(\partial\Omega)}\end{aligned}$$

$$\lesssim \|1/f\|_{L^\infty} \|F_{\partial\Omega}\| \Big(\|\operatorname{tr}_+ f\vec{W}\|_{L^2(\partial\Omega)} + \|\mathcal{H}_f[\varphi_0]\|_{L^2(\partial\Omega)} \Big)$$
$$+ \|\varphi_0\|_{H^{1/2}(\partial\Omega)}$$
$$\lesssim \|1/f\|_{L^\infty} \|F_{\partial\Omega}\| \|\operatorname{tr}_+ f\vec{W}\|_{L^2(\partial\Omega)}$$
$$+ \Big(\|1/f\|_{L^\infty} \|F_{\partial\Omega}\| \|\mathcal{H}_f\| + 1 \Big) \|\varphi_0\|_{H^{1/2}(\partial\Omega)}.$$

□

By the proof of Theorem 6.4.4, we see that $f(W - Z) \in H^2(\Omega) \subseteq \mathfrak{M}^2(\Omega) \subseteq L^2(\Omega, \mathbb{H})$ and $Z \in W^{1,2}(\Omega, \mathbb{H}) \subseteq L^2(\Omega, \mathbb{H})$. This gives us the following.

Proposition 6.4.5 $\widehat{H}^2_f(\Omega) \subseteq \widehat{\mathfrak{M}}^2_f(\Omega)$.

Theorem 6.4.4 is stronger than Proposition 6.4.1 for the range of validity of p; actually the main tool in the proof of 6.4.4 is the Vekua-Hilbert transform $\mathcal{H}_f$ for the scalar boundary data as well as its vector extension provided by Proposition 6.4.2. Recall the definition (6.4.7).

Theorem 6.4.6 *Let Ω be a bounded Lipschitz domain and let $f \in W^{2,\infty}(\Omega, \mathbb{R})$ be a proper conductivity. Then*

(a) *$\widehat{H}^2_f(\Omega)$ is closed in $\widehat{L}^2(\Omega, \mathbb{H})$;*
(b) *$\operatorname{tr}_+ \widehat{H}^2_f(\Omega)$ is closed in $\widehat{L}^2(\partial\Omega, \mathbb{H}) = H^{1/2}(\partial\Omega, \mathbb{R}) + L^2(\partial\Omega, \mathbb{R}^3)$;*
(c) *$\big(fI + (1/f)\mathcal{H}_f\big)\, H^{1/2}(\partial\Omega, \mathbb{R}) \subseteq \operatorname{tr}_+ \widehat{H}^2_f(\Omega)$.*

Proof Let $\{W_n\} \subseteq \widehat{H}^2_f(\Omega)$ be a sequence such that $W_n \to W$ in $\widehat{L}^2(\Omega, \mathbb{H})$. By definition, $\{\mathcal{N} W_n\} \subseteq L^2(\partial\Omega, \mathbb{R})$. Let $\{Z_n\} \subseteq \widehat{H}^2_f(\Omega)$ be the sequence constructed through the Vekua-Hilbert transform $\mathcal{H}_f$; that is, $Z_n = W_{0,n} + \vec{Z}_n$, $\operatorname{tr} f\vec{Z}_n = \mathcal{H}_f[\varphi_{0,n}]$ where $\varphi_{0,n} = \operatorname{tr}(W_{0,n}/f)$. Since f is bounded, we have $\{f(W_n - Z_n)\} \subseteq H^2(\Omega)$. Let $\varphi_0 = \operatorname{tr}(W_0/f)$. Then

$$\|f(\varphi_{0,n} - \varphi_0)\|_{H^{1/2}(\partial\Omega)} \le \|\operatorname{tr}\| \, \|W_{0,n} - W_0\|_{W^{1,2}(\Omega)},$$

and $W_{0,n} \to W_0$ in $W^{1,2}(\Omega, \mathbb{R})$, so $f(W_n - Z_n) \to f(W - Z)$, where $Z = W_0 + \vec{Z} \in H^2_f(\Omega)$ is also given by Proposition 6.4.3; i.e., $\operatorname{tr} f\vec{Z} = \mathcal{H}_f[\varphi_0]$. Using that $H^2(\Omega)$ is closed in $L^2(\Omega, \mathbb{H})$, we see that $f(W - Z) \in H^2(\Omega)$. Since $1/f$ is bounded, $W \in \widehat{H}^2_f(\Omega)$.

The proof of part (b) is straightforward from the closedness of $\widehat{H}^2_f(\Omega)$ and by (6.4.8). Part (c) is a consequence of Proposition 6.4.3 and Theorem 6.4.4. □

6.5 Closing Remarks

We have pointed out the close relationship of the spaces $\widehat{\mathfrak{M}}^p_f(\Omega)$ and $\widehat{H}^p_f(\Omega)$ to the three-dimensional Vekua-Hardy spaces, and have shown that they are Banach spaces. Theorem 6.4.4 can be regarded as a Maximum Principle for functions in the Vekua-Hardy subspace $\widehat{H}^p_f(\Omega)$. For the Vekua-Hardy space in complex numbers, the Maximum Principle is straightforward, because in the planar case the Similarity Principle holds, and many properties are thus inherited automatically from the holomorphic Hardy space. Even though we do not know whether there exists a Similarity Principle for the three-dimensional main Vekua equation (6.2.1), we have been able to compensate for this somewhat via the operator (6.3.8) and the construction of f^2-hyperconjugates through the Vekua-Hilbert transform $\mathcal{H}_f$, which has made possible results such as (6.4.8). A full theory could depend on the development of the ideas of pseudoanalytic function theory for the Vekua equation, and in particular any result implying that the operation of evaluation at a point is a continuous operation in the L^p norm. Currently we do not even know whether $H^p_f(\Omega) \subseteq \mathfrak{M}^p_f(\Omega)$.

In [14], the Vekua-Hilbert transform $\mathcal{H}_f$ associated to the main Vekua equation (6.2.1) was introduced with the aid of a explicit solution of the div-curl system in bounded Lipschitz domains. One can also consider the somewhat more general Vekua equation

$$DW = (Dg/g)W + (Df/f)\overline{W} \tag{6.5.1}$$

with proper conductivities $f, g \in W^{1,\infty}(\Omega, \mathbb{R})$. This is equivalent to $U = W_0/(fg) + (f/g)\vec{W}$ satisfying the "quaternionic Beltrami equation"

$$DU = \frac{1-f^2}{1+f^2} D\overline{U}. \tag{6.5.2}$$

Thus W satisfies (6.5.1) if and only if W_0 and $\vec{W}$ satisfy

$$\operatorname{div}\left(\frac{f}{g}\vec{W}\right) = 0, \quad \operatorname{curl}\left(\frac{f}{g}\vec{W}\right) = -f^2\nabla\left(\frac{W_0}{fg}\right). \tag{6.5.3}$$

This is again a homogeneous div-curl system, and when $g \equiv 1$, (6.5.3) reduces to (6.2.3). The construction of the Vekua-Hilbert transform [14] and the results of this work on the Vekua-Hardy spaces can readily be generalized for this more general context.

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Chapter 7
Radial and Angular Derivatives of Distributions

Fred Brackx

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract When expressing a distribution in Euclidean space in spherical co-ordinates, derivation with respect to the radial and angular co-ordinates is far from trivial. Exploring the possibilities of defining a radial derivative of the delta distribution $\delta(\underline{x})$ (the angular derivatives of $\delta(\underline{x})$ being zero since the delta distribution is itself radial) led to the introduction of a new kind of distributions, the so-called *signumdistributions*, as continuous linear functionals on a space of test functions showing a singularity at the origin. In this paper we search for a definition of the radial and angular derivatives of a general standard distribution and again, as expected, we are inevitably led to consider signumdistributions. Although these signumdistributions provide an adequate framework for the actions on distributions aimed at, it turns out that the derivation with respect to the radial distance of a general (signum)distribution is still not yet unambiguous.

Keywords Distribution · Radial derivative · Angular derivative · Signumdistribution

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7.1 Introduction

Let us consider a scalar-valued distribution $T(\underline{x}) \in \mathcal{D}'(\mathbb{R}^m)$ expressed in terms of spherical co-ordinates: $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} = \sum_{j=1}^m e_j\,\omega_j \in \mathbb{S}^{m-1}$, $(e_1, e_2, \ldots, e_m)$ being an orthonormal basis of $\mathbb{R}^m$ and $\mathbb{S}^{m-1}$ being the unit sphere in $\mathbb{R}^m$. The aim of this paper is to search for an adequate definition of the radial and angular derivatives $\partial_r\, T$ and $\partial_{\omega_j}\, T,\ j = 1, \ldots, m$. This problem was treated in [2] for the special and interesting case of the delta distribution $\delta(\underline{x})$, the following spherical co-ordinates expression of which is often encountered in physics texts:

$$\delta(\underline{x}) = \frac{1}{a_m}\,\frac{\delta(r)}{r^{m-1}} \tag{7.1.1}$$

where $a_m = \frac{2\pi^{m/2}}{\Gamma(m/2)}$ is the area of the unit sphere $\mathbb{S}^{m-1}$ in $\mathbb{R}^m$. Apparently expression (7.1.1) can mathematically be explained in the following way. Write the action of the delta distribution as an integral:

$$\begin{aligned}
\varphi(0) = \langle\, \delta(\underline{x}), \varphi(\underline{x})\,\rangle &= \int_{\mathbb{R}^m} \delta(\underline{x})\,\varphi(\underline{x})\,dV(\underline{x}) \\
&= \int_0^\infty r^{m-1}\delta(\underline{x})\,dr \int_{\mathbb{S}^{m-1}} \varphi(r\,\underline{\omega})\,dS_{\underline{\omega}} \\
&= a_m \int_0^\infty r^{m-1}\,\delta(\underline{x})\,\Sigma^0[\varphi](r)\,dr
\end{aligned}$$

introducing the so-called *spherical mean* of the test function φ given by

$$\Sigma^0[\varphi](r) = \frac{1}{a_m}\int_{\mathbb{S}^{m-1}} \varphi(r\,\underline{\omega})\,dS_{\underline{\omega}}.$$

As it is easily seen that $\Sigma^0[\varphi](0) = \varphi(0)$, it follows that

$$a_m \int_0^\infty r^{m-1}\,\delta(\underline{x})\,\Sigma^0[\varphi](r)\,dr = \int_0^\infty \delta(r)\,\Sigma^0[\varphi](r)\,dr = \langle\,\delta(r), \Sigma^0[\varphi](r)\,\rangle$$

which explains (7.1.1). However we prefer to interpret this expression as

$$\varphi(0) = \langle\,\delta(\underline{x}), \varphi(\underline{x})\,\rangle = \langle\,\delta(r), \Sigma^0[\varphi](r)\,\rangle = \Sigma^0[\varphi](0). \tag{7.1.2}$$

Straightforward successive derivation with respect to r of (7.1.1) leads to

$$\partial_r^{2\ell}\,\delta(\underline{x}) = \frac{1}{(2\ell)!}(m)(m+1)\cdots(m+2\ell-1)\,\frac{1}{a_m}\,\frac{\delta^{(2\ell)}(r)}{r^{m-1}}, \tag{7.1.3}$$

$$\partial_r^{2\ell+1}\,\delta(\underline{x}) = \frac{1}{(2\ell+1)!}(m)(m+1)\cdots(m+2\ell)\,\frac{1}{a_m}\,\frac{\delta^{(2\ell+1)}(r)}{r^{m-1}}. \quad (7.1.4)$$

Expression (7.1.3) then is interpreted as

$$\langle\, \partial_r^{2\ell}\,\delta(\underline{x}), \varphi(\underline{x})\,\rangle = \frac{1}{(2\ell)!}(m)(m+1)\cdots(m+2\ell-1)\,\langle\, \delta^{(2\ell)}(r), \Sigma^0[\varphi](r)\,\rangle$$

which is meaningful and which can serve as the definition of the even order derivatives with respect to r of the delta distribution. However expression (7.1.4) makes no sense since the spherical mean $\Sigma^0[\varphi](r)$ is an even function of r, whence its odd order derivatives vanish at the origin:

$$\langle -\partial_r^{2\ell+1}\,\delta(r), \Sigma^0[\varphi](r)\,\rangle = \{\partial_r^{2\ell+1}\,\Sigma^0[\varphi](r)\}|_{r=0} = 0.$$

How to explain this fact that, proceeding stepwise by derivation with respect to r, the even order derivatives of $\delta(\underline{x})$ apparently make sense, while its odd order derivatives are zero distributions, in this way violating the basic requirement of any derivation procedure that $\partial_r\,\partial_r$ should equal ∂_r^2? Let us to that end have a quick look at the functional analytic background of this phenomenon; for a more systematic treatment we refer to [2].

When expressing a scalar-valued test function $\varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m)$ in spherical co-ordinates, one obtains a function $\widetilde{\varphi}(r,\underline{\omega}) = \varphi(r\underline{\omega}) \in \mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1})$, but it is evident that not all functions $\widetilde{\varphi}(r,\underline{\omega}) \in \mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1})$ stem from a test function in $\mathcal{D}(\mathbb{R}^m)$. However a one-to-one correspondence may be established between the usual space of test functions $\mathcal{D}(\mathbb{R}^m)$ and a specific subspace of $\mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1})$.

Lemma 7.1.1 (See [5]) *There is a one-to-one correspondence* $\varphi(\underline{x}) \leftrightarrow \widetilde{\varphi}(r,\underline{\omega}) = \varphi(r\underline{\omega})$ *between the spaces* $\mathcal{D}(\mathbb{R}^m)$ *and* $\mathcal{V} = \{\phi(r,\underline{\omega}) \in \mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1}) : \phi$ *is even, i.e.* $\phi(-r,-\underline{\omega}) = \phi(r,\underline{\omega})$, *and* $\{\partial_r^n\,\phi(r,\underline{\omega})\}|_{r=0}$ *is a homogeneous polynomial of degree* n *in* $(\omega_1,\ldots,\omega_m)$, $\forall n \in \mathbb{N}\}$.

Clearly $\mathcal{V}$ is a closed (but not dense) subspace of $\mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1})$ and even of $\mathcal{D}_E(\mathbb{R}\times\mathbb{S}^{m-1})$, where the subscript E refers to the even character of the test functions in that space; this space $\mathcal{V}$ is endowed with the induced topology of $\mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1})$. The one-to-one correspondence between the spaces of test functions $\mathcal{D}(\mathbb{R}^m)$ and $\mathcal{V}$ translates into a one-to-one correspondence between the standard distributions $T \in \mathcal{D}'(\mathbb{R}^m)$ and the bounded linear functionals in $\mathcal{V}'$, this correspondence being given by

$$\langle\, T(\underline{x}), \varphi(\underline{x})\,\rangle = \langle\, \widetilde{T}(r,\underline{\omega}), \widetilde{\varphi}(r,\underline{\omega})\,\rangle.$$

By Hahn-Banach's theorem the bounded linear functional $\widetilde{T}(r,\underline{\omega}) \in \mathcal{V}'$ may be extended to the distribution $\mathbb{T}(r,\underline{\omega}) \in \mathcal{D}'(\mathbb{R}\times\mathbb{S}^{m-1})$; such an extension is called a *spherical representation* of the distribution T (see e.g. [9]). However as the subspace $\mathcal{V}$ is not dense in $\mathcal{D}(\mathbb{R}\times\mathbb{S}^{m-1})$, the spherical representation of a distribution is *not* *unique*, but if $\mathbb{T}_1$ and $\mathbb{T}_2$ are two different spherical representations of the same distribution T, their restrictions to $\mathcal{V}$ coincide:

$$\langle\, \mathbb{T}_1(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega})\,\rangle = \langle\, \mathbb{T}_2(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega})\,\rangle = \langle\, \widetilde{T}(r, \underline{\omega}), \varphi(r\underline{\omega})\,\rangle = \langle\, T(\underline{x}), \varphi(\underline{x})\,\rangle.$$

For test functions in $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$ the spherical variables r and $\underline{\omega}$ are ordinary variables, and thus smooth functions. It follows that for distributions in $\mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1})$ multiplication by r and ω_j, $j = 1, \ldots, m$ and differentiation with respect to r and ω_j, $j = 1, \ldots, m$ are well-defined standard operations, whence

$$\langle\, \partial_r\, \mathbb{T}(r, \underline{\omega}),\, \Xi(r, \underline{\omega})\,\rangle = -\,\langle\, \mathbb{T}(r, \underline{\omega}),\, \partial_r\, \Xi(r, \underline{\omega})\,\rangle \tag{7.1.5}$$

for all test functions $\Xi(r, \underline{\omega}) \in \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1})$, and similar expressions for $\partial_{\omega_j}\mathbb{T}$, $r\,\mathbb{T}$ and $\underline{\omega}\,\mathbb{T}$. However if $\mathbb{T}_1$ and $\mathbb{T}_2$ are two different spherical representations of the same distribution $T \in \mathcal{D}'(\mathbb{R}^m)$, then, upon restriction to test functions $\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{V}$, we are stuck with

$$-\,\langle\, \mathbb{T}_1(r, \underline{\omega}),\, \partial_r\, \widetilde{\varphi}(r, \underline{\omega})\,\rangle \neq -\,\langle\, \mathbb{T}_2(r, \underline{\omega}),\, \partial_r\, \widetilde{\varphi}(r, \underline{\omega})\,\rangle$$

because $\partial_r\, \widetilde{\varphi}(r, \underline{\omega})$ does no longer belong to $\mathcal{V}$ (and neither do $\partial_{\omega_j}\, \widetilde{\varphi}(r, \underline{\omega})$, $r\, \widetilde{\varphi}(r, \underline{\omega})$ and $\underline{\omega}\, \widetilde{\varphi}(r, \underline{\omega})$) since it is an odd function in the variables $(r, \underline{\omega})$. And it is also clear that the action (7.1.5) might be unambiguously restricted to testfunctions in $\mathcal{V}$ if the test function Ξ were in a subspace of $\mathcal{D}(\mathbb{R} \times S^{m-1})$ consisting of odd functions. The conclusion is that the concept of spherical representation of a distribution does not allow for an unambiguous definition of the actions proposed. What is more, it becomes apparent that there is a need for a subspace of odd test functions. And at the same time it becomes clear why even order derivatives with respect to r of the delta distribution and of a standard distribution in general, are well-defined instead. Indeed, we have e.g.

$$\langle\, \partial_r^{2\ell}\, \mathbb{T}(r, \underline{\omega}),\, \Xi(r, \underline{\omega})\,\rangle = \langle\, \mathbb{T}(r, \underline{\omega}),\, \partial_r^{2\ell}\, \Xi(r, \underline{\omega})\,\rangle$$

where now $\partial_r^{2\ell}\, \Xi(r, \underline{\omega})$ belongs to $\mathcal{D}_E(\mathbb{R} \times \mathbb{S}^{m-1})$ which enables restriction to test functions in $\mathcal{V}$ in an unambiguous way.

7.2 Preliminaries

In this paper vectors in $\mathbb{R}^m$ will be interpreted as Clifford 1-vectors in the Clifford algebra $\mathbb{R}_{0,m}$, where the basis vectors $(e_j, e_2, \ldots, e_m)$ of $\mathbb{R}^m$, satisfy the relations $e_j^2 = -1$, $e_i \wedge e_j = e_i e_j = -e_j e_i = -e_j \wedge e_i$, $e_i \cdot e_j = 0, i \neq j = 1, \ldots, m$. This allows for the use of the very efficient *geometric* or *Clifford product* of Clifford vectors:

$$\underline{x}\,\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

for which, in particular,

$$\underline{x}\,\underline{x} = \underline{x} \cdot \underline{x} = -\,|\underline{x}|^2$$

$\underline{x}$ being the Clifford 1-vector $\underline{x} = \sum_{j=1}^{m} e_j\, x_j$, whence also

$$\underline{\omega}\,\underline{\omega} = \underline{\omega} \cdot \underline{\omega} = -\,|\underline{\omega}|^2 = -1.$$

For more on Clifford algebras we refer to e.g. [6].

The Dirac operator $\underline{\partial} = \sum_{j=1}^{m} e_j\, \partial_{x_j}$, which may be seen as a Stein-Weiss projection of the gradient operator (see e.g. [8]) and which underlies the higher dimensional theory of monogenic functions (see e.g. [3, 4]), linearizes the Laplace operator: $\underline{\partial}^2 = -\,\Delta$. Its action on a scalar-valued standard distribution $T(\underline{x})$ results into the vector-valued distribution $\underline{\partial}\, T(\underline{x})$ given for all $\varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m)$ by

$$\begin{aligned} \langle\, \underline{\partial}\, T(\underline{x})\,,\, \varphi(\underline{x})\,\rangle &= \sum_{j=1}^{m} e_j\, \langle\, \partial_{x_j}\, T(\underline{x})\,,\, \varphi(\underline{x})\,\rangle = -\sum_{j=1}^{m} e_j\, \langle\, T(\underline{x})\,,\, \partial_{x_j}\, \varphi(\underline{x})\,\rangle \\ &= -\,\langle\, T(\underline{x})\,,\, \underline{\partial}\, \varphi(\underline{x})\,\rangle \end{aligned}$$

which is a meaningful operation since only derivatives with respect to the cartesian co-ordinates are involved.

Two fundamental formulae in monogenic function theory are

$$\{\underline{x}, \underline{\partial}\} = \underline{x}\,\underline{\partial} + \underline{\partial}\,\underline{x} = -\,2\mathbb{E} - m \qquad \text{and} \qquad [\underline{x}, \underline{\partial}] = \underline{x}\,\underline{\partial} - \underline{\partial}\,\underline{x} = m - 2\Gamma$$

where

$$\mathbb{E} = \sum_{j=1}^{m} x_j\, \partial_{x_j}$$

is the scalar Euler operator, and

$$\Gamma = \sum_{j<k} e_j e_k\, L_{jk} = \sum_{j<k} e_j e_k (x_j \partial x_k - x_k \partial_{x_j})$$

is the bivector angular momentum operator. It follows that

$$\underline{x}\,\underline{\partial} = -\mathbb{E} - \Gamma$$

or more precisely

$$\underline{x} \cdot \underline{\partial} = -\mathbb{E} \qquad \text{and} \qquad \underline{x} \wedge \underline{\partial} = -\Gamma.$$

7.3 Signumdistributions

As already observed in the introduction, $\underline{\omega}$ is an ordinary (vector) variable in $\mathbb{R} \times \mathbb{S}^{m-1}$, whence it makes sense to consider the following subspace of vector-valued test functions in $\mathbb{R} \times \mathbb{S}^{m-1}$:

$$\mathcal{W} = \underline{\omega}\,\mathcal{V} \subset \mathcal{D}_O(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m) \subset \mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$$

where now the subscript O refers to the odd character of the test functions under consideration, i.e. $\psi(-r, -\underline{\omega}) = -\,\psi(r, \underline{\omega}), \forall \psi \in \mathcal{D}_O(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$. This space $\mathcal{W}$ is endowed with the induced topology of $\mathcal{D}(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$. By definition there is a one-to-one correspondence between the spaces $\mathcal{V}$ and $\mathcal{W}$.

For each $\mathbb{U}(r, \underline{\omega}) \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1}; \mathbb{R}^m)$ we define $\widetilde{U}(r, \underline{\omega}) \in \mathcal{W}'$ by the restriction

$$\langle\, \widetilde{U}(r, \underline{\omega}), \underline{\omega}\,\widetilde{\varphi}(r, \underline{\omega})\,\rangle = \langle\, \mathbb{U}(r, \underline{\omega}), \underline{\omega}\,\widetilde{\varphi}(r, \underline{\omega})\,\rangle, \quad \forall\, \underline{\omega}\,\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{W}.$$

In $\mathbb{R}^m$ we consider the space $\Omega(\mathbb{R}^m; \mathbb{R}^m) = \{\underline{\omega}\,\varphi(\underline{x}) : \varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m)\}$. Clearly the functions in $\Omega(\mathbb{R}^m; \mathbb{R}^m)$ are no longer differentiable in the whole of $\mathbb{R}^m$, since they are not defined at the origin due to the function $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$. By definition there is a one-to-one correspondence between the spaces $\mathcal{D}(\mathbb{R}^m)$ and $\Omega(\mathbb{R}^m; \mathbb{R}^m)$.

For each $\widetilde{U}(r, \underline{\omega}) \in \mathcal{W}'$ we define ${}^sU(\underline{x})$ by

$$\langle\, {}^sU(\underline{x}), \underline{\omega}\,\varphi(\underline{x})\,\rangle = \langle\, \widetilde{U}(r, \underline{\omega}), \underline{\omega}\,\widetilde{\varphi}(r, \underline{\omega}))\,\rangle, \quad \forall\, \underline{\omega}\,\varphi(\underline{x}) \in \Omega(\mathbb{R}^m; \mathbb{R}^m).$$

Clearly ${}^sU(\underline{x})$ is a bounded linear functional on $\Omega(\mathbb{R}^m; \mathbb{R}^m)$, for which, in [2], we coined the term *signumdistribution*.

Now start with a standard distribution $T(\underline{x}) \in \mathcal{D}'(\mathbb{R}^m)$ and let $\mathbb{T}(r, \underline{\omega}) \in \mathcal{D}'(\mathbb{R} \times \mathbb{S}^{m-1})$ be one of its spherical representations. Put $\mathbb{S}(r, \underline{\omega}) = \underline{\omega}\,\mathbb{T}(r, \underline{\omega})$ which in its turn leads to the signumdistribution ${}^sS(\underline{x}) \in \Omega'(\mathbb{R}^m; \mathbb{R}^m)$. Then we consecutively have

$$\begin{aligned}\langle\, {}^sS(\underline{x}), \underline{\omega}\,\varphi(\underline{x})\,\rangle &= \langle\, \mathbb{S}(r, \underline{\omega}), \underline{\omega}\,\widetilde{\varphi}(r, \underline{\omega})\,\rangle = \langle\, \underline{\omega}\,\mathbb{T}(r, \underline{\omega}), \underline{\omega}\,\widetilde{\varphi}(r, \underline{\omega})\,\rangle \\ &= -\,\langle\, \mathbb{T}(r, \underline{\omega}), \widetilde{\varphi}(r, \underline{\omega})\,\rangle = -\,\langle\, T(\underline{x}), \varphi(\underline{x})\,\rangle\end{aligned}$$

since $\underline{\omega}^2 = -1$. We call ${}^sS(\underline{x})$ a signumdistribution associated to the distribution $T(\underline{x})$ and denote it by $T^\vee(\underline{x})$. It thus holds that for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^m)$

$$\langle\, T^\vee(\underline{x}), \underline{\omega}\,\varphi(\underline{x})\,\rangle = -\,\langle\, T(\underline{x}), \varphi(\underline{x})\,\rangle. \tag{7.3.1}$$

At the same time we call $T(\underline{x})$ the distribution associated to the signumdistribution ${}^sS(\underline{x})$ and we denote this distribution by ${}^sS^\wedge(\underline{x})$. Formula (7.3.1) then also reads

$$\langle\, {}^sS(\underline{x}), \underline{\omega}\,\varphi(\underline{x})\,\rangle = -\,\langle\, {}^sS^\wedge(\underline{x}), \varphi(\underline{x})\,\rangle \tag{7.3.2}$$

and it is clear that

$$T^{\vee\wedge} = T \qquad \text{and} \qquad {}^sS^{\wedge\vee} = {}^sS.$$

At first sight for a given distribution $T(\underline{x})$ the associated signumdistribution $T^{\vee}(\underline{x})$ is not uniquely defined since its construction involves the not uniquely defined spherical representation $\mathbb{T}$ of $T(\underline{x})$. Nevertheless it follows from (7.3.1) that for a given distribution $T(\underline{x})$ its associated signumdistribution $T^{\vee}(\underline{x})$ is unique, what can also be proven directly as follows.

Proposition 7.3.1 *Given the distribution $T(\underline{x})$ its associated signumdistribution $T^{\vee}(\underline{x})$ is uniquely determined.*

Proof Assume that $\mathbb{T}_1$ and $\mathbb{T}_2$ are two different spherical representations of T, i.e. for all test functions $\Xi(r, \underline{\omega}) \in \mathcal{D}(\mathbb{R} \times S^{m-1}; \mathbb{R}^m)$ it holds that

$$\langle\, \mathbb{T}_1, \Xi(r, \underline{\omega}) \,\rangle \;\neq\; \langle\, \mathbb{T}_2, \Xi(r, \underline{\omega}) \,\rangle$$

while for all test functions $\widetilde{\varphi}(r, \underline{\omega}) \in \mathcal{V}$ it holds that

$$\langle\, \mathbb{T}_1, \widetilde{\varphi}(r, \underline{\omega}) \,\rangle \;=\; \langle\, \mathbb{T}_2, \widetilde{\varphi}(r, \underline{\omega}) \,\rangle \;=\; \langle\, \widetilde{T}, \widetilde{\varphi}(r, \underline{\omega}) \,\rangle.$$

Let $T_1^{\vee}$ and $T_2^{\vee}$ be the associated signumdistributions to T through the spherical representations $\mathbb{T}_1$ and $\mathbb{T}_2$ respectively. Then for $j = 1, 2$ it holds that

$$\langle\, T_j^{\vee}, \underline{\omega}\, \varphi(\underline{x}) \,\rangle \;=\; \langle\, \mathbb{T}_j, \widetilde{\varphi}(r, \underline{\omega}) \,\rangle$$

whence $T_1^{\vee} = T_2^{\vee}$ on $\Omega(\mathbb{R}^m; \mathbb{R}^m)$. □

Conversely, for a given signumdistribution ${}^sU \in \Omega'(\mathbb{R}^m; \mathbb{R}^m)$ we define the associated distribution ${}^sU^{\wedge}$ by

$$\langle\, {}^sU^{\wedge}(\underline{x}), \varphi(\underline{x}) \,\rangle = -\langle\, {}^sU(\underline{x}), \underline{\omega}\, \varphi(\underline{x}) \,\rangle \qquad \forall \varphi(\underline{x}) \in \mathcal{D}(\mathbb{R}^m).$$

Clearly it holds that

$$T^{\vee\wedge} = T \qquad \text{and} \qquad {}^sU^{\wedge\vee} = {}^sU.$$

Example As an example consider the distribution $T(\underline{x}) = \delta(\underline{x})$. Our aim is to define the signumdistribution $\delta^{\vee}(\underline{x})$. A spherical representation of the delta distribution is given by

$$\langle\, \mathbb{T}(r, \omega), \Xi(r, \underline{\omega}) \,\rangle = \Sigma^0[\Xi(r, \underline{\omega})]\}|_{r=0}.$$

Indeed, when restricting to the space $\mathcal{V}$ and taking into account property (7.1.2), we obtain

$$\langle\, \mathbb{T}(r,\omega), \widetilde{\varphi}(r,\underline{\omega})\,\rangle = \Sigma^0[\varphi(r\,\underline{\omega})]\}|_{r=0} = \langle\, \delta(\underline{x}), \varphi(\underline{x})\,\rangle.$$

This particular spherical representation of $T(\underline{x})$ induces the signumdistribution associated to $\delta(\underline{x})$, which we define to be $\delta^{\vee}(\underline{x})$. It thus holds that for all test functions $\varphi \in \mathcal{D}(\mathbb{R}^m)$

$$\langle\, \delta^{\vee}(\underline{x}), \underline{\omega}\,\varphi(\underline{x})\,\rangle = -\,\langle\, \delta(\underline{x}), \varphi(\underline{x})\,\rangle. \qquad (7.3.3)$$

For further examples we refer to [2].

7.4 The Dirac Operator in Spherical Co-ordinates

Passing to spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} = \sum_{j=1}^{m} e_j\,\omega_j \in \mathbb{S}^{m-1}$, the Dirac operator takes the form

$$\underline{\partial} = \underline{\partial}_{rad} + \underline{\partial}_{ang}$$

with

$$\underline{\partial}_{rad} = \underline{\omega}\,\partial_r \qquad \text{and} \qquad \underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}}.$$

To give an idea what the angular differential operator $\partial_{\underline{\omega}} = \sum_{j=1}^{m} e_j\,\partial_{\omega_j}$ looks like, let us mention its explicit form in dimension $m = 2$:

$$\partial_{\underline{\omega}} = e_\theta\,\partial_\theta$$

and in dimension $m = 3$:

$$\partial_{\underline{\omega}} = e_\theta\,\partial_\theta + e_\varphi\,\frac{1}{\sin\theta}\,\partial_\varphi,$$

the meaning of the polar co-ordinates θ and φ being straightforward. The operator $\partial_{\underline{\omega}}$ is sometimes called the *spherical Dirac operator*.

Taking into account that $\partial_{\underline{\omega}}$ is orthogonal to $\underline{\omega}$, the Euler operator in spherical co-ordinates then reads:

$$\mathbb{E} = -\,\underline{x}\cdot\underline{\partial} = -\,r\underline{\omega}\cdot\underline{\partial}_{rad} = -\,r\underline{\omega}\cdot\underline{\omega}\,\partial_r = r\,\partial_r$$

while the angular momentum operator Γ takes the form

$$\Gamma = -\underline{x} \wedge \underline{\partial} = -r\underline{\omega} \wedge \underline{\partial}_{ang} = -r\underline{\omega} \wedge \frac{1}{r}\,\partial_{\underline{\omega}} = -\underline{\omega} \wedge \partial_{\underline{\omega}} = -\underline{\omega}\,\partial_{\underline{\omega}}.$$

The question now is how to define, if possible, the action of the operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a standard distribution. To that end both operators should be expressed in terms of cartesian derivatives. This is achieved as follows.

Definition 7.4.1 The actions of the operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a distribution T are given by

$$\underline{\partial}_{rad}\, T = \underline{\omega}\,\partial_r\, T = -\frac{1}{\underline{x}}\,\mathbb{E}\, T$$

and

$$\underline{\partial}_{ang}\, T = \frac{1}{r}\,\partial_{\underline{\omega}}\, T = -\frac{1}{\underline{x}}\,\Gamma\, T.$$

It becomes clear at once that, in this way, the actions of $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ on a standard distribution $T(\underline{x})$ are well-defined but not uniquely defined. Indeed, due to the division by the analytic function $\underline{x}$, both expressions

$$\underline{\partial}_{rad}\, T(\underline{x}) = \underline{\omega}\,\partial_r\, T(r\underline{\omega}) = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\, T(\underline{x})\right] \tag{7.4.1}$$

and

$$\underline{\partial}_{ang}\, T(\underline{x}) = \frac{1}{r}\,\partial_{\underline{\omega}}\, T(r\underline{\omega}) = -\left[\frac{1}{\underline{x}}\,\Gamma\, T(\underline{x})\right] \tag{7.4.2}$$

represent equivalent classes of distributions each two of which differ by a vector multiple of the delta distribution $\delta(\underline{x})$. However if $S_1 = \underline{\partial}_{rad}\, T(\underline{x})$ and $S_2 = \underline{\partial}_{ang}\, T(\underline{x})$ are distributions arbitrarily chosen in the equivalent classes (7.4.1) and (7.4.2) respectively, i.e.

$$\underline{x}\, S_1 = -\,\mathbb{E}\, T(\underline{x}) \qquad \text{and} \qquad \underline{x}\, S_2 = -\,\Gamma T(\underline{x})$$

this choice is not completely arbitrary since S_1 and S_2 always must satisfy the relation

$$S_1 + S_2 = \underline{\partial}_{rad}\, T(\underline{x}) + \underline{\partial}_{ang}\, T(\underline{x}) = \underline{\partial}\, T(\underline{x}) \tag{7.4.3}$$

where the right-hand side, quite naturally, is a known distribution once the distribution T has been given. One could say that the differential operators $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$

are *entangled* in the sense that the results of their actions on a distribution are subject to (7.4.3).

Example Let us give a simple example to illustrate the above phenomenon. Consider the regular distribution $T(\underline{x}) = \underline{x}$. Then $\underline{\partial}\, \underline{x} = -m$, $\mathbb{E}\, \underline{x} = \underline{x}$ and $\Gamma\, \underline{x} = (m-1)\, \underline{x}$, whence

$$(\underline{\omega}\, \partial_r)\, \underline{x} = -1 + \underline{c}_1\, \delta(\underline{x}) \qquad \text{and} \qquad \left(\frac{1}{r}\, \partial_{\underline{\omega}}\right) \underline{x} = 1 - m + \underline{c}_2\, \delta(\underline{x})$$

with the restriction that the vector constants $\underline{c}_1$ and $\underline{c}_2$ always must satisfy the entanglement condition $\underline{c}_1 + \underline{c}_2 = 0$.

Apparently there seems to be no possibility to uniquely define the actions of the $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ operators on a standard distribution by singling out specific distributions in the equivalent classes (7.4.1) and (7.4.2), except for the following two special cases.

(i) If the distribution $T(\underline{x})$ is *radial*, i.e. only depends on $r = |\underline{x}|$, then we put $\frac{1}{r}\, \partial_{\underline{\omega}}\, T = 0$ and $\underline{\omega}\, \partial_r\, T = \underline{\partial}\, T$. This first special case is illustrated by the delta distribution (see also [2]): $\frac{1}{r}\, \partial_{\underline{\omega}}\, \delta(\underline{x}) = 0$ and $\underline{\omega}\, \partial_r\, \delta(\underline{x}) = \underline{\partial}\, \delta(\underline{x})$.

(ii) If the distribution $T(\underline{x})$ is *angular*, i.e. only depends on $\underline{\omega} = \frac{\underline{\omega}}{|\underline{x}|}$, then we put $\underline{\omega}\, \partial_r\, T = 0$ and $\frac{1}{r}\, \partial_{\underline{\omega}}\, T = \underline{\partial}\, T$. This second special case is illustrated by the regular distribution $\underline{\omega}$ for which $\underline{\omega}\, \partial_r\, \underline{\omega} = 0$ and $\frac{1}{r}\, \partial_{\underline{\omega}}\, \underline{\omega} = \underline{\partial}\, \underline{\omega} = -(m-1)\frac{1}{r}$.

In Sect. 7.6 we will expose two other cases where the actions of the $\underline{\partial}_{rad}$ and $\underline{\partial}_{ang}$ operators are uniquely defined.

7.5 The Laplace Operator in Spherical Co-ordinates

As was already observed in Sect. 7.2, the Dirac operator factorizes the Laplace operator: $-\Delta = \underline{\partial}^2$. As the Laplace operator is a scalar operator it holds that

$$\Delta = -\underline{\partial} \cdot \underline{\partial} = |\underline{\partial}|^2.$$

Passing to spherical co-ordinates we obtain, in view of

$$\begin{aligned}
\underline{\partial}_{rad}\, \underline{\partial}_{rad} &= -\, \partial_r^2 \\
\underline{\partial}_{rad}\, \underline{\partial}_{ang} &= -\frac{1}{r^2}\, \underline{\omega}\, \partial_{\underline{\omega}} + \frac{1}{r}\, \underline{\omega}\, \partial_{\underline{\omega}}\, \partial_r \\
\underline{\partial}_{ang}\, \underline{\partial}_{rad} &= -\,(m-1)\frac{1}{r}\, \partial_r - \frac{1}{r}\, \partial_r\, \underline{\omega}\, \partial_{\underline{\omega}} \\
\underline{\partial}_{ang}\, \underline{\partial}_{ang} &= \frac{1}{r^2}\, \partial_{\underline{\omega}}^2
\end{aligned}$$

the following expression for the Laplace operator:

$$\begin{aligned}\Delta &= -\,(\underline{\partial}_{rad} + \underline{\partial}_{ang})^2 \\ &= \partial_r^2 + (m-1)\frac{1}{r}\,\partial_r + \frac{1}{r^2}\,(\underline{\omega}\,\partial_{\underline{\omega}} - \partial_{\underline{\omega}}^2) \\ &= \partial_r^2 + (m-1)\frac{1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta^*\end{aligned}$$

where

$$\Delta^* = \underline{\omega}\,\partial_{\underline{\omega}} - \partial_{\underline{\omega}}^2$$

is the Laplace-Beltrami operator, sometimes denoted by Δ_0. The Laplace-Beltrami operator is a purely angular scalar operator; as $\underline{\omega}\,\partial_{\underline{\omega}} = -\,\Gamma$ is a bivector operator, it follows that

$$\Delta^* = -\,\partial_{\underline{\omega}} \cdot \partial_{\underline{\omega}} = |\partial_{\underline{\omega}}|^2 \qquad \text{and} \qquad \underline{\omega}\,\partial_{\underline{\omega}} = \partial_{\underline{\omega}} \wedge \partial_{\underline{\omega}} = -\,\Gamma.$$

It is a nice observation that while the Laplace operator Δ is the normsquared of the Dirac operator, the spherical Laplace or Laplace-Beltrami operator is the normsquared of the spherical Dirac operator.

As is the case for the Laplace operator $\Delta = \sum_{j=1}^{m} \partial_{x_j}^2$, also the Laplace-Beltrami operator may be expressed in terms of derivatives with respect to the cartesian co-ordinates.

Proposition 7.5.1 *The angular differential operators $\partial_{\underline{\omega}}^2$ and Δ^* may be written in terms of cartesian co-ordinates as*

$$\partial_{\underline{\omega}}^2 = \Gamma^2 - (m-1)\,\Gamma$$

and

$$\Delta^* = (m-2)\,\Gamma - \Gamma^2.$$

Proof One has

$$\begin{aligned}\Gamma^2 &= (-\,\underline{\omega}\,\partial_{\underline{\omega}})^2 = \underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega}\,\partial_{\underline{\omega}} \\ &= \underline{\omega}\,((1-m) - \underline{\omega}\,\partial_{\underline{\omega}})\partial_{\underline{\omega}} \\ &= (1-m)\,\underline{\omega}\,\partial_{\underline{\omega}} + \partial_{\underline{\omega}}^2 \\ &= (m-1)\,\Gamma + \partial_{\underline{\omega}}^2\end{aligned}$$

and

$$\begin{aligned}\Delta^* &= \underline{\omega}\,\partial_{\underline{\omega}} - \partial_{\underline{\omega}}^2 \\ &= -\,\Gamma - \Gamma^2 + (m-1)\,\Gamma \\ &= (m-2)\,\Gamma - \Gamma^2.\end{aligned}$$

□

There is a second, and, quite naturally, equivalent, way to write the Laplace-Beltrami operator by means of cartesian derivatives. It only needs a straightforward calculation to prove the following result.

Proposition 7.5.2 *The Laplace-Beltrami operator may be written as*

$$\Delta^* = \sum_{j<k} L_{jk}^2 = \sum_{j<k} (x_j\,\partial_{x_k} - x_k\,\partial_{x_j})^2.$$

The actions of the Laplace operator and the Laplace-Beltrami operator on a distribution being uniquely well-defined, the question arises how to define the actions on a distribution of the three parts of the Laplace operator expressed in spherical co-ordinates. It turns out that these actions are well-defined, though not uniquely, through equivalent classes of distributions.

Proposition 7.5.3 *Let T be a scalar distribution. One has*

(i) $\partial_r^2\, T = S_2 + \delta(\underline{x})\, c_2 - \sum_{j=1}^m c_{1,j}\,\partial_{x_j}\delta(\underline{x})$
for arbitrary constants c_2 and $c_{1,j}$, $j = 1, \ldots, m$ and any distribution S_2 such that $\underline{x}\, S_2 = \mathbb{E}\,\underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$

(ii) $\frac{1}{r}\,\partial_r\, T = S_3 + \frac{1}{m}\sum_{j=1}^m c_{1,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$
for arbitrarily constant c_3 and any distribution S_3 such that $\underline{x}\, S_3 = \underline{S}_1$

(iii) $\frac{1}{r^2}\,\Delta^*\, T = S_4 + c_4\,\delta(\underline{x}) + \sum_{j=1}^m c_{5,j}\,\partial_{x_j}\delta(\underline{x})$
for arbitrary constants c_4 and $c_{5,j}$, $j = 1, \ldots, m$ and any distribution S_4 such that $r^2\, S_4 = \Delta^*\, T$

Proof

(i) From Sect. 7.4 we know that

$$(\underline{\omega}\,\partial_r)\, T = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] = \underline{S}_1 + \delta(\underline{x})\,\underline{c}_1$$

with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$. It follows that

$$\begin{aligned}
\partial_r^2\,T &= -\,(\underline{\omega}\,\partial_r)^2\,T \\
&= -\,(\underline{\omega}\,\partial_r)\,(\underline{S}_1 + \delta(\underline{x})\,\underline{c}_1) \\
&= \left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{S}_1\right] - \underline{\partial}\delta(\underline{x})\,\underline{c}_1 \\
&= S_2 + \delta(\underline{x})\,c_2 - \underline{\partial}\delta(\underline{x})\,\underline{c}_1
\end{aligned}$$

with $\underline{x}\,S_2 = \mathbb{E}\,\underline{S}_1$.

(ii) We have consecutively

$$\begin{aligned}
\frac{1}{r}\,\partial_r\,T &= \frac{1}{\underline{x}}\,(\underline{\omega}\,\partial_r)\,T \\
&= \frac{1}{\underline{x}}\,(\underline{S}_1 + \delta(\underline{x})\,\underline{c}_1) \\
&= S_3 + \frac{1}{\underline{x}}\,\delta(\underline{x})\,\underline{c}_1 \\
&= S_3 + \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x})\,\underline{c}_1 + \delta(\underline{x})\,c_3
\end{aligned}$$

with $\underline{x}\,S_3 = \underline{S}_1$.

(iii) The distribution $\Delta^*\,T$ is uniquely defined and r^2 is an analytic function with a second order zero at the origin. The result follows immediately.

□

Remark 7.5.4 The operators ∂_r^2, $\frac{1}{r}\,\partial_r$ and $\frac{1}{r^2}\,\Delta^*$ are *entangled* in the sense that, given a distribution T and having chosen appropriately the distributions $\underline{S}_1$, S_2, S_3 and S_4, all arbitrary constants appearing in the expressions of Proposition 7.5.3 should satisfy the entanglement condition

$$\partial_r^2\,T + (m-1)\frac{1}{r}\,\partial_r\,T + \frac{1}{r^2}\,\Delta^*\,T = \Delta\,T$$

the distribution at the right-hand side being uniquely determined.

Example Proposition 7.5.3 may be generalised to distributions which are e.g. vector valued. Let us illustrate this by considering the distribution $T = \underline{x}^3 = -\,r^3\,\underline{\omega}$, for which, by a direct computation, $\Delta\,T = \Delta(\underline{x}^3) = -\,2\,(m+2)\,\underline{x}$, and $\Delta^*\,T = \Delta^*\,(\underline{x}^3) = (m-1)\,r^2\,\underline{x} = -\,(m-1)\,\underline{x}^3$.

As $\mathbb{E}\,T = \mathbb{E}\,(\underline{x}^3) = 3\,\underline{x}^3$, we chose $S_1 = -\,3\,\underline{x}^2 = 3\,r^2$ satisfying $\underline{x}\,S_1 = -\,3\,\underline{x}^3$. As $\mathbb{E}\,S_1 = \mathbb{E}\,(-3\underline{x}^2) = -\,6\,\underline{x}^2 = 6\,r^2$, we chose $\underline{S}_2 = -\,6\,\underline{x}$ satisfying $\underline{x}\,\underline{S}_2 =$

$-6\,\underline{x}^2$, and $\underline{S}_3 = -3\,\underline{x}$ satisfying $\underline{x}\,\underline{S}_3 = -3\,\underline{x}^2$. Finally we chose $\underline{S}_4 = (m-1)\,\underline{x}$, satisfying $r^2\,\underline{S}_4 = \Delta^*\,T = (m-1)\,r^2\,\underline{x}$. This leads to:

(i) $\partial_r^2\,T = \partial_r^2\,(\underline{x}^3) = -6\,\underline{x} + \delta(\underline{x})\,c_2 - \sum_{j=1}^m c_{1,j}\,\partial_{x_j}\delta(\underline{x})$
(ii) $\frac{1}{r}\,\partial_r\,T = \frac{1}{r}\,\partial_r\,(\underline{x}^3) = -3\,\underline{x} + \frac{1}{m}\,\sum_{j=1}^m c_{1,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$
(iii) $\frac{1}{r^2}\,\Delta^*\,T = \frac{1}{r^2}\,\Delta^*\,(\underline{x}^3) = (m-1)\,\underline{x} + c_4\,\delta(\underline{x}) + \sum_{j=1}^m c_{5,j}\,\partial_{x_j}\delta(\underline{x})$

provided that the arbitrary constants should satisfy the entanglement conditions

$$\begin{cases} c_2 + (m-1)\,c_3 + c_4 = 0 \\ -\frac{1}{m}\,c_{1,j} + c_{5,j} = 0, \quad j = 1, \ldots, m. \end{cases}$$

7.6 Radial and Angular Derivatives of Distributions

In Sect. 7.1 we explained why it is impossible to define the radial derivative $\partial_r\,T$ and the vector angular derivative $\partial_{\underline{\omega}}\,T$ of a distribution T within the class of distributions. Neither is it possible to multiply a distribution by the non-analytic functions r and $\underline{\omega}$. For *legitimizing* those *forbidden actions* we have to take the signumdistributions into consideration instead.

Definition 7.6.1 The product of a scalar-valued distribution T by the function $\underline{\omega}$ is the signumdistribution $T^\vee$ associated to T, and it holds that

$$\langle\, \underline{\omega}\,T\,,\ \underline{\omega}\,\varphi\,\rangle = \langle\, T^\vee\,,\ \underline{\omega}\,\varphi\,\rangle = -\,\langle\, T\,, \varphi\,\rangle.$$

Definition 7.6.2 The product of a scalar-valued distribution T by the function r is the signumdistribution $r\,T = (-\,\underline{x}\,T)^\vee$ given by

$$\langle\, r\,T\,,\ \underline{\omega}\,\varphi\,\rangle = \langle\, \underline{x}\,T\,, \varphi\,\rangle = \langle\, T\,,\ \underline{x}\,\varphi\,\rangle$$

according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc} \mathbf{T} & \overset{-\underline{x}}{\longrightarrow} & -\,\underline{\mathbf{x}}\,\mathbf{T} \\ {\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle -r,\ r} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\ T^\vee = \underline{\omega}\,T & \underset{-\underline{x}}{\longrightarrow} & \mathbf{r}\,\mathbf{T} \end{array}$$

Remark 7.6.3 In the above commutative diagram, and in all the commutative diagrams in the sequel of this paper as well, the row above is situated in the world of distributions, while the objects in the row below are signumdistributions. Vertical transition from the distributions to the signumdistributions and vice versa is executed by the multiplication operators $\underline{\omega}$ and $-\underline{\omega}$ respectively. Each of the horizontally acting operators between distributions, has its counterpart in the world of signumdistributions, and vice versa; e.g. in the above commutative diagram the multiplication operator $-\underline{x}$ between the distributions T and $-\underline{x}\,T$ corresponds with the multiplication operator $-\underline{x}$ between the signumdistributions $T^\vee$ and $(-\underline{x}\,T)^\vee = r\,T$. In fact this implies the definition of the multiplication of the signumdistribution $T^\vee = \underline{\omega}\,T$ by the function $\underline{x}$ resulting in the signumdistribution $-r\,T$.

Definition 7.6.4 The derivative with respect to the radial distance r of a scalar-valued distribution T is the equivalent class of signumdistributions

$$[\partial_r T] = \left[-\underline{\omega}\,\partial_r T\right]^\vee = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right]^\vee = \left(S + \underline{c}\,\delta(\underline{x})\right)^\vee = \underline{\omega}\,S + \underline{\omega}\,\delta(\underline{x})\,\underline{c}$$

for any vector distribution S satisfying $\underline{x}\,S = \mathbb{E}\,T$, according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{T} & \overset{-\underline{\omega}\partial_r}{\longrightarrow} & \left[\frac{\mathbf{1}}{\underline{\mathbf{x}}}\,\mathbb{E}\,\mathbf{T}\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{matrix}\nearrow -\partial_r \\ \searrow \partial_r\end{matrix} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
T^\vee = \underline{\omega}\,T & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & [\partial_\mathbf{r}\mathbf{T}]
\end{array}$$

Remark 7.6.5 In the special case of a scalar-valued radial distribution T^{rad}, its radial derivative $\partial_r\,T^{rad}$ is uniquely determined as the signumdistribution $\partial_r\,T^{rad} = (-\underline{\partial}\,T^{rad})^\vee$ given by

$$\langle\, \partial_r\,T^{rad}\,,\ \underline{\omega}\,\varphi\,\rangle = \langle\, \underline{\omega}\partial_r\,T^{rad}\,,\ \varphi\,\rangle = \langle\, \underline{\partial}\,T^{rad}\,,\ \varphi\,\rangle$$

according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{T^{rad}} & \overset{-\underline{\omega}\partial_r}{\longrightarrow} & -\underline{\partial}\,\mathbf{T^{rad}} \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \begin{matrix}\nearrow -\partial_r \\ \searrow \partial_r\end{matrix} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\,T^{rad} & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & \partial_\mathbf{r}\mathbf{T^{rad}} = -\underline{\omega}\,\underline{\partial}\,\mathbf{T^{rad}}
\end{array}$$

Remark 7.6.6 The commutative diagram of Definition 7.6.4 implies the definition of the action of the operator $\underline{\partial}_{rad} = \underline{\omega}\,\partial_r$ on the signumdistribution $T^\vee = \underline{\omega}\,T$ resulting in the (equivalence class of) signumdistributions $-[\partial_r\,T]$. In the special case where the distribution T is radial: $T = T^{rad}$, the action of the operator $\underline{\partial}_{rad} = \underline{\omega}\,\partial_r$ on $\underline{\omega}\,T^{rad}$ is the uniquely determined signumdistribution

$$(\underline{\omega}\,\partial_r)\underline{\omega}\,T^{rad} = -\,\partial_r\,T^{rad} = \underline{\omega}(\underline{\omega}\,\partial_r)T = \underline{\omega}\,\underline{\partial}\,T = (\underline{\partial}\,T)^\vee$$

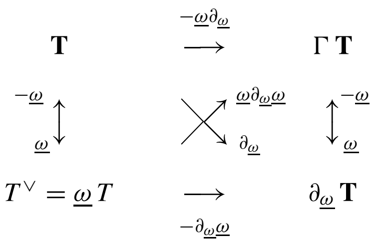
and for all test functions $\underline{\omega}\,\varphi$ it holds that

$$\begin{aligned}\langle\,-\underline{\omega}\,\partial_r\,T^\vee\,,\;\underline{\omega}\varphi\,\rangle &= \langle\,(\underline{\omega}\,\partial_r\,T^\vee)^\wedge\,,\;\varphi\,\rangle = \langle\,\partial_r\,T^\vee\,,\;\varphi\,\rangle\\ &= \langle\,-(\partial_r\,T)^\wedge\,,\;\varphi\,\rangle = \langle\,\partial_r\,T\,,\;\underline{\omega}\,\varphi\,\rangle.\end{aligned}$$

Definition 7.6.7 The angular $\partial_{\underline{\omega}}$-derivative of a scalar-valued distribution T is the signumdistribution $\partial_{\underline{\omega}}\,T = (\Gamma\,T)^\vee$ given by

$$\langle\,\underline{\omega}\varphi\,,\;\partial_{\underline{\omega}}\,T\,\rangle = \langle\,\varphi\,,\;\underline{\omega}\,\partial_{\underline{\omega}}\,T\,\rangle = \langle\,\varphi\,,\;-\Gamma\,T\,\rangle$$

according to (the boldface part of) the commutative diagram



Remark 7.6.8 The commutative diagram of Definition 7.6.7 implies the definition of the action of the operator $\partial_{\underline{\omega}}\,\underline{\omega}$ on the signumdistribution $T^\vee = \underline{\omega}\,T$ resulting in the signumdistribution $-\,\partial_{\underline{\omega}}\,T$, which in its turn implies the definition of the action of the Γ-operator on the signumdistribution $T^\vee = \underline{\omega}\,T$ resulting in the signumdistribution

$$\Gamma(\underline{\omega}\,T) = (m-1)\,\underline{\omega}\,T - \partial_{\underline{\omega}}\,T$$

since

$$\partial_{\underline{\omega}}\,\underline{\omega} = (1-m)\mathbf{1} - \underline{\omega}\,\partial_{\underline{\omega}} = (1-m)\mathbf{1} + \Gamma.$$

7.7 Actions on Signumdistributions

Definition 7.7.1 The product of a scalar-valued signumdistribution ${}^sU$ by the function $\underline{\omega}$ is the distribution $-\,{}^sU^\wedge$ associated to $-\,{}^sU$, and it holds that

$$\langle\, \underline{\omega}\,{}^sU \;,\; \varphi\,\rangle = \langle\, -\,{}^sU^\wedge \;,\; \varphi\,\rangle = \langle\, {}^sU \;,\; \underline{\omega}\varphi\,\rangle .$$

Definition 7.7.2 The product of a scalar-valued signumdistribution ${}^sU$ by the function r is the distribution $r\,{}^sU = \underline{x}\,({}^sU)^\wedge$ given by

$$\langle\, r\,{}^sU \;,\; \varphi\,\rangle = \langle\, \underline{x}\,(-\underline{\omega}\,{}^sU) \;,\; \varphi\,\rangle = \langle\, -\underline{\omega}\,{}^sU \;,\; \underline{x}\,\varphi\,\rangle = \langle\, {}^sU \;,\; -\underline{\omega}\,(\underline{x}\,\varphi)\,\rangle .$$

according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc} \mathbf{{}^sU^\wedge = -\underline{\omega}\,{}^sU} & \overset{\underline{x}}{\longrightarrow} & \mathbf{r\,{}^sU} \\ {}^{-\underline{\omega}}\updownarrow{}_{\underline{\omega}} & \times\;{}^{r}_{-r} & \updownarrow{}^{-\underline{\omega}}_{\underline{\omega}} \\ \mathbf{{}^sU} & \underset{\underline{x}}{\longrightarrow} & \underline{x}\,{}^sU \end{array}$$

Remark 7.7.3 The commutative diagram of Definition 7.7.2 implies the definition of the multiplication of the signumdistribution ${}^sU$ by the function $\underline{x}$ resulting in the signumdistribution $\underline{x}\,{}^sU$ given by

$$\underline{x}\,{}^sU = (r\,{}^sU)^\vee = \underline{\omega}\,(\underline{x}\,{}^sU^\wedge) = \underline{\omega}\,\left(\underline{x}\,(-\underline{\omega})\,{}^sU\right).$$

Definition 7.7.4 The derivative with respect to the radial distance r of a scalar-valued signumdistribution ${}^sU$ is the equivalent class of distributions

$$\left[\partial_r\,{}^sU\right] = \left[\underline{\omega}\,\partial_r\,{}^sU^\wedge\right] = \left[-\frac{1}{\underline{x}}\,\mathbb{E}\,{}^sU^\wedge\right] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{\omega}\,{}^sU\right] = T + c\,\delta(\underline{x})$$

for any scalar distribution T satisfying $\underline{x}\,T = -\,\mathbb{E}\,{}^sU^\wedge = \mathbb{E}\,\underline{\omega}\,{}^sU$, according to (the bold face part of) the commutative diagram

$$\begin{array}{ccc} \mathbf{{}^sU^\wedge = -\underline{\omega}\,{}^sU} & \overset{\underline{\omega}\partial_r}{\longrightarrow} & \left[\partial_{\mathbf{r}}\,\mathbf{{}^sU}\right] \\ {}^{-\underline{\omega}}\updownarrow{}_{\underline{\omega}} & \times\;{}^{\partial_r}_{-\partial_r} & \updownarrow{}^{-\underline{\omega}}_{\underline{\omega}} \\ \mathbf{{}^sU} & \underset{\underline{\omega}\partial_r}{\longrightarrow} & \underline{\omega}\;[\partial_r\,{}^sU] \end{array}$$

Remark 7.7.5 As we have now at our disposal the definitions of the multiplication by r (Definition 7.7.2) and of the radial derivative ∂_r (Definition 7.7.4) of a signumdistribution, we are able to define the action of the Euler operator $\mathbb{E} = r\,\partial_r$ on the signumdistribution ${}^sU$, resulting into the unique signumdistribution $\mathbb{E}\,{}^sU$ given by

$$\mathbb{E}\,{}^sU = (r\,\partial_r)\,{}^sU = r\,(\partial_r\,{}^sU) = \underline{\omega}\,(-\underline{x}\,[\partial_r\,{}^sU]) = \underline{\omega}\,(\mathbb{E}\,{}^sU^\wedge) = \underline{\omega}\,(-\underline{x}\,T) = r\,T$$

for any distribution T satisfying $\underline{x}\,T = -\mathbb{E}\,{}^sU^\wedge$, according to the commutative diagram

$$\begin{array}{ccccc}
{}^sU^\wedge = -\underline{\omega}\,{}^sU & \xrightarrow{\underline{\omega}\partial_r} & [\partial_r\,{}^sU] & \xrightarrow{-\underline{x}} & \mathbb{E}\,{}^sU^\wedge \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle \partial_r,\ -\partial_r} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} & \times\ {\scriptstyle -r,\ r} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
{}^sU & \xrightarrow[\underline{\omega}\partial_r]{} & \underline{\omega}\,[\partial_r\,{}^sU] & \xrightarrow[-\underline{x}]{} & \underline{\omega}\,\mathbb{E}\,{}^sU^\wedge
\end{array}$$

Remark 7.7.6 The commutative diagram of Definition 7.7.4 implies the definition of the action of the operator $\underline{\partial}_{rad} = \underline{\omega}\,\partial_r$ on the signumdistribution ${}^sU$ resulting in the signumdistribution $\underline{\omega}\,\partial_r\,{}^sU$ given by the equivalence class

$$[\underline{\omega}\,\partial_r\,{}^sU] = \underline{\omega}\,\left[\partial_r\,{}^sU\right] = \underline{\omega}\left[-\frac{1}{\underline{x}}\,\mathbb{E}\,{}^sU^\wedge\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{\omega}\,{}^sU\right] = \left[-\frac{1}{\underline{x}}\,\mathbb{E}\,{}^sU\right].$$

In particular, when ${}^sU$ is a radial signumdistribution: ${}^sU = {}^sU^{rad}$, we define the action of the Dirac operator $\underline{\partial}$ on ${}^sU^{rad}$ to be

$$\underline{\partial}\,{}^sU^{rad} = \left[\underline{\omega}\,\partial_r\,{}^sU^{rad}\right] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,\underline{\omega}\,{}^sU^{rad}\right].$$

Definition 7.7.7 The angular $\partial_{\underline{\omega}}$-derivative of a scalar-valued signumdistribution ${}^sU$ is the distribution $\partial_{\underline{\omega}}\,{}^sU = \partial_{\underline{\omega}}\,\underline{\omega}\,{}^sU^\wedge$ according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{{}^sU^\wedge = -\underline{\omega}\,{}^sU} & \xrightarrow{\partial_{\underline{\omega}}\underline{\omega}} & \mathbf{\partial_{\underline{\omega}}\,{}^sU} \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times\ {\scriptstyle \partial_{\underline{\omega}},\ \underline{\omega}\partial_{\underline{\omega}}\underline{\omega}} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\mathbf{{}^sU} & \xrightarrow[\underline{\omega}\partial_{\underline{\omega}}]{} & \underline{\omega}\,\partial_{\underline{\omega}}\,{}^sU
\end{array}$$

Remark 7.7.8 The commutative diagram of Definition 7.7.7 implies the definition of the action of the Γ-operator on the signumdistribution ${}^{s}U$ resulting in the signumdistribution $\Gamma\, {}^{s}U$ given by

$$-\,\Gamma\, {}^{s}U = \underline{\omega}\, \partial_{\underline{\omega}}\, {}^{s}U = \underline{\omega}\, (\partial_{\underline{\omega}}\, {}^{s}U) = \underline{\omega}\, (\partial_{\underline{\omega}}\, \underline{\omega}\, {}^{s}U^{\wedge}) = (\partial_{\underline{\omega}}\, \underline{\omega}\, {}^{s}U^{\wedge})^{\vee}.$$

7.8 Composite Actions of Two Operators

In the preceding sections we were able to define the actions on (signum-) distributions of the operators r, $\underline{\omega}$, ∂_r, and $\partial_{\underline{\omega}}$. In Sect. 7.1 it was argued that the composite action by any two of those operators should lead to a *legal* action on distributions. Let us find out now if this is indeed the case.

1. Multiplication of a distribution T by the analytic function $r^2 = -\underline{x}^2 = \sum_{j=1}^{m} x_j^2$ is well defined. Through the following commutative diagram it is shown that $r(rT) = r^2\, T$:

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{x}} & -\underline{x}\, T & \xrightarrow{\underline{x}} & r^2\, T \\
{\scriptstyle -\underline{\omega}} \updownarrow {\scriptstyle \underline{\omega}} & \searrow {\scriptstyle r} & \updownarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} & \nearrow {\scriptstyle r} & \updownarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\, T & \xrightarrow[-\underline{x}]{} & r\, T & \xrightarrow[\underline{x}]{} & \underline{\omega}\, r^2 T
\end{array}$$

2. Multiplication of a distribution T by the analytic function $\underline{x} = r\, \underline{\omega}$ is well defined. Through the commutative diagram of Definition 7.6.2 it is shown that $r(\underline{\omega}\, T) = \underline{x}\, T$.
3. The action of the Euler operator $\mathbb{E} = \sum_{j=1}^{m} x_j\, \partial_{x_j}$ on a distribution is well defined. Through the following commutative diagram it is shown that $r(\partial_r\, T) = \mathbb{E}\, T$:

$$\begin{array}{ccccc}
T & \xrightarrow{-\,\underline{\omega}\, \partial_r} & [-\,\underline{\omega}\, \partial_r\, T] = \left[\frac{1}{\underline{x}}\, \mathbb{E}\, T\right] & \xrightarrow{\underline{x}} & \mathbb{E}\, T \\
{\scriptstyle -\underline{\omega}} \updownarrow {\scriptstyle \underline{\omega}} & \searrow {\scriptstyle \partial_r} & \updownarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} & \nearrow {\scriptstyle r} & \updownarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\, T & \xrightarrow[-\,\underline{\omega}\, \partial_r]{} & [\partial_r\, T] = \underline{\omega}\left[\frac{1}{\underline{x}}\, \mathbb{E}\, T\right] & \xrightarrow[\underline{x}]{} & \underline{\omega}\, \mathbb{E}\, T
\end{array}$$

4. The action of the operator $\underline{x}\,\Gamma = \underline{x}\left(\sum_{j<k} e_j e_k (x_j \partial x_k - x_k \partial_{x_j})\right)$ on a distribution is well defined. Through the following commutative diagram it is shown that $r(\partial_{\underline{\omega}}\, T) = \underline{x}\,\Gamma\, T$:

$$\begin{array}{ccccc}
T & \xrightarrow{-\,\underline{\omega}\,\partial_{\underline{\omega}}} & \Gamma\, T & \xrightarrow{\underline{x}} & \underline{x}\,\Gamma\, T \\
{\scriptstyle -\underline{\omega}}\updownarrow{\scriptstyle \underline{\omega}} & \searrow^{\partial_{\underline{\omega}}} & \updownarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} & \nearrow^{r} & \updownarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\, T & \xrightarrow[-\,\partial_{\underline{\omega}}\,\underline{\omega}]{} & \partial_{\underline{\omega}}\, T & \xrightarrow[\underline{x}]{} & \underline{x}\,\partial_{\underline{\omega}}\, T
\end{array}$$

5. It is clear that $\underline{\omega}\,(\underline{\omega}\, T) = -T$.
6. The action of the operator $\underline{\omega}\,\partial_r$ on a distribution is well defined, albeit not uniquely but through an equivalence class instead, see (7.4.1). Definition 7.6.4 implies that $\underline{\omega}\,[\partial_r\, T] = [(\underline{\omega}\,\partial_r)\, T]$.
7. The action of the operator $\underline{\omega}\,\partial_{\underline{\omega}} = -\,\Gamma$ on a distribution is well defined. Definition 7.6.7 implies that $\underline{\omega}\,(\partial_{\underline{\omega}}\, T) = -\,\Gamma\, T$.
8. The action of the operator ∂_r^2 on a distribution was defined in Sect. 7.5 by the equivalence class

$$\partial_r^2\, T = \left[-\,(\underline{\omega}\,\partial_r)^2\, T\right] = S_2 + \delta(\underline{x})\, c_2 - \sum_{j=1}^{m} c_{1,j}\,\partial_{x_j}\delta(\underline{x})$$

for arbitrary constants c_2 and $c_{1,j}$, $j = 1, \ldots, m$ and any distribution S_2 such that $\underline{x}\, S_2 = \mathbb{E}\,\underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\, T$, which is in complete agreement with the commutative diagram

$$\begin{array}{ccccc}
T & \xrightarrow{-\,\underline{\omega}\,\partial_r} & [-\,\underline{\omega}\,\partial_r\, T] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\, T\right] & \xrightarrow{\underline{\omega}\,\partial_r} & \left[-\,(\underline{\omega}\,\partial_r)^2\, T\right] \\
{\scriptstyle -\underline{\omega}}\updownarrow{\scriptstyle \underline{\omega}} & \searrow^{\partial_r} & \updownarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} & \nearrow^{\partial_r} & \updownarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\, T & \xrightarrow[-\,\underline{\omega}\,\partial_r]{} & [\partial_r\, T] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\mathbb{E}\, T\right] & \xrightarrow[\underline{\omega}\,\partial_r]{} & \underline{\omega}\left[-\,(\underline{\omega}\,\partial_r)^2\, T\right]
\end{array}$$

9. Start with the observation that for a distribution T,

$$\partial_r\,\partial_{\underline{\omega}}\, T = \underline{\omega}\,\partial_r\,(-\,\underline{\omega}\,\partial_{\underline{\omega}})\, T = -\left[\frac{1}{\underline{x}}\,\mathbb{E}\,\Gamma\, T\right]$$

to see that the action of the operator $\partial_r\, \partial_{\underline{\omega}}$ on a distribution is well-defined, though not uniquely. Then the commutative diagram

$$\begin{array}{ccccc}
T & \overset{-\,\underline{\omega}\,\partial_{\underline{\omega}}}{\longrightarrow} & \Gamma\, T & \overset{\underline{\omega}\,\partial_r}{\longrightarrow} & \partial_r\, \partial_{\underline{\omega}}\, T \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow{\scriptstyle \partial_{\underline{\omega}}} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} & \nearrow{\scriptstyle \partial_r} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\, T & \underset{-\,\partial_{\underline{\omega}}\,\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}\, T & \underset{\underline{\omega}\,\partial_r}{\longrightarrow} & -\,\partial_r\, \Gamma\, T
\end{array}$$

shows that indeed $\partial_r\, (\partial_{\underline{\omega}}\, T) = \partial_r\, \partial_{\underline{\omega}}\, T$.

10. Start with the observation that for a distribution T,

$$\partial_{\underline{\omega}}^2\, T = \partial_{\underline{\omega}}\, \underline{\omega}\, (-\,\underline{\omega}\, \partial_{\underline{\omega}})\, T = (1-m)\, \Gamma\, T + \Gamma^2\, T$$

to see that the action of the operator $\partial_{\underline{\omega}}$ on a distribution is well-defined. Applying twice the commutative diagram of Definition 7.6.7 we obtain

$$\begin{array}{ccccc}
T & \overset{-\,\underline{\omega}\,\partial_{\underline{\omega}}}{\longrightarrow} & \Gamma\, T & \overset{\partial_{\underline{\omega}}\,\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}^2\, T \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow{\scriptstyle \partial_{\underline{\omega}}} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} & \nearrow{\scriptstyle \partial_{\underline{\omega}}} & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\underline{\omega}\, T & \underset{-\,\partial_{\underline{\omega}}\,\underline{\omega}}{\longrightarrow} & \partial_{\underline{\omega}}\, T & \underset{\underline{\omega}\,\partial_{\underline{\omega}}}{\longrightarrow} & \underline{\omega}\, \partial_{\underline{\omega}}^2\, T
\end{array}$$

showing that indeed $\partial_{\underline{\omega}}\, (\partial_{\underline{\omega}}\, T) = \partial_{\underline{\omega}}^2\, T$.

7.9 Division of (Signum)Distributions by r

Division of a standard distribution T by an analytic function $\alpha(\underline{x})$ resulting in an equivalent class of distributions S such that $\alpha(\underline{x})\, S = T$, we expect the division of a standard distribution by the non-analytic function r to lead to an equivalence class of signumdistributions. Let us make this precise.

Definition 7.9.1 The quotient of a scalar distribution T by the radial distance r is the equivalence class of signumdistributions

$$\left[\frac{1}{r}\, T\right] = \underline{\omega}\, \left[\frac{1}{\underline{x}}\, T\right] = \underline{\omega}\, (\underline{S} + \delta(\underline{x})\, \underline{c}) = \underline{\omega}\, \underline{S} + \underline{\omega}\, \delta(\underline{x})\, \underline{c} = \underline{S}^{\vee} + \delta(\underline{x})^{\vee}\, \underline{c}$$

for any vector-valued distribution $\underline{S}$ for which $\underline{x}\,\underline{S} = T$, according to (the boldface part of) the commutative diagram

$$\begin{array}{ccc}
\mathbf{T} & \xrightarrow{\frac{1}{\underline{x}}} & \left[\frac{\mathbf{1}}{\underline{\mathbf{x}}}\,\mathbf{T}\right] \\
{\scriptstyle -\underline{\omega}}\ \updownarrow\ {\scriptstyle \underline{\omega}} & \times\ {\scriptstyle -\frac{1}{r}}\ {\scriptstyle \frac{1}{r}} & {\scriptstyle -\underline{\omega}}\ \updownarrow\ {\scriptstyle \underline{\omega}} \\
T^{\vee} = \underline{\omega}\,T & \xrightarrow[\frac{1}{\underline{x}}]{} & \left[\frac{\mathbf{1}}{\mathbf{r}}\,\mathbf{T}\right]
\end{array}$$

Example Let us illustrate Definition 7.9.1 by the case of the delta-distribution: $T = \delta(\underline{x})$. As $\underline{x}\,\underline{\partial}\,\delta(\underline{x}) = m\,\delta(\underline{x})$ and $\underline{x}\,\delta(\underline{x}) = 0$ we have

$$\frac{1}{\underline{x}}\,\delta(\underline{x}) = \frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}) + \delta(\underline{x})\,\underline{c}_0$$

with $\underline{c}_0$ an arbitrary constant vector. It then follows that

$$\begin{aligned}
\left[\frac{1}{r}\,\delta(\underline{x})\right] &= \underline{\omega}\left[\frac{1}{\underline{x}}\,\delta(\underline{x})\right] = \underline{\omega}\left[\frac{1}{m}\,\underline{\partial}\,\delta(\underline{x}) + \delta(\underline{x})\,\underline{c}_0\right] \\
&= \frac{1}{m}\,\underline{\omega}\,\underline{\partial}\,\delta(\underline{x}) + \underline{\omega}\,\delta(\underline{x})\,\underline{c}_0 = \frac{1}{m}\,(\underline{\partial}\,\delta(\underline{x}))^{\vee} + \delta(\underline{x})^{\vee}\,\underline{c}_0
\end{aligned}$$

or, in view of the definition of $\partial_r\,\delta(\underline{x})$,

$$\left[\frac{1}{r}\,\delta(\underline{x})\right] = -\frac{1}{m}\,\partial_r\,\delta(\underline{x}) + \underline{\omega}\,\delta(\underline{x})\,\underline{c}_0.$$

However in this particular case of the delta-distribution it turns out that $\frac{1}{r}\,\delta(\underline{x})$ is uniquely determined. Indeed, as $\partial_r\,\delta(\underline{x})$ is a radial signumdistribution and as we expect the signumdistribution $\frac{1}{r}\,\delta(\underline{x})$ to be $\mathrm{SO}(m)$-invariant as well, the arbitrary vector constant $\underline{c}_0$ should be zero, eventually leading to

$$\frac{1}{r}\,\delta(\underline{x}) = -\frac{1}{m}\,\partial_r\,\delta(\underline{x}).$$

For the general case of the division of the delta-distribution by natural powers of r we refer to [2].

Remark 7.9.2 The commutative diagram of Definition 7.9.1 implies the definition of the quotient of the signumdistribution $T^\vee = \underline{\omega}\, T$ by r, viz. the equivalence class of distributions

$$\left[\frac{1}{r}\,(\underline{\omega}\, T)\right] = \left[-\frac{1}{\underline{x}}\, T\right]$$

as well as the quotient of the same signumdistribution by $\underline{x}$, viz. the equivalence class of signumdistributions

$$\left[\frac{1}{\underline{x}}\,(\underline{\omega}\, T)\right] = \left[\frac{1}{r}\, T\right].$$

It is also interesting and useful to define the division by r of a signumdistribution, because it will lead to the definition of the action of the angular part $\underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}}$ of the Dirac operator on a signumdistribution, leading in its turn to the definition of the action of the Dirac operator on a signumdistribution.

Definition 7.9.3 The quotient of a scalar-valued signumdistribution ${}^sU$ by the radial distance r is the equivalence class of distributions

$$\left[\frac{1}{r}\,{}^sU\right] = \left[\frac{1}{\underline{x}}\,\underline{\omega}\,{}^sU\right] = S + \delta(\underline{x})\, c$$

for any scalar-valued distribution S for which $\underline{x}\, S = \underline{\omega}\,{}^sU$, according to (the boldface part of) the commutative diagram

$$\begin{array}{ccccc}
 & & -\frac{1}{\underline{x}} & & \\
-\underline{\omega}\,{}^{\mathbf{s}}\mathbf{U} & & \longrightarrow & & \left[\frac{\mathbf{1}}{\mathbf{r}}\,{}^{\mathbf{s}}\mathbf{U}\right] \\
-\underline{\omega} & \updownarrow & \nearrow\, \frac{1}{r} \quad \searrow\, -\frac{1}{r} & \updownarrow & -\underline{\omega} \\
\underline{\omega} & & & & \underline{\omega} \\
{}^{\mathbf{s}}\mathbf{U} & & \longrightarrow & & \underline{\omega}\left[\frac{1}{\underline{x}}\,\underline{\omega}\,{}^sU\right] \\
 & & -\frac{1}{\underline{x}} & &
\end{array}$$

Remark 7.9.4 The commutative diagram of Definition 7.9.3 implies the definition of the quotient of the signumdistribution ${}^sU$ by $\underline{x}$, viz. the equivalence class of signumdistributions

$$\left[\frac{1}{\underline{x}}\,{}^sU\right] = -\,\underline{\omega}\left[\frac{1}{\underline{x}}\,\underline{\omega}^s U\right]$$

as well as the quotient of the distribution $\underline{\omega}\,{}^s U$ by r, viz. the equivalence class of signumdistributions

$$\left[\frac{1}{r}\,(\underline{\omega}\,{}^s U)\right] = \left[-\frac{1}{\underline{x}}\,{}^s U\right].$$

Now as we know how to act with the operator $\partial_{\underline{\omega}}$ on a distribution (see Definition 7.6.7) and how to act with the operator $\frac{1}{r}$ on a signumdistribution (see Definition 7.9.3) we are now in the position to check the action on a distribution of the composition of both operators, viz. the angular part $\underline{\partial}_{ang}$ of the Dirac operator. The outcome should match expression (7.4.2); that this is indeed the case is shown by the following commutative diagram:

$$\begin{array}{ccccc}
T & \xrightarrow{-\underline{\omega}\partial_{\underline{\omega}}} & \Gamma\, T & \xrightarrow{-\frac{1}{\underline{x}}} & \left[\frac{1}{r}\,\partial_{\underline{\omega}}\, T\right] = \left[-\frac{1}{\underline{x}}\,\Gamma\, T\right] \\
{}^{-\underline{\omega}}\updownarrow_{\underline{\omega}} & \searrow^{\partial_{\underline{\omega}}} & \updownarrow^{-\underline{\omega}}_{\underline{\omega}} & \nearrow^{\frac{1}{r}} & \updownarrow^{-\underline{\omega}}_{\underline{\omega}} \\
\underline{\omega}\, T & \xrightarrow[-\partial_{\underline{\omega}}\underline{\omega}]{} & \partial_{\underline{\omega}}\, T & \xrightarrow[-\frac{1}{\underline{x}}]{} & \underline{\omega}\left[-\frac{1}{\underline{x}}\,\Gamma\, T\right]
\end{array}$$

In the same order of ideas we can define the action of $\underline{\partial}_{ang} = \frac{1}{r}\,\partial_{\underline{\omega}}$ on a signumdistribution through the commutative diagram

$$\begin{array}{ccccc}
{}^s U^{\wedge} = -\underline{\omega}\,{}^s U & \xrightarrow{\partial_{\underline{\omega}}\underline{\omega}} & \partial_{\underline{\omega}}\,{}^s U & \xrightarrow{\frac{1}{\underline{x}}} & \left[\frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\,{}^s U\right] \\
{}^{-\underline{\omega}}\updownarrow_{\underline{\omega}} & \nearrow^{\partial_{\underline{\omega}}} & \updownarrow^{-\underline{\omega}}_{\underline{\omega}} & \searrow_{\frac{1}{r}} & \updownarrow^{-\underline{\omega}}_{\underline{\omega}} \\
{}^s U & \xrightarrow[\underline{\omega}\partial_{\underline{\omega}}]{} & \underline{\omega}\,\partial_{\underline{\omega}}\,{}^s U & \xrightarrow[\frac{1}{\underline{x}}]{} & \left[\frac{1}{r}\,\partial_{\underline{\omega}}\,{}^s U\right]
\end{array}$$

in other words

$$\left[\underline{\partial}_{ang}\,{}^s U\right] = \left[\frac{1}{r}\,\partial_{\underline{\omega}}\,{}^s U\right] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\,{}^s U\right] = \left[-\frac{1}{\underline{x}}\,\Gamma\,{}^s U\right].$$

Combining the actions on a signumdistribution of the radial and angular parts of the Dirac operator, we are able to define the action of the Dirac operator itself on a signumdistribution.

Definition 7.9.5 The action of the Dirac operator $\underline{\partial}$ on the signumdistribution ${}^sU$ is given by the equivalence class of signumdistributions

$$\begin{aligned}
\left[\underline{\partial}\,{}^sU\right] &= \left[\left(\underline{\omega}\,\partial_r + \frac{1}{r}\,\partial_{\underline{\omega}}\right){}^sU\right] \\
&= \left[-\frac{1}{\underline{x}}\,\mathbb{E}\,{}^sU\right] + \left[-\frac{1}{\underline{x}}\,\Gamma\,{}^sU\right] \\
&= \left[-\frac{1}{\underline{x}}\,(\mathbb{E}+\Gamma)\,{}^sU\right] \\
&= \left[\frac{1}{\underline{x}}\,(\underline{x}\,\underline{\partial})\,{}^sU\right]
\end{aligned}$$

according to the commutative diagram

$$\begin{array}{ccc}
{}^sU^{\wedge} = -\underline{\omega}\,{}^sU & \xrightarrow{D} & \left[\partial_r\,{}^sU + \frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\,{}^sU\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & & \uparrow\downarrow{\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
{}^sU & \xrightarrow[\underline{\partial}]{} & \left[\underline{\partial}\,{}^sU\right]
\end{array}$$

where D stands for the operator

$$\begin{aligned}
D &= \underline{\omega}\,\partial_r + \frac{1}{\underline{x}}\,\partial_{\underline{\omega}}\,\underline{\omega} \\
&= \underline{\omega}\,\partial_r - \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega} \\
&= \underline{\omega}\,\partial_r - \frac{1}{r}\,\partial_{\underline{\omega}} + (m-1)\frac{1}{r}\,\underline{\omega}.
\end{aligned}$$

Example Let us illustrate Definition 7.9.5 with the following simple example; Sect. 7.10 will offer more elaborated ones. Consider the signumdistribution $\underline{x}$ defined by

$$\langle\, \underline{x}\,,\ \underline{\omega}\varphi\,\rangle = \langle\, \underline{x}\,\underline{\omega}\,,\ \varphi\,\rangle = \langle\, -r\,,\ \varphi\,\rangle = \int_{\mathbb{R}^m} r\,\widetilde{\varphi}(r,\underline{\omega})\,d\underline{x}$$

for which

$$\mathbb{E}\,\underline{x} = \underline{x}$$

and

$$\Gamma\,\underline{x} = (m-1)\underline{x}$$

whence

$$\left[\underline{\partial}\,\underline{x}\right] = \left[-\frac{1}{\underline{x}}\,m\,\underline{x}\right] = [-\,m] = -\,m + \delta(\underline{x})\,c$$

As $\underline{x}^{\wedge} = r$ and $D\,r = \left[m\,\underline{\omega}\right]$, this result fits into the following commutative diagram:

$$\begin{array}{ccc}
r & \overset{D}{\longrightarrow} & \left[m\,\underline{\omega}\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & & \uparrow\downarrow{\scriptstyle -\underline{\omega}\,/\,\underline{\omega}} \\
\underline{x} & \underset{\underline{\partial}}{\longrightarrow} & [-\,m]
\end{array}$$

Remark 7.9.6 The commutative diagram of Definition 7.9.5 shows that the Dirac operator acting on signumdistributions, corresponds with the operator D acting on distributions. We can wonder which operator acting on signumdistributions corresponds to the Dirac operator $\underline{\partial} = \underline{\partial}_{rad} + \underline{\partial}_{ang} = \underline{\omega}\,\partial_r + \frac{1}{r}\,\partial_{\underline{\omega}}$ acting on distributions. From the commutative diagram in Definition 7.6.4 we learn that $\underline{\omega}\,\partial_r$ corresponds with $\underline{\omega}\,\partial_r$, while we saw above that $\frac{1}{r}\,\partial_{\underline{\omega}}$ corresponds with $-\frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega}$. It follows that the Dirac operator acting on distributions corresponds with the operator $\underline{\omega}\,\partial_r - \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\,\underline{\omega}$, which is precisely the operator D, acting on signumdistributions.

Finally, as we know how to act with the multiplication operator $\frac{1}{r}$ on a signumdistribution, we can check the action on a distribution T of the composite operator $(\frac{1}{r} \circ \partial_r)\,T = \frac{1}{r}\,(\partial_r\,T)$ which should coincide with the action $(\frac{1}{r}\,\partial_r)\,T$, defined, though not uniquely, in Proposition 7.5.3 by $\frac{1}{r}\,\partial_r\,T = S_3 + \frac{1}{m}\,\sum_{j=1}^{m} c_{1,j}\,\partial_{x_j}\delta(\underline{x}) + c_3\,\delta(\underline{x})$ for arbitrarily constant c_3 and any distribution S_3 such that $\underline{x}\,S_3 = \underline{S}_1$ with $\underline{x}\,\underline{S}_1 = -\,\mathbb{E}\,T$. That this is indeed the case is shown by the following commutative diagram:

$$\begin{array}{ccccc}
T & \overset{-\,\underline{\omega}\,\partial_r}{\longrightarrow} & [-\,\underline{\omega}\,\partial_r\,T] = \left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] & \overset{-\,\frac{1}{\underline{x}}}{\longrightarrow} & \left[\frac{1}{r^2}\,\mathbb{E}\,T\right] \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \searrow{\scriptstyle \partial_r} & \uparrow\downarrow{\scriptstyle -\underline{\omega}\,/\,\underline{\omega}} & \nearrow{\scriptstyle \frac{1}{r}} & \uparrow\downarrow{\scriptstyle -\underline{\omega}\,/\,\underline{\omega}} \\
\underline{\omega}\,T & \underset{-\,\underline{\omega}\,\partial_r}{\longrightarrow} & [\partial_r\,T] = \underline{\omega}\left[\frac{1}{\underline{x}}\,\mathbb{E}\,T\right] & \underset{-\,\frac{1}{\underline{x}}}{\longrightarrow} & \underline{\omega}\left[\frac{1}{r^2}\,\mathbb{E}\,T\right]
\end{array}$$

7.10 Two Families of Specific (Signum)Distributions

In the context of Clifford analysis a number of families of distributions were thoroughly studied, see e.g. [1]. Of particular importance are the families T_λ and U_λ, λ being a complex parameter. They are defined as follows.

$$\langle\, T_\lambda, \varphi(\underline{x})\,\rangle := a_m \,\langle\, \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^0[\varphi](r)\,\rangle_r$$

$$\langle\, U_\lambda, \varphi(\underline{x})\,\rangle := a_m \,\langle\, \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^1[\varphi](r)\,\rangle_r$$

where the so-called spherical means Σ^0 and Σ^1 are given by

$$\Sigma^0[\varphi](r) = \frac{1}{a_m}\int_{S^{m-1}} \varphi(r\underline{\omega})\, dS(\underline{\omega})$$

$$\Sigma^1[\varphi](r) = \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\,\varphi(r\underline{\omega})\, dS(\underline{\omega})$$

and $\mathrm{Fp}\, r_+^\mu$ stands for the *finite part* distribution on the one-dimensional r-axis.

The distributions T_λ are standard distributions in harmonic analysis; as functions of $\lambda \in \mathbb{C}$ they show simple poles at $\lambda = -m, -m-2, -m-4, \ldots$. The most important distribution is this family is $T_{-m+2} = \frac{1}{r^{m-2}}$, which is, up to a constant, the fundamental solution of the Laplace operator Δ.

The distributions U_λ form a typical Clifford analysis construct; they show simple poles at $\lambda = -m-1, -m-3, -m-5, \ldots$. The most important distribution in this family is $U_{-m+1} = \frac{\underline{\omega}}{r^{m-1}}$ which is, up to a constant, the fundamental solution of the Dirac operator $\underline{\partial}$ (see Sect. 7.4).

Both families of distributions are intertwined by the action of the Dirac operator $\underline{\partial}$, viz.

$$\underline{\partial}\, T_\lambda = \lambda\, U_{\lambda-1} \quad \lambda \neq -m, -m-2, -m-4, \ldots$$

and

$$\underline{\partial}\, U_\lambda = -(\lambda+m-1)\, T_{\lambda-1} \quad \lambda \neq -m+1, -m-1, -m-3, \ldots.$$

In the setting of spherical co-ordinates these formulae take the form:

$$\underline{\omega}\,\partial_r\, T_\lambda = \lambda\, U_{\lambda-1} \qquad \frac{1}{r}\,\partial_{\underline{\omega}}\, T_\lambda = 0 \qquad \lambda \neq -m, -m-2, -m-4, \ldots \tag{7.10.1}$$

and

$$\underline{\omega}\,\partial_r\, U_\lambda = -\lambda\, T_{\lambda-1} \qquad \frac{1}{r}\,\partial_{\underline{\omega}}\, U_\lambda = -(m-1)\, T_{\lambda-1} \qquad \lambda \neq -m+1, -m-1, \ldots. \tag{7.10.2}$$

When restricted to the half-plane $\mathbb{R}e\,\lambda > -m$ the distributions T_λ and U_λ are regular, i.e. locally integrable functions. We know from [2] that a locally integrable function can also be seen as a signumdistribution. Whence the definition of the following two families of signumdistributions:

$$\langle\, {}^sT_\lambda, \underline{\omega}\, \varphi(\underline{x})\,\rangle := \quad a_m\, \langle\, \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^1[\varphi](r)\,\rangle_r$$

$$\langle\, {}^sU_\lambda, \underline{\omega}\, \varphi(\underline{x})\,\rangle := -\, a_m\, \langle\, \mathrm{Fp}\, r_+^{\lambda+m-1}, \Sigma^0[\varphi](r)\,\rangle_r$$

It becomes clear at once that:

$$T_\lambda^\vee = \quad {}^sU_\lambda$$

and

$$U_\lambda^\vee = -\, {}^sT_\lambda$$

Moreover ${}^sT_\lambda$ inherits the simple poles of U_λ, viz. $\lambda = -m-1, -m-3, \ldots$, while ${}^sU_\lambda$ inherits the simple poles of T_λ, viz. $\lambda = -m, -m-2, \ldots$.

Invoking the commutative diagrams of Sect. 7.6 we are now able to compute the radial derivative of the distributions T_λ and U_λ, which at the time [1] and related papers were written, we were not yet able to achieve. We obtain:

$$\begin{array}{ccccc}
-\frac{1}{\lambda} T_\lambda & \xrightarrow{-\underline{\omega}\partial_r} & U_{\lambda-1} & \xrightarrow{-\underline{\omega}\partial_r} & (\lambda-1) T_{\lambda-2} \\
{\scriptstyle -\underline{\omega}} \uparrow\downarrow {\scriptstyle \underline{\omega}} & \times\; {\scriptstyle -\partial_r \nearrow,\ \partial_r \searrow} & \uparrow\downarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} & \times\; {\scriptstyle -\partial_r \nearrow,\ \partial_r \searrow} & \uparrow\downarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
-\frac{1}{\lambda} {}^sU_\lambda & \xrightarrow[-\underline{\omega}\partial_r]{} & -\,{}^sT_{\lambda-1} & \xrightarrow[-\underline{\omega}\partial_r]{} & (\lambda-1)\, {}^sU_{\lambda-2}
\end{array}$$

whence, for general λ, i.e. λ not in the simple poles mentioned above:

$$\partial_r\, T_\lambda = \lambda\, {}^sT_{\lambda-1} \qquad \partial_r\, U_\lambda = \lambda\, {}^sU_{\lambda-1}$$

formulae one should expect right from the start where it not that the results are no longer distributions but signumdistributions instead.

For the exceptional values of the parameter λ, in particular for those values which give rise to the fundamental solutions of the Dirac and Laplace operators, we obtain, in a similar manner, the following commutative diagram:

$$\begin{array}{ccccc}
\frac{T_{-m+2}}{m-2} & \xrightarrow{-\underline{\omega}\partial_r} & U_{-m+1} & \xrightarrow{-\underline{\omega}\partial_r} & -(m-1)T_{-m} + a_m \delta(\underline{x}) \\
{\scriptstyle -\underline{\omega}}\uparrow\downarrow{\scriptstyle \underline{\omega}} & \times \; {\scriptstyle -\partial_r,\ \partial_r} & \uparrow\downarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} & \times \; {\scriptstyle -\partial_r,\ \partial_r} & \uparrow\downarrow {\scriptstyle -\underline{\omega},\ \underline{\omega}} \\
\frac{{}^sU_{-m+2}}{m-2} & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & -{}^sT_{-m+1} & \underset{-\underline{\omega}\partial_r}{\longrightarrow} & -(m-1){}^sU_{-m} + a_m \underline{\omega}\delta\underline{x}
\end{array}$$

Notice that at in this way we proved the formula

$$\underline{\omega}\partial_r\, U_{-m+1} = (m-1)T_{-m} - a_m \delta(\underline{x})$$

which is additional to (7.10.2) and refines the traditional formula

$$\underline{\partial}\, U_{-m+1} = -a_m\, \delta(\underline{x}).$$

We also proved

$$(\underline{\omega}\, \partial_r)\, {}^sT_{-m+1} = -\,(m-1)\, {}^sU_{-m} + a_m\, \underline{\omega}\, \delta(\underline{x}).$$

Compared with the similar formula at the distribution level, viz.

$$(\underline{\omega}\, \partial_r)\, T_{-m+1} = -\,(m-1)\, U_{-m},$$

the appearance of the signumdeltadistribution $\underline{\omega}\, \delta(\underline{x})$ might be surprising, but is well understood when realizing that $\lambda = -m$ is a regular point for U_{-m} while it is a simple pole for ${}^sU_{-m}$.

7.11 Conclusion

In his famous and seminal book [7] Laurent Schwartz writes on page 51: *Using co-ordinate systems other than the cartesian ones should be done with the utmost care* [our translation]. And right he is! Indeed, just consider the delta distribution $\delta(\underline{x})$: it is pointly supported at the origin, it is rotation invariant: $\delta(A\, \underline{x}) = \delta(\underline{x}),\ \forall A \in \mathrm{SO}(m)$, it is even: $\delta(-\underline{x}) = \delta(\underline{x})$ and it is homogeneous of order $(-m)$: $\delta(a\underline{x}) = \frac{1}{|a|^m}\, \delta(\underline{x})$. So in a first, naive, approach, one could think of its radial derivative $\partial_r\, \delta(\underline{x})$ as a distribution which remains pointly supported at the origin, rotation invariant, even and homogeneous of degree $(-m-1)$. Temporarily leaving aside the even character, on the basis of the other cited characteristics the distribution $\partial_r\, \delta(\underline{x})$ should take the following form:

$$\partial_r\, \delta(\underline{x}) = c_0\, \partial_{x_1} \delta(\underline{x}) + \cdots + c_m\, \partial_{x_m} \delta(\underline{x})$$

and it becomes immediately clear that this approach to the radial derivation of the delta distribution is impossible since all distributions appearing in the sum at the right-hand side are odd and not rotation invariant, whereas $\partial_r\, \delta(\underline{x})$ is assumed to be even and rotation invariant. It could be that $\partial_r\, \delta(\underline{x})$ is either the zero distribution or is no longer pointly supported at the origin, but both those possibilities are unacceptable. So from the start we are warned by this example that introducing spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in \mathbb{S}^{m-1}$ makes derivation of distributions in $\mathbb{R}^m$ a far from trivial action, as are, in principle "forbidden", actions such as multiplication by the non-analytic functions r and ω_j, $j = 1, \ldots, m$. But there is more: functional analytic considerations on the space $\mathcal{D}(\mathbb{R}^m)$ of compactly supported smooth test functions expressed in spherical co-ordinates, forced us to introduce a new space of continuous linear functionals on a auxiliary space of test functions showing a singularity at the origin, for which, in [2], we coined the term *signumdistributions*, bearing in mind that $\underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$ may be interpreted as the higher dimensional counterpart to the *signum* function on the real line. It turns out that the actions by r, $\underline{\omega}$, ∂_r and $\partial_{\underline{\omega}}$ map a distribution to a signumdistribution and vice versa. The basic idea behind the definition of these actions on a distribution $T \in \mathcal{D}'(\mathbb{R}^m)$, is to express the resulting signumdistributions as appropriate and "legal" actions on T. So, for example, we put $\langle rT, \underline{\omega}\varphi\rangle = \langle r\underline{\omega}T, \varphi\rangle = \langle \underline{x}T, \varphi\rangle, \forall\varphi \in \mathcal{D}(\mathbb{R}^m)$. This idea may seem to be rather simple, but it is backed up by the functional analytic considerations of Sect. 7.1, and it paves the way for easy to handle calculus rules as established in [2].

Of the four aforementioned actions only the radial derivative $\partial_r\, T$ escapes, in general, from an unambiguous definition, but leads to an equivalent class of signumdistributions instead. Still we are able to define unambiguously $\partial_r\, T$ in two particular cases: (i) when the given distribution T is radial, i.e. rotation invariant, and (ii) when $T = U^\wedge$ is the associated distribution to a given radial signumdistribution U, these two particular cases being quite interesting since they correspond to two families of frequently used distributions such as the fundamental solutions of the Laplace and the Dirac operator, in Clifford analysis.

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Chapter 8
Applications of Parabolic Dirac Operators to the Instationary Viscous MHD Equations on Conformally Flat Manifolds

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In honor of Professor Sprößig's 70th birthday

Abstract In this paper we apply classical and recent techniques from quaternionic analysis using parabolic Dirac type operators and related Teodorescu and Cauchy-Bitzadse type operators to set up some analytic representation formulas for the solutions to the time dependent incompressible viscous magnetohydrodynamic equations on some conformally flat manifolds, such as cylinders and tori associated with different spinor bundles. Also in this context a special variant of hypercomplex Eisenstein series related to the parabolic Dirac operator serve as kernel functions.

Keywords Quaternionic integral operator calculus · Instationary incompressible viscous magnetohydrodynamics equations · Parabolic Dirac operators · Fundamental solutions · Conformally flat manifolds · PDE on spin manifolds

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8.1 Introduction

The magnetohydrodynamic equations (MHD) represent a combination of the Navier-Stokes system with the Maxwell system. They describe fluid dynamical processes under the influence of an electromagnetic field and have been the subject of investigation of numerous authors since more than 20 years. As classical references we emphasize [26] among others.

In general, there is a distinction made between the inviscid and the viscous MHD equations. On the one hand, the inviscid MHD equations play an important role in the description of the dynamic of astrophysical plasmas, for instance in the description of the magnetic phenomena of the heliosphere and in the prediction of the distribution of the solar wind density, see for example [16] and the references therein. On the other hand, the viscous MHD equations have attracted a growing interest by mathematicians and physicists over the last three decades. This topic is in the main focus of recent interest, see for instance [2, 15, 23, 29], where new criteria concerning the existence of global solutions and global well-posedness for particular geometrical settings, in particular axially symmetric settings are being developed. Also, it has recently been applied to medicine, such as in modelling of hydromagnetic blood flows [25]. More classical results can be found in [17].

In this paper we revisit the three dimensional instationary incompressible viscous MHD equations

$$-\frac{1}{Re}\Delta\mathbf{u}+\frac{\partial\mathbf{u}}{\partial t}+(\mathbf{u}\text{ grad})\,\mathbf{u}+\text{grad }p=\frac{1}{\mu_0}\text{rot}\mathbf{B}\times\mathbf{B}\ \text{ in } G \tag{8.1.1}$$

$$-\frac{1}{Rm}\Delta\mathbf{B}+\frac{\partial\mathbf{B}}{\partial t}+(\mathbf{u}\text{ grad})\,\mathbf{B}-(\mathbf{B}\text{ grad})\mathbf{u}=0\ \text{ in } G \tag{8.1.2}$$

$$\text{div }\mathbf{u}=0\ \text{ in } G \tag{8.1.3}$$

$$\text{div }\mathbf{B}=0\ \text{ in } G \tag{8.1.4}$$

$$\mathbf{u}=\mathbf{0},\ \ \mathbf{B}=\mathbf{h}\ \text{ at } \partial G. \tag{8.1.5}$$

In the context of this paper G is some arbitrary time-varying Lipschitz domain $G \subset \mathbb{R}^3 \times \mathbb{R}^+$. The symbol $\mathbf{u}$ represents the velocity of the flow, p the pressure, $\mathbf{B}$ the magnetic field, μ_0 is magnetic permeability of the vacuum and Re and Rm the fluid mechanical resp. magnetic Reynolds number. The first equation basically resembles the time dependent Navier-Stokes equation—the external force however is an unknown magnetic entity that also needs to be computed. Together with the second equation the dynamics of the magnetic field, the velocity, and the pressure, is described. The third equation manifests the incompressibility of the flow. The forth equation states the non-existence of magnetic monopoles. The remaining equations represent the measured (known) data at the boundary $\Gamma = \partial G$ of the domain G.

In [9, 14, 24] some global existence criteria for the weak solutions to the instationary 3D MHD equations have been presented. These works use modern harmonic analysis techniques as proposed in [4] for the incompressible Navier-Stokes equations. However, many theoretical questions concerning existence, uniqueness and regularity in the framework of general domains still remain open problems. In particular, one is interested in improving the explicitness of these criteria and in obtaining explicit analytic representation formulas for the solutions as well as for the Lipschitz contraction constant being valid in all kinds of Lipschitz domains—independently of the particular geometry of the domain.

Furthermore, we observed that in many cases dealing with large temporal distances, the classical time stepping methods (like the Rothe method) are valid for only small periods of time and, therefore, they often do not lead to the desired result. These obstacles motivate us to develop alternative methods.

Over the last three decades the quaternionic operator calculus proposed by K. Gürlebeck, W. Sprößig, M. Shapiro, V.V. Kravchenko, P. Cerejeiras, U. Kähler and by their collaborators, see for example [5, 7, 18, 20], provides an alternative analytic toolkit to treat the Navier-Stokes system, the Maxwell system and many other elliptic PDE. The quaternionic calculus leads to further new explicit criteria for the regularity, the existence and the uniqueness of the solutions. Moreover, it turned out to be also suitable to tackle strongly time dependent problems very elegantly. Based on the new theoretical results also new numerical algorithms could be developed, see for instance [13]. Also fully analytic representation formulas for the solutions to the Navier-Stokes equations and for the Maxwell and Helmholtz systems could be established for some special classes of domains, cf. [10, 11]. An important advantage of the quaternionic calculus is that the formulas hold universally for all bounded Lipschitz domains, independently of its particular geometry.

As shown already by Sijue Wu in [27], quaternionic analytic methods could also be applied to deal the well posedness problem in Sobolev spaces of the full 3D water wave problem, where previously well established methods did not lead to any success.

Since the quaternionic calculus provided an added value both in the treatment of the Navier-Stokes system and of the Maxwell system, it is natural to expect similar insightful results for the MHD system, since the latter one is a coupling of both systems. In [19] we explained how we can compute the solutions of the time independent stationary incompressible viscous MHD system with the quaternionic integral operator calculus. Recently complex quaternions have also been used by M. Tanisli, S. Demir, and T. Tolan to describe the dynamics of dyonic plasmas in an elegante way. In future work we plan to address the fully time-dependent incompressible viscous MHD equations using parabolic versions of the Dirac operator for modelling these type of equations independent of particular geometric constraints—except of regularity conditions on the boundaries

The aim of this paper is to exploit another advantage of quaternionic methods—namely that they are naturally predestinated to also address analogous MHD problems in the more general context of conformally flat spin manifolds that arise by factoring out some simply connected domain by a discrete Kleinian group. In this paper we specifically look at MHD problems on several kinds of conformally flat spin cylinders and tori as these are the most illustrative examples. In particular, this paper provides a generalization of the idea used in [8] were we addressed the "simpler" Navier-Stokes equations on these kind of manifolds without the influence of a magnetic field.

It is worth to mention that in the same way how we treat flat spin cylinders or tori we can also address their non-oriented conformally flat twisted analogues—namely the Möbius strip and the Kleinian bottle—where we have pin- instead of spin-structures.

The construction methods can easily be adapted by replacing the corresponding integral kernels. In this paper we explain how to explicit construct the integral kernels and how these are used in the resolution schemes for our specific MHD problem on the cylinders tori. We finalize with a brief look at particular rotation-invariant variants of these varieties and explain how our construction can easily be transferred to this setting.

8.2 Preliminaries

8.2.1 The Quaternionic Operator Calculus

By e_1, e_2, e_3 we denote the usual vector space basis $\mathbb{R}^3$. To introduce a multiplication operation on $\mathbb{R}^3$, we embed it into the algebra of Hamiltonian quaternions $\mathbb{H}$. A quaternion has the form $x = x_0 + \mathbf{x} := x_0 + x_1e_1 + x_2e_2 + x_3e_3$ where $x_0, \ldots, x_3$ are real numbers. Furthermore, x_0 is called the real part of the quaternion and will be denoted by $\Re(x)$. $\mathbf{x}$ is the vector part of x, also denoted by $\mathrm{Vec}(x)$. In the quaternionic setting the standard basis vectors play the role of imaginary units, we have $e_i^2 = -1$ for $i = 1, 2, 3$. Their mutual multiplication coincides with the usual vector product, i.e., $e_1e_2 = e_3, e_2e_3 = e_1, e_3e_1 = e_2$ and $e_ie_j = -e_je_i$ for $i \neq j$. We also need the quaternionic conjugation defined by $\overline{ab} = \overline{b}\,\overline{a}$, $\overline{e_i} = -e_i$, $i = 1, 2, 3$. The usual Euclidean norm extends to a norm on the whole quaternionic algebra, i.e. $|a| := \sqrt{\sum_{i=0}^{3} a_i^2}$.

The additional multiplicative structure of the quaternions allows us to describe all C^1-functions $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy both div $\mathbf{f} = 0$ and rot $\mathbf{f} = 0$ equivalently in a compact form as null-solutions to one single differential operator. The latter is the three-dimensional Euclidean Dirac operator $\mathbf{D} := \sum_{i=1}^{3} \frac{\partial}{\partial x_i} e_i$. In spin geometry this operator is also known as the Atiyah-Singer-Dirac operator. It naturally arises from the Levi-Civita connection in the context of general Riemannian spin manifolds, reducing to the above stated simple form in the flat case. In turn, the Euclidean

Dirac operator coincides with the usual gradient operator when this one is applied to a scalar-valued function. If $U \subseteq \mathbb{R}^3$ is an open subset, then a real differentiable function $f : U \to \mathbb{H}$ is called left quaternionic holomorphic or left monogenic in U, if $\mathbf{D} f = 0$. In the quaternionic calculus, the square of the Euclidean Dirac operator gives the Euclidean Laplacian up to a minus sign; we have $\mathbf{D}^2 = -\Delta$. Consequently, every real component of a left monogenic function is harmonic. This property allows us to treat harmonic functions with the function theory of the Dirac operator offering generalizations of many powerful theorems used in complex analysis. For deeper insight, we refer the reader for instance to [12, 18].

To treat time dependent problems in $\mathbb{R}^3$ we follow the ideas of [7] and introduce the "parabolic" basis elements $\mathfrak{f}$ and $\mathfrak{f}^\dagger$ which act in the following way

$$\mathfrak{f}\mathfrak{f}^\dagger + \mathfrak{f}^\dagger\mathfrak{f} = 1,$$
$$\mathfrak{f}^2 = (\mathfrak{f}^\dagger)^2 = 0,$$
$$\mathfrak{f} e_j = e_j \mathfrak{f} = 0,$$
$$\mathfrak{f}^\dagger e_j = e_j \mathfrak{f}^\dagger = 0.$$

The associated parabolic Dirac operators have the form

$$D^{\pm}_{\mathbf{x},t} := \sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j} + \mathfrak{f}\frac{\partial}{\partial t} \pm \mathfrak{f}^\dagger$$

and satisfy $(D^{\pm}_{\mathbf{x},t})^2 = -\Delta \pm \frac{\partial}{\partial t}$. The fundamental solution to $D^{+}_{\mathbf{x},t}$ has the form

$$G(\mathbf{x}, t) = \frac{H(t)\exp(-\frac{|\mathbf{x}|^2}{4t})}{(2\sqrt{\pi t})^3}\Big(\frac{1}{2t}\sum_{j=1}^{3} e_j x_j + \mathfrak{f}(\frac{3}{2t} + \frac{|\mathbf{x}|^2}{4t^2}) + \mathfrak{f}^\dagger\Big),$$

where $H(\cdot)$ stands for the usual Heaviside function. Solutions satisfying $D^{\pm}_{\mathbf{x},t} f = 0$ are called left parabolic monogenic (resp. antimonogenic).

For our needs we need the more general parabolic Dirac type operator, used for instance in [1, 6], having the form

$$D^{\pm}_{\mathbf{x},t,k} := \sum_{j=1}^{3} e_j \frac{\partial}{\partial x_j} + \mathfrak{f}\frac{\partial}{\partial t} \pm k\mathfrak{f}^\dagger$$

for a positive real $k \in \mathbb{R}$. This operator factorizes the second order operator

$$(D^{\pm}_{\mathbf{x},t,k})^2 = -\Delta \pm k^2 \frac{\partial}{\partial t}$$

and has very similar properties as the previously introduced one. Its nullsolutions are called left parabolic k-monogenic (resp. left parabolic k-antimonogenic) functions.

Adapting from [1, 6], the fundamental solution to $D^+_{\mathbf{x},t,k}$ turns out to have the form

$$E(\mathbf{x},t;k) = \sqrt{k}\,\frac{H(t)\exp(-\frac{k|\mathbf{x}|^2}{4t})}{(2\sqrt{\pi t})^3}\Big(\frac{k}{2t}\sum_{j=1}^{3} e_j x_j + \mathfrak{f}(\frac{3}{2t} + \frac{k|\mathbf{x}|^2}{4t^2}) + k\mathfrak{f}^\dagger\Big).$$

Suppose that G is in general a space-time varying bounded Lipschitz domain $G \subset \mathbb{R}^3 \times \mathbb{R}^+$. In what follows $W_2^{k,l}(G)$ denotes the parabolic Sobolev spaces of $L_2(G)$ where k is the regularity parameter with respect to $\mathbf{x}$ and l the regularity parameter with respect to t. For our needs we recall, cf. e.g. [1, 6, 7]

Theorem 8.2.1 (Borel-Pompeiu Integral Formula) *Let $G \subset \mathbb{R}^3 \times \mathbb{R}^+$ be a bounded or unbounded Lipschitz domain with a strongly Lipschitz boundary $\Gamma = \partial D$. Then for all $u \in W_2^{1,1}(G)$*

$$\int_\Gamma E(\mathbf{x}-\mathbf{y},t-t_0;k)d\sigma_{\mathbf{x},t}u(\mathbf{x},t) = u(\mathbf{y},t_0) + \int_G E(\mathbf{x}-\mathbf{y},t-t_0;k)D^+_{\mathbf{x},t;k}(u(\mathbf{x},t))dV dt,$$

where $d\sigma_{\mathbf{x},t} = D_{\mathbf{x},t}\rfloor dV dt$. The differential form $d\sigma_{\mathbf{x},t} = D_{\mathbf{x},t}\rfloor dV dt$ is the contraction of the operator $D_{\mathbf{x},t}$ with the volume element $dV dt$.

For $g \in \operatorname{Ker} D^+_{\mathbf{x},\mathbf{t};\mathbf{k}}$ one obtains the following version of Cauchy's integral formula for left parabolic k-monogenic functions in the form

$$\int_\Gamma E(\mathbf{x}-\mathbf{y},t-t_0;k)d\sigma_{\mathbf{x},t}u(\mathbf{x},t) = u(\mathbf{y},t_0).$$

Again, following the above cited works, one can introduce the parabolic Teodorescu transform and the Cauchy transform by

$$T_G u(\mathbf{y},t_0) = \int_G E(\mathbf{x}-\mathbf{y},t-t_0;k)u(\mathbf{x},t)dV dt$$

$$F_\Gamma u(\mathbf{y},t_0) = \int_\Gamma E(\mathbf{x}-\mathbf{y},t-t_0;k)d\sigma_{\mathbf{x},t}u(\mathbf{x},t).$$

Analogously to the Euclidean case one can rewrite the Borel-Pompeiu formula in the form

Lemma 8.2.2 *Let $u \in W_2^{1,0}(G)$. Then $T_G D^+_{\mathbf{x},t;k}u = u - F_\Gamma u$.*

On the other hand one has $D^+_{\mathbf{x},t:k}T_G u = u$. So, the parabolic Teodorescu operator is the right inverse to the parabolic Dirac operator.

The following direct decomposition of the space $L_2(G)$ into the subspace of functions that are square-integrable and left parabolic k-monogenic in the inside of G and its complement will be applied in this paper.

Theorem 8.2.3 (Hodge Decomposition) *Let $G \subseteq \mathbb{R}^3 \times \mathbb{R}^+$ be a bounded or unbounded Lipschitz domain. Then $L_2(G) = B(G) \oplus D^+_{\mathbf{x},t;k} \overset{\circ}{W}{}^{1,1}_2(G)$ where $B(G) := L_2(G) \cap \mathrm{Ker}\, D^+_{\mathbf{x},t;k}$ is the Bergman space of left parabolic k-monogenic functions, and where $\overset{\circ}{W}{}^{1,1}_2(G)$ is the subset of $W^{1,1}_2(G)$ with vanishing boundary data.*

Proofs of the above statements can be found for example in [1, 6, 7].

In what follows $\mathbf{P} : L_2(G) \to B(G)$ denotes the orthogonal Bergman projection while $\mathbf{Q} : L_2(G) \to D^+_{\mathbf{x},t} \overset{\circ}{W}{}^{1,1}_2(G)$ stands for the projection into the complementary space in all that follows. One has $\mathbf{Q} = \mathbf{I} - \mathbf{P}$, where $\mathbf{I}$ stands for the identity operator.

8.3 The Incompressible In-Stationary MHD Equations Revisited in the Quaternionic Calculus

In the classical vector analysis calculus the in-stationary viscous incompressible MHD equations have the form

$$-\frac{1}{Re}\Delta \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\ \mathrm{grad})\,\mathbf{u} + \mathrm{grad}\ p = \frac{1}{\mu_0}\mathrm{rot}\mathbf{B} \times \mathbf{B} \ \text{ in } G \quad (8.3.1)$$

$$-\frac{1}{Rm}\Delta \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} - (\mathbf{u}\ \mathrm{grad})\,\mathbf{B} + (\mathbf{B}\ \mathrm{grad})\mathbf{u} = 0 \ \text{ in } G \quad (8.3.2)$$

$$\mathrm{div}\ \mathbf{u} = 0 \ \text{ in } G \quad (8.3.3)$$

$$\mathrm{div}\ \mathbf{B} = 0 \ \text{ in } G \quad (8.3.4)$$

$$\mathbf{u} = \mathbf{0},\ \mathbf{B} = \mathbf{h} \ \text{ at } \partial G \quad (8.3.5)$$

with given boundary data $\mathbf{u}|_{\partial G} = \mathbf{g} = \mathbf{0}$ and $\mathbf{B}|_{\partial G} = \mathbf{h}$. To apply the quaternionic integral operator calculus to solve these equations we first express this system in the quaternionic language.

First we recall that we have for a time independent quaternionic function $f : \mathbb{R}^4 \to \mathbb{R}^4$, where $(x_0 + \mathbf{x}) \to f(x_0 + \mathbf{x}) = f_0(x_0 + \mathbf{x}) + \mathbf{f}(x_0 + \mathbf{x})$, the relation $\mathcal{D} f = \mathrm{grad}\ f_0 + \mathrm{rot}\ \mathbf{f} - \mathrm{div}\ \mathbf{f}$. Here $f_0 = \Re(f)$ is the scalar part of f while $\mathbf{f} = \mathrm{Vec}(f) \in \mathbb{R}^3$ represents the vectorial part of f, and $\mathcal{D} := \sum_{i=0}^{3} e_i \frac{\partial}{\partial x_i}$ is the quaternionic Cauchy-Riemann operator. Its vector part, denoted by $\mathbf{D}$, is the three dimensional Euclidean Dirac operator introduced in the previous section. In the case where $\mathbf{f}$ is a vector valued function, i.e. a function defined in an open subset of $\mathbb{R}^3$

with values in $\mathbb{R}^3$ we have $\mathbf{Df} = \text{rot } \mathbf{f} - \text{div } \mathbf{f}$. If p is a scalar valued function defined in an open subset of $\mathbb{R}^3$, then we have $\mathbf{D}p = \text{grad } p$.

When applying these rules to the magnetic vector field $\mathbf{B} \in \mathbb{R}^3$ we obtain that $\mathbf{DB} = \text{rot } \mathbf{B} - \text{div } \mathbf{B}$. In view of Eq. (8.1.4) which expresses that there are no magnetic monopoles, this equation reduces to $\mathbf{DB} = \text{rot } \mathbf{B}$. Furthermore, we can express $(\mathbf{DB}) \times \mathbf{B} = \text{Vec}((\mathbf{DB}) \cdot \mathbf{B})$ in terms of the quaternionic product $\cdot$. The divergence of an $\mathbb{R}^3$-valued vector field $\mathbf{f}$ can be expressed as $\text{div } \mathbf{f} = \Re(\mathbf{Df})$. The three-dimensional Euclidean Laplacian $\Delta = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$ can be expressed in terms of the Dirac operator as $\Delta = -\mathbf{D}^2$, applying the rule $e_i^2 = -1$ for all $i = 1, 2, 3$.

Let us next assume that our functions are also dependent on the time variable t. Applying the formulas from the preceding section allow us to express the entities $-\frac{1}{Re}\Delta \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t}$ and $-\frac{1}{Rm}\Delta \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t}$ in the form

$$-\frac{1}{Re}\Delta \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} = (D^{+}_{\mathbf{x},t,Re})^2 \mathbf{u}$$

$$-\frac{1}{Rm}\Delta \mathbf{B} + \frac{\partial \mathbf{B}}{\partial t} = (D^{+}_{\mathbf{x},t,Rm})^2 \mathbf{B}.$$

with

$$D^{+}_{\mathbf{x},t,Re} \mathbf{u} = \frac{1}{\sqrt{Re}} \mathbf{Du} + \mathfrak{f}\partial_t \mathbf{u} + \mathfrak{f}^{\dagger} \mathbf{u}$$

$$D^{+}_{\mathbf{x},t,Rm} \mathbf{B} = \frac{1}{\sqrt{Rm}} \mathbf{DB} + \mathfrak{f}\partial_t \mathbf{B} + \mathfrak{f}^{\dagger} \mathbf{B}$$

Thus, the previous system (together with the mentioned restrictions) can be reformulated in quaternionic form in the following way:

$$(D^{+}_{\mathbf{x},t,Re})^2 \mathbf{u} + \Re(\mathbf{u}\,\mathbf{D})\,\mathbf{u} + \mathbf{D}\,p = \frac{1}{\mu_0} \text{Vec}((\mathbf{DB}) \cdot \mathbf{B}) \text{ in } G \quad (8.3.6)$$

$$(D^{+}_{\mathbf{B},t,Rm})^2 \mathbf{B} - \Re(\mathbf{u}\,\mathbf{D})\,\mathbf{B} + \Re(\mathbf{B}\,\mathbf{D})\mathbf{u} = 0 \text{ in } G \quad (8.3.7)$$

$$\Re(\mathbf{Du}) = 0 \text{ in } G \quad (8.3.8)$$

$$\Re(\mathbf{DB}) = 0 \text{ in } G \quad (8.3.9)$$

$$\mathbf{u} = \mathbf{0}, \ \mathbf{B} = \mathbf{h} \text{ at } \partial G. \quad (8.3.10)$$

The aim is now to apply the previously introduced hypercomplex integral operators in order to get computation formulas for the magnetic field $\mathbf{B}$, the velocity $\mathbf{u}$, and the pressure p.

We remark that whenever we fix the magnetic field $\mathbf{B}$ in the stationary version of Eq. (8.3.6) we obtain (in the weak sense) the pressure p and the velocity $\mathbf{u}$, c.f. [28]. In a similar way, given $(\mathbf{u}, \mathbf{p})$ in Eq. (8.3.7) we can recover the magnetic field $\mathbf{B}$.

Moreover, the solution for magnetic field is unique if the operator is hypoelliptic. These results hold for the in-stationary case.

8.4 The MHD Equations in the More General Context of Some Conformally Flat Spin 3-Manifolds

Due to the conformal invariance of the Dirac operator, the related quaternionic differential and integral operator calculus canonically provides a simple access to easily transfer the results and representation formulas summarized in the previous section to the context of addressing analogous boundary value problems within the more general context of conformally flat spin manifolds.

As a consequence of the famous Liouville theorem, in dimensions $n \geq 3$ conformally flat manifolds are explicitly only those that possess atlasses whose transition functions are Möbius transformations, because these are the only conformal transformations in $\mathbb{R}^n$ whenever $n \geq 3$. The treatment with quaternions (or with Clifford numbers in general) allow us to represent Möbius transformations in the compact form $f(x) = (ax + b)(cx + d)^{-1}$ where a, b, c, d are quaternions satisfying to certain constraints, cf. [3].

Already the classical paper [21] mentions one possibility to construct a number of examples of conformally flat manifolds, namely by factoring out a subdomain $\mathcal{U}$ of $\mathbb{R}^3$ by a torsion-free subgroup Γ of the group of Möbius transformations Γ, under the additional condition that the latter acts strongly discontinuously on $\mathcal{U}$.

The topological quotient $\mathcal{U}/\Gamma$ then is a conformally flat manifold. Of course, this construction just addresses a subclass of all conformally flat manifolds. However, this subclass can be characterized in an intrinsic way. As shown in [21], the class of conformally flat manifolds of the form U/Γ are exactly those for which the universal cover of this manifold admits a local conformal diffeomorphism into S^3 which is a covering map $\tilde{\mathcal{U}} \to \mathcal{U} \subset S^3$.

The most popular examples are 3-tori, cylinders, real projective (rotation invariant) space and the hyperbolic manifolds considered in [3] that arise by factoring upper half-spaces, cones or positivity domains by arithmetic subgroups of higher dimensional generalizations of the modular or Fuchsian group [3].

In order to generalize and to apply the representation formulas and the results that we obtained in the previous sections for the instationary MHD system to the context of analogous instationary boundary value problems on conformally manifolds we only need to introduce the properly adapted analogues of the parabolic Dirac operator as well as the other hypercomplex integral operators on these manifolds. From the geometric point of view one is particularly interested in those conformally flat manifolds that have a spin structure, that means those that admit the construction of at least one spinor bundle over such a manifold. In many cases one gets more than just one spin structure which leads to the consideration of (several) spinor sections, in our case quaternionic spinor sections. For the geometric background we refer to [22].

We explain the method at the simplest non-trivial example dealing with conformally flat spin 1,2-cylinders and 3-tori with inequivalent spinor bundles. This special example illustrates in a nice way how one can transfer the results and construction method to other examples of conformally flat (spin) manifolds that again are constructed by factoring out a connected domain by a discrete arithmetic group of some higher dimensional modular groups, such as those roughly outlined above.

For the sake of simplicity, let $\Omega_3 := \mathbb{Z}e_1 + \mathbb{Z}e_2 + \mathbb{Z}e_3$ be the orthonormal lattice in $\mathbb{R}^3$. Then the topological quotient space $\mathbb{R}^3/\Omega_3$ represents a three-dimensional conformally flat compact torus denoted by T_3, over which one can construct exactly eight different conformally inequivalent spinor bundles over T_3. With the additional time coordinate $t > 0$, this leads to the consideration of a toroidal time half-cylinder of the form $\Omega_3 \times [0, \infty)$ which then represents a non-compact manifold with boundary in upper half space of $\mathbb{R}^4 \cong \mathbb{H}, t > 0$, denoted by $\mathbb{H}^+$. The invariance group is an abelian subgroup of the hypercomplex modular group $SL(2, \mathbb{H}^+)$ just acting on the space coordinates. More generally, we can also factor out sublattices of the form $\Omega_p := \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_p$ where $1 \leq p \leq 3$. The topological quotients $\mathbb{R}^3/\Omega_p$ are 1-resp. 2-cylinders in the cases $p = 1$ and $p = 2$ respectively, having infinite extensions also in x_3- (resp. also in the x_2-) coordinate direction.

We recall that in general different spin structures on a spin manifold M are detected by the number of distinct homomorphisms from the fundamental group $\Pi_1(M)$ to the group $\mathbb{Z}_2 = \{0, 1\}$. In the case of the 3-torus we have $\Pi_1(T_3) = \mathbb{Z}^3$. There are two homomorphisms of $\mathbb{Z}$ to $\mathbb{Z}_2$. The first one is $\theta_1 : \mathbb{Z} \to \mathbb{Z}_2 : \theta_1(n) = 0$ mod 2 while the second one is the homomorphism $\theta_2 : \mathbb{Z} \to \mathbb{Z}_2 : \theta_2(n) = 1$ mod 2. Consequently there are 2^3 distinct spin structures on T_3, or more generally, 2^p different spin structures on T_p with $p \leq 3$.

For the sake of generality, in what follows let $p \in \{1, 2, 3\}$. It is very easy to construct all conformally inequivalent different spinor bundles over T_p. To describe them let l be an integer in the set $\{1, 2, 3\}$, and consider the sublattice $\mathbb{Z}^l = \mathbb{Z}e_1 + \ldots + \mathbb{Z}e_l$ where $(0 \leq l \leq p)$. For $l = 0$ we put $\mathbb{Z}^0 := \emptyset$. There is also the remainder lattice $\mathbb{Z}^{p-l} = \mathbb{Z}e_{l+1} + \ldots + \mathbb{Z}e_p$. In this case $\mathbb{Z}^p = \{\underline{m} + \underline{n} : \underline{m} \in \mathbb{Z}^l$ and $\underline{n} \in \mathbb{Z}^{p-l}\}$. Let us now assume that $\underline{m} = m_1e_1 + \ldots + m_le_l$. We identity $(\mathbf{x}, X)$ with $(\mathbf{x} + \underline{m} + \underline{n}, (-1)^{m_1+\ldots+m_l}X)$ where $\mathbf{x} \in \mathbb{R}^3$ and $X \in \mathbb{H}$. This identification gives rise to a quaternionic spinor bundle $E^{(l)}$ over T_p.

Clearly, $\mathbb{R}^3$ is the universal covering space of T_p. Thus, there is a well-defined projection map $\mathcal{P} : \mathbb{R}^3 \times \mathbb{R}^+ \to T_p \times \mathbb{R}^+$, by identifying $(\mathbf{x} + \omega, t)$ with all equivalent points of the form $(\mathbf{x} \bmod \Omega_p, t)$.

As explained for example in [3] every p-fold periodic resp. anti-periodic open set $\mathcal{U} \subset \mathbb{R}^3$ and every p-fold periodic resp. anti-periodic section $f : \mathcal{U}' \times [0, \infty) \to E^{(l)}$, which satisfies $f(\mathbf{x}, t) = (-1)^{m_1+\cdots+m_l}(\mathbf{x} + \omega, t)$ for all $\omega \in \mathbb{Z}^l \oplus \mathbb{Z}^{p-l}$, descends to a well-defined open set $U' := \mathcal{P}(\mathcal{U}) \times [0, \infty) \subset T_p \times [0, \infty)$ (associated with that particularly chosen spinor bundle) and a well-defined spinor section $f' := \mathcal{P}(f) : U' \subset T_p \times [0, \infty) \to E^{(l)} \subset \mathbb{H}$, respectively.

The projection $\mathcal{P} : \mathbb{R}^3 \times [0, \infty) \to T_p \times [0, \infty)$ induces well-defined cylindrical resp. toroidal modified parabolic Dirac operators on $T_p \times \mathbb{R}^+$ by $\mathcal{P}(D^{\pm}_{\mathbf{x},t,k}) =: \mathcal{D}^{\pm}_{\mathbf{x},t,k}$ acting on spinor sections of $T_p \times \mathbb{R}^+$. Sections defined on open sets U of $T_p \times \mathbb{R}^+$ are called cylindrical resp. toroidal k-left parabolic monogenic if $\mathcal{D}^{\pm}_{\mathbf{x},t,k} s = 0$ holds in U. By $\tilde{D} := \mathcal{P}(\mathbf{D})$ we denote the projection of the time independent Euclidean Dirac operator down to the cylinder resp. torus T_p.

We denote the projections of the p-fold (anti-)periodization of the function $E(\mathbf{x}, t; k)$ by

$$\mathcal{E}(\mathbf{x}, t; k) := \sum_{\omega \in \mathbb{Z}^p \oplus \mathbb{Z}^{p-l}} (-1)^{m_1+\cdots+m_l} E(\mathbf{x} + \omega, t; k).$$

This generalized parabolic monogenic Eisenstein type series provides us with the fundamental section to the cylindrical resp. toroidal parabolic modified Dirac operator $\mathcal{D}^{\pm}_{\mathbf{x},t,k}$ acting on the corresponding spinor bundle of the space cylinder resp. space torus T_p. Indeed, the function $\mathcal{E}(\mathbf{x}, t; k)$ can be regarded as the canonical generalization of the classical elliptic Weierstraß $\wp$-function to the context of the modified Dirac operator $D^{+}_{\mathbf{x},t,k}$ in three space variables x_1, x_2, x_3 and the positive time variable $t > 0$.

To show that $\mathcal{E}(\mathbf{x}, t; k)$ is well-defined parabolic monogenic spinor section on the manifold $T_p \times [0, \infty)$, we have to show that this series actually converges. The regularity behavior then is guaranteed by the application of the Weierstraß convergence theorem.

Theorem 8.4.1 *Let* $1 \leq p \leq 3$. *Then the function series*

$$\mathcal{E}(\mathbf{x}, t; k) = \sum_{\omega \in \mathbb{Z}^p \oplus \mathbb{Z}^{p-l}} (-1)^{m_1+\cdots+m_l} E(\mathbf{x} + \omega, t; k)$$

converges uniformally on any compact subset of $\mathbb{R}^3 \times \mathbb{R}^+$.

Proof The simplest way to prove the convergence is to decompose the full lattice $\mathbb{Z}^p$ into the the following particular union of lattice points $\Omega = \bigcup_{m=0}^{+\infty} \Omega_m$ where

$$\Omega_m := \{\omega \in \mathbb{Z}^p \mid |\omega|_{max} = m\}.$$

Next one defines

$$L_m := \{\omega \in \mathbb{Z}^p \mid |\omega|_{max} \leq m\}.$$

The subset L_m contains exactly $(2m + 1)^p$ points. Hence, the cardinality of Ω_m precisely is $\sharp\Omega_m = (2m + 1)^p - (2m - 1)^p$. Notice that this particular construction admits that Euclidean distance between the set Ω_{m+1} and the Ω_m is exactly $d_m := dist_2(\Omega_{m+1}, \Omega_m) = 1$. This is the motivation for this particular decomposition.

Next, as a standard calculus argument one fixes a compact subset $\mathcal{K} \subset \mathbb{R}^3$ and one considers $t > 0$ as an arbitrary but fixed value. Then there exists a $r \in \mathbb{R}$ such that all $\mathbf{x} \in \mathcal{K}$ satisfy $|\mathbf{x}|_{max} \leq |\mathbf{x}|_2 < r$.

Let $\mathbf{x} \in \mathcal{K}$. For the convergence it suffice to consider those points with $|\omega|_{max} \geq [r] + 1$.

As a consequence of the standard argumentation

$$|\mathbf{x} + \omega|_2 \geq |\omega|_2 - |\mathbf{x}|_2 \geq |\omega|_{max} - |\mathbf{x}|_2 = m - |\mathbf{x}|_2 \geq m - r$$

one may arrive at

$$\begin{aligned}
&\sum_{m=[r]+1}^{+\infty} \sum_{\omega \in \Omega_m} |E(\mathbf{x}, t; k)(\mathbf{x} + \omega)|_2 \\
&\leq \frac{k}{(2\sqrt{\pi t})^3} \sum_{m=[r]+1}^{+\infty} \sum_{\omega \in \Omega_m} \exp(-k|\mathbf{x} + \omega|_2/4t)\Big(\frac{k}{2t}|\mathbf{x} + \omega|_2 + \mathfrak{f}(\frac{3}{2t} + \frac{k|\mathbf{x} + \omega|_2^2}{4t^2}) + k\mathfrak{f}^\dagger\Big) \\
&\leq \frac{k}{(2\sqrt{\pi t})^3} \sum_{m=[r]+1}^{+\infty} \Big([(2m+1)^p - (2m-1)^p]\big(\frac{k(r+m)}{2t} + \mathfrak{f}(\frac{3}{2t} + \frac{k(r+m)^2}{4t^2}) + k\mathfrak{f}^\dagger\big) \\
&\qquad \times \ \exp(\frac{-k(m-r)^2}{4t})\Big),
\end{aligned}$$

in view of $m - r \geq [r] + 1 - r > 0$. This sum is absolutely uniformly convergent because of the exponential decreasing term which dominates the polynomial expressions in m. Due to the absolute convergence, the series

$$\mathcal{E}(\mathbf{x}, t; k) := \sum_{\omega \in \mathbb{Z}^l \oplus \mathbb{Z}^{p-l}} (-1)^{m_1 + \cdots + m_l} E(\mathbf{x} + \omega, t; k),$$

which can be can be rearranged in the requested form

$$\mathcal{E}(\mathbf{x}, t; k) := \sum_{m=0}^{+\infty} \sum_{\omega \in \Omega_m} (-1)^{m_1 + \cdots + m_l} E(\mathbf{x} + \omega, t; k),$$

converges normally on $\mathbb{R}^3 \times \mathbb{R}^+$. Since $E(\mathbf{x} + \omega, t; k)$ belongs to Ker $D^+_{\mathbf{x},t,k}$ in each $(\mathbf{x}, t) \in \mathbb{R}^3 \times \mathbb{R}^+$ the series $\mathcal{E}(\mathbf{x}, t; k)$ satisfies $D^+_{\mathbf{x},t,k}\mathcal{E}(\mathbf{x}, t; k) = 0$ in each $\mathbf{x} \in \mathbb{R}^3 \times \mathbb{R}^+$, which, as mentioned previously, follows from the classical standard Weierstraß convergence argument. ■

Obviously, by a direct rearrangement argument, one obtains that

$$\mathcal{E}(\mathbf{x}, t; k) = (-1)^{m_1+\cdots+m_l}\mathcal{E}(\mathbf{x}+\omega, t; k) \quad \forall \omega \in \Omega$$

which shows that the projection of this kernel correctly descends to a section with values in the spinor bundle $E^{(l)}$. The projection $\mathcal{P}(\mathcal{E}(\mathbf{x}, t; k))$ denoted by $\tilde{\mathcal{E}}(\mathbf{x}, t; k)$ is the fundamental section of the cylindrical (resp. toroidal) modified parabolic Dirac operator $\tilde{D}^{+}_{\mathbf{x},t,k}$. For a time-varying Lipschitz domain $G \subset T_3 \times \mathbb{R}^+$ with a strongly Lipschitz boundary Γ we can now proceed to define, similarly to our description in the previous sections, the canonical analogue of the Teodorescu and of the Cauchy-Bitzadse transform for toroidal k-monogenic parabolic quaternionic spinor valued sections by

$$\tilde{T}_G u(\mathbf{y}, t_0) = \int_G \tilde{\mathcal{E}}(\mathbf{x}-\mathbf{y}, t-t_0; k)u(\mathbf{x}, t)dVdt$$

$$\tilde{F}_\Gamma u(\mathbf{y}, t_0) = \int_\Gamma \tilde{\mathcal{E}}(\mathbf{x}-\mathbf{y}, t-t_0; k)d\sigma_{\mathbf{x},t}u(\mathbf{x}, t).$$

To transfer the integral operator calculus from the flat Euclidean space setting to our setting we introduce the following norms on the manifolds and on the sections with values in the associated spinor bundles. Let $(\mathbf{x}', t)$ be an arbitrary point on $T_p \times [0, \infty)$. Then we put for $1 \le q \le \infty$:

$$\|(\mathbf{x}', t)\|_{T_p,q} := \|\mathcal{P}^{-1}(\mathbf{x}', t)\|_q := \min_{\omega \in \Omega_p} \|(\mathbf{x}+\omega, t)\|_q$$

where $\|\cdot\|_q$ is the usual q-norm on $\mathbb{R}^3 \times [0, \infty)$.

Next we define the L_q-norm on an arbitrary quaternionic spinor section $f' : U' := \mathcal{U} \times [0, \infty) \subset T_p \times [0, \infty) \to E^{(l)} \subset \mathbb{H}$ with values in one of the previously described spinor bundles $E^{(l)}$ by:

$$\|f'\|_{L_q(U')} := \sqrt[q]{\int_U \min_{\omega \in \Omega_p} \{\|\mathcal{P}^{-1} f((\mathbf{x}+\omega, t))\|^q\} d\mathbf{x}dt}$$

Similarly, for $q < \infty$ we may introduce the adequate Sobolev spaces of derivative degree up to a fixed $k \ge 1$ by:

$$\|f'\|_{W_q^k(U')} := \left(\|f\|^q_{L^2(U')} + \sum_{0 < \|\alpha\| + \beta \le k} \left\| \frac{\partial^{|\alpha|+\beta}}{\partial \mathbf{x}^\alpha \partial t^\beta} \right\|^q_{L^2(U')} \right)^{1/q}.$$

An important property is the L_1-boundedness of the cylindrical (toroidal) fundamental solution $\tilde{\mathcal{E}}(\mathbf{x}', t)$ in the norm $\|\cdot\|_{L_1}$. To justify this we note that in view of using the particular definition of the norm $\| \cdot \|_{T_p,1}$ we obtain:

$$\|\tilde{\mathcal{E}}\|_{L_1} = \int_{U'} \|\tilde{\mathcal{E}}(\mathbf{x}', t)\|_{T_p,1} d\mathbf{x}' dt$$

$$= \int_{U} \min_{\omega \in \Omega_p} \|E(\mathbf{x} + \omega, t)\|_1 d\mathbf{x} dt < \infty,$$

since the fundamental solution E is an L_1-function over any bounded domain U in $\mathbb{R}^3 \times \mathbb{R}^+$ according to [7]. This allows us directly to establish

Proposition 8.4.2 *Let $1 \leq q < \infty$. Let $G' \subset T_p \times [0, \infty)$ be a bounded domain. Then the operator $\tilde{T}_{G'}$ is bounded from $L_q(G')$ to $L_q(G')$.*

Proof In view of Young's inequality we have

$$\|\tilde{T}_{G'} g\|_{L_q(G')} = \|\tilde{\mathcal{E}} * g\|_{L_q(G)'} \leq \|\tilde{\mathcal{E}}\|_{L_1(G')} \cdot \|g\|_{L_q(G')}.$$

Since $\|\tilde{\mathcal{E}}\|_{L_1(G')}$ is a finite expression whenever G' is bounded, as shown previously, we obtain the L_q-boundedness of $\tilde{T}_{G'}$. □

As furthermore shown in [7] also the partial derivatives of $E(\mathbf{x}, t)$ are L_1-bounded under the condition that G is a bounded domain, we directly obtain by a similar argument the following

Proposition 8.4.3 *Let $1 \leq q < \infty$. Let $G' \subset T_p \times [0, \infty)$ be a bounded domain. Then the partial derivatives of the operator $\tilde{T}_{G'}$ with respect to x_k ($k = 1, 2, 3$) satisfy the mapping property:*

$$\partial_{x_k}(\tilde{T}_{G'} g) : L_q(G') \to L_q(G'), \; k = 1, 2, 3$$

and are bounded.

To the proof one again only needs to apply Young's inequality leading to

$$\|\partial_{x_k}(\tilde{T}_{G'} g)\|_{L_q(G')} = \|(\partial_{x_k}\tilde{\mathcal{E}}) * g\|_{L_q(G)'} \leq \|\partial_{x_k}\tilde{\mathcal{E}}\|_{L_1(G')} \cdot \|g\|_{L_q(G')}.$$

As a direct consequence of these two propositions we may now establish the important result

Theorem 8.4.4 *Let $p \in \{1, 2, 3\}$, $1 \leq q < \infty$ and let $k \in \mathbb{N}$. Let G' be a bounded domain in the time p-cylinder (torus) $T_p \times [0, \infty)$. Then the operator $\tilde{T}_{G'} : L_q(G') \to W_q^k(G')$ is continuous.*

This property together with the Borel-Pompeiu formula presented in Sect. 8.2 also implies that the operator

$$\tilde{F}_\Gamma : W_q^{k-1/q}(\Gamma) \rightarrow W_q^k(G')$$

is continuous.

To complete the quaternionic integral calculus toolkit, the associated Bergman projection can be introduced by

$$\tilde{\mathbf{P}} = \tilde{F}_\Gamma (tr_\Gamma \tilde{T}_G \tilde{F}_\Gamma)^{-1} tr_\Gamma \tilde{T}_G.$$

and $\tilde{\mathbf{Q}} := \tilde{\mathbf{I}} - \tilde{\mathbf{P}}$.

Now, adapting from [11] we obtain a direct analogy of Theorem 1, Lemma 1 and Lemma 2 on these conformally flat time cylinders rep. time tori using these time cylindrical (toroidal) versions $\tilde{T}_G$, $\tilde{F}_\Gamma$ and $\tilde{\mathbf{P}}$ of operators introduced in Sect. 8.2. Suppose next that we have to solve an MHD problem of the form (1)–(5) within a Lipschitz domain $G \subset T_3 \times \mathbb{R}^+$ with values in the spinor bundle $E^{(l)} \times \mathbb{R}^+$. Then, imposing certain regularity conditions, which will be discussed in very detail in our future work, we can compute its solutions by simply applying the following adapted iterative algorithm

$$\begin{aligned}
\mathbf{u}_n &= \frac{Re}{\mu_0} \tilde{T}_G \tilde{\mathbf{Q}} \tilde{T}_G \Big[\operatorname{Vec}((\tilde{D}\mathbf{B}_{n-1}) \cdot \mathbf{B}_{n-1}) - \Re(\mathbf{u}_{n-1}\tilde{D})\mathbf{u}_{n-1} \Big] \\
&\quad - Re^2 \tilde{T}_G \tilde{\mathbf{Q}} \tilde{T}_G \tilde{D} p_n \\
\Re(\tilde{\mathbf{Q}} \tilde{T}_G \tilde{D} p_n) &= \frac{1}{\mu_0} \Re \Big[\tilde{\mathbf{Q}} \tilde{T}_G \operatorname{Vec}((\tilde{D}\mathbf{B}_{n-1}) \cdot \mathbf{B}_{n-1}) - \Re(\mathbf{u}_{n-1}\tilde{D})\mathbf{u}_{n-1} \Big] \\
\mathbf{B}_n &= Rm^2 \tilde{T}_G \tilde{\mathbf{Q}} \tilde{T}_G \Big[\Re(\mathbf{B}_n \tilde{D})\mathbf{u}_n - \Re(\mathbf{u}_n \tilde{D})\mathbf{B}_n \Big]. \\
\mathbf{B}_n{}^{(i)} &= Rm^2 \tilde{T}_G \tilde{\mathbf{Q}} \tilde{T}_G \Big[\Re(\mathbf{B}_n{}^{(i-1)} \tilde{D})\mathbf{u}_n - \Re(\mathbf{u}_n \tilde{D})\mathbf{B}_n{}^{(i-1)} \Big]
\end{aligned}$$

Again, in our future work, we will address a number of concrete existence and uniqueness criteria for the solutions computed by this fixed point algorithm involving some a priori estimate conditions.

Anyway, it is now clear how this approach even carries over to more general conformally flat spin manifolds that arise by factoring out a simply connected domain U by a discrete Kleinian group Γ. The Cauchy-kernel is constructed by the projection of the Γ-periodization (involving eventually automorphy factors like in [3]) of the fundamental solution $E(\mathbf{x}; t; k)$. With this fundamental solution we construct the corresponding integral operators on the manifold. In terms of these integral operators we can express the solutions of the corresponding MHD boundary value problem on these manifolds, simply by replacing the usual hypercomplex integral operators by its adequate analogies on the manifold. In this framework, of course one has to introduce the adequated norms and to consider the adequated function spaces accordingly.

This again underlines the highly universal character of our approach to treat the MHD equations but also many other complicated elliptic, parabolic, hypoelliptic and hyperbolic PDE systems with the quaternionic operator calculus using Dirac operators. Furthermore, the representation formulas and results also carry directly over to the n-dimensional case in which one simply replaces the corresponding quaternionic operators by Clifford algebra valued operators, such as suggested in [7, 11].

To round off we establish a further result on the invariance behavior of the kernel functions under rotations of S^3 applied to the spatial coordinates. More precisely, we have:

Theorem 8.4.5 *Let $a \in S^3 := \{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x}\| = 1\}$. Then the Cauchy kernel of the parabolic Dirac operator satisfies the invariance property $\overline{a}E(a\mathbf{x}\overline{a}, t; k)a = E(\mathbf{x}, t; k)$ for all $a \in S^3$.*

Proof Let us consider the expression:

$$\begin{aligned}\overline{a}E(a\mathbf{x}\overline{a}, t; k)a &= \overline{a}\left(\frac{H(t)\exp(-\frac{|a\mathbf{x}\overline{a}|^2}{4t})}{(2\sqrt{\pi t})^3}\left(\frac{1}{2t}a\mathbf{x}\overline{a} + \mathfrak{f}(\frac{3}{2t} + \frac{|a\mathbf{x}\overline{a}|^2}{4t^2}) + \mathfrak{f}^\dagger\right)\right)a \\ &= \frac{H(t)\exp(-\frac{|\mathbf{x}|^2}{4t})}{(2\sqrt{\pi t})^3}\left(\frac{1}{2t}\overline{a}a\mathbf{x}\overline{a}a + \overline{a}\mathfrak{f}(\frac{3}{2t} + \frac{|\mathbf{x}|^2}{4t^2})a + \overline{a}\mathfrak{f}^\dagger a\right) \\ &= E(\mathbf{x}, t; k)\end{aligned}$$

where we applied the properties that $a\overline{a} = \|a\|^2 = 1$, $\overline{a}\mathfrak{f}a = \mathfrak{f}$ and $\overline{a}\mathfrak{f}^\dagger a = \mathfrak{f}^\dagger$. □

This property opens the door to treat a class of S^3-invariant manifolds. More precisely, by identifying all points of the form $(a\mathbf{x}\overline{a}, t)$ with $(\mathbf{x}, t)$ we can construct a class of rotation invariant projective orbifolds which under certain constraints on **a** will be manifolds again.

Notice also the cylindrical and toroidal kernels $\mathcal{E}(\mathbf{x}', t)$ exhibit this rotation invariance behavior. This is due to the fact that each single term in the series itself exhibits this rotation invariance property, so that the whole series turn out to have this property.

Moreover, this new identification can additionally be combined with the cylindrical (toroidal) translation invariance where one applies the identification of all Ω_p-equivalent points. This gives rise to an identification of all points of the time cylinder (torus) $(a\mathbf{x}'\overline{a}, t)$ with $(\mathbf{x}', t)$. The associated orbifold resulting from this identification that has both a translation and a rotation invariant structure. In some dimensions we even obtain manifolds.

In the case where we restrict to those points from the unit sphere $a \in S^3$ such that there is a finite number $n \in \mathbb{N}$ with $a^n = 1$ which yields a finite cyclic group of rotations $\mathcal{A} := \{a, a^2, \ldots, a^n\}$, then the corresponding Cauchy kernel can again be constructed by an Eisenstein type series. The latter then has the explicit form

$$\mathcal{E}_{\mathcal{A}}(\mathbf{x}, t; k) = \sum_{a \in \mathcal{A}} \sum_{\omega \in \mathbb{Z}^p \oplus \mathbb{Z}^{p-l}} (-1)^{m_1 + \cdots + m_l} \overline{a} E(a\mathbf{x}\overline{a} + \omega, t; k) a$$

which then descends to a projective rotational variant of the cylinders/tori discussed previously. Since $\mathcal{A}$ only has a finite cardinality, the convergence of this series is guaranteed by the argument of Theorem 8.4.1.

Once one has that the kernel function, one again can introduce the corresponding Teodorescu and Cauchy Bitzadse operators involving these explicit kernels in the same way as performed previously to also address the corresponding boundary value problems in these kinds of geometries introducing the norms properly. This once more underlines the geometric universality of our approach where we do nothing else than exploiting the conformal invariance of the Dirac operator.

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Chapter 9
Generalized Riesz Transforms, Quasi-Monogenic Functions and Frames

Swanhild Bernstein and Sandra Schufmann

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract Monogenic functions can be extended to quasi-monogenic functions using Fourier multipliers. It turns out that the whole class of quasi-monogenic signals has similar properties than the monogenic signal based on the Riesz transforms. Quasi-monogenic Riesz transforms can be used to construct frames. We use quasi-monogenic functions to construct a linearized Riesz transform in $\mathbb{R}^3$ that allows to define quasi-monogenic shearlets in the cone in $\mathbb{R}^3$. We further prove that Riesz transforms and linearized Riesz transforms are L^p multipliers (for $1 < p < \infty$ and similarly for $p = 1$).

Keywords Quasi-monogenic functions · Riesz transforms · Fourier symbol · Generalized Riesz transform

Mathematics Subject Classification (2010) Primary 30G35; Secondary 44A35, 47B34, 92C55

9.1 Introduction

Clifford analysis [5–7, 11, 12] is a refinement of harmonic analysis and function theory. Complex function theory has a lot of applications. One application in signal theory is the analytic signal introduced by Gabor [9], which is a quadrature filter and can be mathematically described as values of an analytic function. The importance and relevance of the analytic signal have forced a search for higher dimensional

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versions. One is the higher dimensional analytic signal by Hahn [13], which describes boundary values of an analytic function in $\mathbb{C}^n$ and the monogenic signal by Felsberg and Sommer [8], which are boundary values of a monogenic function. At the same time the monogenic signal was introduced, Larkin [19] developed the same theory from observations in optics.

The analytic signal is based on the Hilbert transform whereas the monogenic signal is based on the Riesz transforms. Hilbert and Riesz transforms play an important role in mathematics and optics [1, 3, 4]. Because the Hilbert transforms of wavelets are wavelets, the connection of Riesz transforms and wavelets has been studied intensively in the context of monogenic wavelets [17] and monogenic curvelets [22]. The Riesz transforms themselves can be used to construct several useful frames [20, 23]. Constructing specific monogenic shearlets was not possible with the standard Riesz transforms because they commute with rotations but not with shearings. That led to the so-called linearized Riesz transforms in $\mathbb{R}^2$. It turns out that not only the Riesz transforms and linearized Riesz transforms can be used to construct frames but a huge class of singular integral operators [24].

Embedding these ideas into Clifford analysis, Fourier multipliers were used in [2] to construct quasi-monogenic functions in the Fourier domain. The notion quasi-monogenic represents the fact that these functions share a lot of similar properties with monogenic functions. The main development in this paper is the construction of linearized Riesz transforms that commute with shearings in $\mathbb{R}^3$. That is more complicated than in $\mathbb{R}^2$ [16], but it also gives an idea of how to do it in $\mathbb{R}^n$. Furthermore, we prove that the linearized Riesz transforms in $\mathbb{R}^3$ are $L^p, 1 < p < \infty$, multipliers and fulfil similar properties for L^1 based on the Mikhlin-Hörmander multiplier theorem.

This chapter is organized as follows. After this introduction and the mathematical preliminaries, we consider in Sect. 9.3 quasi-monogenic functions and some of their properties. Section 9.4 is devoted to the Riesz transforms and linearized Riesz transforms in $\mathbb{R}^3$ and the proof of commutation and multiplier properties. Section 9.5 is concerned with frames, wavelets and shearlets.

9.2 Preliminaries

9.2.1 Clifford Algebras

Let $C\ell_n$ be the Clifford algebra over the field of complex numbers generated by $e_1, \ldots, e_n$, and e_0 the unit element of the Clifford algebra which fulfill

$$e_0^2 = 1, \quad e_0 e_j = e_j e_0, \quad e_j^2 = -1, \quad e_i e_j = -e_j e_i, \; i, j = 1, \ldots, n.$$

Because e_0 is the unit element of the algebra, we identify e_0 with 1. An arbitrary element of the complex Clifford algebra can be represented as

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{C}.$$

where $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$, $A = (i_1, i_2, \ldots, i_h) \subset \{0, 1, \ldots, n\} \subset \mathbb{N}$, and $0 \leq i_1 < i_2 < \cdots < i_h \leq n$.

We denote the scalar part by $\mathrm{Sc}(a) = a_0 e_0 = a_0$, the real part by $\mathrm{Re}(a) = \sum_A \mathrm{Re}\,(a_A) e_A$, the imaginary part by $\mathrm{Im}(a) = \sum_A \mathrm{Im}\,(a_A) e_A$. We also have several conjugations, the complex conjugation of a complex number $\overline{z}^{\mathbb{C}} = \mathrm{Re}\,(z) - i\mathrm{Im}\,(z)$, and $\overline{a}^{\mathbb{C}} = \sum_A \overline{a_A}^{\mathbb{C}} e_A$, the Clifford conjugation $\overline{e}_0^{C\ell} = e_0$, $\overline{e}_j^{C\ell} = -e_j$ and $\overline{e_A e_B}^{C\ell} = \overline{e_B}^{C\ell} \overline{e_A}^{C\ell}$ and a combination of both $\overline{a} = \sum_A \overline{a_A}^{\mathbb{C}} \overline{e_A}^{C\ell}$. The norm or length of a complex quaternion is given by

$$|a|^2 = \mathrm{Sc}(\overline{a} a) = \sum_A \overline{a_A}^{\mathbb{C}} a_A.$$

Furthermore we define

$$|a|_*^2 = \overline{a}^{C\ell} a = a \overline{a}^{C\ell} = |\mathrm{Re}(a)|^2 - |\mathrm{Im}(a)|^2 + 2i \langle \mathrm{Re}(a),\ \mathrm{Im}(a) \rangle.$$

Not all complex quaternions are invertible, since this algebra has zero divisors. A complex quaternion is invertible if and only if $|a|_*^2 \neq 0$. The inverse quaternion is uniquely determined and given by $a^{-1} = \dfrac{\overline{a}^{C\ell}}{|a|_*^2}$. Let

$$C\ell_n^{(k)} = \mathrm{Span}_{\mathbb{C}} \{ e_{\alpha_1} \cdots e_{\alpha_k} : 1 \leq \alpha_1 < \cdots < \alpha_k \leq n \}$$

be the space of k-multivectors spanned by the reduced products $e_\alpha = e_{\alpha_1} \cdots e_{\alpha_k}$ of length k in the complex Clifford algebra $C\ell_n$. Then $\mathbb{C} = C\ell_n^{(0)}$ and $\mathbb{C}^n = C\ell_n^{(1)}$ and the space of paravectors $C\ell_n^{(0)} \oplus C\ell_n^{(1)}$ will be thought of as $\mathbb{C}^{n+1}$.

9.2.2 Function Spaces

A Clifford valued function $u = \sum_A u_A e_A$ belongs to $L^p(D, C\ell_n)$, where D is an open domain in $\mathbb{R}^n$, if all components $u_A \in L^p(D)$, $1 \leq p < \infty$. The norm $2^{1-n} \sum_A \left(\int_D |u_A|^p \, d\underline{x} \right)^{1/p}$ is equivalent to the norm

$$||u||_{L^p} = \left(\int_D |u|^p \, d\underline{x} \right)^{1/p}.$$

For $p = 2$ the norm is induced by the scalar product

$$\langle u,\ v\rangle := \mathrm{Sc} \int_D \overline{u} v \, d\underline{x} = \sum_A \overline{u_A}^{\mathbb{C}} v_A .$$

The function space $L^2(\mathbb{R}^n, C\ell_n^{(0)} \oplus C\ell_n^{(1)})$ consists of all Clifford-valued function $f(x) = e_0 f_0(x) + \sum_{j=1} e_j f_j(x)$, where $f_j(x) \in L^2(\mathbb{R}^n), j = 0, 1, \ldots, n$. Furthermore, we equip $L^2(\mathbb{R}^n, C\ell_n^{(0)} \oplus C\ell_n^{(1)})$ with the scalar product

$$\langle f, g\rangle = \sum_{j=0}^{n} \overline{f}_j^{\mathbb{C}} g_j, \quad f, g \in L^2(\mathbb{R}^n, C\ell_n^{(0)} \oplus C\ell_n^{(1)}).$$

For all other spaces, the Clifford valued function u belongs to the function space F if an only if all components u_A belong to the scalar-value space F.

More on Clifford analysis and hypercomplex analysis can be found in [5–7, 11, 12].

Given u in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ of rapidly decreasing functions, we define the Fourier transform of u as

$$(\mathcal{F}u)(\underline{\xi}) = \widehat{u}(\underline{\xi}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\underline{x})\, e^{-i\langle \underline{\xi}, \underline{x}\rangle}\, d\underline{x}.$$

The inverse Fourier transform is given by

$$\mathcal{F}^{-1}(u) = u^{\vee}(\underline{x}) := \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(\underline{\xi})\, e^{i\langle \underline{x}, \underline{\xi}\rangle}\, d\underline{\xi}.$$

Definition 9.2.1 **(L^p-Multiplier [10])** Given $1 \le p < \infty$, we denote by $\mathcal{M}_p(\mathbb{R}^n)$ the space of all bounded functions m on $\mathbb{R}^n$ such that the operator

$$T_m(f) = \left(\hat{f} m\right)^{\vee}, \quad f \in \mathcal{S}(\mathbb{R}^n),$$

is bounded on $L^p(\mathbb{R}^n)$(or is initially defined in a dense subspace of $L^p(\mathbb{R}^n)$ and has a bounded extension on the whole space). The norm of m in $\mathcal{M}_p(\mathbb{R}^n)$ is defined by

$$||m||_{\mathcal{M}_p} = ||T_m||_{L^p \to L^p}.$$

The function m is the Fourier symbol of the operator T_m. A function $m \in \mathcal{M}_p(\mathbb{R}^n)$ is called an L^p- or **Fourier multiplier** and m is also called **Fourier symbol** of the operator T_m.

Theorem 9.2.2 (Hörmander-Mikhlin[1] Multiplier Theorem [10]) *Let $m(\underline{\xi})$ be a complex-valued bounded function on $\mathbb{R}^n \backslash \{0\}$ that satisfies either*

(a) *Mikhlin's condition*

$$|\partial_{\underline{\xi}}^{\alpha} m(\underline{\xi})| \leq A|\underline{\xi}|^{-|\alpha|}$$

for all multi-indices $|\alpha| \leq \left[\frac{n}{2}\right] + 1$,

(b) *Hörmander's condition*

$$\sup_{R>0} R^{-n+2|\alpha|} \int_{R<|\underline{\xi}|<2R} |\partial_{\underline{\xi}}^{\alpha} m(\underline{\xi})| \, d\underline{\xi} \leq A^2 < \infty$$

for all multi-indices $|\alpha| \leq \left[\frac{n}{2}\right] + 1$.

Then for all $1 < p < \infty$, *m lies in* $\mathcal{M}_p(\mathbb{R}^n)$ *and the following estimate is valid:*

$$||m||_{\mathcal{M}_p} \leq C_n \max(p, (p-1)^{-1})(A + ||m||_{L^\infty}).$$

Moreover, the operator $f \mapsto (\hat{f} m)^{\vee}$ *maps* $L^1(\mathbb{R}^n)$ *to* $L^{1,\infty}(\mathbb{R}^n)$, *with norm at most a dimensional constant multiple of* $A + ||m||_{L^\infty}$.

Remark 9.2.3 The space $L^{1,\infty}(\mathbb{R}^n)$ is a Lorentz space and its definition can be found in [10], p. 48.

9.3 Quasi-Monogenic Functions

Driven from some application of "generalized Riesz transforms" we are looking for operators that fulfill the following conditions. Let be $Q_1, \ldots, Q_n \in L^2(\mathbb{R}^n, \mathbb{R}) \to L^2(\mathbb{R}^n, \mathbb{R})$ an n-tuple of linear independent, bounded, linear operators. Then $Q = \sum_{j=1}^{n} e_j Q_j$ is called a quadrature operator of order m if the following conditions are fulfilled:

(i) Q is invariant under translation,
(ii) Q is invariant under positive dilations,
(iii) Q is self-inverting, i.e. $Q^2 = I$,
(iv) Q_i is anti-selfadjoint, i.e. $Q_i^* = -Q_i$, $i = 1, 2, \ldots, m$.

Such an operator Q gives raise to the quasi-monogenic signal.

[1] Mikhlin is a transliteration of the Russian name, there are also other versions like Mihlin or Michlin.

Definition 9.3.1 (Quasi-Monogenic Signal) The quasi-monogenic signal in $L^2(\mathbb{R}^n, C\ell_n^{(0)} \oplus C\ell_n^{(1)})$ is defined as

$$Q_m f = f + Qf = f + \sum_{j=1}^{n} e_j Q_j f,$$

with amplitude

$$|Q_m f| = \sqrt{|f|^2 + \sum_{i=1}^{n} |Q_i f|^2},$$

and local phase

$$\varphi = \arccos\left(\frac{f}{|Q_m f|}\right), \quad \varphi \in [0, \pi],$$

and local orientation

$$\underline{q} = \frac{Qf}{|Qf|}.$$

Any kind of convolution operator with Fourier symbol $m(\underline{\xi})$ fulfills (i) and (ii). For it to be self-inverting, we assume that $(m(\underline{\xi}))^2 = 1$ a. e. and to be anti-selfadjoint, we assume that $\overline{m_j(\underline{\xi})}^{\mathbb{C}} = -m_j(\underline{\xi})$, $j = 1, \ldots, n$.

Definition 9.3.2 Let m be a Clifford vector, i.e. $m \in C\ell_n^{(1)}$, invertible with $(m(\underline{\xi}))^2 = 1$ for all $\underline{\xi} \in \mathbb{R}^n \backslash \{0\}$ and an L^p- multiplier, $1 < p < \infty$. We define the **Riesz transforms** R_{m_j} and the **Riesz-Hilbert transform** $\mathcal{H}_m$ associated with m as

$$R_{m_j} u(\underline{x}) = \mathcal{F}^{-1}_{\underline{\xi} \to \underline{x}} \left(m_j(\underline{\xi}) \hat{u}(\underline{\xi}) \right), \quad \mathcal{H}_m u(\underline{x}) = \mathcal{F}^{-1}_{\underline{\xi} \to \underline{x}} \left(m(\underline{\xi}) \hat{u}(\underline{\xi}) \right) = \sum_{j=1}^{n} e_j R_{m_j} u(\underline{x}),$$

as well as the **Dirac operator**

$$D_{\mathcal{H}_m} = |D| \mathcal{H}_m = \mathcal{F}^{-1}_{\underline{\xi} \to \underline{x}} (|\underline{\xi}| m(\underline{\xi})).$$

Next, we define quasi-monogenic functions in the upper and lower half space. Let be

$$\chi_+(\underline{\xi}) = \tfrac{1}{2}(1 + m(\underline{\xi})), \quad \chi_-(\underline{\xi}) = \tfrac{1}{2}(1 - m(\underline{\xi})),$$

which satisfy

$$\chi_+^2(\underline{\xi}) = \chi_+(\underline{\xi}), \quad \chi_-^2(\underline{\xi}) = \chi_-(\underline{\xi}), \quad \chi_+(\underline{\xi})\chi_-(\underline{\xi}) = \chi_-(\underline{\xi})\chi_+(\underline{\xi}) = 0,$$

and decompose accordingly

$$L^\infty(\mathbb{R}^n, C\ell_n) = L_\infty^+(\mathbb{R}^n, C\ell_n) \oplus L_\infty^-(\mathbb{R}^n, C\ell_n)$$

into the subspaces

$$L_\infty^+(\mathbb{R}^n, C\ell_n) = \{u \in L^\infty(\mathbb{R}^n, C\ell_n) : u\chi_- = 0\} = \{u \in L^\infty(\mathbb{R}^n, C\ell_n) : u\chi_+ = u\},$$
$$L_\infty^-(\mathbb{R}^n, C\ell_n) = \{u \in L^\infty(\mathbb{R}^n, C\ell_n) : u\chi_+ = 0\} = \{u \in L^\infty(\mathbb{R}^n, C\ell_n) : u\chi_- = u\}.$$

Theorem 9.3.3 ([2]) *Let $\hat{u} \in L_\infty^+(\mathbb{R}^n, C\ell_n)$ and define U_+ on $\mathbb{R}_+^{n+1}$ by*

$$U_+(x) = U_+(x_0e_0 + \underline{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(\underline{\xi})e^{i\langle \underline{x}, \underline{\xi}\rangle}\, e^{-x_0|\underline{\xi}|}\, d\underline{\xi},$$
$$\text{when } x_0 > 0 \text{ and } \underline{x} \in \mathbb{R}^n.$$

Analoguously,
$$U_-(x) = U_-(x_0e_0 + \underline{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \hat{u}(\underline{\xi})e^{i\langle \underline{x}, \underline{\xi}\rangle}\, e^{x_0|\underline{\xi}|}\, d\underline{\xi},$$
$$\text{when } x_0 < 0 \text{ and } \underline{x} \in \mathbb{R}^n.$$

Then the integral is absolutely convergent, and $|U_\pm(x)| \leq \frac{c}{|x_0|^n}\,||\hat{u}||_\infty$. Furthermore,

(1) $\dfrac{\partial U_\pm}{\partial x_0}(x_0e_0+\underline{x}) + D_{\mathcal{H}_m}U_\pm(x_0e_0+\underline{x}) = 0, \quad x_0e_0+\underline{x} \in \mathbb{R}_\pm^{n+1}$, *or in other words, the functions $U_\pm$ are left* ***quasi-monogenic*** *on their respective half-spaces.*
(2) $\lim_{x_0\to 0\pm} U_\pm(x_0e_0 + \underline{x}) = P^\pm U(\underline{x})$ *for almost all $\underline{x} \in \mathbb{R}^n$. (Plemelj-Sochotzki formulae)*
(3) $\lim_{x_0\to\pm\infty} U_\pm(x_0e_0 + \underline{x}) = 0$ *for all $\underline{x} \in \mathbb{R}^n$.*

Then U_+ is the quasi-monogenic Cauchy kernel

$$k(x) = k(x_0e_0 + \underline{x}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \chi_+(\underline{\xi})e^{i\langle \underline{x}, \underline{\xi}\rangle}\, e^{-x_0|\underline{\xi}|}\, d\underline{\xi}.$$

Let $\underline{x} \in \mathbb{R}^n$ and $\omega_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$ be the surface area of the $(n-1)$ dimensional unit sphere. We denote by $P_t(\underline{x})$ the Poisson kernel for $\mathbb{R}^n$,

$$P_t(\underline{x}) = \frac{2}{\omega_{n+1}}\frac{t}{(t^2 + |\underline{x}|^2)^{\frac{n}{2}}} \quad \text{with Fourier transform} \quad \hat{P}_t(\underline{\xi}) = e^{-t|\underline{\xi}|}.$$

Then the Cauchy integral becomes

$$\mathcal{C}_m f(t, \underline{x}) = (P_t * \tfrac{1}{2}(I + \mathcal{H}_m) f)(\underline{x}), \quad (t, \underline{x}) \in \mathbb{R}^{n+1}_+$$

and we obtain that $\frac{1}{2}(I + \mathcal{H}_m) f(\underline{x}) = \frac{1}{2}(I + \sum_{j=1}^n e_j R_j^m) f(\underline{x})$ are the boundary values of a quasi-monogenic function with Fourier symbol m in the upper half space.

9.4 L^p Multiplier

9.4.1 Riesz Transforms

It is easily seen that all Riesz transforms $R_k^m, k = 1, \ldots, n$, are bounded linear operators in $L^2(\mathbb{R}^n)$. To construct frames, we need specific symbols to be Fourier multipliers. We start with the well known classical Riesz transforms and show that they are Fourier multipliers by using their Fourier symbols.

Theorem 9.4.1 *For all $1 < p < \infty$ the Riesz transforms are $L^p(\mathbb{R}^n)$ multipliers and map $L^1(\mathbb{R}^n)$ to $L^{1,\infty}(\mathbb{R}^n)$.*

Proof The theorem follows from the boundedness of the Fourier symbol m and from the Hörmander-Mikhlin Multiplier theorem [10]. Hence, we need to show for each $m_k, k = 1, \ldots, n$, that for all multi-indices α with $|\alpha| \leq \left[\frac{n}{2}\right] + 1$, there is an $A \in \mathbb{R}^+$ so that

$$|\partial^\alpha m_k(\underline{\xi})| \leq A \left|\underline{\xi}\right|^{-|\alpha|}. \tag{9.4.1}$$

To show (9.4.1), we need to find $\partial^\alpha m_k(\underline{\xi}) = \partial^\alpha \frac{\xi_k}{|\xi|}$.

Since $\frac{\xi_k}{\left|\underline{\xi}\right|} = \partial_k \left|\underline{\xi}\right|$ for all $k \in \{1, \ldots, n\}$ and $|\xi| = \sqrt{\xi_1^2 + \cdots + \xi_n^2}$, we will begin by constructing

$$\partial^\beta g\left(h\left(\underline{\xi}\right)\right),$$

where $g(t) = \sqrt{t}$, $h(\underline{x}) = x_1^2 + \cdots + x_n^2$ and β is a multi-index.

We will later need the derivatives of g:

$$g'(x) = \frac{1}{2}\, x^{-\frac{1}{2}},$$

$$g''(x) = -\frac{1}{2^2}\, x^{-\frac{3}{2}},$$

$$g'''(x) = \frac{1 \cdot 3}{2^3}\, x^{-\frac{5}{2}},$$

$$\vdots$$

$$g^{(t)} = (-1)^{t+1} \frac{(2t-2)!}{(t-1)!2^{2t-1}} x^{-\frac{2t-1}{2}}. \tag{9.4.2}$$

To differentiate the chain-function $g\left(h\left(\xi\right)\right)$, we use Faá di Bruno's formula [14]:

$$\partial^{\beta} g\left(h\left(\underline{\xi}\right)\right) = \sum_{\pi \in \Pi} g^{(|\pi|)}(h) \prod_{B \in \pi} \frac{\partial^{|B|} h(\underline{\xi})}{\prod_{j \in B} \partial x_j}, \tag{9.4.3}$$

where Π is the set of all partitions of $\{1, \ldots, |\beta|\}$, and

$$x_1 = x_2 = \cdots = x_{\beta_1} = \xi_1,$$
$$x_{\beta_1+1} = \cdots = x_{\beta_1+\beta_2} = \xi_2,$$
$$\vdots$$
$$\cdots = x_{|\beta|} = \xi_n.$$

We know that $\partial_i h(\underline{x}) = 2x_i$ and $\partial_{ii} h(\underline{x}) = 2$ for all $i \in \{1, \ldots, n\}$. All other partial derivatives of h vanish. Therefore it suffices to look at those partitions in Π, in which all elements are of the form $\{p\}$ or $\{p, q\}$ with $x_p = x_q$, $p, q \in \{1, \ldots, |\beta|\}$. For all other partitions the summands in (9.4.3) vanish.

We will now look at *different* partitions, i.e. partitions that lead to different summands in (9.4.3): For each partition let b_j be the number of elements of the form $\{p\}$, with $x_p = \xi_j$, and c_j the number of elements of the form $\{p, q\}$, with $x_p = x_q = \xi_j$, for all $j = \{1, \ldots, n\}$. It follows that $b_j + 2c_j = \beta_j$ for all $j \in \{1, \ldots, n\}$.

For each set of $b_j + c_j$ elements there are $\frac{\beta_j!}{b_j! c_j! 2^{c_j}}$ partitions of $\{i \in \{1, \ldots, |\beta|\} : x_i = \xi_j\}$. Hence, the set $\{b_1, c_1, \ldots, b_j, c_j, \ldots, b_n, c_n\}$ encompasses

$$\prod_{j=1}^{n} \frac{\beta_j!}{b_j! c_j! 2^{c_j}} \tag{9.4.4}$$

partitions.

Using (9.4.4) in (9.4.3), we get

$$\partial^\beta g\left(h\left(\underline{\xi}\right)\right) = \sum_{\substack{b_1+2c_1=\beta_1\\ \vdots \\ b_n+2c_n=\beta_n}} \prod_{j=1}^{n} \frac{\beta_j!}{b_j!c_j!2^{c_j}} g^{(b+c)}(h(\underline{\xi})) \prod_{j=1}^{n} \left(\frac{\partial^2}{\partial \xi_j^2} h(\underline{\xi})\right)^{c_j} \left(\frac{\partial}{\partial \xi_j} h(\underline{\xi})\right)^{b_j},$$

where $b = \sum_{j=1}^{n} b_j$ and $c = \sum_{j=1}^{n} c_j$. With (9.4.2), this leads to

$$\begin{aligned}\partial^\beta g\left(h\left(\underline{\xi}\right)\right) &= \sum_{\substack{b_1+2c_1=\beta_1\\ \vdots \\ b_n+2c_n=\beta_n}} \prod_{j=1}^{n} \frac{\beta_j!}{b_j!c_j!2^{c_j}} (-1)^{b+c+1} \frac{(2b+2c-2)!}{(b+c-1)!2^{2b+2c-1}} |\underline{\xi}|^{-2b-2c+1} \\ &\qquad \cdot 2^c \cdot 2^b \cdot \prod_{j=1}^{n} \xi_j^{b_j} \\ &= \sum_{\substack{b_1+2c_1=\beta_1\\ \vdots \\ b_n+2c_n=\beta_n}} \prod_{j=1}^{n} \frac{\beta_j!}{b_j!c_j!} (-1)^{b+c+1} \frac{(2b+2c-2)!}{2^{|\beta|-1}(b+c-1)!} \frac{\prod_{j=1}^{n} \xi_j^{b_j}}{|\underline{\xi}|^{2b+2c-1}}\end{aligned}$$

We will now use this formula to show (9.4.1): Since $|\xi_j| \leq |\underline{\xi}|$ for all $j \in \{1, \ldots, n\}$, it follows that

$$\prod_{j=1}^{n} |\xi_j|^{b_j} \leq |\underline{\xi}|^b, \quad \text{so}$$

$$\frac{\prod_{j=1}^{n} |\xi_j|^{b_j}}{\|\xi\|^{2b+2c-1}} \leq |\underline{\xi}|^{-b-2c+1} = |\underline{\xi}|^{-|\beta|+1}.$$

Hence,

$$\left|\partial^\beta g\left(h\left(\underline{\xi}\right)\right)\right| \leq \underbrace{\sum_{\substack{b_1+2c_1=\beta_1\\ \vdots \\ b_n+2c_n=\beta_n}} \left| \prod_{j=1}^{n} \frac{\beta_j!}{b_j!c_j!} (-1)^{b+c+1} \frac{(2b+2c-2)!}{2^{|\beta|-1}(b+c-1)!} \right|}_{=:A(\beta)} |\underline{\xi}|^{-|\beta|+1}.$$

Since our goal is to show (9.4.1) for $\partial^\alpha \frac{\xi_k}{|\underline{\xi}|}$, let $\beta = (\alpha_1, \dots, \alpha_{k-1}, \alpha_k + 1, \alpha_{k+1}, \dots, \alpha_n)$. Then $|\alpha| = |\beta| - 1$ and

$$\partial^\alpha \frac{\xi_k}{|\underline{\xi}|} = \partial^\beta |\underline{\xi}| \leq A(\beta)|\underline{\xi}|^{-|\alpha|}.$$

□

9.4.2 Linearized Riesz Transforms in $\mathbb{R}^2$

The standard Riesz transforms are invariant under rotations, and due to Proposition 2 in [21], p. 58, the Riesz transforms generate all rotational invariant operators. The Riesz transforms are rotational invariant but not invariant under shears. To get shear invariant operators we represent $\frac{\underline{\xi}}{|\underline{\xi}|}$ in trigonometric form. We have

$$\begin{aligned}
\underline{\xi} &= \xi_1 e_1 + \xi_2 e_2 = |\underline{\xi}| \left(\frac{\xi_1}{|\underline{\xi}|} e_1 + \frac{\xi_2}{|\underline{\xi}|} e_2 \right) = |\underline{\xi}| e_1 \left(\frac{\xi_1}{|\underline{\xi}|} + \frac{\xi_2}{|\underline{\xi}|} \overline{e}_1 e_2 \right) \\
&= |\underline{\xi}| e_1 \left(\frac{\xi_1}{|\underline{\xi}|} + \frac{\xi_2}{|\underline{\xi}|} \overline{e}_3 \right) = |\underline{\xi}| e_1 \exp(\overline{e}_3 \theta(\underline{\xi})) = |\underline{\xi}| e_1 \left(\cos(\theta(\underline{\xi})) + \overline{e}_3 \sin(\theta(\underline{\xi})) \right) \\
&= |\underline{\xi}| \left(\cos(\theta(\underline{\xi})) e_1 + \sin(\theta(\underline{\xi})) e_2 \right),
\end{aligned}$$

where $\theta(\underline{\xi}) = \arctan 2(\underline{\xi}) = \arctan 2\left(\frac{\xi_2}{\xi_1}\right)$. Because the aim for the construction of the modified Riesz transform is to have a transform that commutes with shears, $\theta(\underline{\xi})$ will be linearized to $\theta_L(\underline{\xi})$ (cf. [16]) defined as

$$\theta_L(\underline{\xi}) := \begin{cases} (1 - \operatorname{sign}(\xi_1)) \operatorname{sign}(\xi_2) \frac{\pi}{2} + \frac{\pi}{4} \frac{\xi_2}{\xi_1} & \text{if} \quad (\xi_1, \xi_2) \in C^h, \\ \operatorname{sign}(\xi_2) \frac{\pi}{2} - \frac{\pi}{4} \frac{\xi_1}{\xi_2} & \text{if} \quad (\xi_1, \xi_2) \in C^v, \end{cases}$$

$$= \begin{cases} \frac{\pi}{4} \frac{\xi_2}{\xi_1} & \text{if} \quad (\xi_1, \xi_2) \in C^h_+, \\ \frac{\pi}{2} - \frac{\pi}{4} \frac{\xi_1}{\xi_2} & \text{if} \quad (\xi_1, \xi_2) \in C^v_+, \\ \operatorname{sign}(\xi_2)\pi + \frac{\pi}{4} \frac{\xi_2}{\xi_1} & \text{if} \quad (\xi_1, \xi_2) \in C^h_-, \\ -\frac{\pi}{2} - \frac{\pi}{4} \frac{\xi_1}{\xi_2} & \text{if} \quad (\xi_1, \xi_2) \in C^v_-, \end{cases} \tag{9.4.5}$$

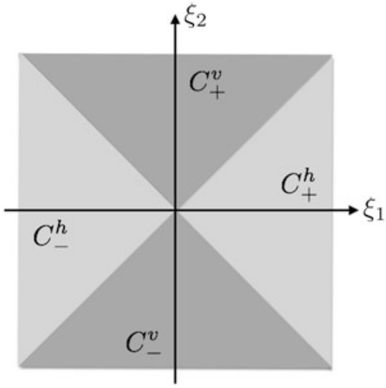

Fig. 9.1 Cones

where the cones are defined as (Fig. 9.1)

$$C_+^h := \left\{(\xi_1,\ \xi_2) \in \mathbb{R}^2 : \left|\frac{\xi_2}{\xi_1}\right| \le 1, \xi_1 \ge 0\right\},$$

$$C_-^h := \left\{(\xi_1,\ \xi_2) \in \mathbb{R}^2 : \left|\frac{\xi_2}{\xi_1}\right| \le 1, \xi_1 \le 0\right\},$$

$$C_+^v := \left\{(\xi_1,\ \xi_2) \in \mathbb{R}^2 : \left|\frac{\xi_1}{\xi_2}\right| \le 1, \xi_2 \ge 0\right\},$$

$$C_-^v := \left\{(\xi_1,\ \xi_2) \in \mathbb{R}^2 : \left|\frac{\xi_1}{\xi_2}\right| \le 1, \xi_2 \le 0\right\},$$

$$\text{and} \quad C^h = C_+^h \cup C_-^h, \quad C^v = C_+^v \cup C_-^v,$$

The Linearized Riesz transforms $R_{L,1}, R_{L,2}$ were introduced by Häuser et al. [16].

Definition 9.4.2 (Linearized Riesz Transforms [16]) The linearized Riesz transforms $R_{L,1},\ R_{L,2},\ L^2(\mathbb{R}^2, \mathbb{R}) \to L^2(\mathbb{R}^2, \mathbb{R})$ are defined by

$$R_{L,1}u(\underline{x}) := \mathcal{F}^{-1}(-i\cos(\theta_L(\underline{\xi}))\widehat{u}(\underline{\xi})), \quad R_{L,2}u(\underline{x}) := \mathcal{F}^{-1}(-i\sin(\theta_L(\underline{\xi}))\widehat{u}(\underline{\xi}))$$

and $\mathcal{H}_L = e_1 R_{L,1} + e_2 R_{L,2}$.

Theorem 9.4.3 ([2]) *For all* $1 < p < \infty$*, the linearized Riesz transforms* $\mathcal{R}_{L,1},\ \mathcal{R}_{L,2}$ *are* $L^p(\mathbb{R}^2)$*-multipliers and map* $L^1(\mathbb{R}^2)$ *to* $L^{1,\infty}(\mathbb{R}^2)$.

In [16] it is shown that the linearized Riesz transforms are invariant under shears. Let

$$S_s := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad s \in \mathbb{R},$$

be the shear operator.

Lemma 9.4.4 ([16]) *Let q be a scalar-valued filter function such that $\widehat{g}$ is supported in $\{\underline{\xi} \in \mathbb{R}^2 : \left|\frac{\xi_2}{\xi_1}\right| \leq \alpha\}$, where $0 \leq \alpha < 1$. Then, for $\alpha - 1 \leq s \leq 1 - \alpha$,*

$$R_{L,\,j}(g(S_s^{-1}\underline{x})) = e^{is\frac{\pi}{4}}(R_{L,\,j}g)(S_s^{-1}\underline{x}), \quad j = 1, 2.$$

The linearized Riesz transform can be used to construct quasi-monogenic shearlets.

9.4.3 Linearized Riesz Transforms in $\mathbb{R}^3$

We will extend the definition of the linearized Riesz transforms to three dimensions. Recall that the Fourier multipliers of the Riesz Transform are $m_j(\underline{\xi}) = -i\frac{\xi_j}{|\underline{\xi}|}$. For $n = 3$ we will represent ξ in polar coordinates, $\underline{\xi} = \left(r, \Phi_2(\underline{\xi}), \Phi_1(\underline{\xi})\right)$ with $\Phi_2 = \Phi_2(\underline{\xi}) \in [0, 2\pi]$, $\Phi_1 = \Phi_1(\underline{\xi}) \in [0, \pi]$, where

$$\tan \Phi_2 = \frac{\xi_3}{\xi_2},$$

$$\tan \Phi_1 = \frac{\sqrt{\xi_3^2 + \xi_2^2}}{\xi_1}.$$

Then the Fourier multipliers can be written as

$$m_1 = -i \cos \Phi_1,$$

$$m_2 = -i \sin \Phi_1 \cos \Phi_2,$$

$$m_3 = -i \sin \Phi_1 \sin \Phi_2,$$

which are combined to

$$\widehat{Rf}(\underline{\xi}) = \sum_{j=1}^{3} e_j m_j(\underline{\xi}) \hat{f}(\underline{\xi}).$$

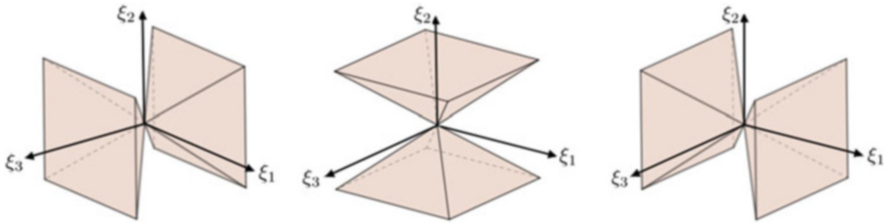

Fig. 9.2 Partition of $\mathbb{R}^3$ into six cones

For the following definition, the angles Φ_1 and Φ_2 are redefined in a way, that has its roots in the concept of pseudo-polar coordinates. First, we will divide the space $\mathbb{R}^3$ into three pairs of pyramids, which are in this context also called *cones*, compare Fig. 9.2.

$$\mathcal{C}_1 = \left\{\xi \in \mathbb{R}^3 : \left|\frac{\xi_2}{\xi_1}\right| \leq 1, \left|\frac{\xi_3}{\xi_1}\right| \leq 1\right\},$$

$$\mathcal{C}_2 = \left\{\xi \in \mathbb{R}^3 : \left|\frac{\xi_1}{\xi_2}\right| \leq 1, \left|\frac{\xi_3}{\xi_2}\right| \leq 1\right\},$$

$$\mathcal{C}_3 = \left\{\xi \in \mathbb{R}^3 : \left|\frac{\xi_1}{\xi_3}\right| \leq 1, \left|\frac{\xi_2}{\xi_3}\right| \leq 1\right\}.$$

From this definition it follows that for all $\underline{\xi} \in \mathcal{C}_k$, we have $|\xi_k| = \max\{|\xi_1|, |\xi_2|, |\xi_3|\}$ for $k = 1, 2, 3$. The following definition depends on the value of k, i. e. on the cone that it is applied to.

Definition 9.4.5 (Linear Riesz Transform) The linearized Riesz transforms $R_{L,1},\ R_{L,2},\ R_{L,3} : L^2(\mathbb{R}^3, \mathbb{R}) \to L^2(\mathbb{R}^3, \mathbb{R})$ are defined by

$$R_{L,j}u(\underline{x}) := \mathcal{F}^{-1}(m_j(\underline{\xi}))\widehat{u}(\underline{\xi})), \quad j = 1, 2, 3,$$

and $\mathcal{H}_L = e_1 R_{L,1} + e_2 R_{L,2} + e_3 R_{L,3}$, where for $\underline{\xi} \in \mathcal{C}_k$

$$m_k(\underline{\xi}) = -i \sin \Phi_1(\underline{\xi}),$$

$$m_{k_2}(\underline{\xi}) = -i \cos \Phi_1(\underline{\xi}) \sin \Phi_2(\underline{\xi}),$$

$$m_{k_3}(\underline{\xi}) = -i \cos \Phi_1(\underline{\xi}) \cos \Phi_2(\underline{\xi})$$

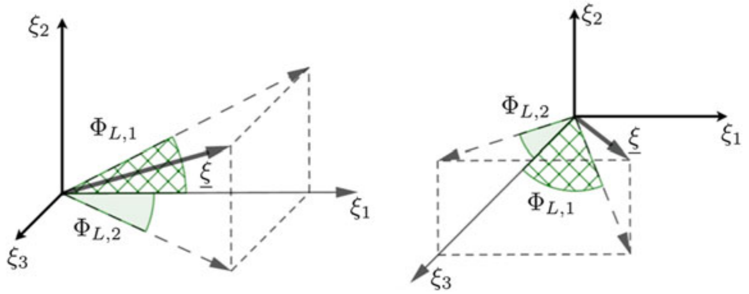

Fig. 9.3 Definition of angles for the cases $\xi \in \mathcal{C}_1, \xi_1 > 0$ (left) and $\xi \in \mathcal{C}_3, \xi_3 > 0$ (right)

with

$$(k_2, k_3) = \begin{cases} (2,3) & \text{if } k = 1 \\ (3,1) & \text{if } k = 2 \\ (1,2) & \text{if } k = 3 \end{cases}.$$

Depending on the cone that $\underline{\xi}$ is in, the angles are defined in the following way (Fig. 9.3).

$$\Phi_{L,1}(\underline{\xi}) = \frac{\xi_{k_2}}{|\xi_k|}\,\frac{\pi}{4},$$

$$\Phi_{L,2}(\underline{\xi}) = \begin{cases} \frac{\xi_{k_3}}{\xi_k}\,\frac{\pi}{4} & \text{if } \xi_k > 0 \\ \pi + \frac{|\xi_{k_3}|}{\xi_k}\,\frac{\pi}{4} & \text{if } \xi_k < 0 \end{cases}.$$

Then

$$\begin{aligned} |m_k|^2 + |m_{k_2}|^2 + |m_{k_3}|^2 &= \sin^2\Phi_{L,1} + \cos^2\Phi_{L,1}\sin^2\Phi_{L,2} + \cos^2\Phi_{L,1}\cos^2\Phi_{L,2} \\ &= \sin^2\Phi_{L,1} + \cos^2\Phi_{L,1}\left(\sin^2\Phi_{L,2} + \cos^2\Phi_{L,2}\right) \\ &= \sin^2\Phi_{L,1} + \cos^2\Phi_{L,1} \\ &= 1. \end{aligned}$$

From the definition of the $\Phi_{L,k}$, $k = 1, 2$ it follows that $\Phi_{L,k}(a\underline{\xi}) = \Phi_{L,k}(\underline{\xi})$, and so $m_k(a\underline{\xi}) = m_k(\underline{\xi})$ for all $k = 1, 2, 3$, $a > 0$ and $\underline{\xi} \neq 0$.

Theorem 9.4.6 *For $1 < p < \infty$, the Fourier multipliers of the linearized Riesz transforms $m_{L,k}$, $k = 1, 2, 3$, are $L^p(\mathbb{R}^3)$ multipliers and map $L^1(\mathbb{R}^3)$ to $L^{1,\infty}(\mathbb{R}^3)$.*

Proof Since all $m_{L,k}$ are bounded, we only need to show that Mihlin's condition [10] holds, i.e. for all multi-indices α with $|\alpha| \leq \left[\frac{n}{2}\right] + 1$, there is an $A \in \mathbb{R}^+$ so that

$$|\partial^\alpha m(\underline{\xi})| \leq A \cdot |\underline{\xi}|^{-|\alpha|}. \tag{9.4.6}$$

Since the Linear Riesz Transforms are symmetrical for each of the six cones in $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$, it suffices to show (9.4.6) for $\underline{\xi} \in \mathcal{C}_1, \xi_1 > 0$.

In $\mathbb{R}^3$, (9.4.6) needs to be shown for $|\alpha| \leq \left[\frac{3}{2}\right] + 1 = 2$. Therefore, we will approximate the first and second derivatives of $m \in \mathcal{M}_L$.

Each of m_1, m_2, m_3 can be written as

$$m_l(\underline{\xi}) = -i \operatorname{trig}_1(\underline{\xi}) \cdot \operatorname{trig}_2(\underline{\xi}),$$

$l = 1, 2, 3$, with

$$\operatorname{trig}_1(\underline{\xi}) = \begin{cases} \sin \Phi_1(\underline{\xi}) & \text{if } l = 1 \\ \cos \Phi_1(\underline{\xi}) & \text{if } l = 2, 3 \end{cases},$$

$$\operatorname{trig}_2(\underline{\xi}) = \begin{cases} 1 & \text{if } l = 1 \\ \sin \Phi_2(\underline{\xi}) & \text{if } l = 2 \\ \cos \Phi_2(\underline{\xi}) & \text{if } l = 3 \end{cases}.$$

Differentiating m_l in the jth coordinate, $j = 1, 2, 3$, gives

$$\partial_j m_l(\underline{\xi}) = -i\partial_j \Phi_1(\underline{\xi}) \operatorname{trig}_1'(\underline{\xi}) \operatorname{trig}_2(\underline{\xi}) - i\partial_j \Phi_2(\underline{\xi}) \operatorname{trig}_1(\underline{\xi}) \operatorname{trig}_2'(\underline{\xi}), \tag{9.4.7}$$

where trig_1' and trig_2' are the first derivatives of the corresponding trigonometric functions. Since $|\operatorname{trig}_k(x)|, |\operatorname{trig}_k'(x)| \leq 1$ for all $x \in \mathbb{R}$, $k = 1, 2$, we can approximate (9.4.7) by

$$|\partial_j m_l(\xi)| \leq |\partial_j \Phi_1(\xi)| + |\partial_j \Phi_2(\xi)|.$$

The first derivatives of Φ_1 and Φ_2 are

$$\partial_j \Phi_1(\xi) = \begin{cases} -\frac{\xi_2}{\xi_1^2} \frac{\pi}{4} & \text{if } j = 1 \\ \frac{1}{\xi_1} \frac{\pi}{4} & \text{if } j = 2 \\ 0 & \text{if } j = 3 \end{cases},$$

$$\partial_j \Phi_2(\xi) = \begin{cases} -\frac{\xi_3}{\xi_1^2} \frac{\pi}{4} & \text{if } j = 1 \\ 0 & \text{if } j = 2 \\ \frac{1}{\xi_1} \frac{\pi}{4} & \text{if } j = 3 \end{cases}.$$

Since $|\xi_2|, |\xi_3| \leq |\xi_1|$ and $||\xi||^2 = \xi_1^2 + \xi_2^2 + \xi_3^2 \leq 3\xi_1^2$, i. e. $|\xi_1|^{-1} \leq \sqrt{3}||\xi||^{-1}$, the derivatives can be approximated by

$$\begin{aligned} |\partial_j \Phi_1(\xi)|, |\partial_j \Phi_2(\xi)| &\leq \frac{1}{|\xi_1|} \frac{\pi}{4} \\ &\leq \frac{\sqrt{3}\pi}{4} ||\xi||^{-1}. \end{aligned} \tag{9.4.8}$$

With that we have

$$|\partial_j m_l(\xi)| \leq \frac{\sqrt{3}\pi}{2} \|\xi\|^{-1}.$$

Now we will examine the case $|\alpha| = 2$:
For the second derivatives of Φ_1 and Φ_2 we get

$$\partial_{jr}\Phi_1(\xi) = \begin{cases} \frac{\xi_2}{\xi_1^3} \frac{\pi}{2} & \text{if } j = r = 1 \\ -\frac{1}{\xi_1^2} \frac{\pi}{4} & \text{if } (j, r) = (1, 2), (2, 1) \\ 0 & \text{else} \end{cases},$$

$$\partial_{jr}\Phi_2(\xi) = \begin{cases} \frac{\xi_3}{\xi_1^3} \frac{\pi}{2} & \text{if } j = r = 1 \\ -\frac{1}{\xi_1^2} \frac{\pi}{4} & \text{if } (j, r) = (1, 3), (3, 1) \\ 0 & \text{else} \end{cases}.$$

Because of $|\xi_2|, |\xi_3| \leq |\xi_1|$ and $|\xi_1|^{-1} \leq \sqrt{3}||\xi||^{-1}$, this means

$$\begin{aligned} |\partial_{jr} \Phi_1(\xi)|, |\partial_{jr} \Phi_2(\xi)| &\leq \frac{1}{\xi_1^2} \frac{\pi}{2} \\ &\leq \frac{3\pi}{2} \, ||\xi||^{-2}. \end{aligned} \tag{9.4.9}$$

Differentiating (9.4.7) in the rth coordinate, $r = 1, 2, 3$, gives

$$\begin{aligned} \partial_{jr} m_l(\xi) = &- i\partial_{jr}\Phi_1(\xi)\text{trig}_1'(\xi)\text{trig}_2(\xi) - i\partial_j\Phi_1(\xi)\partial_r\Phi_1(\xi)\text{trig}_1''(\xi)\text{trig}_2(\xi) \\ &- \partial_j\Phi_1(\xi)\partial_r\Phi_2(\xi)\text{trig}_1'(\xi)\text{trig}_2'(\xi) - i\partial_{jr}\Phi_2(\xi)\text{trig}_1(\xi)\text{trig}_2'(\xi) \\ &- i\partial_j\Phi_2(\xi)\partial_r\Phi_1(\xi)\text{trig}_1'(\xi)\text{trig}_2'(\xi) - i\partial_j\Phi_2(\xi)\partial_r\Phi_2(\xi)\text{trig}_1(\xi)\text{trig}_2''(\xi), \end{aligned}$$

where trig_1'' and trig_2'' are the second derivatives of the corresponding trigonometric functions. With (9.4.8), (9.4.9) and $|\mathrm{trig}_k''(x)| \le 1$ for all $x \in \mathbb{R}$, $k = 1, 2$, the above can be approximated by

$$\begin{aligned}|\partial_{jr} m_l(\xi)| &\le 2 \cdot \frac{3\pi}{2} ||\xi||^{-2} + 4 \cdot \left(\frac{\sqrt{3}\pi}{4} \, ||\xi||^{-1} \right)^2 \\ &= \left(3\pi + \frac{3}{4}\pi^2 \right) ||\xi||^{-2}.\end{aligned}$$

□

We will now examine the behavior of the linearized Riesz transforms combined with shear mappings. Shearings do not generally form a group under composition. Sets of shearings along a fixed axis, however, form an abelian group. Thus, we will only look at shearings along one axis, here the ξ_3-axis.

Let S be the shear operator

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ s_1 & s_2 & 1 \end{pmatrix}.$$

The following theorem shows that the Linearized Riesz Transform commutes with shearings. Here, we generalize a result from [16].

Theorem 9.4.7 *Let $g \in L_2\left(\mathbb{R}^3, \mathbb{R}\right)$ be a filter function such that $\hat{g}$ is supported in $\left\{\xi \in \mathbb{R}^3 : \left|\frac{\xi_k}{\xi_3}\right| \le \alpha_k, k = 1, 2\right\}$, where $0 \le \alpha_k \le 1$, $k = 1, 2$. Then, for $\alpha_k - 1 \le s_k \le 1 - \alpha_k$, the relation*

$$(R_L g)\left(S^{-1}\cdot\right) = D_1^{s_1 \frac{\pi}{4}} D_2^{s_2 \frac{\pi}{4}} R_L g \left(S^{-1}\cdot\right)$$

where the rotations D_k^α are defined as below, holds true. In other words, up to a set of rotations by $s_2\frac{\pi}{4}$ and $s_1\frac{\pi}{4}$, the linearized Riesz transform of the sheared filter equals the sheared linearized Riesz transform of the filter.

Proof First we need to consider rotations in $\mathbb{R}^3$. Any rotation can be defined by a plain, in which the rotation is to take place, and an angle α. The plain is well-defined as the span of two orthogonal unit vectors $\vec{g}_1$, $\vec{g}_2$; the orientation of the rotation is given by the order of those two vectors. We will denote the described rotation with $D^\alpha_{\vec{g}_1,\vec{g}_2}$.

Let $\vec{g}_1$ and $\vec{g}_2$ be given in polar representation, i.e. $(1, \Phi_2, \Phi_1)$ with $\Phi_2 \in [0, 2\pi]$, $\Phi_1 \in [0, \pi]$. We will denote a rotation where $\vec{g}_1 = (1, \Phi_2, \Phi_1)$ and $\vec{g}_2 = \left(1, \Phi_2 + \frac{\pi}{2}, \Phi_1\right)$ for all possible Φ_k, $k = 1, 2$, as D_2^α. Similarly, D_1^α is to be the rotation with $\vec{g}_1$ as above and $\vec{g}_2 = \left(1, \Phi_2, \Phi_1 + \frac{\pi}{2}\right)$. Note that a specific D_k^α is actually not one distinct rotation but a set of rotations that are dependent on the angles Φ_2 and Φ_1, and so on the vector that is to be rotated.

It can easily be seen that if

$$\Phi_2 + \alpha \in [0, 2\pi] \text{ for } k = 2 \text{ and } \Phi_1 + \alpha \in [0, \pi] \text{ for } k = 1, \tag{9.4.10}$$

then D_k^α is a well-defined mapping of $\mathbb{R}^3$ to itself,

$$D_1^\alpha : (r, \Phi_2, \Phi_1) \mapsto (r, \Phi_2, \Phi_1 + \alpha)\,.$$
$$D_2^\alpha : (r, \Phi_2, \Phi_1) \mapsto (r, \Phi_2 + \alpha, \Phi_1)\,.$$

In contrast to standard rotations, D_k^α even commutes, i.e.

$$D_k^\alpha \circ D_l^\beta = D_l^\beta \circ D_k^\alpha,$$

as long as the condition in (9.4.10) is kept. This property can quickly be confirmed by the fact that D_k^α only changes one polar angle of the vector it is applied to, and leaves the radius and other angle the same.

We can now show (9.4.7): Since S is a linear, invertible operator and $\det(S) = 1$, the General Stretch Theorem gives us

$$\begin{aligned} \widehat{g\left(S^{-1}\cdot\right)}(\xi) &= |\det S|\hat{g}\left(S^T\xi\right) \\ &= \hat{g}(\xi_1 + s_1\xi_3, \xi_2 + s_2\xi_3, \xi_3). \end{aligned}$$

By the support assumption on g, this function becomes zero if one of the conditions

$$-\alpha_k \le \frac{\xi_k + s_k\xi_3}{\xi_3} \le \alpha_k, \quad k = 1, 2$$

is not fulfilled.

Because of the restriction on s_k, we have

$$-1 \le -\alpha_k - s_k \le \frac{\xi_k}{\xi_3} \le \alpha_k - s_k \le 1.$$

Together with the definition of R_L, we get for $\xi_3 > 0$,

$$\begin{aligned} \left(R_L g\left(S^{-1}\cdot\right)\right)\hat{}(\xi) &= -i\left(1, \Phi_{L,2}(\xi), \Phi_{L,1}(\xi)\right)\hat{g}(S^T\xi) \\ &= -i\left(1, \frac{\xi_2}{\xi_3}\frac{\pi}{4}, \frac{\xi_1}{\xi_3}\frac{\pi}{4}\right)\hat{g}(S^T\xi). \end{aligned}$$

On the other hand,

$$\left|\frac{\left[S^T\xi\right]_k}{\left[S^T\xi\right]_3}\right| = \left|\frac{\xi_k + s_k\xi_3}{\xi_3}\right| = \left|\frac{\xi_k}{\xi_3} + s_k\right|$$
$$\leq \left|\frac{\xi_k}{\xi_3}\right| + |s_k| \leq \alpha_k + (1-\alpha_k) = 1, \quad k = 1, 2,$$

so we get

$$\begin{aligned}\left((R_L g)\left(S^{-1}\cdot\right)\right)\hat{}(\xi) &= \widehat{R_L g}\left(S^T\xi\right)\\ &= -i\left(1, \Phi_{L,2}(S^T\xi), \Phi_{L,1}(S^T\xi)\right)\hat{g}(S^T\xi)\\ &= -i\left(1, \frac{\xi_2}{\xi_3}\frac{\pi}{4} + s_2\frac{\pi}{4}, \frac{\xi_1}{\xi_3}\frac{\pi}{4} + s_1\frac{\pi}{4}\right)\hat{g}(S^T\xi)\\ &= -i\, D_1^{s_1\frac{\pi}{4}} D_2^{s_2\frac{\pi}{4}}\left(1, \frac{\xi_2}{\xi_3}\frac{\pi}{4}, \frac{\xi_1}{\xi_3}\frac{\pi}{4}\right)\hat{g}(S^T\xi)\\ &= D_1^{s_1\frac{\pi}{4}} D_2^{s_2\frac{\pi}{4}}\left(R_L g\left(S^{-1}\cdot\right)\right)\hat{}(\xi).\end{aligned}$$

Similarly we can conclude for $\xi_3 < 0$.

The assertion follows by taking the inverse Fourier transform and the Fourier rotation theorem,

$$(R_L g)\left(S^{-1}\cdot\right) = D_1^{s_1\frac{\pi}{4}} D_2^{s_2\frac{\pi}{4}} R_L g\left(S^{-1}\cdot\right).$$

□

9.5 Frames

Theorem 9.5.1 *The quasi-monogenic Riesz transform* $R_m = \sum_{j=1}^n e_j R_{m_j}$ *maps* $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n, C\ell_n^{(1)})$ *and the adjoint operator* $(R_m)^* F = -\sum_{j=1}^n R_{m_j} f_j$*, where* $F = \sum_{j=1}^n e_j f_j$*, maps* $L^2(\mathbb{R}^n, C\ell_n^{(1)}) \to L^2(\mathbb{R}^n, \mathbb{R})$.

Proof We have already proven that R_k^m maps $L^2(\mathbb{R}^n)$ into $L^2(\mathbb{R}^n)$. We compute the adjoint operator.

$$\begin{aligned}\langle R_m f, G\rangle &= \sum_{j=1}^{n} \int \overline{R_{m_j} f(x)}^{\mathbb{C}} G_j(\underline{x})\, d\underline{x} = \sum_{j=1}^{n} \int \overline{m_j(\underline{\xi})\hat{f}}^{\mathbb{C}} \hat{G}_j(\underline{\xi})\, d\underline{\xi} \\ &= \int \overline{\hat{f}}^{\mathbb{C}} \left(\sum_{j=1}^{n} \overline{m_j(\underline{\xi})}^{\mathbb{C}} \hat{G}_j(\underline{\xi}) \right) d\xi = - \int \overline{f}^{\mathbb{C}} \left(\sum_{j=1}^{n} \overline{f}^{\mathbb{C}} R_{m_j} R_m G_j(\underline{x}) \right) d\underline{x},\end{aligned}$$

i.e. $(R_m)^* G(\underline{x}) = -\sum_{j=1}^{n} R_{m_j} G_j(\underline{x})$. □

It follows that

$$(R_m)^* R_m f(\underline{x}) = -\sum_{j=1}^{n} R_{m_j} R_{m_j} f(\underline{x}) = f(\underline{x}).$$

Definition 9.5.2 (Frame, Parseval Frame) A family $\{f_j\}_{j\in\mathbb{Z}}$ of elements of a Hilbert space $\mathcal{H}$ is a frame if there exist positive constants A and B such that

$$A||f||^2 \leq \sum_{n\in\mathbb{Z}} |\langle f, f_n\rangle|^2 \leq B||f||^2 \quad \text{for all } f \in \mathcal{H}.$$

The frame is called tight if $A = B$ and it is a Parseval frame if $A = B = 1$.

Theorem 9.5.3 *Suppose that* $\{\varphi_k : k \in \mathbb{Z}\}$ *is a Parseval frame for* $L^2(\mathbb{R}^n)$*. Then the quasi-monogenic Riesz transform generates a frame:*

$$\{\psi_{k,j} = R_{m_j}(\varphi_k) : k \in \mathbb{Z}, j = 1, \ldots, n\},$$

with

$$f = \sum_{j=1}^{n} \sum_{k} \langle f, \psi_{k,j}\rangle \psi_{k,j}.$$

Proof From Theorem 9.5.1 it follows $f = (R_m)^* R_m f$, then $R_{m_j} f$ can be expanded in the original frame:

$$R_{m_j} f = \sum_{k\in\mathbb{Z}} \langle R_{m_j} f, \varphi_k\rangle \varphi_k = -\sum_{k\in\mathbb{Z}} \langle f, R_{m_j}\varphi_k\rangle \varphi_k$$

and hence

$$f = (R_m)^* \left(-\sum_{k\in\mathbb{Z}} e_j \langle f, R_{m_j}\varphi_k\rangle \varphi_k \right) = -\sum_{j=1}^{n} R_{m_j} \left(-\sum_{k\in\mathbb{Z}} \langle f, R_{m_j}\varphi_k\rangle \varphi_k \right)$$
$$= \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} \langle f, \psi_{k,j}\rangle \psi_{k,j}.$$

It is a tight frame due to

$$||R_m f||_2^2 = \sum_{j=1}^{n} ||R_{m_j} f||_2^2 = \sum_{j=1}^{n} \sum_{k\in\mathbb{Z}} |\langle R_{m_j} f, \varphi_k\rangle|^2 = \sum_{k\in\mathbb{Z}} |\langle f, \psi_{k,j}\rangle|^2.$$

□

9.5.1 Riesz Wavelet Frames

We can get more results if we construct wavelet frames from a primal wavelet. These wavelet frames are families of functions $\{\psi_{i,\underline{k}}\}_{\underline{k}\in\mathbb{Z}^k, i\in\mathbb{Z}}$ that are indexed by a pair $(\underline{k}, i)$ of location and scale indices. The wavelet $\psi_{\underline{k},i}(\underline{x})$ is a dilated and translated version of the mother wavelet $\psi = \psi_{0,0}$, i.e. $\psi_{\underline{k},i}(\underline{x}) = D^i T_{\underline{k}} \psi(\underline{x}) = \det(D)^{i/2}\psi(D^{-i}\underline{x} - \underline{k})$, where D is dilation matrix with positive determinant. The standard dyadic wavelets correspond to $D = 2I$. Theorem 9.5.1 implies the decomposition/reconstruction formula

$$f = \sum_{j=1}^{n} \sum_{\underline{k}\in\mathbb{Z}^d} \sum_{i\in\mathbb{Z}} \langle f, \psi_{i,\underline{k},j}\rangle \psi_{i,\underline{k},j}, \quad \forall f \in L^2(\mathbb{R}^n),$$

the coefficients of which can be obtained from coefficients of the primal wavelet:

$$\langle f, \psi_{i,\underline{k},j}\rangle = \langle f, R_{m_j}\psi_{i,\underline{k}}\rangle = \langle R^*_{m_j} f, \psi_{i,\underline{k}}\rangle.$$

Because R_{m_j} is a convolution operator and $\psi_{i,\underline{k}} = D^i T_{\underline{k}}\psi$, we obtain $\langle f, \psi_{i,\cdot,j}\rangle = \langle R^*_{m_j} f, \psi_{i,\cdot}\rangle = R^*_{m_j}\langle f, D^i\psi(\underline{x} - \cdot)\rangle$.

The first example is the classical Riesz transforms. Not only do Riesz transforms form a frame, they are also invariant with respect to rotations and this property is unique in the following sense:

Theorem 9.5.4 ([21]) *Let $T = (T_1, T_2, \ldots, T_n)$ be an n-tuple of bounded transforms on $L^2(\mathbb{R}^n)$. Suppose*

(a) *Each T_j commutes with translations of $\mathbb{R}^n$,*
(b) *Each T_j commutes with (isotropic) dilations of $\mathbb{R}^n$,*
(c) *For every rotation $\rho = (\rho_{jk})$ of $\mathbb{R}^n$, $\rho T_j \rho^{-1} f = \sum_k \rho_{jk} T_k f$.*

Then the T_j are a constant multiple of the Riesz transforms, i.e. there exists a constant c, so that $T_j = cR_j$, $j = 1, \ldots, n$.

An immediate consequence is the steerability of the Riesz transforms:

Definition 9.5.5 Let $\underline{u} \in \mathbb{R}^n : |\underline{u}| = 1$ and let $\rho \in SO(n) : \underline{u} = \rho e_1$. Then the Riesz transform in direction $\underline{u}$ is given by

$$R_{\underline{u}} f(\underline{x}) = \rho^{-1} R_1 \rho f(\underline{x}) = \sum_{l=1}^{n} \rho_{1,l} R_l f(\underline{x}), \quad \forall f \in L^2(\mathbb{R}^n), \underline{x} \in \mathbb{R}^n,$$

where $\rho_{k,l}$ are the entries of the matrix ρ.

Hence the Riesz transform is steerable, since the Riesz transform with respect to any direction $\underline{u}$ is a linear combination of the n Riesz transforms R_j with respect to the basis directions.

But we can get even more:

Theorem 9.5.6 *Let $\psi \in L^2(\mathbb{R}^n)$ and let $\mathcal{D}$ be a rotated dilation, i.e. $\mathcal{D} := D_d \rho$, where $\rho \in SO(n)$. Then the monogenic wavelet transform is generated by the monogenic mother wavelet*

$$\psi_m := \psi + R\psi = \psi + \sum_{l=1}^{n} e_k R_l \psi$$

and satisfies

$$W_{\psi_m} f(\underline{t}, j) = \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle + \sum_{l,k=1}^{n} e_l (\rho^j)_{k,l} \langle R_k^* f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle.$$

Proof

$$\begin{aligned} W_{\psi_m} f(\underline{t}, j) &= \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle + \sum_{l=1}^{n} e_l \langle f, \mathcal{D}_d^j T_{\underline{t}} R_l \psi_m \rangle \\ &= \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle + \sum_{l=1}^{n} e_l \langle f, D_{d^j} \rho^j T_{\underline{t}} R_l \psi_m \rangle \end{aligned}$$

$$
\begin{aligned}
&= \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle + \sum_{l,k=1}^{n} e_l (\rho^j)_{k,l} \langle f, R_k D_{d^j} \rho^j T_{\underline{t}} \psi_m \rangle \\
&= \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle + \sum_{l,k=1}^{n} e_l (\rho^j)_{k,l} \langle R_k^* f, D_{d^j} \rho^j T_{\underline{t}} \psi_m \rangle \\
&= \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle + \sum_{l,k=1}^{n} e_l (\rho^j)_{k,l} \langle R_k^* f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle
\end{aligned}
$$

□

Because of $\langle R_k^* f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle = R_k^* \langle f, \mathcal{D}_d^j T_{\underline{t}} \psi_m \rangle$ the coefficients of the monogenic wavelet transform with the mother wavelet ψ_m can be directly computed from the coefficients of the wavelet transform with the mother wavelet ψ.

Next, we would like to have a quasi-monogenic Riesz transform that interacts the same way with shearings as Riesz transforms do with rotations.

9.5.2 Shearlets

Let $\psi \in L^2(\mathbb{R}^2, \mathbb{R})$ be a function that is composed of a wavelet ψ_1 and a bump function in Fourier domain $\hat{\psi}_2$ with supp $\hat{\psi}_2 \subseteq [-1, 1]$:

$$
\hat{\psi}(\underline{\xi}) = \hat{\psi}_1(\xi_1) \hat{\psi}_2 \left(\frac{\xi_2}{\xi_1} \right).
$$

Such a function fulfills the admissibility condition and will be a mother shearlet.

$$
\int_{\mathbb{R}^2} \frac{|\hat{\psi}(\underline{\xi})|^2}{\xi_1^2} \, d\underline{\xi} < \infty
$$

Additionally, the shearlets have a scaling function φ. For specific constructions see [16, 18]. Let the parabolic scaling matrix A_α be defined by

$$
A_a = \begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}, \quad a \in \mathbb{R}^+.
$$

Then the shearlets $\psi_{a,s,t}$ are defined by dilation, shearing and translation

$$
\psi_{a,s,t}(\underline{x}) := a^{-\frac{3}{4}} \psi(A_a^{-1} S_s^{-1} (\underline{x} - \underline{t})).
$$

The continuous shearlet transform $SH_\psi(f)$ of a function $f \in L^2(\mathbb{R}^2)$ is defined by

$$SH_\psi(f)(a, s, t) := \langle f, \psi_{a,s,t} \rangle = \langle \hat{f}, \hat{\psi}_{a,s,t} \rangle.$$

Classical continuous shearlet systems do exhibit a directional bias. This problem can be resolved by partitiong the Fourier domain into four conic regions and considering the low frequencies separately. Therefore, we define the restricted horizontal and vertical cones by

$$C^h := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \leq \frac{1}{2}, |\xi_2| < |\xi_1| \right\},$$

$$C^v := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_2| \leq \frac{1}{2}, |\xi_2| > |\xi_1| \right\},$$

respectively, and the "intersection" of the two cones and the low frequency set by

$$C^x := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| \leq \frac{1}{2}, |\xi_2| \leq \frac{1}{2},\ |\xi_1| = |\xi_2| \right\},$$

$$C^0 := \left\{ (\xi_1, \xi_2) \in \mathbb{R}^2 : |\xi_1| < 1, |\xi_2| < 1 \right\}.$$

To obtain discrete shearlets on the cone, the discrete scaling and shear parameters are chosen to be

$$a_j := 2^{-2j} = \frac{1}{4^j}, j = 0, \ldots, j_0 - 1, \quad s_{j,k} := k2^{-j}, -2^j \leq k \leq 2^j, k \in \mathbb{Z}$$

and the translation parameters $t_{\underline{l}}, \underline{l} \in \mathbb{Z}^2$. The domain is chosen to be a discrete square

$$\mathcal{D} := \{\underline{l} = (l_1, l_2) : l_i = 0, \ldots, N-1, i = 1, 2\}$$

and assume periodic continuation over the boundary. Then the Fourier domain is

$$\Omega := \{\underline{\xi} = (\xi_1, \xi_2) : \xi_i = -[\frac{N}{2}], \ldots, [\frac{N}{2}] - 1, i = 1, 2\}.$$

We deal only with a finite number of scales $j = 0, 1, \ldots, j_0$, where $j_0 := [\frac{1}{2} \log_2 N]$ and choose the translation parameters $t_{\underline{l}} = \frac{\underline{l}}{N}, \underline{l} \in \mathcal{D}$. Then the discrete shearlets are defined as

$$\psi_{j,k,\underline{l}}(\underline{x}) := 2^{-\frac{3j}{2}} \psi_{a_j, s_{jk}, t_{\underline{l}}}(\underline{x}) = \psi(A_{a_j}^{-1}).$$

The shearlets ψ^h on the horizontal con $\mathcal{C}^h$ are given by

$$\psi_{j,k,\underline{l}}(\underline{x}) = \psi(A_{a_j}^{-1} S_{s_{jk}}^{-1}(\underline{x} - \underline{t}))\chi_{\mathcal{C}^h},$$

where $\chi_{\mathcal{C}^h}$ is the characteristic function of the cone $\mathcal{C}^h$. Analoguously, the shearlets ψ^v on the vertical cone are defined by changing the roles of ξ_1 and ξ_2. Moreover,

$$\psi^{hxv}_{j,\pm 2^j,\underline{l}} := \psi^h_{j,\pm 2^j,\underline{l}}\chi_{\mathcal{C}^x},$$

with the characteristic function $\chi_{\mathcal{C}^x}$ of $\mathcal{C}^x$. In [15] it is shown that the set

$$\{\psi^h_{j,k,\underline{l}}, \psi^v_{j,k,\underline{l}}, \psi^{hxv}_{j,\pm 2^j,\underline{l}} : j = 0, \ldots, j_0 - 1, -2^j + 1 \le k \le 2^j - 1, \underline{l} \in \mathcal{D}\} \\ \cup \{\varphi_{\underline{l}} : \underline{l} \in \mathcal{D}\}$$

For the discrete setting we define the quasi-monognic Riesz transforms with the discrete Fourier transform and restrict everything to the discrete domains. I.e. for a function $\mathcal{D} \to \mathbb{R}$ the linearized Riesz transform is defined using DFT as

$$\widehat{R_L f}(\underline{\xi}) := -i e^{i\varphi_L(\underline{\xi})} \hat{f}(\underline{\xi}), \quad \underline{\xi} \in \Omega.$$

and $\mathcal{H}'_L f := (f, R_{L,1} f, R_{L,2} f)$.

Definition 9.5.7 The discrete quasi-monogenic shearlet transform on the cone is defined by

$$\mathcal{B} := \{\mathcal{H}'_L \psi^h_{j,k,\underline{l}}, \mathcal{H}'_L \psi^v_{j,k,\underline{l}}, \mathcal{H}'_L \psi^{hxv}_{j,\pm 2^j,\underline{l}} : \\ j = 0, \ldots, j_0 - 1, -2^j + 1 \le k \le 2^j - 1, \underline{l} \in \mathcal{D}\} \cup \{\mathcal{H}'_L \varphi_{\underline{l}} : \underline{l} \in \mathcal{D}\}. \tag{9.5.1}$$

and the quasi-monogenic discrete shearlet transform by

$$\mathcal{MSH}(f)(\kappa, j, k, \underline{l}) := \begin{cases} \langle f, \varphi_{\underline{l}} \rangle & \text{for } \kappa = 0, \\ \langle f, \mathcal{H}'_L \psi^\kappa_{j,k,\underline{l}} \rangle & \text{for } \kappa \in \{h, v\}, \\ \langle f, \mathcal{H}'_L \psi^\kappa_{j,k,\underline{l}} \rangle & \text{for } \kappa = x, |k| = 2^j, \end{cases}$$

where $j = 0, \ldots, j_0, -2^j + 1 \le k \le 2^j - 1$, and $\underline{l} \in \mathcal{D}$.

To prove that the definition actually defines shearlets, it is important that the following property is fulfilled. Because $\hat{\psi}_{j,0,\underline{0}}$ is supported in $\{\underline{\xi} \in \mathbb{R}^2 : \left|\frac{\xi_2}{\xi_1}\right| \le$

$2^{-j}\}$, using $\alpha = 2^{-j}$ and $s = 2^{-j}k$ in Lemma 9.4.4, we obtain

$$R_{L,i}(\psi_{j,0,\underline{0}}(S_{s_{jk}}^{-1}\cdot)) = e^{2^{-j}k\frac{\pi}{4}}(R_{L,i}\psi_{j,0,\underline{0}})(S_{s_{jk}}^{-1}\cdot), \quad i = 1, 2.$$

Furthermore, the set $\mathcal{B}$ of quasi-monogenic shearlets defined in (9.5.1) forms a tight frame for $L^2(\mathcal{D})$ with frame bound $A = 2$.

To compute the quasi-monogenic shearlet transform we use

$$\begin{aligned}\langle f, \mathcal{H}_L'\psi_{j,k,\underline{l}}^{\kappa}\rangle &= (\langle f, \psi_{j,k,\underline{l}}^{\kappa}\rangle, \langle f, R_{L,1}\psi_{j,k,\underline{l}}^{\kappa}\rangle, \langle f, R_{L,2}\psi_{j,k,\underline{l}}^{\kappa}\rangle)\\ &= (\langle f, \psi_{j,k,\underline{l}}^{\kappa}\rangle, \langle R_{L,1}^{*}f, \psi_{j,k,\underline{l}}^{\kappa}\rangle, \langle R_{L,2}^{*}f, \psi_{j,k,\underline{l}}^{\kappa}\rangle).\end{aligned}$$

It follows with $\psi_{j,k,\cdot}^{\kappa} := \left(\psi_{j,k,\underline{l}}^{\kappa}\right)_{\underline{l}\in\mathcal{D}}$ that

$$\langle f, \mathcal{H}_L'\psi_{j,k,\cdot}^{\kappa}\rangle = (\langle f, \psi_{j,k,\cdot}^{\kappa}\rangle, R_{L,1}^{*}\langle f, \psi_{j,k,\cdot}^{\kappa}\rangle, R_{L,2}^{*}\langle f, \psi_{j,k,\cdot}^{\kappa}\rangle).$$

This means that the adjoint Riesz transform can simply be applied to the shearlet coefficients to obtain the monogenic coefficients. Similarly, 3d monogenic shearlets on the cone can be built using the linearized Riesz transforms and the construction of shearlets in [18].

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Part III
Monogenic Polynomials and Numerical Methods

Chapter 10
Quaternionic Operator Calculus for Boundary Value Problems of Micropolar Elasticity

Klaus Gürlebeck and Dmitrii Legatiuk

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract Micropolar elasticity is a refined version of the classical elasticity. Equations of micropolar elasticity are not given only by a single differential equation w.r.t. a vector field of displacement, but by a coupled system of differential equations connecting fields of displacements and rotations. However, construction of solution methods for boundary value problems of micropolar elasticity is still an open mathematical task, mostly due to the coupled nature of the resulting system of partial differential equations. Especially, only few results are available for spatial problems of micropolar elasticity. Therefore, in this paper, we present a quaternionic operator calculus-based approach to construct general solutions to three-dimensional problems of micropolar elasticity. Moreover, we prove solvability of the boundary value problem of micropolar elasticity, as well as we provide an explicit estimate for the difference between the classical elasticity and the micropolar model.

Keywords Micropolar elasticity · Operator calculus · Quaternionic analysis · Representation formulae · Modelling error

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10.1 Introduction

Original ideas for the extension of classical elasticity theory to account microeffects of a continuum go back to the work [1] of Cosserat brothers, where they introduced a new theory called the *Cosserat continuum*. The introduced theory grabbed attention of many scientists. Among others, works of Eringen [2], and Nowacki [3] significantly supported further development of the theory. Especially Eringen introduced micro-inertia in the theory, which has led to renaming of the theory to the *micropolar elasticity*. From practical point of view, the micropolar theory models not only displacements of a continuum, as in the classical theory of elasticity, but also its rotations. Therefore, the micropolar theory assures a more precise description of composites, materials with cellular structure, materials with fibers, and human bones. Additional to micromodelling of materials, micropolar elasticity can be used on macrolevel for the modelling of masonry structures and objects having similar cellular-like structure.

The development of solution methods for boundary value problems of micropolar elasticity was essentially based on the methods of complex function and potential theories, since the two theories were widely and successfully applied in the classical elasticity theory. Generalisations of the Papkovich-Neuber approach to micropolar elasticity were presented in [4, 5]. Representation formulae based on complex analysis were presented in [6], where the approach similar to the classical Kolosov-Muskhelishvili formulae was used. Due to the progress in the field of modern materials, a growing interest appeared in recent years to the development of solution techniques for boundary value problems of micropolar elasticity. Especially, for problems containing stress concentrations, such as crack, see for example [7–9] and references therein. However, only two-dimensional problems have been considered so far. Therefore, development of solution methods for spatial problems of micropolar elasticity is an open task.

In this paper we introduce representation formulae for the solution of spatial boundary value problems of micropolar elasticity. The representation formulae are constructed in the framework of quaternionic analysis, which is a natural extension of the classical complex analysis to higher dimensions. The main toolbox for constructing representation formulae for problems of mathematical physics in hypercomplex analysis is the co-called quaternionic operator calculus, which has been introduced in [10], see [11] for applications and recent advances. The essential ingredient is the T-operator (Teodorescu transform), which is a right inverse to the generalised Cauchy-Riemann operator. Accomplishing the T-operator with the F-operator (Cauchy-Bitsadze operator), the higher-dimensional generalisation of the classical Borel-Pompeiu formula can be obtained, which is the core of applications of the operator calculus to boundary value problems of mathematical physics. However, problems of micropolar elasticity have not been considered so far in the hypercomplex setting. Thus, in this paper we study equations of micropolar elasticity by tools of quaternionic operator calculus. Moreover, by working with general operator equations we discuss the solvability of Dirichlet boundary value problems for micropolar elasticity in a strong sense, while previous results are related to variational approaches, see for example [12, 13]. Additionally, we provide an estimate for the difference between the classical elasticity theory and the micropolar elasticity. This difference is estimated by help of the corresponding operator norms.

10.2 Preliminaries and Notations

10.2.1 Basics of Quaternionic Analysis

Let $1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ be an orthonormal basis of the Euclidean vector space $\mathbb{R}^4$. As usual we identify the basis vector $\mathbf{e}_0$ with 1. We introduce an associative multiplication of the basis vectors subject to the multiplication rules:

$$\mathbf{e}_1^2 = \mathbf{e}_2^2 = \mathbf{e}_3^2 = -1, \quad \mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_3.$$

This non-commutative product generates the algebra of real quaternions denoted by $\mathbb{H}$. The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying the element $\mathbf{a} = (a_0, a_1, a_2, a_3) \in \mathbb{R}^4$ with the element

$$\mathbf{a} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathbb{H}.$$

The real number $\mathbf{Sc}\,\mathbf{a} := a_0$ is called the scalar part of $\mathbf{a}$ and $\mathbf{Vec}\,\mathbf{a} := a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$ is the vector part of $\mathbf{a}$, or the *pure quaternion*. Analogous to the complex case, the conjugate of $\mathbf{a} := a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \in \mathbb{H}$ is the quaternion $\bar{\mathbf{a}} := a_0 - a_1\mathbf{e}_1 - a_2\mathbf{e}_2 - a_3\mathbf{e}_3$. The norm of $\mathbf{a}$ is given by $|\mathbf{a}| = \sqrt{\mathbf{a}\bar{\mathbf{a}}}$ and coincides with the corresponding Euclidean norm of $\mathbf{a}$, as a vector in $\mathbb{R}^4$. Finally, the real vector space $\mathbb{R}^3$ will be embedded in $\mathbb{H}$ by identifying the element $\mathbf{a} = (a_0, a_1, a_2) \in \mathbb{R}^3$ with the corresponding pure quaternion, i.e. $\mathbf{a} = a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + a_2\mathbf{e}_3 \in \mathbb{H}$.

Let Ω be an open subset of $\mathbb{R}^3$ with a sufficiently smooth boundary. An $\mathbb{H}$-valued function is a mapping

$$f : \Omega \mapsto \mathbb{H} \text{ with } f(\mathbf{x}) = \sum_{k=0}^{3} f^k(\mathbf{x})\mathbf{e}_k, \quad \mathbf{x} \in \Omega.$$

The coordinates f^k are real-valued functions defined in Ω, i.e.

$$f^k : \Omega \mapsto \mathbb{R}, \quad k = 0, 1, 2, 3.$$

Continuity, differentiability or integrability of f are defined coordinate-wisely.

Definition 10.2.1 For continuously real-differentiable functions $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$, which we will denote for simplicity by $f \in C^1(\Omega, \mathbb{H})$, the operator

$$D := \sum_{k=1}^{3} \mathbf{e}_k \partial_{x_k}$$

is called the Dirac operator.

Additionally, we need to introduce two integral operators [11]:

Definition 10.2.2 Let $\Omega \subset \mathbb{R}^3$, $u \in C(\Omega)$. Then the linear integral operator

$$(T\, u)\,(\mathbf{x}) := -\int_{\Omega} E(\mathbf{y}-\mathbf{x})u(\mathbf{y})d\sigma_{\mathbf{y}}$$

with

$$E(\mathbf{x}) = \frac{1}{4\pi}\,\frac{\bar{\omega}(\mathbf{x})}{|\mathbf{x}|^3}, \quad \omega(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|},$$

is called the Teodorescu transform over Ω. We also define the operator

$$(F_\Gamma u)\,(\mathbf{x}) := \int_{\Gamma} E(\mathbf{y}-\mathbf{x})d\mathbf{y}^* u(\mathbf{y})$$

that is called Cauchy-Bitsadze operator.

Finally, by using the introduced operators, the Borel-Pompeiu formula can be written in the form

$$(F_\Gamma u)\,(\mathbf{x}) + (T\,D\,u)\,(\mathbf{x}) = \begin{cases} u(\mathbf{x}), & \mathbf{x} \in \Omega, \\ 0, & \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega}, \end{cases}$$

or shortly $F + TD = I$ for $\mathbf{x} \in \Omega$.

For the treatment of boundary value problems it is important to know the boundary behaviour of the Cauchy-Bitsadze operator.

Theorem 10.2.3 (Plemelj-Sokhotzki Formulae) *Let $u \in C^{0,\beta}(\Gamma, \mathbb{H})$, $0 < \beta \leq 1$. Then we have for each regular point $\mathbf{x}_0 \in \Gamma$*

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}_0 \\ \mathbf{x}\in G^{\pm},\mathbf{x}_0\in\Gamma}} (F_\Gamma u)\,(\mathbf{x}) = \frac{1}{2}\left[\pm u(\mathbf{x}_0) + (S_\Gamma u)\,(\mathbf{x}_0)\right],$$

where $G^+ := G$ and $G^- := \mathbb{R}^n \setminus \overline{G^+}$, the limit has to be taken as a non-tangential limit, and S_Γ is the singular integral operator defined by

$$(S_\Gamma u)(\mathbf{x}) := 2 \int_\Gamma E(\mathbf{y}-\mathbf{x}) d\mathbf{y}^* u(\mathbf{y}), \quad \mathbf{x} \in \Gamma.$$

The limits define the Plemelj projections $P_\Gamma := \frac{1}{2}(I + S_\Gamma)$ and $Q_\Gamma := \frac{1}{2}(I - S_\Gamma)$. These operators can be extended to Sobolev spaces. For details, see [10].

Additionally to the classical version of operators, we need to introduce *modified* operators, which will be used later during the factorisation of equations of micropolar elasticity.

Definition 10.2.4 For continuously real-differentiable functions $f : \Omega \subset \mathbb{R}^3 \to \mathbb{H}$, which we will denote for simplicity by $f \in C^1(\Omega, \mathbb{H})$, the operator

$$D_\alpha := \alpha + \sum_{k=1}^{3} \mathbf{e}_k \partial_{x_k}, \quad \alpha \in \mathbb{C}$$

is called a modified Dirac operator.

Definition 10.2.5 Let $\Omega \subset \mathbb{R}^3$, $u \in C(\Omega)$. Then the weakly singular integral operator

$$(T_\alpha u)(\mathbf{x}) := -\int_G e_\alpha(\mathbf{y}-\mathbf{x}) u(\mathbf{y}) d\sigma_{\mathbf{y}}, \quad \mathbf{x} \in \Omega,$$

is called the *modified Teodorescu transform*; further, the operator

$$(F_\alpha u)(\mathbf{x}) := \int_\Gamma e_\alpha(\mathbf{y}-\mathbf{x}) d\mathbf{y}^* u(\mathbf{y}), \quad \mathbf{x} \in \mathbb{R}^3 \setminus \overline{\Omega},$$

acting of functions $\mathbf{u} \in C^1(\Omega) \cap C(\overline{\Omega})$, is called the *modified Cauchy-Bitsadze operator*. The kernel e_α is given by

$$e_\alpha(\mathbf{x}) = -\frac{1}{4\pi|\mathbf{x}|^3} \left(\alpha|\mathbf{x}|^2 + (i\alpha|\mathbf{x}| + 1) \sum_{k=1}^{3} \mathbf{e}_x x_k \right) e^{-i\alpha|\mathbf{x}|}.$$

Theorem 10.2.6 (Modified Plemelj-Sokhotzki Formulae) *Let* $u \in C^{0,\beta}(\Gamma, \mathbb{H})$, $0 < \beta \leq 1$. *Then we have*

$$\lim_{\substack{y \to x \in \Gamma \\ y \in \Omega}} (F_\alpha u)(x) = (P_\alpha u)(x) = \frac{1}{2}(I + S_\alpha)\, u(x),$$

$$\lim_{\substack{y \to x \in \Gamma \\ y \in \mathbb{R}^n \setminus \overline{\Omega}}} (F_\alpha u)(x) = -(Q_\alpha u)(x) = -\frac{1}{2}(I - S_\alpha)\, u(x),$$

where the singular integral operator S_α *is defined by*

$$(S_\alpha u)(\mathbf{x}) := \int_\Gamma e_\alpha(\mathbf{y} - \mathbf{x}) d\mathbf{y}^* u(\mathbf{y}), \quad \mathbf{x} \in \Gamma.$$

Details of the modified operators and study of their properties can be found in [11].

Important ingredients for the representation formulas which will be used later on in the paper are the mapping properties of the integral operators. These properties have been studied precisely in [10, 11, 14]:

$$T : W^{k,p}(\Omega) \to W^{k+1,p}(\Omega), \text{ and } T_{\pm\alpha} : W^{k,p}(\Omega) \to W^{k+1,p}(\Omega),$$

as well as the facts

$$\partial_i T\mathbf{u} \in W^{k,p}(\Omega), \text{ and } \partial_i T_{\pm\alpha}\mathbf{u} \in W^{k,p}(\Omega) \text{ if } \mathbf{u} \in W^{k,p}(\Omega),$$

with ∂_i denoting partial derivatives w.r.t. coordinate x_i, $i = 1, 2, 3$.

10.2.2 Equations of Micropolar Elasticity

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain with a sufficiently smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. A boundary value problem of the micropolar elasticity is formulated as follows

$$(\lambda + 2\mu + \kappa)\, \nabla\nabla \cdot \mathbf{u} - (\mu + \kappa)\, \nabla \times \nabla \times \mathbf{u} = -\kappa \nabla \times \boldsymbol{\varphi}, \tag{10.2.1}$$

$$(\alpha + \beta + \gamma)\, \nabla\nabla \cdot \boldsymbol{\varphi} - \gamma \nabla \times \nabla \times \boldsymbol{\varphi} - 2\kappa\boldsymbol{\varphi} = -\kappa \nabla \times \mathbf{u}, \tag{10.2.2}$$

with boundary conditions

$$\begin{cases} \mathbf{u} = \mathbf{g}_1 \text{ on } \Gamma_0, \\ \boldsymbol{\varphi} = \mathbf{g}_2 \text{ on } \Gamma_0, \end{cases} \quad \text{and} \quad \begin{cases} t_{lk} n_l = t_{(\mathbf{n})k} \text{ on } \Gamma_1, \\ m_{lk} n_l = m_{(\mathbf{n})k} \text{ on } \Gamma_1, \end{cases} \tag{10.2.3}$$

where $\mathbf{u}$ is the displacement vector, $\boldsymbol{\varphi}$ is the vector of micropolar rotation, t_{lk} is the stress tensor, m_{lk} is the couple stress tensor, ρ is the material density, j is a rotational inertia, λ and μ are the Lamé parameters, $\kappa, \alpha, \beta, \gamma$ are material parameters of micropolar theory, n_j are components of the unit outer normal vector, $t_{(\mathbf{n})k}$ are given surface forces, and $m_{(\mathbf{n})k}$ are given surface moments. See [15] and reference therein for the details related to mechanical meaning of additional material constants, as well as their experimental derivations. However, since it is not the purpose of this paper to discuss practical applicability of the micropolar theory, we will not address the issue of identification of material parameters here.

Equations (10.2.1) and (10.2.2) are general equations of micropolar elasticity in the static case. For now, we do not specify the function spaces for $\mathbf{u}$ and $\boldsymbol{\varphi}$, since during the construction of representation formulae in the next section the regularity requirements will become clear.

10.3 Application of Quaternionic Operator Calculus to Micropolar Elasticity Equations

We start with a hypercomplex reformulation of Eqs. (10.2.1) and (10.2.2), which is given in the following Proposition:

Proposition 10.3.1 *Considering the displacement field $\mathbf{u} \in C^2(\Omega)$ and micropolar rotations $\boldsymbol{\varphi} \in C^2(\Omega)$ as pure quaternions, i.e. $\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$, $\boldsymbol{\varphi} = \varphi_1\mathbf{e}_1 + \varphi_2\mathbf{e}_2 + \varphi_3\mathbf{e}_3$, equations of micropolar elasticity (10.2.1) and (10.2.2) can be written as follows*

$$\begin{aligned} D\, M_1\, D\,\mathbf{u} + \kappa \mathrm{Vec}\, D\boldsymbol{\varphi} &= 0, \\ \left(D - i\sqrt{\tfrac{2\kappa}{\gamma}}\right) M_2 \left(D + i\sqrt{\tfrac{2\kappa}{\gamma}}\right) \boldsymbol{\varphi} + \kappa \mathrm{Vec}\, D\mathbf{u} &= 0, \end{aligned} \tag{10.3.1}$$

where the operators M_1 and M_2 are defined by

$$\begin{aligned} M_1\mathbf{w} &:= -(\lambda + 2\mu + \kappa)w_0 - (\mu + \kappa)w_1\mathbf{e}_1 - (\mu + \kappa)w_2\mathbf{e}_2 \\ &\quad -(\mu + \kappa)w_3\mathbf{e}_3, \\ M_2\mathbf{w} &:= -(\alpha + \beta + \gamma)w_0 - \gamma\, w_1\mathbf{e}_1 - \gamma\, w_2\mathbf{e}_2 - \gamma\, w_3\mathbf{e}_3, \end{aligned}$$

for a quaternion-valued function $\mathbf{w} = w_0 + w_1\mathbf{e}_1 + w_2\mathbf{e}_2 + w_3\mathbf{e}_3$.

Proof The proof can be done by straight-forward calculations. □

We will reformulate the system as a system of operator equations. This form allows it in a better way to study directly the questions of existence, regularity, stablity and uniqueness as well as the formulation of some basic ideas for the approximate solution.

10.3.1 Representation Formulae

Theorem 10.3.2 *The system of equations*

$$\begin{cases} D\,M_1\,D\,\mathbf{u} + \kappa \operatorname{Vec} D\boldsymbol{\varphi} = 0, \\ \left(D - i\sqrt{\frac{2\kappa}{\gamma}}\right) M_2 \left(D + i\sqrt{\frac{2\kappa}{\gamma}}\right) \boldsymbol{\varphi} + \kappa \operatorname{Vec} D\mathbf{u} = 0, \end{cases} \tag{10.3.2}$$

with Dirichlet boundary conditions

$$\begin{cases} \mathbf{u} = \mathbf{g}_1 \ \textit{on}\ \Gamma_0, \\ \boldsymbol{\varphi} = \mathbf{g}_2 \ \textit{on}\ \Gamma_0, \end{cases}$$

is equivalent to the system of operator equations

$$\begin{cases} \mathbf{u} = F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma (\operatorname{tr} T M_1^{-1} F_\Gamma)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1 \\ \qquad -\kappa\, T M_1^{-1} T \operatorname{Vec} D\boldsymbol{\varphi}, \\ \\ \boldsymbol{\varphi} = F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha}\right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2 \\ \qquad -\kappa\, T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D\mathbf{u}, \end{cases} \tag{10.3.3}$$

where $\tilde{\mathbf{g}}_1 = \mathbf{g}_1 + \kappa \operatorname{tr} T M_1^{-1} T \operatorname{Vec} D\boldsymbol{\varphi}$ *and* $\tilde{\mathbf{g}}_2 = \mathbf{g}_1 + \kappa \operatorname{tr} T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D\mathbf{u}$.

Proof At first we construct the representation formula for $\mathbf{u}$, depending on $\boldsymbol{\varphi}$, i.e. we consider the following boundary value problem

$$\begin{cases} D\,M_1\,D\,\mathbf{u} = -\kappa \operatorname{Vec} D\boldsymbol{\varphi}, \ \text{in } \Omega \\ \mathbf{u} = \mathbf{g}_1, \ \text{on } \Gamma. \end{cases} \tag{10.3.4}$$

Taking into account the Borel-Pompeiu formula and that $DT = I$, the solution of the non-homogeneous boundary value problem (10.3.4) can be written in the form

$$\mathbf{u} = F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma \left(\operatorname{tr}\ T M_1^{-1} F_\Gamma\right)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1 - \kappa\, T M_1^{-1} T \operatorname{Vec} D\boldsymbol{\varphi}.$$

Assuming for the moment that the inverse operator $\left(\operatorname{tr}\ T M_1^{-1} F_\Gamma\right)^{-1}$ is correctly defined it is a straight forward calculation to show that $\mathbf{u}$ solves the boundary value problem (10.3.4). The crucial point is the study of

$$\operatorname{tr} T M_1^{-1} F_\Gamma : \operatorname{im} P_\Gamma \cap W^{k+\frac{1}{p},p}(\Gamma) \mapsto W^{k+1+\frac{1}{p},p}(\Gamma) \cap \operatorname{im} Q_\Gamma .$$

The regularity properties follow from the mapping properties of the single operators. From the Borel-Pompeiu formula we get that the boundary values of Tf for any f belong to the image of Q_Γ.

The operator under consideration is a one-to-one mapping. Assuming $\operatorname{tr} T M_1^{-1} F_\Gamma \mathbf{w} = 0$ and $\operatorname{tr} \mathbf{w} = 0$ we get that $\mathbf{v} = T M_1^{-1} F_\Gamma \mathbf{w} = 0$ because the Dirichlet problem for the Lamé equation is uniquely solvable and $\mathbf{v}$ is a solution. Applying D and then M_1 we get $F_\Gamma \mathbf{w} = 0$ implying that $\mathbf{w} \in \operatorname{im} Q_\Gamma \cap \operatorname{im} P_\Gamma$ and $\mathbf{w} = 0$.

If, conversely, $\mathbf{w} \in \operatorname{im} Q_\Gamma$, then due to the solvability of the Lamé system a function $\mathbf{u} \in \ker D M_1 D$ with $\operatorname{tr} \mathbf{u} = \mathbf{w}$ exists. Applying the Borel-Pompeiu formula, M_1^{-1} and once again the Borel-Pompeiu formula we obtain $\mathbf{w} = \operatorname{tr} T M_1^{-1} F_\Gamma \mathbf{s}$ with $\mathbf{s} = \operatorname{tr} M_1 D \mathbf{u} \in \operatorname{im} P_\Gamma$ and therefore, the mapping $\operatorname{tr} T M_1^{-1} F_\Gamma : \operatorname{im} P_\Gamma \cap W^{k+\frac{1}{p},p}(\Gamma) \mapsto W^{k+1+\frac{1}{p},p}(\Gamma) \cap \operatorname{im} Q_\Gamma$ is surjective and an isomorphism between the mentioned subspaces.

Now we consider the boundary value problem for $\boldsymbol{\varphi}$ depending on $\mathbf{u}$.

$$\begin{cases} \left(D - i\sqrt{\frac{2\kappa}{\gamma}}\right) M_2 \left(D + i\sqrt{\frac{2\kappa}{\gamma}}\right) \boldsymbol{\varphi} = -\kappa \operatorname{Vec} D\mathbf{u}, \text{ in } \Omega \\ \boldsymbol{\varphi} = \mathbf{g}_2, \text{ on } \Gamma. \end{cases} \tag{10.3.5}$$

The technique used to construct the representation formula for the solution of (10.3.5) is similar to the one we have presented for $\mathbf{u}$. The principal difference here is the factorisation of the second order differential operator, acting on $\boldsymbol{\varphi}$, by operators of the type

$$D_\alpha := \alpha + D, \quad D_{-\alpha} := \alpha - D.$$

The study of such operators has been performed in [16], which is adopted to our case by considering $\alpha = i\beta$, $\beta \in \mathbb{R}$. Applying the modified Borel-Pompeiu formula and taking into account that $D_{\pm\alpha} T_{\pm\alpha} = I$ we get the following representation formula for the solution of the non-homogeneous boundary value problem

$$\boldsymbol{\varphi} = F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha}\right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2 - \kappa\, T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D\mathbf{u}.$$

Finally, the validity of the constructed representation formulae can be checked by a direct substitution into (10.3.5). Similar to the case of $\mathbf{u}$ we study now

$$\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha} : \operatorname{im} P_{-\alpha} \cap W^{k+\frac{1}{p},p}(\Gamma) \mapsto W^{k+1+\frac{1}{p},p}(\Gamma) \cap \operatorname{im} Q_\alpha .$$

Taking into account the regularity properties of the single operators, and the fact that the boundary values of $T_\alpha f$ for any f belong to the image of Q_α, we conclude that the operator under consideration is a one-to-one mapping. Applying the same reasoning as for (10.3.4) for the modified operators, we finally obtain that the mapping $\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha} : \operatorname{im} P_{-\alpha} \cap W^{k+\frac{1}{p},p}(\Gamma) \mapsto W^{k+1+\frac{1}{p},p}(\Gamma) \cap \operatorname{im} Q_\alpha$ is surjective and an isomorphism between the mentioned subspaces, which finishes the proof. □

10.3.2 Uniqueness of Solution

For the discussion related to practical use of representation formulae (10.3.3) we provide the following obvious corollary:

Corollary 10.3.3 *Representation formulae (10.3.3) can be written as follows*

$$\begin{cases} \mathbf{u} = \mathbf{A}_1 \boldsymbol{\varphi} + \mathbf{f}_1, \\ \boldsymbol{\varphi} = \mathbf{A}_2 \mathbf{u} + \mathbf{f}_2, \end{cases} \tag{10.3.6}$$

where the operators $\mathbf{A}_1$ *and* $\mathbf{A}_2$ *are defined by*

$$\mathbf{A}_1 := -\kappa\, T M_1^{-1} T \operatorname{Vec} D, \quad \mathbf{A}_2 := -\kappa\, T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D,$$

together with additional terms $\mathbf{f}_1$, $\mathbf{f}_2$

$$\begin{aligned} \mathbf{f}_1 &:= F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma (\operatorname{tr} T M_1^{-1} F_\Gamma)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1, \\ \mathbf{f}_2 &:= F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha} \right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2. \end{aligned}$$

Clearly, system of equations (10.3.6) is a coupled system. Two general strategies can be considered to solve such a coupled system of equations:

(i) For decoupling the system (10.3.6) we can consider the iteration scheme

$$\begin{cases} \mathbf{u}_n = \mathbf{A}_1 \boldsymbol{\varphi}_{n-1} + \mathbf{f}_1, \\ \boldsymbol{\varphi}_n = \mathbf{A}_2 \mathbf{u}_{n-1} + \mathbf{f}_2, \end{cases}$$

for $n = 0, 1, \ldots$ with given initial conditions $\mathbf{u}_0$ and $\boldsymbol{\varphi}_0$.

(ii) Perform a *direct decoupling* of system (10.3.6). The decoupling strategy is particularly beneficial for the purpose of studying the difference between models of classical theory of elasticity and micropolar elasticity. Additionally, the decoupling strategy provides a higher flexibility in practical realisation of the solution procedure, since only one equation has to be solved by implementing the operator representation, while the second can be calculated by standard methods knowing one of two functions $\mathbf{u}$ or $\boldsymbol{\varphi}$. Thus, we discuss the decoupling strategy in the sequel.

Remark 10.3.4 We would like to remark, that regardless of the chosen solution strategy, boundary conditions are automatically satisfied by the use of operator calculus.

Using the representation formula for $\boldsymbol{\varphi}$ from (10.3.3) in the first equation of (10.3.6), and correspondingly, the representation formula for $\mathbf{u}$ from (10.3.3) in the second equation of (10.3.6), we obtain

$$\begin{cases} \mathbf{u} = \mathbf{A}_1 \Big(F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha} \right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2 \\ \qquad -\kappa\, T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D\mathbf{u} \Big) + \mathbf{f}_1, \\ \boldsymbol{\varphi} = \mathbf{A}_2 \Big(F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma (\operatorname{tr} T M_1^{-1} F_\Gamma)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1 \\ \qquad -\kappa\, T M_1^{-1} T \operatorname{Vec} D\boldsymbol{\varphi} \Big) + \mathbf{f}_2. \end{cases}$$

By help of the new notations defined by

$$\mathbf{B}_1 := \mathbf{A}_1 \left(-\kappa\, T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D \right),$$
$$\mathbf{B}_2 := \mathbf{A}_2 \left(-\kappa\, T M_1^{-1} T \operatorname{Vec} D \right),$$

and

$$\mathbf{f}_1^* = \mathbf{f}_1 + \mathbf{A}_1 \left(F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha} \right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2 \right),$$
$$\mathbf{f}_2^* = \mathbf{f}_2 + \mathbf{A}_2 \left(F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma (\operatorname{tr} T M_1^{-1} F_\Gamma)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1 \right),$$

we finally obtain the following decoupled system

$$\begin{cases} \mathbf{u} = \mathbf{B}_1 \mathbf{u} + \mathbf{f}_1^*, \\ \boldsymbol{\varphi} = \mathbf{B}_2 \boldsymbol{\varphi} + \mathbf{f}_2^*. \end{cases} \tag{10.3.7}$$

Decoupled system (10.3.7) can be equivalently written as

$$\begin{cases} (\mathbf{I} - \mathbf{B}_1)\, \mathbf{u} = \mathbf{f}_1^*, \\ (\mathbf{I} - \mathbf{B}_2)\, \boldsymbol{\varphi} = \mathbf{f}_2^*, \end{cases} \tag{10.3.8}$$

where $\mathbf{I}$ is the identity operator.

Solvability of decoupled equations (10.3.8) can be shown via existence of the bounded inverse operators $(\mathbf{I} - \mathbf{B}_1)^{-1}$ and $(\mathbf{I} - \mathbf{B}_2)^{-1}$. These inverse operators exist if $\|B_j\| < 1$ and $B_j : H_j \to H_j$, where H_j are the corresponding Banach spaces for $j = 1, 2$. For the sake of readability, we provide complete definition of operators

$\mathbf{B}_j$, $j = 1, 2$:

$$\begin{aligned}\mathbf{B}_1 &:= \kappa^2 T M_1^{-1} T \operatorname{Vec} D \left(T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D\right),\\ \mathbf{B}_2 &:= \kappa^2 T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D \left(T M_1^{-1} T \operatorname{Vec} D\right).\end{aligned} \tag{10.3.9}$$

Taking into account the mapping properties of operators used in (10.3.9) we obtain the following corollary:

Corollary 10.3.5 *The operators B_1 and B_2 defined in (10.3.9) are continuous mapping from $W^{k,p}(\Omega)$ to $W^{k,p}(\Omega)$.*

For the discussion regarding the norms of operators $\mathbf{B}_j$ we refer again to works [10, 11, 14], where the estimates for norms together with explicit formulae for constants have been presented. However, for the purpose of studying only the solvability of (10.3.8), we notice additionally that the norms of operators (10.3.9) are, in fact, controlled by the parameter κ, which is a material constant coming from the micropolar model, we write the norm estimates as follows

$$\mathbf{B_1} \leq \kappa^2 C_1(T, TD, T_{-\alpha} D_\alpha), \quad \mathbf{B_2} \leq \kappa^2 C_2(T, TD, T_{-\alpha} D_\alpha),$$

where C_1 and C_2 are constants depending on sharp estimates of operators used in (10.3.9). Thus, assuming that κ is sufficiently small, we get that $\|B_j\| < 1$.

Finally, by using the inverse mapping theorem, we can formulate the following proposition:

Proposition 10.3.6 *For given boundary conditions $\mathbf{g}_{1,2} \in W^{k+\frac{3}{2},2}(\Gamma)$ and sufficiently small κ, the solution $\mathbf{u}, \boldsymbol{\varphi} \in W^{k,2}(\Omega)$ of decoupled problem (10.3.8) is unique, the problem is well-posed, and the solution can be estimated as follows*

$$\|\mathbf{u}\| \leq \|\,(\mathbf{I} - \mathbf{B}_1)^{-1}\,\| \|\mathbf{f}_1^*\|, \quad \|\boldsymbol{\varphi}\| \leq \|\,(\mathbf{I} - \mathbf{B}_2)^{-1}\,\| \|\mathbf{f}_2^*\|,$$

with $\mathbf{f}_1^*$ and $\mathbf{f}_2^*$ explicitly given by

$$\begin{aligned}\mathbf{f}_1^* &= F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma (\operatorname{tr} T M_1^{-1} F_\Gamma)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1\\ &\quad -\kappa\, T M_1^{-1} T \operatorname{Vec} D \left(F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha}\right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2\right),\\ \mathbf{f}_2^* &= F_\alpha \tilde{\mathbf{g}}_2 + T_\alpha M_2^{-1} F_{-\alpha} \left(\operatorname{tr} T_\alpha M_2^{-1} F_{-\alpha}\right)^{-1} Q_\alpha \tilde{\mathbf{g}}_2\\ &\quad -\kappa\, T_\alpha M_2^{-1} T_{-\alpha} \operatorname{Vec} D \left(F_\Gamma \tilde{\mathbf{g}}_1 + T M_1^{-1} F_\Gamma (\operatorname{tr} T M_1^{-1} F_\Gamma)^{-1} Q_\Gamma \tilde{\mathbf{g}}_1\right).\end{aligned}$$

Remark 10.3.7 The assumption of small values of κ naturally limits the class of practical problems covered by the proposed approach. However, it is indeed important to discuss it, since the case of $\kappa = 0$ in (10.2.1) corresponds to the classical Lamé equation. Thus, the case of small κ can be classified as the class of "boundary" models, where it is not a-priori clear if the classical elasticity or the micropolar model should be used. Therefore, our interest is to provide an estimate between the two models for the case of small κ.

10.3.3 Difference Between the Models

Finally, we study now the difference between the classical theory of elasticity and the micropolar theory. Our goal is to provide the estimate for

$$\|\mathbf{u}_e - \mathbf{u}_m\|,$$

in $W^{2,1}(\Omega)$ with $\mathbf{u}_e$ denoting the elasticity solution, and $\mathbf{u}_m$ denoting the micropolar solution. Using representation formula for the displacement $\mathbf{u}$ from (10.3.3) and taking into account that elasticity solution $\mathbf{u}_e$ can be obtained from that formula by setting κ to zero, we get the following estimate:

$$\|\mathbf{u}_e - \mathbf{u}_m\| \leq \|\kappa\, T M_1^{-1} T \,\mathrm{Vec}\, D\boldsymbol{\varphi}\|.$$

Using now in the last inequality solution for $\boldsymbol{\varphi}$ provided by (10.3.8), we get

$$\|\mathbf{u}_e - \mathbf{u}_m\| \leq \|\kappa\, T M_1^{-1} T \,\mathrm{Vec}\, D\left[(\mathbf{I} - \mathbf{B}_2)^{-1}\,\mathbf{f}_2^{*}\right]\|. \tag{10.3.10}$$

This estimate depends on Dirichlet boundary data given in terms of displacement and micropolar rotations. In practice it means, that a boundary value problem of the classical linear elasticity can be formulated at first, then micropolar rotations can be measured at the boundary of a domain; after that estimate (10.3.10) can be used in order to decide if a coupled model has to be considered, or if it is sufficient to work in the framework of the classical theory.

10.4 Summary and Outlook

Micropolar elasticity is a refined version of the classical elasticity. Equations of micropolar elasticity are not given only by a single differential equation w.r.t. a vector field of displacement, but by a coupled system of differential equations connecting fields of displacements and rotations. However, construction of solution methods for boundary value problems of micropolar elasticity is still an open mathematical task, mostly due to the coupled nature of the resulting system of partial differential equations. Particularly, spatial problems of micropolar elasticity were not addressed in full generality. Therefore, in this paper we have proposed a solution strategy of spatial boundary value problems of micropolar elasticity by means of quaternionic operator calculus.

By using the tools of quaternionic operator calculus, the solvability of boundary value problems of micropolar elasticity with Dirichlet boundary data has been proved. Moreover, the solvability has been proved in a strong sense, while results available in literature are related to variational approaches. Additionally, an explicit estimate for the difference between the classical theory of linear elasticity and the micropolar model has been obtained. The estimate is constructed for a specific class of models with small parameter κ, where it is not a-priori clear if the classical elasticity or the micropolar model should be used. Thus, the presented estimate supports practical use the micropolar theory.

The scope of future research is related to practical realisation of the proposed operator calculus approach. Particularly, since the discrete operator calculus and the discrete function theory are gaining a growing attention in recent years, it is attractive to transfer the presented results to the discrete level.

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Chapter 11
Constructive Orthonormalisation of Monogenic Polynomials on a Finite Cylinder

Sebastian Bock and Dmitrii Legatiuk

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract Orthogonal series expansions are widely used in different fields of mathematics, as well as in numerous practical applications. Especially from a computational point of view, orthogonal functions systems provide numerical efficiency and stability. Therefore, the construction of such systems is important not only for theoretical studies, but rather for practical applications, particularly in the context of hypercomplex analysis, where extensive computations related to functions systems appear frequently. In recent years, systems of orthogonal monogenic polynomials for some canonical domains, such as spheres, balls and infinite cylinders, have been developed by several authors. However, from the practical point of view, a finite cylinder is one of the very important domains, particularly for applications in spatial elasticity theory, which still lacks an orthogonal system. Thus, the objective of this paper is to provide an orthogonal system of monogenic polynomials for a finite cylinder. To this end, we present an adaptive orthonormalisation scheme with explicit formulae for the calculation of inner products allowing a simple and efficient construction of an orthonormal Appell system for a finite cylinder. Finally, we present some numerical results for the comparison with a non-orthonormalised polynomial basis, which underline the practical relevance of the proposed orthonormal system.

Keywords Monogenic functions · Appell polynomials · Finite cylinder · Constructive orthonormalisation

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11.1 Introduction

Hypercomplex analysis is a natural extension of complex function theory to higher dimensions. In this connection, monogenic functions generalise the concept of holomorphic functions in $\mathbb{C}$ and are functions in the kernel of a generalised Cauchy-Riemann operator. In recent years, monogenic functions appeared more and more frequently in different areas of applications. Similar to the complex case, where holomorphic functions can be defined through orthogonal series expansions in $\mathbb{C}$ in terms of integer powers of the complex variable z, monogenic functions in the hypercomplex setting can be represented by complete orthogonal systems of monogenic polynomials (see [1, 2]). These systems of monogenic polynomials generalise several properties of the complex monomials to higher dimensions, such as, for instance, the Appell property [3] and the orthogonality w.r.t. the unit ball, which are essential to define higher dimensional orthogonal Taylor- and Fourier series expansions [4]. However, with an eye on practical applications, it is valuable to construct orthogonal systems also for other three-dimensional domains. Particularly, a finite cylinder is one of very important domains for applications in the field of spatial elasticity theory. So far, no orthonormal system of monogenic polynomials has been constructed for a finite cylinder. Therefore, the objective of this paper is to provide a constructive orthonormalisation scheme for monogenic polynomials over a finite cylinder. More precisely, we consider a cylindrical domain defined in the classical way by

$$\mathcal{C} := \left\{(x_0, r, \varphi) \,\middle|\, x_0 \in [-\xi, \xi], r \in [0, \rho], \varphi \in [0, 2\pi)\right\}, \ \xi, \rho \in \mathbb{R}^+.$$

In [5] the authors claim to construct an orthogonal Appell system of monogenic polynomials in a cylinder. However, only the well-known classical Appell polynomials were studied here and the orthogonality w.r.t. the domain of a finite cylinder was proven only for the polynomial subspaces for a fixed degree of homogeneity and not for the whole space. Thus, this is not an orthogonal system in the classical sense. Another orthogonal system has been constructed in [6] for a domain

$$\mathcal{C}_a := \left\{|z| \le a, \ \rho > 0, \ 0 \le \varphi < 2\pi \ : \ a \in \mathbb{R}^+\right\},$$

which the authors refer to as an infinite cylinder. Thus, the construction of a complete orthonormal system for a finite cylinder is still an open problem.

To construct an orthonormal system of monogenic polynomials over a finite cylinder, we propose an adaptive orthonormalisation scheme which takes advantage of the used Appell basis leading to explicit formulae for the calculation of inner products. Based on this result, we show that the resulting Gram matrix is real and sparse. Additionally, a rearrangement of the basis elements leads to a diagonal block structure of the Gram matrix. Consequently, the proposed adaptive orthonormalisation scheme can be easily implemented using methods from real linear algebra and do not suffer from numerical instability.

The article is structured as follows. Section 11.2 introduces some necessary fundamentals and notations of the algebra $\mathbb{H}$ of real quaternions. Section 11.3 recalls some known results about orthogonal Appell polynomials, which are needed in the following sections. In Sect. 11.4 we first give a closed-form representation for the monogenic Appell polynomials in cylindrical coordinates. Using this representation, we get explicit formulas for the inner product of two arbitrary functions w.r.t. the domain of a finite cylinder. These results are then used to provide an adaptive orthonormalisation scheme for the system of monogenic Appell polynomials on the finite cylinder. Finally, we demonstrate the practical need for orthonormalisation by evaluating the condition number of the Gram matrix of the non-orthonormalised Appell polynomials.

11.2 Preliminaries and Notations

Let $\mathbb{H}$ be the algebra of real quaternions with the standard basis $\{\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ subject to the multiplication rules

$$\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}\,\mathbf{e}_0,\ \ i, j = 1, 2, 3,$$
$$\mathbf{e}_1\mathbf{e}_2 = \mathbf{e}_3,\ \ \mathbf{e}_0\mathbf{e}_i = \mathbf{e}_i\mathbf{e}_0 = \mathbf{e}_i,\ \ i = 0, 1, 2, 3.$$

The real vector space $\mathbb{R}^4$ will be embedded in $\mathbb{H}$ by identifying the element $\boldsymbol{a} = [a_0, a_1, a_2, a_3]^{\mathrm{T}} \in \mathbb{R}^4$ with the quaternion $\boldsymbol{a} = a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3$, $a_i \in \mathbb{R}$, $i = 0, 1, 2, 3$, and $\mathbf{e}_0 = [1, 0, 0, 0]^{\mathrm{T}}$ is the multiplicative unit element of the algebra $\mathbb{H}$. Further, we denote by

(i) $\mathbf{Sc}(\boldsymbol{a}) := a_0$ the scalar part, $\mathbf{Vec}(\boldsymbol{a}) = \underline{\boldsymbol{a}} := \sum_{i=1}^{3} a_i\mathbf{e}_i$ the vector part of $\boldsymbol{a}$,
(ii) $\overline{\boldsymbol{a}} := a_0 - \underline{\boldsymbol{a}}$ the conjugate of $\boldsymbol{a}$,
(iii) $\widehat{\boldsymbol{a}} := -\mathbf{e}_3\boldsymbol{a}\mathbf{e}_3$ the $\mathbf{e}_3$-involution of $\boldsymbol{a}$,
(iv) $|\boldsymbol{a}| := \sqrt{\boldsymbol{a}\,\overline{\boldsymbol{a}}}$ the norm of $\boldsymbol{a}$,
(v) $\boldsymbol{a}^{-1} := \frac{\overline{\boldsymbol{a}}}{|\boldsymbol{a}|^2}$, $\boldsymbol{a} \neq 0$ the inverse of $\boldsymbol{a}$.

Throughout the article we will often represent elements from $\mathbb{H}$ (coefficients, functions, differential operators etc.) in the component form

$$\boldsymbol{a} = (a_0 + a_3\mathbf{e}_3) + \mathbf{e}_1(a_1 - a_2\mathbf{e}_3) =: \boldsymbol{a}^{03} + \mathbf{e}_1\boldsymbol{a}^{12}.$$

The multiplication of the components $\boldsymbol{a}^{03}, \boldsymbol{a}^{12} \in \operatorname{span}\{\mathbf{e}_0, \mathbf{e}_3\}$ commutes with $\mathbf{e}_0, \mathbf{e}_3$ and anti-commutes with $\mathbf{e}_1, \mathbf{e}_2$. Accordingly, for $(i, j) = \{(0, 3), (1, 2)\}$ we have

$$\mathbf{e}_k\,\boldsymbol{a}^{ij} = \begin{cases} \boldsymbol{a}^{ij}\mathbf{e}_k : k = 0, 3, \\ \overline{\boldsymbol{a}^{ij}}\mathbf{e}_k : k = 1, 2\,. \end{cases}$$

Moreover, the components of $\boldsymbol{a}$ can be calculated by help of the relations

$$\boldsymbol{a}^{03} = \frac{1}{2}\left(\boldsymbol{a} + \widehat{\boldsymbol{a}}\right) \quad \text{and} \quad \boldsymbol{a}^{12} = -\frac{\mathbf{e}_1}{2}\left(\boldsymbol{a} - \widehat{\boldsymbol{a}}\right).$$

Now, let us consider the subset $\mathcal{A} := \operatorname{span}_{\mathbb{R}}\{1, \mathbf{e}_1, \mathbf{e}_2\}$. The real vector space $\mathbb{R}^3$ will be embedded in $\mathcal{A}$ by the identification of $\boldsymbol{x} = [x_0, x_1, x_2]^T \in \mathbb{R}^3$ with the *reduced quaternion*

$$\boldsymbol{x} = x_0 + \mathbf{e}_1\boldsymbol{\zeta} \in \mathcal{A} \quad \text{with} \quad \boldsymbol{\zeta} := x_1 - \mathbf{e}_3 x_2.$$

As a consequence, the symbol $\boldsymbol{x}$ is often used to represent a point in $\mathbb{R}^3$ as well as to represent the corresponding reduced quaternion. Note that $\mathcal{A}$ is only a real vector space but not a sub-algebra of $\mathbb{H}$.

Let now Ω be an open subset of $\mathbb{R}^3$ with a piecewise smooth boundary. An $\mathbb{H}$-valued function is a mapping

$$\boldsymbol{f} : \Omega \longrightarrow \mathbb{H} \quad \text{such that} \quad \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}^{03}(\boldsymbol{x}) + \mathbf{e}_1\boldsymbol{f}^{12}(\boldsymbol{x}) := \sum_{i=0}^{3} f^i(\boldsymbol{x})\,\mathbf{e}_i.$$

The coordinates $f^i(\boldsymbol{x})$, $\boldsymbol{x} \in \Omega$ are real-valued functions defined in Ω, i.e., $f^i(\boldsymbol{x}) : \Omega \longrightarrow \mathbb{R}, i = 0, 1, 2, 3$. Continuity, differentiability or integrability of $\boldsymbol{f}$ are defined coordinate-wise. Due to the non-commutativity of the algebra all functions will be considered in the right $\mathbb{H}$-linear Hilbert space of square-integrable $\mathbb{H}$-valued functions denoted by $L^2(\Omega; \mathbb{H})$ and equipped with the $\mathbb{H}$-valued inner product

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle_{L^2(\Omega;\mathbb{H})} := \int_\Omega \overline{\boldsymbol{f}}\,\boldsymbol{g}\,dV.$$

Here dV denotes the Lebesgue measure in $\mathbb{R}^3$. Let us now denote by ∂_j the partial derivatives w.r.t. x_j, $j = 0, 1, 2$, the operator

$$\overline{\boldsymbol{\partial}} := \partial_0 + 2\mathbf{e}_1\boldsymbol{\partial}_{\overline{\zeta}} \quad \text{with} \quad 2\boldsymbol{\partial}_{\overline{\zeta}} := \partial_1 - \mathbf{e}_3\partial_2$$

is called *generalised Cauchy-Riemann operator*. The corresponding *adjoint generalised Cauchy-Riemann operator* is defined by

$$\boldsymbol{\partial} := \partial_0 - 2\mathbf{e}_1\boldsymbol{\partial}_{\overline{\zeta}}.$$

Throughout this article the introduced differential operators are considered as operators acting from the left and analogously denoted as in complex analysis (see, e.g. [7]) contrary to the originally introduced notation in Clifford analysis (see, e.g. [8]). This leads to the following definition:

Definition 11.2.1 A function $\boldsymbol{f} \in C^1(\Omega; \mathbb{H})$ is called *monogenic* in $\Omega \subset \mathbb{R}^3$ if $\overline{\boldsymbol{\partial}}\boldsymbol{f} = 0$ in Ω. Conversely, a function $\boldsymbol{g} \in C^1(\Omega; \mathbb{H})$ is called *anti-monogenic* in $\Omega \subset \mathbb{R}^3$ if $\boldsymbol{\partial}\boldsymbol{g} = 0$ in Ω.

A relationship between monogenic and anti-monogenic functions is given by the $\mathbf{e}_3$-involution as follows:

Corollary 11.2.2 ([9]) *Let $\boldsymbol{f} = f^{03} + \mathbf{e}_1 f^{12} \in C^1(\Omega; \mathbb{H})$ be a monogenic function in $\Omega \subset \mathbb{R}^3$. The $\mathbf{e}_3$-involution of $\boldsymbol{f}$, given by $\widehat{\boldsymbol{f}} = f^{03} - \mathbf{e}_1 f^{12}$, is an anti-monogenic function in Ω.*

Let us remark that in the complex one-dimensional case the conjugation of a holomorphic function $f \in C^1(\Omega; \mathbb{C})$ gives directly the corresponding anti-holomorphic function $\widehat{f}$ and thus $\overline{f} \equiv \widehat{f}$. For $\mathbb{H}$-valued monogenic functions this property does not hold in general as Corollary 11.2.2 shows. The subset of $\mathcal{A}$-valued monogenic functions is an exception to this.

Finally, we need the concept of the hypercomplex derivative (see, e.g. the first works [10, 11] or for a survey [12]). The main result of [11] is summarised in the following definition:

Definition 11.2.3 (Hypercomplex Derivative) Let $\boldsymbol{f} \in C^1(\Omega; \mathbb{H})$ be monogenic in Ω. The expression $\boldsymbol{\partial}_{\boldsymbol{x}} \boldsymbol{f} := \frac{1}{2}\boldsymbol{\partial}\boldsymbol{f}$ is called the *hypercomplex derivative* of $\boldsymbol{f}$ in Ω.

As a consequence of Definition 11.2.3, we introduce a special subset of monogenic functions characterised by a vanishing first derivative.

Definition 11.2.4 (Monogenic Constant) A C^1-function belonging to $\ker \boldsymbol{\partial}_{\boldsymbol{x}} \cap \ker \overline{\boldsymbol{\partial}}$ is called a *monogenic constant*.

11.3 Orthogonal Appell Basis on the Unit Ball

In this section we give a brief overview of the extensively studied orthogonal Appell basis of monogenic polynomials with respect to the unit ball $\mathbb{B}$ in $\mathbb{R}^3$. This polynomial basis is a natural generalisation of the holomorphic z-monomials to $\mathbb{R}^3$ having special properties with regard to the hypercomplex derivation and primitivation. In the following, only some basic properties of the Appell polynomials are recalled. For a detailed description see [1, 2] and [9].

The monogenic Appell polynomials in $\mathbb{R}^3$ can be defined by a three-term recurrence relation given in the following theorem.

Theorem 11.3.1 ([2]) *The system $\left\{\boldsymbol{A}_n^l : l = 0, \ldots, n\right\}_{n \in \mathbb{N}_0}$ is an orthogonal Appell basis in $L^2(\mathbb{B}; \mathbb{H}) \cap \ker \overline{\boldsymbol{\partial}}$ whose elements satisfy the three-term recurrence* *relation*

$$\begin{aligned} &2(n-l+1)(n+l+2)\,\boldsymbol{A}_{n+1}^{l}(\boldsymbol{x}) \\ &\quad = (n+1)\left[\Big((2n+3)\boldsymbol{x}+(2n+1)\overline{\boldsymbol{x}}\Big)\boldsymbol{A}_{n}^{l}(\boldsymbol{x}) - 2n\,\boldsymbol{x}\overline{\boldsymbol{x}}\,\boldsymbol{A}_{n-1}^{l}(\boldsymbol{x})\right] \end{aligned} \tag{11.3.1}$$

with

$$\boldsymbol{A}_{l+1}^{l}(\boldsymbol{x}) \;=\; \frac{1}{4}\big[(2l+3)\boldsymbol{x}+(2l+1)\overline{\boldsymbol{x}}\big]\,\boldsymbol{A}_{l}^{l}(\boldsymbol{x}) \quad \text{and} \quad \boldsymbol{A}_{l}^{l}(\boldsymbol{x}) \;=\; \boldsymbol{\zeta}^{l}.$$

Furthermore, for each $n \in \mathbb{N}$ *it holds that*

$$\boldsymbol{\partial}_{\boldsymbol{x}}\boldsymbol{A}_{n}^{l}(\boldsymbol{x}) = \begin{cases} n\,\boldsymbol{A}_{n-1}^{l}(\boldsymbol{x}) & : n \neq l, \\ 0 & : n = l \end{cases} \quad \text{and} \quad \boldsymbol{\partial}_{\boldsymbol{\zeta}}\boldsymbol{A}_{n}^{n}(\boldsymbol{x}) = n\,\boldsymbol{A}_{n-1}^{n-1}(\boldsymbol{x}).$$

Figure 11.1 presents the structural scheme of the orthogonal Appell basis given in Theorem 11.3.1. Firstly, for a fixed degree $n = l + p$, $n \in \mathbb{N}_0$ we obtain $n+1$ Appell polynomials of order $l = 0, \ldots, n$ leading to a triangular scheme of basis functions. Secondly, the three-term recurrence relation (11.3.1) relates Appell polynomials with the same polynomial order l, where the initial functions are defined by the set $\left\{\boldsymbol{A}_{n}^{n}(\boldsymbol{x})\right\}_{n\in\mathbb{N}_0}$ of monogenic constants. The set $\left\{\boldsymbol{A}_{n}^{n}(\boldsymbol{x})\right\}_{n\in\mathbb{N}_0}$ is a basis in the orthogonal subspace of monogenic constants in $L^2(\mathbb{B};\mathbb{H})$ [1]. Thirdly, it has been proved in [2] that the basis polynomials have the Appell property, i.e. that the application of the hypercomplex derivative $\boldsymbol{\partial}_{\boldsymbol{x}}$ (see Definition 11.2.3) to an arbitrary Appell polynomial $\boldsymbol{A}_{n}^{l}(\boldsymbol{x})$ yields $n\boldsymbol{A}_{n-1}^{l}(\boldsymbol{x})$ if $\boldsymbol{A}_{n}^{l}(\boldsymbol{x}) \in \ker\overline{\boldsymbol{\partial}} \setminus (\ker\boldsymbol{\partial}_{\boldsymbol{x}} \cap \ker\overline{\boldsymbol{\partial}})$ or $\boldsymbol{0}$ if $\boldsymbol{A}_{n}^{l}(\boldsymbol{x}) \in \ker\boldsymbol{\partial}_{\boldsymbol{x}} \cap \ker\overline{\boldsymbol{\partial}}$ is a monogenic constant. Note that the application of $\boldsymbol{\partial}_{\boldsymbol{x}}$ only changes the degree n and not the order l of the Appell polynomial and thus acts along the columns of the triangular scheme shown in Fig. 11.1.

	0	1	2	3	4	$l = 0, \ldots, n$
$n = l + p$	$\boldsymbol{A}_n^0$	$\boldsymbol{A}_n^1$	$\boldsymbol{A}_n^2$	$\boldsymbol{A}_n^3$	$\boldsymbol{A}_n^4$	$\ldots\ \boldsymbol{A}_n^n$
4	$\boldsymbol{A}_4^0$	$\boldsymbol{A}_4^1$	$\boldsymbol{A}_4^2$	$\boldsymbol{A}_4^3$	$\boldsymbol{A}_4^4$	
3	$\boldsymbol{A}_3^0$	$\boldsymbol{A}_3^1$	$\boldsymbol{A}_3^2$	$\boldsymbol{A}_3^3$		
2	$\boldsymbol{A}_2^0$	$\boldsymbol{A}_2^1$	$\boldsymbol{A}_2^2$			
1	$\boldsymbol{A}_1^0$	$\boldsymbol{A}_1^1$				
0	$\boldsymbol{A}_0^0$					

Fig. 11.1 Structural scheme of the orthogonal Appell basis

Furthermore, in [9] it has been proved that in spherical coordinates

$$x_0 = r\cos\theta \quad \text{and} \quad \boldsymbol{\zeta} = r\sin\theta(\cos\psi - \mathbf{e}_3\sin\psi),$$

where $r > 0$, $\theta \in (0,\pi]$, $\psi \in (0, 2\pi]$, the orthogonal Appell polynomials defined in Theorem 11.3.1 have the closed form representation

$$\boldsymbol{A}_n^l(r,\boldsymbol{\omega}) = 2\,\boldsymbol{\Upsilon}^{n,l}(r,\boldsymbol{\omega}) + \mathbf{e}_1\,\boldsymbol{\Upsilon}^{n,l+1}(r,\boldsymbol{\omega}), \tag{11.3.2}$$

where

$$\boldsymbol{\Upsilon}^{n,l}(r,\boldsymbol{\omega}) = \frac{r^n\,2^{l-1}\,n!}{(n+l)!}\left[P_n^l(\cos\theta)\,\boldsymbol{e}^{-\mathbf{e}_3 l\psi}\right] \text{ and } \boldsymbol{e}^{-\boldsymbol{e}_3 l\psi} = \cos(l\psi) - \boldsymbol{e}_3\sin(l\psi).$$

Here, $\boldsymbol{\omega} = \cos\theta + \mathbf{e}_1\sin\theta(\cos\psi - \mathbf{e}_3\sin\psi)$ and $P_n^l(t)$ denote the *associated Legendre function of the first kind* (see, e.g., [13]), given by

$$P_n^l(t) = (1-t^2)^{\frac{l}{2}}\sum_{k=0}^{\lfloor\frac{n-l}{2}\rfloor}\frac{(-1)^k\,(2n-2k)!}{2^n\,k!\,(n-k)!\,(n-2k-l)!}\,t^{n-l-2k}.$$

Note that, up to a normalisation factor, the components of the Appell polynomials $A_n^l(r,\boldsymbol{\omega})$ are the classical spherical harmonics (see, e.g., [14]). Spherical harmonics are a well-known and extensively studied orthogonal function system w.r.t. the unit ball and play an important role in many theoretical and practical applications. Representation formula (11.3.2) gives therefore an interesting insight to the structure of hypercomplex polynomials and related series expansions.

For the purpose of this article, we provide representation (11.3.2) also in Cartesian coordinates. Using the relations

$$r = |\boldsymbol{x}|,\ \cos\theta = \frac{x_0}{|\boldsymbol{x}|} \quad \text{and} \quad \boldsymbol{e}^{-\boldsymbol{e}_3\psi} = \frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|},$$

a closed form representation of the Appell polynomials in Cartesian coordinates is given by

$$\boldsymbol{A}_n^l(\boldsymbol{x}) = \left[2\,\boldsymbol{\Upsilon}^{n,l}(\boldsymbol{x}) + \mathbf{e}_1\,\boldsymbol{\Upsilon}^{n,l+1}(\boldsymbol{x})\right], \tag{11.3.3}$$

where

$$\boldsymbol{\Upsilon}^{n,l}(\boldsymbol{x}) = \frac{2^{l-1}\,n!}{(n+l)!}\,|\boldsymbol{x}|^n\,P_n^l(t)\left(\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}\right)^l \quad \text{and} \quad t = \frac{x_0}{|\boldsymbol{x}|}.$$

11.4 Appell Polynomials on the Cylinder and a Constructive Orthonormalisation Procedure

In this section we propose a system of monogenic Appell polynomials for a finite cylinder, and additionally, we develop a methodology for the constructive orthonormalisation of these polynomials on the finite cylinder. Finally, we demonstrate the practical need for orthonormalisation by evaluating the conditioning of Gram's matrix of the non-orthonormalised Appell polynomials.

11.4.1 Appell Polynomials in Cylindrical Coordinates

Let us introduce cylindrical coordinates as follows

$$x_0 = x_0, \quad x_1 = r\cos\varphi, \quad x_2 = r\sin\varphi \quad \text{with} \quad r > 0,\ \varphi \in [0, 2\pi), \qquad (11.4.1)$$

and define, as pointed out in the introduction, the domain of the finite cylinder

$$\mathcal{C} := \left\{ (x_0, r, \varphi) \,\middle|\, x_0 \in [-\xi, \xi], r \in [0, \rho], \varphi \in [0, 2\pi) \right\}, \ \xi, \rho \in \mathbb{R}^+.$$

Hence, the reduced quaternion variable transforms to

$$\boldsymbol{x} = x_0 + \mathbf{e}_1 \boldsymbol{\zeta} = x_0 + \mathbf{e}_1 r \boldsymbol{e}^{-\mathbf{e}_3 \varphi} \quad \text{with} \quad \boldsymbol{e}^{-\mathbf{e}_3 \varphi} = \cos\varphi - \mathbf{e}_3 \sin\varphi.$$

Applying the coordinate transform (11.4.1) to (11.3.3) and using the identities

$$|\boldsymbol{x}| = \sqrt{x_0^2 + r^2}, \quad \left(\frac{\boldsymbol{\zeta}}{|\boldsymbol{\zeta}|}\right)^l = \boldsymbol{e}^{-\mathbf{e}_3 l \varphi} \quad \text{and} \quad \sqrt{1 - \left(\frac{x_0}{|\boldsymbol{x}|}\right)^2} = \frac{r}{\sqrt{x_0^2 + r^2}},$$

we obtain by a straightforward calculation a system $\left\{ \boldsymbol{A}_{l+p}^l(x_0, r, \varphi) : l, p \in \mathbb{N}_0 \right\}$ of monogenic Appell polynomials in cylindrical coordinates, given by

$$\boldsymbol{A}_{l+p}^l(x_0, r, \varphi) = \left[\sum_{h=0}^{\lfloor \frac{p}{2} \rfloor} r^{2h+l} x_0^{p-2h-1} \left(c_{l,p,h}\, x_0 + d_{l,p,h}\, \mathbf{e}_1\, r \boldsymbol{e}^{-\mathbf{e}_3 \varphi} \right) \right] \boldsymbol{e}^{-\mathbf{e}_3 l \varphi}, \qquad (11.4.2)$$

where

$$c_{l,p,h} = \frac{(-1)^h (l+p)!}{2^{2h} h! (l+h)! (p-2h)!} \quad \text{and} \quad d_{l,p,h} = \frac{(-1)^h (l+p)!}{2^{2h+1} h! (l+h+1)! (p-2h-1)!}.$$

The closed-form representation (11.4.2) for the monogenic Appell polynomials in cylindrical coordinates is used in the following section for the explicit calculation of the inner product of two arbitrary Appell polynomials w.r.t. the domain of a finite cylinder.

11.4.2 Explicit Calculation of Inner Products

Let us consider now the inner product in $L^2(\mathcal{C};\mathbb{H})$ of two arbitrary cylindrical Appell polynomials (11.4.2), i.e.

$$\left\langle \boldsymbol{A}_{l_1+p_1}^{l_1}, \boldsymbol{A}_{l_2+p_2}^{l_2} \right\rangle = \int_{-\xi}^{\xi} \int_0^{2\pi} \int_0^{\rho} \overline{\boldsymbol{A}_{l_1+p_1}^{l_1}}\, \boldsymbol{A}_{l_2+p_2}^{l_2}\, r\, dr\, d\varphi\, dx_0.$$

First, we evaluate the integrand using the component form of the cylindrical monogenic polynomials (11.4.2). For this we denote the components of the polynomial by $\boldsymbol{A}_{l_j+p_j}^{l_j} =: \boldsymbol{A}_j^{03} + \mathbf{e}_1 \boldsymbol{A}_j^{12}$, $j = 1, 2$ and compute

$$\begin{aligned}\overline{\boldsymbol{A}_{l_1+p_1}^{l_1}}\, \boldsymbol{A}_{l_2+p_2}^{l_2}\, r &= \left(\overline{\boldsymbol{A}_1^{03}} - \mathbf{e}_1 \boldsymbol{A}_1^{12}\right)\left(\boldsymbol{A}_2^{03} + \mathbf{e}_1 \boldsymbol{A}_2^{12}\right) r \\ &= \left[\left(\overline{\boldsymbol{A}_1^{03}}\boldsymbol{A}_2^{03} + \overline{\boldsymbol{A}_1^{12}}\boldsymbol{A}_2^{12}\right) + \mathbf{e}_1\left(\boldsymbol{A}_1^{03}\boldsymbol{A}_2^{12} - \boldsymbol{A}_1^{12}\boldsymbol{A}_2^{03}\right)\right] r.\end{aligned}$$

To shorten the notations we further denote the real coefficients by $c_j := c_{l_j,p_j,h_j}$ and $d_j := d_{l_j,p_j,h_j}$. Thus, we obtain for the (03)-component of the integrand

$$\begin{aligned}\mathcal{I}^{03} &= \left(\overline{\boldsymbol{A}_1^{03}}\boldsymbol{A}_2^{03} + \overline{\boldsymbol{A}_1^{12}}\boldsymbol{A}_2^{12}\right) r \\ &= \boldsymbol{e}^{\mathbf{e}_3(l_1-l_2)\varphi} \sum_{h_1=0}^{\lfloor \frac{p_1}{2} \rfloor} \sum_{h_2=0}^{\lfloor \frac{p_2}{2} \rfloor} r^{2h_1+2h_2+l_1+l_2+1}\, x_0^{p_1+p_2-2h_1-2h_2-2} \left(c_1 c_2\, x_0^2 + d_1 d_2\, r^2\right)\end{aligned}$$

and accordingly for the (12)-component of the integrand

$$\begin{aligned}\mathcal{I}^{12} &= \left(\boldsymbol{A}_1^{03}\boldsymbol{A}_2^{12} - \boldsymbol{A}_1^{12}\boldsymbol{A}_2^{03}\right) r \\ &= \boldsymbol{e}^{-\mathbf{e}_3(l_1+l_2+1)\varphi} \sum_{h_1=0}^{\lfloor \frac{p_1}{2} \rfloor} \sum_{h_2=0}^{\lfloor \frac{p_2}{2} \rfloor} r^{2h_1+2h_2+l_1+l_2+1}\, x_0^{p_1+p_2-2h_1-2h_2-1} \left(c_1 d_2 - d_1 c_2\right).\end{aligned}$$

Now consider the integration of the components $\mathcal{I}^{03}$ and $\mathcal{I}^{12}$ w.r.t. φ. For each $k \in \mathbb{N}_0$ holds

$$\int_0^{2\pi} \boldsymbol{e}^{-\mathbf{e}_3 k\varphi} \mathrm{d}\varphi = \begin{cases} 2\pi & : k = 0, \\ 0 & : k \neq 0, \end{cases}$$

and thus leading to the cases

(i) $l_1 \neq l_2$

$$\int_0^{2\pi} \overline{\boldsymbol{A}_{l_1+p_1}^{l_1}}\, \boldsymbol{A}_{l_2+p_2}^{l_2}\, r\, d\varphi = 0\,,$$

(ii) $l_1 = l_2 = l$

$$\int_0^{2\pi} \overline{\boldsymbol{A}_{l+p_1}^{l}}\, \boldsymbol{A}_{l+p_2}^{l}\, r\, d\varphi$$
$$= 2\pi \sum_{h_1=0}^{\lfloor \frac{p_1}{2} \rfloor} \sum_{h_2=0}^{\lfloor \frac{p_2}{2} \rfloor} r^{2h_1+2h_2+2l+1}\, x_0^{p_1+p_2-2h_1-2h_2-2} \left(c_1 c_2\, x_0^2 + d_1 d_2\, r^2 \right).$$

In the next step we consider the case (ii) and integrate w.r.t. the radius r that gives

$$\int_0^{\rho} \int_0^{2\pi} \overline{\boldsymbol{A}_{l+p_1}^{l}}\, \boldsymbol{A}_{l+p_2}^{l}\, r\, d\varphi dr$$
$$= \pi \sum_{h_1=0}^{\lfloor \frac{p_1}{2} \rfloor} \sum_{h_2=0}^{\lfloor \frac{p_2}{2} \rfloor} \rho^{2(h_1+h_2+l+1)}\, x_0^{p_1+p_2-2h_1-2h_2-2} \left(\frac{c_1 c_2\, x_0^2}{h_1+h_2+l+1} + \frac{d_1 d_2\, \rho^2}{h_1+h_2+l+2} \right).$$

After all, we compute for the last result the integral w.r.t. x_0 by applying

$$\int_{-\xi}^{\xi} x^a dx = \begin{cases} \dfrac{2\tau^{a+1}}{a+1} & : a \in \mathbb{N}_0, \text{even}, \\ 0 & : a \in \mathbb{N}, \text{odd}, \end{cases}$$

and finally end up with the main result of this section summarised in the following theorem:

Theorem 11.4.1 *The inner product in* $L^2(\mathcal{C}; \mathbb{H})$ *of two arbitrary elements of the Appell basis (s. Theorem 11.3.1) is given by*

(a) $l_1 \neq l_2 \;\vee\; p_1, p_2$ *have different parity*

$$\int_{-\xi}^{\xi} \int_0^{\rho} \int_0^{2\pi} \overline{\boldsymbol{A}_{l_1+p_1}^{l_1}}\, \boldsymbol{A}_{l_2+p_2}^{l_2}\, r\, d\varphi\, dr\, dx_0 = 0,$$

(b) $l_1 = l_2 \ \wedge \ p_1, p_2$ *have same parity*

$$\int_{-\xi}^{\xi} \int_0^{\rho} \int_0^{2\pi} \overline{\mathbf{A}_{l+p_1}^{l}} \, \mathbf{A}_{l+p_2}^{l} \, r \, d\varphi \, dr \, dx_0$$

$$= 2\pi \sum_{h_1=0}^{\lfloor \frac{p_1}{2} \rfloor} \sum_{h_2=0}^{\lfloor \frac{p_2}{2} \rfloor} \frac{c_{l,p_1,h_1} c_{l,p_2,h_2} \, \rho^{2(h_1+h_2+l+1)} \, \xi^{p_1+p_2-2(h_1-h_2)+1}}{(h_1+h_2+l+1)(p_1+p_2-2h_1-2h_2+1)}$$

$$+ \frac{d_{l,p_1,h_1} d_{l,p_2,h_2} \, \rho^{2(h_1+h_2+l+2)} \, \xi^{p_1+p_2-2(h_1-h_2)-1}}{(h_1+h_2+l+2)(p_1+p_2-2h_1-2h_2-1)}, \qquad (11.4.3)$$

where the real constants are defined by

$$c_{l,p,h} = \frac{(-1)^h (l+p)!}{2^{2h} h!(l+h)!(p-2h)!} \quad \text{and} \quad d_{l,p,h} = \frac{(-1)^h (l+p)!}{2^{2h+1} h!(l+h+1)!(p-2h-1)!}.$$

On the basis of the results in Theorem 11.4.1, it can be clearly seen that the polynomials for every fixed degree of homogeneity $n \in \mathbb{N}_0$ are already orthogonal w.r.t. the finite cylinder. In addition, the classical Appell polynomials with even (or odd, respectively) degree of homogeneity are also orthogonal to each other. For all other cases, the inner product is distinct from zero, which, contrary to [5], means that the classical Appell system is not an orthogonal system for the finite cylinder. Furthermore, it should be pointed out here that all inner products in $L^2(\mathcal{C}; \mathbb{H})$ of two arbitrary Appell polynomials are real, and hence, the corresponding Gram matrix for a finite system of basis functions is real and sparse. As a consequence, all methods of real linear algebra can be used to orthonormalise the remaining non-orthonormal polynomials.

11.4.3 Adaptive Orthonormalisation Scheme

Orthonormal systems are the best choice for the purpose of approximation. However, it is well known that direct application of the Gram-Schmidt process to orthonormalise a given system of functions typically leads to numerical stability problems. Therefore, explicit formulae to construct an orthonormal system are of particular interest in practical applications. As it has been already shown in the previous section, the proposed system of Appell polynomials contains functions which are already orthogonal in $L^2(\mathcal{C}; \mathbb{H})$, but a remaining part of the system has still to be orthonormalised.

As a first step we represent the basis up to a maximal polynomial degree $\nu \in \mathbb{N}$ as an ordered set

$$\begin{aligned}
&\left\{ \boldsymbol{A}_n^l \,:\, l = 0, \ldots, n \right\}_{n=0}^{\nu} \\
&\qquad = \left\{ \left\{ \boldsymbol{A}_n^0 \right\}_{\substack{n=0, \\ n \text{ even}}}^{\nu}, \left\{ \boldsymbol{A}_n^0 \right\}_{\substack{n=0, \\ n \text{ odd}}}^{\nu}, \ \ldots\ , \ \ldots\ , \left\{ \boldsymbol{A}_n^\nu \right\}_{\substack{n=\nu, \\ n \text{ even}}}^{\nu}, \left\{ \boldsymbol{A}_n^\nu \right\}_{\substack{n=\nu, \\ n \text{ odd}}}^{\nu} \right\} \\
&\qquad = \left\{ \left\{ \boldsymbol{A}_n^k \right\}_{\substack{n=k, \\ n \text{ even}}}^{\nu}, \left\{ \boldsymbol{A}_n^k \right\}_{\substack{n=k, \\ n \text{ odd}}}^{\nu} \right\}_{k=0}^{\nu}.
\end{aligned}$$

To shorten the notations we denote the respective subsets as follows

$$\left\{ \boldsymbol{A}_n^l \,:\, l = 0, \ldots, n \right\}_{n=0}^{\nu} \ =: \ \left\{ \mathcal{S}_{2k}^{\nu}, \mathcal{S}_{2k+1}^{\nu} \right\}_{k=0}^{\nu}.$$

The proposed notation indicates that for each column $k \in \mathbb{N}_0$ in Fig. 11.1 we have two subsets $\mathcal{S}_{2k}^{\nu}$ (even degree of homogeneity) and $\mathcal{S}_{2k+1}^{\nu}$ (odd degree of homogeneity) in the construction. This rearrangement of the basis polynomials follows directly from the fact that each subset contains polynomials which are already orthogonal to the polynomials of the other subsets. Thus, we need to orthonormalise only the polynomials in each subset. The exact number of elements in each subset is given by the formulae

$$\left| \mathcal{S}_{2k}^{\nu} \right| = \begin{cases} \left\lceil \frac{\nu-k+1}{2} \right\rceil \,:\, \nu \text{ is even}, \\ \left\lfloor \frac{\nu-k+1}{2} \right\rfloor \,:\, \nu \text{ is odd} \end{cases} \quad \text{and} \quad \left| \mathcal{S}_{2k+1}^{\nu} \right| = \begin{cases} \left\lfloor \frac{\nu-k+1}{2} \right\rfloor \,:\, \nu \text{ is even}, \\ \left\lceil \frac{\nu-k+1}{2} \right\rceil \,:\, \nu \text{ is odd} \end{cases}$$

and depends on the maximal polynomial degree ν and the order k. Taking into account the proposed ordering of the basis elements, the Gram matrix has a diagonal block structure (see Fig. 11.2). For a fixed $\nu \in \mathbb{N}$, the square blocks on the main diagonal are the Gram matrices

$$\boldsymbol{G}_j^{\nu} := \left[\left\langle \varphi_{j,p}^{\nu}, \varphi_{j,q}^{\nu} \right\rangle_{L^2(\mathcal{C},\mathbb{H})} \right]_{p,q=1,\ldots,|\mathcal{S}_j^{\nu}|}, \quad j = 0, \ldots, 2\nu + 1$$

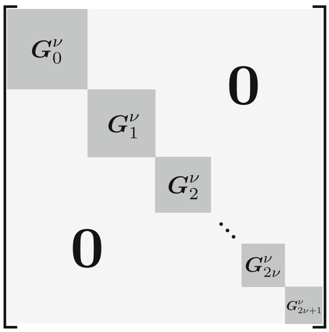

Fig. 11.2 Diagonal block structure of the Gram matrix

of the basis elements $\varphi^{\nu}_{j,p} \in \mathcal{S}^{\nu}_j$, $p = 1, \ldots, |\mathcal{S}^{\nu}_j|$ of the respective subsets $\mathcal{S}^{\nu}_j$. The entries of the Gram matrices $\boldsymbol{G}^{\nu}_j$ are explicitly given by relation (11.4.3) and therefore do not require high computational costs.

An adaptive orthogonalisation procedure taking advantage of the diagonal block structure of the Gram matrix could then be realised as follows. Due to the fact that the matrices $\boldsymbol{G}^{\nu}_j$ are real, symmetric and positive definite, we used the Cholesky decomposition for the orthonormalisation of the basis polynomials in the respective subsets $\mathcal{S}^{\nu}_j$. Of course, other matrix decompositions or the application of the classical Gram-Schmidt orthonormalisation process would also be possible here.

Finally, it should be estimated to what extent the computational costs are reduced by the explicit knowledge of the inner products and the resulting diagonal block structure of the Gram matrix. To that end we evaluate the complexity of the computational methods used in Listing 11.1 by considering only multiplications and divisions. Given a real square matrix of size $m \in \mathbb{N}$, it is well known that the Cholesky decomposition has complexity $\frac{1}{6}m^3 + \mathcal{O}(m^2)$, the inversion of an upper triangular matrix has complexity $m^3 + \mathcal{O}(m^2)$ and the matrix-vector multiplication $\mathcal{O}(m^2)$. Thus, the computational costs for the orthonormalisation of $m \in \mathbb{N}$ polynomials with regard to the algorithm presented in Listing 11.1 can be estimated by

$$\Lambda(m) = \frac{7}{6}m^3 + \mathcal{O}(m^2).$$

From this we conclude directly that for a fixed maximal polynomial degree $\nu \in \mathbb{N}$ and a total number of basis functions

$$\kappa = \frac{1}{2}(\nu + 1)(\nu + 2)$$

Listing 11.1 Adaptive orthogonalisation scheme

```
define ∀ j = 0, ..., 2ν + 1, ν ∈ ℕ fixed
    Φ_j^ν := [φ_{j,k}^ν]_{k=1,...,|S_j^ν|}, φ_{j,k}^ν ∈ S_j^ν;  and  G_j^ν := [⟨φ_{j,p}^ν, φ_{j,q}^ν⟩_{L²(C,ℍ)}]_{p,q=1,...,|S_j^ν|};
for j from 0 to 2ν+1 do
    compute L_j^ν such that G_j^ν = L_j^ν (L_j^ν)^T;
    compute C_j^ν := ((L_j^ν)^T)^{-1};
    compute Φ_j^{ν,⋆} := C_j^ν Φ_j^ν;  //orthonormalised polynomials of S_j^ν
end do;
```

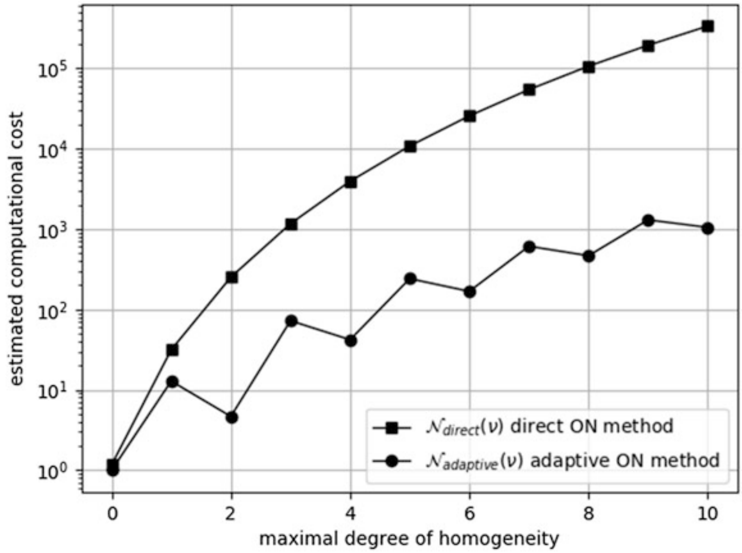

Fig. 11.3 Estimated computational costs of direct and adaptive orthonormalisation method

the computational costs of the orthonormalisation of the Appell polynomials (11.4.2) w.r.t. the domain of the finite cylinder, be it with or without utilising the Gram matrix structure, can be estimated by

$$\mathcal{N}_{direct}(\nu) \approx \frac{7}{48}\left(\nu^2 + 3\nu + 2\right)^3 \quad \text{and} \quad \mathcal{N}_{adaptive}(\nu) \approx \sum_{j=0}^{2\nu+1} \Lambda\left(|\mathcal{S}_j^\nu|\right).$$

A comparison of the respective computational costs is shown in Fig. 11.3. Thus, the proposed adaptive orthonormalisation procedure leads to a significant reduction in computational costs compared to the direct application of the orthonormalisation process without using any prior knowledge of the Gram matrix structure.

11.4.4 Conditioning of the Gram Matrix of Non-orthonormalised Appell Basis

In this concluding subsection, we will examine the practical relevance of an orthonormal system for the cylinder. Let us now consider the numerical properties of classical Appell polynomials w.r.t. the finite cylinder. To that end we consider, for a fixed maximal polynomial degree $\nu \in \mathbb{N}$, the subset $\left\{\boldsymbol{\varphi}_n^l(\boldsymbol{x}) \,:\, l = 0, \ldots, n\right\}_{n=0}^{\nu}$ of Appell polynomials normalised w.r.t. the domain of a ball $\mathbb{B}_\rho$ of radius $\rho \in \mathbb{R}^+$

(see [2] for the details), given explicitly by

$$\boldsymbol{\varphi}_n^l(\boldsymbol{x}) \;=\; \frac{1}{2^{l+1}\,n!}\sqrt{\frac{(2n+3)\,(n-l)!\,(n+l+1)!}{\pi\,\rho_{sph}^{2n+3}}}\;\boldsymbol{A}_n^l(\boldsymbol{x}),$$

and calculate the condition number of the Gram matrix for different cylinder geometries $\mathcal{C}_{\rho,\tau}$. For the numerical evaluation, consider the case where the radius of the ball $\mathbb{B}_\rho$ and the cylinder $\mathcal{C}_{\rho,\tau}$ are the same, i.e, $\rho_{sph} = \rho_{cyl}$, and the height $\tau \in \mathbb{R}^+$ of the cylinder varies (see Fig. 11.4a). The other case where $\rho_{sph} = \tau$ and the radius ρ_{cyl} of the cylinder varies leads to similar results. Figure 11.4b shows the condition numbers of the Gram matrix of subsets $\left\{\boldsymbol{\varphi}_n^l(\boldsymbol{x}) \,:\, l = 0,\ldots,n\right\}_{n=0}^{\nu}$ for different maximal polynomial degrees ν and different ratios τ/ρ_{cyl} of the cylinder parameters. The evaluation clearly shows that the non-orthonormalised Appell system w.r.t. a finite cylinder only has good numerical properties (relatively small condition numbers) for ratios $\tau/\rho_{cyl} = 1$, which is the case when $\mathbb{B}_\rho$ is the insphere of $\mathcal{C}_{\rho,\tau}$, i.e., $\rho_{sph} = \rho_{cyl} = \tau$. For ratios where τ is much larger or smaller than the radius ρ_{cyl}, the Gram matrix is ill-conditioned due to the exponential growth of the condition numbers. For these reasons it can be concluded that, in particular for practical problems where the considered domains are either long thin cylinders (e.g. supports with circular cross section) or very flat cylinders (e.g. circular plates), the orthonormalisation of the Appell system is indispensable.

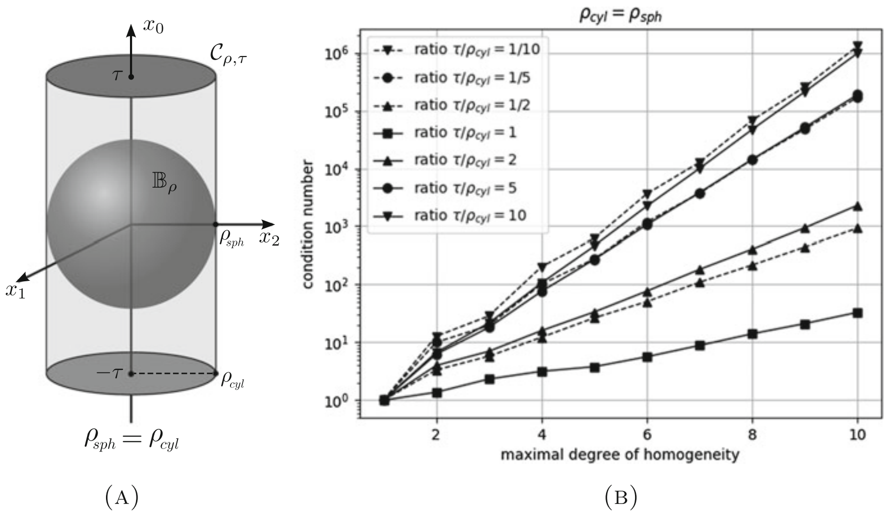

Fig. 11.4 Numerical evaluation of the condition number of the Gram matrix of the non-orthonormalised Appell polynomials $\boldsymbol{\varphi}_n^l(\boldsymbol{x})$ w.r.t. different cylinder geometries. (**a**) Geometrical setting. (**b**) Condition number for different ratios

11.5 Conclusions

In this article we have examined the orthogonality properties of the classical Appell polynomials w.r.t. a finite cylinder. Based on an explicit calculation of the inner products of two arbitrary Appell polynomials it has been shown that most of the functions are already orthogonal. This results for a given finite subset of functions in a real and sparse Gram matrix. The structural properties of the Gram matrix have been used to propose an adaptive orthonormalisation method, which significantly reduces the computational costs. Finally, the practical necessity of orthonormalising the Appell system has been illustrated by numerically evaluating the condition number of the Gram matrix for the non-orthonormal Appell polynomials. Here it has been clearly verified that an orthonormalisation of the Appell polynomials for cylindrical geometries with ratios $\tau/\rho \ll 1$ or $\tau/\rho \gg 1$ is beneficial.

Acknowledgements We wish Prof. Wolfgang Sprößig all the best for the anniversary and would like to express our utmost respect and gratitude to his scientific work on this occasion.

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Chapter 12
Comments on an Orthogonal Family of Monogenic Functions on Spheroidal Domains

Joaõ Morais

Celebrating Wolfgang Sprößig 70th birthday

Abstract The problem of building an orthogonal basis for the space of square-integrable harmonic functions defined in a spheroidal (either oblate or prolate) domain leads to special functions, which provide an elegant analysis of a variety of physical problems. Many generalizations of these ideas in the context of Quaternionic Analysis possess a similar elegant mathematical structure. A brief descriptive review is given of these developments.

Keywords Quaternionic analysis · Spherical harmonics · Spheroidal harmonics · Monogenic functions

Mathematics Subject Classification (2010) Primary 30G35, Secondary 30C65

12.1 Introduction

The origins behind the study of orthogonal bases of polynomials for the spaces of square-integrable harmonic functions defined in a prolate or oblate spheroid are to be found in [16]. The orthogonality was taken with respect to certain inner products, each of which lead to the discussion of a PDE by means of the kernel of the orthogonal system corresponding to that inner product. As regards treatises on the subject, we add the names of Laplace [28], Lamé [27], Heine [23], Liouville [34], Thomson and Tait [51], Hilbert [24], Niven [47], Klein [26], Lindemann [33], Stieltjes [49], Darwin [11], Ferrers [15], Féjer [14], Whittaker and Watson [52], among others, while more general aspects of their theory were given by Hobson

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[25], Szegö [50], Byerly [6], Sansone [48], Lebedev [31], and Dassios [12]. In this connection, recently in [17] the spheroidal harmonics were defined following [16], with a rescaling factor which permits including the unit ball as a limit of both the prolate and oblate cases, combined into a single one-parameter family.

Multi-dimensional extensions of the prolate spheroidal harmonics to the framework of Quaternionic Analysis were originally developed in [36] and subsequently in [37], which provided many of their properties and have subsequently attracted special attention. In [43] it was shown that the underlying prolate spheroidal monogenics play an important role in defining and studying the monogenic Szegö kernel function for prolate spheroids. In [46] the authors developed an orthogonal basis of oblate spheroidal monogenics and some recurrence formulae were found. It was shown that in the case of an oblate spheroid a basis can only be either orthogonal or Appell basis. Some aspects on generating monogenic functions that are orthogonal in a region outside a prolate spheroid were considered in [44]. Generalization of these results has been recently done in [38].

The object of the present note is twofold: to review the construction of a single one-parameter family of spheroidal harmonics with special emphasis on those orthogonal in the L_2-Hilbert space structure; and to construct an orthogonal basis of spheroidal monogenics, whose elements are parametrized by the shape of the corresponding spheroid. We observe that this analysis cannot be done with models in which the unit ball only is approximated as a degenerate case and requires a separate, yet completely analogous, treatment for prolate and oblate spheroids [16, 25]. The proofs of the main results are simplified, in accordance with developments of the theory later in date than the original proofs; other results are given in a form more general than that in which they were first discovered. The references given are to be regarded solely as indicating sources of information from which I have drawn, or where more detailed information on the various topics is to be found.

12.2 Background on Spheroidal Harmonics

We consider the family of coaxial spheroidal domains Ω_μ, scaled so that the major axis is of length 2:

$$\Omega_\mu = \{x \in \mathbb{R}^3 |\ x_0^2 + \frac{x_1^2 + x_2^2}{e^{2\nu}} < 1\}, \tag{12.2.1}$$

where $\nu \in \mathbb{R}$ and $\mu = (1 - e^{2\nu})^{\frac{1}{2}}$ will be useful in later formulas. This follows the notation in [17]. The equations relating the Cartesian coordinates of a point $x = (x_0, x_1, x_2)$ inside Ω_μ to *spheroidal coordinates* $(\eta, \vartheta, \varphi)$ are

$$x_0 = \mu \cosh \eta \cos \vartheta, \quad x_1 = \mu \sinh \eta \sin \vartheta \cos \varphi, \quad x_2 = \mu \sinh \eta \sin \vartheta \sin \varphi, \tag{12.2.2}$$

where in the case of the *prolate spheroid* ($\nu < 0$) the coordinates range over $\eta \in [0, \pi]$, $\vartheta \in [0, \operatorname{arctanh} e^{\nu}]$, $\varphi \in [0, 2\pi)$, and $0 < \mu < 1$ is the eccentricity, while for the *oblate spheroid* ($\nu > 0$) we have $\eta \in [0, \pi]$ and $\vartheta \in [0, \operatorname{arccoth} e^{\nu}]$, $\varphi \in [0, 2\pi)$ and μ is imaginary, $\mu/i > 0$. The spheroids reduce to the unit ball for $\nu = 0$, $\mu = 0$: $\Omega_0 = \{x \in \mathbb{R}^3 : |x|^2 < 1\}$.

In terms of the coordinates (12.2.2), *the spheroidal harmonics* are

$$U^{\pm}_{l,m}[\mu](x) := U_{l,m}[\mu](\eta, \vartheta)\, \Phi^{\pm}_m(\varphi), \tag{12.2.3}$$

where

$$U_{l,m}[\mu](\eta, \vartheta) = \alpha_{l,m}\, \mu^l\, P^m_l(\cos\vartheta) P^m_l(\cosh\eta) \tag{12.2.4}$$

for $\mu \neq 0$. Here P^m_l are the *associated Legendre functions of the first kind* (see [25, Ch. III]) of degree l and order m, and we write $\Phi^+_m(\varphi) = \cos(m\varphi)$, $\Phi^-_m(\varphi) = \sin(m\varphi)$, and

$$\alpha_{l,m} = \frac{(l-m)!}{(2l-1)!!} \tag{12.2.5}$$

with use of the symbol $n!! = \prod_{k=0}^{\lceil n/2\rceil - 1}(n-2k)$ for the double factorial. To avoid repetition, we state once and for all that $U^-_{l,m}[\mu]$ is only defined for $m \geq 1$, i.e. $U^-_{l,0}[\mu]$ is expressly excluded from all statements of theorems.

It was shown in [17] that with the scale factor (12.2.5), the $U^{\pm}_{l,m}[\mu]$ are polynomials in the variables x_0, x_1, x_2, which are normalized so that the limiting case $\mu \to 0$ gives the classical *solid spherical harmonics* [45, 48],

$$U^{\pm}_{l,m}[0](x) = |x|^l P^m_l\left(\frac{x_0}{|x|}\right) \Phi^{\pm}_m(\varphi), \tag{12.2.6}$$

where we employ spherical coordinates $x_0 = \rho\cos\theta$, $x_1 = \rho\sin\theta\cos\varphi$, and $x_2 = \rho\sin\theta\sin\varphi$.

Moreover, in [16] it was shown that while the $U^{\pm}_{l,m}[\mu]$ are orthogonal in the Dirichlet norm on Ω_μ, the closely related functions, which we will call the *Garabedian spheroidal harmonics*,

$$V^{\pm}_{l,m}[\mu](x) = \frac{\partial}{\partial x_0} U^{\pm}_{l+1,m}[\mu](x) \tag{12.2.7}$$

form an orthogonal basis for $L_2(\Omega_\mu) \cap \mathrm{Har}(\Omega_\mu)$, the set of harmonic functions in $L_2(\Omega_\mu)$. This property makes the $V^{\pm}_{l,m}[\mu]$ of greater interest for many considerations.

In accordance with the notation already employed, we shall use $V^{\pm}_{l,m}[\mu] = V_{l,m}[\mu]\Phi^{\pm}_m$ when the factors $\Phi^{\pm}_m$ are not of interest. It will be convenient, before proceeding, to investigate the algebraical forms of the $V_{l,m}[\mu]$. We will assume that $\nu < 0$, because the case $\nu > 0$ is similar. From differentiating (12.2.2),

$$\frac{\partial}{\partial x_0} = \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)}\left(\cos\vartheta\,\sinh\eta\,\frac{\partial}{\partial\eta} - \sin\vartheta\,\cosh\eta\,\frac{\partial}{\partial\vartheta}\right),$$

from which the definition (12.2.7) gives

$$\frac{(\cosh^2\eta - \cos^2\vartheta)}{\alpha_{l+1,m}\,\mu^l}V_{l,m}[\mu] = \cos\vartheta\,\sinh^2\eta\,P^m_{l+1}(\cos\vartheta)(P^m_{l+1})'(\cosh\eta) + \sin^2\vartheta\,\cosh\eta\,P^m_{l+1}(\cosh\eta)(P^m_{l+1})'(\cos\vartheta). \tag{12.2.8}$$

There are many well-known recurrence relations for the associated Legendre functions (see for example [25, Ch. III]). The relation

$$(1-t^2)(P^m_{l+1})'(t) = (l+m+1)P^m_l(t) - (l+1)t\,P^m_{l+1}(t) \tag{12.2.9}$$

yields that (12.2.8) is equal to $(l+m+1)$ times

$$\cosh\eta\;P^m_l(\cos\vartheta)P^m_{l+1}(\cosh\eta) - \cos\vartheta\;P^m_{l+1}(\cos\vartheta)P^m_l(\cosh\eta).$$

It follows, then, that

$$V_{l,m}[\mu] = \frac{\alpha_{l+1,m}(l+m+1)\mu^l}{(\cosh^2\eta - \cos^2\vartheta)}\big[\cosh\eta\;P^m_l(\cos\vartheta)P^m_{l+1}(\cosh\eta) - \cos\vartheta\;P^m_l(\cosh\eta)P^m_{l+1}(\cos\vartheta)\big], \tag{12.2.10}$$

with the initial values

$$V_{l,l}[\mu] = (2l+1)U_{l,l}[\mu],$$
$$V_{l+1,l}[\mu] = 2(l+1)U_{l+1,l}[\mu].$$

In order to avoid the difficulties usually attendant on manipulations like those of the formulas (12.2.10), it will here be convenient to prove very simple recurrence relations for the functions $V_{l,m}[\mu]$. The following will be key in the proof of Theorem 12.3.1 and it is based on the results of [36].

Proposition 12.2.1 *For each $l \geq 2$, the functions $V_{l,m}[\mu]$ satisfy the recurrence relations*

$$V_{l,m}[\mu] = (l+m+1)U_{l,m}[\mu] + \frac{\mu^2(l+m+1)(l+m)}{(2l+1)(2l-1)}V_{l-2,m}[\mu]. \tag{12.2.11}$$

Proof Equation (12.2.10) together with the further relation

$$(l-m+1)P_{l+1}^m(t) = (2l+1)tP_l^m(t) - (l+m)P_{l-1}^m(t) \tag{12.2.12}$$

show that

$$\begin{aligned} V_{l+1,m}[\mu] &= (l+m+1)U_{l,m}[\mu] \\ &\quad + \frac{\alpha_{l,m}\,\mu^l(l+m+1)(l+m)}{(\cosh^2\eta-\cos^2\vartheta)(2l+1)}[\cos\vartheta\; P_{l-1}^m(\cos\vartheta)P_l^m(\cosh\eta) \\ &\quad - \cosh\eta\; P_l^m(\cos\vartheta)P_{l-1}^m(\cosh\eta)], \end{aligned}$$

with

$$\alpha_{l,m} = \frac{2l+1}{l-m+1}\,\alpha_{l+1,m}.$$

Using again (12.2.12), we obtain

$$\begin{aligned} V_{l+1,m}[\mu] &= (l+m+1)U_{l,m}[\mu] \\ &\quad + \frac{\alpha_{l-1,m}\,\mu^l(l+m+1)(l+m)(l+m-1)}{(\cosh^2\eta-\cos^2\vartheta)(2l-1)(2l+1)} \\ &\quad \times [\cosh\eta\; P_{l-2}^m(\cos\vartheta)P_{l-1}^m(\cosh\eta) \\ &\quad - \cos\vartheta\; P_{l-1}^m(\cos\vartheta)P_{l-2}^m(\cosh\eta)]. \end{aligned}$$

The result now follows. □

Since the basic harmonics $U_{l,m}^{\pm}[\mu]$ of [16] are polynomials of degree l, it is clear that the operations of rescaling by $1/\mu$ or i/μ and multiplying by μ^l implied in (12.2.4) assure that the $V_{l,m}^{\pm}[\mu]$ are polynomials in μ. By Eq. (12.2.11) it is clear that $-\mu$ produces the same results as μ, so the only powers of μ which appear are even.

In this regard, from (12.2.11) we note that for spherical harmonics,

$$\frac{\partial}{\partial x_0}U_{l+1,m}^{\pm}[0](x) = (l+m+1)U_{l,m}^{\pm}[0](x), \tag{12.2.13}$$

whereas $V_{l,m}^{\pm}[\mu]$ is not so simply related to $U_{l,m}^{\pm}[\mu]$ for $\mu \neq 0$, as was proved in [36]:

Theorem 12.2.2 *Let $l \geq 0$, $0 \leq m \leq l$. The non-vanishing coefficients $v_{l,m,k}$ in the relation*

$$V_{l,m}^{\pm}[\mu] = \sum_{k=0}^{\left[\frac{l-m}{2}\right]} v_{l,m,k}\,\mu^{2k}U_{l-2k,m}^{\pm}[\mu] \tag{12.2.14}$$

are given by

$$v_{l,m,k} = \frac{(l+m+1)!(2l+1-4k)!!}{(l+m-2k)!(2l+1)!!}. \tag{12.2.15}$$

Proof Suppose inductively that the formula of theorem is true when l is replaced by $l' < l$. Then

$$V^{\pm}_{l,m}[\mu] = (l+m+1)U^{\pm}_{l,m}[\mu] + \frac{(l+m+1)(l+m)}{(2l+1)(2l-1)} \sum_{k=0}^{\left[\frac{l-2-m}{2}\right]} v_{l-2,m,k}\, \mu^{2(k+1)} U^{\pm}_{l-2(k+1),m}[\mu].$$

Since by (12.2.15)

$$v_{l,m,0} = l+m+1,$$
$$v_{l,m,k+1} = \frac{(l+m+1)(l+m)}{(2l+1)(2l-1)}\, v_{l-2,m,k},$$

we find that the stated formula is also true, completing the proof. □

An important result of [16] regarding the orthogonailty of the $V^{\pm}_{l,m}[\mu]$ in the L_2-Hilbert space can be restated as follows.

Theorem 12.2.3 *For a fixed μ, the functions $V^{\pm}_{l,m}[\mu]$ $(l \geq 0)$ form a complete orthogonal family in the closed subspace $L_2(\Omega_\mu) \cap \mathrm{Har}(\Omega_\mu)$ of $L_2(\Omega_\mu)$ with the norms*

$$\| V^{\pm}_{l,m}[\mu] \|^2_{L_2(\Omega_\mu)} = 2\pi(1+\delta_{0,m})\mu^{2l+3}\gamma_{l,m} I_{l,m}(\mu), \tag{12.2.16}$$

where $I_{l,m}(\mu)$ is defined by

$$I_{l_1,m}(\mu) := \int_1^{\frac{1}{\mu}} P^m_{l_1}(t) P^m_{l_1+2}(t)dt, \tag{12.2.17}$$

and

$$\gamma_{l,m} = \frac{(l+m+1)(l+2-m)!(l+m+1)!}{(2l+1)!!(2l+3)!!}. \tag{12.2.18}$$

For the limiting case, $\mu = 0$,

$$\| V^{\pm}_{l,m}[0] \|^2_{L_2(\Omega_0)} = \frac{2\pi(1+\delta_{0,m})(l+m+1)(l+m+1)!}{(2l+1)(2l+3)(l-m)!}. \tag{12.2.19}$$

12.3 An Orthogonal Basis of Spheroidal Monogenics

The standard bases for spheroidal harmonics have their counterparts for the corresponding spaces of monogenic functions taking values in $\mathbb{R}^3$. These monogenic polynomials are defined by regarding $\mathbb{R}^3$ as the subset of the *quaternions* $\mathbb{H} := \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}\}$ for which $x_3 = 0$. Although this subspace is not closed under the quaternionic multiplication (which is defined, as usual, so that $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$ and $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$), it is possible to carry out a great deal of the analysis analogous to that of complex numbers [13, 21, 35, 39, 40, 42].

Consider the Cauchy-Riemann (or Fueter) operators

$$\overline{\partial} = \frac{\partial}{\partial x_0} + \mathbf{i}\frac{\partial}{\partial x_1} + \mathbf{j}\frac{\partial}{\partial x_2}, \quad \partial = \frac{\partial}{\partial x_0} - \mathbf{i}\frac{\partial}{\partial x_1} - \mathbf{j}\frac{\partial}{\partial x_2}. \tag{12.3.1}$$

Define the set of *monogenic functions*

$$\mathcal{M}(\Omega_\mu) := \left\{\mathbf{f} = [\mathbf{f}]_0 + [\mathbf{f}]_1\mathbf{i} + [\mathbf{f}]_2\mathbf{j} \in C^1(\Omega_\mu) : \overline{\partial}\mathbf{f} = 0\right\}.$$

Monogenic functions are harmonic, but not vice-versa. The *hypercomplex derivative* is simply denoted by $(1/2)\partial\mathbf{f}$ [18].

A basis of polynomials spanning the square-integrable solutions of $\overline{\partial}\mathbf{f} = 0$ was given in [36] (cf. [37]) for prolate spheroids and another in [46] for oblate spheroids, via explicit formulas. Note that the latter prolate and oblate spheroidal monogenics can be obtained as a special case of the present theory by appropriate interpretation. In the following, we consider the prolate and oblate cases of spheroids simultaneously.

In analogy to (12.2.7) the *basic monogenic spheroidal polynomials* are constructed as

$$\mathbf{X}^{\pm}_{l,m}[\mu] = \partial U^{\pm}_{l+1,m}[\mu]. \tag{12.3.2}$$

It was noted in [39] that Sc $\mathbf{X}^{\pm}_{l,m}[0]$ is equal to $V^{\pm}_{l,m}[0] = (l+m+1)U^{\pm}_{l,m}[0]$, and an explicit expression for the vector part was written out, which was later generalized from the sphere to the spheroid in [36].

Using (12.2.10) and further properties of the Legendre functions, we can verify that

$$V_{l,-1}[\mu] = \begin{cases} -\frac{1}{(l+1)(l+2)} V_{l,1}[\mu] & l = 1, 2, \ldots, \\ 0 & l = 0. \end{cases} \tag{12.3.3}$$

These functions will appear in the representation (12.3.4) for the case of zero-order monogenic polynomials (see Theorem 12.3.1 below). Similar results can be found in [36].

Theorem 12.3.1 *For each $l \geq 0$ and $0 \leq m \leq l+1$, the basic spheroidal monogenic polynomials* (12.3.2) *are equal to*

$$\mathbf{X}^{\pm}_{l,m}[\mu] = V^{\pm}_{l,m}[\mu] + \frac{\mathbf{i}}{2}\left[(l+m+1)V^{\pm}_{l,m-1}[\mu] - \frac{1}{l+m+2}V^{\pm}_{l,m+1}[\mu]\right]$$
$$\mp \frac{\mathbf{j}}{2}\left[(l+m+1)V^{\mp}_{l,m-1}[\mu] + \frac{1}{l+m+2}V^{\mp}_{l,m+1}[\mu]\right], \qquad (12.3.4)$$

where the harmonic polynomials $V^{\pm}_{l,m}[\mu]$ are defined by (12.2.7). *Moreover, the set $\{\mathbf{X}^{\pm}_{l,m}[\mu] : l \geq 0,\ 0 \leq m \leq l+1\}$ is orthogonal in the sense of the scalar product defined by*

$$\langle \mathbf{f}, \mathbf{g} \rangle_{[\mu]} = \mathrm{Sc} \iiint_{\Omega_\mu} \bar{\mathbf{f}}\mathbf{g}\, dx. \qquad (12.3.5)$$

Their norms are given by

$$\|\mathbf{X}^{\pm}_{l,m}[\mu]\|^2_{L_2(\Omega_\mu)} = \frac{\pi\, \mu^{2l+3}}{(l+2)(l+m+2)(2l+1)!!(2l+3)!!}$$
$$\big[(l+2)(l+m)(l+m+1)(l-m+3)!(l+m+2)!I_{l,m-1}$$
$$+ 2\delta_{0,m}(l+m+2)(l+1)!(l+2)!I_{l,1}$$
$$+ (l+2)(l-m+1)!(l+m+2)!\big(I_{l,m+1}$$
$$+ 2(l-m+2)(l+m+1)(1+\delta_{0,m})I_{l,m}\big)\big],$$

where $I_{l,m}(\mu)$ is defined by (12.2.17). *For the limiting case, $\mu = 0$,*

$$\|\mathbf{X}^{\pm}_{l,m}[0]\|^2_{L_2(\Omega_0)} = \frac{2\pi(1+\delta_{0,m})(l+1)(l+1+m)!}{(2l+3)(l+1-m)!}.$$

Proof The full operator (12.3.1) in spheroidal coordinates (12.2.2) is

$$\partial = \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)}\left(\cos\vartheta \sinh\eta \frac{\partial}{\partial\eta} - \sin\vartheta\cosh\eta\frac{\partial}{\partial\vartheta}\right)$$
$$- \frac{1}{\mu(\cosh^2\eta - \cos^2\vartheta)}(\cos\varphi\mathbf{i} + \sin\varphi\mathbf{j})\left(\sin\vartheta\cosh\eta\frac{\partial}{\partial\eta} + \cos\vartheta\sinh\eta\frac{\partial}{\partial\vartheta}\right)$$
$$- \frac{1}{\mu\sin\vartheta\sinh\eta}(-\sin\varphi\mathbf{i} + \cos\varphi\mathbf{j})\frac{\partial}{\partial\varphi}.$$

The first line of this expression applied to $U^{\pm}_{l+1,m}[\mu]$ produces the scalar part of $\mathbf{X}^{\pm}_{l,m}[\mu]$ in (12.3.4) and was calculated in [36]. For the non-scalar part, we use the relation (12.2.9) to obtain

$$
\begin{aligned}
&\frac{2}{\mu^{l+1}\alpha_{l+1,m}\Phi^{\pm}_m}\left(\cos\vartheta\,\sinh\eta\,\frac{\partial}{\partial\vartheta}+\sin\vartheta\,\cosh\eta\,\frac{\partial}{\partial\eta}\right)U^{\pm}_{l+1,m}[\mu]\\
&=(l+m+1)(l-m+2)\Big[\sin\vartheta\,\cosh\eta\,P^m_{l+1}(\cos\vartheta)\,P^{m-1}_{l+1}(\cosh\eta)\\
&\quad-\cos\vartheta\,\sinh\eta\,P^{m-1}_{l+1}(\cos\vartheta)\,P^m_{l+1}(\cosh\eta)\Big]\\
&\quad+\sin\vartheta\,\cosh\eta\,P^m_{l+1}(\cos\vartheta)\,P^{m+1}_{l+1}(\cosh\eta)\\
&\quad+\cos\vartheta\,\sinh\eta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^m_{l+1}(\cosh\eta).
\end{aligned}
$$

Next, we use the relation

$$
\sqrt{1-t^2}\,P^m_{l+1}(t)=(l-m)t\,P^{m-1}_{l+1}(t)-(l+m)\,P^{m-1}_l(t)
$$

(valid for $|t|<1$, and replacing $1-t^2$ with t^2-1 for $|t|>1$) produces

$$
\begin{aligned}
-\frac{(\cosh^2\eta-\cos^2\vartheta)}{\mu^l\alpha_{l+1,m-1}}\,V_{l,m-1}[\mu]&=\sin\vartheta\,\cosh\eta\,P^m_{l+1}(\cos\vartheta)\,P^{m-1}_{l+1}(\cosh\eta)\\
&\quad-\cos\vartheta\,\sinh\eta\,P^m_{l+1}(\cosh\eta)\,P^{m-1}_{l+1}(\cos\vartheta).
\end{aligned}
$$

Furthermore, using the expression

$$
(1-t^2)^{1/2}\,P^m_{l+1}(t)=\frac{1}{2l+3}\,(P^{m+1}_{l+2}(t)-P^{m+1}_l(t)),
$$

and its counterpart for $|t|>1$, and then applying (12.2.12), we arrive at

$$
\begin{aligned}
&\cosh\eta\,\sin\vartheta\,P^m_{l+1}(\cos\vartheta)\,P^{m+1}_{l+1}(\cosh\eta)+\sinh\eta\,\cos\vartheta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^m_{l+1}(\cosh\eta)\\
&=\frac{(\cosh^2\eta-\cos^2\vartheta)}{(l+1-m)(l+2+m)\mu^l\alpha_{l+1,m+1}}\,V_{l,m+1}[\mu].
\end{aligned}
$$

With these calculations at hand, we have

$$
\begin{aligned}
&-\frac{1}{\mu(\cosh^2\eta-\cos^2\vartheta)}\left(\sin\vartheta\,\cosh\eta\frac{\partial}{\partial\eta}+\cos\vartheta\,\sinh\eta\frac{\partial}{\partial\vartheta}\right)U^{\pm}_{l+1,m}[\mu]\\
&=\frac{(l+1+m)}{2}V_{l,m-1}[\mu]\Phi^{\pm}_m-\frac{1}{2(l+2+m)}V_{l,m+1}[\mu]\Phi^{\pm}_m.
\end{aligned}
$$

Similarly, one can prove that

$$
\begin{aligned}
&\frac{1}{\sin\vartheta\,\sinh\eta}\,\frac{\partial}{\partial\varphi}U^{\pm}_{l+1,m}[\mu]\\
&= \mp\,\frac{m\,\mu^{l+1}\alpha_{l+1,m}}{\cosh^2\eta-\cos^2\vartheta}\,\Phi^{\mp}_m\\
&\quad\times\left[\frac{\sinh\eta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^{m}_{l+1}(\cosh\eta)}{\sin\vartheta}+\frac{\sin\vartheta\,P^{m+1}_{l+1}(\cos\vartheta)\,P^{m}_{l+1}(\cosh\eta)}{\sinh\eta}\right]\\
&= \pm\,\frac{\mu}{2}\left[\frac{1}{l+2+m}V_{l,m+1}[\mu]+(l+1+m)V_{l,m-1}[\mu]\right]\Phi^{\mp}_m.
\end{aligned}
$$

Combining these three formulas one straightforward obtains the desired expressions for $(\partial/\partial x_1)U^{\pm}_{l+1,m}[\mu]$ and $(\partial/\partial x_2)U^{\pm}_{l+1,m}[\mu]$.

In the sequel, we will denote by $[\mathbf{f}]_i$ $(i=0,1,2)$ the components of a function $\mathbf{f}\colon \Omega_\mu\to\mathbb{R}^3$. By definition of the integral (12.3.5) it follows that

$$
\begin{aligned}
&\langle \mathbf{X}^{\pm}_{l_1,m_1}[\mu],\mathbf{X}^{\pm}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)}\\
&=\iiint_{\Omega_\mu}\Big([\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_0[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_0+[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_1\\
&\quad+[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_2\Big)\,dx.
\end{aligned}
$$

By Eqs. (12.3.3) and (12.3.4), and Theorem 12.2.3 we have

$$
\iiint_{\Omega_\mu}[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_0[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_0\,dx=\|V^{\pm}_{l_1,m_1}[\mu]\|^2_{L_2(\Omega_\mu)}\,\delta_{l_1,l_2}\delta_{l_1,l_2} \tag{12.3.6}
$$

and

$$
\begin{aligned}
&\iiint_{\Omega_\mu}\Big([\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_1+[\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_2\Big)\,dx\\
&=\frac{\pi p_1(l_2+m_1+1)\delta_{m_1,m_2}}{2}\int_0^{\operatorname{arctanh}e^\nu}\int_0^\pi V_{l_1,m_1-1}[\mu]V_{l_2,m_1-1}[\mu]\,dR\\
&\quad\pm\frac{\pi}{(l_1+2)(l_2+2)}\delta_{m_1,0}\int_0^{\operatorname{arctanh}e^\nu}\int_0^\pi V_{l_1,1}[\mu]V_{l_2,1}[\mu]\,dR\\
&\quad+\frac{\pi}{2p_1(l_2+m_1+1)}\delta_{m_1,m_2}\int_0^{\operatorname{arctanh}e^\nu}\int_0^\pi V_{l_1,m_1+1}[\mu]V_{l_2,m_1+1}[\mu]\,dR,
\end{aligned}
$$

where $dR=\mu^3(\cosh^2\eta-\cos^2\vartheta)\sin\vartheta\,\sinh\eta\,d\vartheta\,d\eta$.

Using Proposition 12.2.1, and applying again the orthogonality of Theorem 12.2.3, we are left with

$$\iiint_{\Omega_\mu} \Big([\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_1[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_1 + [\mathbf{X}^{\pm}_{l_1,m_1}[\mu]]_2[\mathbf{X}^{\pm}_{l_2,m_2}[\mu]]_2\Big)\, dx$$
$$= \frac{\pi\mu^{2l_1+3}}{(l_1+2)(2l_1+1)!!(2l_1+3)!!}$$
$$\times \Big[(l_1+2)(l_1+m_1+1)!$$
$$\Big((l_1+m_1)(l_1+m_1+1)(l_1-m_1+3)!I_{l_1,m_1-1}$$
$$+ (l_1-m_1+1)!I_{l_1,m_1+1}\Big) + 2(l_1+1)!(l_1+2)!I_{l_1,1}\delta_{0,m}\Big]\delta_{m_1,m_2}\delta_{l_1,l_2} \tag{12.3.7}$$

with $I_{l,m}$ defined in (12.2.17). Combining (12.3.6) and (12.3.7), we conclude that

$$\langle \mathbf{X}^{+}_{l_1,m_1}[\mu], \mathbf{X}^{+}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)} = 0$$

when $l_1 \neq l_2$ or $m_1 \neq m_2$. Similarly, $\langle \mathbf{X}^{-}_{l_1,m_1}[\mu], \mathbf{X}^{-}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)} = 0$ when $l_1 \neq l_2$ or $m_1 \neq m_2$.

Using once more the orthogonality of the system $\{\Phi^{\pm}_m\}$ on $[0, 2\pi]$, we conclude that

$$\langle \mathbf{X}^{\pm}_{l_1,m_1}[\mu], \mathbf{X}^{\mp}_{l_2,m_2}[\mu]\rangle_{L_2(\Omega_\mu)} = 0$$

when the indices do not coincide. The calculation of the norms comes from taking $l_1 = l_2$ and $m_1 = m_2$ in (12.3.7) and adding the expression (12.2.16). By the symmetric form taken by $\mathbf{X}^{\pm}_{l,m}[\mu]$ in (12.3.4), we know that when $m \neq 0$,

$$\|\mathbf{X}^{+}_{l,m}[\mu]\|_{L_2(\Omega_\mu)} = \|\mathbf{X}^{-}_{l,m}[\mu]\|_{L_2(\Omega_\mu)}.$$

The limiting case, $\mu = 0$, follows with the use of Eq. (12.2.19). □

The solid spherical monogenics $\mathbf{X}^{\pm}_{l,m}[0]$ are embedded generically in this one-parameter family of spheroidal monogenics. In contrast, in treatments such as [16, 25, 36, 37, 44, 46], the spheroidal monogenics degenerate as the eccentricity of the spheroid decreases.

For a general orientation, the reader is urged to read some of the existing works where the spherical monogenics emerged [7, 9, 10]. It is worth mentioning that at the time of the publications [8–10] a closed-form representation corresponding to the $\mathbf{X}^{\pm}_{l,m}[0]$ in terms of the basic solid spherical harmonics (12.2.6), originally stated in [39], were not at disposal for the investigation of some basic properties of these functions. They played a fundamental role in [19, 20, 22, 39, 40] (cf. [35]) in the study of higher-dimensional counterparts of the well-known Bohr theorem, Borel-Carathéodory's theorem and Hadamard real part theorems on the majorant of a Taylor's series, as well as Bloch's theorem, in the context of Quaternionic Analysis, where they were investigated in detail. In a different context, orthogonal Appell bases of monogenic polynomials were constructed in [1], [4] and [41] (cf. [2, 3]) using a basis of quaternionic-valued spherical monogenics orthogonal with respect to the quaternionic inner product

$$\langle \mathbf{f}, \mathbf{g} \rangle_{[0]} = \iiint_{\Omega_0} \bar{\mathbf{f}} \mathbf{g} \, dx.$$

These bases were rediscovered in [29] (cf. [5, 30]) using a different algebraic approach based on *Gelfand-Tsetlin schemes*.

In [32] it is shown that the dimension of the space $\mathcal{M}^{(l)}$ of homogeneous monogenic polynomials of degree l in x_0, x_1, x_2 is $2l + 3$ (this does not depend on the domain). Since the polynomials $\mathbf{X}^{\pm}_{l,m}[\mu]$ are not homogeneous, we consider the space

$$\mathcal{M}^{(l)}_{*} = \bigcup_{0 \le k \le l} \mathcal{M}^{(l)}$$

of monogenic polynomials of degree l, a class which is not altered by adding monogenic polynomials of lower degree. Thus

$$\dim \mathcal{M}^{(l)}_{*} = \sum_{k=0}^{l} (2k + 3) = (l + 3)(l + 1). \qquad (12.3.8)$$

Consider the collections of $2k + 3$ polynomials

$$\mathbf{B}_k[\mu] := \{\mathbf{X}^{+}_{k,m}[\mu],\ 0 \le m \le k + 1\} \cup \{\mathbf{X}^{-}_{k,m}[\mu],\ 1 \le m \le k + 1\}.$$

By Theorem 12.3.1 and (12.3.8), the union

$$\bigcup_{0 \le k \le l} \mathbf{B}_k[\mu] \qquad (12.3.9)$$

is an orthogonal basis for $\mathcal{M}^{(l)}_{*}$. In addition, $\mathcal{M}^{(l)}_{*}$ is dense in $L_2(\Omega_\mu) \cap \mathcal{M}(\Omega_\mu)$. Therefore the following result, which will be of use in the further discussion, can now be established:

Proposition 12.3.2 *For a fixed μ, the function set (12.3.9) forms an orthogonal basis of $L_2(\Omega_\mu) \cap \mathcal{M}(\Omega_\mu)$.*

Furthermore, it would be useful in practice if the foregoing orthogonal basis (12.3.9) has the *Appell property* also. It was shown in [46] that there does not exist an orthogonal Appell basis in the case of spaces of solid oblate spheroidal monogenics. We shall proceed in such a manner that we compute the hypercomplex derivative of a spheroidal monogenic of degree l and show, as expected, that the obtained polynomial is not a member of the family with degree $l-1$ like in cases of Appell bases [4, 7, 8, 10]. We find that the hypercomplex derivative of a basic spheroidal monogenic is a combination of $[(l-m)/2]+1$ spheroidal monogenics of lower degrees. Basically, it can be represented by all polynomials of degree at most $l-1$.

Theorem 12.3.3 *For a fixed μ, the hypercomplex derivative of $\mathbf{X}^{\pm}_{l,m}[\mu]$ has the form:*

$$(\frac{1}{2}\partial)\mathbf{X}^{\pm}_{l,m}[\mu] = \sum_{k=0}^{[\frac{l-m}{2}]} v_{l,m,k}\, \mu^{2k}\mathbf{X}^{\pm}_{l-1-2k,m}[\mu], \tag{12.3.10}$$

where the constants $v_{l,m,k}$ are given by (12.2.15)*.*

Proof Since $\partial/\partial x_0$ is a linear operator, we find, by Theorem 12.2.2, the relation:

$$\frac{\partial}{\partial x_0} V^{\pm}_{l,m}[\mu] = \sum_{k=0}^{[\frac{l-m}{2}]} v_{l,m,k}\, \mu^{2k} V^{\pm}_{l-1-2k,m}[\mu].$$

The rest of the proof is straightforward. □

An advantage of Eq. (12.3.10) is that it furnishes a concise expression for the hypercomplex derivatives of the basic monogenic spheroidal polynomials by means of which many of their properties may be easily investigated.

The next proposition shows that there are two *hyperholomorphic constants* among the basic spheroidal monogenic polynomials, i.e., functions whose hypercomplex derivative is identically zero.

Proposition 12.3.4 *For a fixed μ, $\mathbf{X}^{\pm}_{l,l+1}[\mu]$ are hyperholomorphic constants.*

Proof The proof is a consequence of Theorem 12.3.3. □

It can be further shown that $\mathbf{X}^{\pm}_{l,l+1}[\mu] = \mathbf{X}^{\pm}_{l,l+1}[0]$; that is, the hyperholomorphic constants $\mathbf{X}^{\pm}_{l,l+1}[\mu]$ do not depend on the parameter μ.

The hypercomplex derivatives of the prescribed monogenic polynomials in its extended signification being thus computed, no difficulties can arise in restricting it to a particular limiting case. In fact, when $\mu = 0$, we have readily [7, 10]:

$$(\frac{1}{2}\partial)\mathbf{X}^{\pm}_{l,m}[0] = (l+m+1)\mathbf{X}^{\pm}_{l-1,m}[0]. \tag{12.3.11}$$

The reader might find without any additional work that, using (12.3.11) and setting for each $l \geq 0, 0 \leq m \leq l+1$,

$$\mathbf{Y}^{\pm}_{l,m} := \frac{l!(m+1)!}{(l+m+1)!}\mathbf{X}^{\pm}_{l,m}[0], \tag{12.3.12}$$

the equality follows:

$$(\frac{1}{2}\partial)\mathbf{Y}^{\pm}_{l,m} = l\,\mathbf{Y}^{\pm}_{l-1,m}. \tag{12.3.13}$$

Thus the application of the hypercomplex derivative to $\mathbf{Y}^{\pm}_{l,m}$ results again in a real multiple of the similar function one degree lower [41]. The special normalization (12.3.13) is called *Appell property*. In [8] it is proved that the solid spherical monogenics (12.3.12) form, indeed, an *orthogonal Appell basis* for $\mathcal{M}^{(l)}(\Omega_0)$, $l \geq 0$. In [1] and [3], fundamental recursion formulas were obtained for the elements of the prescribed Appell basis.

We turn now to show that the Appell property holds for a part of the $\mathbf{X}^{\pm}_{l,m}[\mu]$ (providing the prescribed normalization (12.3.12)).

Corollary 12.3.5 *Let μ be fixed. For $l - m = 0, 1$, the hypercomplex derivatives of $\mathbf{X}^{\pm}_{l,m}[\mu]$ follow the rule*

$$(\frac{1}{2}\partial)\mathbf{X}^{\pm}_{l,m}[\mu] = (l+1+m)\mathbf{X}^{\pm}_{l-1,m}[\mu].$$

Proof It is an immediate consequence of Theorem 12.3.3. □

One of our leading results is that the three-dimensional spherical monogenics considered, e.g., in [4, 8, 10] are embedded in the prescribed one-parameter family of internal spheroidal monogenics. Hence, the latter can be naturally seen as an extention of the former functions to arbitrarily spheroidal domains. Further investigations on this topic are now under investigation and will be reported in a forthcoming paper.

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Chapter 13
Newton's Approach to General Algebraic Equations over Clifford Algebras

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Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract There is a short section describing how Newton's method works for algebraic problems over Clifford algebras. There are two applications. Zeros of unilateral polynomials over a Clifford algebra in $\mathbb{R}^8$ and solutions of a Riccati equation over all eight Clifford algebras in $\mathbb{R}^4$.

Keywords Clifford algebras · Newton's method · Algebraic equations over Clifford algebras · Riccati equation

Mathematics Subject Classification (2010) Primary 15A66; Secondary 12E10, 1604

13.1 Introduction

Let $p : \mathbb{R}^N \to \mathbb{R}^N$ be a mapping, where $\mathbb{R}^N$ stands for any Clifford algebra. For introduction to Clifford Algebras, see [1], for algebraic and analytic properties of coquaternion algebra, see [6]. Examples for p are polynomials of a general type, including unilateral and bilateral polynomials. Other examples are matrix equations

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like the Riccati equation

$$p(\mathbf{X}) := \mathbf{A} + \mathbf{B}\mathbf{X} + \mathbf{X}\mathbf{C} + \mathbf{X}\mathbf{D}\mathbf{X} = \mathbf{0}, \tag{13.1.1}$$

where the matrix entries are elements from a Clifford algebra in $\mathbb{R}^N$. If $\mathbf{X}$ is a matrix of size $m \times n$, then, in this case, $p : \mathbb{R}^{mnN} \to \mathbb{R}^{mnN}$. The Riccati equation will be treated separately in the last section. For other examples see also [2]. In order to find the solutions of $p(z) = 0$ by Newton's method, the linear system for h

$$p(z) + p'(z)h = \mathbf{0} \tag{13.1.2}$$

has to be solved where in the beginning, z has to be replaced by an arbitrary *guess* $z \in \mathbb{R}^N$ and after having found h as the solution of (13.1.2) the guess has to be replaced by $z := z + h$. In a paper by Lauterbach and Opfer [5], it was shown that $p'(z)h$ is the linear part of $p(z + h)$ with respect to h. To mention an example let $p(z) = z^2 + a_1 z + a_0$. Then,

$$p(z+h) = (z+h)^2 + a_1(z+h) + a_0 = z^2 + hz + zh + h^2 + a_1 z + a_1 h + a_0$$

and the linear part of this expression with respect to h is

$$p'(z)h = hz + zh + a_1 h. \tag{13.1.3}$$

In all cases and independent of the algebra the linear part always consist of a sum with terms of the form ahb. Since, by definition, $p'(z)h$ is a real, linear mapping $\mathbb{R}^N \to \mathbb{R}^N$ in h, the linear part must have a matrix representation

$$p'(z)h = \mathbf{M}_z h \tag{13.1.4}$$

where $\mathbf{M}_z$ is a real $N \times N$ matrix and h is now a real column vector of length N. The matrix $\mathbf{M}_z$ is called the *Jacobi matrix* of (13.1.2). In comparison with the numerical Jacobi matrix which is constructed by replacing $p'(z)h$ by partial derivatives, the Jacobi matrix we use is exact and in addition easy to compute. Even for general, nonunilateral polynomials the matrix representation of $p'(z)h$ is given in [5, Section 4]. And the underlying algebra is not of large importance. In the MATLAB program given in Table 13.4 one can see how to compute the matrix $\mathbf{M}_z$ given in (13.1.4) for the linear term ahb. Let $y = p(z)$. We call the euclidean $\mathbb{R}^N$ norm $||y||$ the *error* of z. Thus, our aim is to find all z with error zero.

13.2 Application of Newton's Method to Polynomials with Clifford Coefficients

Let p now be a unilateral polynomial in any $\mathbb{R}^N$ algebra. In order to find zeros of p we have to use (13.1.2) together with (13.1.4). As an example we use a Clifford algebra in $\mathbb{R}^8$ which we call C_4 and as examples we will be looking for zeros of a quadratic polynomial and a polynomials of degree 7.

Table 13.1 Multiplication table of C_4 for the canonical unit vectors u_k in $\mathbb{R}^8$, $1 \le k \le 8$

u_1	u_2	u_3	u_4	u_5	u_6	u_7	u_8
u_2	$-u_1$	u_4	$-u_3$	u_6	$-u_5$	u_8	$-u_7$
u_3	$-u_4$	$-u_1$	u_2	u_7	$-u_8$	$-u_5$	u_6
u_4	u_3	$-u_2$	$-u_1$	u_8	u_7	$-u_6$	$-u_5$
u_5	$-u_6$	$-u_7$	u_8	$-u_1$	u_2	u_3	$-u_4$
u_6	u_5	$-u_8$	$-u_7$	$-u_2$	$-u_1$	u_4	u_3
u_7	u_8	u_5	u_6	$-u_3$	$-u_4$	$-u_1$	$-u_2$
u_8	$-u_7$	u_6	$-u_5$	$-u_4$	u_3	$-u_2$	u_1

In order to describe the multiplication rules of the algebra C_4 we abbreviate the eight unit vectors units(k) in $\mathbb{R}^8$ also by $u_k,\ k = 1, 2, \ldots, 8$, and the multiplication rules are listed in Table 13.1.

The elements in the C_4 algebra for use in the MATLAB program, which is printed at the end of this section as Table 13.4, page 273, will have the name

$$a = c4([a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8]), a_j \in \mathbb{R}, j = 1, 2, \ldots, 8 \tag{13.2.1}$$

or in short

$$a = c4(h), h \in \mathbb{R}^8.$$

In order to apply the program, an initial guess `xold` is needed. For a systematic search a random guess is very practical. We always used random integer values as components of `xold`. This can be done by the MATLAB command `xold=c4(round(40*(rand(1,8)-0.5)))`, where `rand` is the MATLAB command for random (uniformly distributed) numbers in [0, 1]. The given expression for `xold` produces an integer row vector with eight entries in $[-20, 20]$. We have applied the program to two examples, a quadratic polynomial and a polynomial of degree 7. In all cases there are several solutions.

Example Let the quadratic polynomial p be defined by

$$p(z) = z^2 + c4([2, 3, 5, 7, 11, 13, 17, 19])z + u_8. \tag{13.2.2}$$

We found the following four zeros (see Table 13.2) all in less than 20 steps by applying Newton's method. We never found a singular Jacobi matrix. Whether there are more solutions as given, it is an open question.

Example We selected the following polynomial

$$p(z) = u_1 z^7 + u_2 z^6 + u_3 z^5 + u_4 z^4 + u_5 z^3 + u_6 z^2 + u_7 z + u_8. \tag{13.2.3}$$

Table 13.2 Zeros of p, a quadratic polynomial defined in (13.2.2)

−1.981359548776446,	−2.983544918358173,	−5.012933206449555,	−6.989595272862536,
−10.993692804341773,	−13.005505685207023,	−16.997886278535510,	−18.997172393977728.
−0.018640451223553,	−0.016455081641829,	0.012933206449553,	−0.010404727137464,
−0.006307195658224,	0.005505685207025,	−0.002113721464491,	−0.002827606022272.
8.497172393977728,	−9.997886278535509,	4.005505685207025,	−8.993692804341777,
−8.989595272862536,	−3.987066793550447,	−9.983544918358172,	−8.518640451223554.
−10.497172393977728,	6.997886278535509,	−9.005505685207025,	1.993692804341777,
−2.010404727137465,	−9.012933206449553,	−7.016455081641829,	−10.481359548776446.

Two consecutive lines, separated by a dot . and a small skip define one zero

We can guess already one solution, namely $z = u_4$. If we look at Table 13.1 we see that $u_4^2 = -1 \Rightarrow u_4^4 = 1 \Rightarrow u_4^5 = u_4 \Rightarrow u_4^6 = u_4^2 = -1 \Rightarrow u_4^7 = u_4^3 = -u_4$, thus,

$$p(u_4) = -u_4 - u_2 + u_3u_4 + u_4 - u_5u_4 - u_6 + u_7u_4 + u_8 =$$
$$= -u_4 - u_2 + u_2 + u_4 - u_8 - u_6 + u_6 + u_8 = 0.$$

We applied the given program to (13.2.3) and in less than 40 Newton iterations for each zero we found the solutions given in Table 13.3. It contains 18 zeros of p. These are found by using random integer guesses. Whether there are more zeros than indicated, it is an open problem. We did not encounter one example which did not converge. The convergence behavior was typical for Newton's method. If the error was under a certain limit, say 10^{-2}, then there were only few remaining steps so that almost machine precision was reached. Therefore, in the two Tables 13.2, 13.3 the original MATLAB results are presented. The advantage of using Newton's method for finding zeros of polynomials with arbitrary algebra coefficients is its easy use and its high precision. There is another advantage. If with some other method a zero is computed it is easy to apply few additional Newton steps in order to reduce the error. The disadvantage is, that we do not know whether there are still other, nondiscovered zeros. We note, that the frequently appearing number 0.353553390593274 in Table 13.3 is $\frac{\sqrt{2}}{4}$ (Table 13.4).

13.3 Application to Riccati Equation

In the Riccati equation over a Clifford algebra in $\mathbb{R}^N$, given in (13.1.1), we can choose the sizes of the corresponding matrices as follows:

$$\mathbf{X} : m \times n \Rightarrow \mathbf{A} : m \times n,\ \mathbf{B} : m \times m,\ \mathbf{C} : n \times n,\ \mathbf{D} : n \times m.$$

The linear part of $p(\mathbf{X} + \mathbf{H})$ is easy to compute:

$$p'(\mathbf{X})\mathbf{H} = \mathbf{H}(\mathbf{C} + \mathbf{D}) + (\mathbf{B} + \mathbf{XD})\mathbf{H} = \mathbf{M}_1\mathbf{H} + \mathbf{M}_2\mathbf{H} = (\mathbf{M}_1 + \mathbf{M}_2)\mathbf{H}.$$

In this case the matrices $\mathbf{M}_1,\ \mathbf{M}_2$ are of order mnN and $\mathbf{H}$ is a column vector of dimension mnN. For more details see also [5], Theorem 8.1 and [3]. In [5], also quotations to the relevant literature is given. We will use an example defined in all eight $\mathbb{R}^4$ algebras which have the names and notations:

1. Quaternions, $\mathbb{H}$,
2. Coquaternions, $\mathbb{H}_{coq}$,
3. Tessarines, $\mathbb{H}_{tes}$,

Table 13.3 Table of zeros of p, a polynomial of degree 7, defined in (13.2.3)

$\pm$(0,	0,	0,	0.923879532511287,
0,	0,	0,	−0.382683432365090).
0.353553390593274,	0.353553390593274,	0,	0.5,
−0.5,	0,	0.353553390593274,	−0.353553390593274.
$\pm$(0.382683432365090,	0,	0,	0,
0.923879532511287,	0,	0,	0).
0.191341716182520,	0,	0,	0.961939766255643,
−0.038060233744357,	0,	0,	−0.191341716182520.
0,	0,	0,	1,
0,	0,	0,	0.
−0.191341716182545,	0,	0,	0.038060233744357,
−0.961939766255643,	0,	0,	0.191341716182545.
$\pm$(0,	$\sqrt{0.5}$,	0,	0,
0,	0,	0,	$-\sqrt{0.5}$).
0.191341716182545,	0,	0,	0.038060233744357,
0.961939766255643,	0,	0,	0.191341716182545.
−0.191341716182545,	0,	0,	0.961939766255643,
0.038060233744357,	0,	0,	−0.191341716182545.
−0.353553390593272,	0.353553390593274,	0,	0.5,
0.5,	0,	−0.353553390593275,	−0.353553390593272.
$\pm$(0.544895106775818,	0.353553390593275,	0,	−0.461939766255644,
0.461939766255644,	0,	0.353553390593272,	−0.162211674410729).
$\pm$(-0.162211674410729,	0.353553390593274,	0,	0.461939766255643,
0.461939766255643,	0,	−0.353553390593274,	−0.544895106775819).
−0.353553390593272,	−0.353553390593274,	0,	0.5,
−0.5,	0,	−0.353553390593275,	0.353553390593272.

Two consecutive lines, separated by a dot . and a small skip define one or two zeros

Table 13.4 MATLAB program for computing zeros of polynomials over coefficients from $\mathbb{R}^8$ algebra C_4 by Newton's method

```
%[xnew,J]=c4_newton(c,xold); c presents the vector of polynomial
%coefficients and xold is an arbitrary initial guess. Computed
%are J, the exact Jacobi matrix at xold, and xnew, the result
%of the application of one Newton step.
%%%
%Basis for the program:
%R. Lauterbach - G. Opfer, AACA 24(2014), pp. 1059 - 1073.
%The polynomials have the form (highest coefficient first)
%%%
%p(x) = c_1x^n + c_2x^{n-1} + ... + c_{n+1}.
%%%
%This programm works in principle for all geometric R^N algebras,
%provided the corresponding algebraic rules have been transmitted
%to MATLAB by a technique called "overloading".
%%%%=============================================================
%function [xnew,J]=c4_newton(c,xold); %J is the exact Jacobi matrix
%  n=length(c)-1;
%  global N;
%  dim=8;
%  N=dim;
%  J=zeros(N,N);
%  for ell=n:-1:1
%    J=J+derivative_of_xpowerj(ell,xold,c(n-ell+1));
%  end;
%  y=Polyval(c,xold);
%  if abs(det(J))>=1e-10
%    h=-J\col(y);
%    xnew=xold+c4(h);
%    else
%    error('Jacobi matrix J is near to singular');
%    %This happens almost never!
%  end;
%
%%%%=============================================================
%function M=derivative_ahb_in_matrixform(a,b); %a,b depend on x
%global N;
%M=[];
%for k=1:N
%M=[M,col(a*units(k)*b)]; %M is real NxN matrix
%end;
%
%%%%=============================================================
%function M=derivative_of_xpowerj(j,x,factor);
%  %M is  real NxN Matrix.
%  %factor stands for polynomial coefficients.
%  global N;
%  M=zeros(N,N);
%  for k=1:j
%    a=factor*x^(k-1); b=x^(j-k);
%    M=M+derivative_ahb_in_matrixform(a,b);
%  end;}
%
%%%%=============================================================
```

Table 13.5 Definition of the coefficients of algebraic Riccati equation

$$\mathbf{A} := \begin{bmatrix} (\ \ 1,-4,-2,\ \ 0), & (-1,\ \ 4,\ \ 3,\ \ 5), & (\ \ 2,-3,\ \ 2,-5) \\ (-1,-1,\ \ 3,\ \ 3), & (\ \ 2,-3,-4,\ \ 0), & (\ \ 4,\ \ 5,\ \ 0,-4) \end{bmatrix};$$

$$\mathbf{B} := \begin{bmatrix} (\ \ 3,-3,\ \ 4,-2), & (-3,-2,\ \ 1,\ \ 0) \\ (-1,\ \ 3,\ \ 1,\ \ 0), & (\ \ 4,-2,\ \ 3,\ \ 3) \end{bmatrix};$$

$$\mathbf{C} := \begin{bmatrix} (-1,\ \ 1,-4,-4), & (\ \ 0,\ \ 3,\ \ 4,-4), & (\ \ 1,\ \ 0,-5,-2) \\ (-3,\ \ 3,-2,\ \ 0), & (-3,\ \ 1,-2,\ \ 2), & (\ \ 2,\ \ 2,\ \ 0,-4) \\ (-3,\ \ 4,-3,\ \ 3), & (\ \ 0,\ \ 5,-4,-1), & (-4,\ \ 5,-5,\ \ 3) \end{bmatrix};$$

$$\mathbf{D} := \begin{bmatrix} (\ \ 5,\ \ 5,-3,\ \ 5), & (-3,\ \ 0,-1,\ \ 1) \\ (\ \ 2,-5,\ \ 3,\ \ 4), & (\ \ 5,-2,\ \ 1,-3) \\ (-2,-5,-4,\ \ 3), & (-4,-2,\ \ 3,-2) \end{bmatrix}.$$

4. Cotessarines, $\mathbb{H}_{\text{cotes}}$,
5. Nectarines, $\mathbb{H}_{\text{nec}}$,
6. Conectarines, $\mathbb{H}_{\text{con}}$,
7. Tangerines, $\mathbb{H}_{\text{tan}}$,
8. Cotangerines, $\mathbb{H}_{\text{cotan}}$.

It should be noted, that the algebras numbered 3, 4, 7, 8 are commutative. About the names and the multiplication rules see [4, 7]. We quote an example from [5].

Example We choose $m = 2, n = 3$ and define the algebraic Riccati equation by $\mathbf{A} \in \mathcal{A}^{2\times 3}, \mathbf{B} \in \mathcal{A}^{2\times 2}, \mathbf{C} \in \mathcal{A}^{3\times 3}, \mathbf{D} \in \mathcal{A}^{3\times 2}$, where the matrix entries are given in Table 13.5.

The data of Table 13.5 are randomly chosen integers in $[-5, 5]$. We choose $\mathbf{X} = \mathbf{0}$ as initial guess with the exception of Algebra 6, where another guess is used. Newton's method converges then in all 8 cases. The solutions are given in Table 13.6. These solutions are not necessarily the only solutions.

Table 13.6 Solutions $\mathbf{X} \in \mathcal{A}^{2\times 3}$ of the algebraic Riccati equation in all eight $\mathbb{R}^4$ algebras

Pos.	Quaternions $\mathbb{H}$	Coquaternions $\mathbb{H}_{coq}$
x_{11}	(−0.1045, −0.1495, 0.2660, 0.0712)	(0.1544, 0.6186, −0.0883, 0.3512)
x_{21}	(0.3071, 0.3399, −0.6350, 0.1765)	(0.2732, 0.3379, −0.5506, 0.2582)
x_{12}	(0.2731, −0.1913, 0.1003, 0.6167)	(0.1612, 0.0891, −0.6692, −0.2445)
x_{22}	(0.5842, 0.7970, −0.0900, −0.4788)	(0.5916, 1.3237, 0.1198, 0.6950)
x_{13}	(0.9956, 0.1306, 0.3851, 0.6174)	(−0.1351, 0.2386, −0.6664, 0.8003)
x_{23}	(−0.5317, −0.3326, −0.5753, −0.3978)	(−1.2738, 0.5379, 0.6102, 1.0119)

Pos.	Tessarines $\mathbb{H}_{tes}$	Cotessarines $\mathbb{H}_{cotes}$
x_{11}	(−0.8748, −0.1261, 0.0210, 0.0205)	(0.0749, 0.5097, 0.8189, −0.7321)
x_{21}	(−0.1044, −0.4106, 1.0738, −0.9254)	(0.1177, −0.0515, −0.7539, 0.0017)
x_{12}	(0.1632, 0.0329, −0.1188, −0.4736)	(−1.7670, 0.0836, 0.3282, −1.2972)
x_{22}	(0.0282, 0.4299, 0.3035, 0.4502)	(0.3680, 0.1663, 0.7493, 1.4654)
x_{13}	(−0.6189, 0.5054, −0.8030, −0.4800)	(1.0919, −1.0135, −0.5820, 1.0441)
x_{23}	(0.8554, −0.0146, 0.2011, −1.1011)	(−0.5413, −0.2660, 0.0956, −0.2739)

Pos.	Nectarines $\mathbb{H}_{nec}$	Conectarines $\mathbb{H}_{con}$
x_{11}	(0.1490, −0.1557, 0.6250, 0.3262)	(1.3763, −0.5053, 0.8822, −1.0762)
x_{21}	(0.1598, −0.5805, −0.9796, −1.2966)	(0.4270, 0.0322, 0.5237, 0.8558)
x_{12}	(0.5285, 0.1447, −0.4479, 0.0074)	(1.5484, −4.3525, −1.4404, 5.4397)
x_{22}	(−0.2607, −0.4621, −0.8051, −0.4240)	(−0.5729, 1.4386, 1.0001, −1.7677)
x_{13}	(−0.2660, −0.3224, 0.3163, 0.3754)	(1.7588, −0.5349, −0.3498, 0.8295)
x_{23}	(−0.7931, −0.9394, 0.4466, 0.6899)	(−0.3289, 0.4151, −0.0512, −0.3035)

Pos.	Tangerines $\mathbb{H}_{tan}$	Cotangerines $\mathbb{H}_{cotan}$
x_{11}	(0.0238, 0.1553, −0.6713, 0.3590)	(−0.4374, −0.8677, −0.4367, 0.8265)
x_{21}	(0.3589, −0.1427, −0.2777, −0.6520)	(−0.4075, 0.6499, 0.5927, −0.0469)
x_{12}	(−0.1408, −0.0282, 0.2520, 0.8164)	(0.2330, −0.6134, −0.4203, 0.2160)
x_{22}	(−0.1875, −0.0557, 0.5904, 0.4577)	(−0.4898, −0.1886, 0.2013, 0.4882)
x_{13}	(0.1099, −0.4139, 0.0090, −0.1696)	(0.2155, −0.2357, −0.4028, 0.0678)
x_{23}	(−0.1393, −0.2064, 0.0938, 0.0396)	(0.2917, 0.1488, 0.3577, −0.2748)

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Part IV
Differential Geometry

Chapter 14
Connections in Euclidean and Non-commutative Geometry

Viktor Abramov and Olga Liivapuu

To the 70th birthday of professor Wolfgang Sprössig

Abstract In this paper, we trace the development of concepts of differential geometry such as a first order differential calculus, an algebra of differential forms and a connection from Euclidean geometry to noncommutative geometry. We begin with basic structures of differential geometry in n-dimensional Euclidean space such as vector field, differential form, connection and then, having explained a general idea of non-commutative geometry, we show how these notions can be developed in the assumption that the algebra of smooth functions on Euclidean space is replaced by its non-commutative analog and the differential graded algebra of differential forms is replaced by a q-differential graded algebra, where q is a primitive Nth root of unity and a differential d of this algebra satisfies the equation $d^N = 0$.

Keywords Vector fields · Differential forms · Connections · Differential graded algebra · q-Differential graded algebra · Non-commutative differential calculus · Quantum hyperplane

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14.1 Introduction

A theory of connections is very important part of modern differential geometry. The discovery of interconnection between this theory and the gauge field theories has given a powerful impetus to development of the entire theory. In the present paper we track the development of a concept of connection from the simplest case of the canonical connection in the n-dimensional Euclidean space $\mathbb{R}^n$ to its generalizations in non-commutative geometry. The present paper is written on the basis of lectures delivered to the doctoral level students of Division of Applied Mathematics of University of Mälardalen within the framework of NordPlus Higher Education Program 2017.

In the first subsections of Sect. 14.2 we describe the structures of elementary differential geometry of n-dimensional Euclidean space $\mathbb{R}^n$ such as vector fields and differential 1-forms. We lay particular stress on algebraic aspects of these structures and bring a reader to an idea of algebraic structure which is called a first order differential calculus over an algebra. Then we explain a basic idea of non-commutative geometry, where the commutative algebra of smooth functions $C^\infty(\mathbb{R})^n$ on $\mathbb{R}^n$ is replaced by a non-commutative algebra. We show that in order to construct the differential 1-forms on a non-commutative space we should have a coordinate first order differential calculus with right partial derivatives over a non-commutative algebra. As an example of a coordinate first order differential calculus with right partial derivatives we consider the differential calculus on the quantum hyperplane. In Sect. 14.3 we describe the algebra of differential forms with exterior differential in the n-dimensional Euclidean space $\mathbb{R}^n$ and its generalization in an approach of non-commutative geometry, which is called a higher order differential calculus over an algebra. Particularly we explain the notion of the universal differential graded algebra. In Sect. 14.4 we describe the canonical connection in n-dimensional Euclidean space and derive its Cartan's structure equations. In Sect. 14.5 we develop a generalization of the theory of connection on modules with the help of the concept of q-differential graded algebra, where q is a primitive Nth root of unity.

14.2 Vector Fields, Differential 1-Forms in $\mathbb{R}^n$ and Non-commutative First Order Differential Calculus

In this section we describe the Lie algebra of smooth vector fields in n-dimensional Euclidean space $\mathbb{R}^n$ and an approach of non-commutative geometry to a first order differential calculus in a non-commutative space. In what follows we will use the Einstein summation convention over repeated subscript and superscript.

14.2.1 Vector Fields in the n-Dimensional Euclidean Space $\mathbb{R}^n$

Consider the n-dimensional space $\mathbb{R}^n$. This space has the structure of n-dimensional vector space with the component-wise addition of two vectors and the component-wise multiplication by real numbers. If we consider an element $(v^1, v^2, \ldots, v^n)$ of $\mathbb{R}^n$ as a vector then it will be denoted by $\vec{v} = (v^1, v^2, \ldots, v^n)$. For any two vectors $\vec{v} = (v^1, v^2, \ldots, v^n)$, $\vec{w} = (w^1, w^2, \ldots, w^n)$ we have the inner product $< \vec{v}, \vec{w} >= \sum_i v^i\, w^i$, which determines the Euclidean structure of $\mathbb{R}^n$. If we do not use the vector space structure of $\mathbb{R}^n$ then an element $(p^1, p^2, \ldots, p^n)$ will be called a point of $\mathbb{R}^n$ and denoted by $p = (p^1, p^2, \ldots, p^n)$. The coordinate functions of $\mathbb{R}^n$ will be denoted by $x^1, x^2, \ldots, x^n$ and by definition $x^i(p) = p^i$. The canonical basis for the Euclidean vector space $\mathbb{R}^n$ will be denoted by $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$, where the ith component of $\vec{e}_i$ is 1 and the others are zeros.

Let $U \subset \mathbb{R}^n$ be an open subset. A real-valued function $f : U \to \mathbb{R}$ is called a smooth function if it has continuous partial derivative of any order. The set of all smooth functions on the n-dimensional Euclidean space $\mathbb{R}^n$ will be denoted by $C^\infty(\mathbb{R}^n)$. The set $C^\infty(\mathbb{R}^n)$ endowed with the pointwise addition of smooth functions and the multiplication of a smooth function by a real number is the infinite dimensional vector space. We remind that a vector space $\mathscr{A}$ is said to be a *unital associative algebra* if $\mathscr{A}$ is equipped with a product $a \cdot b$, where $a, b \in \mathscr{A}$, such that $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (associativity), and this product has the identity element e satisfying $a \cdot e = e \cdot a = a$. If, in addition to associativity, the product $a \cdot b$ of any two elements is commutative, i.e. $a \cdot b = b \cdot a$, a unital associative algebra $\mathscr{A}$ is called *commutative*. The vector space $C^\infty(\mathbb{R}^n)$ of smooth functions endowed with the product fg of two smooth functions f, g, which is defined by $(fg)(p) = f(p)\, g(p)$, is the commutative unital associative algebra, where the identity element is the function, whose value at any point of the space is 1. If we ignore the multiplication of smooth functions by scalars (real numbers) then $C^\infty(\mathbb{R}^n)$ is the *commutative unital associative ring*.

A *tangent vector* to the n-dimensional Euclidean space $\mathbb{R}^n$ at a point $p \in \mathbb{R}^n$ is a pair $(p; \vec{v}) \in \mathbb{R}^n \times \mathbb{R}^n$, which will be denoted by $\vec{v}_p$, i.e. $\vec{v}_p = (p; \vec{v})$. A tangent space of all tangent vectors to $\mathbb{R}^n$ at a point p will be denoted by $\mathrm{T}_p\mathbb{R}^n$. The vector space and Euclidean structure of $\mathbb{R}^n$ can be extended to any tangent space $\mathrm{T}_p\mathbb{R}^n$ in the natural way

$$\vec{v}_p + \vec{w}_p = (p; \vec{v} + \vec{w}), \quad a\vec{v}_p = (p; a\vec{v}), \quad < \vec{v}_p, \vec{w}_p >=< \vec{v}, \vec{w} >, \quad a \in \mathbb{R}.$$

Then any tangent vector $\vec{v}_p = (p; \vec{v}) = (p; v^1, v^2, \ldots, v^n)$ can be expressed as $\vec{v}_p = v^i\, \vec{e}_{p,i}$, where $\vec{e}_{p,i} = (p; \vec{e}_i)$. The disjoint union of tangent spaces

$$\mathrm{T}\mathbb{R}^n = \cup_{p\in\mathbb{R}^n} \mathrm{T}_p\mathbb{R}^n, \tag{14.2.1}$$

will be referred to as the *tangent bundle over the n-dimensional Euclidean space* $\mathbb{R}^n$. The projection $\pi : \mathrm{T}\mathbb{R}^n \to \mathbb{R}^n$ is defined by $\pi(\vec{v}_p) = p$. A *section* of the tangent bundle $\mathrm{T}\mathbb{R}^n$ is a smooth mapping $X : \mathbb{R}^n \to \mathrm{T}\mathbb{R}^n$ such that $\pi \circ X = \mathrm{id}_{\mathbb{R}^n}$. A smooth section X of the tangent bundle $\mathrm{T}\mathbb{R}^n$ is called a *vector field* in the Euclidean space $\mathbb{R}^n$. Obviously any vector field X is uniquely determined by n smooth functions $X^1, X^2, \ldots, X^n$ such that

$$X : p \in \mathbb{R}^n \mapsto X_p = (p; X^1(p), X^2(p), \ldots, X^n(p)) \in \mathrm{T}_p\mathbb{R}^n.$$

The functions $X^1, X^2, \ldots, X^n$ will be called the components of a vector field X. The vector space structure of a tangent space $\mathrm{T}_p\mathbb{R}^n$ induces the vector space structure in the set $\mathfrak{D}$ of all vector fields.

Let $\mathscr{M}$ be an Abelian group and $\mathscr{A}$ be a unital associative ring. We remind that $\mathscr{M}$ together with a mapping $(a, u) \in \mathscr{A} \times \mathscr{M} \mapsto a \cdot u \in \mathscr{M}$ satisfying

$$(a+b)\cdot u = a\cdot u + b\cdot u, a\cdot(u+v) = a\cdot u + a\cdot v, (ab)\cdot u = a\cdot(b\cdot u), e\cdot u = u,$$

where $a, b \in \mathscr{A}$, $u, v \in \mathscr{M}$ and e is the identity element of $\mathscr{A}$, is called a left $\mathscr{A}$-module. A notion of right $\mathscr{A}$-module is defined in a similar manner. $\mathscr{A}$*-bimodule* is an Abelian group $\mathscr{M}$, which is both left and right $\mathscr{A}$-module and $(a\cdot u)\cdot b = a\cdot(u\cdot b)$. One can extend the notion of a module (left, right or bimodule) over a ring to a notion of a module over a unital associative algebra assuming that in this case $\mathscr{A}$, $\mathscr{M}$ are vector spaces and scalars commute with everything. A left $\mathscr{A}$-module $\mathscr{M}$ is referred to as *finitely generated* if there is a set $\{u_1, u_2, \ldots, u_n\}$ of elements of $\mathscr{M}$ such that any element u of $\mathscr{M}$ can be written as $u = a_1\, u_1 + a_2\, u_2 + \ldots + a_n\, u_n$, where $a_1, a_2, \ldots, a_n \in \mathscr{A}$. If $\{u_1, u_2, \ldots, u_n\}$ are linearly independent (over $\mathscr{A}$) then a finitely generated left $\mathscr{A}$-module $\mathscr{M}$ is called a *free module with a basis*. If a ring $\mathscr{A}$ has an invariant basis number then the cardinality of any basis for free left $\mathscr{A}$-module is called the *rank of a free module*.

It turns out that the concept of a module is applicable to the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ and the vector space of vector fields $\mathfrak{D}$, and plays an important role in constructing noncommutative generalizations of vector fields. Indeed given a smooth function f and a vector field X one can define the product fX (left multiplication of vector fields by smooth functions) as the vector field $fX : p \mapsto f(p)\, X_p$. It is easy to check that this product defines the structure of left $C^\infty(\mathbb{R}^n)$-module in $\mathfrak{D}$. Analogously one can define a structure of right $C^\infty(\mathbb{R}^n)$-module in $\mathfrak{D}$, and then from $f(p)\, X_p = X_p\, f(p)$ (numbers commute with vectors) it follows that in the case of the Euclidean space $\mathbb{R}^n$ functions commute with vector fields $fX = Xf$.

Let us define the vector fields $E_i, i = 1, 2, \ldots, n$ by the formula $E_i(p) = \vec{e}_{p,i}$, where $p \in \mathbb{R}^n$. Making use of the left multiplication of vector fields by smooth functions one can express any vector field X, whose components are functions $X^1, X^2, \ldots, X^n$, in the form $X = X^i\, E_i$. Evidently the vector fields $E_i, i = 1, 2, \ldots, n$ are linearly independent (over the ring of smooth functions). The ring of smooth functions $C^\infty(\mathbb{R}^n)$ is commutative, hence it has an invariant basis number, and consequently the formula $X = X^i\, E_i$ shows that $\mathfrak{D}$ is the free left $C^\infty(\mathbb{R}^n)$-module of rank n. We can consider the basis $\{E_1, E_2, \ldots, E_n\}$ for the free left

$C^\infty(\mathbb{R}^n)$-module $\mathfrak{D}$ as the *frame field*, i.e. as the mapping which attaches to each point p of the Euclidean space the frame $\{\vec{e}_{p,i}\}_{i=1}^n$ of tangent space $\mathrm{T}_p\mathbb{R}^n$. We will denote this frame field by E and call it the *canonical frame field* for the tangent bundle $\mathrm{T}\mathbb{R}^n$. Obviously the canonical frame field E is orthonormal, i.e. $< E_i, E_j >= \delta_{ij}$.

14.2.2 Vector Field as the Directional Derivative

Let f be a smooth function and X be a vector field in $\mathbb{R}^n$. A vector field X at a point p is the tangent vector $X_p = (p; \vec{v}) \in \mathrm{T}_p\mathbb{R}^n$. How we can measure a rate of change of a function f at a point p in the direction of tangent vector X_p? For this purpose we can use the *directional derivative of a function*. Evidently there is a parametrized curve $\alpha : I \to \mathbb{R}^n$, where $I \subset \mathbb{R}$ is an open interval, such that $\alpha(0) = p$ (curve passes through a point p), and $\vec{\alpha}'(0) = X_p$ (the tangent vector to a curve at a point p is X_p). For instance one can take the straight line $\alpha(t) = p + t\,\vec{v}$. Then the directional derivative Xf at a point p is defined by

$$Xf\,(p) = \frac{d}{dt}(f \circ \alpha(t))|_{t=0}. \tag{14.2.2}$$

The directional derivative of a function f determines a new smooth function Xf, whose value at any point of the Euclidean space is defined by (14.2.2). Hence a vector field X induces the directional derivative and can be considered as a linear mapping $X : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$, which satisfies the Leibniz rule $X(fg) = (Xf)\;g + f\;(Xg)$. An approach to a vector field X as the directional derivative is very useful because it makes clear an algebraic nature of a vector field. We remind that a linear mapping $\delta : \mathscr{A} \to \mathscr{A}$ of an algebra $\mathscr{A}$ is said to be a *derivation* if it satisfies $\delta(ab) = \delta(a)\,b + a\,\delta(b)$, where $a, b \in \mathscr{A}$. Thus a vector field considered as a directional derivative of a function is a derivation of the algebra $C^\infty(\mathbb{R}^n)$. It can be proved that the vector space of all derivations of the algebra $C^\infty(\mathbb{R}^n)$ coincides with the vector space of vector fields $\mathfrak{D}$, but generally if we consider the algebra of Nth order differentiable functions this is not the case.

It can be shown that if we consider a vector field X in the Euclidean space $\mathbb{R}^n$ as the derivation of the algebra $C^\infty(\mathbb{R}^n)$ then we can identify a vector field X with the first order differential operator

$$X = X^i\,\frac{\partial}{\partial x^i}. \tag{14.2.3}$$

This formula is equivalent to $X = X^i\,E_i$, because the constant vector field E_i, considered as the directional derivative, can be identified with partial derivative $\frac{\partial}{\partial x^i}$. Consequently the vector fields $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ form the basis for the free left $C^\infty(\mathbb{R}^n)$-module $\mathfrak{D}$ of vector fields. From this it follows that at any point p of the

n-dimensional Euclidean space there is the canonical basis $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \ldots, \frac{\partial}{\partial x^n}|_p$ for the tangent space $T_p\mathbb{R}^n$.

Next algebraic structure, which plays an important role in the theory of vector fields, is a Lie algebra. We remind that a vector space $\mathfrak{g}$ is said to be a *Lie algebra* if it is equipped with a *Lie bracket* $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which for any $x, y, z \in \mathfrak{g}$ satisfies

- $[x, y] = -[y, x]$ (*skew-symmetry*),
- $[[x, y], z] + [[y, z], x] + [[z, x], y] = 0$ (*Jacobi identity*).

Consider the *commutator* of two vector fields $[X, Y] = X \circ Y - Y \circ X$. It is easy to verify that because of the symmetry (Schwarz's theorem) of second order partial derivatives the commutator of two vector fields is the vector field. Indeed if $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ then

$$[X, Y] = (X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}) \frac{\partial}{\partial x^j}. \tag{14.2.4}$$

Clearly the commutator is skew-symmetric, and it can be checked by straightforward calculation that it satisfies the Jacobi identity. Hence the commutator of two vector fields is the Lie bracket, and it determines the structure of Lie algebra in $\mathfrak{D}$. It should be pointed out that this Lie algebra is infinite-dimensional.

14.2.3 Differential 1-Forms in the Euclidean Space $\mathbb{R}^n$

A calculus of differential forms is dual to the calculus of vector fields. Let $T_p^*\mathbb{R}^n$ be the dual or *cotangent space* of the tangent space $T_p\mathbb{R}^n$ at a point p. We remind that in the case of finite dimensional vector spaces the dual space V^* of a vector space V is the vector space of all linear $\mathbb{R}$-valued functions on V. We will call the elements of dual space *covectors*. We write an element of the cotangent space $T_p^*\mathbb{R}^n$ at a point p as the pair $(p; \phi)$, where p is a point of Euclidean space and $\phi : \mathbb{R}^n \to \mathbb{R}$ is a linear function, and define $(p; \phi)(\vec{v}_p) = \phi(\vec{v})$, where $\vec{v}_p = (p; \vec{v}) \in T_p\mathbb{R}^n$. Consider the disjoint union

$$T^*\mathbb{R}^n = \cup_{p\in\mathbb{R}^n} T_p^*\mathbb{R}^n,$$

which will be referred to as the *cotangent bundle over the Euclidean space* $\mathbb{R}^n$. Define the projection $\tilde{\pi} : T^*\mathbb{R}^n \to \mathbb{R}^n$ by $\tilde{\pi}(p; \phi) = p$. Then a differential form of degree 1 or 1-form ω is a smooth section of the cotangent bundle $\omega : \mathbb{R}^n \to T^*\mathbb{R}^n$, i.e. it satisfies $\tilde{\pi} \circ \omega = \mathrm{id}_{\mathbb{R}^n}$. Hence a differential 1-form is a smooth mapping $\omega : p \mapsto \omega_p \in T_p^*\mathbb{R}^n$.

The vector space structure of $T_p^*\mathbb{R}^n$ induces the vector space structure in the set of all 1-forms, and this vector space will be denoted by $\Omega^1(\mathbb{R}^n)$. The infinite-dimensional vector space of 1-forms can be considered as a bimodule over the algebra of smooth functions. Indeed given a smooth function f and a 1-form ω one can define the product $f \cdot \omega$ (left multiplication of 1-forms by functions) as the 1-form such that $(f \cdot \omega)_p = f(p)\,\omega_p$, where p is a point of $\mathbb{R}^n$. Since real numbers commute with covectors the right-hand side of this formula can be written as $\omega_p\, f(p)$, which means that we can define the product $\omega \cdot f$ (right multiplication of 1-forms by functions) by simply setting it equal to $f \cdot \omega$. These two products $f \cdot \omega$ and $\omega \cdot f$ determine the $C^\infty(\mathbb{R}^n)$-bimodule structure of $\Omega^1(\mathbb{R}^n)$. Next we define the value of differential 1-form ω on a vector field X as the function $f = \omega(X)$, whose value at a point p is defined by $f(p) = \omega_p(X_p)$. Evidently for any functions g, h and any vector fields X, Y it holds $\omega(gX + hY) = f\,\omega(X) + g\,\omega(Y)$. This shows that a differential 1-form ω determines the homomorphism $\omega : \mathfrak{D} \to C^\infty(\mathbb{R}^n)$ of $C^\infty(\mathbb{R}^n)$-bimodules. Two differential forms ω_1, ω_2 are equal $\omega_1 \equiv \omega_2$ iff for any vector field X it holds $\omega_1(X) = \omega_2(X)$. Hence a 1-form ω is uniquely determined if we show how to compute its value on any vector field X (this dependence on a vector field should be $C^\infty(\mathbb{R}^n)$-linear). We can apply this way of constructing differential 1-forms to functions. Indeed given a function $f \in C^\infty(\mathbb{R}^n)$ we can define the differential 1-form df by means of the formula

$$df(X) = Xf, \tag{14.2.5}$$

where X is a vector field. Hence any smooth function f induces the differential 1-form df, i.e. we have the linear mapping $f \in C^\infty(\mathbb{R}^n) \to df \in \Omega^1(\mathbb{R}^n)$. Because a vector field X is the derivation of the algebra of smooth functions, it holds

$$d(fg)(X) = X(fg) = (Xf)\,g + f\,(Xg) = df(X)\,g + f\,dg(X),$$

or, omitting a vector field X and making use of $C^\infty(\mathbb{R}^n)$-bimodule structure of $\Omega^1(\mathbb{R}^n)$, we can write

$$d(fg) = df \cdot g + f \cdot dg. \tag{14.2.6}$$

We see that the formula $df(X) = Xf$ defines the linear mapping $d : C^\infty(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ from the algebra to bimodule over this algebra, which satisfies (14.2.6).

Now our aim is to find an expression for a 1-form ω in the coordinates x^i of the n-dimensional Euclidean space $\mathbb{R}^n$. For this purpose we remind that if V^* is the dual space of a finite dimensional vector space V, $\vec{e}_i$ is a basis for V then the elements e^i of the dual space V^* defined by $e^i(\vec{e}_j) = \delta^i_j$ form the basis for V^*, which is called the dual basis of $\vec{e}_i$. Any element (covector) ϕ of the dual space V^* can be expressed in terms of dual basis as $\phi = \phi_i\, e^i$, where ϕ_i are real numbers. We also remind that x^i are regarded as the coordinate functions on the Euclidean space $\mathbb{R}^n$. Hence each coordinate function x^i induces the 1-form dx^i, and, according to the definition, we have

$$dx^i(\frac{\partial}{\partial x^j}) = \frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

This shows that at any point p the covectors $dx^1_p, dx^2_p, \dots, dx^n_p$ form the basis for the cotangent space $\mathrm{T}^*_p\mathbb{R}^n$, which is dual to the canonical basis $\frac{\partial}{\partial x^1}|_p, \frac{\partial}{\partial x^2}|_p, \dots, \frac{\partial}{\partial x^n}|_p$. From this it is easy to conclude that any 1-form ω can be expressed as follows $\omega = \omega_i\, dx^i$, where ω_i are smooth functions. This also shows that the 1-forms dx^i form the basis for the bimodule of 1-forms $\Omega^1(\mathbb{R}^n)$ over the algebra $C^\infty(\mathbb{R}^n)$. Hence $\Omega^1(\mathbb{R}^n)$ is free left (or right) $C^\infty(\mathbb{R}^n)$-module of rank n.

14.2.4 Non-commutative First Order Differential Calculus

Now we can draw some conclusions from the previous considerations. The important conclusion is that the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ is the basic structure for the calculus of vector fields and differential 1-forms. Indeed we see that a vector field can be identified with the derivation of this algebra and differential 1-forms $\Omega^1(\mathbb{R}^n)$ can be considered as elements of the bimodule over this algebra. This observation underlies an approach used in non-commutative geometry. The algebra $C^\infty(\mathbb{R}^n)$ is commutative, but we can consider a non-commutative algebra, which by its properties should be close, in a sense, to $C^\infty(\mathbb{R}^n)$. This non-commutative algebra will mimic an algebra of functions on our space, and, making use of this algebra, we can then develop structures of differential geometry such as a calculus of vector fields, differential forms and so on. Peculiar property of this approach to geometry is that we do not need a notion of a point of our space because the only thing which we use to develop a differential geometry is an algebra of functions. It is worth to mention that our main goal is the algebraic aspect of noncommutative geometry approach, that is, we ignore the topological questions of functional spaces.

Let $\mathscr{A}$ be a unital associative algebra, which is not necessarily commutative. In order to be able to model the algebraic aspect of the calculus of differential forms developed in the previous subsection, we should have a bimodule over this algebra. We will denote this $\mathscr{A}$-bimodule by $\mathscr{M}$. Now we can give a general definition of a *first order differential calculus over a unital associative algebra* [14]. A triple $(\mathscr{A}, d, \mathscr{M})$, where $\mathscr{A}$ is a unital associative algebra, $\mathscr{M}$ is an $\mathscr{A}$-bimodule and $d : \mathscr{A} \to \mathscr{M}$ is a linear mapping, is said to be a first order differential calculus over an algebra $\mathscr{A}$ if d satisfies the Leibniz rule $d(ab) = da \cdot b + a \cdot db$, where $a, b \in \mathscr{A}$. If $\mathscr{A}$ is a commutative (non-commutative) algebra then a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ is called a commutative (non-commutative) first order differential calculus. Particularly the triple $(C^\infty(\mathbb{R}^n), d, \Omega^1(\mathbb{R}^n))$, where d is defined by (14.2.5), is the commutative first order differential calculus over the algebra of smooth functions in the n-dimensional Euclidean space $\mathbb{R}^n$.

Given a unital associative algebra $\mathscr{A}$ one can construct a universal first order differential calculus. Indeed the tensor product $\mathscr{A}^{\otimes 2} = \mathscr{A} \otimes \mathscr{A}$ is the $\mathscr{A}$-bimodule, where the left and right multiplications by elements of $\mathscr{A}$ are defined by

$$a \cdot (b \otimes c) = (ab) \otimes c, \quad (b \otimes c) \cdot a = b \otimes (ca).$$

For any $a \in \mathscr{A}$ define the linear mapping $d : \mathscr{A} \to \mathscr{A}^{\otimes 2}$ by $da = e \otimes a - a \otimes e$, where e is the identity element of $\mathscr{A}$. Applying this mapping to product of two elements

$$d(ab) = e \otimes (ab) - (ab) \otimes e = e \otimes (ab) - a \otimes b + a \otimes b - (ab) \otimes e$$
$$= (e \otimes a - a \otimes e)\, b + a\, (e \otimes b - b \otimes e) = da \cdot b + a \cdot db,$$

we see that d satisfies the Leibniz rule and thus $(\mathscr{A}, d, \mathscr{A}^{\otimes 2})$ is the first order differential calculus, which is referred to as the *universal first order differential calculus* over $\mathscr{A}$ [7].

A first order differential calculus over an algebra is very general algebraic concept and in order to make it more close to differential 1-forms in the n-dimensional Euclidean space we can use the fact that $\Omega^1(\mathbb{R}^n)$ is the free left (or right) $C^\infty(\mathbb{R}^n)$-module of rank n and the set $\{dx^i\}_{i=1}^n$ of 1-forms can be taken as the basis for this module. Let $(\mathscr{A}, d, \mathscr{M})$ be a non-commutative first order differential calculus over an algebra $\mathscr{A}$. Assume that the right $\mathscr{A}$-module $\mathscr{M}$ is free module of rank n and $\xi^i, i = 1, 2, \ldots, n$ is a basis for this module. In analogy with the differential calculus in the Euclidean space $\mathbb{R}^n$ one can define the right partial derivatives $\partial_i : \mathscr{A} \to \mathscr{A}$ by $da = \xi^i\, \partial_i a$. In this case a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ is called a first order differential calculus *with right partial derivatives* (r.p.d.). Now the left $\mathscr{A}$-module structure of $\mathscr{M}$ induces the mappings $H^i_j : a \in \mathscr{A} \mapsto H^i_j(a) \in \mathscr{A}$ defined by the formula $a\, \xi^i = \xi^j\, H^i_j(a)$, where a is an element of algebra $\mathscr{A}$. It can be proved [4] that the right partial derivatives ∂_i satisfy the *twisted Leibniz rule*

$$\partial_i (ab) = (\partial_i a)\, b + H^j_i(a)\, (\partial_j b), \;\; a, b \in \mathscr{A}. \tag{14.2.7}$$

We can compose the nth order matrix $H(a) = (H^j_i(a))$ over $\mathscr{A}$ by positioning the element $H^j_i(a)$ at the intersection of jth column and ith row. It can be proved that $H(ab) = H(a)\, H(b)$, which shows that H is the homomorphism from an algebra $\mathscr{A}$ to the algebra of nth order matrices $\mathrm{Mat}_n(\mathscr{A})$ over $\mathscr{A}$.

Next assume that a unital associative algebra $\mathscr{A}$ is generated by variables $x^i, i = 1, 2, \ldots, n$ which obey the relations $f_\alpha(x^1, x^2, \ldots, x^n) = 0, \alpha = 1, 2, \ldots, m$, where each $f_\alpha(x^1, x^2, \ldots, x^n)$ is the finite polynomial of variables $x^1, x^2, \ldots, x^n$ and $dx^i = \xi^i$. In this case a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ with r.p.d. is called a *coordinate first order differential calculus* [4]. Clearly in this case the generators $x^1, x^2, \ldots, x^n$ can be viewed as analogs of coordinate functions.

14.2.5 Two Dimensional Quantum Space

A well known example of first order non-commutative differential calculus can be constructed in the case of the quantum hyperplane [13]. As it was mentioned before, in non-commutative geometry approach one constructs and studies various structures of differential geometry in a non-commutative space by means of algebra of functions on this space. The algebra of functions on the *quantum hyperplane* is the algebra of finite polynomials over $\mathbb{C}$ generated by variables $x^1, x^2, \ldots, x^n$, which obey relations

$$x^i x^j = q\, x^j x^i, \tag{14.2.8}$$

where q is a non-zero complex number. The generators of the algebra $x^1, x^2, \ldots, x^n$ can be considered as the coordinate functions on the quantum hyperplane. Particularly if $n = 2$ then the algebra of functions generated by x, y, which obey the relations

$$xy = q\, yx, \tag{14.2.9}$$

will be referred to as the algebra of functions on the *quantum plane* and denoted by $\mathfrak{C}_q$. Our aim in this subsection is to construct a first order differential calculus over the algebra of functions on the quantum plane. It is useful to write the relation (14.2.9) in the form $r(x, y) = 0$, where $r(x, y) = xy - q\, yx$.

According to the notion of first order differential calculus over an algebra explained in the previous subsection, we have to construct a $\mathfrak{C}_q$-bimodule $\mathfrak{M}_q$ together with a differential $d : \mathfrak{C}_q \to \mathfrak{M}_q$, which satisfies the Leibniz rule. For this purpose we consider the right $\mathfrak{C}_q$-module $\mathfrak{M}_q$ freely generated by ξ, η. We define the $\mathfrak{C}_q$-bimodule structure of $\mathfrak{M}_q$ by putting

$$x\xi = \xi\, H^1_1(x) + \eta\, H^1_2(x),\;\; x\eta = \xi\, H^2_1(x) + \eta\, H^2_2(x), \tag{14.2.10}$$

$$y\xi = \xi\, H^1_1(y) + \eta\, H^1_2(y),\;\; y\eta = \xi\, H^2_1(y) + \eta\, H^2_2(y), \tag{14.2.11}$$

where

$$H : x \mapsto \begin{pmatrix} H^1_1(x) & H^2_1(x) \\ H^1_2(x) & H^2_2(x) \end{pmatrix}, \quad H : y \mapsto \begin{pmatrix} H^1_1(y) & H^2_1(y) \\ H^1_2(y) & H^2_2(y) \end{pmatrix}, \tag{14.2.12}$$

is a homomorphism from the algebra of functions $\mathfrak{C}_q$ to the algebra of 2×2-matrices over $\mathfrak{C}_q$ defined on the generators. Hence for any two functions $f, g \in \mathfrak{C}_q$ it holds $H(fg) = H(f)H(g)$. Now we can define a differential $d : \mathfrak{C}_q \to \mathfrak{M}_q$. Since differential is a linear mapping and it satisfies the Leibniz rule, it suffices to define it on the generators x, y. In order to have a coordinate first order differential calculus with r.p.d. we put $dx = \xi, dy = \eta$. As it is shown in the previous subsection the differential d induces the right partial derivatives

$$df = dx\, \partial_x f + dy\, \partial_y f, \quad f \in \mathfrak{C}_q, \tag{14.2.13}$$

which satisfy the twisted Leibniz rule (14.2.7)

$$\partial_x(fg) = (\partial_x f)\, g + H_1^1(f)\, \partial_x g + H_1^2(f)\, \partial_y g, \tag{14.2.14}$$

$$\partial_y(fg) = (\partial_y f)\, g + H_2^1(f)\, \partial_x g + H_2^2(f)\, \partial_y g. \tag{14.2.15}$$

Thus our first order differential calculus is coordinate differential calculus with r.p.d.

Since we defined the differential d by $dx = \xi, dy = \eta$, the relations (14.2.10) and (14.2.11) can be considered as commutation relations between coordinate functions x, y and their differentials dx, dy. It is worth to mention that two matrices $H(x), H(y)$ completely determine the coordinate first order differential calculus with r.p.d. over the algebra of functions on the quantum plane. Hence the matrices (14.2.12) can be considered as parameters of a possible differential calculus. Obviously these matrices should be compatible with the defining relation of quantum plane $r(x, y) = xy - q\, yx = 0$. Hence the matrices $H(x), H(y)$ have to satisfy the following conditions

$$\partial_x r(x, y) = 0,\ \ \partial_y r(x, y) = 0,\ \ H(r(x, y)) = 0. \tag{14.2.16}$$

We find

$$\partial_x(xy - q\, yx) = y + H_1^2(x) - q\, H_1^1(y),\ \ \partial_y(xy - q\, yx) = H_2^2(x) - qx - q\, H_2^1(y).$$

Hence the conditions $\partial_x r(x, y) = 0,\ \partial_y r(x, y) = 0$ imply

$$H_1^2(x) = q H_1^1(y) - y,\quad H_2^2(x) = q H_2^1(y) + qx, \tag{14.2.17}$$

which can be written as

$$H^2(x) = q H^1(y) + \begin{pmatrix} -y \\ qx \end{pmatrix}, \tag{14.2.18}$$

where $H^i(x)(H^i(y))$ is the ith column of the matrix $H(x)\ (H(y))$.

From $H(r(x, y)) = 0$ it follows

$$H^2(y)\, H_2^i(x) = q^{-1} H(x)\, H^i(y) - H^1(y)\, H_1^i(x),\ \ i = 1, 2. \tag{14.2.19}$$

The simplest case is when the matrices $H(x), H(y)$ depend linearly on the coordinates x, y. Hence we assume

$$H(x) = Ax + By,\ \ H(y) = Cx + Dy,$$

where A, B, C, D are complex matrices. It follows from (14.2.18) that these matrices must satisfy

$$A^2 = q\, C^1 + \begin{pmatrix} 0 \\ q \end{pmatrix}, \quad B^2 = q\, D^1 + \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \tag{14.2.20}$$

From (14.2.19) it follows that

$$C^2 A_2^i - q^{-1} A C^i + C^1 A_1^i = 0, \tag{14.2.21}$$

$$D^2 A_2^i - A D^i + D^1 A_1^i + q C^2 B_2^i - q^{-1} B C^i + q C^1 B_1^i = 0, \tag{14.2.22}$$

$$D^2 B_2^i - q^{-1} B D^i + D^1 B_1^i = 0. \tag{14.2.23}$$

Since $H(y) = Cx + Dy$ it is natural to seek a solution on the assumption $C = 0$. Then the first condition in (14.2.20) immediately gives

$$A^2 = \begin{pmatrix} 0 \\ q \end{pmatrix}.$$

Thus $A_1^2 = 0$, $A_2^2 = q$. Now the condition (14.2.21) is identically satisfied and the condition (14.2.22) takes the form

$$D^2 A_2^i - A D^i + D^1 A_1^i = 0. \tag{14.2.24}$$

The second natural assumption is that the matrices A, D are diagonal, i.e.

$$A = \begin{pmatrix} q^2 & 0 \\ 0 & q \end{pmatrix}, \quad D = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

where $\alpha_1, \alpha_2, \gamma_1, \gamma_2$ are different from zero. Then from the second formula in (14.2.20) we get $B_1^2 = q\gamma_1 - 1$, $B_2^2 = 0$. It is easy to verify that now the condition (14.2.24) is identically satisfied, while Eq. (14.2.24) gives

$$(1 - q^{-1})\gamma_1 B_1^1 = 0, \ (\gamma_2 - q^{-1}\gamma_1) B_2^1 = 0, \ (\gamma_1 - q^{-1}\gamma_2) B_1^2 = 0. \tag{14.2.25}$$

Since we assume $\gamma_1 \neq 0$, it follows from the first relation that $B_1^1 = 0$. The second and third relations have a symmetric form and we can solve them either by putting $\gamma_2 - q^{-1}\gamma_1 = 0$, $B_1^2 = 0$ or $\gamma_1 - q^{-1}\gamma_2 = 0$, $B_2^1 = 0$ (other choices lead either to restriction of q, which is unacceptable, or to $B = 0$, which makes the whole construction very indeterminate). In order to be more specific, we take $\gamma_1 - q^{-1}\gamma_2 = 0$, $B_2^1 = 0$ and fix $\gamma_1 = q$. Then $\gamma_2 = q^2$, and we finally obtain the well known first order coordinate differential calculus with r.p.d. on the quantum plane

$$x\, dx = q^2 dx\, x, \quad x\, dy = (q^2 - 1)\, dx\, y + q\, dy\, x, \tag{14.2.26}$$

$$y\, dx = q\, dx\, y, \quad y\, dy = q^2\, dy\, y. \tag{14.2.27}$$

14.3 Algebra of Differential Forms and Differential Graded Algebra

In this section we describe the higher order differential forms in the n-dimensional Euclidean space. We lay particular stress on an algebraic structure of differential forms and bring a reader to idea of a notion of differential graded algebra. We analyze in details the structure of differential graded algebra by pointing out that it contains the first order differential calculus. We explain the notion of universal differential graded algebra over a first order differential calculus.

14.3.1 Algebra of Differential Forms in $\mathbb{R}^n$

In the previous section we showed how one can construct the calculus of differential 1-forms in the n-dimensional Euclidean space and its possible generalizations within the framework of noncommutative geometry. In order to continue this construction to higher degree differentials forms in $\mathbb{R}^n$ we attach to each point p of the Euclidean space a vector space of totally skew-symmetric multilinear real-valued k-forms $\wedge^k(\mathrm{T}^*_p\mathbb{R}^n)$. The disjoint union $\wedge^k(\mathrm{T}^*\mathbb{R}^n) = \cup_p \wedge^k (\mathrm{T}^*_p\mathbb{R}^n)$ is referred to as the *vector bundle of exterior k-forms* over the Euclidean space $\mathbb{R}^n$. An element of this bundle can be written in the form $(p; \varphi)$, where p is a point of the Euclidean space and φ is a totally skew-symmetric multilinear k-form on the vector space $\mathbb{R}^n$, that is

$$\varphi : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \ (k \text{ times}) \to \mathbb{R},$$

which for any permutation $\sigma = (i_1, i_2, \ldots, i_k)$ of integers $(1, 2, \ldots, k)$ satisfies

$$\varphi(\vec{v}_{i_1}, \vec{v}_{i_2}, \ldots, \vec{v}_{i_k}) = (-1)^{|\sigma|}\, \varphi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k),$$

where $|\sigma|$ is the parity of a permutation. We consider the pair $\varphi_p = (p; \varphi) \in \wedge^k(\mathrm{T}^*_p\mathbb{R}^n)$ as the k-form on the tangent space $T_p\mathbb{R}^n$, where

$$\varphi_p(\vec{v}_{p;1}, \vec{v}_{p;2}, \ldots, \vec{v}_{p;k}) = \varphi(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k), \quad \vec{v}_{p;i} = (p; \vec{v}_i).$$

The projection $\pi_{(k)} : \wedge^k(\mathrm{T}^*\mathbb{R}^n) \to \mathbb{R}^n$ is defined in the natural way $\pi_{(k)}\, \varphi_p = p$. A smooth section $\theta : \mathbb{R}^n \to \wedge^k(\mathrm{T}^*\mathbb{R}^n)$ of the vector bundle of exterior k-forms is referred to as a differential k-form. The vector space of all differential k-forms will be denoted by $\Omega^k(\mathbb{R}^n)$ and the degree of a differential k-form θ will be denoted by $|\theta|$, i.e. $|\theta| = k$. Similar to differential 1-forms we can define the products $f\theta, \theta f$, where f is a smooth function, by means of point-wise multiplication and since scalars commute with vectors we have $f\theta = \theta f$. Hence the vector space $\Omega^k(\mathbb{R}^n)$ can be regarded as the bimodule over the algebra

$C^\infty(\mathbb{R}^n)$. The value of a differential k-form on vector fields $X_1, X_2, \ldots, X_n$ is the function $f = \theta(X_1, X_2, \ldots, X_n)$, whose value at a point p is defined by $f(p) = \theta_p((X_1)_p, (X_2)_p, \ldots, (X_k)_p)$.

Let ω be a differential k-form and θ be a differential l-form. The wedge product $\omega \wedge \theta$ of two differential forms ω, θ is the differential $(k+l)$-form, which is defined by

$$\omega \wedge \theta(X_1, X_2, \ldots, X_{k+l}) = \sum_\sigma (-1)^{|\sigma|} \omega(X_{i_1}, X_{i_2}, \ldots, X_{i_k}) \times \theta(X_{j_1}, X_{j_2}, \ldots, X_{j_l}), \quad (14.3.1)$$

where $\sigma = (i_1, i_2, \ldots, i_k, j_1, j_2, \ldots, j_l)$ is a permutation of integers $(1, 2, \ldots, k + l)$ such that $i_1 < i_2 < \ldots < i_k$, $j_1 < j_2 < \ldots < j_l$ and sum is taken over all such permutations. It can be proved that the wedge product of differential forms has the following properties:

(i) $\omega \wedge \theta = (-1)^{|\omega||\theta|} \theta \wedge \omega$,
(ii) $(\omega \wedge \theta) \wedge \chi = \omega \wedge (\theta \wedge \chi)$, i.e. the wedge product of differential forms is associative.

It is useful to add the algebra of smooth functions $C^\infty(\mathbb{R}^n)$ to the sequence $\Omega^k(\mathbb{R}^n)$, $k = 1, 2, \ldots$, of the vector spaces of differential forms by assigning degree zero to functions. Hence we identify the vector space of differential 0-forms with $C^\infty(\mathbb{R}^n)$, i.e. $\Omega^0(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$. In order to complete the construction of algebra of differential forms we introduce the direct sum of vector spaces $\Omega(\mathbb{R}^n) = \oplus_i \Omega^i(\mathbb{R}^n)$. Evidently $\Omega(\mathbb{R}^n)$ is closed under the wedge product of differential forms and hence it is the associative unital algebra, which is called the algebra of differential forms in the n-dimensional Euclidean space. By unital we mean that the constant function **1**, whose value at any point is one, can be taken as the identity element of the algebra of differential forms.

Now we remind the notion of a graded algebra. A unital associative algebra $\mathscr{A}$ is called a *graded algebra* if $\mathscr{A} = \oplus_{k \in \mathbb{Z}} \mathscr{A}^k$ and for any elements $u \in \mathscr{A}^i$, $v \in \mathscr{A}^j$ it holds $u \cdot v \in \mathscr{A}^{i+j}$. If $u \in \mathscr{A}^i$ then u is an element of degree i and we will denote its degree by $|u|$. An element of graded algebra, which has the certain degree, is called homogeneous. A graded algebra is said to be a graded commutative if for any two homogeneous elements $u, v \in \mathscr{A}$ it holds $u \cdot v = (-1)^{|u||v|} v \cdot u$. It is useful to introduce the *graded commutator* $[u, v]$ of two homogeneous elements u, v by $[u, v] = u \cdot v - (-1)^{|u||v|} v \cdot u$. Then the condition of graded commutativity can be given in the form $[u, v] = 0$.

Making use of the notion of graded algebra, we can say that the algebra of differential forms is the graded algebra because for any two homogeneous forms ω, θ it holds $|\omega \wedge \theta| = |\omega| + |\theta|$. Moreover, because of the first property of the wedge product, the algebra of differential forms is graded commutative.

A differential 1-form ω can be expressed in terms of coordinate functions x^i of the n-dimensional Euclidean space as $\omega = \omega_i \, dx^i$, where the coefficients ω_i are the smooth functions and the 1-forms dx^i form the basis for the bimodule $\Omega^1(\mathbb{R}^n)$. It follows from the properties of the wedge product that $dx^i \wedge dx^j = -dx^j \wedge dx^i$ or, equivalently, $dx^i \wedge dx^i = 0$. It is easy to show that differential 2-forms $dx^i \wedge dx^j$, where $i < j$, form the basis for the bimodule of 2-forms $\Omega^2(\mathbb{R}^n)$ and any differential 2-form θ can be written as follows

$$\theta = \frac{1}{2}\theta_{ij} dx^i \wedge dx^j,$$

where indices i, j run independently from 1 to n and the functions θ_{ij} satisfy $\theta_{ij} = -\theta_{ji}$. Analogously any differential k-form can be written as follows

$$\theta = \frac{1}{k!}\theta_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots dx^{i_k}, \tag{14.3.2}$$

where the functions $\theta_{i_1 i_2 \ldots i_k}$ are totally skew-symmetric under permutations of subscripts. From the expression for a differential k-form (14.3.2) and the property $dx^i \wedge dx^i = 0$ it follows that the highest degree of non-trivial differential form in the n-dimensional Euclidean space is n. Hence $\Omega(\mathbb{R}^n) = \oplus_{k=0}^n \Omega^k(\mathbb{R}^n)$.

Finally we would like to point out that at any fixed point p of the Euclidean space the wedge product of differential forms induces the wedge products of covectors $dx^1|_p, dx^2|_p, \ldots, dx^n|_p$, which are subjected to the commutation relations

$$dx^i|_p \wedge dx^j|_p = -dx^j|_p \wedge dx^i|_p.$$

These relations show that $dx^i|_p$ are the generators of *Grassmann algebra* $\wedge(\mathrm{T}^*_p\mathbb{R}^n) = \oplus_k \wedge^k (\mathrm{T}^*_p\mathbb{R}^n)$, which is called the *exterior algebra of the cotangent space* $\mathrm{T}^*_p\mathbb{R}^n$.

The exterior differential d of the algebra of differential forms is defined as follows:

(i) if f is a smooth function then the exterior differential df is the 1-form defined for any vector field X by $df(X) = Xf$,
(ii) for any differential k-form θ the exterior differential $d\theta$ is the differential $(k+1)$-form defined by the formula

$$\begin{aligned} d\theta(X_1, X_2, .., X_{k+1}) = &\sum_{i=1}^{k+1}(-1)^{i+1} X_i\theta(X_1, X_2, .., \hat{X}_i, .., X_{k+1}) \\ &+ \sum_{i<j}(-1)^{i+j}\theta([X_i, X_j], X_1, X_2, .., \hat{X}_i, .., \hat{X}_j, .., X_{k+1}), \end{aligned}$$

where hat over X_i means that this vector field is omitted.

It can be proved that the exterior differential has the following properties:

1. the exterior differential has the degree 1, i.e. $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$,
2. for any homogeneous forms ω, θ it holds $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{|\omega|}\omega \wedge d\theta$, and this property is referred to as the *graded Leibniz rule*,
3. $d^2 = 0$.

It should be mentioned that the exterior differential d is uniquely determined by the above properties. The last property is very important and it is the key property for a concept of *de Rham cohomology*.

14.3.2 Non-commutative Higher Order Differential Calculus

In the previous subsection it was shown that the triple $(C^\infty(\mathbb{R}^n), d, \Omega^1(\mathbb{R}^n))$ is the commutative first order differential calculus in the Euclidean space $\mathbb{R}^n$ and this calculus was included as the subalgebra of the algebra of differential forms $\Omega(\mathbb{R}^n)$ by assigning degree zero to smooths functions, i.e. $\Omega^0(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$. Hence we can look at the algebra of differential forms with exterior differential d as the extension of the first order differential calculus $(C^\infty(\mathbb{R}^n), d, \Omega^1(\mathbb{R}^n))$ to higher degree differentials forms, which satisfies the listed above properties of exterior differential.

A general approach to this kind of extensions of first order differential calculus is provided by the notion of differential graded algebra. A *differential graded algebra* (DGA) $\mathscr{G}$ is a graded algebra $\mathscr{G} = \oplus_k \mathscr{G}^k$ endowed with a linear mapping $d : \mathscr{G}^k \to \mathscr{G}^{k+1}$ of degree 1, which satisfies the graded Leibniz rule $d(uv) = du\, v + (-1)^{|u|}u\, dv$, where $u, v \in \mathscr{G}$, $|u|$ is the degree of u, and $d^2 = 0$. Hence the algebra of differential forms in the Euclidean space $\mathbb{R}^n$ is the commutative differential graded algebra.

Firstly it follows from the definition of a DGA that the subspace of elements of degree zero $\mathscr{G}^0$ is the *subalgebra* of $\mathscr{G}$. Indeed for $u, v \in \mathscr{G}^0$ we have $|uv| = |u| + |v| = 0$, which means that the product of two degree zero elements uv is the element of degree zero, hence $uv \in \mathscr{G}^0$. Secondly any subspace $\mathscr{G}^k$ of elements of degree $k \geq 0$ is the $\mathscr{G}^0$-bimodule. Indeed if we multiply an element $w \in \mathscr{G}^k$ of degree k by an element u of degree zero either from the left or from the right then the degree of products is $|w|+|u| = |w| = k$, and thus the products are the elements of $\mathscr{G}^k$. Consequently the multiplication by elements of degree zero determines the mappings $\mathscr{G}^0 \times \mathscr{G}^k \to \mathscr{G}^k$, $\mathscr{G}^k \times \mathscr{G}^0 \to \mathscr{G}^k$, and it is easy to verify that all axioms of bimodule are fulfilled. Thirdly the triple $(\mathscr{G}^0, d, \mathscr{G}^1)$ is the first order differential calculus over the algebra of elements of degree zero $\mathscr{G}^0$, because the graded Leibniz rule in the case of zero degree elements reduces to ordinary Leibniz rule. This suggests the following definition: If $(\mathscr{A}, d, \mathscr{M})$ is a first order differential calculus and $\mathscr{G}$ is a DGA with differential d' such that $\mathscr{G}^0 \equiv \mathscr{A}, \mathscr{G}^1 \equiv \mathscr{M}$ and d' coincides with d, when restricted to $\mathscr{G}^0$, then a DGA $\mathscr{G}$ will be referred to as a *higher order differential calculus over a first order differential calculus* $(\mathscr{A}, d, \mathscr{M})$.

Assume $(\mathscr{A}, d, \mathscr{M})$ is a first order differential calculus and $\mathscr{G}$ is a higher order differential calculus over $\mathscr{A}$. A first order differential calculus, where $\mathscr{A}$-bimodule $\mathscr{M}$ does not contain unnecessary elements, is of most interest in a theory of higher order calculus over an algebra. Hence we are most interested in the case, where $\mathscr{M}$ is generated by elements of $\mathscr{A}$ and their differentials, i.e. $\mathscr{M} = \mathscr{A}\, d\mathscr{A}\, \mathscr{A}$. But it immediately follows from the Leibniz rule that $\mathscr{M} = \mathscr{A}\, d\mathscr{A}\, \mathscr{A} = d\mathscr{A}\, \mathscr{A}$. Indeed we have $a\, db = d(ab) - da\, b$, which implies $a\, db\, c = (d(ab) - da\, b)\, c = d(ab)\, c - da\, (bc)$. It can be proved [12] that if $(\mathscr{A}, d, \mathscr{M})$ is a first order differential calculus, where $\mathscr{M} = d\mathscr{A}\, \mathscr{A}$, then there exists a DGA $\mathscr{G}$ generated by $\mathscr{G}^0 = \mathscr{A}$ such that its differential coincides with d, when restricted to $\mathscr{A}$. This DGA is usually referred to as the *universal differential graded algebra* of $(\mathscr{A}, d, \mathscr{M})$.

The structure of the universal differential graded algebra of a first order differential calculus $(\mathscr{A}, d, \mathscr{M})$ with right partial derivatives, where $\mathscr{A}$ is generated by a set of variables $x^i, i \in I$ (with relations $f_\alpha = 0$) and the right $\mathscr{A}$-module is freely generated by $\omega^k, k \in K$, is of interest because it is similar to algebra of differential forms in the Euclidean space $\mathbb{R}^n$. Hence we have for the generators x^i of $\mathscr{A}$ and for the $\mathscr{A}$-bimodule structure of $\mathscr{M}$ the following relations

$$f_\alpha(x^i) = 0, \tag{14.3.3}$$

$$x^i \omega^k = \omega^l\, H_l^{ik}, \tag{14.3.4}$$

where $f_\alpha(x^i)$ are finite polynomials and $H_l^{ik} = H_l^k(x^i)$. Now let $\bar{\mathscr{G}}$ be the algebra generated by variables x^i, ω^k, which obey relations (14.3.3), (14.3.4). In order to consider a case more general than a coordinate calculus we assume $dx^i = \omega^k g_k^i$, where $\omega^k \in d\mathscr{A}$, i.e. $d\omega^k = 0$ (ω^k are closed "differential 1-forms"). Extending a differential d to elements of $\mathscr{M}$ by the graded Leibniz rule and differentiating (14.3.4) and $dx^i = \omega^k g_k^i$ we get

$$\omega^l \omega^m \big(H_m^k(g_l^i) + \partial_m H_l^{ik}\big) = 0, \quad \omega^k \omega^m\, \partial_m g_k^i = 0. \tag{14.3.5}$$

Now consider the algebra $\mathscr{G}$ generated by x^i, ω^k, which are subjected to the relations (14.3.3), (14.3.4), (14.3.5). This algebra is endowed with differential $d : \mathscr{A} \to \mathscr{M}$. From (14.3.4) it follows that any element of $\mathscr{G}$ can be expressed as follows

$$\omega^{k_1} \omega^{k_2} \dots \omega^{k_n} h_{k_1 k_2 \dots k_n}, \quad h_{k_1 k_2 \dots k_n} \in \mathscr{A}. \tag{14.3.6}$$

This implies that $\mathscr{G}$ is the graded algebra with the degree of a homogeneous element determined by the number of ω^k in the expression (14.3.6). Now it can be proved [4] that if we extend a differential $d : \mathscr{A} \to \mathscr{M}$ to the algebra $\mathscr{G}$ by means of the formula

$$d(\omega^{k_1} \omega^{k_2} \dots \omega^{k_n} h_{k_1 k_2 \dots k_n}) = (-1)^n \omega^{k_1} \omega^{k_2} \dots \omega^{k_n} \omega^k\, \partial_k h_{k_1 k_2 \dots k_n}, \tag{14.3.7}$$

then $\mathscr{G}$ is the universal differential graded algebra over coordinate first order differential calculus $(\mathscr{A}, d, \mathscr{M})$.

14.4 Connection in Euclidean Space

A concept of connection arises when we consider the problem of *parallel translation of a vector* in the n-dimensional Euclidean space $\mathbb{R}^n$. Assume $\alpha : I \to \mathbb{R}^n, I \subset \mathbb{R}$ is a parametrized curve which passes through a point $p = \alpha(0)$. Let $\vec{v}_p = (p; \vec{v}) \in \mathrm{T}_p\mathbb{R}^n$ be a tangent vector at a point $p \in \mathbb{R}^n$ of the space and our goal is to move this vector in parallel way along a curve to some other point of a curve $q = \alpha(t_0), t_0 \in I$. Because we know what does it mean the parallelism in the Euclidean space $\mathbb{R}^n$ the solution is easy. In order to extend a tangent vector $\vec{v}_p$ in parallel way along a curve α we construct the constant vector field $V(\alpha(t)) = (\alpha(t); \vec{v})$ along α. But this problem becomes less trivial and leads to interesting geometric structure if we consider this problem of parallel translation of a vector in curvilinear coordinates. Let us assume that U is an open subset of the Euclidean space $\mathbb{R}^n$ and $x'^1, x'^2, \ldots, x'^n$ are curvilinear coordinates determined in U. We also assume that these curvilinear coordinates can be expressed in terms of the Cartesian coordinates $x^1, x^2, \ldots, x^n$ by means of smooth functions, i.e. $x'^i = x'^i(x^1, x^2, \ldots, x^n)$, and vice versa the Cartesian coordinates can be expressed in terms of curvilinear coordinates by means of smooth functions $x^i = x^i(x'^1, x'^2, \ldots, x'^n)$. We also assume that the coordinate lines of curvilinear coordinates are orthogonal and hence we can construct the orthonormal frame field $E' = \{E'_1, E'_2, \ldots, E'_n\}$ by means of the vector fields $\frac{\partial}{\partial x'^i}$ (normalizing them if necessary). This orthonormal frame can be expressed in terms of the canonical frame field $E = \{E_1, E_2, \ldots, E_n\}$ as follows $E'_i = g_i^j\, E_j$, where the matrix $G = (g_i^j)$ depends on a point of U and for any $G \in \mathrm{SO}(n)$, i.e. $G\,G^T = I$, $\mathrm{Det}\, G = 1$ and I is the unit matrix.

Now we can write the constant vector field $V(\alpha(t))$ as follows

$$V(\alpha(t)) = V^i(\alpha(t))\, E'_i(\alpha(t)),$$

and our aim is to find unknown functions $V^i(\alpha(t))$. Differentiating both sides with respect to t we get zero at the left-hand side because $V(\alpha(t))$ is the constant vector field. The right-hand side can be written as follows

$$\frac{d}{dt}\big(V^i(\alpha(t))\, E'_i(\alpha(t))\big) = \frac{d}{dt}\big(V^i(\alpha(t))\big)\, E'_i(\alpha(t)) + V^i(\alpha(t))\, \frac{d}{dt}\big(E'_i(\alpha(t))\big).$$

Making use of the definition of directional derivative of a function we can interpret the coefficients $\frac{d}{dt}\big(V^i(\alpha(t))\big)$ in the first sum as the directional derivatives of functions $V^i(\alpha(t)$ in the direction of the tangent vector field $X(\alpha(t)) = (\alpha(t); \vec{\alpha}'(t))$ along a curve, i.e. we can write them as XV^i. The derivatives in the second sum

$\frac{d}{dt}\big(E_i'(\alpha(t))\big)$ can be regarded as analogs of directional derivatives for vector fields, and this suggests us to introduce a new derivative for vector fields, which is called a covariant derivative.

Let X, Y be two vector fields, $p \in \mathbb{R}^n$ be a point, $\alpha : I \to \mathbb{R}^n$ be a curve such that $\alpha(0) = p, \vec{\alpha}'(0) = X_p$. The *covariant derivative of a vector field Y with respect to* a vector field X at a point p is the tangent vector $(D_X Y)_p \in \mathrm{T}_p\mathbb{R}^n$, which is defined by

$$(D_X Y)_p = \frac{d}{dt}\big(Y|_{\alpha(t)}\big)|_{t=0}.$$

Hence the covariant derivative determines the vector field $p \mapsto (D_X Y)_p$, which will be denoted by $D_X Y$. From this definition it follows that in any curvilinear coordinate system x'^i and at any point p we have

$$(D_X Y)^i_p = \frac{\partial Y^i}{\partial x'^j}|_p \frac{dx'^j}{dt}|_{t=0} = X^j(p)\frac{\partial Y^i}{\partial x'^j}|_p = (XY^i)(p). \tag{14.4.1}$$

Hence $D_X Y$ is the vector field which in curvilinear coordinates x'^i can be written as follows

$$D_X Y = (XY^i)\,\frac{\partial}{\partial x'^i}. \tag{14.4.2}$$

From (14.4.2) it follows that covariant derivative has the following properties:

(i) $D_{X_1+X_2}Y = D_{X_1}Y + D_{X_2}Y,\ D_{fX}Y = f\,D_X Y$;
(ii) $D_X(Y_1 + Y_2) = D_X Y_1 + D_X Y_2,\ D_X(fY) = (Xf)\,Y + f\,D_X Y$;
(iii) $D_X Y - D_Y X = [X, Y]$;
iv) $X < Y, Z > = < D_X Y, Z > + < Y, D_X Z >$.

The property (i) shows that the covariant derivative $D_X Y$ depends linearly on a vector field X, and this clearly suggests that we can describe the structure of covariant derivative in the terms of differential 1-forms. Indeed the formula (14.4.2) can be written in the form

$$D_X Y = dY^i(X)\,\frac{\partial}{\partial x'^i}, \tag{14.4.3}$$

where dY^i is the exterior differential of the function Y^i. Now our aim is to get rid of a vector field X in the above formula and to use differential 1-forms. For this purpose we consider the right-hand side of (14.4.3) as a vector field valued differential 1-form. In order to assign a geometric meaning to these words we attach to each point p of the space $\mathbb{R}^n$ the tensor product $\wedge^k(\mathrm{T}^*_p\mathbb{R}^n) \otimes \mathrm{T}_p\mathbb{R}^n$ and consider the vector bundle $\wedge^k(\mathrm{T}^*\mathbb{R}^n) \otimes \mathrm{T}\mathbb{R}^n = \cup_p \wedge^k (\mathrm{T}^*_p\mathbb{R}^n) \otimes \mathrm{T}_p\mathbb{R}^n$ with obvious projection. A smooth section of this bundle is referred to as a *vector field valued differential k-form* in the Euclidean space $\mathbb{R}^n$. The vector space of vector field valued differential

k-forms will be denoted by $\Omega^k(\mathbb{R}^n, \mathfrak{D})$. It is worth to point out that this vector space can be considered as the left $C^\infty(\mathbb{R}^n)$-module, i.e. we can multiply vector field valued forms by functions from the left. For instance any vector field valued 1-form ω can be written in the form

$$\omega = \omega^i_j \, dx'^j \otimes \frac{\partial}{\partial x'^i} = \omega^i \otimes \frac{\partial}{\partial x'^i} = dx'^j \otimes X_j,$$

where ω^i_j are smooth functions, $\omega^i = \omega^i_j \, dx'^j$ are $\mathbb{R}$-valued differential 1-forms and $X_j = \omega^i_j \frac{\partial}{\partial x'^i}$ are vector fields. If X is a vector field and a vector field valued differential 1-form is written as $\omega = \omega^i \otimes \frac{\partial}{\partial x'^i}$ then its value on a vector field X is the vector field defined by

$$\omega(X) = \omega^i(X) \frac{\partial}{\partial x'^i}.$$

Now making use of vector field valued differential forms we can omit a vector field X in the formula (14.4.3) and write it in the equivalent form

$$DY = dY^i \otimes \frac{\partial}{\partial x'^i}. \tag{14.4.4}$$

Clearly for any vector field X we have $DY(X) = D_X Y$. Thus starting with the covariant derivative $D_X Y$ we constructed the mapping $Y \mapsto DY$, which assigns to any vector field the vector field valued differential 1-form. What are the properties of this mapping? Now the property (i) of the covariant derivative is obvious, because DY is the differential 1-form. The property (ii) of covariant derivative shows that $D : \mathfrak{D} \to \Omega^1(\mathbb{R}^n, \mathfrak{D})$ is the linear mapping of vector spaces. The second part of this properties gives

$$D(fY) = df \otimes Y + f DY. \tag{14.4.5}$$

In order to write the property (iv) in the terms of D we must extend the scalar product of vector fields to vector field valued differential form and we can do this by means of the formula

$$< \omega \otimes X, \theta \otimes Y >=< X, Y > \omega \wedge \theta.$$

Particularly the scalar product of vector field valued 1-form $\omega \otimes X$ and a vector field Y is the 1-form $< \omega \otimes X, Y >=< X, Y > \omega$. Now the property (iv) implies

$$d < Y, Z >=< DY, Z > + < Y, DZ >, \tag{14.4.6}$$

and this property is usually referred to as the condition of consistency of the covariant derivative with the metric (inner product) of the Euclidean space $\mathbb{R}^n$. Till now we used the frame field $\{\frac{\partial}{\partial x'^i}\}_{i=1}^n$ for the module of vector fields and the basis $\{dx'^i\}_{i=1}^n$ for the module of differential forms to obtain formulae for the covariant derivatives. Now our aim is to study the structure of covariant derivative with the help of the frame field $E' = \{E'_i\}_{i=1}^n$. Let $\{\theta^i\}_{i=1}^n$ be the dual basis, where θ^i are differential 1-forms, which satisfy $\theta^i(E'_j) = \delta^i_j$. We can easily find the expression for these differential forms in the terms of Cartesian coordinates dx^i. Indeed if we denote $\theta^i = \theta^i_j\, dx^j$ then

$$\delta^i_k = \theta^i(E'_k) = \theta^i(g^j_k\, E_j) = \theta^i_m\, g^j_k\, dx^m(E_j) = \theta^i_m\, g^m_k.$$

Thus $\theta^i_m = (G^{-1})^i_m$ and $\theta^i = (G^{-1})^i_m\, dx^m$.

Given a vector field X we can write it in the frame field E' induced by curvilinear coordinates as follows $X = X^i\, E'_i$. Making use of the properties of covariant derivative we find

$$DX = D(X^i\, E'_i) = dX^i \otimes E'_i + X^i\, DE'_i. \tag{14.4.7}$$

The covariant derivative DE'_i is the vector field valued 1-form and hence it can be expanded as $\omega^j_i \otimes E'_j$, where ω^j_i are the differential 1-forms. The matrix $\omega = (\omega^j_i)$, whose elements are differential 1-forms, is referred to as the *matrix of connection*. Thus if we fix a frame field (a basis for the module of vector fields) then the covariant derivative induces the matrix of connection, which depends on a choice of a frame field. Before we compute the matrix of connection, we can derive its very important property from the consistency with the Euclidean metric (14.4.6). For two vector fields E'_i, E'_j of the frame field the consistency condition (14.4.6) takes on the form

$$d < E'_i, E'_j >=< DE'_i, E'_j > + < E'_i, DE'_j > .$$

Taking into account that $< E'_i, E'_j >= \delta_{ij}$ and substituting $DE'_i = \omega^k_i \otimes E'_k$, we obtain $\omega^i_j + \omega^j_i = 0$. Thus the matrix of connection is the skew-symmetric matrix $\omega + \omega^T = 0$. If we analyze the origin of this property of the matrix of connection we can see that the reason lies in the orthogonality of the attitude matrix $G = (g^i_j)$, which determines the transformation of the canonical frame field E into the orthonormal frame field E', induced by curvilinear coordinates. We remind that the Lie algebra $\mathrm{so}(n)$ of the special orthogonal group $\mathrm{SO}(n)$ is the vector space of skew-symmetric matrices, i.e.

$$\mathrm{so}(n) = \{h \in \mathrm{Mat}_n(\mathbb{R}) : h + h^T = 0\}.$$

Consequently we conclude that if we use the matrix group $\mathrm{SO}(n)$ for a transition from one frame field to another, or, by other words, we consider the action of the matrix group $\mathrm{SO}(n)$ on the set of orthonormal frames of the tangent space $\mathrm{T}_p\mathbb{R}^n$ at any point $p \in U$ then the matrix of connection is so(n)-valued differential 1-form, i.e. Lie algebra valued 1-form.

The matrix of connection depends on a choice of a frame field. Let us find how the matrix of connection transforms when we pass from one frame field to another. Let $\{E'_i\}, \{E''_i\}$ be two orthonormal frame fields and an orthogonal matrix $G = (g^j_i) \in \mathrm{SO}(n)$ be a transition matrix from $\{E'_i\}$ to $\{E''_i\}$, i.e. $E''_i = g^j_i\, E'_j$. We will write this symbolically as $E'' = G \cdot E'$. It is worth to mention that if we consider the previous formula at a fixed point p, i.e. $(E''_i)_p = g^j_i(p)\,(E'_j)_p$ (symbolically $E''_p = G(p)\cdot E'_p$), then it determines the action of the orthogonal group SO(n) on the set of all orthonormal frames of the tangent space $\mathrm{T}_p\mathbb{R}^n$. This suggests us to attach to each point p of the Euclidean space the set $\mathscr{F}_p$ of all orthonormal frames for $\mathrm{T}_p\mathbb{R}^n$ and to consider the disjoint union $\mathscr{F}(U) = \cup_p \mathscr{F}_p$. We will refer to $\mathscr{F}(U)$ as the *bundle of orthonormal frames over an open subset* U of the Euclidean space $\mathbb{R}^n$, and to $\mathscr{F}_p$ as the *fiber* of this bundle at a point p. The *projection* $\pi : \mathscr{F}(U) \to U$ is defined in the obvious way and any orthonormal frame field is a smooth section of the bundle $\mathscr{F}(U)$. The special orthogonal group SO(n) acts on the bundle of orthonormal frames from the left as it is shown above, i.e. $E_p \to G\cdot E_p$, and we will denote this *left action* by $L : (G, E_p) \mapsto G \cdot E_p$, i.e. $L : \mathrm{SO}(n) \times \mathscr{F}(U) \to \mathscr{F}(U)$. This action is

(i) *transitive*, i.e. for any two $E'_p, E''_p \in \mathscr{F}_p$ there exists $G \in \mathrm{SO}(n)$ such that $E''_p = G \cdot E'_p$,
(ii) *effective*, i.e. $G \cdot E'_p = E'_p$ implies $G = I$.

Now let E', E'' be two orthonormal frame fields, i.e. two sections of the bundle of orthonormal frames, and $G = (g^j_i) : U \to \mathrm{SO}(n)$ be the SO(n)-valued function such that $E'' = G\cdot E'$. Following the terminology used in a gauge field theory we can call this transformation (from one frame field to another) the *gauge transformation of first kind*. Hence $E''_i = g^j_i\, E'_j$, where g^j_i depend smoothly on a point $x \in U$. Let $\tilde{\omega}, \omega$ be the matrices of connection in a frame fields E'', E' respectively. Then $DE''_i = \tilde{\omega}^k_i \otimes E''_k$, $DE'_i = \omega^k_i \otimes E'_k$. On the one hand $DE''_i = \tilde{\omega}^k_i \otimes (g^m_k\, E'_m) = (g^m_k\, \tilde{\omega}^k_i) \otimes E'_m$. On the other hand

$$DE''_i = dg^m_i \otimes E'_m + g^j_i\, DE'_j = (dg^m_i + \omega^m_j\, g^j_i) \otimes E'_m,$$

and we get

$$g^m_k \tilde{\omega}^k_i = dg^m_i + \omega^m_j\, g^j_i,$$

or, written in the matrix form

$$\tilde{\omega} = G^{-1}\omega\, G + G^{-1}dG. \tag{14.4.8}$$

We derived the transformation rule of the matrix of connection and this is usually called in a gauge field theory the *gauge transformation of second kind*.

Particularly in the case of $E' = G \cdot E$, where E is the canonical frame field, let us denote the connection matrix in the frame field E' by ω and the connection matrix in the canonical frame field E by ω_0. Since the canonical frame field E consists of constant vector fields, the gauge transformation (14.4.8) takes the form $\omega = G^{-1}d\,G$, because in the case of the canonical frame field $DE_i = 0$ (E_i is the constant vector field) and hence $\omega_0 = 0$. In a gauge field theory the connection $\omega = G^{-1}d\,G$ is referred to as the *pure gauge*. It is easy to show that $\omega = G^{-1}d\,G$ is the so(n)-valued differential 1-form. Indeed we have $G^{-1}\,G = I$ and, differentiating both sides, we obtain

$$dG^{-1}\,G + G^{-1}dG = 0. \tag{14.4.9}$$

But the first term can be written $dG^{-1}\,G = (dG)^T(G^{-1})^T = (G^{-1}dG)^T = \omega^T$ and we conclude $\omega + \omega^T = 0$.

We remind that the dual 1-forms θ^i for E_i' are $\theta^i = (G^{-1})^i_j\,dx^j$. Differentiating and making use of (14.4.9) written in the form $dG^{-1} = -G^{-1}dG\,G^{-1}$, we get

$$d\theta^i = d(G^{-1})^i_j\wedge dx^j = -(G^{-1}dG\,G^{-1})^i_j\wedge dx^j = -(G^{-1}dG)^i_k\wedge((G^{-1})^k_j\wedge dx^j),$$

or

$$d\theta^i = -\omega^i_k \wedge \theta^k. \tag{14.4.10}$$

Equation (14.4.10) is called the *first Cartan's structure equation*. Analogously computing the exterior differential of the matrix of connection $d\omega$, we obtain

$$d\omega = d(G^{-1}dG) = dG^{-1} \wedge dG = -(G^{-1}dG) \wedge (G^{-1}dG),$$

or

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j \quad \Leftrightarrow \quad d\omega = -\omega \wedge \omega. \tag{14.4.11}$$

Equation (14.4.11) is called the *second Cartan's structure equation*. Remind that the matrix of connection ω can be considered as the so(n)-valued differential 1-form. Let $\mathfrak{f}_\alpha$, where $\alpha = 1, 2, \ldots, \frac{n(n-1)}{2}$, be a basis for the Lie algebra so(n). Then $\omega = \omega^\alpha\,\mathfrak{f}_\alpha$ or $\omega^i_j = \omega^\alpha(\mathfrak{f})^i_j$, where ω^α is the differential 1-form. Define

$$[\omega, \omega] = \omega^\alpha \wedge \omega^\beta\,[\mathfrak{f}_\alpha, \mathfrak{f}_\beta]. \tag{14.4.12}$$

From this it follows

$$[\omega, \omega]^i_j = \omega^\alpha \wedge \omega^\beta \left((\mathfrak{f}_\alpha)^i_k(\mathfrak{f}_\beta)^k_j - (\mathfrak{f}_\beta)^i_k(\mathfrak{f}_\alpha)^k_j\right) = 2\,\omega^i_k \wedge \omega^k_j.$$

Now the second Cartan's structure equation can be written in the matrix form as follows

$$d\omega = -\frac{1}{2}[\omega, \omega]. \qquad (14.4.13)$$

14.5 q-Differential Graded Algebra and N-Connection

In this section we describe a generalization of the notion of connection which arises in the framework of non-commutative geometry. First of all we would like to remind a reader that the notion of a connection in the Euclidean space $\mathbb{R}^n$, described in the previous section, can be extended to a vector bundle over a smooth manifold. A smooth n-dimensional manifold M is a Hausdorff topological space, which is locally homeomorphic to open subset of the n-dimensional Euclidean space $\mathbb{R}^n$ (this is called a local chart), and the smooth structure of M is determined by the condition that the transition functions of any two local charts must be smooth. A vector bundle $\mathfrak{V}$ over a manifold M is a triple $(\mathfrak{V}, \pi, M)$, where $\mathfrak{V}$ is a $(n+r)$-dimensional manifold, $\pi : \mathfrak{V} \to M$ is a differentiable map, which is called a *projection*, such that for any point x of a manifold M the fiber $\pi^{-1}(x)$ is an r-dimensional vector space. Additionally it is required that locally a vector bundle is trivial, i.e. for any point $x \in M$ there exists its neighborhood $U \subset M$ such that $\pi^{-1}(U)$ is diffeomorphic to $U \times \mathbb{R}^r$. A *section* of a vector bundle $\mathfrak{V}$ is a differentiable map $s : M \to \mathfrak{V}$ such that $\pi \circ s = \mathrm{id}_M$. Let $\mathfrak{C} = C^\infty(M)$ be the algebra of smooth functions on M, $\Omega(M) = \oplus_i \Omega^i(M)$ be the algebra of differential forms on M, $\mathscr{E}(\mathfrak{V})$ be the vector space of smooth sections of a vector bundle $\mathfrak{V}$. This vector space of sections $\mathscr{E}(\mathfrak{V})$ and the algebra of differential forms $\Omega(M)$ can be endowed with the structure of module over the algebra of functions $\mathfrak{C}$ by means of pointwise multiplication. Now we can extend the notion of a vector field valued differential form to a notion of vector bundle valued differential form by considering the tensor product $\Omega(\mathfrak{V}) = \Omega(M) \otimes_{\mathfrak{C}} \mathscr{E}(\mathfrak{V})$. It is important here that the first factor in this tensor product is the DGA. Now in accordance with the formula (14.4.5) we can define a connection in a vector bundle $\mathfrak{V}$ as a linear mapping $D : \mathscr{E}(\mathfrak{V}) \to \Omega^1(\mathfrak{V})$, which assigns to each section of a vector bundle the vector bundle valued 1-form and satisfies

$$D(f \cdot s) = df \otimes s + f \cdot Ds.$$

Hence we see that important ingredient in the structure of connection is the DGA of differential forms $\Omega(M)$. In this section we describe a generalization of the notion of connection which can be constructed if instead of a DGA we consider a more general structure, which is called a q-differential graded algebra (q-DGA), where q is a primitive Nth root of unity.

14.5.1 q-Differential Graded Algebra

A basic algebraic structure used in the theory of connections on modules is a DGA. Therefore if we consider a generalization of a DGA, where the basic property of differential $d^2 = 0$ is given in a more general form $d^N = 0,\ N \geq 2$ and the graded Leibniz rule is replaced by the graded q-Leibniz rule, where q is a primitive Nth root of unity, we can develop a generalization of the theory of connections on modules.

A notion of q-differential graded algebra was introduced in [5] and studied in the series of papers [1, 6, 9, 11]. Let $N \geq 2$, q be a primitive Nth root of unity and $\mathscr{G}_q = \oplus_k {\mathscr{G}_q}^k$ be an associative unital $\mathbb{Z}_N$-graded algebra over a field of complex numbers. An algebra $\mathscr{G}_q$ is said to be a *q-differential graded algebra* (q-DGA) if it is endowed with a linear mapping d of degree one, satisfying the graded q-Leibniz rule

$$d(u\,v) = d(u)\,v + q^k u\,d(v), \tag{14.5.1}$$

where $u \in {\mathscr{G}_q}^k$, $v \in \mathscr{G}_q$, and the N-nilpotency condition

$$d^N = 0. \tag{14.5.2}$$

A concept of q-DGA is related to a monoidal structure introduced in [11] for a category of N-complexes. It is proved in [8] that the monoids of the category of N-complexes can be determined as the q-DGA. In agreement with the terminology developed in [5] we shall call d the N-differential of q-DGA $\mathscr{G}_q$.

Clearly in the case $N = 2$ and $q = -1$ we get a notion of DGA, which allows us to consider a concept of a q-DGA as a generalization of a DGA.

Let $\mathscr{G}_q$ be a q-DGA and $\mathfrak{A}$ be an unital associative algebra over the field of complex numbers. The subspace $\mathscr{G}_q^0 \subset \mathscr{G}_q$ of elements of degree zero is the subalgebra of an algebra $\mathscr{G}_q$. Obviously the triple $(\mathfrak{A}, d, \mathscr{G}_q^1)$ is the first order differential calculus over the the algebra $\mathfrak{A}$ provided that $\mathfrak{A} = \mathscr{G}_q^0$. The triple $(\mathfrak{A}, d, \mathscr{G}_q)$ is said to be an N-differential calculus over the algebra $\mathfrak{A}$. Every subspace $\mathscr{G}_q^k$ can be viewed as the bimodule over the algebra $\mathscr{G}_q^0$ if we determine the structure of a bimodule with the mappings $\mathscr{G}_q^0 \times \mathscr{G}_q^k \to \mathscr{G}_q^k$ and $\mathscr{G}_q^k \times \mathscr{G}_q^0 \to \mathscr{G}_q^k$ defined by $(u, w) \mapsto uw$ and $(w, v) \mapsto wv$, where $u, v \in \mathscr{G}_q^0$ and $w \in \mathscr{G}_q^k$. Hence we have the following sequence of bimodules over the algebra $\mathscr{G}_q^0$

$$\ldots \xrightarrow{d} \mathscr{G}_q^{k-1} \xrightarrow{d} \mathscr{G}_q^k \xrightarrow{d} \mathscr{G}_q^{k+1} \xrightarrow{d} \ldots \tag{14.5.3}$$

The sequence (14.5.3) can be considered as a cochain N-complex of modules or simply N-complex with N-differential d [6]. The generalized cohomologies of this

N-complex are defined by the formula $H^k_m(\mathscr{G}_q) = Z^k_m(\mathscr{G}_q)/B^k_m(\mathscr{G}_q)$, where

$$Z^k_m(\mathscr{G}_q) = \{u \in \mathscr{G}^k_q : d^m u = 0\} \subset \mathscr{G}^k_q,$$
$$B^k_m(\mathscr{G}_q) = \{u \in \mathscr{G}^k_q : \exists\, v \in \mathscr{G}^{k+m-N}_q, u = d^{N-m} v\} \subset Z^k_m(\mathscr{G}_q).$$

Given a q-DGA $\mathscr{G}_q$ one can associate to it the generalized homologies $H_m(\mathscr{G}_q) = \oplus_{k\in\mathbb{Z}_N} H^k_m(\mathscr{G}_q)$ of the corresponding N-complex (14.5.3).

Next we give the statement of theorem which allows us to construct various N-complexes. Let $\mathscr{G} = \oplus_{k\in\mathbb{Z}_N}\mathscr{G}^k$ be an associative unital $\mathbb{Z}_N$-graded algebra over the field of complex numbers and e be the identity element of this algebra. The graded subspace $\mathscr{Z}(\mathscr{G}) \subset \mathscr{G}$ generated by homogeneous elements $u \in \mathscr{G}^k$, which for any $v \in \mathscr{G}^l$ satisfy $uv = (-1)^{kl} vu$, is called a *graded center* of an algebra $\mathscr{G}$.

Let us generalize the notions of graded commutator and graded derivation of a graded algebra with the help of q-deformations. In general q may be any complex number different from one but for the structures we construct we need q to be a primitive Nth root of unity. *The graded q-commutator* $[\ ,\ ]_q : \mathscr{G}^k \otimes \mathscr{G}^l \to \mathscr{G}^{k+l}$ is defined by

$$[u, v]_q = uv - q^{kl} vu,$$

where $u \in \mathscr{G}^k, \mathscr{G}^l$ are homogeneous elements and q is a primitive Nth root of unity. *A graded q-derivation of degree m* of a graded algebra $\mathscr{G}$ is a linear mapping $\delta : \mathscr{G} \to \mathscr{G}$ of degree m with respect to the graded structure of $\mathscr{G}$, i.e. $\delta : \mathscr{G}^k \to \mathscr{G}^{k+m}$ satisfying the graded q-Leibniz rule

$$\delta(u\, v) = \delta(u)\, v + q^{ml} u\, \delta(v),$$

where $u \in \mathscr{G}^l$.

The following theorem [2] can be used to construct the structure of a q-DGA for a certain class of graded associative unital algebra.

Theorem 14.5.1 *If there exists an element $v \in \mathscr{G}^1$ of degree one which satisfies the condition $v^N \in \mathscr{Z}(\mathscr{G})$, where $N \geq 2$, then an algebra $\mathscr{G}$ equipped with the linear mapping $d : \mathscr{G} \to \mathscr{G}$ defined by the formula $d(u) = [v, u]_q$, $u \in \mathscr{G}$ is the q-DGA and d is its N-differential.*

14.5.2 Connection on Module

In this section we propose a notion of N-connection, which can be viewed as a generalization of a concept of connection on modules. In our generalization we use an algebraic approach based on the concept of q-DGA to define a notion of N-connection and show that in the case of $N = 2$ we get the algebraic analog of a classical connection. A theory of connection on modules can be found in an review [7]. We study the structure of an N-connection, define its curvature and prove the Bianchi identity [1, 2]. We begin this section by recalling the notion of connection on modules given in [7] and called Ω-connection. Suppose that $\mathfrak{A}$ is an unital associative algebra over the field of complex numbers and $\mathcal{E}$ is a left module over $\mathfrak{A}$. Let Ω be a DGA with differential d, such that $\Omega^0 = \mathfrak{A}$, it means that the triple $(\mathfrak{A}, d, \Omega^1)$ is the first oder differential calculus over $\mathfrak{A}$. Since an subspace of elements of grading one can be viewed as a $(\mathfrak{A}, \mathfrak{A})$-bimodule, the tensor product $\Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ clearly has the structure of left $\mathfrak{A}$-module.

A linear map $\nabla : \mathcal{E} \to \Omega^1 \otimes_{\mathfrak{A}} \mathcal{E}$ is called an Ω-*connection* if it satisfies

$$\nabla(us) = du \otimes_{\mathfrak{A}} s + u\nabla(s)$$

for any $u \in \mathfrak{A}$ and $s \in \mathcal{E}$. Similarly to the case of connections on vector bundles, this map has a natural extension $\nabla : \Omega \otimes_{\mathfrak{A}} \mathcal{E} \to \Omega \otimes_{\mathfrak{A}} \mathcal{E}$ by setting

$$\nabla(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + (-1)^p \omega\nabla(s),$$

where $\omega \in \Omega^p$ and $s \in \mathcal{E}$.

We will generalize a notion of Ω-connection taking q-DGA Ω_q instead of DGA Ω. Let $\mathfrak{A}$ be an unital associative algebra over a field of complex numbers, Ω_q is a q-DGA with N-differential d and $\mathfrak{A} = \Omega_q^0$. Let $\mathcal{E}$ be a left $\mathfrak{A}$-module. Considering algebra Ω_q as the $(\mathfrak{A}, \mathfrak{A})$-bimodule we take the tensor product of left $\mathfrak{A}$-modules $\Omega_q \otimes_{\mathfrak{A}} \mathcal{E}$ which has the structure of left $\mathfrak{A}$-module. To minimize the notation, we denote this left $\mathfrak{A}$-module by $\mathfrak{F}$. Taking into account that an algebra Ω_q can be viewed as the direct sum of $(\mathfrak{A}, \mathfrak{A})$-bimodules Ω_q^k we can split the left $\mathfrak{A}$-module $\mathfrak{F}$ into the direct sum of the left $\mathfrak{A}$-modules $\mathfrak{F}^k = \Omega_q^k \otimes_{\mathfrak{A}} \mathcal{E}$, i.e. $\mathfrak{F} = \oplus_k \mathfrak{F}^k$, which means that $\mathfrak{F}$ inherits the graded structure of algebra Ω_q, and $\mathfrak{F}$ is the graded left $\mathfrak{A}$-module. It is worth noting that the left $\mathfrak{A}$-submodule $\mathfrak{F}^0 = \mathfrak{A} \otimes_{\mathfrak{A}} \mathcal{E}$ of elements of grading zero is isomorphic to a left $\mathfrak{A}$-module $\mathcal{E}$, where isomorphism $\varphi : \mathcal{E} \to \mathfrak{F}^0$ can be defined for any $s \in \mathcal{E}$ by $\varphi(s) = e \otimes_{\mathfrak{A}} s$, where e is the identity element of algebra $\mathfrak{A}$. Since a graded q-DGA Ω_q can be viewed as the (Ω_q, Ω_q)-bimodule, the left $\mathfrak{A}$-module $\mathfrak{F}$ can be also considered as the left Ω_q-module and we will use this structure to describe a concept of N-connection. Let us mention that multiplication by elements of Ω^k, where $k \neq 0$, does not preserve the graded structure of the left Ω_q-module $\mathfrak{F}$.

The tensor product $\mathfrak{F} = \Omega_q \times \mathcal{E}$ as the tensor product of two vector spaces has also the structure of the vector space over $\mathbb{C}$. Obviously $\mathfrak{F}$ has a graded structure, i.e. $\mathfrak{F} = \oplus_k \mathfrak{F}^k$, where $\mathfrak{F}^k = \Omega_q^k \otimes_{\mathbb{C}} \mathcal{E}$. Due to the structure of vector space of $\mathfrak{F}$ we can introduce the notion of linear operator on $\mathfrak{F}$. We denote the vector space of linear operators on $\mathfrak{F}$ by $\mathrm{Lin}(\mathfrak{F})$. The structure of the graded vector space of $\mathfrak{F}$ induces the structure of a graded vector space on $\mathrm{Lin}(\mathfrak{F})$, and we shall denote the subspace of homogeneous linear operators of degree k by $\mathrm{Lin}^k(\mathfrak{F})$.

An *N-connection* on the left Ω_q-module $\mathfrak{F}$ is a linear operator $\nabla_q : \mathfrak{F} \to \mathfrak{F}$ of degree one satisfying the condition

$$\nabla_q(\omega \otimes_{\mathfrak{A}} s) = d\omega \otimes_{\mathfrak{A}} s + q^{|\omega|}\, \omega\, \nabla_q(s), \tag{14.5.4}$$

where $\omega \in \Omega_q^i$, $s \in \mathcal{E}$, and $|\omega|$ is the grading of the homogeneous element of algebra Ω_q.

It is worth to mention that if $N = 2$ then $q = -1$, and in this particular case we get the algebraic analog of a classical connection. A connection on vector bundle can be viewed as a linear map on a left module of sections of vector bundle, taking values a algebra of differential 1-forms with values in this vector bundle, which clearly has a structure of a left module over an algebra of smooth functions on a base manifold. Therefore a concept of a N-connection can be viewed as a generalization of a classical connection.

We use the following proposition proved in [1] to define the curvature of N-connection.

Proposition 14.5.2 *The N-th power of any N-connection ∇_q is the endomorphism of degree N of the left Ω_q-module $\mathfrak{F}$.*

The endomorphism $F = \nabla_q^N$ of degree N of the left Ω_q-module $\mathfrak{F}$ is said to be the *curvature of an N-connection* ∇_q.

Let us show that the curvature of an N-connection satisfies Bianchi identity. We proceed to show that the graded vector space $\mathrm{Lin}(\mathfrak{F})$ has a structure of graded algebra. To this end, we take the product $A \circ B$ of two linear operators A, B of the vector space $\mathfrak{F}$ as an algebra multiplication. If $A : \mathfrak{F} \to \mathfrak{F}$ is a homogeneous linear operator than we can extend it to the linear operator $L_A : \mathrm{Lin}(\mathfrak{F}) \to \mathrm{Lin}(\mathfrak{F})$ on the whole graded algebra of linear operators $\mathrm{Lin}(\mathfrak{F})$ by means of the graded q-commutator: $L_A(B) = [A, B]_q = A \circ B - q^{|A||B|} B \circ A$, where B is a homogeneous linear operator. It makes allowable to extend an N-connection ∇_q to the linear operator on the vector space $\mathrm{Lin}(\mathfrak{F})$

$$\nabla_q(A) = [\nabla_q, A]_q = \nabla_q \circ A - q^{|A|}\, A \circ \nabla_q, \tag{14.5.5}$$

where A is a homogeneous linear operator. N-connection ∇_q is the linear operator of degree one on the vector space $\mathrm{Lin}(\mathfrak{F})$, i.e. $\nabla_q : \mathrm{Lin}^k(\mathfrak{F}) \to \mathrm{Lin}^{k+1}(\mathfrak{F})$, and ∇_q satisfies the graded q-Leibniz rule with respect to the algebra structure of $\mathrm{Lin}(\mathfrak{F})$. Consequently the curvature F of an N-connection can be viewed as the linear operator of degree N on the vector space $\mathfrak{F}$, i.e. $F \in \mathrm{Lin}^N(\mathfrak{F})$. Therefore one can act on F by N-connection ∇_q, and it holds that for any N-connection ∇_q the curvature F of this connection satisfies the Bianchi identity

$$\nabla_q(F) = 0. \tag{14.5.6}$$

14.5.3 Local Structure of N-Connection

Connection on the vector bundle of finite rank over a finite dimensional smooth manifold can be studied locally by choosing a local trivialization of the vector bundle and this leads to the basis for the module of sections of this vector bundle.

In order to construct an algebraic analog of the local structure of an N-connection ∇_q we assume $\mathcal{E}$ to be a finitely generated free left $\mathfrak{A}$-module. Let $\mathfrak{e} = \{\mathfrak{e}_\mu\}_{\mu=1}^r$ be a basis for a left module $\mathcal{E}$. This basis induces the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, where $\mathfrak{f}_\mu = e \otimes_{\mathfrak{A}} \mathfrak{e}_\mu$, for the left $\mathfrak{A}$-module $\mathfrak{F}^0 \cong \mathcal{E}$. For any $\xi \in \mathfrak{F}^0$ we have $\xi = \xi^\mu \mathfrak{f}_\mu$. Taking into account that $\mathfrak{F}^0 \subset \mathfrak{F}$ and $\mathfrak{F}$ is the left Ω_q-module we can multiply the elements of the basis $\mathfrak{f}$ by elements of an q-DGA Ω_q. It is easy to see that if $\omega \in \Omega_q^k$ then for any μ we have $\omega \mathfrak{f}_\mu \in \mathfrak{F}^k$. Consequently we can express any element of the $\mathfrak{F}^k$ as a linear combination of $\mathfrak{f}_\mu$ with coefficients from Ω_q^k. Indeed let $\omega \otimes_{\mathfrak{A}} s$ be an element of $\mathfrak{F}^k = \Omega^k \otimes_{\mathfrak{A}} \mathcal{E}$. Then

$$\begin{aligned}\omega \otimes_{\mathfrak{A}} s &= (\omega\, e) \otimes_{\mathfrak{A}} (s^\mu \mathfrak{e}_\mu) = (\omega\, e\, s^\mu) \otimes_{\mathfrak{A}} \mathfrak{e}_\mu \\ &= (\omega s^\mu\, e) \otimes_{\mathfrak{A}} \mathfrak{e}_\mu = \omega s^\mu\, (e \otimes_{\mathfrak{A}} \mathfrak{e}_\mu) = \omega^\mu \mathfrak{f}_\mu,\end{aligned}$$

where $\omega^\mu = \omega s^\mu \in \Omega_q^k$.

Let $\mathfrak{F}^0$ be a finitely generated free module with a basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$, and $s = s^\mu \mathfrak{f}_\mu \in \mathfrak{F}^0$, where $s^\mu \in \mathfrak{A}$. Since N-connection ∇_q is a linear operator of degree one, it follows that $\nabla_q(s) \in \mathfrak{F}^1$, and making use of q-Leibniz rule we can express the element $\nabla_q(s)$ as follows: $\nabla_q(s) = \nabla_q(s^\mu \mathfrak{f}_\mu)$

Denote by $\mathfrak{M}_r(\Omega_q)$ be the vector space of square matrices of order r whose entries are the elements of an q-DGA Ω_q. If each entry of a matrix $\Theta = (\theta_\mu^\nu)$ is an element of a homogeneous subspace Ω_q^k, i.e. $\theta_\mu^\nu \in \Omega_q^k$ then Θ will be refereed to as a homogeneous matrix of degree k and we shall denote the vector space of such matrices by $\mathfrak{M}_r^k(\Omega_q)$. Obviously $\mathfrak{M}_r(\Omega_q) = \oplus_k \mathfrak{M}_r^k(\Omega_q)$. The vector space $\mathfrak{M}_r(\Omega_q)$ of $r \times r$-matrices becomes the associative unital graded algebra if we define the product of two matrices $\Theta = (\theta_\mu^\nu)$, $\Theta' = (\theta'^\nu_\mu) \in \mathfrak{M}_r(\Omega_q)$ by $(\Theta\,\Theta')_\mu^\nu = \theta_\mu^\sigma\, \theta'^\nu_\sigma$.

If $\Theta, \Theta' \in \mathfrak{M}_r(\Omega_q)$ are homogeneous matrices then we define the graded q-commutator by $[\Theta, \Theta']_q = \Theta\,\Theta' - q^{|\Theta||\Theta'|}\Theta'\,\Theta$. We extend the N-differential d of an q-DGA Ω_q to the algebra $\mathfrak{M}_r(\Omega_q)$ as follows $d\Theta = d(\theta_\mu^\nu) = (d\theta_\mu^\nu)$.

Since any element of a left $\mathfrak{A}$-module $\mathfrak{F}^1$ can be expressed in terms of the basis $\mathfrak{f} = \{\mathfrak{f}_\mu\}_{\mu=1}^r$ with coefficients from Ω_q^1, we have

$$\nabla_q(\mathfrak{f}_\mu) = \theta_\mu^\nu\, \mathfrak{f}_\nu, \tag{14.5.7}$$

where $\theta_\mu^\nu \in \Omega_q^1$. An $r \times r$-matrix $\Theta = (\theta_\mu^\nu)$, whose entries θ_μ^ν are the elements of Ω_q^1 i.e. $\Theta \in \mathrm{Mat}_r^1(\Omega_q)$, is said to be a *matrix of an N-connection* ∇_q with respect to the basis $\mathfrak{f}$ of the left $\mathfrak{A}$-module $\mathfrak{F}^0$. Using the definition of N-connection we obtain

$$\nabla_q(s) = (ds^\mu + s^\nu \theta_\nu^\mu)\,\mathfrak{f}_\mu. \tag{14.5.8}$$

Let $\mathfrak{f}' = \{\mathfrak{f}'_\mu\}_{\mu=1}^r$ be another basis for the left $\mathfrak{A}$-module $\mathfrak{F}^0$ with the same number of elements (this will always be the case if $\mathfrak{A}$ is a division algebra or if $\mathfrak{A}$ is commutative). Then $\mathfrak{f}'_\mu = g_\mu^\nu \mathfrak{f}_\nu$, where $G = (g_\mu^\nu) \in \mathrm{Mat}_r^0(\Omega_q)$ is a transition matrix from the basis $\mathfrak{f}$ to the basis $\mathfrak{f}'$. It is well known [10] that in the case of finitely generated free module transition matrix is an invertible matrix. If we denote by θ'^μ_ν the coefficients of ∇_q with respect to a basis $\mathfrak{f}'$ and $\tilde{g}_\nu^\mu$ are the entries of the inverse matrix G^{-1} then

$$\theta'^\mu_\nu = dg_\nu^\sigma\,\tilde{g}_\sigma^\mu + g_\nu^\sigma\,\theta_\sigma^\tau\,\tilde{g}_\tau^\mu,$$

and this clearly shows that the components of ∇_q with respect to different bases of module $\mathfrak{F}^0$ are related by the gauge transformation.

Our next aim is to express the components of the curvature F of a N-connection ∇_q in the terms of the entries of the matrix Θ of an N-connection ∇_q. Computation in successive steps allows us to introduce polynomials $\psi_\nu^{l,\mu} \in \Omega_q^l$ on the entries of the matrix of N-connection and their differentials. We have

$$\begin{aligned}\nabla_q(s) &= (ds^\mu + s^\nu \theta_\nu^\mu)\,\mathfrak{f}_\mu,\\ \psi_\nu^{1,\mu} &:= \theta_\nu^\mu,\end{aligned}$$

$$\begin{aligned}\nabla_q^2(s) &= (d^2 s^\mu + [2]_q ds^\nu \theta_\nu^\mu + s^\nu(d\theta_\nu^\mu + q\theta_\nu^\sigma\theta_\sigma^\mu))\mathfrak{f}_\mu,\\ \psi_\nu^{2,\mu} &:= d\theta_\nu^\mu + q\,\theta_\nu^\sigma\theta_\sigma^\mu,\end{aligned} \tag{14.5.9}$$

$$\begin{aligned}\nabla_q^3(s) &= \Big(d^3 s^\mu + [3]_q d^2 s^\nu \theta_\nu^\mu + [3]_q ds^\nu(d\theta_\nu^\mu + q\theta_\nu^\sigma\theta_\sigma^\mu)\\ &\quad + s^\nu(d^2\theta_\nu^\mu + (q+q^2)d\theta_\nu^\sigma\theta_\sigma^\mu + q^2\theta_\nu^\sigma d\theta_\sigma^\mu + q^3\theta_\nu^\tau\theta_\tau^\sigma\theta_\sigma^\mu)\Big)\mathfrak{f}_\mu,\\ \psi_\nu^{(3,k)\mu} &:= d^2\theta_\nu^\mu + (q+q^2)\,d\theta_\nu^\sigma\theta_\sigma^\mu + q^2\,\theta_\nu^\sigma\,d\theta_\sigma^\mu + q^3\,\theta_\nu^\tau\theta_\tau^\sigma\theta_\sigma^\mu\end{aligned} \tag{14.5.10}$$

Therefore, the kth power of N-connection ∇_q has the following form

$$\begin{aligned}\nabla_q^k(s) &= \sum_{l=0}^k \begin{bmatrix} k\\ l\end{bmatrix}_q d^{k-l}s^\mu\;\psi_\mu^{l,\nu}\;\mathfrak{f}_\nu\\ &= (d^k s^\mu\;\psi_\mu^{0,\nu} + [k]_q\,d^{k-1}s^\mu\;\psi_\mu^{1,\nu} + \ldots + s^\mu\;\psi_\mu^{k,\nu})\,\mathfrak{f}_\nu,\end{aligned} \tag{14.5.11}$$

We can calculate the polynomials $\psi_\mu^{l,\nu}$ by means of the following recursion formula

$$\psi_\mu^{l,\nu} = d\psi_\mu^{l-1,\nu} + q^{l-1}\,\psi_\mu^{l-1,\sigma}\,\theta_\sigma^\nu, \tag{14.5.12}$$

or in the matrix form

$$\Psi^l = d\Psi^{l-1} + q^{l-1}\,\Psi^{l-1}\,\Theta, \tag{14.5.13}$$

We begin with the polynomial $\psi_\mu^{0,\nu} = \delta_\mu^\nu\, e \in \mathfrak{A}$, and e is the identity element of $\mathfrak{A} \subset \Omega_q$. From (14.5.11) it follows that if $k = N$ then the first term $d^N\xi^\mu\ \psi_\mu^{(0,N)\nu}$ in this expansion vanishes because of the N-nilpotency of the N-differential d, and the next terms corresponding to the l values from 1 to $N-1$ also vanish because of the property of q-binomial coefficients. Hence if $k = N$ then the formula (14.5.11) takes on the form

$$\nabla_q^N(s) = s^\mu\,\psi_\mu^{(N,N)\nu}\,\mathfrak{f}_\nu. \tag{14.5.14}$$

In order to simplify the notations and assuming that N is fixed we shall denote $\psi_\mu^\nu = \psi_\mu^{(N,N)\nu}$.

An $(r \times r)$-matrix $\Psi = (\psi_\mu^\nu)$, whose entries are the elements of degree N of a graded q-differential algebra Ω_q, is said to be the *curvature matrix* of a N-connection ∇_q.

Obviously $\Psi \in \mathfrak{M}_r^N(\Omega_q)$. In new notations the formula (14.5.14) can be written as follows $\nabla_q^N(s) = s^\mu\,\psi_\mu^\nu\,\mathfrak{f}_\nu$, and it shows that ∇_q^N is the endomorphism of degree N of the left Ω_q-module $\mathfrak{F}$.

Let us consider the expressions for curvature in the case when $N = 2$. If $N = 2$ then $q = -1$, and a graded q-differential algebra Ω_q is a graded differential algebra with differential d satisfying $d^2 = 0$. This is a classical case, and if we assume that Ω_q is the algebra of differential forms on a smooth manifold M with exterior differential d and exterior multiplication $\wedge$, $\mathcal{E}$ is the module of smooth sections of a vector bundle E over M, ∇_q is a connection on E, $\mathfrak{e}$ is a local frame of a vector bundle E then Θ is the matrix of 1-forms of a connection ∇_q and we have for the components of curvature $\psi_\mu^\nu = d\theta_\mu^\nu - \theta_\mu^\sigma\theta_\sigma^\nu$. In this case Ω_q is super-commutative algebra and we can put the expressions for components of curvature into the form $\psi_\mu^\nu = d\theta_\mu^\nu + \theta_\sigma^\nu\theta_\mu^\sigma$. or by means of matrices $\Psi = d\Theta + \Theta\cdot\Theta$ in which we recognize the classical expression for the curvature.

From the previous section it follows that the curvature of a N-connection satisfies the Bianchi identity. If θ_ν^μ, ψ_ν^μ are the components of an N-connection ∇_q and its curvature F with respect to a basis $\mathfrak{f}$ for the module $\mathfrak{F}$ then the Bianchi identity takes on the form

$$d\psi_\nu^\mu = \theta_\mu^\sigma\psi_\sigma^\nu - \psi_\mu^\sigma\theta_\sigma^\nu.$$

Let us consider now the structure of N-connection forms and their curvature. We apply the algebra of polynomials $\mathfrak{P}[\mathfrak{d}, a]$ over $\mathbb{C}$, constructed in the paper [3] to study the structure of N-curvature. Let Ω_q be an q-DGA We will call an element of degree one $\Theta \in \Omega_q^1$ an *N-connection form* in a graded q-differential algebra Ω_q. The linear operator of degree one $\nabla_q = d + \Theta$ will be referred to as a *covariant N-differential* induced by a N-connection form Θ.

We remind that d is an N-differential which means that $d^k \neq 0$ for $1 \leq k \leq N-1$ and if we successively apply it to an N-connection form Θ we get the sequence of elements $\Theta, d\Theta, d^2\Theta, \ldots, d^{N-1}\Theta$, where $d^k\Theta \in \Omega_q^{k+1}$. Let us denote

$$\Theta_1 = \Theta$$
$$\Theta_2 = d\Theta$$
$$\vdots$$
$$\Theta_N = d^{N-1}\Theta.$$

We denote by $\Omega_q[\Theta]$ the graded subalgebra of Ω_q generated by elements $\Theta_1, \Theta_2, \ldots, \Theta_N$. For any integer $k = 1, 2, \ldots, N$ we define the polynomial $F_k \in \Omega_q[\Theta]$ by the formula $F_k = \nabla_q^{k-1}(\Theta)$. Evidently the subalgebra $\Omega_q[\Theta]$ is isomorphic to the q-DGA $\mathfrak{P}_q[a]$ of [3] if we identify $\Theta_k \rightarrow a_k$. Then the polynomials F_k are identified with the polynomials f_k and we can apply all formulae proved in the case of $\mathfrak{P}_q[a]$ to study the structure of $\Omega_q[\Theta]$.

It follows from [3] that for any integer $1 \leq k \leq N$ the k th power of the covariant N-differential ∇_q can be expanded as follows

$$(\nabla_q)^k = \sum_{i=0}^{k} \begin{bmatrix} k \\ i \end{bmatrix}_q F_{(i)}\, d^{k-1} = d^k + [k]_q F_1 d^{k-1} + \ldots + [n]_q F_{k-1} d + F_k,$$

where $F_k = (\nabla_q)^{k-1}(\Theta)$. Particularly if $k = N$ then the Nth power of the covariant N-differential ∇_q is the operator of multiplication by the element F_N of grading zero. It makes possible to define the curvature of an N-connection form Θ : the *N-curvature form* of an N-connection form Θ is the element of grading zero $F_N \in \mathfrak{A}$.

We get the explicit power expansion formula for N-curvature form of an N-connection

$$F_k = \sum_{\sigma \in \Upsilon_k} \begin{bmatrix} k_2 - 1 \\ k_1 \end{bmatrix}_q \begin{bmatrix} k_3 - 1 \\ k_2 \end{bmatrix}_q \cdots \begin{bmatrix} k - 1 \\ k_{r-1} \end{bmatrix}_q \Theta_{i_1} \Theta_{i_2} \ldots \Theta_{i_r},$$

where Υ_k is the set of all compositions of an integer $1 \leqslant k \leqslant N$, $\sigma = (i_1, i_2, \ldots, i_r)$ is composition of an integer k in the form of a sequence of strictly positive integers, where $i_1 + i_2 + \ldots + i_r = N$, and

$$\begin{aligned} k_1 &= i_1, \\ k_2 &= i_1 + i_2, \\ k_3 &= i_1 + i_2 + i_3, \\ &\ldots \\ k_{r-1} &= i_1 + i_2 + \ldots + i_{r-1}. \end{aligned}$$

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Chapter 15
Conformal Parametrisation of Loxodromes by Triples of Circles

Vladimir V. Kisil and James Reid

Dedicated to Prof. Wolfgang Sprößig on the occasion of his 70th birthday

Abstract We provide a parametrisation of a loxodrome by three specially arranged cycles. The parametrisation is covariant under fractional linear transformations of the complex plane and naturally encodes conformal properties of loxodromes. Selected geometrical examples illustrate the usage of parametrisation. Our work extends the set of objects in Lie sphere geometry—circle, lines and points—to the natural maximal conformally-invariant family, which also includes loxodromes.

Keywords Loxodrome · Fractional linear transformations · Logarithmic spiral · Cycle · Lie geometry · Möbius map · Fillmore–Springer–Cnops construction

Mathematics Subject Classification (2010) Primary 51B10; Secondary 51B25, 51N25, 30C20, 30C35

15.1 Introduction

It is easy to come across shapes of logarithmic spirals, as on Fig. 15.1a, looking either on a sunflower, a snail shell or a remote galaxy. It is not surprising since the fundamental differential equation $\dot{y} = \lambda y$, $\lambda \in \mathbb{C}$ serves as a first approximation to many natural processes. The main symmetries of complex analysis are build on the

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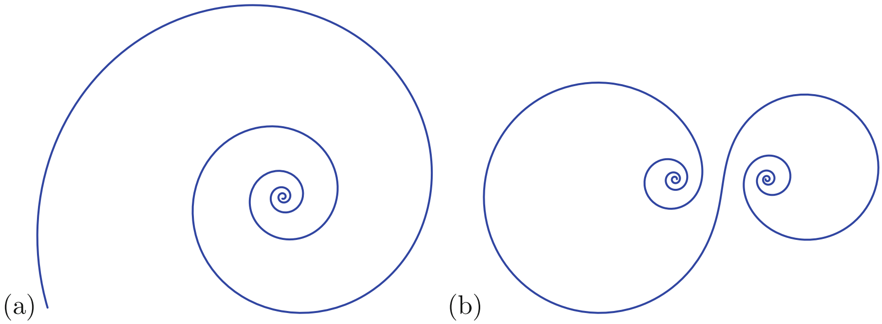

Fig. 15.1 A logarithmic spiral (**a**) and its image under a fractional linear transformation—loxodrome (**b**)

fractional linear transformation (FLT):

$$\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} : z \mapsto \frac{\alpha z + \beta}{\gamma z + \delta}, \quad \text{where } \alpha, \beta, \gamma, \delta \in \mathbb{C} \text{ and } \det \begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix} \neq 0. \qquad (15.1)$$

Thus, images of logarithmic spirals under FLT, called loxodromes, as on Fig. 15.1b shall not be rare. Indeed, they appear in many occasions from the stereographic projection of a rhumb line in navigation to a preferred model of a Carleson arc in the theory singular integral operators [5, 7]. Furthermore, loxodromes are orbits of one-parameter continuous groups of FLT of loxodromic type [3, § 4.3]; [42, § 9.2]; [44, § 9.2].

This setup motivates a search for effective tools to deal with FLT-invariant properties of loxodromes. They were studied from a differential geometry point of view in many papers [6, 37–40, 43], see also [35, § 2.7.6]. In this work we develop a "global" description which matches the Lie sphere geometry framework, see Remark 15.2.3.

The outline of the paper is as follows. After preliminaries on FLT and invariant geometry of cycles (Sect. 15.2) we review the basics of logarithmic spirals and loxodromes (Sect. 15.3). A new parametrisation of loxodromes is introduced in Sect. 15.4 and several examples illustrate its usage in Sect. 15.5. Section 15.6 frames our work within a wider approach [29–31], which extends Lie sphere geometry. A brief list of open questions concludes the paper.

15.2 Preliminaries: Fractional Linear Transformations and Cycles

In this section we provide some necessary background in Lie geometry of circles, fractional-linear transformations and Fillmore–Springer–Cnops construction (FSCc). Regretfully, the latter remains largely unknown in the context of complex numbers despite of its numerous advantages. We will have some further discussion of this in Remark 15.2.3 below.

The right way [42, § 9.2] to think about FLT (15.1) is through the *projective complex line* $P\mathbb{C}$. It is the family of cosets in $\mathbb{C}^2 \setminus \{(0,0)\}$ with respect to the equivalence relation $\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \sim \begin{pmatrix} \alpha w_1 \\ \alpha w_2 \end{pmatrix}$ for all nonzero $\alpha \in \mathbb{C}$. Conveniently $\mathbb{C}$ is identified with a part of $P\mathbb{C}$ by assigning the coset of $\begin{pmatrix} z \\ 1 \end{pmatrix}$ to $z \in \mathbb{C}$. Loosely speaking $P\mathbb{C} = \mathbb{C} \cup \{\infty\}$, where ∞ is the coset of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The pair $[w_1 : w_2]$ with $w_2 \neq 0$ gives *homogeneous coordinates* for $z = w_1/w_2 \in \mathbb{C}$. Then, the linear map $\mathbb{C}^2 \to \mathbb{C}^2$

$$M : \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \mapsto \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} \alpha w_1 + \beta w_2 \\ \gamma w_1 + \delta w_2 \end{pmatrix}, \text{ where } M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}) \tag{15.2}$$

factors from $\mathbb{C}^2$ to $P\mathbb{C}$ and coincides with (15.1) on $\mathbb{C} \subset P\mathbb{C}$.

Generic equations of cycle in real and complex coordinates $z = x + \mathrm{i}y$ are:

$$k(x^2 + y^2) - 2lx - 2ny + m = 0 \qquad \text{or} \qquad kz\bar{z} - \bar{L}z - L\bar{z} + m = 0\,, \tag{15.3}$$

where $(k, l, n, m) \in \mathbb{R}^4$ and $L = l + \mathrm{i}n$. This includes lines (if $k = 0$), points as zero-radius circles (if $l^2 + n^2 - mk = 0$) and proper circles otherwise. Homogeneity of (15.3) suggests that (k, l, m, n) shall be considered as homogeneous coordinates $[k : l : m : n]$ of a point in three-dimensional projective space $P\mathbb{R}^3$.

The homogeneous form of cycle's equation (15.3) for $z = [w_1 : w_2]$ can be written[1] using matrices as follows:

$$kw_1\bar{w}_1 - \bar{L}w_1\bar{w}_2 - L\bar{w}_1 w_2 + mw_2\bar{w}_2 = \begin{pmatrix} -\bar{w}_2 & \bar{w}_1 \end{pmatrix} \begin{pmatrix} \bar{L} & -m \\ k & -L \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = 0. \tag{15.4}$$

From now on we identify a cycle C given by (15.3) with its 2×2 matrix $\begin{pmatrix} \bar{L} & -m \\ k & -L \end{pmatrix}$, this is called the *Fillmore–Springer–Cnops construction* (FSCc). Again, C shall be treated up to the equivalence relation $C \sim tC$ for all real $t \neq 0$. Then, the linear action (15.2) corresponds to some action on 2×2 cycle matrices by the intertwining

[1] Of course, this is not the only possible presentation. However, this form is particularly suitable to demonstrate FLT-invariance (15.8) of the cycle product below.

identity:

$$\begin{pmatrix}-\bar{w}_2' & \bar{w}_1'\end{pmatrix}\begin{pmatrix}\bar{L}' & -m' \\ k' & -L'\end{pmatrix}\begin{pmatrix}w_1' \\ w_2'\end{pmatrix} = \begin{pmatrix}-\bar{w}_2 & \bar{w}_1\end{pmatrix}\begin{pmatrix}\bar{L} & -m \\ k & -L\end{pmatrix}\begin{pmatrix}w_1 \\ w_2\end{pmatrix}. \tag{15.5}$$

Explicitly, for $M \in \mathrm{GL}_2(\mathbb{C})$ those actions are:

$$\begin{pmatrix}w_1' \\ w_2'\end{pmatrix} = M\begin{pmatrix}w_1 \\ w_2\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}\bar{L}' & -m' \\ k' & -L'\end{pmatrix} = \bar{M}\begin{pmatrix}\bar{L} & -m \\ k & -L\end{pmatrix}M^{-1}, \tag{15.6}$$

where $\bar{M}$ is the component-wise complex conjugation of M. Note, that FLT M (15.1) corresponds to a linear transformation $C \mapsto M(C) := \bar{M}CM^{-1}$ of cycle matrices in (15.6). A quick calculation shows that $M(C)$ indeed has real off-diagonal elements as required for a FSCc matrix.

This paper essentially depends on the following

Proposition 15.2.1 *Define a* cycle product *of two cycles C and C' by:*

$$\langle C, C'\rangle := \mathrm{tr}(C\bar{C}') = L\bar{L}' + \bar{L}L' - mk' - km'. \tag{15.7}$$

Then, the cycle product is FLT-invariant:

$$\langle M(C), M(C')\rangle = \langle C, C'\rangle \qquad \text{for any } M \in \mathrm{SL}_2(\mathbb{C}). \tag{15.8}$$

Proof Indeed:

$$\begin{aligned}\langle M(C), M(C')\rangle &= \mathrm{tr}(M(C)\overline{M(C')}) \\ &= \mathrm{tr}(\bar{M}CM^{-1}M\bar{C}'\bar{M}^{-1}) \\ &= \mathrm{tr}(\bar{M}C\bar{C}'\bar{M}^{-1}) \\ &= \mathrm{tr}(C\bar{C}') \\ &= \langle C, C'\rangle,\end{aligned}$$

using the invariance of trace. □

Note that the cycle product (15.7) is *not* positive definite, it produces a Lorentz-type metric in $\mathbb{R}^4$. Here are some relevant examples of geometric properties expressed through the cycle product:

Example 15.2.2

1. If $k = 1$ (and C is a proper circle), then $\langle C, C\rangle/2$ is equal to the square of radius of C. In particular $\langle C, C\rangle = 0$ indicates a zero-radius circle representing a point.
2. If $\langle C_1, C_2\rangle = 0$ for non-zero radius cycles C_1 and C_2, then they intersects at the right angle.

3. If $\langle C_1, C_2 \rangle = 0$ and C_2 is zero-radius circle, then C_1 passes the point represented by C_2.

In general, a combination of (15.6) and (15.8) yields that a consideration of FLT in $\mathbb{C}$ can be replaced by linear algebra in the space of cycles $\mathbb{R}^4$ (or rather $P\mathbb{R}^3$) with an indefinite metric, see [12] for the latter.

A spectacular (and needed later) illustration of this approach is orthogonal pencils of cycles. Consider a collection of all cycles passing two different points in $\mathbb{C}$, it is called an *elliptic pencil*. A beautiful and non-elementary fact of the Euclidean geometry is that cycles orthogonal to every cycle in the elliptic pencil fill the entire plane and are disjoint, the family is called a *hyperbolic pencil*. The statement is obvious in the standard arrangement when the elliptic pencil is formed by straight lines—cycles passing the origin and infinity. Then, the hyperbolic pencil consists of the concentric circles, see Fig. 15.2. For the sake of completeness, a *parabolic pencil* (not used in this paper) formed by all circles touching a given line at a given point, [25, Ex. 6.10] contains further extensions and illustrations. See [44, § 11.8] for an example of cycle pencils' appearance in operator theory.

This picture trivialises a bit in the language of cycles. A pencil of cycles (of any type!) is a linear span $tC_1 + (1-t)C_2$ of two arbitrary different cycles C_1 and C_2 from the pencil. Again, this is easier to check for the standard pencils. A pencil is elliptic, parabolic or hyperbolic depending on which inequality holds [25, Ex. 5.28.ii]:

$$\langle C_1, C_2 \rangle^2 \lesseqqgtr \langle C_1, C_1 \rangle \langle C_2, C_2 \rangle . \tag{15.9}$$

Then, the orthogonality of cycles on the plane is exactly their orthogonality as vectors with respect to the indefinite cycle product (15.7). For cycles in the

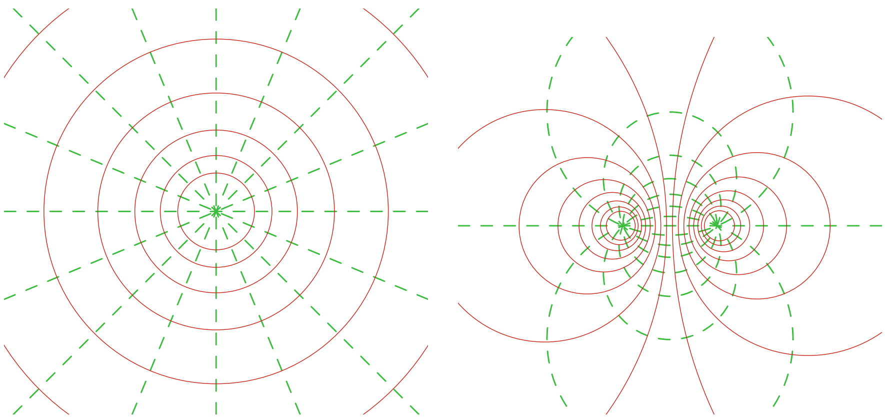

Fig. 15.2 Orthogonal elliptic (green-dashed) and hyperbolic (red-solid) pencils of cycles. Left drawing shows the standard case and the right—generic, which is the image of the standard pencils under FLT

standard pencils this is immediately seen from the explicit expression of the product $\left\langle C, C' \right\rangle = L\bar{L}' + \bar{L}L' - mk' - km'$ in cycle components. Finally, linearization (15.6) of FLT in the cycle space shows that a pencil (i.e. a linear span) is transformed to a pencil and FLT-invariance (15.8) of the cycle product guarantees that the orthogonality of two pencils is preserved.

Remark 15.2.3 A sketchy historic overview (we apologise for any important omission!) starts from the concept of Lie sphere geometry, see [4, Ch. 3] for a detailed presentation. It unifies circles, lines and points, which all are called cycles in this context (analytically it is already in (15.3)). The main invariant property of Lie sphere geometry is *tangential contact*. The first radical advance came from the observation that cycles (through their parameters in (15.3)) naturally form a linear or projective space, see [36]; [41, Ch. 1]. The second crucial step is the recognition that the cycle space carries out the FLT-invariant indefinite metric [4, Ch. 3]; [18, § F.4]. At the same time some presentations of cycles by 2×2 matrices were used [42, § 9.2]; [41, Ch. 1]; [18, § F.4]. Their main feature is that FLT in $\mathbb{C}$ corresponds to a some sort of linear transform by matrix conjugation in the cycle space. However, the metric in the cycle space was not expressed in terms of those matrices.

All three ingredients—matrix presentation with linear structure and the invariant product—came happily together as Fillmore–Springer–Cnops construction (FSCc) in the context of Clifford algebras [8, Ch. 4]; [9]. Regretfully, FSCc have not yet propagated back to the most fundamental case of complex numbers, cf. [42, § 9.2] or somewhat cumbersome techniques used in [4, Ch. 3]. Interestingly, even the founding fathers were not always strict followers of their own techniques, see [10].

A combination of all three components of Lie cycle geometry within FSCc facilitates further development. It was discovered that for the smaller group $\mathrm{SL}_2(\mathbb{R})$ there exist more types—elliptic, parabolic and hyperbolic–of invariant metrics in the cycle space [19, 23, 25, Ch. 5]. Based on the earlier work [18], the key concept of Lie sphere geometry—*tangency of two cycles* C_1 and C_2—was expressed through the cycle product (15.7) as [25, Ex. 5.26.ii]:

$$\langle C_1 + C_2, C_1 + C_2 \rangle = 0$$

for C_1, C_2 normalised such that $\langle C_1, C_1 \rangle = \langle C_2, C_2 \rangle = 1$. Furthermore, $C_1 + C_2$ is the zero-radius cycle representing the point of contact.

FSCc is useful in consideration of the Poincaré extension of Möbius maps [29] and continued fractions [28]. In theoretical physics FSCc nicely describes conformal compactifications of various space-time models [15, 16, 21, 25, § 8.1]. Last but not least, FSCc is behind the Computer Algebra System (CAS) operating in Lie sphere geometry [20, 30]. FSCc equally well covers not only the field of complex numbers but rings of dual and double numbers as well [25]. New usage of FSCc will be given in the following sections in applications to loxodromes.

15.3 Fractional Linear Transformations and Loxodromes

In aiming for a covariant description of loxodromes we start from the following definition.

Definition 15.3.1 A *standard logarithmic spiral* (SLS) with parameter $\lambda \in \mathbb{C}$ is the orbit of the point 1 under the (disconnected) one-parameter subgroup of FLT of diagonal matrices

$$D_\lambda(t) = \begin{pmatrix} \pm e^{\lambda t/2} & 0 \\ 0 & e^{-\lambda t/2} \end{pmatrix}, \qquad t \in \mathbb{R}. \tag{15.10}$$

Remark 15.3.2 Our SLS is a *union* of two branches, each of them is a logarithmic spiral in the common sense. The three-cycle parametrisation of loxodromes presented below will becomes less elegant if those two branches need to be separated. Yet, we draw just one "positive" branch on Fig. 15.3 to make it more transparent.

SLS is the solution of the differential equation $z' = \lambda z$ with the initial value $z(0) = \pm 1$ and has the parametric equation $z(t) = \pm e^{\lambda t}$. Furthermore, we obtain the same orbit for λ_1 and $\lambda_2 \in \mathbb{C}$ if $\lambda_1 = a\lambda_2$ for real $a \neq 0$ through a re-parametrisation of the time $t \mapsto at$. Thus, SLS is identified by the point $[\Re(\lambda) : \Im(\lambda)]$ of the real projective line $P\mathbb{R}$. Thereafter the following classification is useful:

Definition 15.3.3 SLS is

- *positive*, if $\Re(\lambda) \cdot \Im(\lambda) > 0$;
- *degenerate*, if $\Re(\lambda) \cdot \Im(\lambda) = 0$;
- *negative*, if $\Re(\lambda) \cdot \Im(\lambda) < 0$.

Informally: a positive SLS unwinds counterclockwise, a negative—clockwise. Degenerate SLS is the unit circle if $\Im(\lambda) \neq 0$ and the punctured real axis $\mathbb{R} \setminus \{0\}$ if $\Re(\lambda) \neq 0$. If $\Re(\lambda) = \Im(\lambda) = 0$ then SLS is the single point 1.

Definition 15.3.4 A *logarithmic spiral* is the image of a SLS under a complex affine transformation $z \mapsto \alpha z + \beta$, with $\alpha, \beta \in \mathbb{C}$. A *loxodrome* is an image of a SLS under a generic FLT (15.1).

Obviously, a complex affine transformation is FLT with the upper triangular matrix $\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$. Thus, logarithmic spirals form an affine-invariant (but not FLT-invariant) subset of loxodromes. Thereafter, loxodromes (and their degenerate forms—circles, straight lines and points) extend the notion of cycles from the Lie sphere geometry, cf. Remark 15.2.3.

By the nature of Definition 15.3.4, the parameter λ and the corresponding classification from Definition 15.3.3 remain meaningful for logarithmic spirals and loxodromes. FLTs eliminate distinctions between circles and straight lines, but for degenerate loxodromes ($\Re(\lambda) \cdot \Im(\lambda) = 0$) we still can note the difference between two cases of $\Re(\lambda) \neq 0$ and $\Im(\lambda) \neq 0$: orbits of former are whole circles (straight lines) while latter orbits are only arcs of circles (segments of lines).

The immediate consequence of Definition 15.3.4 is

Proposition 15.3.5 *The collection of all loxodromes is a FLT-invariant family. Degenerate loxodromes—(arcs of) circles and (segments) of straight lines—form a FLT-invariant subset of loxodromes.*

As mentioned above, SLS is completely characterised by the point $[\Re(\lambda) : \Im(\lambda)]$ of the real projective line $P\mathbb{R}$ extended by the additional point $[0 : 0]$.[2] In the standard way, $[\Re(\lambda) : \Im(\lambda)]$ is associated with the real value $\tilde{\lambda} := 2\pi\Re(\lambda)/\Im(\lambda)$ extended by ∞ for $\Im(\lambda) = 0$ and symbol $\frac{0}{0}$ for the $\Re(\lambda) = \Im(\lambda) = 0$ cases. Geometrically, $a = \exp(\tilde{\lambda}) \in \mathbb{R}_+$ represents the next point after 1, where the given SLS branch meets the real positive half-axis after one full counterclockwise turn. Obviously, $a > 1$ and $a < 1$ for positive and negative SLS, respectively. For a degenerate SLS:

1. with $\Im(\lambda) \neq 0$ we obtain $\tilde{\lambda} = 0$ and $a = 1$;
2. with $\Re(\lambda) \neq 0$ we consistently *define* $a = \infty$.

In essence, a loxodrome Λ is defined by the pair $(\tilde{\lambda}, M)$, where M is the FLT mapping Λ to SLS with the parameter $\tilde{\lambda}$. While $\tilde{\lambda}$ is completely determined by Λ, a map M is not.

Proposition 15.3.6

1. *The subgroup of FLT which maps SLS with the parameter $\tilde{\lambda}$ to itself consists of products $D_{\tilde{\lambda}}(t)R^{\varepsilon}$, $\varepsilon = 0, 1$ of transformations $D_{\tilde{\lambda}}(t) = D_{\lambda}(t)$, $\lambda = \tilde{\lambda} + 2\pi\mathrm{i}$* (15.10) *and branch-swapping reflections:*

$$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : z \mapsto -z^{-1}. \tag{15.11}$$

2. *Pairs $(\tilde{\lambda}, M)$ and $(\tilde{\lambda}', M')$ define the same loxodrome if and only if*
   a. $\tilde{\lambda} = \tilde{\lambda}'$;
   b. $M = D_{\tilde{\lambda}}(t)R^{\varepsilon}M'$ *for* $\varepsilon = 0, 1$ *and* $t \in \mathbb{R}$.

Remark 15.3.7 Often loxodromes appear as orbits of one-parameter continuous subgroups of loxodromic FLT, which are characterised by non-real traces [3, § 4.3]; [42, § 9.2]; [44, § 9.2]. In the above notations such a subgroup is $MD_{\tilde{\lambda}}(t)M^{-1}$, thus the common presentation is not much different from the above $(\tilde{\lambda}, M)$-parametrisation. Furthermore, we need to pick up any point on a loxodrome to present it as an orbit.

[2] Pedantic consideration of the trivial case $\Re(\lambda) = \Im(\lambda) = 0$ will be often omitted in the following discussion.

15.4 Three-Cycle Parametrisation of Loxodromes

Although pairs $(\tilde{\lambda}, M)$ provide a parametrisation of loxodromes, the following alternative is more operational. It is inspired by the orthogonal pairs of elliptic and hyperbolic pencils described in Sect. 15.2.

Definition 15.4.1 A *three-cycle parametrisation* $\{C_1, C_2, C_3\}$ of a non-degenerate SLS $\tilde{\lambda}$ satisfies the following conditions:

1. C_1 is the straight line passing the origin;
2. C_2 and C_3 are two circles with their centres at the origin;
3. Λ passes the intersection points $C_1 \cap C_2$ and $C_1 \cap C_3$; and
4. A branch of Λ makes one full counterclockwise turn between intersection points $C_1 \cap C_2$ and $C_1 \cap C_3$ belonging to a ray in C_1 from the origin.

We say that three-cycle parametrisation is *standard* if C_1 is the real axis and C_2 is the unit circle, then $C_3 = \{z : |z| = \exp(\tilde{\lambda})\}$. A three-cycle parametrisation can be consistently extended to a degenerate SLS Λ as follows:

$\tilde{\lambda} = 0$: any straight line C_1 passing the origin and the unit circles $C_2 = C_3 = \Lambda$;
$\tilde{\lambda} = \infty$: the real axis as $C_1 = \Lambda$, the unit circle as C_2 and $C_3 = (0, 0, 0, 1)$ being the zero-radius circle at infinity.

Since cycles are elements of the projective space, the following *normalised cycle product*:

$$[C_1, C_2] := \frac{\langle C_1, C_2\rangle}{\sqrt{\langle C_1, C_1\rangle \langle C_2, C_2\rangle}} \tag{15.12}$$

is more meaningful than the cycle product (15.7) itself. Note that, $[C_1, C_2]$ is defined only if neither C_1 nor C_2 is a zero-radius cycle (i.e. a point). Also, the normalised cycle product is $GL_2(\mathbb{C})$-invariant in comparison to $SL_2(\mathbb{C})$-invariance in (15.8).

A reader will instantly recognise the familiar pattern of the cosine of angle between two vectors appeared in (15.12). Simple calculations show that this geometric interpretation is very explicit in two special cases of our interest.

Lemma 15.4.2

1. *Let C_1 and C_2 be two straight lines passing the origin with slopes $\tan\phi_1$ and $\tan\phi_2$ respectively. Then $C_2 = D_{x+\mathrm{i}y}(1)C_1$ for transformation (15.10) with any $x \in \mathbb{R}$ and $y = \phi_2 - \phi_1$ satisfying the relations:*

$$[C_1, C_2] = \cos y\,. \tag{15.13}$$

2. *Let C_1 and C_2 be two circles centred at the origin and radii r_1 and r_2 respectively. Then $C_2 = D_{x+\mathrm{i}y}(1)C_1$ for transformation (15.10) with any $y \in \mathbb{R}$ and $x =$*

$\log(r_2) - \log(r_1)$ *satisfying the relations:*

$$[C_1, C_2] = \cosh x\,. \tag{15.14}$$

Note the explicit elliptic-hyperbolic analogy between (15.13) and (15.14). By the way, both expressions produce real x and y due to inequality (15.9) for the respective types of pencils. Now we can deduce the following properties of three-cycle parametrisation.

Proposition 15.4.3 *For a given SLS* Λ *with a parameter* λ*:*

1. *Any transformation* (15.10) *maps a three-cycle parametrisation of* Λ *to another three-cycle parametrisation of* Λ.
2. *For any two three-cycle parametrisations* $\{C_1, C_2, C_3\}$ *and* $\{C_1', C_2', C_3'\}$ *there exists* $t_0 \in \mathbb{R}$ *such that* $C_j' = D_\lambda(t_0)C_j$ *for* $D_\lambda(t_0)$ (15.10) *and* $j = 1, 2, 3$.
3. *The parameter* $\tilde{\lambda} = 2\pi\Re(\lambda)/\Im(\lambda)$ *of SLS can be recovered form its three-cycle parametrisation by the relation:*

$$\tilde{\lambda} = \operatorname{arccosh}[C_2, C_3] \quad \textit{and} \quad \lambda \sim \tilde{\lambda} + 2\pi \mathrm{i}\,. \tag{15.15}$$

Proof The first statement is obvious. For the second we take $D_\lambda(t_0) : \Lambda \to \Lambda$ which maps $C_1 \cap C_2$ to $C_1' \cap C_2'$, this transformation maps $C_j \mapsto C_j'$ for $j = 1, 2, 3$. Finally, the last statement follows from (15.14). □

Note that expression (15.15) is FLT-invariant. Since any loxodrome is an image of SLS under FLT we obtain a three-cycle parametrisation of loxodromes as follows.

Proposition 15.4.4

1. *Any three-cycle parametrisation* $\{C_1, C_2, C_3\}$ *of SLS has the following FLT-invariant properties:*
   a. C_1 *is orthogonal to* C_2 *and* C_3*;*
   b. C_2 *and* C_3 *either disjoint or coincide.*[3]
2. *For any FLT* M *and a three-cycle parametrisation* $\{C_1', C_2', C_3'\}$ *of SLS, three cycles* $C_j = M(C_j')$*,* $j = 1, 2, 3$ *satisfy the above conditions* (1a) *and* (1b).
3. *For any triple of cycles* $\{C_1, C_2, C_3\}$ *satisfying the above conditions* (1a) *and* (1b) *there exist a FLT* M *such cycles* $\{M(C_1), M(C_2), M(C_3)\}$ *provide a three-cycle parametrisation of SLS with the parameter* $\tilde{\lambda}$ (15.15)*. FLT* M *is uniquely defined by the additional condition that* $\{M(C_1), M(C_2), M(C_3)\}$ *is a* standard *parametrisation of SLS.*

Proof The first statement is obvious, the second follows because properties (1a) and (1b) are FLT-invariant.

[3] Recall that if $C_2 = C_3$, then SLS is degenerate and coincide with $C_2 = C_3$.

For (3) in the degenerate case $C_2 = C_3$: any M which sends $C_2 = C_3$ to the unit circle will do the job. If $C_2 \neq C_3$ we explicitly describe below the procedure, which produces FLT M mapping the loxodrome to SLS. □

Procedure 15.4.5 Two disjoint cycles C_2 and C_3 span a hyperbolic pencil H as described in Sect. 15.2. Then C_1 belongs to the elliptic E pencil orthogonal to H. Let C_0 and C_∞ be the two zero-radius cycles (points) from the hyperbolic pencil H. Every cycle in E, including C_1, passes C_0 and C_∞, we label those two in such a way that

- for a positive $\tilde{\lambda}$ cycle C_3 is between C_2 and C_∞; and
- for a negative $\tilde{\lambda}$ cycle C_3 is between C_2 and C_0.

Here "between" for cycles means "between" for their intersection points with C_1. Finally, let C_u be any of two intersection points $C_1 \cap C_2$. Then, there exists the unique FLT M such that $M : C_0 \mapsto 0$, $M : C_u \mapsto 1$ and $M : C_\infty \mapsto \infty$. We will call M the *standard FLT associated* to the three-cycle parametrisation $\{C_1, C_2, C_3\}$ of the loxodrome.

Remark 15.4.6 To complement the construction of the standard FLT M associated to the three-cycle parametrisation $\{C_1, C_2, C_3\}$ from Procedure 15.4.5, we can describe the inverse operation. For the loxodrome, which is the image of SLS with the parameter λ under FLT M, we define the *standard three-cycle parametrisation* $\{M(\mathbb{R}), M(C_u), M(C_\lambda)\}$ as the image of the standard parametrisation of the SLS under M. Here $\mathbb{R}$ is the real axis, $C_u = \{z : |z| = 1\}$ is the unit circle and $C_\lambda = \{z : |z| = \exp(\tilde{\lambda})\}$.

In essence, the previous proposition says that a three-cycle and (λ, M) parametrisations are equivalent and delivers an explicit procedure producing one from another. However, three-cycle parametrisation is more geometric, since it links a

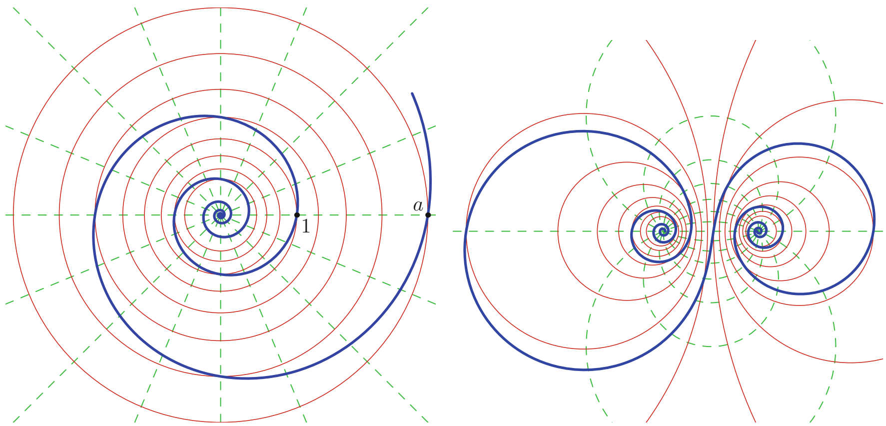

Fig. 15.3 Logarithmic spirals (left) and loxodrome (right) with associated pencils of cycles. This is a combination of Figs. 15.1 and 15.2

loxodrome to a pair of orthogonal pencils, see Fig. 15.3. Furthermore, cycles C_1, C_2, C_3 (unlike parameters λ and M) can be directly drawn on the plane to represent a loxodrome, which may be even omitted.

15.5 Applications of Three-Cycle Parametrisation

Now we present some examples of the usage of three-cycle parametrisation of loxodromes. Any parametrisation mentioned in this paper has some arbitrariness. For pairs $(\tilde{\lambda}, M)$ that is described in Proposition 15.3.6. Characterisation as orbits from Remark 15.3.7 seems to be most ambiguous: besides of the previous freedom in the one-parameter subgroup choice, we shall pick up any point of the loxodrome as well. Now we want to resolve non-uniqueness in the three-cycle parametrisation. Recall, that a triple $\{C_1, C_2, C_3\}$ is non-degenerate if $C_2 \neq C_3$ and C_3 is not zero-radius.

Proposition 15.5.1 *Two non-degenerate triples $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$ parameterise the same loxodrome if and only if all the following conditions are satisfied:*

1. *Pairs $\{C_2, C_3\}$ and $\{C_2', C_3'\}$ span the same hyperbolic pencil. That is cycles C_2' and C_3' are linear combinations of C_2 and C_3 and vise versa.*
2. *Pairs $\{C_2, C_3\}$ and $\{C_2', C_3'\}$ define the same parameter $\tilde{\lambda}$:*

$$[C_2, C_3] = \left[C_2', C_3'\right]. \tag{15.16}$$

3. *The elliptic-hyperbolic identity holds:*

$$\frac{\operatorname{arccosh}\left[C_j, C_j'\right]}{\operatorname{arccosh}\left[C_2, C_3\right]} \equiv \frac{1}{2\pi} \arccos\left[C_1, C_1'\right] \pmod 1, \tag{15.17}$$

where j is either 2 *or* 3.

Proof Necessity of (1) is obvious, since hyperbolic pencils spanned by $\{C_2, C_3\}$ and $\{C_2', C_3'\}$ shall be both the image of concentric circles centred at origin under FLT M defining the loxodrome. Necessity of (2) is also obvious since $\tilde{\lambda}$ is uniquely defined by the loxodrome. Necessity of (3) follows from the analysis of the following demonstration of sufficiency.

For sufficiency, let M be FLT constructed through Procedure 15.4.5 from $\{C_1, C_2, C_3\}$. Then (1) implies that $M(C_2')$ and $M(C_3')$ are also circles centred at origin. Then Lemma 15.4.2 implies that the transformation $D_{x+\mathrm{i}y}(1)$, where $x = \operatorname{arccosh}\left[C_2, C_2'\right]$ and $y = \arccos\left[C_1, C_1'\right]$ maps C_1' and C_2' to C_1 and C_2 respectively. Furthermore, from identity (15.16) follows that the same $D_{x+\mathrm{i}y}(1)$ maps C_3' to C_3. Finally, condition (15.17) means that $x + \mathrm{i}(y + 2\pi n) = a(\tilde{\lambda} + 2\pi \mathrm{i})$

for $a = x/\tilde{\lambda}$ and some $n \in \mathbb{Z}$. In other words $D_{x+\mathrm{i}y}(1) = D_{\tilde{\lambda}}(a)$, thus $D_{x+\mathrm{i}y}(1)$ maps SLS with the parameter $\tilde{\lambda}$ to itself. Since $\{M(C_1), M(C_2), M(C_3)\}$ and $\{M(C'_1), M(C'_2), M(C'_3)\}$ are two three-cycle parametrisations of the same SLS, $\{C_1, C_2, C_3\}$ and $\{C'_1, C'_2, C'_3\}$ are two three-cycle parametrisations of the same loxodrome. □

Equivalent triples of cycles parametrising the same loxodrome are shown on Fig. 15.4 (an animation is available with the electronic version of this paper at arXiv [32]). Relation (15.17), which correlates elliptic and hyperbolic rotations

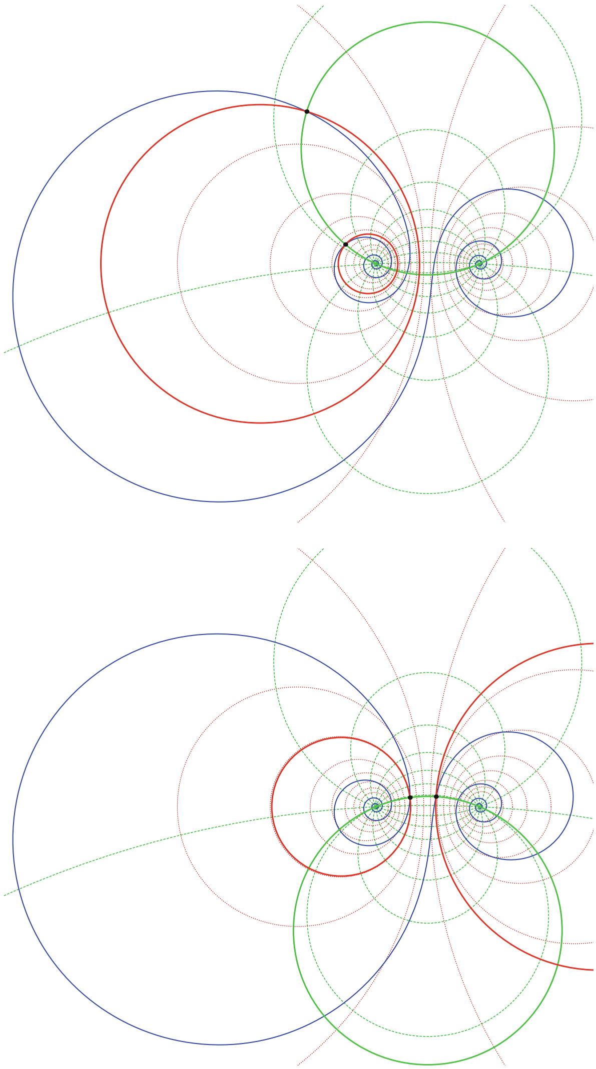

Fig. 15.4 Two equivalent parametrisations of the same loxodrome by different triples of cycles. The green cycle is C_1, two red circles are C_2 and C_3. Full animation of different parametrisations can be seen at [27]

for loxodrome, regularly appears in this context. The next topic provides another illustration of this.

Procedure 15.5.2 To verify whether a loxodrome parametrised by three cycles $\{C_1, C_2, C_3\}$ passes a point parametrised by a zero-radius cycle C_0 we perform the following steps:

1. Define the cycle

$$C_h = tC_2 + (1-t)C_3\,, \qquad \text{where } t = -\frac{\langle C_0, C_3\rangle}{\langle C_0, C_2 - C_3\rangle}\,, \tag{15.18}$$

   which belongs to the hyperbolic pencil spanned by $\{C_2, C_3\}$ and is orthogonal to C_0, that is, passes the respective point.
2. Find cycle C_e from the elliptic pencil orthogonal to $\{C_2, C_3\}$ which passes C_0. C_e is the solution of the system of three linear (with respect to parameters of C_e) equations, cf. Example 15.2.2:

$$\langle C_e, C_0\rangle = 0\,,$$
$$\langle C_e, C_2\rangle = 0\,,$$
$$\langle C_e, C_3\rangle = 0\,.$$

3. Verify the elliptic-hyperbolic relation:

$$\frac{\operatorname{arccosh}[C_h, C_2]}{\operatorname{arccosh}[C_2, C_3]} \equiv \frac{1}{2\pi}\arccos[C_e, C_1] \pmod 1\,. \tag{15.19}$$

Proof Let M be the standard FLT associated to $\{C_1, C_2, C_3\}$ from Procedure 15.4.5. The point C_0 belongs to the loxodrome if the transformation $D_{\tilde{\lambda}}(t)$ for some t moves $M(C_0)$ to the intersection $M(C_1) \cap M(C_2)$. But $D_{x+\mathrm{i}y}(1)$ with $x = \operatorname{arccosh}[C_h, C_2]$ and $y = \arccos[C_e, C_1]$ maps $M(C_h) \to M(C_2)$ and $M(C_e) \to M(C_1)$, thus it also maps $M(C_0) \subset M(C_h) \cap M(C_e)$ to $M(C_1) \cap M(C_2)$. Condition (15.19) guaranties that $D_{x+\mathrm{i}y}(1) = D_{\tilde{\lambda}}(x/\tilde{\lambda})$, as in the previous Prop. □

Our final example considers two loxodromes which may have completely different associated pencils.

Procedure 15.5.3 Let two loxodromes are parametrised by $\{C_1, C_2, C_3\}$ and $\{C_1', C_2', C_3'\}$. Assume they intersect at some point parametrised by a zero-radius cycle C_0 (this can be checked by Procedure 15.5.2, if needed). To find the angle of intersection we perform the following steps:

1. Use (15.18) to find cycles C_h and C_h' belonging to hyperbolic pencils, spanned by $\{C_2, C_3\}$ and $\{C_2', C_3'\}$ respectively, and both passing C_0.

2. The intersection angle is

$$\arccos\left[C_h, C_h'\right] - \arctan\left(\frac{\tilde{\lambda}}{2\pi}\right) + \arctan\left(\frac{\tilde{\lambda}'}{2\pi}\right), \tag{15.20}$$

where $\tilde{\lambda}$ and $\tilde{\lambda}'$ are determined by (15.15).

Proof A loxodrome intersects any cycle from its hyperbolic pencil with the fixed angle $\arctan(\tilde{\lambda}/(2\pi))$. This is used to amend the intersection angle $\arccos\left[C_h, C_h'\right]$ of cycles from the respective hyperbolic pencils. □

Corollary 15.5.4 *Let a loxodrome parametrised by $\{C_1, C_2, C_3\}$ passes a point parametrised by a zero-radius cycle C_0 as in Procedure 15.5.2. A non-zero radius cycle C is tangent to the loxodrome at C_0 if and only if two conditions holds:*

$$\begin{aligned} \langle C, C_0\rangle &= 0\,, \\ \arccos\left[C, C_h\right] &= \arctan\left(\frac{\tilde{\lambda}}{2\pi}\right), \end{aligned} \tag{15.21}$$

where C_h is given by (15.18) and is $\tilde{\lambda}$ is determined by (15.15).

Proof The first condition simply verifies that C passes C_0, cf. Example 15.2.2. Cycle C, as a degenerated loxodrome, is parametrised by $\{C_e, C, C\}$, where C_e is any cycle orthogonal to C and C_e is not relevant in the following. The hyperbolic pencil spanned by two copies of C consists of C only. Thus we put $C_h' = C$, $\tilde{\lambda}' = 0$ in (15.20) and equate it to 0 to obtain the second identity in (15.21). □

15.6 Discussion and Open Questions

It was mentioned at the end of Sect. 15.4 that a three-cycle parametrisation of loxodromes is more geometrical than their presentation by a pair (λ, M). Furthermore, three-cycle parametrisation reveals the natural analogy between elliptic and hyperbolic features of loxodromes, see (15.17) as an illustration. Examples in Sect. 15.5 show that various geometrical questions are explicitly answered in term of three-cycle parametrisation. Thus, our work extends the set of objects in Lie sphere geometry—circle, lines and points—to the natural maximal conformally-invariant family, which also includes loxodromes. In practical terms, three-cycle parametrisation allows to extend the library `figure` for Möbuis invariant geometry [30, 31] to operate with loxodromes as well.

It is even more important, that the presented technique is another implementation of a general framework [28–31], which provides a significant advance in Lie sphere geometry. The Poincaré extension of FLT from the real line to the upper half-plane was performed by a pair of orthogonal cycles in [29]. A similar extension of FLT from the complex plane to the upper half-space can be done by a *triple of pairwise orthogonal cycles*. Thus, triples satisfying FLT-invariant properties (1a) and (1b) of Proposition 15.4.4 present another non-equivalent class of cycle ensembles in the sense of [29]. In general, Lie sphere geometry *can be enriched by consideration of cycle ensembles* interrelated by a list of FLT-invariant properties [29]. Such ensembles become new objects in the extended Lie spheres geometry and can be represented by points in a *cycle ensemble space*.

There are several natural directions to extend this work further, here are just few of them:

1. Link our "global" parametrisation of loxodromes with differential geometry approach from [6, 37, 40]. Our last Corollary 15.5.4 can be a first step in this direction.
2. Consider all FLT-invariant non-equivalent classes of three-cycle ensembles on $\mathbb{C}$: pairwise orthogonal triples (representing points in the upper half-space [29]), triples satisfying properties (1a) and (1b) of Proposition 15.4.4 (representing loxodromes), etc.
3. Extend this consideration for quaternions or Clifford algebras [13, 33]. The previous works [38, 39] and availability of FSCc in this setup [8, Ch. 4]; [9] make it rather promising.
4. Consider Möbius transformations in rings of dual and double numbers [2, 22–26, 29, 34]. There are enough indications that the story will not be quite the same as for complex numbers.
5. Explore further connections of loxodromes with
   - Carleson curves and microlocal properties of singular integral operators [1, 5, 7]; or
   - applications in operator theory [42, 44].

Some combinations of those topics shall be fruitful as well.

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Graphics for this paper was prepared using `Asymptote` software [14]. We supply an `Asymptote` library `cycle2D` for geometry of planar cycles under GPL licence [11] with the copy of this paper at arXiv [32].

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Chapter 16
Automorphic Forms and Dirac Operators on Conformally Flat Manifolds

Rolf Sören Kraußhar

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract In this paper we present a summarizing description of the connection between Dirac operators on conformally flat manifolds and automorphic forms based on a series of joint work with John Ryan over the last 15 years. We also outline applications to boundary value problems.

Keywords Dirac operators · Automorphic forms · Conformally flat manifolds

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16.1 Introduction

A natural generalization to $\mathbb{R}^n$ of the classical Cauchy-Riemann operator has proved to be the Euclidean Dirac operator D. Here $\mathbb{R}^n$ is considered as embedded in the real 2^n-dimensional Clifford algebra Cl_n satisfying the relation $x^2 = -\|x\|^2$ for each $x \in \mathbb{R}^n$. The elements $e_1, \ldots, e_n$ of the standard orthonormal basis of $\mathbb{R}^n$ satisfy the relation $e_ie_j + e_je_i = -2\delta_{ij}$. The Dirac operator is defined to be $\sum_{j=1}^n e_j \frac{\partial}{\partial x_j}$. Clifford algebra valued functions f and g that satisfy $Df = 0$ respectively $gD = 0$ are often called left (right) monogenic functions.

Its associated function theory together with its applications is known as Clifford analysis and can be regarded as a higher dimensional generalization of complex function theory in the sense of the Riemann approach. Indeed, associated to this

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operator there is a higher dimensional direct analogue of Cauchy's integral formula and other nice analogues, cf. [8]. As in complex analysis, also the Euclidean Dirac factorizes the higher dimensional Euclidean Laplacian viz $D^2 = -\Delta$. Indeed, the Euclidean Dirac operator has been used successfully in understanding boundary value problems and aspects of classical harmonic analysis in $\mathbb{R}^n$. See for instance [14, 27].

On the other hand Dirac operators have proved to be extremely useful tools in understanding geometry over spin and pin manifolds. Basic aspects of Clifford analysis over spin manifolds have been developed in [2, 4]. Further in [16–19] and elsewhere it is illustrated that the context of conformally flat manifolds provide a useful setting for developing Clifford analysis.

Conformally flat manifolds are those manifolds which possess an atlas whose transition functions are Möbius transformations. Under this viewpoint conformally flat manifolds can be regarded as higher dimensional generalizations of Riemann surfaces.

Following the classical work of Kuiper [22], one can construct examples of conformally flat manifolds by factoring out a subdomain $U \subseteq \mathbb{R}^n$ by a torsion-free Kleinian group Γ acting totally discontinuously on U.

Examples of conformally flat manifolds include spheres, hyperbolas, real projective space, cylinders, tori, the Möbius strip, the Kleinian bottle and the Hopf manifolds $S^1 \times S^{n-1}$. The oriented manifolds among them are also spin manifolds. In [16–18] explicit Clifford analysis techniques, including Cauchy and Green type integral formulas, have been developed for these manifolds.

Finally, in one of our follow-up papers [1] we also looked at a class of hyperbolic manifolds namely those that arise from factoring out upper half-space in $\mathbb{R}^n$ by a torsion-free congruence subgroup, H, of the generalized modular group Γ_p. Γ_p is the arithmetic group that is generated by p translation matrices $(p < n)$ and the inversion matrix. In two real variables these are k-handled spheres. Notice that the group Γ_p is not torsion-free, as it contains the negative identity matrix. Consequently, the topological quotient of upper half-space with Γ_p has only the structure of an orbifold. To overcome this problem we deal with congruence subgroups of level $N \geq 2$, which are going to be introduced later on.

In this paper we present an overview about some of our most important joint results. In the final section of this paper we also outline some applications addressing boundary value problems modelling stationary flow problems on these classes of manifolds where we adapt the techniques from [14] to this more general geometric context.

16.2 Clifford Algebras and Spin Geometry

16.2.1 Clifford Algebras and Orthogonal Transformations

As mentioned in the introduction, we embed the $\mathbb{R}^n$ into the real Clifford algebra Cl_n generated by the relation $x^2 = -\|x\|^2$. For details, see [4, 8, 14]. This relation defines the multiplication rules $e_i^2 = -1$, $i = 1, \ldots, n$ and $e_i e_j = -e_j e_i$ $\forall i \neq j$. A vector space basis for Cl_n is given by $1, e_1, \ldots, e_n, e_1 e_2, \ldots, e_{n-1} e_n, \ldots, e_1 \cdots e_n$. Each $x \in \mathbb{R}^n \backslash \{0\}$ has an inverse of the form $x^{-1} = -\frac{x}{\|x\|^2}$. We also consider the reversion anti-automorphism defined by $\tilde{ab} = \tilde{b}\tilde{a}$, where $\tilde{e_j} = e_j$ $\forall j = 1, \ldots, n$ and the conjugation defined by $\overline{ab} = \overline{b}\,\overline{a}$, where $\overline{e_j} = -e_j$ $\forall j = 1, \ldots, n$.

Notice that $e_1 x e_1 = -x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$. The multiplication of e_1 from the left and from the right realizes in a simple form a reflection in the e_1-direction. More generally, one can say: If $O \in O(n)$, then there are reflections $R_1, \ldots, R_m$ such that $O = R_1 \cdots R_m$. In turn for each R_j there exists a $y_j \in S^{n-1}$ such that $R_j x = y_j x y_j$ for all $x \in \mathbb{R}^n$. Summarizing, one can represent a general transformation of $O(n)$ in the way $Ox = y_1 \cdots y_m x y_m \cdots y_1$, so $Ox = ax\tilde{a}$ with $a = y_1 \cdots y_m$.

This motivates the definition of the pin group as

$$Pin(n+1) = \{a \in Cl_n \mid a = y_1 \cdots y_m,\ y_i \in S^n\}.$$

Each transformation of $O(n)$ can be written as $Ox = ax\tilde{a}$ with an $a \in Pin(n)$. In view of $ax\tilde{a} = (-a)x(-\tilde{a})$, $Pin(n)$ is a double cover of $O(n)$. A subgroup of index 2 is the spin group defined by

$$Spin(n) := \{a = y_1 \cdots y_m \in Pin(n) \mid m \equiv 0(2)\}.$$

Again, $Spin(n)$ is the double cover of $SO(n)$.

16.2.2 Spin Geometry

Here we summarize some basic results from [2, 9, 22].

Let M be a connected orientable Riemannian manifold with Riemann metric g_{ij}.

Consider for $x \in M$ all orthonormal-bases of the tangential space TM_x, which again are mapped to orthonormal-bases of TM_x by the action of the $SO(n)$. This gives locally rise to a fiber bundle.

Gluing together all these fiber bundles gives rise to a principal bundle P over M with a copy of $SO(n)$.

This naturally motivates the question whether it is possible to lift each fiber to $Spin(n)$ in a continuous way to obtain a new principal bundle S that covers P.

However, the ambiguity caused by the sign may give a problem. If $s : U \to U \times SO(n)$ is a section then there are two options of lifting s to a spinor bundle $s^* : U \to U \times Spin(n)$, namely s^* and $-s^*$. So, it may happen that:

- It is not always possible to choose the sign in order to construct in a unique way a bundle S over M, such that each fiber is a copy of $Spin(n)$.
- There also might be several possibilities.
- The different spin structures are described by the cohomology group $H^1(M, \mathbb{Z}_2)$.

16.2.3 The Atiyah-Singer-Dirac Operator

Let M be a Riemannian spin manifold. Let Γ be the Levi-Civita connection. Then $\Gamma g_{ij} = 0$. Stokes's theorem tells us that

$$\int_{\partial V} \langle s_1(x), n(x)s_2(x)\rangle_S d\sigma(x)$$
$$= \int_{V} (\langle s_1(x)D, s_2(x)\rangle_S + \langle s_1(x), Ds_2(x)\rangle_S)dV$$

The arising differential operator here is the Atiyah-Singer-Dirac operator. In a local orthonormal basis $e_1(x), \dots e_n(x)$ the latter has the form

$$D = \sum_{j=1}^{n} e_j(x)\Gamma^*_{e_j(x)}.$$

16.2.4 The Dirac Operator in $\mathbb{R}^n$ and $\mathbb{R} \oplus \mathbb{R}^n$

Following [8] and others, the Dirac operator in $\mathbb{R}^n$ has the simple form $D = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} e_j$. In the so-called space of paravectors $\mathbb{R} \oplus \mathbb{R}^n$ it particularly has the form $D = \frac{\partial}{\partial x_0} + \sum_{j=1}^{n} \frac{\partial}{\partial x_j} e_j$. The latter naturally generalizes the well-known Cauchy-Riemann operator $\frac{\partial}{\partial x_0} + \frac{\partial}{\partial x_1} i$ in a straight forward way to higher dimensions by adding the additional basis elements.

As mentioned in the introduction, functions $f : U \to Cl_n$ (where $U \subseteq \mathbb{R}^n$ resp. $U \subseteq \mathbb{R} \oplus \mathbb{R}^n$) that are in the kernel of D are often called monogenic.

16.3 Analysis on Manifolds

16.3.1 Conformally Flat Manifolds in $\mathbb{R}^{n+1}$

Following for example [22], conformally flat manifolds are Riemannian manifolds with vanishing Weyl tensor. In turn these are exactly those Riemannian manifolds which have an atlas whose transition functions are conformal maps. In $\mathbb{R}^2$ conformal maps are exactly the (anti-)holomorphic functions. So, one deals with classical Riemann surfaces. Up from space dimension $n \geq 3$ in $\mathbb{R}^{\geq 3}$ however, the set of conformal maps coincides with the set of Möbius transformations, cf. [4] The latter set of functions then represent reflections at spheres and hyperplanes. This seems to be quite restrictive at the first glance. However, this is not the case as the abundance of classical important examples will show as outlined in the following.

So, let us now turn to an explicit construction principle of conformally flat manifolds in $\mathbb{R}^n$ with $n \geq 3$. To proceed in this direction take a torsion free discrete subgroup $\mathcal{G}$ of the orthogonal group $O(n+1)$ that acts totally discontinuously on a simply connected domain $\mathcal{D}$. Next define a group action $\mathcal{G} \times \mathcal{D} \to \mathcal{D}$. Then the topological quotient space $\mathcal{D}/\mathcal{G}$ is a conformally flat manifold. As also shown in the original work of N.H. Kuiper in 1949 [22], the universal cover of a conformally flat manifold possesses a local diffeomorphism to S^n. Conformally flat manifolds of the form $\mathcal{D}/\mathcal{G}$ are exactly those for which this local diffeomorphism is a covering map $\mathcal{D} \to \mathcal{D} \subset S^n$.

Let us present a few elementary examples:

- Take $\mathcal{G} = \mathcal{T}_p := \mathbb{Z} + \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_{p-1}$, $\mathcal{D} = \mathbb{R}^{n+1}$ and consider the action

$$(m_0, \ldots, m_n) \circ (x_0, \ldots, x_n) \mapsto (x_0 + m_0, \ldots, x_{p-1} + m_{p-1}, x_p, \ldots, x_n).$$

  Then the topological quotients $\mathcal{D}/\mathcal{G}$ represent oriented p cylinders. These are spin manifolds with 2^p many different spinor bundles. In the particular case $p = n+1$ one deals with a compact oriented $n+1$-torus, cf. [17, 18].
- Take again as group $\mathcal{T}_{n+1}$ and the same domain, but consider a different action of the form

$$(m_0, \ldots, m_{n-1}, m_n) \circ (x_0, \ldots, x_{n-1}, x_n)$$
$$\mapsto (x_0 + m_0, \ldots, x_{n-1} + m_{n-1}, (-1)^{m_n} x_n + m_n).$$

  Now $\mathcal{D}/G$ is the non-orientable Kleinian bottle in $\mathbb{R}^{n+1}$. Due to the lack of orientability it is not spin. However, it is a pin manifold with 2^{n+1} many pinor bundles. For some Clifford analysis on these manifolds we refer the reader to our recent works [16, 20].

Further interesting examples can be constructed by forming quotients with non-abelian subgroups of Möbius transformations in $\mathbb{R}^n$, in particular with arithmetic subgroups of the so-called Ahlfors-Vahlen group discussed for instance in [10] and many other papers. To make the paper self-contained we recall its definition:

Definition 16.3.1 (Ahlfors-Vahlen Group) A Clifford algebra valued matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat(2, Cl_n)$ belongs to the special Ahlfors Vahlen group $SAV(n)$, if:

- a, b, c, d are products of paravectors from $\mathbb{R} \oplus \mathbb{R}^n$
- $a\tilde{c}, c\tilde{d}, d\tilde{b}, b\tilde{c} \in \mathbb{R} \oplus \mathbb{R}^n$ and $a\tilde{d} - b\tilde{c} = 1$.

The use of Clifford algebras allow us to describe Möbius transformations in $\mathbb{R} \oplus \mathbb{R}^n$ in the simple way $M < x >= (ax+b)(cx+d)^{-1}$, similarly to complex analysis. For our needs we consider the special hypercomplex modular group [15] defined by

$$\Gamma_p := \Big\langle \underbrace{\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & e_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & e_p \\ 0 & 1 \end{pmatrix}}_{=:\mathcal{T}_p}, \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{=:J} \Big\rangle$$

If $p < n$, then by applying the same arguments as in [12], Γ_p acts totally discontinuously on upper half-space discussed in [10]

$$H^+(\mathbb{R} \oplus \mathbb{R}^n) := \{x_0 + x_1 e_1 + \cdots + x_n e_n \in \mathbb{R} \oplus \mathbb{R}^n \mid x_n > 0\}.$$

In two dimensions it coincides with the classical group $SL(2, \mathbb{Z})$. To get a larger amount of examples we look at the following arithmetic congruence subgroups:

$$\Gamma_p[N] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_p \mid a-1, b, c, d-1 \in N\mathcal{O}_p \right\},$$

where $\mathcal{O}_p := \sum_{A \subseteq P(\{1,\ldots,p\})} \mathbb{Z} e_A$ are the standard orders in Cl_n.

If we now take $\mathcal{D} = H^+(\mathbb{R}^{n+1})$, $G = \Gamma_p[N]$ with $N \geq 2$ and if we consider the action

$$(M, x) \mapsto M < x >:= (ax+b) \cdot (cx+d)^{-1},$$

where $\cdot$ is the Clifford-multiplication, then for $N \geq 2$ the topological quotient $\mathcal{D}/\mathcal{G}$ realizes a class of conformally flat orientable manifold with spin structures generalizing k-tori to higher dimensions, cf. [1].

16.3.2 Sections on Conformally Flat Manifolds

In the sequel let us make the general assumption that $\mathcal{M} := \mathcal{D}/\mathcal{G}$ is an orientable conformally flat manifold. Let furthermore $f : \mathcal{D} \to \mathbb{R}^{n+1}$ be a function with

$f(G < x >)) = j(G, x)f(x)\ \forall G \in \mathcal{G}$ where $j(G, x)$ is an automorphic factor satisfying a certain co-cycle relation. In the simplest case $j(G, x) \equiv 1$ one deals with a totally invariant function under the group action of G. Then the canonical projection $p : \mathcal{D} \to \mathcal{M}$ induces a well-defined function $f' := p(f)$ on the quotient manifold $\mathcal{M}$. Now let $D := \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} e_i$ be the Euclidean Dirac- or Cauchy-Riemann operator and let Δ be the usual Euclidean Laplacian. The canonical projection $p : \mathcal{D} \to \mathcal{M}$ in turn induces a Dirac operator $D' = p(D)$ resp. Laplace operator $\Delta' = p(D)$ on $\mathcal{M}$.

Consequence: A $\mathcal{G}$-invariant function on $\mathcal{D}$ from Ker D (resp. from Ker Δ) induces functions on $\mathcal{M}$ in Ker D' resp. in Ker Δ'. More generally, one considers for f' monogenic resp. harmonic spinor sections with values in certain spinor bundles.

16.4 Automorphic Forms

16.4.1 Basic Background

Following classical literature on automorphic forms, for instance [12], let $\mathcal{G}$ be a discrete group that acts totally discontinuously on a domain $\mathcal{D}$ by $\mathcal{G} \times \mathcal{D} \to \mathcal{D},\ \ (g, d) \to d^*$, Roughly speaking, automorphic forms on $\mathcal{G}$ are functions on $\mathcal{D}$ that are quasi-invariant under the action of $\mathcal{G}$. The fundamental theory of holomorphic automorphic forms in one complex variable was founded around 1890, mainly by H. Poincaré, F. Klein and R. Fricke. The associated quotient manifolds are holomorphic Riemann surfaces.

The simplest examples are holomorphic periodic functions. They serve as example of automorphic functions on discrete translation groups. Further very classical examples are the so-called Eisenstein series

$$G_n(\tau) := \sum_{(c,d)\in\mathbb{Z}^2\setminus\{(0,0)\}} (c\tau + d)^{-n} \quad n \equiv 0(\bmod 2) \;\; n \geq 4$$

They are holomorphic functions on $H^+(\mathbb{C}) := \{z = x + iy \in \mathbb{C} \mid y > 0\}$. For all $z \in H^+(\mathbb{C})$ they satisfy :

$$f(z) = (f|_n M)(z) \;\; \forall M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

where $(f|_n M)(z) := (cz + d)^{-n} f\left(\frac{az+b}{cz+d}\right)$.

The Eisenstein series G_n are the simplest non-vanishing holomorphic automorphic forms on $SL(2, \mathbb{Z})$. Further important examples are Poincaré series: For $n > 2,\ n \in 2\mathbb{N}$ let

$$P_n(z, w) = \sum_{M \in SL(2,\mathbb{Z})} (cz + d)^{-n}(w + M < z >)^{-n}$$

These functions have the same transformation behavior as the previously described Eisenstein series, namely:

$$P_n(w, z) = P_n(z, w) = (cz + d)^{-n} P_n(M < z >, w).$$

In contrast to the Eisenstein series, these Poincaré series have the special property that they vanish at the singular points of the quotient manifold resp. orbifold.

16.4.2 Higher Dimensional Generalizations in n Real Variables

Already in the 1930s C.L. Siegel developed higher dimensional analogues of the classical automorphic forms in the framework of holomorphic functions in several complex variables. The context is Siegel upper half-space where one considers the action of discrete subgroups of the symplectic group.

More closely related to our intention is the line of generalization initiated by H. Maaß and extended by Elstrodt et al. [10] and Krieg [21] and many followers in the period of 1985–1990 and onwards.

These authors looked at higher dimensional generalizations of Maaß forms which are non-holomorphic automorphic forms on discrete subgroups of the Ahlfors-Vahlen group (including for example the Picard group and the Hurwitz quaternions) on upper half-space $H^+(\mathbb{R}^n)$. As important analytic properties they exhibit to be complex-valued eigensolutions to the scalar-valued Laplace-Beltrami operator

$$\Delta_{LB} = x_n^2\Big(\sum_{i=0}^{n} \frac{\partial^2}{\partial x_i^2}\Big) - (n-1)x_n \frac{\partial}{\partial x_n}$$

In the case $n = 1$ one has $\Delta_{LB} = x_1^2 \Delta$. Therefore, in the one-dimensional case holomorphic automorphic forms simply represent a special case of Maaß forms.

A crucial disadvantage of Maaß forms behind the background of our particular intentions consists in the fact that they do not lie in the kernel of the Euclidean Dirac or Laplace operator. Additionally, they are only scalar-valued.

16.4.3 Clifford-Analytic Automorphic Forms

16.4.3.1 Some Milestones in the Literature

To create a theory of Clifford algebra valued automorphic forms that are in kernels of Dirac operators remained a challenge for a long time. A serious obstacle has been to overcome the problem that neither the multiplication nor the composition of monogenic functions are monogenic again. However, the set of monogenic functions is quasi-invariant under the action of Möbius transformations up to an automorphic factor that fortunately obeys a certain co-cycle relation. The latter actually provides the fundament to build up a theory of automorphic forms in the Clifford analysis setting.

The first contribution in the Clifford analysis setting is the paper [25] where J. Ryan constructed n-dimensional monogenic analogues of the Weierstraß $\wp$-function and the Weierstraß ζ-function built as summations of the monogenic Cauchy kernel over an n-dimensional lattice. Here, the invariance group is an abelian translation group with n linearly independent generators.

In the period of 1998–2004 the author developed the fundamentals for a more general theory of Clifford holomorphic automorphic forms on more general arithmetic subgroups of the Ahlfors-Vahlen group, cf. [15]. The geometric context is again upper half-space. However, the functions are in general Clifford algebra valued and are null-solutions to iterated Dirac equations. In fact, the framework of null-solutions to the classical first order Dirac operator is too small for a large theory of automorphic forms, because the Dirac operator admits only the construction of automorphic forms to one weight factor only. To consider automorphic forms with several automorphic weight factors a more appropriate framework is the context of iterated Dirac equations of the form $D^m f = 0$ to higher order integer powers m. In fact, higher order Dirac equations can also be related to k-hypermonogenic functions [24] which allows us to relate the theory of Maaß forms to Clifford holomorphic automorphic forms. This is successfully explained in [5, 7]. In this context it was possible to generalize Eisenstein- and Poincaré series construction which gave rise to the construction of spinor sections with values in certain spinor bundles on the related quotient manifolds.

In fact, as shown in [7], it is possible to decompose the Clifford module of Clifford holomorphic automorphic forms as an orthogonal direct sum of Clifford holomorphic Eisenstein- and Poincaré series. As shown in our recent paper [13], both modules turn out to be finitely generated.

The applications to solve boundary value problems on related spin manifolds started with our first joint paper [17] where we applied multiperiodic Eisenstein series on translation groups to construct Cauchy and Green kernels on conformally flat cylinders and tori associated to the trivial spinor bundle.

In [18] we extended our study to address the other spinor bundles on these manifolds. Furthermore, we also looked at dilation groups instead of translation groups, too, and managed to give closed representation formulas for the Cauchy kernel of the Hopf manifold $S^1 \times S^{n-1}$. Already in this paper we outlined the construction of spinor sections on some hyperbolic manifolds of genus ≥ 2. However, to obtain the Cauchy kernel for these manifolds we needed to construct Poincaré series which was a hard puzzle piece to find. Eisenstein series were easy to construct and they lead to non-trivial spinor sections on these manifolds. However, the Cauchy kernel must have the property to vanish at the cusps of the group—and that construction was hard to do. A breakthrough in that direction could be established in our joint paper [1] where we were able to fill in that gap.

Finally, in [6] and [7] we were able to extend these constructions to the framework of k-hypermonogenic functions [11] and holomorphic Clifford functions addressing null-solutions to $D\Delta^m f = 0$. This is the function class considered by G. Laville and I. Ramadanoff introduced in [23].

Summarizing, the theory of Clifford holomorphic automorphic form provides us with a toolbox to solve boundary value problems related to the Euclidean Dirac or Laplace operator on conformally flat spin manifolds or more generally on manifolds generalizing classical modular curves.

It also turned out to be possible to make analogous constructions for some non-orientable conformally flat manifolds with pin structures. Belonging to that context, in [18] we addressed real projective spaces and in [16, 20] we carried over our constructions to the framework of higher dimensional Möbius strips and the Klein bottle.

16.4.4 Some Concrete Examples

- The simplest non-trivial examples of Clifford-holomorphic automorphic forms on the translation groups $\mathcal{T}_p$ are given by the series

$$\epsilon_{\mathbf{m}}^{(p)}(x) := \sum_{\omega \in \Omega_p} q_{\mathbf{m}}(x+\omega), \qquad \Omega_p := \mathbb{Z}\omega_0 + \mathbb{Z}\omega_1 + \dots \mathbb{Z}\omega_p$$

which are normally convergent, if $|\mathbf{m}| \geq p+2$. Here, $q_{\mathbf{m}}(x) := \frac{\partial^{|\mathbf{m}|}}{\partial x^{\mathbf{m}}} q_{\mathbf{0}}(x)$ where $q_{\mathbf{0}}(x) := \frac{\overline{x}}{\|x\|^{m+1}}$ and where $\mathbf{m} := (m_1, \cdots, m_n)$ is a multi-index and $|\mathbf{m}| := m_1 + m_2 + \cdots + m_n$ and $x^{\mathbf{m}} := x_1^{m_1} \cdots x_n^{m_n}$ is used as in usual multi-index notation.

These series $\epsilon_{\mathbf{m}}^{(p)}(x)$ can be interpreted as Clifford holomorphic generalizations of the trigonometric functions (with singularities) and of the Weierstraß $\wp$-function. The projection $p(\epsilon_{\mathbf{m}}^{(p)}(x))$ then induces a well-defined spinor section on the cylinder resp. torus $\mathbb{R}^{n+1}/\Omega_p$ with values in the trivial spinor bundle. Other spinor bundles can be constructed by introducing proper minus signs and the following decomposition of the period lattice in the way $\Omega_p := \Omega_l \oplus \Omega_{p-l}$ where $0 \leq l \leq p$. The proper analogues of $\epsilon_{\mathbf{m}}^{(p)}(x)$ then are defined by

$$\epsilon_{\mathbf{m}}^{(p,l)}(x) := \sum_{\omega \in \Omega_l \oplus \Omega_{p-l}} (-1)^{m_0+\cdots+m_l} q_{\mathbf{m}}(x+\omega), \ \Omega_p := \mathbb{Z}\omega_0 + \mathbb{Z}\omega_1 + \ldots \mathbb{Z}\omega_p.$$

In total one can construct 2^{p+1} different spinor bundles. The Cauchy kernel is obtained by the series that one obtains in the case $\mathbf{m} = \mathbf{0}$. In that case it may happen that the associated series above does not converge. To obtain convergence in those cases one needs to apply special convergence preserving terms. For the technical details we refer to our papers [17–19].

- Let us now turn to examples of hyperbolic manifolds that are generated by quotient forming with non-abelian groups. The simplest non-trivial example in that context are the monogenic Eisenstein series for $\Gamma_{n-1}[N]$ (and also for $\Gamma_p[N]$) with $p < n-1$ which have been introduced in [5].

$$E(z) = \lim_{\sigma \to 0^+} \sum_{M:\mathcal{T}_{n-1}[N] \backslash \Gamma_{n-1}[N]} \left(\frac{x_n}{\|cx+d\|^2} \right)^{\sigma} q_{\mathbf{0}}(cx+d).$$

Notice that we here applied the Hecke trick (cf. [12]) to get well-definedness. In fact, these Eisenstein series provide us with the simplest examples of non-vanishing spinor sections defined on the hyperbolic quotient manifolds $H^+(\mathbb{R}^{n+1})/\Gamma_p[N]$. However, these series do not vanish at the cusps of the group. They serve as examples but they don't reproduce the Cauchy integral. This property can be achieved by the following monogenic Poincaré series, introduced in our papers [1, 5]. The latter have the form

$$P_p(x,w) = \lim_{\sigma \to 0^+} \sum_{M \in \Gamma_p[N]} \left(\frac{x_n}{\|cx+d\|^2} \right)^{\sigma} q_{\mathbf{0}}(cx+d) q_{\mathbf{0}}(w + M\langle x \rangle).$$

The series $P_p(x,w)$ are indeed monogenic cusp forms, in particular

$$\lim_{x_n \to \infty} P_p(x_n e_n) = 0.$$

Its projection down to the manifold induce the Cauchy kernel. In the following section we want to outline how the explicit knowledge of the Cauchy kernel allows us to solve boundary value problems on these manifolds.

16.5 Applications to BVP on Manifolds

Some practical motivations for the following studies are to understand weather forecast and flow problems on spheres, cylindrical ducts or other geometric contexts fitting into the line of investigation of [3, 26].

16.5.1 The Cauchy Kernel on Spin Manifolds

Monogenic generalization of the Weierstraß $\wp$-function $\varepsilon_n^{(p)}$ give rise to monogenic sections on p-cylinders $\mathbb{R}^{n+1}/\mathcal{T}_p$. So, monogenic automorphic forms on $\Gamma_p(\mathcal{I})[N]$ define spinor sections on the k-tori $H^+(\mathbb{R}^{n+1})/\Gamma_p(\mathcal{I})[N]$. The Poincaré series give us the Cauchy kernel on these manifolds. Summarizing, for $p < n-1$ the Cauchy kernel has the concrete and explicit form

$$C(x, y) = \sum_{M \in \Gamma_p(\mathcal{I})[N]} \frac{\overline{cx+d}}{|cx+d|^n} \frac{(\overline{y - M < x >})}{\| y - M < x > \|^{n+1}}.$$

Each monogenic section f' on $\mathcal{M}$ then satisfies

$$f'(y') = \int_{\partial S'} C'(x', y') d\sigma'(x') f'(x')$$

which is the reproduction of the Cauchy integral, cf. [1].

16.5.2 The Stationary Stokes Flow Problem on Some Conformally Flat Spin Manifolds

Let $\mathcal{M} := \mathcal{D}/G$ be a conformally flat spin manifold for which we know the Cauchy kernel $C(x, y)$ to the Dirac operator (concerning a fixed chosen spinor bundle F).

Next let $E \subset \mathcal{M}$ be a domain with sufficiently smooth boundary $\Gamma := \partial E$.

Now we want to consider the following Stokes problem on $\mathcal{M}$:

$$\begin{aligned} -\Delta u + \frac{1}{\eta} p &= F \quad \text{in } E \qquad (16.5.1)\\ \operatorname{div} u &= 0 \quad \text{in } E \\ u &= 0 \quad \text{on } \Gamma, \end{aligned}$$

where $u \in W_2^1(E, F)$ is the velocity of the flow and where $p \in W_2^1(E, \mathbb{R})$ is the hydrostatic pressure. The explicit knowledge of the Cauchy kernel on $\mathcal{M}$ allows us to set up explicit analytic representation formulas for the solutions.

To meet this end we define in close analogy of [14] the Teodorescu transform and the Cauchy transform on $\mathcal{M}$ by

$$(T_E f)(x) := -\int_E C(x, y) f(y) dV(y)$$
$$(F_\Gamma f)(x) := \int_\Gamma C(x, y) d\sigma(y) f(y).$$

where now $C(x, y)$ stands for the Cauchy kernel associated to the chosen spinor bundle F on the manifold. The associated Bergman projection $P : L^2(E) \to L^2(E) \cap Ker(D)$ then can be represented in the usual form

$$P = F_\Gamma (tr_\Gamma T_E F_\Gamma)^{-1} tr_\Gamma T_E.$$

An application of the Clifford analysis methods provide us with explicit analytic formulas for the velocity and the pressure of the fluid running over this manifold. An application of T_E to (16.5.1) yields:

$$(T_E D)(Du) + \frac{1}{\eta} T_E Dp = T_E F.$$

Next, an application of the Borel-Pompeiu formula (Cauchy-Green formula) leads to:

$$Du - F_\Gamma Du + \frac{1}{\eta} p - \frac{1}{\eta} F_\Gamma p = T_E F$$

If we apply the Pompeiu projection $Q := I - P$, then one obtains

$$QDu - QF_\Gamma Du + \frac{1}{\eta} Qp - \frac{1}{\eta} QF_\Gamma p = QT_E F \quad (*)$$

Since $F_\Gamma Du, F_\Gamma p \in \text{Ker } D$, one further gets

$$QF_\Gamma Du = 0 \quad \text{and} \quad QF_\Gamma p = 0.$$

Thus,

$$QDu + \frac{1}{\eta} Qp = QT_E F. \tag{16.5.2}$$

Further, another application of T_E and the property $QDu = Du$ leads to

$$T_E Du + \frac{1}{\eta} T_E Qp = T_E Q T_E F$$

$$\Leftrightarrow u - \underbrace{F_\Gamma u}_{=0} + \frac{1}{\eta} T_E Qp = T_E Q T_E F.$$

In view of $u|_\Gamma = 0$ we obtain the following representation formula for the velocity:

$$u = T_E(I - P)T_E F - \frac{1}{\eta} T_E(I - P)p.$$

Finally, the condition div $u = 0$ allows us to determinate the pressure

$$(\Re(I - P))p = \eta \Re((I - P)T_E(I - P)F)$$

where $\Re(\cdot)$ stands for the scalar part of an element from the Clifford algebra.

Remark An extension of this approach to the parabolic setting in which instationary flow problems are considered are treated in our follow up paper [3].

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Chapter 17
Higher Order Fermionic and Bosonic Operators

Chao Ding, Raymond Walter, and John Ryan

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract This paper studies a particular class of higher order conformally invariant differential operators and related integral operators acting on functions taking values in particular finite dimensional irreducible representations of the Spin group. The differential operators can be seen as a generalization to higher spin spaces of kth-powers of the Euclidean Dirac operator. To construct these operators, we use the framework of higher spin theory in Clifford analysis, in which irreducible representations of the Spin group are realized as polynomial spaces satisfying a particular system of differential equations. As a consequence, these operators act on functions taking values in the space of homogeneous harmonic or monogenic polynomials depending on the order. Moreover, we classify these operators in analogy with the quantization of angular momentum in quantum mechanics to unify the terminology used in studying higher order higher spin conformally invariant operators: for integer and half-integer spin, these are respectively bosonic and fermionic operators. Specifically, we generalize arbitrary powers of the Dirac and Laplace operators respectively to spin-$\frac{3}{2}$ and spin-1.

Keywords Higher order fermionic and bosonic operators · Conformal invariance · Fundamental solutions · Intertwining operators · Ellipticity

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17.1 Introduction

Classical Clifford analysis started as a generalization of aspects of one variable complex analysis to m-dimensional Euclidean spaces. At the heart of this theory is the study of the Dirac operator D_x on $\mathbb{R}^m$, a conformally invariant first order differential operator which generalizes the role of the Cauchy-Riemann operator. Moreover, this operator is related to the Laplace operator with $D_x^2 = -\Delta_x$. The classical theory is centered around the study of functions on $\mathbb{R}^m$ and taking values in a spinor space [2, 4], and abundant results have been found. See for instance [4, 11, 23, 28, 32, 33].

P.A.M. Dirac constructed a first order relativistically covariant equation describing the dynamics of an electron by using Clifford modules; hence differential operators constructed using Clifford modules are called Dirac operators. In the presence of an electromagnetic field, the Dirac Hamiltonian for an electron acquires an additional contribution formally analogous to internal angular momentum called spin, from which the Spin group and related notions take their name; for the electron, spin has the value $\frac{1}{2}$ [25]. Indeed, in dimension four with appropriate signature, null-solutions of the Dirac operator D_x from classical Clifford analysis correspond to solutions for the relativistically covariant dynamical equation of a massless particle of spin $\frac{1}{2}$, also called the Weyl equation. The Dirac equation for the electron, which has mass, may be considered an inhomogeneous equation satisfied by the Dirac operator D_x. The Dirac equation is not only relativistically covariant, but also conformally invariant. The construction of conformally invariant massless wave equations, in terms of invariant operators with conformal weights over spin fields, is well described by Eelbode and Roels [17]. The general importance of conformal invariance in physics has long been recognized [21].

Rarita and Schwinger [30] introduced a simplified formulation of the theory of particles of arbitrary half-integer spin $k + \frac{1}{2}$ and in particular considered its implications for particles of spin $\frac{3}{2}$. In the context of Clifford analysis, the so-called *higher spin theory* was first introduced through the Rarita-Schwinger operator [7], which is named analogously to the Dirac operator and reproduces the wave equations for a massless particle of arbitrary half-integer spin in four dimensions with appropriate signature [31]. (The solutions to these wave equations may not be physical [39, 40].) The higher spin theory studies generalizations of classical Clifford analysis techniques to higher spin spaces [5, 7, 9, 15, 17, 26]. This theory concerns the study of the operators acting on functions on $\mathbb{R}^m$, taking values in arbitrary irreducible representations of $Spin(m)$. These arbitrary representations are defined in terms of polynomial spaces that satisfy certain differential equations, such as j-homogeneous monogenic polynomials (half-integer spin) or j-homogeneous harmonic polynomials (integer spin). More generally, one can consider the highest weight vector of the spin representation as a parameter [12], but this is beyond our present scope. The present paper contributes to the study of conformally invariant operators in the higher spin theory.

In principle, all conformally invariant differential operators on locally conformally flat manifolds in higher spin theory are classified by Slovák [35], see also [37]. This classification is non-constructive, showing only between which vector bundles these operators exist and what is their order; explicit expressions of these operators are still being found. Eelbode and Roels [17] point out that the Laplace operator Δ_x is not conformally invariant anymore when it acts on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, where $\mathcal{H}_1$ is the degree 1 homogeneous harmonic polynomial space (correspondingly $\mathcal{M}_1$ for monogenic polynomials). They construct a second order conformally invariant operator on $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$, the (generalized) Maxwell operator. In dimension four with appropriate signature it reproduces the Maxwell equation, or the wave equation for a massless spin-1 particle (the massless Proca equation) [17]. De Bie and his co-authors [9] generalize this Maxwell operator from $C^\infty(\mathbb{R}^m, \mathcal{H}_1)$ to $C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ to provide the higher spin Laplace operators, the second order conformally invariant operators generalizing the Laplace operator to arbitrary integer spins. Their arguments also suggest that D_x^k is not conformally invariant in the higher spin theory. This raises the following question: what operators generalize kth-powers of the Dirac operator in the higher spin theory? We know these operators exist, with even order operators taking values in homogeneous harmonic polynomial spaces and odd order operators in homogeneous monogenic polynomial spaces [35]. This paper explicitly answers the question with the condition that the target space is a degree 1 homogeneous polynomial space, encompassing the spin-1 and spin-$\frac{3}{2}$ cases. More generally, one can consider bosonic and fermionic operators corresponding to either integer or half-integer spins, taking values in polynomial spaces of appropriate degree of homogeneity that are either harmonic or monogenic. The general case of arbitrary order and spin is addressed in [14] using a different method.

The paper is organized as follows. We briefly introduce Clifford algebras, Clifford analysis, and representation theory of the Spin group in Sect. 17.2. In Sect. 17.3, we introduce the kth-order higher spin operators $\mathcal{D}_{1,k}$ as the generalization of D_x^k when acting on $C^\infty(\mathbb{R}^m, U)$, where $U = \mathcal{H}_1$ (spin-1) or $U = \mathcal{M}_1$ (spin-$\frac{3}{2}$) depending on whether k is even or odd. We overview classification, existence, and uniqueness results for higher spin operators. Nomenclature is given for the higher order higher spin operators that we consider: bosonic and fermionic operators. The construction and conformal invariance of the operators $\mathcal{D}_{1,k}$ are given with the help of the concept of *generalized symmetry*, as in [9, 17]. Then we provide the intertwining operators for these operators, which also reveal that they are conformally invariant. These intertwining operators are special cases of Knapp-Stein intertwining operators [8, 24] in higher spin theory. Section 17.4 presents the fundamental solutions (up to a multiplicative constant) of $\mathcal{D}_{1,k}$ with the help of Schur's Lemma from representation theory. We provide the value of the constant here, but it is derived from a different technique in [14]. In this paper, since we use the fact that the only conformal transformations are Möbius transformations when dimension of the space $m \geq 3$, we will not see logarithm functions involved in the fundamental solutions, which corresponds to $m = 2$. More details can be found in Sect. 17.4. We also present an argument that these fundamental solutions when seen as a type of convolution operator are also conformally invariant. These convolution type operators can also be recovered as Knapp-Stein operators [8, 24] in higher spin theory. Further, the expressions of the fundamental solutions also suggest that $\mathcal{D}_{1,k}$ is a generalization of D_x^k to low spins. With the observation that the bases of the target spaces U have simple expressions, we also prove that $\mathcal{D}_{1,k}$ is an elliptic operator in Sect. 17.5.

17.2 Preliminaries

17.2.1 Clifford Algebra

A real Clifford algebra, $\mathcal{C}l_m$, can be generated from $\mathbb{R}^m$ by considering the relationship

$$\underline{x}^2 = -\|\underline{x}\|^2$$

for each $\underline{x} \in \mathbb{R}^m$. We have $\mathbb{R}^m \subseteq \mathcal{C}l_m$. If $\{e_1, \ldots, e_m\}$ is an orthonormal basis for $\mathbb{R}^m$, then $\underline{x}^2 = -\|\underline{x}\|^2$ tells us that

$$e_i e_j + e_j e_i = -2\delta_{ij},$$

where δ_{ij} is the Kronecker delta function. An arbitrary element of the basis of the Clifford algebra can be written as $e_A = e_{j_1} \cdots e_{j_r}$, where $A = \{j_1, \cdots, j_r\} \subset \{1, 2, \cdots, m\}$ and $1 \leq j_1 < j_2 < \cdots < j_r \leq m$. Hence for any element $a \in \mathcal{C}l_m$, we have $a = \sum_A a_A e_A$, where $a_A \in \mathbb{R}$. Similarly, the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ is defined as the complexification of the real Clifford algebra

$$\mathcal{C}l_m(\mathbb{C}) = \mathcal{C}l_m \otimes \mathbb{C}.$$

We consider real Clifford algebra $\mathcal{C}l_m$ throughout this subsection, but in the rest of the paper we consider the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ unless otherwise specified.

The Pin and Spin groups play an important role in Clifford analysis. The Pin group can be defined as

$$Pin(m) = \{a \in \mathcal{C}l_m : a = y_1 y_2 \ldots y_p,\ y_1, \ldots, y_p \in \mathbb{S}^{m-1}, p \in \mathbb{N}\},$$

where $\mathbb{S}^{m-1}$ is the unit sphere in $\mathbb{R}^m$. $Pin(m)$ is clearly a group under multiplication in $\mathcal{C}l_m$.

Now suppose that $a \in \mathbb{S}^{m-1} \subseteq \mathbb{R}^m$, if we consider axa, we may decompose

$$x = x_{a\|} + x_{a\perp},$$

where $x_{a\|}$ is the projection of x onto a and $x_{a\perp}$ is the rest, perpendicular to a. Hence $x_{a\|}$ is a scalar multiple of a and we have

$$axa = ax_{a\|}a + ax_{a\perp}a = -x_{a\|} + x_{a\perp}.$$

So the action axa describes a reflection of x in the direction of a. By the Cartan-Dieudonné Theorem each $O \in O(m)$ is the composition of a finite number of reflections. If $a = y_1 \cdots y_p \in Pin(m)$, we define $\tilde{a} := y_p \cdots y_1$ and observe that $ax\tilde{a} = O_a(x)$ for some $O_a \in O(m)$. Choosing $y_1, \ldots, y_p$ arbitrarily in $\mathbb{S}^{m-1}$, we see that the group homomorphism

$$\theta : Pin(m) \longrightarrow O(m) : a \mapsto O_a, \tag{17.2.1}$$

with $a = y_1 \cdots y_p$ and $O_a x = ax\tilde{a}$ is surjective. Further $-ax(-\tilde{a}) = ax\tilde{a}$, so $1, -1 \in Ker(\theta)$. In fact $Ker(\theta) = \{1, -1\}$, see [29]. The Spin group is defined as

$$Spin(m) = \{a \in Cl_m : a = y_1 y_2 \ldots y_{2p};\ y_1, \ldots, y_{2p} \in \mathbb{S}^{m-1}, p \in \mathbb{N}\}$$

and it is a subgroup of $Pin(m)$. There is a group homomorphism

$$\theta : Spin(m) \longrightarrow SO(m),$$

which is surjective with kernel $\{1, -1\}$. It is defined by (1). Thus $Spin(m)$ is the double cover of $SO(m)$. See [29] for more details.

For a domain U in $\mathbb{R}^m$, a diffeomorphism $\phi : U \longrightarrow \mathbb{R}^m$ is said to be conformal if, for each $x \in U$ and each $\mathbf{v}, \mathbf{w} \in TU_x$, the angle between $\mathbf{v}$ and $\mathbf{w}$ is preserved under the corresponding differential at x, $d\phi_x$. For $m \geq 3$, a theorem of Liouville tells us the only conformal transformations are Möbius transformations. Ahlfors and Vahlen show that given a Möbius transformation on $\mathbb{R}^m \cup \{\infty\}$ it can be expressed as $y = (ax + b)(cx + d)^{-1}$ where $a, b, c, d \in Cl_m$ and satisfy the following conditions [1]:

1. a, b, c, d *are all products of vectors in* $\mathbb{R}^m$;
2. $a\tilde{b}, c\tilde{d}, \tilde{b}c, \tilde{d}a \in \mathbb{R}^m$;
3. $a\tilde{d} - b\tilde{c} = \pm 1$.

Since $y = (ax+b)(cx+d)^{-1} = ac^{-1} + (b - ac^{-1}d)(cx+d)^{-1}$, a conformal transformation can be decomposed as compositions of translation, dilation, reflection and inversion. This gives an *Iwasawa decomposition* for Möbius transformations. See [26] for more details.

The Dirac operator in $\mathbb{R}^m$ is defined to be

$$D_x := \sum_{i=1}^{m} e_i \partial_{x_i}.$$

Note $D_x^2 = -\Delta_x$, where Δ_x is the Laplacian in $\mathbb{R}^m$. A $\mathcal{C}l_m$-valued function $f(x)$ defined on a domain U in $\mathbb{R}^m$ is left monogenic if $D_x f(x) = 0$. Since multiplication of Clifford numbers is not commutative in general, there is a similar definition for right monogenic functions. Sometimes we will consider the Dirac operator D_u in vector u rather than x.

Let $\mathcal{M}_j$ denote the space of $\mathcal{C}l_m$-valued monogenic polynomials, homogeneous of degree j. Note that if $h_j(u) \in \mathcal{H}_j$, the space of $\mathcal{C}l_m$-valued harmonic polynomials homogeneous of degree j, then $D_u h_j(u) \in \mathcal{M}_{j-1}$, but $D_u u p_{j-1}(u) = (-m-2j+2)p_{j-1}(u)$, where $p_{j-1} \in \mathcal{M}_{j-1}$so

$$\mathcal{H}_j = \mathcal{M}_j \oplus u\mathcal{M}_{j-1},\ h_j = p_j + up_{j-1}.$$

This is an *Almansi-Fischer decomposition* of $\mathcal{H}_j$. See [15] for more details. In this Almansi-Fischer decomposition, we define P_j as the projection map

$$P_j : \mathcal{H}_j \longrightarrow \mathcal{M}_j.$$

Suppose again U is a domain in $\mathbb{R}^m$. Consider a differentiable function $f : U \times \mathbb{R}^m \longrightarrow \mathcal{C}l_m$, such that for each $x \in U$, $f(x,u)$ is a left monogenic polynomial homogeneous of degree j in u, then the Rarita-Schwinger operator [7, 15] is defined by

$$R_j f(x,u) := P_j D_x f(x,u) = \left(\frac{uD_u}{m+2j-2} + 1\right) D_x f(x,u).$$

Though we have presented the Almansi-Fischer decomposition, the Dirac operator, and the Rarita-Schwinger operator here in terms of functions taking values in the real Clifford algebra $\mathcal{C}l_m$, they can all be realized in the same way for spinor-valued functions in the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$; we discuss spinors in the next section.

17.2.2 Irreducible Representations of the Spin Group

The following three representation spaces of the Spin group are frequently used as the target spaces in Clifford analysis. The spinor representation is the most commonly used spin representation in classical Clifford analysis and the other two polynomial representations are often used in higher spin theory.

17.2.2.1 Spinor Representation of $Spin(m)$

Consider the complex Clifford algebra $\mathcal{C}l_m(\mathbb{C})$ with even dimension $m = 2n$. Then $\mathbb{C}^m$ or the space of vectors is embedded in $\mathcal{C}l_m(\mathbb{C})$ as

$$(x_1, x_2, \cdots, x_m) \mapsto \sum_{j=1}^{m} x_j e_j : \mathbb{C}^m \hookrightarrow \mathcal{C}l_m(\mathbb{C}).$$

Define the *Witt basis* elements of $\mathbb{C}^{2n}$ as

$$f_j := \frac{e_j - ie_{j+n}}{2}, \quad f_j^\dagger := -\frac{e_j + ie_{j+n}}{2}.$$

Let $I := f_1 f_1^\dagger \ldots f_n f_n^\dagger$. The space of *Dirac spinors* is defined as

$$\mathcal{S} := \mathcal{C}l_m(\mathbb{C})I.$$

This is a representation of $Spin(m)$ under the following action

$$\rho(s)I := sI, \ \ for \ s \in Spin(m).$$

Note that $\mathcal{S}$ is a left ideal of $\mathcal{C}l_m(\mathbb{C})$. For more details, we refer the reader to [11]. An alternative construction of spinor spaces is given in the classical paper of Atiyah et al. [2].

17.2.2.2 Homogeneous Harmonic Polynomials on $\mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$

It is a well-known fact that the space of complex-valued harmonic polynomials defined on several vector variables is invariant under the action of $Spin(m)$, since the Laplacian Δ_m is an $SO(m)$ invariant operator. But it is not irreducible for $Spin(m)$. It can be decomposed into the infinite sum of j-homogeneous harmonic polynomials, $0 \leq j < \infty$. Each of these spaces is irreducible for $Spin(m)$. This brings us the most familiar representations of $Spin(m)$: spaces of j-homogeneous complex-valued harmonic polynomials defined on $\mathbb{R}^m$, henceforth denoted by $\mathcal{H}_j := \mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$. The following action has been shown to be an irreducible representation of $Spin(m)$ [38]:

$$\rho : Spin(m) \longrightarrow Aut(\mathcal{H}_j), \ s \longmapsto (f(x) \mapsto \tilde{s} f(sx\tilde{s})s).$$

This can also be realized as follows

$$\begin{aligned} Spin(m) &\xrightarrow{\theta} SO(m) \xrightarrow{\rho} Aut(\mathcal{H}_j); \\ a &\longmapsto O_a \longmapsto \big(f(x) \mapsto f(O_a x)\big), \end{aligned}$$

where θ is the double covering map and ρ is the standard action of $SO(m)$ on a function $f(x) \in \mathcal{H}_j$ with $x \in \mathbb{R}^m$. The function $\phi(z) = (z_1 + iz_m)^j$ is the highest weight vector for $\mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$ having highest weight $(j, 0, \cdots, 0)$ (for more details, see [23]). Accordingly, the spin representations given by $\mathcal{H}_j(\mathbb{R}^m, \mathbb{C})$ are said to have integer spin j; we can either specify an integer spin j or the degree of homogeneity j of harmonic polynomials.

17.2.2.3 Homogeneous Monogenic Polynomials on $\mathcal{C}l_m$

In $\mathcal{C}l_m$-valued function theory, the previously mentioned Almansi-Fischer decomposition shows that we can also decompose the space of j-homogeneous harmonic polynomials as follows

$$\mathcal{H}_j = \mathcal{M}_j \oplus u\mathcal{M}_{j-1}.$$

If we restrict $\mathcal{M}_j$ to the spinor valued subspace, we have another important representation of $Spin(m)$: the space of j-homogeneous spinor-valued monogenic polynomials on $\mathbb{R}^m$, henceforth denoted by $\mathcal{M}_j := \mathcal{M}_j(\mathbb{R}^m, \mathcal{S})$. More specifically, the following action has been shown to be an irreducible representation of $Spin(m)$:

$$\pi\ :\ Spin(m) \longrightarrow Aut(\mathcal{M}_j),\ s \longmapsto (f(x) \mapsto \tilde{s} f(sx\tilde{s})).$$

When m is odd, in terms of complex variables $z_s = x_{2s-1} + ix_{2s}$ for all $1 \leq s \leq \frac{m-1}{2}$, the highest weight vector is $\omega_j(x) = (\bar{z_1})^j I$ for $\mathcal{M}_j(\mathbb{R}^m, \mathcal{S})$ having highest weight $(j + \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$, where $\bar{z_1}$ is the conjugate of z_1, $\mathcal{S}$ is the Dirac spinor space, and I is defined as in Sect. 17.2.2.1; for details, see [38]. Accordingly, the spin representations given by $\mathcal{M}_j(\mathbb{R}^m, \mathcal{S})$ are said to have half-integer spin $j + \frac{1}{2}$; we can either specify a half-integer spin $j + \frac{1}{2}$ or the degree of homogeneity j of monogenic spinor-valued polynomials.

17.3 The Higher Order Higher Spin Operator $\mathcal{D}_{1,k}$

17.3.1 Motivation

We have mentioned that the Laplace operator (acting on a $\mathbb{C}$-valued field) is related to the Dirac operator (acting on a spinor-valued field) and they are both conformally invariant operators [33]. Moreover, the kth-power of the Dirac operator D_x^k for k a positive integer, is shown also to be conformally invariant in the spinor-valued function theory [33]. However, the Dirac operator D_x and the Laplace operator are no longer conformally invariant when acting on functions taking values in the higher spin spaces, in the sense explained in the next paragraph; see [9, 17], and

[13] for the Dirac operator case. The first generalization of the Dirac operator to higher spin spaces is instead the so-called Rarita-Schwinger operator [7, 15], and the generalization of the Laplace operator to higher spin spaces is the so-called higher spin Laplace or Maxwell operator given in [9, 17].

Let us look deeper into this lack of conformal invariance of the Dirac operator D_x when acting on functions taking values in the higher spin spaces. Given a function $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_j)$ such that $D_x f(x,u) = 0$, we apply inversion $x \longmapsto \frac{x}{||x||^2}$ to it. There is also a reflection of u in the direction x given by $\frac{xux}{||x||^2}$; this reflection involves x, which changes the conformal invariance of D_x such that $D_x f(x,u) = 0$ does not hold in general. This explanation also applies for the Laplace operator Δ_x in the higher spin theory. The explanation we just mentioned further implies that the kth-power of the Dirac operator D_x^k is not conformally invariant in the higher spin theory. In this section, we will provide the generalization of D_x^k when it acts on $C^\infty(\mathbb{R}^m, U)$, where $U = \mathcal{H}_1$ or $U = \mathcal{M}_1$ depending on the order. We provide nomenclature for these higher order operators in higher spin theory. We begin by examining existence and uniqueness of conformally invariant differential operators in higher spin spaces.

17.3.2 Existence of Conformally Invariant Operators

There is a well developed literature on the existence of conformally invariant operators [6, 19, 35–37]. In [35], Slovák demonstrated the existence of conformally invariant differential operators in higher spin spaces. Then Souček considered parts of Slovák's results in a form more suitable for Clifford analysis. Indeed, Souček's results only cover differential operators with intertwining properties, which also imply conformal invariance properties. In this section, we review Souček's results. For more details, we refer the reader to [37].

Let $M = \mathbb{R}^m \cup \{\infty\}$ be the conformal compactification of $\mathbb{R}^m$. $\Gamma_m = \{x \in \mathcal{C}l_m : xv\tilde{x} \in \mathbb{R}^m \text{ for all } v \in \mathbb{R}^m\}$ is the so-called Clifford group, and $V(m)$ be the group of Ahlfors-Vahlen matrices. We know that all conformal transformations in $\mathbb{R}^m$, $m > 2$ can be expressed in the form $\varphi(x) = (ax+b)(cx+d)^{-1}$ with $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(m)$. Let G denote the identity component of the group $V(m)$. The group G acts transitively on M and the isotropic group of the point $0 \in \mathbb{R}^m$ is clearly the subgroup H of all matrices in G with the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Hence $M \cong G/H$.

For a matrix $A \in H$, the element $a \in \Gamma_m$ has a nonzero norm and can be written as the product of $\frac{a}{||a||} \in Spin(m)$ and $||a|| \in \mathbb{R}^+$. If λ is a dominant integral weight for $Spin(m)$ with the corresponding irreducible representation V_λ and $\omega \in \mathbb{C}$ is a conformal weight, we denote $\rho_\lambda(\omega)$ the irreducible representation of H on V_λ

given by

$$\rho_\lambda(\omega)(h)[v] = ||a||^{-2\omega}\rho_\lambda\left(\frac{a}{||a||}\right)[v];\ \ v \in V_\lambda,\ \ h \in H;\ \ h = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}.$$

Below we discuss differential operators acting on sections of homogeneous vector bundles over $M = G/H$. We shall consider only bundles associated to irreducible representations of the isotropic group H. Hence they are specified by a highest weight λ giving an irreducible representation of $Spin(m)$ and by a conformal weight $\omega \in \mathbb{C}$. Such a bundle will be denoted by $V_\lambda(\omega)$. The following lemma gives the action of G on $C^\infty(\mathbb{R}^m, V_\lambda(\omega))$.

Lemma 17.3.1 ([37]) *The action of G on $C^\infty(\mathbb{R}^m, V_\lambda(\omega))$ is given by*

$$[g \cdot f](x) = ||cx + d||^{-2\omega}\rho_\lambda\left(\frac{\widetilde{cx + d}}{||cx + d||}\right) f((ax + b)(cx + d)^{-1}),$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$. "$\cdot$" stands for the action of g on function f.

We now consider conformally invariant differential operators between sections of $\Gamma(M, V_\lambda(\omega))$ and $\Gamma(M, V_{\lambda'}(\omega'))$ of order $\omega' - \omega$, separately for the even and odd dimension cases. Sections 3.2.1 and 3.2.2 are quoted from [37]. These only cover generalizations of kth-power of the Dirac operator, which possess intertwining properties, in higher spin spaces. For instance, twistor and dual twistor operators are not included. A complete list can be found in [35].

17.3.2.1 Even Dimension $m = 2n$

The description in [35], Chapt. 8, uses a different notation for irreducible representations of H, namely coefficients written in Dynkin diagrams over individual simple roots are used there. There are $n + 1$ such coefficients, and in [37], the author uses the symbols B; D_i, $i = 1, \cdots, n - 2$; A, C to denote them. The number B can be any integer, all others should be positive integers.

To relate this notation to one used standardly in Clifford analysis, we shall use the following labels for irreducible representations of $Spin(m)$:

$$\{\lambda_i = (1, \cdots, 1, 0, \cdots, 0) :\ i = 1, 2, \cdots, n - 2, \text{the first } i \text{ entries are } 1\}$$

are highest weights of fundamental representations $\wedge^i(\mathbb{C}^m)$ and

$$\sigma^\pm = \left(\frac{1}{2}, \frac{1}{2}, \cdots, \pm\frac{1}{2}\right)$$

are highest weights of the basic spinor representations $\mathcal{S}^{\pm}$ of $Spin(m)$ [23]. Then the $(n+1)$-tuple (B, D_i, A, C) specifies the irreducible representation $\rho_\lambda(\omega)$ for $Spin(m)$, where

$$\lambda = \sum_{i=1}^{n-2}(D_i - 1)\lambda_i + (A-1)\sigma^+ + (C-1)\sigma^- \tag{17.3.1}$$

and the conformal weight is given by

$$\omega = n - \left[B + \sum_{i=1}^{n-2} D_i + \frac{A+C}{2}\right]. \tag{17.3.2}$$

Let us now state Souček's theorem on classification of nonstandard operators (lower arrows in the corresponding diagrams in Theorem 8.13 in [35]), in the even dimension case.

Theorem 17.3.2 *[37] Let (λ, ω) and (λ', ω') be computed using Eqs. (17.3.1) and (17.3.2), where the positive integers $D_i, A, C, D_i', A', C', i = 1, 2, \cdots, n-2$, may adopt any values in the columns to their right in the following table, and where (λ', ω') are determined by primed coefficients. In the table, $a, b, c, d_i, i = 1, 2, \cdots, n-2$ are nonnegative integers, $d = \sum_i d_i$, and the integer e is defined by $e = a + b + c + d$.*

B	$-b-d$	$-b-d+d_{n-2}$	$\cdots$	$-b$	b
B'	$-e$	$-e-d_{n-2}$	$\cdots$	$-e-d$	$-e-d$
$D_1 = D_1'$	b	b	$\cdots$	$b+d_1$	d_1
$D_2 = D_2'$	d_1	d_1	$\cdots$	d_2	d_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D_{n-3} = D_{n-3}'$	d_{n-4}	d_{n-4}	$\cdots$	d_{n-3}	d_{n-3}
$D_{n-2} = D_{n-2}'$	d_{n-3}	$d_{n-3}+d_{n-2}$	$\cdots$	d_{n-2}	d_{n-2}
$A = C'$	$a+d_{n-2}$	a	$\cdots$	a	a
$C = A'$	$c+d_{n-2}$	c	$\cdots$	c	c

Then there exists (up to a multiple) unique nontrivial conformally invariant differential operators between sections of $\Gamma(M, V_\lambda(\omega))$ and $\Gamma(M, V_{\lambda'}(\omega'))$; its order is equal to $\omega' - \omega$. This is a complete list of the so-called nonstandard conformally invariant differential operators on spaces of even dimension.

17.3.2.2 Odd Dimension $m = 2n + 1$

In odd dimension, the highest weights of fundamental representations $\Lambda^i(\mathbb{C}^m)$ are

$$\{\lambda_i = (1, \cdots, 1, 0, \cdots, 0) : \ i = 1, \cdots, n-1, \text{the first } i \text{ entries are } 1\},$$

and highest weights of the basic spinor representation $\mathcal{S}$ of $Spin(m)$ [23] are n-tuples

$$\sigma = \left(\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2}\right).$$

Then the $(n+1)$-tuple (B, D_i, A, C) specifies the irreducible representation $\rho_\lambda(\omega)$ for $Spin(m)$, where

$$\lambda = \sum_{i=1}^{n-1} (D_i - 1)\lambda_i + (A - 1)\sigma \tag{17.3.3}$$

and the conformal weight is given by

$$\omega = \frac{2n+1}{2} - \left[B + \sum_{i=1}^{n-1} D_i + \frac{A}{2}\right]. \tag{17.3.4}$$

Let us now state Souček's theorem on classification of nonstandard operators, now in the odd dimension case.

Theorem 17.3.3 *[37] Let (λ, ω) and (λ', ω') be computed using Eqs. (17.3.3) and (17.3.4), where the positive integers $D_i, A, C, D_i', A', C', i = 1, 2, \cdots, n-1$, may adopt any values in the columns to their right in the following table, and where (λ', ω') are determined by primed coefficients. In the table, $a, b, c, d_i, i = 1, 2, \cdots, n-1$ are nonnegative half-integers or integers (at least one of them being half integral), $d = \sum_i d_i$, and the integer e is defined by $e = a + b + d$.*

B	$-b-d$	$-b-d+d_{n-1}$	$\cdots$	$-b$	b
B'	$-e$	$-e-d_{n-1}$	$\cdots$	$-e-d$	$-e-d$
$D_1 = D_1'$	b	b	$\cdots$	$b+d_1$	d_1
$D_2 = D_2'$	d_1	d_1	$\cdots$	d_2	d_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$D_{n-2} = D_{n-2}'$	d_{n-3}	d_{n-3}	$\cdots$	d_{n-2}	d_{n-2}
$D_{n-1} = D_{n-1}'$	d_{n-2}	$d_{n-2}+d_{n-1}$	$\cdots$	d_{n-1}	d_{n-1}
$A = A'$	$a+d_{n-2}$	a	$\cdots$	a	a

Then there exists (up to a multiple) unique nontrivial conformally invariant differential operators between sections of $\Gamma(M, V_\lambda(\omega))$ and $\Gamma(M, V_{\lambda'}(\omega'))$; its order is equal to $\omega' - \omega$. This is a complete list of the so-called nonstandard conformally invariant differential operators on spaces of odd dimension.

17.3.2.3 Applications to Our Cases

Theorems 17.3.2 and 17.3.3 show existence of conformally invariant differential operators as follows.

Theorem 17.3.4 *[37] Let* (λ, ω) *and* (λ', ω') *be one of couples for which there is a (nontrivial) invariant differential operator*

$$D: \ \Gamma(M, V_\lambda(\omega)) \longrightarrow \Gamma(M, V_{\lambda'}(\omega'))$$

(the nonstandard operators are listed in Theorems 17.3.2 and 17.3.3; the complete list is in [35]).

Let $T_{\lambda,\omega}(g),\ g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ *(similarly for* $T_{\lambda',\omega'}(g)$*) be the operator acting on smooth maps from* $\mathbb{R}^m$ *to* V_λ *by*

$$\big[T_{\lambda,\omega}(g) f\big](x) = ||cx+d||^{-2\omega} \rho_\lambda\left(\frac{\widetilde{cx+d}}{||cx+d||}\right)\big[f((ax+b)(cx+d)^{-1})\big].$$

Then

$$D(T_{\lambda,\omega}(g) f) = T_{\lambda',\omega'}(g)(Df),\ g \in G,\ f \in C^\infty(\mathbb{R}^m, V_\lambda).$$

This paper considers differential operators acting on functions $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ or $f(x,u) \in C^\infty(\mathbb{R}^m, \mathcal{M}_j)$. Here we only show existence of conformally invariant differential operators on spaces of *even dimension* m; the odd dimensional case is similar. We work out the allowable highest weights, conformal weights, and orders on operators acting on these function spaces.

From Theorem 17.3.2, we notice that highest weight λ is determined by $D_i,\ A$ and C. From the table, we also have $\lambda = \lambda'$. In other words, conformally invariant operators exist between $C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ and itself or $C^\infty(\mathbb{R}^m, \mathcal{M}_j)$ and itself. We consider each in turn.

Integer Spin Case: $C^\infty(\mathbb{R}^m, \mathcal{H}_j) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_j)$
As an irreducible representation of $Spin(m)$, $\mathcal{H}_j$ has highest weight of n-tuple $\lambda = \lambda' = (j, 0, \cdots, 0)$. The group action ρ_λ is defined in Sect. 17.2.2.2. From the table in Theorem 17.3.2 and

$$\lambda = \sum_{i=1}^{n-2} (D_i - 1)\lambda_i + (A-1)\sigma^+ + (C-1)\sigma^-,$$
$$\lambda_i = (1, 1, \cdots, 1, 0, \cdots, 0),\ i = 1, 2, \cdots, n-2,$$
$$\sigma^\pm = \left(\frac{1}{2}, \frac{1}{2}, \cdots, \pm\frac{1}{2}\right),$$

we know that $D_1 = j+1,\ D_i = 1,\ i = 2, \cdots, n-2$ and $A = C = 1$. There is exactly one possibility for all entries but the last column in the table, for which there is a sequence of possibilities indexed by a nonnegative integer b. The last column corresponds to the $(2b+2n+2j-2)$-th order conformally invariant differential operator, with

$$d = j + n - 2,\ a = c = 1,\ e = b + j + n,\ B = b,\ B' = -b - 2j - 2n + 2,$$

and conformal weights $\omega = 1 - b - j$ and $\omega' = b + j + 2n - 1$. Hence, we have

$$\mathcal{D}_{1,2b+2n+2j-2} T_{\lambda,1-b-j} = T_{\lambda,b+j+2n-1} \mathcal{D}_{1,2b+2n+2j-2},$$

in other words,

$$\mathcal{D}_{1,2b+2n+2j-2} ||cx + d||^{2j+2b-2} = ||cx + d||^{-2b-2j-4n+2} \mathcal{D}_{1,2b+2n+2j-2}.$$

To see the above intertwining operators coincide with the forms of the intertwining operators we have at the end of Sect. 17.3, we let $2b + 2n + 2j - 2 = 2s$ and since $m = 2n$, we have

$$\mathcal{D}_{1,2s} ||cx + d||^{2s-m} = ||cx + d||^{-m-2s} \mathcal{D}_{1,2s}.$$

Similar considerations apply for spin $j > 1$.

Half-Integer Spin Case: $C^\infty(\mathbb{R}^m, \mathcal{M}_j) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_j)$
As an irreducible representation of $Spin(m)$, $\mathcal{M}_j$ has highest weight as n-tuple $\lambda = \lambda' = \left(j + \frac{1}{2}, \frac{1}{2}, \cdots, \pm\frac{1}{2}\right)$. The group action ρ_λ is the action defined as in Sect. 17.2.2.3. From the table and

$$\lambda = \sum_{i=1}^{n-2} (D_i - 1)\lambda_i + (A - 1)\sigma^+ + (C - 1)\sigma^-,$$
$$\lambda_i = (1, 1, \cdots, 1, 0, \cdots, 0),\ i = 1, 2, \cdots, n - 2,$$
$$\sigma^\pm = \left(\frac{1}{2}, \frac{1}{2}, \cdots, \pm\frac{1}{2}\right),$$

we can find that $D_1 = j + 1, D_i = 1, i = 2, \cdots, n - 2$ and $A = 1, C = 0$(or $A = 0, C = 1$ depending on the last entry of λ is $\frac{1}{2}$ or $-\frac{1}{2}$). Similar as the previous case, there is just one possibility for all but the last column in the table and there is a sequence of possibilities indexed by a nonnegative integer b for the last column. The last column corresponds to the $(2j + 2n + 2b - 3)$-th order conformally invariant differential operator, with

$$d = j + n - 2,\ a = 1, c = 0\ (or\ a = 0,\ c = 1),\ e = b + j + n - 1,$$
$$B = b,\ B' = -2j - 2n - b + 3,$$

and conformal weights $\omega = -b - j + \frac{3}{2}$ and $\omega' = j + 2n + b - \frac{3}{2}$. Hence, we have

$$\mathcal{D}_{1,2j+2n+2b-3} T_{\lambda,-b-j+\frac{3}{2}} = T_{\lambda,j+2n+b-\frac{3}{2}} \mathcal{D}_{1,2j+2n+2b-3},$$

in other words,

$$\begin{aligned} &\mathcal{D}_{1,2j+2n+2b-3} ||cx+d||^{2b+2j-3} \frac{\widetilde{cx+d}}{||cx+d||} \\ &= ||cx+d||^{-2j-4n-2b+3} \frac{\widetilde{cx+d}}{||cx+d||} \mathcal{D}_{1,2j+2n+2b-3}. \end{aligned}$$

To see the above intertwining operators coincide with the intertwining operators we have at the end of Sect. 17.3, we let $2j + 2n + 2b - 3 = 2s + 1$ and since $m = 2n$, we have

$$\mathcal{D}_{1,2s+1} \frac{\widetilde{cx+d}}{||cx+d||^{m-2s}} = \frac{\widetilde{cx+d}}{||cx+d||^{m+2s+2}} \mathcal{D}_{1,2s+1}.$$

Similar considerations apply for spin $j > \frac{3}{2}$.

Similar arguments apply for the odd dimensional cases. This establishes existence of the conformally invariant differential operators we wish to consider. Further, even order conformally invariant differential operators only exist between $C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ and odd order ones only exist between $C^\infty(\mathbb{R}^m, \mathcal{M}_j)$. Intertwining operators of conformally invariant differential operators in Theorem 17.3.21 can also be recovered. Once we establish conformal invariance of the operators that we construct between the desired higher spin spaces, uniqueness up to multiplicative constant of these higher order higher spin operators is established by the preceding theorems.

17.3.3 Construction and Conformal Invariance

We have established by arguments of Slovák [35] and Souček [37], for integers $j \geq 0$ and $k > 0$ there exist conformally invariant differential operators in the higher spin setting

$$\mathcal{D}_{j,k} : C^\infty(\mathbb{R}^m, U) \longrightarrow C^\infty(\mathbb{R}^m, U),$$

where $U = \mathcal{H}_j$ if k is even and $U = \mathcal{M}_j$ if k is odd. Note that the target space U here is a function space. Then any element in $C^\infty(\mathbb{R}^m, U)$ is of the form $f(x, u)$, where $f(x, u) \in U$ for each fixed $x \in \mathbb{R}^m$ and x is the variable on which $\mathcal{D}_{1,k}$ acts.

We introduce some nomenclature suggestive of massless spin fields in mathematical physics, which we hope is adopted by others studying higher spin theory in Clifford analysis. As a Spin representation $\mathcal{H}_j$ is associated with integer spin j and particles of integer spin are called bosons, so the operators $\mathcal{D}_{j,k}$: $C^\infty(\mathbb{R}^m, \mathcal{H}_j) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_j)$ are named *bosonic operators*. Thus in the spin 0 case we have the Laplace operator and its k-powers, the spin 1 case the Maxwell operator and its generalization to order $k = 2n$, and general higher spin Laplace operators and their generalization to order $k = 2n$. Correspondingly, as a Spin representation $\mathcal{M}_j$ is associated with half-integer spin $j + \frac{1}{2}$ and particles of half-integer spin are called fermions, so the operators $\mathcal{D}_{j,k}$: $C^\infty(\mathbb{R}^m, \mathcal{M}_j) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_j)$ are named *fermionic operators*. Thus in the spin $\frac{1}{2}$ case we have the Dirac operator and its $k = 2n + 1$ powers, the spin $\frac{3}{2}$ case the simplest Rarita-Schwinger operator and its generalization to order $k = 2n + 1$, and general Rarita-Schwinger operators and their generalization to order $k = 2n + 1$. Note that our notation indexes according to degree of homogeneity of the target space j and differential order k, so fractions are not used in the notation; if we indexed according to spin, fractional spins would need to be used for odd order operators.

Thus, we proceed to construct the bosonic operators of spin-1 and even order, followed by the fermionic operators of spin-$\frac{3}{2}$ and odd order.

k **Even,** $k = 2n, n > 1$ **(The Bosonic Case)**

Theorem 17.3.5 *For positive integer n, the unique $2n$-th order conformally invariant differential operator of spin-1 $\mathcal{D}_{1,2n} : C^\infty(\mathbb{R}^m, \mathcal{H}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_1)$ has the following form, up to a multiplicative constant:*

$$\mathcal{D}_{1,2n} = \Delta_x^n - \frac{4n}{m + 2n - 2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}.$$

This is the $2n$-th order fermionic operator of spin-1. For the case $n = 1$, we retrieve the Maxwell operator from [17]. Our proof of conformal invariance of these operators follows closely the method of [17]. In order to explain what conformal invariance means, we begin with the concept of a generalized symmetry (see [16]):

Definition 17.3.6 An operator η_1 is a generalized symmetry for a differential operator $\mathcal{D}$ if and only if there exists another operator η_2 such that $\mathcal{D}\eta_1 = \eta_2\mathcal{D}$. Note that for $\eta_1 = \eta_2$, this reduces to a definition of a (proper) symmetry: $\mathcal{D}\eta_1 = \eta_1\mathcal{D}$.

One determines the first order generalized symmetries of an operator, which span a Lie algebra [17, 27]. In this case, the first order symmetries will span a Lie algebra isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m + 1)$; in this sense, the operators we consider are conformally invariant. The operator $\mathcal{D}_{1,2n}$ is $\mathfrak{so}(m)$-invariant (rotation-invariant) because it is the composition of $\mathfrak{so}(m)$-invariant (rotation-invariant) operators, which means the angular momentum operators $L_{ij}^x + L_{i,j}^u$ that generate these rotations are proper symmetries of $\mathcal{D}_{1,2n}$. The infinitesimal translations $\partial_{x_j}, j = 1, \cdots, n$, corresponding to linear momentum operators are proper symmetries of $\mathcal{D}_{1,2n}$; this is an alternative way to say that $\mathcal{D}_{1,2n}$ is invariant under translations that are generated by these infinitesimal translations. Readers familiar with quantum mechanics will recognize the connection to isotropy and homogeneity of space, the rotational and translational invariance of Hamiltonian, and the conservation of angular and linear momentum [34]; see also [4] concerning Rarita-Schwinger operators.

The remaining two of the first order generalized symmetries of $\mathcal{D}_{1,2n}$ are the Euler operator and special conformal transformations. The Euler operator $\mathbb{E}_x$ that measures degree of homogeneity in x is a generalized symmetry because $\mathcal{D}_{1,2n}\mathbb{E}_x = (\mathbb{E}_x + 2n)\mathcal{D}_{1,2n}$; this is an alternative way to say that $\mathcal{D}_{1,2n}$ is invariant under dilations, which are generated by the Euler operator. The special conformal transformations are defined in Lemma 17.3.8 in terms of harmonic inversion for $\mathcal{H}_1$-valued functions; harmonic inversion is defined in Definition 17.3.7 and is an involution mapping solutions of $\mathcal{D}_{1,2n}$ to $\mathcal{D}_{1,2n}$. Readers familiar with conformal field theory will recognize that invariance under dilation corresponds to scale-invariance and that special conformal transformations are another class of conformal transformations arising on spacetime [20]. An alternative method of proving conformal invariance of $\mathcal{D}_{1,2n}$ is to prove the invariance of $\mathcal{D}_{1,2n}$ under those finite transformations generated by these first order generalized symmetries (rotations, dilations, translations, and special conformal transformations) to show invariance of $\mathcal{D}_{1,2n}$ under actions of the conformal group; this may be phrased in terms of Möbius transformations and the Iwasawa decomposition. However, the first-order generalized symmetry method emphasizes the connection to mathematical physics and is more amenable to our proof of a certain property of harmonic inversion. It is also that used by earlier authors [9, 17].

Definition 17.3.7 The harmonic inversion is a conformal transformation defined as

$$\begin{aligned}\mathcal{J}_{2n} &: C^\infty(\mathbb{R}^m, \mathcal{H}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_1),\\ f(x,u) &\mapsto \mathcal{J}_{2n}[f](x,u) := ||x||^{2n-m} f\left(\frac{x}{||x||^2}, \frac{xux}{||x||^2}\right).\end{aligned}$$

Note that this inversion consists of Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u \omega$ acting on the dummy variable u (where $x = ||x||\omega$) and a multiplication by a conformal weight term $||x||^{2n-m}$; it satisfies $\mathcal{J}_{2n}^2 = 1$.

Then we have the special conformal transformation defined in the following lemma. The definition is an infinitesimal version of the fact that finite special conformal transformations consist of a translation preceded and followed by an inversion [20]: an infinitesimal translation preceded and followed by harmonic inversion. The second equality in the lemma shares some terms in common with the generators of special conformal transformations in conformal field theory [20], and is a particular case of a result in [18].

Lemma 17.3.8 *The special conformal transformation given by $\mathcal{C}_{2n} := \mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n}$ satisfies*

$$\mathcal{C}_{2n} = 2\langle u, x\rangle\partial_{u_j} - 2u_j\langle x, D_u\rangle + ||x||^2\partial_{x_j} - x_j(2\mathbb{E}_x + m - 2n).$$

Proof A similar calculation as in *Proposition A.1* in [9] will show the conclusion. □

Then, we have the main proposition as follows.

Proposition 17.3.9 *The special conformal transformations $\mathcal{C}_{2n}$ are generalized symmetries of $\mathcal{D}_{1,2n}$, where $j \in \{1, 2, \ldots, m\}$. More specifically,*

$$[\mathcal{D}_{1,2n}, \mathcal{C}_{2n}] = -4nx_j\mathcal{D}_{1,2n}.$$

In particular, this shows that

$$\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = ||x||^{4n}\mathcal{D}_{1,2n}, \tag{17.3.5}$$

which is the generalization of the case of the classical higher order Laplace operator Δ_x^n [3]. This also implies $\mathcal{D}_{1,2n}$ is invariant under inversion.

If the main proposition holds, then the conformal invariance can be summarized in the following theorem:

Theorem 17.3.10 *The first order generalized symmetries of $\mathcal{D}_{1,2n}$ are given by:*

1. *The infinitesimal rotation $L_{i,j}^x + L_{i,j}^u$, with $1 \le i < j \le m$.*
2. *The shifted Euler operator $\mathbb{E}_x + \dfrac{m-2n}{2}$.*
3. *The infinitesimal translations ∂_{x_j}, with $1 \le j \le m$.*
4. *The special conformal transformations $\mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n}$, with $1 \le j \le m$.*

These operators span a Lie algebra which is isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m+1)$, whereby the Lie bracket is the ordinary commutator.

Proof The proof is similar as in [18] via transvector algebras. Notice that the shift in the shifted Euler operator $\mathbb{E}_x + \omega$ defines the conformal weight (defined in Sect. 17.3.2) $\omega = \dfrac{m-2n}{2}$. □

Detailed Proof of Proposition 17.3.9

First, let us prove a few technical lemmas. It is worth pointing out that since we are dealing with degree-1 homogeneous polynomials in u, terms involving second derivatives with respect to u disappear.

Lemma 17.3.11 *For all $1 \le j \le m$, we have*

$$[\Delta_x^n, \mathcal{C}_{2n}] = -4nx_j\Delta_x^n + 4n\langle u, D_x\rangle\partial_{u_j}\Delta_x^{n-1} - 4nu_j\langle D_u, D_x\rangle\Delta_x^{n-1}.$$

Proof We prove this by induction. First, we have [9]

$$[\Delta_x, \mathcal{C}_2] = -4x_j\Delta_x + 4\langle u, D_x\rangle\partial_{u_j} - 4u_j\langle D_u, D_x\rangle.$$

Assuming the lemma is true for Δ^{n-1}, applying the fact that for general operators $A,\ B$ and C

$$[AB, C] = A[B, C] + [A, C]B$$

and

$$\mathcal{C}_{2n} = \mathcal{C}_2 + (2n-2)x_j,$$

we have

$$[\Delta_x^n, \mathcal{C}_{2n}] = \Delta_x^{n-1}[\Delta_x, \mathcal{C}_{2n}] + [\Delta_x^{n-1}, \mathcal{C}_{2n}]\Delta_x.$$

Since

$$\mathcal{C}_{2n} = \mathcal{C}_{2n-2} + 2x_j,$$

a straightforward calculation leads to the conclusion. □

Lemma 17.3.12 *For all* $1 \leq j \leq m$, *we have*

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}, \mathcal{C}_{2n}] = -4nx_j\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}$$
$$+(m+2n-2)\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big)\Delta_x^{n-1}.$$

Proof First, we have [9]:

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle, \mathcal{C}_2] = 2||u||^2\partial_{u_j}\langle D_u, D_x\rangle - 4x_j\langle u, D_x\rangle\langle D_u, D_x\rangle$$
$$+\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big)(2\mathbb{E}_u + m - 2)$$
$$= -4x_j\langle u, D_x\rangle\langle D_u, D_x\rangle + m\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big).$$

Then

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}, \mathcal{C}_{2n}]$$
$$= \langle u, D_x\rangle\langle D_u, D_x\rangle[\Delta_x^{n-1}, \mathcal{C}_{2n}] + [\langle u, D_x\rangle\langle D_u, D_x\rangle, \mathcal{C}_{2n}]\Delta_x^{n-1}$$

together with the previous lemma prove the conclusion. □

With the help of *Lemmas* 17.3.11 and 17.3.12, a straightforward calculation shows that

$$[\mathcal{D}_{1,2n}, \mathcal{C}_{2n}] = -4nx_j\mathcal{D}_{1,2n}.$$

Since $\mathcal{D}_{1,2n}$ is conformally invariant by Theorem 17.3.10 and Slovak's results provide the uniqueness and intertwining operators of conformally invariant differential operators, we have

$$\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = ||x||^{4n}\mathcal{D}_{1,2n}$$

from the intertwining operators under (harmonic) inversion. This is a generalization of the Laplacian case [3].

k **Odd,** $k = 2n - 1, n > 1$ **(The Fermionic Case)**

Theorem 17.3.13 *For positive integer n, the unique* $(2n-1)$*-th order conformally invariant differential operator of spin-*$\frac{3}{2}$ *is* $\mathcal{D}_{1,2n-1} : C^\infty(\mathbb{R}^m, \mathcal{M}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_1)$ *and has the following form, up to a multiplicative constant:*

$$\mathcal{D}_{1,2n-1} = D_x\Delta_x^{n-1} - \frac{2}{m+2n-2}u\langle D_u, D_x\rangle\Delta_x^{n-1} - \frac{4n-4}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x.$$

This is the $(2n-1)$-th order fermionic operator of spin-$\frac{3}{2}$. When $n = 1$, we have the Rarita-Schwinger operator appearing in [7, 15] and elsewhere. The same strategy in the even case applies: we only must show the special conformal transformation defined below is a generalized symmetry of $\mathcal{D}_{1,2n-1}$. We have the definition for monogenic inversion as follows.

Definition 17.3.14 Monogenic inversion is a conformal transformation defined as

$$\mathcal{J}_{2n+1} : C^\infty(\mathbb{R}^m, \mathcal{M}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_1),$$
$$f(x,u) \mapsto \mathcal{J}_{2n+1}[f](x,u) := \frac{x}{||x||^{m-2n}} f(\frac{x}{||x||^2}, \frac{xux}{||x||^2}).$$

Note that this inversion also consists of Kelvin inversion $\mathcal{J}$ on $\mathbb{R}^m$ in the variable x composed with a reflection $u \mapsto \omega u\omega$ acting on the dummy variable u (where $x = ||x||\omega$) and a multiplication of a conformal weight term $\frac{x}{||x||^{m-2n}}$; it satisfies $\mathcal{J}_{2n+1}^2 = -1$ instead. Similarly, monogenic inversion is an involution mapping solutions for $\mathcal{D}_{1,2n-1}$ to solutions for $\mathcal{D}_{1,2n-1}$ [31]. Then we have the following lemma.

Lemma 17.3.15 *The special conformal transformation is defined as*

$$\begin{aligned}\mathcal{C}_{2n-1} &:= \mathcal{J}_{2n-1}\partial_{x_j}\mathcal{J}_{2n-1}\\ &= -e_j x - 2\langle u, x\rangle\partial_{u_j} + 2u_j\langle x, D_u\rangle - ||x||^2\partial_{x_j} + x_j(2\mathbb{E}_x + m - 2n),\end{aligned}$$

$\mathcal{C}_{2n-1} = \mathcal{C}_{2n-3} - 2x_j = -\mathcal{C}_{2n-2} - e_j x - 2x_j.$

Proof As similar calculation as in *Proposition A.1* in [9] will show the conclusion. □

Then we arrive at the main proposition, stating that the special conformal transformations are generalized symmetries of operator $\mathcal{D}_{1,2n-1}$.

Proposition 17.3.16 *The special conformal transformations $\mathcal{C}_{2n-1}$, with $j \in \{1, 2, \dots, m\}$ are generalized symmetries of $\mathcal{D}_{1,2n-1}$ (respectively $\mathcal{D}_{3,j}$). More specifically,*

$$[\mathcal{D}_{1,2n-1}, \mathcal{C}_{2n-1}] = (4n-2)x_j\mathcal{D}_{1,2n-1}.$$

In particular, this shows that $\mathcal{J}_{2n-1}\mathcal{D}_{1,2n-1}\mathcal{J}_{2n-1} = ||x||^{4n-2}\mathcal{D}_{1,2n-1}$, which is the generalization of the case of the classical higher order Dirac operator D_x^{2n-1} [3]. This also implies $\mathcal{D}_{1,2n-1}$ is invariant under inversion.

Theorem 17.3.17 *The first order generalized symmetries of $\mathcal{D}_{1,2n-1}$ are given by:*

1. *The infinitesimal rotation $L^x_{i,j} + L^u_{i,j} - \frac{1}{2}e_i e_j$, with $1 \le i < j \le m$.*
2. *The shifted Euler operator $\mathbb{E}_x + \dfrac{m-2n+1}{2}$.*
3. *The infinitesimal translations ∂_{x_j}, with $1 \le j \le m$.*
4. *The special conformal transformations $\mathcal{J}_{2n-1}\partial_{x_j}\mathcal{J}_{2n-1}$, with $1 \le j \le m$.*

These operators span a Lie algebra which is isomorphic to the conformal Lie algebra $\mathfrak{so}(1, m+1)$, whereby the Lie bracket is the ordinary commutator.

Detailed Proof of Proposition 17.3.16

To prove Proposition 17.3.16, as in the even case, we need a few technical lemmas.

Lemma 17.3.18 *For all $1 \le j \le m$, we have*

$$\begin{aligned}[D_x\Delta_x^{n-1}, \mathcal{C}_{2n-1}] = {}& (4n-4)\big(u_j\langle D_u, D_x\rangle - \langle u, D_x\rangle\partial_{u_j}\big)D_x\Delta_x^{n-2}\\ &-2u\partial_{u_j}\Delta_x^{n-1} + (4n-2)x_j D_x\Delta_x^{n-1}.\end{aligned}$$

Lemma 17.3.19 *For all $1 \le j \le m$, we have*

$$\begin{aligned}&[u\langle D_u, D_x\rangle\Delta_x^{n-1}, \mathcal{C}_{2n-1}] = (4n-2)x_j u\langle D_u, D_x\rangle\Delta_x^{n-1}\\ &-(m+2n-2)u\partial_{u_j}\Delta_x^{n-1} - (2n-2)ue_j\langle D_u, D_x\rangle\Delta_x^{n-2}.\end{aligned}$$

Lemma 17.3.20 *For all* $1 \leq j \leq m$*, we have*

$$[\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x, \mathcal{C}_{2n-1}] = (4n-2)x_j\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x$$
$$-(m+2n-2)\big(\langle u, D_x\rangle\partial_{u_j} - u_j\langle D_u, D_x\rangle\big)\Delta_x^{n-2}D_x$$
$$+ue_j\langle D_u, D_x\rangle\Delta_x^{n-2}D_x.$$

We combine *Lemmas* 17.3.18, 17.3.19 and 17.3.20 to get

$$[\mathcal{D}_{1,2n-1}, \mathcal{C}_{2n-1}] = (4n-2)x_j\mathcal{D}_{1,2n-1}.$$

This implies $\mathcal{J}_{2n-1}\mathcal{D}_{1,2n-1}\mathcal{J}_{2n-1} = ||x||^{4n-2}\mathcal{D}_{1,2n-1}$.

Conformal Invariance and Intertwining Operators, Both Cases
Strictly speaking, Theorem 17.3.4 together with the constructions in this subsection provide the intertwining operators for the bosonic and fermionic operators in this paper. However, for the sake of concreteness and to highlight the alternative approach centering upon Möbius transformations, here we rely on the Iwasawa decomposition for Möbius transformations to determine these intertwining operators. Let $\mathcal{D}_{1,k,x,u}$ and $\mathcal{D}_{1,k,y,w}$ be the higher order higher spin operators with respect to $x,\ u$ and $y,\ w$, respectively and $y = \phi(x) = (ax+b)(cx+d)^{-1}$ is a Möbius transformation. Let

$$J_k = \frac{\widetilde{cx+d}}{||cx+d||^{m-2n}}, \quad for\ k = 2n+1;$$
$$J_k = \frac{1}{||cx+d||^{m-2n}}, \quad for\ k = 2n;$$
$$J_{-k} = \frac{\widetilde{cx+d}}{||cx+d||^{m+2n+2}}, \quad for\ k = 2n+1;$$
$$J_{-k} = \frac{1}{||cx+d||^{m+2n}}, \quad for\ k = 2n,$$

with $n = 1, 2, 3, \cdots$. See [28]. Then we make the following claim.

Theorem 17.3.21 *Let* $y = \phi(x) = (ax+b)(cx+d)^{-1}$ *be a Möbius transformation. Then*

$$J_{-k}\mathcal{D}_{1,k,y,w}f(y,w) = \mathcal{D}_{1,k,x,u}J_kf\bigg(\phi(x), \frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}\bigg),$$

where $w = \dfrac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$*. The exact same result holds for all* $\mathcal{D}_{j,k}$*, including* $\mathcal{D}_{j,3}$ *and* $\mathcal{D}_{j,3}$ *for all integers* $j > 0$*; notably the intertwining operators depend only on the order k, not the spin* j *or* $j+\frac{1}{2}$*..*

We only prove the bosonic (order $k = 2n$) case for spin $j = 1$, as the other bosonic cases ($j > 0$) and the fermionic (order $k = 2n + 1$) cases are similar. According to the Iwasawa decomposition, we need only prove this with respect to orthogonal transformation and inversion, since translation and dilation are trivial. Note that our argument here requires the invariance under harmonic inversion established earlier.

Orthogonal Transformations $a \in Pin(m)$

Lemma 17.3.22 *If $x = ay\tilde{a}$, $u = aw\tilde{a}$, then*

$$\mathcal{D}_{1,2n,x,u} f(x,u) = a\mathcal{D}_{1,2n,y,w}\tilde{a} f(y,w).$$

Proof

$$\begin{aligned}
\mathcal{D}_{1,2n,x,u} f(x,u) &= \left(\triangle_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\right)\Delta_x^{n-1} f(x,u) \\
&= \left(a\triangle_y\tilde{a} - \frac{4n}{m+2n-2}a\langle w, D_y\rangle\tilde{a}a\langle D_w, D_y\rangle\tilde{a}\right)a\Delta_y^{n-1}\tilde{a} f(y,w) \\
&= a\left(\triangle_y - \frac{4n}{m+2n-2}\langle w, D_y\rangle\langle D_w, D_y\rangle\right)\Delta_y^{n-1}\tilde{a} f(y,w) \\
&= a\mathcal{D}_{1,2n,y,w}\tilde{a} f(y,w).
\end{aligned}$$

□

Inversions

Lemma 17.3.23 *Let $x = y^{-1}$ and $u = \dfrac{ywy}{||y||^2}$, then*

$$\mathcal{D}_{1,2n,y,w}||x||^{m-2n} f(y,w) = ||x||^{m+2n}\mathcal{D}_{1,2n,x,u} f(x,u).$$

Proof Recall that after we showed $[\mathcal{D}_{1,2n}, \mathcal{J}_{2n}\partial_{x_j}\mathcal{J}_{2n}] = -4nx_j\mathcal{D}_{1,2n}$ for $\mathcal{J}_{2n}$ the harmonic inversion, we claimed and later showed that $\mathcal{J}_{2n}\mathcal{D}_{1,2n}\mathcal{J}_{2n} = ||x||^{4n}\mathcal{D}_{1,2n}$. This can also be written as

$$\mathcal{D}_{1,2n,y,w}||x||^{m-2n} f(y,w) = ||x||^{m+2n}\mathcal{D}_{1,2n,x,u} f(x,u).$$

□

Theorem 17.3.21 now follows using the Iwasawa decomposition. See [13] for the first order case.

17.4 Fundamental Solutions

To get the fundamental solutions of $\mathcal{D}_{1,k}$, we use techniques from [7]. It is worth pointing out that the reproducing kernels of $\mathcal{M}_1$ and $\mathcal{H}_1$ below have simple expressions, but we insist on using techniques used in [7], since they also work for more general cases when we have $\mathcal{M}_j$ or $\mathcal{H}_j$ instead; indeed, the general case is proved in our manuscript [14] by the present method, but it only provides us the fundamental solutions up to a multiplicative constant. Though we provide the appropriate constant here, it was determined by a different method in [14], based on an iterative procedure that starts from known fundamental solutions of the lowest order operators of arbitrary spin.

k Even, $k = 2n$ (The Bosonic Case)
Recall that the reproducing kernel for j-homogeneous harmonic spherical polynomials $Z_j(u, v)$ is called the *zonal spherical harmonic* of degree j, and is invariant under reflections (and consequently rotations) in the variables u and v [3]. We concern ourselves primarily with $j = 1$, in which case

$$Z_1(u, v) = \frac{(m-2)^2\omega_{m-1}}{m}\langle u, v\rangle$$

is the zonal spherical harmonic of degree 1, where ω_{m-1} is the surface area of the $(m-1)$-dimensional unit sphere and $\langle u, v\rangle$ is the standard inner product in Euclidean space. It can be considered as the identity of $End(\mathcal{H}_1)$ and satisfies

$$P_1(v) = (Z_1(u, v), P_1(u))_u := \int_{\mathbb{S}^{m-1}} \overline{Z_1(u, v)}P_1(u)dS(u),$$

where $(\ ,\ )_u$ denotes the Fischer inner product with respect to u; we define the Fischer inner product of two functions by the integral of their product over the sphere, consistent with other work in higher spin theory [7, 15]. For an explicit characterization of $Z_j(u, v)$, we refer the reader to [9].

A homogeneous $End(\mathcal{H}_1)$-valued C^∞-function $x \to E(x)$ on $\mathbb{R}^m\backslash\{0\}$ satisfying $\mathcal{D}_{1,2n}E(x) = \delta(x)Z_1(u, v)$ is referred to as a fundamental solution for the operator $\mathcal{D}_{1,2n}$. We will show that such a fundamental solution has the form

$$E_{1,2n}(x, u, v) = c_1||x||^{2n-m}Z_1\left(\frac{xux}{||x||^2}, v\right).$$

Since $Z_1(u, v)$ is a trivial solution of $\mathcal{D}_{1,2n}$, according to the invariance of $\mathcal{D}_{1,2n}$ under inversion, we obtain a non-trivial solution $\mathcal{D}_{1,2n}E_{1,2n}(x, u, v) = 0$ in $\mathbb{R}^m\backslash\{0\}$. Clearly the function $E_{1,2n}(x, u, v)$ is homogeneous of degree $2n - m$ in x, so $\mathcal{D}_{1,2n}E_{1,2n}(x, u, v)$ is homogeneous of degree $-m$ in x and it belongs to $L_1^{loc}(\mathbb{R}^m)$. Because $\delta(x)$ is the only (up to a multiple) distribution homogeneous of degree $-m$

with support at the origin, we have in the sense of distributions:

$$\mathcal{D}_{1,2n}E_{1,2n}(x,u,v)=\delta(x)P_1(u,v)$$

for some $P_1(u,v)\in\mathcal{H}_1\otimes\mathcal{H}_1^*$. Then we have

$$\begin{aligned}&\int_{\mathbb{S}^{m-1}}\mathcal{D}_{1,2n}\overline{E_{1,2n}(x,u,v)}Q_1(v)dS(v)\\ &=\delta(x)\int_{\mathbb{S}^{m-1}}\overline{P_1(u,v)}Q_1(v)dS(v).\end{aligned}$$

Now, for all $Q_1\in\mathcal{H}_1$, we have

$$\begin{aligned}&\int_{\mathbb{S}^{m-1}}\mathcal{D}_{1,2n}\overline{E_{1,2n}(x,u,v)}Q_1(v)dS(v)\\ &=\mathcal{D}_{1,2n}\int_{\mathbb{S}^{m-1}}c_1||x||^{2n-m}\overline{Z_1\left(\frac{xux}{||x||^2},v\right)}Q_1(v)dS(v)\\ &=\mathcal{D}_{1,2n}\int_{\mathbb{S}^{m-1}}c_1||x||^{2n-m}\overline{Z_1\left(\frac{xux}{||x||^2},\frac{xv'x}{||x||^2}\right)}Q_1\left(\frac{xv'x}{||x||^2}\right)dS(v'),\end{aligned}$$

where in the last line we made a change of variables in the second argument of Z_1. Since $Z_1(u,v)$ is invariant under reflection and $\frac{xux}{||x||^2}$ is a reflection of variable u in the direction of x, the last line in the last equation becomes

$$\begin{aligned}&\mathcal{D}_{1,2n}\int_{\mathbb{S}^{m-1}}c_1\overline{Z_1(u,v')}||x||^{2n-m}Q_1\left(\frac{xv'x}{||x||^2}\right)dS(v')\\ &=c_1\mathcal{D}_{1,2n}||x||^{2n-m}Q_1\left(\frac{xux}{||x||^2}\right).\end{aligned}$$

Hence, we obtain

$$\delta(x)\int_{\mathbb{S}^{m-1}}\overline{P_1(u,v)}Q_1(v)dS(v)=c_1\mathcal{D}_{1,2n}||x||^{2n-m}Q_1\left(\frac{xux}{||x||^2}\right).$$

As the reproducing kernel $Z_1(u,v)$ is invariant under the $Spin(m)$-representation $H:\ f(u,v)\mapsto\tilde{s}f(su\tilde{s},sv\tilde{s})s$, the kernel $E_{1,2n}(x,u,v)$ is also $Spin(m)$-invariant:

$$\tilde{s}E_{1,2n}(sx\tilde{s},su\tilde{s},sv\tilde{s})s=E_{1,2n}(x,u,v).$$

From this it follows that $P_1(u,v)$ must be also invariant under H. Let now ϕ be a test function with $\phi(0)=1$. Let L be the action of $Spin(m)$ given by $L:\ f(u)\mapsto$

$\tilde{s} f(\tilde{s} u s)s$. Then

$$\begin{aligned}
&\langle \mathcal{D}_{1,2n}\big(c_1||x||^{2n-m} L\Big(\frac{x}{||x||}\Big) L(s) Q_1(u)\big), \phi(x)\rangle \\
&= \int_{\mathbb{S}^{m-1}} \overline{P_1(u,v)} L(s) Q_1(v) dS(v) \\
&= L(s) \int_{\mathbb{S}^{m-1}} \overline{P_1(u,v)} Q_1(v) dS(v) \\
&= \langle L(s)\big(\mathcal{D}_{1,2n} c_1||x||^{2n-m} L\Big(\frac{x}{||x||}\Big) Q_1(u)\big), \phi(x)\rangle.
\end{aligned}$$

In this way we have constructed an element of $End(\mathcal{H}_1)$ commuting with the L-representation of $Spin(m)$ that is irreducible; see Sect. 17.2.2.2. By Schur's Lemma [22], it follows that $P_1(u, v)$ must be the reproducing kernel $Z_1(u, v)$ if we choose c_1 properly. Hence

$$\mathcal{D}_{1,2n} E_{1,2n}(x, u, v) = \delta(x) Z_1(u, v).$$

We summarize these results in a theorem as follows.

Theorem 17.4.1 *The* $2n$*-th order bosonic operator of spin-*1*,*

$$\mathcal{D}_{1,2n} : C^\infty(\mathbb{R}^m, \mathcal{H}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{H}_1),$$

possesses the fundamental solution

$$E_{1,2n}(x, u, v) = c_1||x||^{2n-m} Z_1\Big(\frac{xux}{||x||^2}, v\Big),$$

where $Z_1(u, v)$ *is the reproducing kernel of* $\mathcal{H}_1$ *and the constant* c_1 *is*

$$(-1)^{n-2}\frac{(m-2)\Gamma(\frac{m}{2}-1)}{4(4-m)\pi^{\frac{m}{2}}}\prod_{s=2}^{n}\frac{2s(2s-2)m(m-2)}{m^2-4ms+4s^2+4m-4s-4}$$

that was determined using a different technique in our recent paper [14].

Recall that in harmonic analysis, the fundamental solution of Laplacian equation involves a logarithm function on the plane. However, we require that the dimension of the Euclidean space $m \geq 3$ in this paper, since we used a theorem of Liouville that states the only conformal transformations are Möbius transformations when dimension of the space $m \geq 3$. Hence, logarithm functions are not involved in our fundamental solutions here, as they corresponds to $m = 2$. A detailed calculation for the fundamental solution of the second order conformally invariant differential operator can be found in [9, 17]. Further, in light of familiar results from harmonic function theory, we must remark on dimensionality considerations. One would initially expect when the dimension m is even, we must restrict order $2j$ or $2j-1$ to be less than m. This would be analogous to the powers of the Dirac operator (see [32]), for which, when the order k is greater than m, the fundamental solution contains a logarithm function. For an example of why this is necessary, consider when $m=k=2n$: the usual expression $||x||^{k-m}$ is a constant, so it cannot be the fundamental solution of D_x^k. This does not happen in our case, however, since the reproducing kernel factor term in the fundamental solutions, $Z_j\left(\frac{xux}{||x||^2}, v\right)$, renders this restriction on the order unnecessary for even dimensions.

More specifically, as an example, consider the operator $\mathcal{D}_{1,2n}$. Suppose the dimension is not only even, but that it equals the order of the operator: $m = 2n$. Then the candidate of the fundamental solution for $\mathcal{D}_{1,2n}$ becomes $||x||^{2n-m}Z_1\left(\frac{xux}{||x||^2}, v\right) = Z_1\left(\frac{xux}{||x||^2}, v\right)$. Recall that

$$\mathcal{D}_{1,2n} = \Delta_x^n - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-1}.$$

Now, if we apply Δ_x to $Z_1\left(\frac{xux}{||x||^2}, v\right)$ and let $\xi = \frac{xux}{||x||^2} = u - \frac{2x\langle u, x\rangle}{||x||^2}$, in other words, $\xi_j = u_j - \frac{2x_j\langle u, x\rangle}{||x||^2}$. Then, we will have

$$\Delta_x Z_1\left(\frac{xux}{||x||^2}, v\right) = \sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_j}{\partial x_i}\frac{\partial Z_1(\xi, v)}{\partial \xi_j}\right)$$

$$= \sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{\partial \xi_j}{\partial x_i}\right)\frac{\partial Z_1(\xi, v)}{\partial \xi_j} + \sum_{i,j,k=1}^{m}\frac{\partial \xi_k}{\partial x_i}\frac{\partial \xi_j}{\partial x_i}\frac{\partial^2 Z_1(\xi, v)}{\partial \xi_j\partial_{\xi_k}}.$$

Notice that $Z_1(\xi, v)$ has homogeneity of degree-1 in the variable ξ, so the second sum above vanishes. Hence, we only need to calculate the first sum, which becomes

$$-2\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{(\delta_{ij}\langle u, x\rangle + x_j u_i)||x||^2 - 2x_i x_j\langle u, x\rangle}{||x||^4}\right)\frac{\partial Z_1(\xi, v)}{\partial \xi_j}$$

$$= -2\sum_{i,j=1}^{m}\frac{\partial}{\partial x_i}\left(\frac{\delta_{ij}\langle u, x\rangle}{||x||^2} + \frac{u_i x_j}{||x||^2} - \frac{2x_i x_j\langle u, x\rangle}{||x||^4}\right)\frac{\partial Z_1(\xi, v)}{\partial \xi_j}$$

$$= \left(\frac{-4\langle u, D_\xi\rangle}{||x||^2} + \frac{4m\langle u, x\rangle\langle x, D_\xi\rangle}{||x||^4}\right)Z_1(\xi, v),$$

where $\langle u, D_\xi \rangle = \sum_{i=1}^m u_i \frac{\partial}{\partial_\xi}$ is the standard Euclidean inner product of u and the Dirac operator D_ξ. The $||x||$ terms above allow the possibility of taking derivatives in the distribution sense, which decreases the exponent of $||x||$. Indeed, when $2n = 2$, we already have (see [17])

$$\mathcal{D}_{1,2} \frac{-\Gamma(\frac{m}{2})}{(m-4)2\pi^{\frac{m}{2}}} ||x||^{2-m} Z_1\left(\frac{xux}{||x||^2}, v\right) = \delta(x) Z_1(u, v).$$

Further we also have (see [14])

$$\begin{aligned} &\mathcal{D}_{1,2n} a_{2n} ||x||^{2n-m} Z_1\left(\frac{xux}{||x||^2}, v\right) \\ &= \mathcal{D}_{1,2n-2} a_{2n-2} ||x||^{2n-m-2} Z_1\left(\frac{xux}{||x||^2}, v\right) \\ &= \cdots = \mathcal{D}_{1,2} ||x||^{2-m} \frac{-\Gamma(\frac{m}{2})}{(m-4)2\pi^{\frac{m}{2}}} Z_1\left(\frac{xux}{||x||^2}, v\right) = \delta(x) Z_1(u, v) \end{aligned}$$

in the distribution sense, where for $2 \le j \le n$, a_{2j} is a constant given in [14]. In short, the Laplacian acting on the zonal spherical harmonic (or monogenic) generates the powers of $1/||x||$ needed to overcome the loss of the singularity in the conformal weight factor in the fundamental solution.

k **Odd,** $k = 2n - 1$ **(The Fermionic Case)**
The reproducing kernel $Z_k(u, v)$ for degree k homogeneous monogenic spherical polynomials, those in $\mathcal{M}_k$, is called the *zonal spherical monogenic* [10]. (There should be no confusion using the same notation for zonal spherical harmonics and monogenics.) In our circumstance, for $u, v \in \mathbb{S}^{m-1}$,

$$Z_1(u, v) = \frac{1}{\omega_{m-1}} \left(\frac{2\mu + 1}{2\mu} C_1^\mu(t) + (u \wedge v) C_0^{\mu+1}(t) \right),$$

where $\mu = \frac{m}{2} - 1$, $t = \langle u, v \rangle$, $u \wedge v = uv + \langle u, v \rangle$, and $C_k^\mu(t)$ are the Gegenbauer polynomials [10]. With similar arguments and the fact that $Z_1(u, v)$ is also $Spin(m)$-invariant under the same $Spin(m)$-action as in the even case, one can show that

$$E_{1,2n-1}(x, u, v) = c_1' \frac{x}{||x||^{m-2n+2}} Z_1\left(\frac{xux}{||x||^2}, v\right)$$

is the fundamental solution of $\mathcal{D}_{1,2n-1}$ for some constant c_1' specified in the next theorem. As before, we summarize with a theorem as follows.

Theorem 17.4.2 *The* $(2n-1)$*-th order fermionic operator of spin-*$\frac{3}{2}$, $\mathcal{D}_{1,2n-1}$: $C^\infty(\mathbb{R}^m, \mathcal{M}_1) \longrightarrow C^\infty(\mathbb{R}^m, \mathcal{M}_1)$*, possesses the fundamental solution*

$$E_{1,2n-1}(x, u, v) = c_1' \frac{x}{||x||^{m-2n+2}} Z_1\left(\frac{xux}{||x||^2}, v\right)$$

where $Z_1(u, v)$ *is the reproducing kernel of* $\mathcal{M}_1$ *and the constant* c_1' *has the value*

$$c_1' = \frac{-m}{(m-2)\omega_{m-1}} \prod_{s=1}^{n-1} \frac{m-2s}{(4s^2-2ms-4)(m-2s)+4m}$$

that was determined using a different technique in our recent paper [14].

Conformal Invariance and Intertwiners of Fundamental Solutions, $j=1$
Recall that if $E_{1,k}(x, u, v)$ is the fundamental solution of $\mathcal{D}_{1,k}$, then we have

$$\int_{\mathbb{R}^m} \int_{\mathbb{S}^{m-1}} E_{1,k}(x-y, u, v)\mathcal{D}_{1,k}\psi(x, u)dS(u)dx^m = \psi(y, v),$$

where $\psi(x, u) \in C^\infty(\mathbb{R}^m, U)$ with compact support in x for each $u \in \mathbb{R}^m$, $U = \mathcal{M}_1$ when k is odd and $U = \mathcal{H}_1$ when k is even. Hence, we have $\mathcal{D}_{1,k}E_{1,k} = Id$ and $E_{1,k} = \mathcal{D}_{1,k}^{-1}$ in the distribution sense. Now,

$$J_{-k}\mathcal{D}_{1,k,y,w}\psi(y, w) = \mathcal{D}_{1,k,x,u}J_k\psi\left(\phi(x), \frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}\right),$$

where $y = \widetilde{\phi(x)} = (ax+b)(cx+d)^{-1}$ is a Möbius transformation and $w = \frac{(cx+d)u(\widetilde{cx+d})}{||cx+d||^2}$ as in Theorem 3, we get

$$J_k^{-1}\mathcal{D}_{1,k,x,u}^{-1}J_{-k} = \mathcal{D}_{1,k,y,w}^{-1},$$

alternatively,

$$J_k^{-1}E_{1,k,x,u}J_{-k} = E_{1,k,y,w}.$$

This gives us the intertwiners of the fundamental solution $E_{1,k}$ under Möbious transformations, which also reveals that the fundamental solutions are conformally invariant under Möbius transformations.

17.5 Ellipticity of the Operator $\mathcal{D}_{1,k}$

Notice that the bases of the target space $\mathcal{H}_1$ and $\mathcal{M}_1$ have simple expressions. We can use techniques similar to those in [9, 17] to show that the operators $\mathcal{D}_{1,k}$ are elliptic. First, we introduce the definition for an elliptic operator.

Definition 17.5.1 A linear homogeneous differential operator of k-th order $\mathcal{D}_{1,k} : C^\infty(\mathbb{R}^m, V_\lambda) \longrightarrow C^\infty(\mathbb{R}^m, V_\mu)$ is elliptic if for every non-zero vector $x \in \mathbb{R}^m$ its principal symbol, the linear map $\sigma_x(\mathcal{D}_{1,k}) : V_\lambda \longrightarrow V_\mu$ obtained by replacing its partial derivatives ∂_{x_j} with the corresponding variables x_j, is a linear isomorphism.

Note V_λ stands for a representation space of $Spin(m)$ with a dominant weight λ, see Sect. 17.3.2. Then we prove ellipticity of $\mathcal{D}_{1,k}$ in the even and odd cases individually.

k **Even,** $k = 2n$ **(The Bosonic Case)**

Theorem 17.5.2 *The operator* $\mathcal{D}_{1,2n} := \left(\Delta_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\right)\Delta_x^{n-1}$ *is an elliptic operator when* $m \neq 2n+2$.

Proof In [17] it was shown that the operator $\Delta_x - \frac{4}{m}\langle u, D_x\rangle\langle D_u, D_x\rangle$ is elliptic. In our case, the term in the parentheses is the same as the previous one up to a constant coefficient, so a similar argument shows that $\Delta_x - \frac{4n}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle$ is elliptic when $\frac{4n}{m+2n-2} \neq 1$, in other words, $m \neq 2n+2$. □

k **Odd,** $k = 2n-1$ **(The Fermionic Case)**

Theorem 17.5.3 *The operator*

$$\mathcal{D}_{1,2n-1} := D_x\Delta_x^{n-1} - \frac{2}{m+2n-2}u\langle D_u, D_x\rangle\Delta_x^{n-1} - \frac{4n-4}{m+2n-2}\langle u, D_x\rangle\langle D_u, D_x\rangle\Delta_x^{n-2}D_x$$

is an elliptic operator.

Proof To prove the theorem, we show that, for fixed $x \in \mathbb{R}^m$, the symbol of the operator $\mathcal{D}_{1,2n-1}$, which is given by

$$x||x||^{2n-2} - \frac{2u\langle D_u, x\rangle||x||^{2n-2}}{m+2n-2} - \frac{4n-4}{m+2n-2}\langle u, x\rangle\langle D_u, x\rangle||x||^{2n-4}x,$$

is a linear isomorphism from $\mathcal{M}_1$ to $\mathcal{M}_1$. As the symbol is clearly a linear map, it remains to be proven that the map is injective. Recall that $\mathcal{M}_1$ is actually $\mathcal{M}_1(\mathbb{R}^m, \mathcal{S})$, however, if we can prove the symbol is injective for $\mathcal{M}_1(\mathcal{C}l_m)$, then this also implies that it is injective for $\mathcal{M}_1(\mathcal{S}) \subset \mathcal{M}_1(\mathcal{C}l_m(\mathbb{C}))$. From the Almansi-Fischer decomposition $\mathcal{H}_1 = \mathcal{M}_1 \oplus u\mathcal{M}_0$, it is easy to obtain that $dim\mathcal{M}_1 = m-1$. Since $\{e_ju_m + e_mu_j\}_{j=1}^{m-1}$ are in $\mathcal{M}_1$ and it is also a linearly independent set in $\mathcal{M}_1$. Therefore, it is actually a basis of $\mathcal{M}_1$. Hence, an arbitrary element $\boldsymbol{u} \in \mathcal{M}_1$ can be written as $\boldsymbol{u} = \sum_{j=1}^{m-1} \alpha_j(e_j u_m + e_m u_j)$ with $\alpha_j \in \mathbb{C}$ for all $1 \leq j \leq m-1$. We next show that the following system of equations has a unique solution when $x \neq 0$:

$$\left(x||x||^2 - \frac{2u\langle D_u, x\rangle ||x||^2}{m+2n-2} - \frac{4n-4}{m+2n-2} x\langle u, x\rangle\langle D_u, x\rangle\right)\boldsymbol{u} = 0.$$

With $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_{m-1})$, $c_1 = \frac{2}{m+2n-2}$, $c_2 = \frac{4n-4}{m+2n-2}$, $a_i = (c_1 e_i ||x||^2 + c_2 x x_i)$, $b_j = x_m e_j + x_j e_m$, and $1 \leq i, j \leq m-1$, this equation system can be written in matrix notation as follows:

$$\begin{bmatrix} -x||x||^2 e_m - a_1 b_1 & \ldots & -a_1 b_{m-1} \\ -a_2 b_1 & \ldots & -a_2 b_{m-1} \\ \vdots & \ddots & \vdots \\ -a_{m-1} b_1 & \ldots & -x||x||^2 e_m - a_{m-1} b_{m-1} \end{bmatrix} \boldsymbol{\alpha}^T = 0.$$

Notice that the left side of the equation above is a Clifford-valued number, which implies all coefficients for e_A should be zero. Further, since

$$a_i b_j = c_1 \|x\|^2 x_m e_i e_j + c_1 \|x\|^2 x_j e_i e_m + c_2 x x_i x_m e_j + c_2 x x_i x_j e_m$$

for $1 \leq i, j \leq m-1$, one can observe that the constant is

$$((1+c_1)x_m \|x\|^2 I + 2\boldsymbol{A})\boldsymbol{\alpha}^T = 0,$$

where

$$\boldsymbol{A} = \begin{bmatrix} c_2 x_m x_1^2 & c_2 x_1 x_2 x_m & \ldots & c_2 x_1 x_{m-1} x_m \\ c_2 x_1 x_2 x_m & c_2 x_2^2 x_m & \ldots & c_2 x_2 x_{m-1} x_m \\ c_2 x_1 x_3 x_m & c_2 x_2 x_3 x_m & \ldots & c_2 x_3 x_{m-1} x_m \\ \vdots & \vdots & \ddots & \vdots \\ c_2 x_1 x_{m-1} x_m & c_2 x_2 x_{m-1} x_m & \ldots & c_2 x_m x_{m-1}^2 \end{bmatrix}.$$

In order to show the system above has a unique solution, we need to show the determinant of its coefficient matrix is nonzero. Using the notation $\boldsymbol{x} = (x_1, x_2, \cdots, x_{m-1})$, then the determinant of the coefficient matrix above is equal to

$$\det\left((1+c_1)x_m \|x\|^2 I + 2c_2 x_m \boldsymbol{x}\boldsymbol{x}^T\right) = (1+c_1)x_m \|x\|^2 + 2c_2 x_m \sum_{j=1}^{m-1} x_j^2.$$

Since $1 + c_1$ and c_2 are both positive and $x \neq 0$, then the only possibility for the determinant to be zero is that $x_m = 0$. In this case, $a_i b_j = c_1 \|x\|^2 x_j e_i e_m +$

$c_2 x x_i x_j e_m$. Since $x \neq 0$, without loss of generality, we assume that $x_1 \neq 0$. Since the coefficient of $e_1 e_m$ is zero, one can have

$$\begin{bmatrix} (1+c_1)x_1\|x\|^2 + c_2 x_1^3 & \dots & c_1 x_{m-1}\|x\|^2 + c_2 x_1^2 x_{m-1} \\ c_2 x_1^2 x_2 & \dots & c_2 x_1 x_2 x_{m-1} \\ c_2 x_1^2 x_3 & \dots & c_2 x_1 x_3 x_{m-1} \\ \vdots & \ddots & \vdots \\ c_2 x_1^2 x_{m-1} & \dots & x_1\|x\|^2 + c_2 x_1 x_{m-1}^2 \end{bmatrix} \cdot \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \\ \alpha_{m-1} \end{bmatrix} = 0.$$

Let us denote the coefficient matrix above by A, then if $x_2 = 0$, it is easy to see that $\det A \neq 0$. When $x_2 \neq 0$, using basic row and column operations, one can obtain that $\det A$ is equal to the determinant of the upper block triangular matrix $\begin{bmatrix} A_{2,2} & A_{2,m-3} \\ 0 & A_{m-3,m-3} \end{bmatrix}$, where

$$A_{2,2} = \begin{bmatrix} (1+c_1)x_1\|x\|^2 & c_1 x_2\|x\|^2 - \frac{x_1^2}{x_2}\|x\|^2 - \sum_{j=3}^{m-1} c_1 \frac{x_j^2}{x_2}\|x\|^2 \\ c_2 x_1^2 x_2 & x_1\|x\|^2 + c_2 x_1 \sum_{j=2}^{m-1} x_j^2 \end{bmatrix}$$

and $A_{m-3,m-3} = x_1\|x\|^2 I_{m-3,m-3}$. Therefore, one can easily check that $\det A = \det A_{2,2} \cdot \det A_{m-3,m-3} \neq 0$. Therefore, one obtains that for non-zero x, the only solution for the equation system above is that $\alpha_1 = \cdots = \alpha_{m-1} = 0$, which completes the proof. □

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Chapter 18
Clifford Möbius Geometry

Craig A. Nolder

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract The Riemann sphere is a compactification of the complex plane on which the complex Möbius group naturally acts. This Möbius group is isomorphic to the conformal orthogonal group $SO^+(1, 3)$. Here we give a unified approach to this compactification and the corresponding Möbius groups for the Clifford algebras of dimensions two and four.

Keywords Clifford composition algebras · Compactification · Conformal Mobius groups

Mathematics Subject Classification (2010) Primary 22-06; Secondary 16-06

18.1 Introduction

Quadratic spaces and Clifford algebras are related both algebraically and geometrically. In dimensions two and four, Clifford algebras generate a unique quadratic form. The conformal compactification of the quadratic spaces then gives geometric models for the compactification of corresponding Clifford algebras. The conformal special orthogonal groups, which act on these compactifications, are isomorphic to the corresponding Möbius groups of the Clifford algebras. These Möbius groups are represented by appropriate two by two matrices with entries from the Clifford algebras.

A goal is to understand in a unified way the action of Clifford Möbius groups on a compactification of the algebra. We describe isomorphisms between linear

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Lie groups and corresponding orthogonal Lie groups. In particular we present continuous group isomorphisms. Although well known, these constructions are difficult to find in the literature, see [4]. We use here material from these notes. What is interesting here is that in low dimensions, these groups are isomorphic to the Möbius groups of Clifford algebras which act on the compactifications of the corresponding quadratic spaces. These observations are perhaps new in the case of the split complex numbers and the split quaternions.

18.2 Quadratic Spaces and Orthogonal Groups

Definition 18.2.1 For $x, y \in \mathbb{R}^n$ we define the bilinear form

$$\langle x, y\rangle_{p,q} = x_1 y_1 + \cdots + x_p y_p - x_{p+1} y_{p+1} - \cdots - x_{p+q} y_{p+q}$$

where $p+q = n$. The pair $(\mathbb{R}^{p,q}, \langle\cdot\rangle_{p,q})$ is a real quadratic space when $\mathbb{R}^{p,q} = \mathbb{R}^n$, has the quadratic form

$$\langle x, x\rangle_{p,q} = Q_{p,q}(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

for $x \in \mathbb{R}^n$.

Notice that

$$\langle x, y\rangle_{p,q} = Q(x) + Q(y) - Q(x + y).$$

Definition 18.2.2 The group of linear transformations of $\mathbb{R}^{p,q}$ which preserve $Q_{p,q}$ is denoted $O(p, q)$. Hence if $T \in O(p, q)$, then $Q_{p,q}(Tx) = Q_{p,q}(x)$.

When $p, q \neq 0$, $O(p, q)$ has four connected components and the special orthogonal group $SO(p, q)$, those with determinant one, has two. We denote the connected component of the identity by $SO^+(p, q)$. With $n = p + q$ the dimension of $SO(p, q)$ is $n(n-1)/2$. The group $SO(p, q)$ is isomorphic to $SO(q, p)$. See [6].

18.2.1 Compactification of Quadratic Spaces

Notice

$$\langle x, x\rangle_{p,q} = [\frac{1 + \langle x, x\rangle_{p,q}}{2}]^2 - [\frac{1 - \langle x, x\rangle_{p,q}}{2}]^2. \tag{18.2.1}$$

So we have the identity

$$[\frac{1-\langle x,x\rangle_{p,q}}{2}]^2+\Sigma_{i=1}^{p}x_i^2=\Sigma_{i=p+1}^{p+q}x_i^2+[\frac{1+\langle x,x\rangle_{p,q}}{2}]^2. \tag{18.2.2}$$

We denote by $S(x)$ the common value of the above quantities. We then have an embedding of $\mathbb{R}^{p,q}$ into $\mathbb{R}^{p+1,q+1}$ by the formula

$$\tau(x)=\frac{1}{S(x)}(\frac{1-\langle x,x\rangle_{p,q}}{2},x_1,\ldots,x_n,\frac{1+\langle x,x\rangle_{p,q}}{2}). \tag{18.2.3}$$

Hence the image of $\mathbb{R}^{p,q}$ under τ is a subset of the Cartesian product of spheres $S^p\times S^q$ which lie in the sphere S^{n+1} in $\mathbb{R}^{n+2}$ of radius $\sqrt{2}$.

A conformal compactification of $\mathbb{R}^{p,q}$ is obtained by compactifying the projectivisation of this embedding ($S(x)$ is positive) :

$$(\frac{1-\langle x,x\rangle_{p,q}}{2}:x_1:\ldots:x_n:\frac{1+\langle x,x\rangle_{p,q}}{2}).$$

We denote this compactification by $N^{p,q}$. As such $N^{p,q}$ is the projective product $S^p\times S^q/\tilde{.}$ See [7] for more details.

In dimensions 2 and 4, we discuss this compactification in the context of Clifford algebras.

Definition 18.2.3 The real Clifford algebra $\mathcal{C}\ell_{r,s}$ is the algebra generated over $\mathbb{R}$ by the generators $\{e_1,e_2,\ldots,e_n\}$ where $r+s=n$ with

$$e_i^2=1,\quad i=1,\ldots,r,\quad e_i^2=-1,\quad i=r+1,\ldots,n$$

and

$$e_ie_j=-e_je_i,\quad i,j=1,\ldots,n,\quad i\neq j.$$

Using these relations we can reduce products to the form $e_{i_1}e_{i_2}\cdots e_{i_k}$, $i_1<i_2<\cdots<i_k$. So a Clifford algebra is a graded algebra

$$\mathcal{C}\ell_{r,s}=\oplus_{k=0}^{n}\mathcal{C}\ell_{r,s}^{(k)}.$$

Here $\mathcal{C}\ell_{r,s}^{(k)}$ are the reduced products of length k, moreover $\mathcal{C}\ell_{r,s}^{(0)}=\mathbb{R}$ and $\mathcal{C}\ell_{r,s}^{(1)}=\mathbb{R}^n$ as vector spaces.

The subspace of even terms is a subalgebra.

$$\mathcal{C}\ell_{r,s}^{0}=\oplus_{k\ even}\mathcal{C}\ell_{r,s}^{(k)}.$$

Definition 18.2.4 The group $spin(r, s)$ is defined by

$$spin(r, s) = \{x \in \mathcal{C}\ell^0_{r,s},\ Q_{r,s}(x) = 1 \,|\, xv\overline{x} \in \mathbb{R}^n,\ v \in \mathbb{R}^n,\}$$

The Clifford products here, $xv\overline{x}$, are products of reflections in $\mathbb{R}^n$ and as such are orthogonal transformations. The action is trivial precisely when $x = \pm 1$. As such $spin(r, s)$ is a double cover of $SO^+(r, s)$. The groups $spin(r, s)$ and $spin(s, r)$ are isomorphic. See [5, 6].

When $n = 2, 4$ we identify the Clifford algebras with the corresponding quadratic spaces :

- The complex numbers are

$$\mathbb{C} = \mathcal{C}\ell_{0,1} = \{\zeta = x_0 + x_1 i \,|\, x_0, x_1 \in \mathbb{R}, i^2 = -1\}.$$

  Here the conjugate is $\overline{\zeta} = x_0 - x_1 i$ and so $\zeta\overline{\zeta} = x_0^2 + x_1^2$. As such this corresponds to $\mathbb{R}^{2,0}$.
- The split complex numbers

$$\mathcal{C}\ell_{1,0} = \{\zeta = x_0 + x_1 j \,|x_0, x_1 \in \mathbb{R}, j^2 = 1\}$$

  have conjugates $\overline{\zeta} = x_0 - x_1 j$, so that $\zeta\overline{\zeta} = x_0^2 - x_1^2$ corresponding to $\mathbb{R}^{1,1}$.
- The quaternions

$$\mathbb{H} = \mathcal{C}\ell_{0,2} = \{\zeta = x_0 + x_1 i + x_2 j + x_3 ij \,|\, x_0, x_1, x_2, x_3 \in \mathbb{R}, i^2 = -1,\ j^2 = -1,\ ij = -ji\},$$

$$\overline{\zeta} = x_0 - x_1 i - x_2 j - x_3 ij,\ \zeta\overline{\zeta} = x_0^2 + x_1^2 + x_2^2 + x_3^2 \quad \text{correspond to } \mathbb{R}^{4,0}.$$
- The split quaternions

$$\mathcal{C}\ell_{1,1} \cong \mathcal{C}\ell_{2,0} = \{\zeta = x_0 + x_1 i + x_2 j + x_3 ij |\, x_0, x_1, x_2, x_3 \in \mathbb{R}, i^2 = -1\ j^2 = 1,\ ij = -ji\},$$

$$\overline{\zeta} = x_0 - x_1 i - x_2 j - x_3 ij,\ \zeta\overline{\zeta} = x_0^2 + x_1^2 - x_2^2 - x_3^2, \text{ corresponds to } \mathbb{R}^{2,2}.$$

In each case the corresponding quadratic form is given by the Clifford product $N(\zeta) = \zeta\overline{\zeta}$. The algebras are more complicated in higher dimensions. A Clifford number ζ is invertible when this product is nonzero :

$$\zeta^{-1} = \frac{\overline{\zeta}}{N(\zeta)}.$$

Moreover

$$N(\zeta^{-1}) = 1/N(\zeta).$$

In the cases we consider, the above compactification allows the extension of inversion to all embedded elements of the Clifford algebra.

In $\mathbb{RP}^5$,

$$\zeta^{-1} \to \left(\frac{1-\frac{1}{N(\zeta)}}{2} : \frac{x_0}{N(\zeta)} : \frac{-x_1}{N(\zeta)} : \frac{-x_2}{N(\zeta)} : \frac{-x_3}{N(\zeta)} : \frac{1+\frac{1}{N(\zeta)}}{2}\right)$$

$$= \left(\frac{N(\zeta)-1}{2} : x_0 : -x_1 : -x_2 : -x_3 : \frac{N(\zeta)+1}{2}\right).$$

The embeddings into $\mathbb{RP}^3$ lie in the subspace where $x_2 = x_3 = 0$.

Hence inversion extends to the compactification as the involution :

$$(y_0 : y_1 : y_2 : y_3 : y_4 : y_5) \to (-y_0 : y_1 : -y_2 : -y_3 : -y_4 : y_5).$$

The Möbius groups of the Clifford algebras are isomorphic to the conformal groups of the compactifications. Moreover, these groups are represented by 2 by 2 matrices in the corresponding algebra [1, 2].

Notice that the algebras embed to points where $y_0 + y_5 \neq 0$. The added points for the compactification satisfy $y_0 + y_5 = 0$. The non invertible points embed as :

$$(1/2 : x_0 : x_1 : x_2 : x_3 : 1/2)$$

These invert to

$$(-1/2 : x_0 : -x_1 : -x_2 : -x_3 : 1/2).$$

For example zero embeds as $(1 : 0 : 0 : 0 : 0 : 1)$ and $0^{-1} = (-1 : 0 : 0 : 0 : 0 : 1)$. In the case of the complex numbers and the quaternions, this inverse is only added point in the compactification. The case is different for indefinite signatures.

In general there are other points needed to connect the above described pieces.

Let's specify to $\mathcal{C}\ell_{1,1}$ and suppose that $x_1 = x_3 = 0$ and $x_0 = x_2$. These are non-invertible elements.

Then

$$\lim_{x_0\to\infty}(1/2x_0 : 1 : 0 : 1 : 0 : 1/2x_0) = (0 : 1 : 0 : 1 : 0 : 0).$$

The inverse of this limit is also the limit along the path $x_0 = -x_2$.

$$\lim_{x_0\to\infty}(1/2x_0 : 1 : 0 : -1 : 0 : 1/2x_0) = (0 : 1 : 0 : -1 : 0 : 0).$$

Notice these points also satisfy $y_0 + y_5 = 0$.

18.3 Möbius Groups

18.3.1 Lie Groups and Lie Algebras

We mention some relevant results from [6].

The matrix groups $SL(2, \mathbb{R})$, $SL(2, \mathbb{C})$, $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, $SL(4, \mathbb{R})$ and $SL(4, \mathbb{C})$ are connected Lie groups. The connected component of the identity of the orthogonal group $SO^+(p, q)$ is also a connected Lie group.

Definition 18.3.1 Two Lie groups G, G' with identities e, e', are isomorphic if there exists and analytic isomorphism of G onto G'. The Lie groups are locally isomorphic if there exist neighborhoods U, U' of e, e' and an analytic diffeomorphism f of U onto U' so that $x, y, xy \in U$ implies $f(xy) = f(x)f(y)$ and $x', y', x'y' \in U'$ implies $f^{-1}(x'y') = f^{-1}(x')f^{-1}(y')$.

Theorem 18.3.2 (p. 99. Theorem 1.11, [6]) *Lie groups are locally isomorphic if and only is their Lie algebras are isomorphic.*

Theorem 18.3.3 (p. 107, Theorem 2.6, [6]) *Let G and H be Lie groups and ϕ a continuous homomorphism of G into H. Then ϕ is analytic.*

Below is a table of the Lie groups, along with their corresponding Lie algebras, which we encounter.

Lie group	$SL(2, \mathbb{R})$	$SL(2, \mathbb{C})$	$SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$
Lie algebra	$\mathfrak{sl}(2, \mathbb{R})$	$\mathfrak{sl}(2, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$

$SL(4, \mathbb{R})$	$SL(2, \mathbb{H})$	$SO(2, 1)$	$SO(3, 1)$	$SO(2, 2)$
$\mathfrak{sl}(4, \mathbb{R})$	$\mathfrak{sl}(2, \mathbb{H})$	$\mathfrak{so}(2, 1)$	$\mathfrak{so}(3, 1)$	$\mathfrak{so}(2, 2)$

$SO(5, 1)$	$SO(3, 3)$
$\mathfrak{so}(5, 1)$	$\mathfrak{so}(3, 3)$

Theorem 18.3.4 (pp. 351–353, [6]) *We have the following Lie algebra isomorphisms.*

$$\mathfrak{sl}(2, \mathbb{R}) \approx \mathfrak{so}(2, 1)$$

$$\mathfrak{sl}(2, \mathbb{C}) \approx \mathfrak{so}(3, 1)$$

$$\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R}) \approx \mathfrak{so}(2, 2)$$

$$\mathfrak{sl}(2, \mathbb{H}) \approx \mathfrak{so}(5, 1)$$

$$\mathfrak{sl}(4, \mathbb{R}) \approx \mathfrak{so}(3, 3)$$

18.3.2 The Real Numbers

Theorem 18.3.5 *The Lie groups $PSL(2,\mathbb{R})$ and $SO^+(2,1)$ are isomorphic.*

Define the map

$$\Phi_{2,1} : \mathbb{R}^{2,1} \to M(2,\mathbb{R})$$

by

$$\Phi_{2,1} : x = (x_1, x_2, x_3) \to \Phi_{2,1}(x) = \begin{pmatrix} x_1 & x_3 + x_2 \\ x_3 - x_2 & x_1 \end{pmatrix}.$$

Notice $Q_{2,1}(x) = x_1^2 + x_2^2 - x_3^2 = \det \Phi_{2,1}(x)$.
For $g \in SL(2,\mathbb{R})$ the map $T_g : \mathbb{R}^{2,1} \to \mathbb{R}^{2,1}$ is given by

$$T_g(x) = \Phi_{2,1}^{-1}(g\Phi_{2,1}(x)g^t).$$

It follows that

$$Q_{2,1}(T_g(x)) = \det \Phi_{2,1}(T_g(x)) = \det(g\Phi_{2,1}(x)g^t) =$$

$$\det \Phi_{2,1}(x) = Q_{2,1}(x).$$

As such, $T_g \in SO(2,1)$ for all $g \in SL(2,\mathbb{R})$.
Let $g, h \in SL(2,\mathbb{R})$, then

$$T_g(T_h(x)) = T_g[\Phi_{2,1}^{-1} h\Phi_{2,1}(x)h^t] = \Phi_{2,1}^{-1} g\Phi_{2,1}[\Phi_{2,1}^{-1} h\Phi_{2,1}(x)h^t]g^t$$

$$= T_{gh}(x).$$

As such T is a homomorphism. The kernel of T is plus or minus the identity.
Since $PSL(2,\mathbb{R})$ is connected, the image is in the connected component of the identity $SO^+(2,1)$.
We have the following diagram.

$$\begin{array}{ccc} SL(2,\mathbb{R}) & \xrightarrow{\text{isomorphism}} & spin(2,1) \\ \text{double cover} \downarrow & & \text{double cover} \downarrow \\ PSL(2,\mathbb{R}) & \xrightarrow{\text{isomorphism}} & SO^+(2,1) \end{array}$$

The group $SO^+(2,1)$ acts as the conformal automorphisms of $S^1 \times S^0/\sim$. Notice that $PSL(2,\mathbb{R})$ is isomorphic to $SU(1,1)$, the conformal automorphisms of the disk.
The dimension of both $PSL(2,\mathbb{R})$ and $SO^+(2,1)$ is three.

18.3.3 $\mathbb{C} = \mathcal{C}\ell_{0,1}$

Theorem 18.3.6 *The Lie groups* $PSL(2, \mathbb{C})$ *and* $SO^+(3, 1)$ *are isomorphic.*

Define the map

$$\Phi_{3,1} : \mathbb{R}^{3,1} \to M(2, \mathbb{C})$$

by

$$\Phi_{2,1} : x = (x_1, x_2, x_3, x_4) \to \Phi_{3,1}(x) = \begin{pmatrix} x_1 + ix_2 & x_4 + x_3 \\ x_4 - x_3 & x_1 - ix_2 \end{pmatrix}.$$

Notice $Q_{3,1}(x) = x_1^2 + x_2^2 + x_3^2 - x_4^2 = \det \Phi_{3,1}(x)$.
For $g \in SL(2, \mathbb{C})$ the map $T_g : \mathbb{R}^{3,1} \to \mathbb{R}^{3,1}$ is given by

$$T_g(x) = \Phi_{3,1}^{-1}(g\Phi_{3,1}(x)g^t).$$

It follows that

$$Q_{3,1}(T_g(x)) = \det \Phi_{3,1}(T_g(x)) = \det(g\Phi_{3,1}(x)g^t) =$$

$$\det \Phi_{3,1}(x) = Q_{3,1}(x).$$

As such, $T_g \in SO(3, 1)$ for all $g \in SL(2, \mathbb{C})$.
Let $g, h \in SL(2, \mathbb{C})$, then

$$T_g(T_h(x)) = T_g[\Phi_{3,1}^{-1}h\Phi_{3,1}(x)h^t] = \Phi_{3,1}^{-1}g\Phi_{3,1}[\Phi_{3,1}^{-1}h\Phi_{3,1}(x)h^t]g^t$$

$$= T_{gh}(x).$$

As such T is a homomorphism. The kernel of T is plus or minus the identity.
Since $PSL(2, \mathbb{C})$ is connected, the image is in the connected component of the identity $SO^+(3, 1)$. The dimensions of both $PSL(2, \mathbb{C})$ and $SO^+(3, 1)$ is six. We have the following diagram.

$$\begin{array}{ccc} SL(2,\mathbb{C}) & \xrightarrow{\text{isomorphism}} & spin(3,1) \\ \text{double cover}\downarrow & & \text{double cover}\downarrow \\ PSL(2,\mathbb{C}) & \xrightarrow{\text{isomorphism}} & SO^+(3,1) \end{array}$$

The group $SO^+(3, 1)$ acts as conformal automorphisms on $N^{0,2} = (S^0 \times S^2)/\tilde{}$

18.3.4 $\mathcal{C}\ell_{1,0}$

Theorem 18.3.7 *The Lie groups $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ and $SO^+(2,2)$ are isomorphic.*

Recall that $j^2 = 1$. We write $j_+ = (1+j)/2$ and $j_- = (1-j)/2$. It follows that $j_+^2 = j_+,\ j_-^2 = j_-,\ j_+j_- = j_-j_+ = 0,\ j_+ + j_- = 1$ and $j_+ - j_- = j$.

As such we can rewrite a split complex number

$$\zeta = x + jy = uj_+ + vj_-,$$

where $u = x + y$ and $v = x - y$. We write $\zeta_i = u_i j_+ + v_i j_-$, $i = 1, 2, 3, 4$. It is immediate that

$$\mathcal{A} = \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix} = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} j_+ + \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix} j_- = \mathcal{A}_1 j_+ + \mathcal{A}_2 j_-. \tag{18.3.1}$$

Moreover

$$\mathcal{A}\mathcal{B} = (\mathcal{A}_1 j_+ + \mathcal{A}_2 j_-)(\mathcal{B}_1 j_+ + \mathcal{B}_2 j_-) = \mathcal{A}_1\mathcal{B}_1 j_+ + \mathcal{A}_2\mathcal{B}_2 j_-,$$

and so

$$\mathcal{A}^{-1} = \mathcal{A}_1^{-1} j_+ + \mathcal{A}_2^{-1} j_-.$$

Also a calculation shows that

$$\det \mathcal{A} = \det \mathcal{A}_1 j_+ + \det \mathcal{A}_2 j_-.$$

Hence $\det \mathcal{A} = 1$ if and only if $\det \mathcal{A}_1 = 1$ and $\det \mathcal{A}_2 = 1$.

In this way we see that $SL(2, \mathcal{C}\ell_{1,0})$ is isomorphic to $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

Now we define the map

$$\Phi_{2,2} : \mathbb{R}^{2,2} \to M(2,\mathbb{R})$$

by

$$\Phi_{2,2} : x = (x_1, x_2, x_3, x_4) \to \Phi_{2,2}(x) = \begin{pmatrix} x_1 + x_3 & x_4 + x_2 \\ x_4 - x_2 & x_1 - x_3 \end{pmatrix}.$$

Notice $Q_{2,2}(x) = x_1^2 + x_2^2 - x_3^2 - x_4^2 = \det \Phi_{2,2}(x)$.

For $(g,h) \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ the map $T_{(g,h)} : \mathbb{R}^{2,2} \to \mathbb{R}^{2,2}$ is given by

$$T_{(g,h)}(x) = \Phi_{2,2}^{-1}(g\Phi_{2,2}(x)h^t).$$

It follows that

$$Q_{2,2}(T_{(g,h)}(x)) = \det \Phi_{2,2}(T_{(g,h)}(x)) = \det(g\Phi_{2,2}(x)h^t) =$$

$$\det \Phi_{2,2}(x) = Q_{2,2}(x).$$

As such, $T_{(g,h)} \in SO(2,2)$ for all $g \in SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.
Let $g, h \in SL(2,\mathbb{R})$, then

$$T_{(g_1,h_1)}(T_{(g_2,h_2)}(x)) = T_{(g_1,h_1)}[\Phi_{2,2}^{-1}\, g_2\, \Phi_{2,2}(x)\, h_2^t] = \Phi_{2,2}^{-1}\, g_1\, \Phi_{2,2}[\Phi_{2,2}^{-1}\, g_2\, \Phi_{2,2}(x)\, h_2^t]h_1^t$$

$$= T_{(g_1 g_2, h_1 h_2)}(x).$$

As such T is a homomorphism. The kernel of T is plus or minus the identity.
We have the following diagram. Both $PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R})$ and $SO^+(2,2)$ have dimension six.

$$\begin{array}{ccc}
SL(2, \mathcal{C}\ell_{1,0}) \cong SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) & \xrightarrow{\text{isomorphism}} & spin(2,2) \\
\text{double cover}\big\downarrow & & \text{double cover}\big\downarrow \\
PSL(2, \mathcal{C}\ell_{1,0}) \cong PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) & \xrightarrow{\text{isomorphism}} & SO^+(2,2)
\end{array}$$

The group $SO^+(2,2)$ acts as conformal automorphisms on $N^{1,1} = (S^1 \times S^1)/\tilde{}$ see [1, 7].
In dimension four we use the following inner product.

Lemma 18.3.8 *We use the standard basis* $e_i = (\delta_{i,1}, \delta_{i,2}, \delta_{i,3}, \delta_{i,4})$.
The inner product, with $u, v \in \Lambda^2\mathbb{C}^4$, *defined by*

$$u \wedge v = \langle u, v\rangle\, e_1 \wedge e_2 \wedge e_3 \wedge e_4,$$

is invariant under $g \in SL(4,\mathbb{C})$ *with the action* $g(x \wedge y) = gx \wedge gy$, $x, y \in \mathbb{C}^4$.
Notice this action is trivial if and only if $g = \pm I$.

Proof Let $x, y, z, w \in \mathbb{C}^4$. We have

$$\langle g(x \wedge y), g(z \wedge w)\rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4 = gx \wedge gy \wedge gz \wedge gw =$$

$$(\det g)\, x \wedge y \wedge z \wedge w = \langle x \wedge y, z \wedge w\rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4.$$

□

18.3.5 $\mathbb{H} = \mathcal{C}\ell_{0,2}$

Theorem 18.3.9 *The Lie groups* $PSL(2, \mathbb{H})$ *and* $SO^+(1, 5)$ *are isomorphic.*

A quaternion can be rewritten :

$$\zeta = \alpha + \beta j, \ \alpha = x_0 + x_1 i, \ \beta = x_2 + x_3 i.$$

We have a faithful representation $\phi : \mathbb{H} \to M(2, \mathbb{C})$

$$\phi(\zeta) = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{18.3.2}$$

We use the notation $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We then have the characterization

$$\mathbb{H} = \{h \in M(2, \mathbb{C}) |\, \bar{h} = whw^{-1}\}.$$

Hence we define

$$SL(2, \mathbb{H}) = \{g \in SL(4, \mathbb{C}) |\, \bar{g} = WgW^{-1}\}$$

where $W = \begin{pmatrix} w & 0 \\ 0 & w \end{pmatrix}$.

Explicitly

$$SL(2, \mathbb{H}) = \{ \begin{pmatrix} \alpha_1 & \beta_1 & \alpha_2 & \beta_2 \\ -\bar{\beta_1} & \bar{\alpha_1} & -\bar{\beta_2} & \bar{\alpha_2} \\ \alpha_3 & \beta_3 & \alpha_4 & \beta_4 \\ -\bar{\beta_3} & \bar{\alpha_3} & -\bar{\beta_4} & \bar{\alpha_4} \end{pmatrix} |\, \zeta_i = \alpha_i + \beta_i j, i = 1, 2, 3, 4.\}. \tag{18.3.3}$$

We identify $\mathbb{R}^{1,5}$ with a six dimensional $\mathbb{R}$-subspace V of $\Lambda^2\mathbb{C}^4$. The subspace V consists of elements invariant under the conjugate linear map $J : \Lambda^2\mathbb{C}^4 \to \Lambda^2\mathbb{C}^4$ given by

$$J(u \wedge v) = W\overline{u} \wedge W\overline{v}.$$

An orthonormal basis of V is given by

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e_1 \wedge e_2 + e_3 \wedge e_4 \\ e_1 \wedge e_2 - e_3 \wedge e_4 \\ e_1 \wedge e_3 + e_2 \wedge e_4 \\ ie_1 \wedge e_3 - ie_2 \wedge e_4 \\ e_1 \wedge e_4 - e_2 \wedge e_3 \\ ie_1 \wedge e_4 + ie_2 \wedge e_3 \end{pmatrix} \tag{18.3.4}$$

For $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^{3,3}$ we define $\Phi_{1,5}(x) = \Sigma_{i=1}^{6} x_i v_i$. Then $\langle x, x\rangle_{1,5} = \langle \Phi_{1,5}(x), \Phi_{1,5}(x)\rangle$. Both $PSL(2, \mathbb{H})$ and $SO^+(1, 5)$ have dimension 15.

We display this as follows.

$$\begin{array}{ccc} SL(2,\mathbb{H}) & \xrightarrow{\text{isomorphism}} & spin(1,5) \\ \text{double-cover} \downarrow & & \text{double cover} \downarrow \\ PSL(2,\mathbb{H}) & \xrightarrow{\text{isomorphism}} & SO^+(1,5) \end{array}$$

The group $SO^+(1, 5)$ acts as conformal automorphisms on $N^{0,4} = (S^0 \times S^4)/\tilde{}$.

18.3.6 $\mathcal{C}\ell_{1,1} \cong \mathcal{C}\ell_{2,0}$

Theorem 18.3.10 *The Lie groups $PSL(4, \mathbb{R})$ and $SO(3, 3)^+$ are isomorphic.*

We write $M(2, \mathcal{C}\ell_{1,1})$ is the collection of $2{\times}2$ matrices with entries from $\mathcal{C}\ell_{1,1}$ and $M(4, \mathbb{R})$ are the $4{\times}4$ real matrices. We denote elements of $\mathcal{C}\ell_{1,1}$ by $\zeta_k = \alpha_k + \beta_k i + \gamma_k j + \delta_k ij$, $k = 1, 2, 3, 4$. We write $N(\zeta) = \zeta\bar{\zeta} = \alpha^2 + \beta^2 - \gamma^2 - \delta^2$.

We define the following map, see [3].

$$\phi : \begin{pmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{pmatrix} \to \begin{pmatrix} \alpha_1 + \delta_1 & -\beta_1 + \gamma_1 & \alpha_2 + \delta_2 & -\beta_2 + \gamma_2 \\ \beta_1 + \gamma_1 & \alpha_1 - \delta_1 & \beta_2 + \gamma_2 & \alpha_2 - \delta_2 \\ \alpha_3 + \delta_3 & -\beta_3 + \gamma_3 & \alpha_4 + \delta_4 & -\beta_4 + \gamma_4 \\ \beta_3 + \gamma_3 & \alpha_3 - \delta_3 & \beta_4 + \gamma_4 & \alpha_4 - \delta_4 \end{pmatrix} \tag{18.3.5}$$

We have $\phi(AB) = \phi(A)\phi(B)$ and $\phi(Id) = Id$. In fact, ϕ is an isometric isomorphism of the monoid $M(2, \mathcal{C}\ell_{1,1})$ onto the monoid $M(4, \mathbb{R})$. Notice it follows that $A \in M(2, \mathcal{C}\ell_{1,1})$ is invertible if and only if $\phi(A)$ is invertible in $M(4, \mathbb{R})$. We define the group $SL(2, \mathcal{C}\ell_{1,1})$ as $\phi^{-1}(SL(4, \mathbb{R}))$. See [3].

We identify the quadratic space $(\mathbb{R}^{3,3}, \langle\cdot\rangle_{3,3})$ with $(V, \langle\cdot\rangle)$ where $V = \Lambda^2\mathbb{R}^4$ and $\langle\cdot\rangle$ is defined by

$$x \wedge y = \langle x, y\rangle\, e_1 \wedge e_2 \wedge e_3 \wedge e_4, \quad x, y \in V,$$

where e_1, e_2, e_3, e_4 is the standard basis of $\mathbb{R}^4$. An orthonormal basis of V is defined by

$$\sqrt{2}\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6 \end{pmatrix} = \begin{pmatrix} e_1\wedge e_2 + e_3\wedge e_4\\ e_1\wedge e_3 - e_2\wedge e_4\\ e_1\wedge e_4 + e_2\wedge e_3\\ e_1\wedge e_2 - e_3\wedge e_4\\ e_1\wedge e_3 + e_2\wedge e_4\\ e_1\wedge e_4 - e_2\wedge e_3 \end{pmatrix} \tag{18.3.6}$$

For $x = (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^{3,3}$ we define $\Phi_{3,3}(x) = \Sigma_{i=1}^{6} x_i v_i$. Then $\langle x, x\rangle_{3,3} = \langle \Phi_{3,3}(x), \Phi_{3,3}(x)\rangle$. Both $PSL(4,\mathbb{R})$ and $SO^+(3,3)$ have dimension 15.

We have the following diagram.

$$\begin{array}{ccc} SL(2,\mathcal{C}\ell_{1,1}) \cong SL(4,\mathbb{R}) & \xrightarrow{\text{isomorphism}} & spin(3,3) \\ \text{double-cover}\downarrow & & \text{double cover}\downarrow \\ PSL(2,\mathcal{C}\ell_{1,1}) \cong PSL(4,\mathbb{R}) & \xrightarrow{\text{isomorphism}} & SO^+(3,3) \end{array}$$

The group $SO^+(3,3)$ acts as conformal automorphisms on $N^{2,2} = (S^2 \times S^2)/\tilde{}$.

We display some of the transformations in $SO(3,3)$ which correspond to those in $SL(2,\mathcal{C}\ell_{1,1})$.

- Inversion

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{18.3.7}$$

- Orthogonal, A

$$\begin{pmatrix} 1 & 0 & 0\\ 0 & A & 0\\ 0 & 0 & 1 \end{pmatrix} \tag{18.3.8}$$

- Dilation

$$\begin{pmatrix} 1+\lambda^2 & 0 & 0 & 0 & 0 & 1-\lambda^2 \\ 0 & 2\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 2\lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\lambda & 0 \\ 1-\lambda^2 & 0 & 0 & 0 & 0 & 1+\lambda^2 \end{pmatrix} \tag{18.3.9}$$

See [3, 7].

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Chapter 19
Separation of Variables in the Semistable Range

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Dedicated to Professor Wolfgang Sprößig

Abstract In this paper, we give an alternative proof of separation of variables for scalar-valued polynomials $P : (\mathbb{R}^m)^k \to \mathbb{C}$ in the semistable range $m \geq 2k - 1$ for the symmetry given by the orthogonal group $O(m)$. It turns out that uniqueness of the decomposition of polynomials into spherical harmonics is equivalent to irreducibility of generalized Verma modules for the Howe dual partner $\mathfrak{sp}(2k)$ generated by spherical harmonics. We believe that this approach might be applied to the case of spinor-valued polynomials and to other settings as well.

Keywords Fischer decomposition · Separation of variables · Spherical harmonics · Monogenic polynomials · Dirac equation

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19.1 Introduction

It is well-known that each polynomial P in the Euclidean space $\mathbb{R}^m$ can be uniquely written as a finite sum

$$P = H_0 + r^2 H_1 + \cdots + r^{2j} H_j + \cdots$$

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where $r^2 = x_1^2 + \cdots + x_m^2$ for $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ and H_j are harmonic polynomials in $\mathbb{R}^m$, that is, $\Delta H_j = 0$ for the Laplace operator

$$\Delta = \partial_{x_1}^2 + \cdots + \partial_{x_m}^2.$$

In other words, the space $\mathcal{P}$ of $\mathbb{C}$-valued polynomials on $\mathbb{R}^m$ decomposes as

$$\mathcal{P} = \bigoplus_{n=0}^{\infty} r^{2n} \mathcal{H}$$

where $\mathcal{H} = \mathrm{Ker}(\Delta) \cap \mathcal{P}$ is the space of spherical harmonics in $\mathbb{R}^m$. This result is known as separation of variables or the Fischer decomposition. The underlying symmetry is given by the orthogonal group $O(m)$. The invariant operators Δ, r^2, h generate the Lie algebra $\mathfrak{sl}(2)$ where

$$h = x_1 \partial_{x_1} + \cdots + x_m \partial_{x_m} + m/2$$

is the Euler operator. This is the so-called hidden symmetry in this case. Actually, it is a simple example of Howe duality for Howe dual pair $(O(m), \mathfrak{sl}(2))$.

For separation of variables and Howe duality in various cases and for other symmetry groups, see e.g. [1, 2, 4, 6, 10, 12, 13, 15–17, 19, 20].

For example, spinor valued polynomials in one variable of $\mathbb{R}^m$ decompose into monogenic polynomials [7]. Let $\mathbb{S}$ be an irreducible spin representation of the group $Pin(m)$, the double cover of $O(m)$. The spinor space $\mathbb{S}$ is usually realized inside the complex Clifford algebra $\mathbb{C}_m$ generated by the generators $e_1, \ldots, e_m$ satisfying the relations $e_i e_j + e_j e_i = -2\delta_{ij}$. The Euclidean space $\mathbb{R}^m$ is embedded into $\mathbb{C}_m$ as

$$(x_1, \ldots, x_m) \to x_1 e_1 + \cdots + x_m e_m.$$

A polynomial $P : \mathbb{R}^m \to \mathbb{S}$ is called monogenic if it satisfies the equation $\partial P = 0$ where

$$\partial := \sum_{i=1}^{m} e_i \partial_{x_i}$$

is the Dirac operator in $\mathbb{R}^m$. Then each polynomial $P : \mathbb{R}^m \to \mathbb{S}$ has a unique expression as a finite sum

$$P = M_0 + x M_1 + \cdots + x^j M_j + \cdots$$

where $x = x_1 e_1 + \cdots + x_m e_m$ is the vector variable of $\mathbb{R}^m$ and M_j are monogenic polynomials in $\mathbb{R}^m$. Then the space of spinor-valued polynomials on

$\mathbb{R}^m$ decomposes as

$$\mathcal{P} \otimes \mathbb{S} = \bigoplus_{n=0}^{\infty} x^n \mathcal{M}$$

where $\mathcal{M} = \mathrm{Ker}(\partial) \cap (\mathcal{P} \otimes \mathbb{S})$ is the space of spherical monogenics in $\mathbb{R}^m$. The symmetry is given by the group $Pin(m)$ and the invariant operators ∂, x generate the Lie superalgebra $\mathfrak{osp}(1|2)$. This is Howe duality $(Pin(m), \mathfrak{osp}(1|2))$.

The case of scalar or spinor valued polynomials in more variables is more interesting and involved.

19.1.1 Scalar-Valued Polynomials

Let us start with the scalar case which has been studied for a long time and is well-understood. We consider $\mathbb{C}$-valued polynomials in k vector variables x^i of $\mathbb{R}^m$. Here $x^i = (x^i_1, \ldots, x^i_m) \in \mathbb{R}^m$ for $i = 1, \ldots, k$. We take a natural action of $O(m)$ on the space $\mathcal{P}$ of polynomials $P : (\mathbb{R}^m)^k \to \mathbb{C}$. Then the invariant operators

$$\Delta_{ij} = \partial_{x^i_1}\partial_{x^j_1} + \cdots + \partial_{x^i_m}\partial_{x^j_m}, \quad r^2_{ij} = x^i_1 x^j_1 + \cdots + x^i_m x^j_m$$

$$h_{ij} = x^i_1 \partial_{x^j_1} + \cdots + x^i_m \partial_{x^j_m} + (m/2)\delta_{ij}, \quad i, j = 1, \ldots, k$$

generate the Lie algebra $\mathfrak{sp}(2k)$, and the mixed Euler operators h_{ij} its subalgebra $\mathfrak{gl}(k)$. This case is indeed Howe duality $(O(m), \mathfrak{sp}(2k))$. Spherical harmonics are the polynomials in the kernel of all the mixed laplacians Δ_{ij}. Thus we denote

$$\mathcal{H} = \mathrm{Ker}(\Delta_{ij}, 1 \le i \le j \le k) \cap \mathcal{P}.$$

Let us remark that $\Delta_{ij} = \Delta_{ji}$ and $r^2_{ij} = r^2_{ji}$. Theorem A below describes, in a semistable range $m \ge 2k - 1$, a decomposition of polynomials into spherical harmonics we can view as a proper generalization of the harmonic Fischer decomposition to more variables.

Theorem A *If $m \ge 2k - 1$, then*

$$\mathcal{P} = \bigoplus_{n} r^{2n} \mathcal{H} \quad \text{with} \quad r^{2n} = \prod_{1 \le i \le j \le k} r_{ij}^{2n_{ij}}$$

where the sum is taken over all $n = \{n_{ij},\ 1 \le i \le j \le k\} \subset \mathbb{N}_0$.

This result at least in the stable range $m \geq 2k$ is well-known in invariant theory and theory of Howe duality, see [13]. In the next section, we give an alternative proof and extend the result even to the semistable range $m \geq 2k-1$. The non-stable range is much more complicated and less understood.

19.1.2 Spinor-Valued Polynomials

The form of the Fischer decomposition for spinor-valued polynomials in the semistable range was conjectured by Colombo et al. in 2004 in the book [3]. Before recalling this, let us introduce some notations. On the space $\mathcal{P} \otimes \mathbb{S}$ of polynomials $P : (\mathbb{R}^m)^k \to \mathbb{S}$, there is a natural action of the group $Pin(m)$. In this case, we have k vector variables $x^i \in \mathbb{R}^m$,

$$x^i = e_1 x_1^i + \cdots + e_m x_m^i$$

and k corresponding Dirac operators ∂^i,

$$\partial^i = e_1 \partial_{x_1^i} + \cdots + e_m \partial_{x_m^i}$$

for $i = 1, \ldots, k$. Then the odd invariant operators x^i, ∂^i generate the Lie superalgebra $\mathfrak{osp}(1|2k)$, and its even part $\mathfrak{sp}(2k)$ is generated by the 'scalar' operators $\Delta_{ij}, r_{ij}^2, h_{ij}$ we know from the scalar case. The role of spherical harmonics is played by spherical monogenics, that is, polynomial solutions $P : (\mathbb{R}^m)^k \to \mathbb{S}$ of the system of all the Dirac equations $\partial^i P = 0$ for $i = 1, \ldots, k$. Denote

$$\mathcal{M} = \mathrm{Ker}(\partial^1, \ldots, \partial^k) \cap (\mathcal{P} \otimes \mathbb{S}).$$

Finally, for $J \subset \{1, 2, \ldots, k\}$, put $x^J = x^{j_1} \cdots x^{j_r}$ where $J = \{j_1, \cdots, j_r\}$ and $j_1 < \cdots < j_r$. Here $x^{\emptyset} := 1$. Then we have the following result.

Theorem B *If $m \geq 2k$, then*

$$\mathcal{P} \otimes \mathbb{S} = \bigoplus_{n,J} r^{2n} x^J \mathcal{M}$$

where the sum is taken over all $J \subset \{1, \ldots, k\}$ and $n = \{n_{ij},\ 1 \leq i \leq j \leq k\} \subset \mathbb{N}_0$.

The decomposition of Theorem B was conjectured in [3] but even in the semistable range $m \geq 2k-1$. Theorem B (that is, this decomposition only in the stable range) was recently proved in [17] using the harmonic Fischer decomposition. For the case of two variables, see [20]. Since the harmonic Fischer decomposition is now extended to the semistable range there is a hope that the conjecture could be proved in the full semistable range as well.

It is known [2] that, in $\mathcal{P} \otimes \mathbb{S}$, the isotypic components of $Spin(m)$ form irreducible lowest weight modules for $\mathfrak{osp}(1|2k)$ with lowest weights $(a_1 + (m/2), \ldots, a_k + (m/2))$ for integers $a_1 \geq \cdots \geq a_k \geq 0$ when the dimension m is even. Let us remark that there are not so many known explicit realizations of such modules, see e.g. the paraboson Fock space [18]. For a classification of such modules, we refer to [8, 9].

19.2 Proof of Theorem A

In this section, we prove the harmonic Fischer decomposition in the semistable range. We divide the proof into three steps. The first two steps are quite standard. In the last step, to show uniqueness of the decomposition of polynomials we study irreducibility of generalized Verma modules for the Howe dual partner $\mathfrak{sp}(2k)$ generated by spherical harmonics. This approach seems to be very flexible and to work well in other settings. In particular, we believe that, using this approach, Theorem B might be proved in the full semistable range by studying the structure of generalized Verma modules for $\mathfrak{osp}(1|2k)$.

Step 1: Decomposition into Spherical Harmonics
First we show that each polynomial can be expressed in terms of spherical harmonics. This is easy. But as we shall see the question of uniqueness of such an expression is more difficult.

Lemma 19.1

(i) We have

$$\mathcal{P} = \mathcal{H} \oplus \sum_{1 \leq i \leq j \leq k} r_{ij}^2 \, \mathcal{P}$$

(ii) We have

$$\mathcal{P} = \sum_{n} r^{2n} \mathcal{H} \quad \textit{with} \;\; r^{2n} = \prod_{1 \leq i \leq j \leq k} r_{ij}^{2n_{ij}} \tag{19.1}$$

where the sum is taken over all $n = \{n_{ij},\ 1 \leq i \leq j \leq k\} \subset \mathbb{N}_0$.

Proof

(i) This is an orthogonal decomposition with respect to the Fischer inner product on the space $\mathcal{P}$ of polynomials. Here the Fischer inner product is defined, for $P, Q \in \mathcal{P}$, by

$$(P, Q) = [\overline{P(\partial)}(Q(x))]_{x=0}$$

where $\bar{z}$ is the complex conjugation of $z \in \mathbb{C}$ and $P(\partial)$ denotes the constant coefficient differential operator obtained by substituting derivatives $\partial_{x_j^i}$ for the variables x_j^i.

(ii) Use repeatedly (i).

□

It is known that, in general, the sum (19.1) is not direct. In Step 3, we show that this sum is direct in the semistable range.

Step 2: Decomposition of Spherical Harmonics

It is easy to see that the space $\mathcal{H}$ of spherical harmonics is invariant with respect not only to the symmetry group $O(m)$ but also to the Lie algebra $\mathfrak{gl}(k)$ generated by the mixed Euler operators h_{ij}. Before describing an irreducible decomposition of $\mathcal{H}$ under the joint action of $O(m) \times \mathfrak{gl}(k)$ let us recall some notations.

A partition $a = (a_1, \ldots, a_m)$ of the length at most m is a non-negative integer sequence $a_1 \geq a_2 \geq \cdots \geq a_m \geq 0$. With a partition a we often identify the corresponding Young diagram, that is, the array of square boxes arranged in left-justified horizontal rows and with row i having just a_i boxes. Then there is a one-to-one correspondence between finite-dimensional irreducible representations of $O(m)$ and partitions a satisfying the condition

$$(a')_1 + (a')_2 \leq m. \tag{19.2}$$

Here a' denotes the transpose of the Young diagram a and thus the condition (19.2) means that the sum of the first two columns of the Young diagram a is at most m. See [11, Sections 5.2.2, 10.2.4 and 10.2.5] for details.

Finite-dimensional irreducible representations of $\mathfrak{gl}(k)$ are indexed by highest weights. In what follows, we use the triangular decomposition

$$\mathfrak{gl}(k) = \mathfrak{t}_- \oplus \mathfrak{t}_0 \oplus \mathfrak{t}_+$$

with $\mathfrak{t}_- = \text{span}\{h_{ij}, i < j\}$, $\mathfrak{t}_0 = \text{span}\{h_{ij}, i = j\}$, $\mathfrak{t}_+ = \text{span}\{h_{ij}, i > j\}$.

Theorem 19.1 *Under the joint action of $O(m) \times \mathfrak{gl}(k)$, we have an irreducible decomposition*

$$\mathcal{H} = \bigoplus_a \mathcal{H}_a^S \otimes F_{\tilde{a}}$$

where the sum is taken over all partitions $a = (a_1, \ldots, a_k)$ of the length at most k and satisfying the condition (19.2) above, $\mathcal{H}_a^S$ is $O(m)$-irreducible module with the label a and $F_{\tilde{a}}$ is $\mathfrak{gl}(k)$-irreducible module with the highest weight $\tilde{a} = (a_1 + m/2, \ldots, a_k + m/2)$.

This theorem is true in general, not only in the semistable or stable range. Actually, in the case when $k \geq m$, we can realize each finite-dimensional irreducible representation of $O(m)$ inside the space $\mathcal{H}$ of spherical harmonics. Indeed, we have

$$\mathcal{H}_a^S = \mathcal{H} \cap \mathrm{Ker}(\mathfrak{t}_-) \cap \mathcal{P}_a$$

where $\mathcal{P}_a$ is the subspace of polynomials of $\mathcal{P}$ homogeneous in x^i of degree a_i for each $i = 1, \ldots, k$ and a satisfies the condition (19.2) above. Let us remark that polynomials of $\mathcal{H}^S = \mathcal{H} \cap \mathrm{Ker}(\mathfrak{t}_-)$ are sometimes called simplicial harmonics. For a proof of Theorem 19.1 and for a construction of highest weight vectors, see [13, 3.6, pp. 37–40] and cf. [5].

Step 3: Uniqueness of the Decomposition
In the last step, we show uniqueness of the decomposition of polynomials in the semistable range. By Steps 1 and 2, we know

$$\mathcal{P} = \sum_n r^{2n}\mathcal{H} \ \text{ and } \ \mathcal{H} = \bigoplus_a \mathcal{H}_a^S \otimes F_{\tilde{a}}$$

where the sums are taken over all $n = \{n_{ij},\ i \leq j\} \subset \mathbb{N}_0$ and all partitions a of the length at most k and satisfying the condition (19.2), respectively. Then we have

$$\mathcal{P} = \bigoplus_a \mathcal{H}_a^S \otimes L_{\tilde{a}} \ \text{ with } \ L_{\tilde{a}} = \sum_n r^{2n} F_{\tilde{a}}. \tag{19.3}$$

Here the first sum in (19.3) is direct because this is an isotypic decomposition for the group $O(m)$. Moreover, it is easy to see that $L_{\tilde{a}}$ is a lowest weight module with the lowest weight $\tilde{a}$ for the Howe dual partner $\mathfrak{sp}(2k)$ generated by the invariant operators. Actually, it is well-known that, using Howe duality $(O(m), \mathfrak{sp}(2k))$, the module $L_{\tilde{a}}$ is even irreducible. But we do not need this fact in our argument. But what we really need is to observe that $L_{\tilde{a}}$ is a quotient of a generalized Verma module $V_{\tilde{a}}$ for $\mathfrak{sp}(2k)$. In the next section, we introduce generalized Verma modules V_λ for $\mathfrak{sp}(2k)$ and its parabolic subalgebra suitable for our purposes and find out sufficient conditions on the weight λ under which V_λ is irreducible. It turns out that uniqueness of the decomposition (19.1) is closely related to the structure of the modules $L_{\tilde{a}}$. Indeed, the sum in $L_{\tilde{a}}$ of (19.3) is direct if and only if $L_{\tilde{a}}$ is isomorphic to $V_{\tilde{a}}$. But we show that, in the semistable range $m \geq 2k - 1$, all modules $V_{\tilde{a}}$ are in fact irreducible (see Proposition 19.1 and Example below) and hence

$$\mathcal{P} = \bigoplus_n r^{2n}\mathcal{H},$$

which completes the proof of Theorem A.

In the non-stable range, the module $L_{\tilde{a}}$ is not, in general, isomorphic to $V_{\tilde{a}}$ but it is just a unique irreducible quotient of $V_{\tilde{a}}$. So, even in the non-stable range, the study of the decomposition of polynomials (19.1) is closely related to the structure of the modules $L_{\tilde{a}}$ and representation theory might help much with this task.

19.3 Generalized Verma Modules for $\mathfrak{sp}(2k)$

In this section, we introduce generalized Verma modules for $\mathfrak{sp}(2k)$ we need in Step 3 of the proof of Theorem A. For an account of generalized Verma modules, we refer to [14].

In our setting, the Lie algebra $\mathfrak{sp}(2k)$ is generated by the invariant operators Δ_{ij}, r^2_{ij} and h_{ij}. We have a decomposition $\mathfrak{sp}(2k) = \mathfrak{p}_- \oplus \mathfrak{k} \oplus \mathfrak{p}_+$ where

$$\mathfrak{p}_- = \text{span}\{\Delta_{ij},\ 1 \le i \le j \le k\},\ \ \mathfrak{p}_+ = \text{span}\{r^2_{ij},\ 1 \le i \le j \le k\},$$

$$\mathfrak{k} = \text{span}\{h_{ij},\ 1 \le i, j \le k\}.$$

We take a parabolic subalgebra $\mathfrak{p} = \mathfrak{p}_- \oplus \mathfrak{k}$ with its Levi subalgebra $\mathfrak{k} \simeq \mathfrak{gl}(k)$.

Definition 19.1 Let F_λ be a finite dimensional $\mathfrak{gl}(k)$-irreducible module with the highest weight λ such that the action of $\mathfrak{p}_-$ on F_λ is trivial, that is, $(\mathfrak{p}_-) \cdot F_\lambda = 0$. Then we define the generalized Verma module for $\mathfrak{g} = \mathfrak{sp}(2k)$ and its parabolic subalgebra $\mathfrak{p}$ as the induced module

$$V_\lambda := \text{Ind}^{\mathfrak{g}}_{\mathfrak{p}} F_\lambda.$$

It is well-known that, at least as vector spaces,

$$V_\lambda \simeq \bigoplus_n r^{2n} F_\lambda.$$

The following proposition [14, 9.12, p. 196] gives sufficient conditions on the weight λ under which V_λ is irreducible.

Proposition 19.1 *The generalized Verma module V_λ is irreducible if*

(1) $\lambda_i + \lambda_j - 2k + i + j - 2 \notin -\mathbb{N}$ *for* $1 \le i < j \le k$, *and*
(2) $\lambda_i - k + i - 1 \notin -\mathbb{N}$ *for* $1 \le i \le k$.

Here $\lambda = (\lambda_1, \dots, \lambda_k)$.

Example Let $a = (a_1, \dots, a_k)$ be a partition and $\tilde{a} = (a_1 + m/2, \dots, a_k + m/2)$. Then the conditions of Proposition 19.1 for $\lambda = \tilde{a}$ read as

(1′) $a_i + a_j + m - 2k + i + j - 2 \notin -\mathbb{N}$ for $1 \le i < j \le k$, and
(2′) $a_i + (m/2) - k + i - 1 \notin -\mathbb{N}$ for $1 \le i \le k$.

In particular, in the semistable range $m \ge 2k - 1$, the conditions (1′) and (2′) are always satisfied and hence the corresponding module $V_{\tilde{a}}$ is irreducible.

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Chapter 20
Variety of Idempotents in Nonassociative Algebras

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Dedicated to Professor Wolfgang Sprößig on the occasion of his 70th birthday

Abstract In this paper, we study the variety of all nonassociative (NA) algebras from the idempotent point of view. We are interested, in particular, in the spectral properties of idempotents when algebra is generic, i.e. idempotents are in general position. Our main result states that in this case, there exist at least $n^2 - 1$ nontrivial obstructions (syzygies) on the Peirce spectrum of a generic NA algebra of dimension n. We also discuss the exceptionality of the eigenvalue $\lambda = \frac{1}{2}$ which appears in the spectrum of idempotents in many classical examples of NA algebras and characterize its extremal properties in metrized algebras.

Keywords Idempotents · Nonassociative algebras · Metrized algebras · Peirce spectrum · Axial algebras

20.1 Introduction

The Peirce decomposition is a central tool of nonassociative algebra. In associative algebras (for example in matrix algebras), idempotents are projections onto subspaces, with eigenvalues 1 and 0 and play a distinguished role. In nonassociative algebras the spectrum of an idempotent (which is known also as the Peirce numbers) can be very arbitrarily. Still, many classical examples of nonassociative algebras share the following basic feature: the set of idempotents in algebra is rich enough

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(for example spans or generates the algebra) while the number of possible distinct Peirce numbers is few in comparison with the algebra dimension.

Let A be a commutative nonassociative algebra over a field K of $\mathrm{char}(K) = 0$ unless otherwise stated explicitly. Let $L_u : x \to ux$ be the operator of multiplication by u on A. Any semisimple idempotent $0 \neq c = c^2 \in A$ induces the corresponding Peirce decomposition:

$$A = \bigoplus_{\lambda \in \sigma(c)} A_c(\lambda),$$

where $L_c x = \lambda x$ for any $x \in A_c(\lambda)$ and $\sigma(c)$ is the Peirce spectrum of c, i.e. the multi-set (a set with repeated elements) of all eigenvalues of L_c. The Peirce spectrum $\sigma(A) = \{\lambda_1, \ldots, \lambda_s\}$ of the algebra A is the set of all possible distinct eigenvalues λ_i in $\sigma(c)$, when c runs all idempotents of A. A fusion (or multiplication) rule is the inclusion of the following kind:

$$A_c(\lambda_i)A_c(\lambda_j) \subset \bigoplus_{k \in \mathcal{F}(i,j)} A_c(\lambda_k), \quad \mathcal{F}(i, j) \subset \{1, 2 \ldots, s\}$$

For example, if A is power-associative (for instance, A is a Jordan algebra) then its Peirce spectrum (i.e. the only possible Peirce numbers of A) is $\sigma(A) = \{0, \frac{1}{2}, 1\}$, see [1, 29]. The middle value $\frac{1}{2}$ is crucial for structural properties and classification of formally real Jordan algebras: a Jordan algebra A is simple if and only if the corresponding eigenspace $A_c(\frac{1}{2})$ is nontrivial for any nonzero idempotent $c \in A$ [9, p. 63]. It is also well-known that the fusion rules of a Jordan algebra are $\mathbb{Z}/2$-graded for any idempotent $c \in A$: if $A^0 = A_c(0) \oplus A_c(1)$ and $A^1 = A_c(\frac{1}{2})$ then $A^i A^j \subset A^{i+j \mod 2}$.

Another important example is axial algebras appearing in connection with the Monster sporadic simple group [23]. The most famous example here is the Griess algebra generated by idempotents with Peirce numbers $1, 0, \frac{1}{4}, \frac{1}{32}$ and satisfying the so-called Ising fusion rules [12, 14, 27]. These fusion rules are also $\mathbb{Z}/2$-graded.

We also mention very recent examples of the so-called Hsiang algebras appearing in the classification of cubic minimal cones (the REC-algebras in terminology of [22, Chapter 6]). The Peirce spectrum of such an algebra consists of four numbers: $\sigma(A) = \{1, -1, -\frac{1}{2}, \frac{1}{2}\}$. The Hsiang algebras have nice fusion rules but they are not graded. Furthermore, these algebras share a remarkable property: all idempotents have the same spectrum. As we shall see below, the latter property is closely related to the fact that $\frac{1}{2}$ belongs to the algebra spectrum.

Traditionally, one defines a (nonassociative) algebra structure by virtue of algebra identities (for example, Lie, Jordan and power-associative algebras) or a multiplication table (for example, division algebras or evolution algebras [32]). Also, an algebraic structure can be defined by postulating some distinguished properties of idempotents, as in example of the axial algebras mentioned before.

One of the main goals of the present paper is to support the following conjectural paradigm: *Many essential or invariant properties of a non-associative algebra can* *be recovered directly from its Peirce spectrum.* In other words, the knowledge of the Peirce spectrum of an algebra allows one to determine the most important features of the algebra.

In this connection, the following principal questions arise and will be discussed in this paper.

(a) How arbitrary can the Peirce spectrum be? What kind of obstructions (syzygies) can exist?
(b) What can be said about the structure of an algebra with a prescribed set of the Peirce numbers?
(c) Which Peirce numbers have 'distinguished' properties?
(d) If the Peirce spectrum is known, what can be said about the possible multiplicities of the eigenvalues? For example, when the spectrum is independent of a choice of an idempotent?
(e) Do there exist any obstructions/syzigies on the fusion table?

Thus formulated program is rather ambitious even for characteristic 0. To obtain some significant results, we sometimes assume that K is a subfield of $\mathbb{C}$. We emphasize that in this paper we are more interested in discussing and illustrating some new methods and phenomena with a clear analytical or topological flavor. We outline only some of the possible directions and obtain some particular answers to the above questions.

Our main result here describes syzygies (obstructions) on the idempotent set of a finite dimensional generic commutative nonassociative algebra A under fairly general assumptions. More precisely, we show (see Theorem 20.4.1 below) that

$$\sum_{c\in \mathrm{Idm}_0(A)} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n, \quad \forall t \in \mathbb{R},$$

where $n = \dim A$ and $\chi_c(t)$ is the characteristic polynomial of an idempotent c. In this paper, we only outline some general lines and discuss basic properties of syzygies. The border-line case when an algebra contains 2-nilpotents and the exceptional case when there exists infinitely many idempotents will be considered elsewhere.

The paper is organized as follows. We recall some well-known concepts in Sect. 20.2, then define the concept of a generic algebra and study its basic properties in Sect. 20.3. We explain also here why the presence or absence of eigenvalue $\frac{1}{2}$ in the algebra spectrum plays an exceptional role. The principal syzygies are determined in Sect. 20.4 and some applications and explicit examples are given in Sects. 20.4.2 and 20.4.3. Furthermore, applying the syzygy method we study in Sect. 20.5 some examples of algebras with a prescribed Peirce spectrum. A distinguished subclass of nonassociative algebras is algebras admitting an associative bilinear form, the so-called metrized algebras. This class is in a natural correspondence with the space of all cubic forms on the ground vector space. We discuss the spectral properties of metrized algebras in Sect. 20.6 and establish an extremal property of $\lambda = \frac{1}{2}$ in Sect. 20.6.2. We also show that the presence of $\frac{1}{2}$ in the algebra spectrum yields a fusion rule for the corresponding Peirce eigenspace.

20.2 Preliminaries

In choosing what material to include here, we have tried to concentrate on the class of *commutative* nonassociative algebras over a field of characteristic 0. In fact, many of our results, including the principal syzygies, remain similar in the noncommutative case and for general finite fields but in that case some topics become lengthy and require more careful analysis, and will be treated elsewhere.

Therefore, in what follows by A we mean an (always finite dimensional) commutative nonassociative algebra over a filed K. We point out that 'algebra' always means a nonassociative algebra.

We need to make some additional assumptions on the ground field K. If not explicitly stated otherwise, we shall assume that K is a subfield of $\mathbb{C}$, the field of complex numbers. By $A_{\mathbb{C}}$ we denote the complexification of A obtained in an obvious way by extending the ground field such that $\dim_K A = \dim_{\mathbb{C}} A_{\mathbb{C}}$.

An element c is called idempotent if $c^2 = c$ and 2-nilpotent if $c^2 = 0$. By

$$\mathrm{Idm}(A) = \{0 \neq c \in A : c^2 = c\}$$

we denote the set of all *nonzero* idempotents of A and the complete set of idempotents will be denoted by

$$\mathrm{Idm}_0(A) = \{0\} \cup \mathrm{Idm}(A).$$

The set of all idempotents and 2-nilpotents of A will be denoted by

$$\mathbf{P}(A) = \{x \in A : \text{either } x^2 = x \text{ or } x^2 = 0\}$$

If the algebra A is unital with unit e then given an idempotent $c \in \mathrm{Idm}_0(A)$, its conjugate $\bar{c} := e - c$ is also idempotent:

$$\bar{c}^2 = (e - c)^2 = e - 2c + c = \bar{c}.$$

It is also well known that c and $\bar{c}$ are orthogonal in the sense that $c\bar{c} = 0$.

We follow the standard notation and denote by L_x the multiplication operator (sometimes also called adjoint of x):

$$L_x y = xy = yx.$$

Regarding L_x as an endomorphism in the vector space A, we define the corresponding characteristic polynomial by

$$\chi_x(t) = \det(L_x - tI), \quad t \in K.$$

Let $\sigma(x)$ denote the set of (in general complex) roots of the characteristic equation $\chi_x(t) = 0$ counting multiplicity. By the made assumption, $\sigma(x)$ is well defined and is said to be the *Peirce spectrum* of x. It is easy to see that if $t \in K$ is a root of $\chi_x(t) = 0$ then the corresponding *Peirce subspace*

$$A_x(t) := \ker(L_x - tI)$$

is nontrivial. Thus, any $t \in \sigma(x) \cap K$ is actually an eigenvalue of L_x.

Now suppose that $c \in \mathrm{Idm}_0(A)$ is a nonzero idempotent. Then $t = 1$ is an obvious eigenvalue of L_c (corresponding to c), thus $1 \in \sigma(c)$. Distinct elements of the Peirce spectrum $\sigma(c)$ are called *Peirce numbers*.

An idempotent c is called *semisimple* if A is decomposable as the sum of the corresponding Peirce subspaces:

$$A = \bigoplus_i A_c(\lambda_i),$$

where λ_i are the Peirce numbers of c. We define in this case the corresponding *Peirce dimensions*

$$n_c(\lambda) = \dim\ker(L_c - \lambda I).$$

Note that the number of idempotents in a (finite-dimensional) algebra can not be very arbitrary. Namely, the set of idempotents can be studied by purely algebraic geometry methods, an idea coming back to the classical paper of Segre [30]. More precisely, Segre showed that the set $\mathbf{P}(A)$ can be described as the solution set of a system of quadratic equations over K, actually as intersection of certain quadrics. This in particular implies that a real or complex algebra without nilpotent elements always admits idempotents.

For the following convenience we briefly recall Segre's argument. Let us consider an algebra over K, not necessarily commutative. Let us associate to A with the multiplication map

$$\psi_A(u, v) = uv : A \times A \to A$$

which is naturally identified with a corresponding element $\psi_A \in V^* \otimes V^* \otimes V$. If $\mathbf{e} = \{e_1, \ldots, e_n\}$ is an arbitrary basis in A, where $n = \dim_K A$, then ψ_A induces a K-quadratic polynomial map $\Psi_A : K^n \to K^n$ defined by

$$\psi_A \circ \epsilon = \epsilon \circ \Psi_A, \tag{20.1}$$

where ϵ is the coordinatization map

$$\epsilon(x) := \sum_{i=1}^{n} x_i e_i : K^n \to A, \qquad x = (x_1, \ldots, x_n) \in K^n.$$

In this setting, Ψ_A is a bilinear map on K^n. Then an element $c = \epsilon(x) \in A$ is idempotent if and only if the corresponding $x \in K^n$ is a fixed point of $\Psi_A(x, x)$, i.e.

$$\Psi_A(x, x) - x = 0. \tag{20.2}$$

It is convenient to consider the projectivization of the latter system. Namely, let

$$\Psi_A^{\mathbf{P}}(X) = \Psi_A(x, x) - x_0 x,$$

where $X = (x_0, x_1, \dots, x_n) \in K^{n+1}$. The modified equation

$$\Psi_A^{\mathbf{P}}(X) = 0 \tag{20.3}$$

is homogeneous of degree 2. By the made assumption on K, we can consider both (20.2) and (20.3) as equations over the complex numbers. Furthermore, (20.3) defines a variety in $\mathbb{CP}^n$. Clearly, if x solves (20.2) then $X = (1, x)$ is a solution of (20.3), and, conversely, if $X = (x_0, x)$ solves (20.3) with $x_0 \neq 0$ then $\frac{1}{x_0}x$ is a solution of (20.2). In the exceptional case $x_0 = 0$, one has $\Phi(x) = 0$, i.e. $\epsilon(x)$ is a 2-nilpotent in A.

In summary, there exists a natural bijection (depending on a choice of a basis in A) between the set $\mathbf{P}(A_{\mathbb{C}})$ and all solutions of (20.3) in $\mathbb{CP}^n$. In this picture, 2-nilpotents correspond to the 'infinite' part of solutions of (20.2) (i.e. solutions of (20.3) with $x_0 = 0$).

Then the classical Bezóut's theorem implies the following dichotomy: either there are infinitely many solutions of (20.3) or the number of distinct solutions is less or equal to 2^n, where $n = \dim_K A$. Therefore if the set $\mathbf{P}(A_{\mathbb{C}})$ is finite then necessarily

$$\operatorname{card} \mathbf{P}(A_{\mathbb{C}}) \leq 2^n \tag{20.4}$$

We point out that one should interpret a solution to (20.3) in the projective sense.

Some remarks are in order. First note that the above correspondence makes an explicit bijection between idempotents and 2-nilpotents only in the complexification $A_{\mathbb{C}}$. In general, if $X = (x_0, x)$ is a solution to (20.3) then $x \in \mathbf{P}(A)$ only if $X \in K^n$. This, of course, also yields the corresponding inequality over K:

$$\operatorname{card} \mathbf{P}(A_K) \leq \operatorname{card} \mathbf{P}(A_{\mathbb{C}}) \leq 2^n. \tag{20.5}$$

Note, however, that a priori it is possible that there can exist only finitely many number solutions over K while there can be infinitely many solutions over $\mathbb{C}$.

20.3 Generic Nonassociative Algebras and the Exceptionality of $\frac{1}{2}$

It is well known that a generic (in the Zariski sense) polynomial system has Bézout's number of solutions. In our case, if $K = \mathbb{C}$ then an algebra having exactly $2^{\dim A}$ idempotents (Bézout's number for (20.3)) is generic in the sense that the subset of nonassociative algebra structures on V with exactly $2^{\dim A}$ idempotents is an open Zariski subset in $V^* \otimes V^* \otimes V$. This motivates the following definition.

Definition 20.3.1 An algebra A over K is called a *generic nonassociative algebra*, or generic NA algebra, if its complexification $A_{\mathbb{C}}$ contains exactly 2^n distinct idempotents, where $n = \dim A$.

The definition given above should not be confused with similar definitions of generic subsets for certain distinguished classes of algebras (like a generic division algebra). Namely, our definition distinguish generic algebras in the class of *all* nonassociative algebras. We refer also to [28, p. 196], where the generic phenomenon is essentially interpreted as the absence of 2-nilpotents and the presence of idempotents. This supports our definition.

Remark 20.3.2 Note that Definition 20.3.1 together with Bézout's theorem imply that if A is generic then neither $A_{\mathbb{C}}$ nor A have nonzero 2-nilpotents.

The class of generic NA algebras can be thought of as the most natural model for testing the above program. First note that the definition itself implies certain obstructions on the algebras spectrum. The following criterium shows that the property being a generic for an algebra is essentially equivalent to the fact that the algebra spectrum does not contain $\frac{1}{2}$.

Theorem 20.3.3 *If A is a commutative generic algebra then $\frac{1}{2} \notin \sigma(A)$. In the converse direction: if $\frac{1}{2} \notin \sigma(A)$ and A does not contain 2-nilpotents then A is generic.*

Proof First let us define the associated quadratic map

$$\Psi_A(x) := \Psi_A(x, x) : K^n \to K^n$$

and consider the fixed point equation

$$f_A(x) := \Psi_A(x) - x = 0. \tag{20.6}$$

By the commutativity assumption, the multiplication map

$$\Psi_A(x, y) = \Psi_A(y, x)$$

is symmetric, therefore it is recovered from Ψ_A by polarization:

$$\Psi_A(x, y) = \tfrac{1}{2}(\Psi_A(x + y) - \Psi_A(x) - \Psi_A(y)) = \tfrac{1}{2} D\Psi_A(x)\, y, \tag{20.7}$$

in particular, this yields $\epsilon(L_x y) = \frac{1}{2} D\Psi_A(x)\, y$ for all $x, y \in K^n$, i.e.

$$\epsilon \circ L_x = \tfrac{1}{2} D\Psi_A(x). \tag{20.8}$$

This yields that

$$\begin{aligned} \det(D\Psi_A(c) - I) &= \det(2\epsilon \circ L_c - I) \\ &= 2^n \det(\epsilon \circ L_c - \tfrac{1}{2} I) \\ &= 2^n \chi_c(\tfrac{1}{2}). \end{aligned} \tag{20.9}$$

All the corresponding relations above, of course, are valid as well for $D\Psi_{A_{\mathbb{C}}}$.

Now, suppose that A is generic. Since the number of idempotents in $A_{\mathbb{C}}$ is maximal (equal to Bézout's number 2^n) and all idempotents are distinct, it follows that all solutions of (20.6) are regular points, see [10, Sec. 8], [31, Sec. 4], therefore

$$\det(D\Psi_A(c) - I) \neq 0, \qquad \forall c \in \operatorname{Idm}(A),$$

therefore, it follows from (20.9) that for any idempotent $c \in \operatorname{Idm}(A)$: $\frac{1}{2} \notin \sigma(c)$.

In the converse direction, let $\frac{1}{2} \notin \sigma(A)$ and let $A_{\mathbb{C}}$ does not contain nonzero 2-nilpotents. Arguing by contradiction, let $A_{\mathbb{C}}$ have either (i) multiple idempotents or (ii) infinitely many idempotents. Then (i) and (ii) are respectively equivalent to saying that Eq. (20.6) has (i) multiple solutions and (ii) infinitely many solutions.

Now, if (i) holds then there is a multiple solution x of (20.6) representing a multiple idempotent $c = \epsilon(x) \in \operatorname{Idm}(A_{\mathbb{C}})$. Then $\det f_A(c) = \det(D\Psi_A(c) - I) = 0$, thus (20.9) implies $\chi_c(\frac{1}{2}) = 0$, a contradiction. Next, suppose that (ii) holds and let $E := \{x_k\}_{1 \le k \le \infty} \subset A_{\mathbb{C}}$ be a countable subset of distinct solutions of (20.6). Let us equip $A_{\mathbb{C}}$ with an Euclidean metric $\|x\|$. If the set E is unbounded then there exists a subsequence (we denote it by x_k again) such that $x_k \to \infty$ as $k \to \infty$, therefore we have from (20.6) that $\lim_{k\to\infty} \Psi_A(x_k/\|x_k\|) = 0$. This proves by the standard compactness argument that there exists a unit vector $y \in A_{\mathbb{C}}$, $\|y\| = 1$ (an accumulation point of $x_k/\|x_k\|$) such that $\Psi_A(y) = 0$, i.e. $y^2 = 0$ on the algebra level. The latter means that y is a 2-nilpotent, a contradiction. Finally, if the sequence x_k is bounded then one can find a finite accumulation point, say, $z \in A_{\mathbb{C}}$ which is the limit of a subsequence of x_k. Clearly, z is a solution of (20.6), therefore $\epsilon(z)$ is an idempotent of $A_{\mathbb{C}}$. It also easily follows that z is a *non-isolated* solution of (20.6), hence

$$0 = \det(D\Psi_A(z) - I) = 2^n \chi_z(\tfrac{1}{2})$$

a contradiction again. The theorem is proved. □

Remark 20.3.4 The proof of Proposition 20.3.3 is also valid for the noncommutative case. But in this case one should require that the spectrum of the symmetrized multiplication $L_c + R_c$ does not contain 1.

It is interesting to point out here that the classical examples of nonassociative algebras like Jordan and power-associative algebras are non-generic: indeed they have $\frac{1}{2}$ in the Peirce spectrum. The same property are shared by the Hsiang algebras mentioned in Sect. 20.1. Furthermore, it was recently remarked in [12, 26] that the classification of axial algebras depends very much on the inclusion $\frac{1}{2} \in \sigma(A)$.

It was already pointed out that the non-generic case is essentially equivalent to the inclusion $\frac{1}{2} \in \sigma(A_{\mathbb{C}})$ except for the case when $A_{\mathbb{C}}$ contains nonzero 2-nilpotents. The latter situation is still close to the generic case: indeed, one can prove that the syzygies in Theorem 20.4.1 are also valid with some mild restrictions. But the case $\frac{1}{2} \in \sigma(A_{\mathbb{C}})$ is really peculiar because in that case normally $A_{\mathbb{C}}$ contains multiple or infinite number of idempotents. In fact, it follows from Bezóut's theorem that $A_{\mathbb{C}}$ contains some varieties of idempotents.

Remark 20.3.5 In the case when an algebra A over $\mathbb{R}$ or $\mathbb{C}$ admits a topological structure consistent with the multiplicative structure of A, it is also interesting to study the path connectivity between idempotents, see for example [3, 8]. It turns out that the inclusion $\frac{1}{2} \in \sigma(A)$ is also crucial here. In particularly, in [8], Esterle proves that two homotopic idempotents may always be connected by a polynomial idempotent-valued path.

We illustrate the latter remark by the following simple observation. Let A be a commutative algebra over $\mathbb{R}$ containing a smooth path of idempotents (homotopic idempotents), i.e. $c = c(t) \in \mathrm{Idm}(A)$, $t \in \Delta \subset \mathbb{R}$. Then differentiating $c^2(t) = c(t)$ with respect to t yields $c(t)c'(t) = \frac{1}{2}c'(t)$, thus $\frac{1}{2} \in \sigma(c(t))$ as long as $c(t)$ is regular at t. In fact, a stronger property holds.

Proposition 20.3.6 *Let A be a commutative finite dimensional algebra over a field K. If there are idempotents $c_1, c_2 \in \mathrm{Idm}(A)$ such that $\alpha c_1 + (1-\alpha)c_2 \in \mathrm{Idm}(A)$ for some $\alpha \in K$ with $\alpha(1-\alpha) \neq 0$ then $\alpha c_1 + (1-\alpha)c_2 \in \mathrm{Idm}(A)$ for all $\alpha \in K$ and $c_1 - c_2 \in \mathrm{Nil}_2(A)$. In particular, if $\mathrm{Nil}_2(A) = 0$ then any three distinct nonzero idempotents spans a two-dimensional subspace.*

Proof We have

$$(\alpha c_1 + (1-\alpha)c_2)^2 = \alpha c_1 + (1-\alpha)c_2, \tag{20.10}$$

therefore $2\alpha(1-\alpha)c_1c_2 = \alpha(1-\alpha)(c_1+c_2)$, implying by the made assumption that $c_1 + c_2 = 2c_1c_2$, or equivalently $(c_1-c_2)^2 = 0$, hence $c_1 - c_2 \in \mathrm{Nil}_2(A)$. It also follows that (20.10) holds true for all $\alpha \in K$, as desired. □

20.4 Syzygies in Generic NA Algebras

In this section we show that a commutative algebra cannot have an arbitrary spectrum. More precisely, if $\frac{1}{2} \notin \sigma(A)$ then there exists $n = \dim A$ nontrivial identities on $\sigma(A)$. This remarkable phenomenon sheds a new light on the spectral properties of many well-established examples. We discuss these in more detail in Sect. 20.6 below.

20.4.1 The Principal Syzygies

We need the following version of the celebrated Euler-Jacobi formula which gives an algebraic relation between the critical points of a polynomial map and their indices, see [2, p. 106] (see also Theorem 4.3 in [4]).

Theorem (Euler-Jacobi Formula) *Let $F(x) = (F_1(x), \ldots, F_n(x))$, $x \in K^m$, be a polynomial map and let $\widetilde{F}(x)$ be the polynomial map, whose components are the highest homogeneous terms of the components of $F(x)$. Denote by $S_{\mathbb{C}}(F)$ the set of all complex roots of $F_1(x) = F_2(x) = \ldots = F_n(x) = 0$ and suppose that any root $a \in S_{\mathbb{C}}(F)$ is simple and, furthermore, that $S_{\mathbb{C}}(\widetilde{F}) = \{0\}$. Then, for any polynomial h of degree less than the degree of the Jacobian: $\deg h < N = -n + \sum_{i=1}^{n} \deg F_i$, one has*

$$\sum_{a \in S(F)} \frac{h(a)}{\det[DF(a)]} = 0 \qquad (20.11)$$

where $D(\cdot)$ denotes the Jacobi matrix.

Now, let A be a commutative nonassociative algebra over K. Using the notation of Sect. 20.2, associate to the multiplicative structure on A the bilinear map Ψ_A by (20.1) such that the multiplication in the algebra ϵ-conjugates with the Jacobi map: $\epsilon \circ L_x = \frac{1}{2} D\Psi_A$. In this setting, the coordinatization $x = \epsilon(c)$ of an arbitrary idempotent $c \in \mathrm{Idm}(A)$ is a fixed point of the quadratic map $\Psi_A(x)$ and vice versa, any fixed point of $\Psi_A(x)$ gives rise to an idempotent of A. Then in the notation of the Euler-Jacobi Formula and (20.6) we have

$$\epsilon(\mathrm{Idm}(A)) = S_K(f_A). \qquad (20.12)$$

Similarly, the set of 2-nilpotents of A coincides with the set of solutions of the reduced system $\widetilde{f}_A \equiv \Psi_A$:

$$\epsilon(\mathrm{Nil}_2(A)) = S_K(\widetilde{f}_A) = S_K(\Psi_A).$$

Furthermore, we have from Theorem 20.3.3 that an idempotent $c \in \mathrm{Idm}(A)$ is a regular point of the map f_A if and only if

$$\det Df_A(c) = 2^n \chi_c(\tfrac{1}{2}) \neq 0. \tag{20.13}$$

Now we are ready to prove the main result of this section.

Theorem 20.4.1 *Let A be a generic commutative nonassociative algebra over K,* $\dim A = n$. *Then*

$$\sum_{c \in \mathrm{Idm}_0(A)} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n, \quad \forall t \in \mathbb{R}. \tag{20.14}$$

In particular,

$$\sum_{c \in \mathrm{Idm}_0(A)} \frac{\chi_c^{(k)}(\frac{1}{2})}{\chi_c(\frac{1}{2})} = 0, \quad k = 1, 2, \ldots, n \tag{20.15}$$

where $\chi^{(k)}$ denotes the k-th derivative of χ.

Proof In notation of the Euler-Jacobi Formula, we have $F(x) = f_A(x)$, $\widetilde{F}(x) = \Psi_A(x)$. Since A is generic, it has exactly 2^n distinct idempotents, thus they are all regular points of $f_A(x)$, in particular, (20.13) holds for any $c \in \mathrm{Idm}(A)$. Since A is generic, we also have from (20.12)

$$S_K(f_A) = S_{\mathbb{C}}(f_A) = \epsilon(\mathrm{Idm}(A)) \quad \text{and} \quad S_K(\widetilde{f}_A) = \epsilon(\mathrm{Nil}_2(A)) = \{0\}.$$

Furthermore, the condition on h reads in the present notation as

$$\deg h < N = -n + \sum_{i=1}^{n} \deg F_i = -n + 2n = n.$$

Therefore, combining the Euler-Jacobi Formula with (20.13), we obtain for any polynomial h of degree $\leq n-1$ in the variables $x_1, \ldots, x_n$ that

$$0 = \sum_{c \in \mathrm{Idm}(A)} \frac{h(x_c)}{\det[Df_A(x_c]} = \frac{1}{2^n} \sum_{c \in \mathrm{Idm}(A)} \frac{h(x_c)}{\chi_c(\frac{1}{2})} \tag{20.16}$$

where $x_c \in K^n$ is defined by $\epsilon(x_c) = c$, and c runs over all idempotents in $\mathrm{Idm}(A)$.

Let us rewrite the shifted characteristic polynomial as follows:

$$(-1)^n\chi_c(t-\tfrac{1}{2}) = t^n - a_1t^{n-1} + \ldots + (-1)^n a_n, \qquad a_k = a_k(c).$$

Then each a_k is an elementary symmetric function of the roots $t_1, \ldots, t_n$ of $P(t)$. By Newton's identities, the coefficient a_k is also expressible as a linear combination of power sums

$$p_i = p_i(c) = t_1^i + \ldots + t_n^i.$$

For example,

$$\begin{aligned} a_1 &= T_1(p_1) := p_1 \\ a_2 &= T_2(p_1, p_2) := \tfrac{1}{2}(p_1^2 - p_2) \\ a_3 &= T_3(p_1, p_2, p_3) := \tfrac{1}{6}(p_1^3 - 3p_1p_2 + 2p_3), \ldots \end{aligned}$$

Each polynomial $T_k(p_1, \ldots, p_s)$ has homogeneous degree s in the sense that all monomials $p_1^{m_1} \cdots p_k^{m_k}$ in T_k has the total degree $k = m_1 + 2m_2 + \ldots + km_k$.

Next, the power sums can be evaluated as the successive traces of $L_c - \frac{1}{2}I$:

$$p_k(c) = t_1^k + \ldots + t_n^k = \mathrm{tr}(L_c - \tfrac{1}{2}I)^k =: \tau_k.$$

Therefore, $a_k = T_k(\tau_1, \ldots, \tau_k)$. Now, let us define $h(x)$ in the Euler-Jacobi Formula above by

$$h_k(x) = T_k(\mathrm{tr}\, Df_A(x), \ldots, \mathrm{tr}(Df_A(x))^k), \quad 0 \le k \le n-1.$$

Note that the entries of the Jacobi matrix $Df_A(x)$ are linear functions in the variables x_i, thus $\deg h_k = k$, which is consistent with the degree condition in the Euler-Jacobi Formula for all $0 \le k \le n-1$. By (20.8) and the homogeneity we have

$$h_k(x_c) = T_k(\mathrm{tr}\, Df_A(x_c), \ldots, \mathrm{tr}(Df_A(x_c))^k) = 2^k a_k(c)$$

therefore applying (20.16) we obtain

$$\sum_{c\in \mathrm{Idm}(A)} \frac{h_k(x_c)}{\chi_c(\frac{1}{2})} = \sum_{c\in \mathrm{Idm}(A)} \frac{a_k(c)}{\chi_c(\frac{1}{2})} = 0, \qquad 0 \le k \le n-1. \tag{20.17}$$

Since $a_k(c) = b_k\chi_c^{(k)}(\frac{1}{2})$, where $b_k = (-1)^k/(n-k)!$ does not depend on c, we derive the identities for the derivatives (20.15). Also, using Taylor's expansion $\chi_c(t) = \sum_{k=0}^n \frac{1}{k!}\chi_c^{(k)}(\frac{1}{2})(t-\frac{1}{2})^k$ yields (20.14). □

Theorem 20.4.1 describes the so-called symmetric syzygies, i.e. when the numerator in (20.14) is a symmetric function of eigenvalues of each c. It is also convenient to have general scalar and vector syzygies. These are given in the proposition below.

Proposition 20.4.2 *Under conditions of Theorem 20.4.1, let $H(x) : K^n \to K^s$ be a vector-valued polynomial map ($s \geq 1$) such that for each coordinate $\deg H_i \leq n-1$, $1 \leq i \leq n$. Then*

$$\sum_{c\in \mathrm{Idm}_0(A)} \frac{H(x_c)}{\chi_c(\frac{1}{2})} = 0, \tag{20.18}$$

where $x_c \in K^n$ is defined by $\epsilon(x_c) = c$. In particular,

$$\sum_{c\in \mathrm{Idm}(A)} \frac{c}{\chi_c(\frac{1}{2})} = 0, \tag{20.19}$$

Proof The first identity is just a corollary of (20.16). To prove (20.19), we apply (20.18) for $H(x) = x$ followed by homomorphism ϵ. □

Corollary 20.4.3 *Under conditions of Theorem 20.4.1*

$$\sum_{c\in \mathrm{Idm}(A)} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = 2^n(1-t^n) \tag{20.20}$$

and

$$\sum_{c\in \mathrm{Idm}(A)} \frac{\widetilde{\chi}_c(t)}{\widetilde{\chi}_c(\frac{1}{2})} = 2^{n-1}(1+t+\ldots+t^{n-1}), \qquad \textit{where} \quad \widetilde{\chi}_c(t) = \frac{\chi_c(t)}{t-1}. \tag{20.21}$$

Proof Since $\chi_0(t) = t^n$, (20.20) follows from (20.14). Next, since $1 \in \sigma(c)$ for all idempotents c, one can factorize $\chi_c(t) = (t-1)\widetilde{\chi}_c(t)$ so that (20.20) yields (20.21). □

Remark 20.4.4 Some remarks concerning the number of independent syzygies is in order. Note first that we do not study this question in details because it requires a more careful analysis even in the generic case. Formally, it may be thought that the number of syzygies S is the degree of the polynomial identity in (20.14) minus the tautological identity obtained when $t = \frac{1}{2}$, i.e. $S = n^2 - 1$. This is true, for example, for three scalar syzygies of the generic two-dimensional algebras, see Sect. 20.4.3 below. In fact, the number of nontrivial syzygies is sometimes less than $n^2 - 1$, see the discussion of unital algebras in the next section. In fact, any a priori assumption on the algebra structure such as the existence of unity, an algebra identity etc, of course, decreases the number of possible ('extra') syzygies defined by (20.14) or (20.21). This question deserves a separate study.

20.4.2 Syzygies in Unital Generic Algebras

Let us consider a unital commutative algebra A. Then the unit e is also a (nonzero) idempotent. In fact, as we shall see below, the existence of a unit decreases the number of nontrivial syzygies. This follows from the fact that the spectrum of each idempotent in a unital algebra is partially prescribed. Indeed, first note that there is a natural involution map on the set of idempotents in the algebra A: $\bar{c} := e - c$ is an idempotent if and only c is (the idempotent $\bar{c}$ is called the conjugate to c). Then

$$c\bar{c} = c(e - c) = c - c = 0,$$

i.e. each nontrivial (i.e. distinct from the unit and the zero elements) idempotent has at least the eigenvalues 1 and 0 in its spectrum:

$$\{0, 1\} \subset \sigma(c). \tag{20.22}$$

Furthermore, if $\dim A = n$ then the corresponding characteristic polynomials are obviously related as follows:

$$\chi_{\bar{c}}(t) = (-1)^n \chi_c(1 - t). \tag{20.23}$$

For example, $\chi_0(t) = t^n$ and $\chi_e(t) = (t - 1)^n$.

Suppose now that A is generic. Then it has exactly 2^n distinct idempotents (including the zero and the unit elements). Observe that $\bar{c} \neq c$ because otherwise $c = c^2 = c\bar{c} = 0$ implying $c = 0$, and on the other hand, $c = \bar{c} = e - c = e$, a contradiction. Thus, the conjugation $c \to \bar{c}$ splits up the set of all idempotents $\mathrm{Idm}_0(A)$ into 2^{n-1} *distinct* pairs of idempotents. Let

$$\mathrm{Idm}^+(A) := \{c_0 = 0, c_1, \ldots, c_{2^{n-1}-1}\}$$

be set of some representatives of the pairs (of course, this choice is not unique).

Proposition 20.4.5 *Let A be a unital generic algebra of dimension $n \geq 2$ and let* $\mathrm{Idm}^+(A)$ *be set of some representatives of the pairs. Then*

$$\sum_{c \in \mathrm{Idm}^+(A)} \frac{\chi_c(\frac{1}{2} + s) + \chi_c(\frac{1}{2} - s)}{\chi_c(\frac{1}{2})} = 2^n. \tag{20.24}$$

Proof We have from (20.23) for any $c \in \mathrm{Idm}^+(A)$ that

$$\chi_{\bar{c}}(\tfrac{1}{2} + s) = (-1)^n \chi_c(\tfrac{1}{2} - s), \tag{20.25}$$

and therefore

$$\chi_{\bar{c}}(\tfrac{1}{2}) = (-1)^n \chi_c(\tfrac{1}{2}).$$

Therefore

$$\begin{aligned} P_A(s) &:= -2^n + \sum_{c \in \mathrm{Idm}_0(A)} \frac{\chi_c(t)}{\chi_c(\frac{1}{2})} \\ &= -2^n + \sum_{c \in \mathrm{Idm}^+(A)} \frac{1}{\chi_c(\frac{1}{2})}(\chi_c(\tfrac{1}{2}+s) + \chi_c(\tfrac{1}{2}-s)) \end{aligned}$$

is an *even* polynomial. Also, according to (20.14) $P(s) \equiv 0$, as desired. □

Below we consider some applications for small dimensions. First suppose that $n = 2$. Then $\mathrm{Idm}^+(A) = \{c\}$, where $\mathrm{Idm}_0(A) = \{0, e, c, \bar{c}\}$, and $\chi_c(t) = (t-1)(t-\lambda)$, i.e. $\chi_c(\frac{1}{2}+s) = (s-\frac{1}{2})(s+\frac{1}{2}-\lambda)$. Applying (20.24) yields

$$\frac{\chi_c(\frac{1}{2}-s) + \chi_c(\frac{1}{2}+s)}{\chi_c(\frac{1}{2})} = 4 - \frac{\chi_0(\frac{1}{2}-s) + \chi_0(\frac{1}{2}+s)}{\chi_0(\frac{1}{2})} \equiv 2(1-4s^2),$$

therefore $8s^2\lambda = 0$, hence $\lambda = 0$ and $\chi_c(t) = (t-1)t$. The latter conclusion can also easily be derived directly from (20.22).

Next consider the case $n = 3$. Then $\mathrm{Idm}^+(A) = \{0, c_1, c_2, c_3\}$ and (20.24) yields

$$\sum_{i=1}^{3} \frac{\chi_{c_i}(\frac{1}{2}-s) + \chi_{c_i}(\frac{1}{2}+s)}{\chi_{c_i}(\frac{1}{2})} = 8 - \frac{\chi_0(\frac{1}{2}-s) + \chi_0(\frac{1}{2}+s)}{\chi_0(\frac{1}{2})} \equiv 6(1-4s^2). \tag{20.26}$$

Since $n = 3$ and (20.22), the characteristic polynomials of c_i is $\chi_{c_i}(t) = t(t-1)(t-\alpha_i)$, $i = 1, 2, 3$, $\alpha_i \in \mathbb{C}$. An easy analysis shows that (20.26) holds identically.

This shows that *there are no nontrivial syzygies (on the eigenvalues) in a 3-dimensional unital algebra*. See also (20.44) below for an example of a 3-dimensional Matsuo algebra whose algebra spectrum is $\sigma(A) = \{0, 1, \alpha, 1-\alpha\}$, $\alpha \in \mathbb{C}$.

Finally, consider $n = 4$. Then $\mathrm{Idm}^+(A) = \{0, c_1, c_2, c_3, c_4, c_5, c_6, c_7\}$ and $\chi_{c_i}(t) = t(t-1)(t-\alpha_i)(t-\beta_i)$, $1 \le i \le 7$, $\alpha_i, \beta_i \in \mathbb{C}$, hence (20.24) yields

$$\sum_{i=1}^{7} \frac{\chi_{c_i}(\frac{1}{2}-s) + \chi_{c_i}(\frac{1}{2}+s)}{\chi_{c_i}(\frac{1}{2})} = 2(1-4s^2)(4s^2+7). \tag{20.27}$$

Note that the left hand side (20.27) is an even degree polynomial, therefore (20.27) implies totally three identities. But (20.27) is satisfied identically for $s = \frac{1}{2}$ and $s = 0$. Therefore there exists only one nontrivial syzygy. One can, for example, equate the coefficients of s^4 in (20.27), which yields

$$\sum_{i=1}^{7} \frac{1}{(\frac{1}{2} - \alpha_i)(\frac{1}{2} - \beta_i)} = 4. \tag{20.28}$$

20.4.3 Spectral Theory of Two-Dimensional Algebras

The algebras in dimension two are well-understood and classified, see for example [6, 7, 24, 25, 37]. Below we revisit two-dimensional commutative algebras with emphasis on the idempotent and syzygies aspects. Our main goal is to show that in the two-dimensional case there exists essentially one nontrivial syzygy, which agrees perfectly with Remark 20.4.4 above.

We want to point out that in that case it is possible to derive the principal syzygy (20.14) by pure algebraic argument, without resorting to Bezóut's theorem, and for a field K of any characteristic (non necessarily algebraically closed).

Proposition 20.4.6 *Let A be a nonassociative commutative algebra over a field K,* $\dim_K A = 2$. *Let $c_1 \neq c_2$ and $c_i \in \mathrm{Idm}(A)$. Then either of the following holds:*

(i) $\frac{1}{2} \in \sigma(c_i)$ *for some* $i = 1, 2$*;*
(ii) *there exists exactly three distinct nonzero idempotents in A;*
(iii) *there exists exactly two distinct nonzero idempotents c_1 and c_2 and a nonzero* 2*-nilpotent;*

Proof Since $c_1 \neq c_2$, they are linearly independent, thus form a basis in A. Write $c_1c_2 = \alpha c_1 + \beta c_2$, $\alpha, \beta \in K$. Then $\sigma(c_1) = \{1, \beta\}$ and $\sigma(c_2) = \{1, \alpha\}$.

Let us assume that (i) does not hold, i.e. $\alpha, \beta \neq \frac{1}{2}$. Consider $u = xc_1 + yc_2$, $x, y \in K$. Then

$$u^2 = x(x + 2\alpha y)c_1 + y(2\beta x + y)c_2. \tag{20.29}$$

If the determinant $\begin{vmatrix} 1 & 2\alpha \\ 2\beta & 1 \end{vmatrix} = 1 - 4\alpha\beta = 0$ then $\beta = \frac{1}{4\alpha}$, hence $c_1c_2 = \alpha c_1 + \frac{1}{4\alpha} c_2$, therefore

$$(c_1 - \tfrac{1}{2\alpha} c_2)^2 = 0.$$

Therefore $0 \neq c_1 - \frac{1}{2\alpha}c_2 \in \mathrm{Nil}_2(A)$. Let us show that there are not other idempotents in A except for c_1 and c_2. Indeed, if u is such an idempotent then $xy \neq 0$ (otherwise $u = c_1$ or $u = c_2$), hence (20.29) yields

$$\begin{cases} x + 2\alpha y = 1 \\ 2\beta x + y = 1 \end{cases} \tag{20.30}$$

thus, using $\beta = \frac{1}{4\alpha}$ implies $2\alpha = 1$, a contradiction. Therefore one comes to (iii).

If the determinant is nonzero: $\Delta := 1 - 4\alpha\beta \neq 0$ then there exists a solution

$$(x, y) = \Big(\frac{1 - 2\alpha}{\Delta}, \frac{1 - 2\beta}{\Delta}\Big)$$

of (20.30) which implies $u \in A$ such that $u^2 = u$. Note that by the assumption $(1 - 2\alpha)(1 - 2\beta) \neq 0$, hence $xy \neq 0$, i.e. u is distinct from c_1, c_2. Therefore, we have three distinct idempotents, i.e. (ii). It also follows that in that case there exists exactly three idempotents. □

Now, let us consider the case of a *generic* algebra of dimension 2.

Theorem 20.4.7 *Let A be a nonassociative commutative algebra over a field K with* $\dim_K A = 2$. *Suppose that there exists three distinct nonzero idempotents c_i,* $i = 1, 2, 3$. *Then*

$$4\lambda_1\lambda_2\lambda_3 - \lambda_1 - \lambda_2 - \lambda_3 + 1 = 0, \tag{20.31}$$

where $\sigma(c_i) = \{\lambda_i\}$.

Proof Any pair of distinct idempotents is a basis of A. Decompose $c_i c_j = xc_i + yc_j$ for $i \neq j$. Then $\sigma(c_i) = \{1, y\}$ and $\sigma(c_j) = \{1, x\}$. This yields

$$c_1c_2 = \lambda_2 c_1 + \lambda_1 c_2, \tag{20.32}$$

$$c_2c_3 = \lambda_3 c_2 + \lambda_2 c_3, \tag{20.33}$$

$$c_3c_1 = \lambda_1 c_3 + \lambda_3 c_1, \tag{20.34}$$

for some $\lambda_i \in K$. In particular, this implies that each idempotent is semi-simple and $\sigma(c_i) = \{1, \lambda_i\}$, $i = 1, 2, 3$.

Next, by our assumption c_1, c_2, c_3 are distinct, hence

$$c_3 = a_1c_1 + a_2c_2, \qquad a_1, a_2 \in K,\ \ a_1a_2 \neq 0, \tag{20.35}$$

hence substituting the latter identity in (20.34)

$$(a_1c_1 + a_2c_2)c_1 = \lambda_1(a_1c_1 + a_2c_2) + \lambda_3 c_1,$$

one finds by virtue of (20.32) that $(a_1(1-\lambda_1)+a_2\lambda_2-\lambda_3)c_1=0$ implying

$$\lambda_3=(1-\lambda_1)a_1+\lambda_2 a_2. \tag{20.36}$$

Arguing similarly with (20.33) one arrives at

$$\lambda_3=\lambda_1 a_1+(1-\lambda_2)a_2. \tag{20.37}$$

This yields

$$a_1(1-2\lambda_1)=a_2(1-2\lambda_2).$$

Let first $\lambda_1=\frac{1}{2}$. Then by the assumption $a_2\neq 0$, hence $\lambda_2=\frac{1}{2}$, which obviously satisfies (20.31) for any λ_3. Next, if $\lambda_1\neq\frac{1}{2}$ then $\lambda_2\neq\frac{1}{2}$ and we have

$$\frac{a_2}{a_1}=\frac{1-2\lambda_1}{1-2\lambda_2}. \tag{20.38}$$

On the other hand, rewriting (20.35) as

$$c_1=-\frac{1}{a_1}c_3+\frac{a_2}{a_1}c_2=b_3c_3+b_2c_2,$$

we obtain for symmetry reasons that $-a_2=\frac{b_2}{b_3}=\frac{1-2\lambda_3}{1-2\lambda_2}$, hence from (20.38) we also have $-a_1=\frac{1-2\lambda_3}{1-2\lambda_1}$. Summing up (20.36) and (20.37) we obtain

$$2\lambda_3=a_1+a_2=-(1-2\lambda_3)\left(\frac{1}{1-2\lambda_1}+\frac{1}{1-2\lambda_2}\right) \tag{20.39}$$

which readily yields (20.31). The proposition follows. □

Remark 20.4.8 For any triple $\lambda_1,\lambda_2,\lambda_3$ satisfying (20.31), it is easy to construct an algebra with three idempotents having the spectrum $\sigma(c_i)=\{1,\lambda_i\}$. The relation (20.31) (as well as (20.41) below) appears in the classification of rank three algebras by S. Walcher, see [37, p. 3407]. See also our recent discussion in connection to idempotent geometry in [16].

Now we show that (20.31) is actually equivalent to the principal syzygies (20.15). To this end, note that if $\lambda_i\neq\frac{1}{2}$ then (20.39) yields also another form of (20.31), namely

$$\frac{1}{1-2\lambda_1}+\frac{1}{1-2\lambda_2}+\frac{1}{1-2\lambda_3}=1. \tag{20.40}$$

Taking into account that $\chi_{c_0}(t) = t^2$, where $c_0 = 0$ is the trivial idempotent, the latter equation can be written as

$$\frac{1}{\chi_{c_0}(\frac{1}{2})} + \frac{1}{\chi_{c_1}(\frac{1}{2})} + \frac{1}{\chi_{c_2}(\frac{1}{2})} + \frac{1}{\chi_{c_3}(\frac{1}{2})} = 0. \tag{20.41}$$

This yields the syzygy in (20.15) for $k = n = 2$. Another (the last for $n = 2$) possibility is $k = 1$ when (20.15) becomes

$$\sum_{i=0}^{3} \frac{\chi'_{c_i}(\frac{1}{2})}{\chi_{c_i}(\frac{1}{2})} = 0. \tag{20.42}$$

Since $\chi'_{c_i}(\frac{1}{2}) = -\lambda_i$ for $i \neq 0$ and $\chi'_{c_0}(\frac{1}{2}) = 1$, one readily verifies that (20.42) is in fact equivalent to (20.41).

We have two further corollaries of (20.41).

Corollary 20.4.9 *Let A be a nonassociative commutative algebra over a field K, $\dim_K A = 2$, and let c_i, $i = 1, 2, 3$ be three nonzero idempotents. If $\frac{1}{2} \in \sigma(c_i)$ for some i then at least one of the remained idempotents has the same property.*

Proof Indeed, let $S(\lambda_1, \lambda_2, \lambda_3)$ denote the left hand side of (20.31). Then S is irreducible in $\mathbb{C}[\lambda_1, \lambda_2, \lambda_3]$. But if $\lambda_i = \frac{1}{2}$ for some i, say $\lambda_3 = \frac{1}{2}$ then S factorizes as follows:

$$S(\lambda_1, \lambda_2, \frac{1}{2}) = \frac{1}{2}(2\lambda_1 - 1)(2\lambda_2 - 1)$$

This yields the desired conclusion. □

Corollary 20.4.10 *Given two linearly independent idempotents e_1, e_2 in a two dimensional non associative algebra A over a field K, its multiplication table contains explicitly the Peirce numbers of e_1, e_2.*

Proof It follows immediately from (20.32)–(20.34). □

Remark 20.4.11 Such a multiplication table structure may be recognized as a *diagonal*, where the spectral parameters are presented explicitly.

20.5 Algebras with a Prescribed Peirce Spectrum

The spectrum of any nonzero idempotent contains 1. If algebra is unital then the spectrum of any nontrivial idempotent contains 0 and 1 (see Sect. 20.4.2). There are many algebras sharing a remarkable property: the spectrum of all or a 'large' subset of idempotents is constant or contains some prescribed values. It is interesting to know which common properties such algebras have.

20.5.1 Isospectral and Nearly Isospectral Algebras

The first natural and nontrivial example which can be treated by virtue of the constructed syzygies are generic algebras with constant spectrum.

Definition 20.5.1 An algebra is said to be *isospectral* if all nonzero idempotents have the same spectrum (counting multiplicities).

One such family is the Hsiang (or REC) algebras, see [22, 35, 36, sec. 6]. More precisely, a Hsiang algebra A is a commutative algebra over $\mathbb{R}$ with a symmetric bilinear form $\langle;\rangle$ (see the definition (20.48) below) such that for any element $x \in A$ two following identities hold:

$$\langle x^2; x^3\rangle = \langle x; x^2\rangle\langle x; x\rangle, \quad \operatorname{tr} L_x = 0.$$

It can be proved that the spectrum of *any* idempotent in A is constant and contains only the eigenvalues ± 1 and $\pm\frac{1}{2}$ (with certain multiplicities independent on a choice of an idempotent). However, since $\frac{1}{2} \in \sigma(c)$, Hsiang algebras are *not* generic (any Hsiang algebra contains infinitely many idempotents). Therefore, the above syzygies are not applicable here.

Nevertheless, in two dimensions an algebra with constant spectrum can be easily constructed. Let us consider a commutative two-dimensional algebra generated by e_1 and e_2 with identities

$$e_1^2 = -e_2^2 = e_1, \quad e_1e_2 = e_2e_1 = -e_2.$$

A simple analysis reveals that there is exactly $4 = 2^2$ idempotents:

$$c_0 = 0, \quad c_1 = e_1, \quad \text{and } c_{2,3} = -\tfrac{1}{2}e_1 \pm \tfrac{\sqrt{3}}{2}e_2.$$

In particular, A is a generic algebra over $\mathbb{R}$. It is easily verified that the algebra A possesses the constant spectrum property: the spectrum of any *nonzero* idempotent is the same:

$$\sigma(c_i) = \{1, -1\}, \quad i = 1, 2, 3.$$

One can readily prove that in dimension 2 any commutative algebra with the above property is necessarily isomorphic to A.

In the general case, one has the following observation.

Corollary 20.5.2 *If A is a generic algebra, $n = \dim A$, such that all nonzero idempotents has the same spectrum then for any $c \in \mathrm{Idm}_0(A)$, $\chi_c(t) = (-1)^n(t^n - 1)$. In other words, if A is such an algebra then*

$$\sigma(c) = \{e^{\frac{2\pi k\sqrt{-1}}{n}}, \quad k = 0, 1, 2, \dots, n-1\} \tag{20.43}$$

for any idempotent $c \neq 0$.

Proof Since the characteristic polynomial $\chi_c(t)$ is the same for all nonzero idempotents c then using (20.20) we obtain that

$$\frac{\chi_c(t)}{\chi_c(\frac{1}{2})} = \frac{2^n(1-t^n)}{2^n-1}$$

and the desired conclusion follows immediately from the last identity. □

It is interesting to know whether an algebra satisfying the conditions of Corollary 20.5.2 is realizable for any $n \geq 3$. The following example shows that this holds at least for $n = 3$.

Example 20.5.3 Let us consider a three dimensional commutative algebra over $\mathbb{C}$ generated by *idempotents* c_1, c_2, c_3 with the multiplication table

$$c_i c_j = \alpha c_i + \beta c_j + \gamma c_k, \qquad \text{where } (i, j, k) \text{ is a } \textit{cyclic} \text{ permutation of } (1, 2, 3),$$

with the structure constants

$$\alpha = \gamma - 1, \quad \beta = -\gamma, \quad \text{where } \gamma = \tfrac{1}{4} - \tfrac{1}{4}\sqrt{-7}$$

The algebra A is generic because, except for the basis idempotents c_i, there exists exactly $4 = 7 - 3$ nonzero idempotents, namely

$$\begin{aligned}
c_4 &= -\gamma(c_1 + c_2 + c_3)\\
c_5 &= (\gamma - 1)c_1 - \gamma c_2 + \gamma c_3,\\
c_6 &= \gamma c_1 + (\gamma - 1)c_2 - \gamma c_3,\\
c_7 &= -\gamma c_1 + \gamma c_2 + (\gamma - 1)c_3.
\end{aligned}$$

A straightforward verification shows that all idempotents have the same spectrum

$$\sigma(c_i) = \{1, \tfrac{1}{2}(-1 \pm \sqrt{-3}\}, \quad \forall c_i \in \mathrm{Idm}_0(A).$$

Note that the spectrum is constant and its elements are exactly the three roots of $z^3 - 1 = 0$ given in (20.43). Furthermore, it can be seen the validity of syzygies (20.19):

$$\sum_{i=1}^{7} c_i = 0.$$

One can prove that any three-dimensional generic algebra satisfying conditions of Corollary 20.5.2 is necessarily isomorphic to the above algebra.

Remark 20.5.4 The above expressions for idempotents do not look very illuminating, but closer inspection reveals some structure. In fact, one can see that the multiplication in an algebra constructed in Example 20.5.3 satisfies the *medial quasigroup identity* [18, p. 270]: $(xy)(zw) = (xz)(yw)$. We discuss this phenomenon in more details and show that there exist isospectral algebras in any dimension in a forthcoming paper [17].

Remark 20.5.5 Dropping the requirement that the algebra A is generic, yields many other families of nonassociative isospectral or nearly isospectral algebras. See for example an algebra with finitely many idempotents given in Sect. 20.6.10 below.

Let us relax the constant spectrum property and consider a generic algebra such that all idempotents has a common value in the spectrum.

Corollary 20.5.6 *Let A be a generic isospectral algebra. If $\alpha \in \sigma(c)$ for all nonzero $c \in \mathrm{Idm}_0(A)$ then $\alpha^n = 1$ and $\alpha \neq 1$.*

Proof If $\alpha \neq 1$ is a common value of the spectrum $\sigma(c)$ for all $c \neq 0$ then it is a common root of all $\widetilde{\chi}_c(t)$ in (20.21), hence $t - \alpha$ divides the right hand side of (20.21), implying the desired conclusion. □

20.5.2 Algebras with a Thin Spectrum

It was already mentioned in Sect. 20.1, that many interesting examples of nonassociative algebras share another characteristic property: the spectrum of each idempotent consists only of few distinct prescribed eigenvalues. The classically known model examples here are associative algebras with the Peirce spectrum $\{0, 1\}$ and power-associative and Jordan algebras with the Peirce spectrum $\{0, \frac{1}{2}, 1\}$. In general, if an algebra satisfies an identity of lower degree then its Peirce spectrum is 'thin' and its explicit form can be characterized effectively, see [36].

Another prominent example is the 196,883-dimensional nonassociative commutative Griess-Norton algebra $\mathbb{M}$ [11] whose authomorphism group is the Monster, the largest sporadic simple group. The algebra $\mathbb{M}$ are generated by idempotents with the Peirce spectrum $\{0, 1, \frac{1}{4}, \frac{1}{32}\}$ [13]. There are many interesting subalgebras emerging in the context of $\mathbb{M}$, for example, the Matsuo algebra $3C_\alpha$ considered below. It is a particular case of the so-called Matsuo algebras family [21, 27].

More precisely, let us consider the three-dimensional algebra $A = 3C_\alpha$ over a field K containing α and $\mathrm{char}(K) \neq 2$ spanned by three idempotents e_1, e_2, e_3 subject to the algebra identities

$$e_i e_j = \tfrac{\alpha}{2}(e_i + e_j - e_k), \quad \{i, j, k\} = \{1, 2, 3\}. \tag{20.44}$$

Then a simple analysis reveals that if $\alpha \neq -1$ and $\alpha \neq \frac{1}{2}$ then there exists exactly $7 = 2^3 - 1$ distinct nonzero idempotents, namely

$$\begin{aligned} e_7 &= \tfrac{1}{\alpha+1}(e_1 + e_2 + e_3), \\ e_{3+i} &= \bar{e}_i, \quad i = 1, 2, 3, \end{aligned}$$

where e_7 is the algebra unit and $\bar{c} = e_7 - c$ is the conjugate idempotent. In summary we have (see also [26])

Proposition 20.5.7 *If $\alpha \neq -1, \frac{1}{2}$ then the Matsuo algebra $3C_\alpha$ is a 3-dimensional generic unital algebra. Its spectrum is given as follows:*

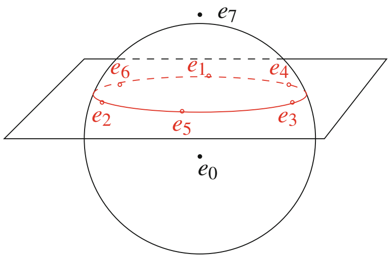
$$\sigma(e_0) = \{0, 0, 0\}, \qquad \sigma(e_7) = \{1, 1, 1\},$$
$$\sigma(e_i) = \{0, \alpha, 1\}, \qquad \sigma(\bar{e}_i) = \{0, 1-\alpha, 1\},$$

where $e_0 = 0$ and $i = 1, 2, 3$.

Remark 20.5.8 In the exceptional case $\alpha = \frac{1}{2}$, there exists an infinite family of idempotents $c_x := x_1e_1 + x_2e_2 + x_3e_3$ on the circle

$$(x_1 - \tfrac{1}{3})^2 + (x_2 - \tfrac{1}{3})^2 + (x_3 - \tfrac{1}{3})^2 = \tfrac{2}{3}, \quad x_1 + x_2 + x_3 = 1. \tag{20.45}$$

Then $e_1, e_2, e_3, e_4, e_5, e_6$ lie on the circle (20.45), while $e_0 = 0$ and the unit e_7 lies outside, see the figure below.



One readily verifies that the Peirce spectrum of all c lying in (20.45) is the same and is equal to $\sigma(c) = \{\frac{1}{2}, 1, 0\}$. In particular, the only Matsuo algebra $3C_{\frac{1}{2}}$ is a power associative.

20.5.3 The Generalized Matsuo Algebras

The last example admits the following generalization. Let us define $A = 3C_{\alpha,\varepsilon}$ being the three-dimensional algebra over K containing α,ε, char $K \neq 2$ and spanned by three idempotents e_1, e_2, e_3 subject to the algebra identities

$$e_ie_j = \frac{\alpha}{2}(e_i + e_j) + \frac{(\varepsilon - \alpha)}{2}e_k, \quad \{i, j, k\} = \{1, 2, 3\}. \tag{20.46}$$

This obviously determines a unique algebra structure on K^3. Note that

$$3C_\alpha = 3C_{\alpha,0},$$

and the new algebra structure can be thought of as a perturbation of the original structure. Under conditions (20.46), the spectrum of each e_i is found to be

$$\sigma(e_i) = \{1,\ \alpha - \tfrac{1}{2}\varepsilon,\ \tfrac{1}{2}\varepsilon\}.$$

Furthermore, a simple analysis reveals that if

$$(\alpha + 1 + \varepsilon)(\varepsilon - 1)\gamma \neq 0, \tag{20.47}$$

$$\gamma := \alpha + 1 + \varepsilon(\varepsilon - 2\alpha - 1),$$

then there exists exactly $7 = 2^3 - 1$ distinct nonzero idempotents in A, i.e. the algebra $3C_{\alpha,\varepsilon}$ is *generic*. In that case the remained four idempotents are

$$\begin{aligned} e_7 &= \frac{1}{\alpha + 1 + \varepsilon}(e_1 + e_2 + e_3), \\ e_{3+i} &= \frac{(1-\varepsilon)(\alpha + 1 + \varepsilon)}{\gamma} e_7 - \frac{\alpha + 1 - 2\varepsilon}{\gamma} e_i, \quad i = 1, 2, 3. \end{aligned}$$

One also can readily see that the idempotent e_7 is the unity element in A if and only if $\varepsilon = 0$ (in which case, the algebra $3C_{\alpha,0}$ is isomorphic to the Matsuo algebra $3C_\alpha$). The Peirce spectrum then is found to be

$$\sigma(e_7) = \{1, \mu, \mu\}, \quad \text{where } \mu = 1 - \frac{3\varepsilon}{2(1 + \alpha + \varepsilon)}$$

and

$$\sigma(e_{i+3}) = \{1,\ \frac{\varepsilon(2 - \alpha - \varepsilon)}{2\gamma},\ \frac{2(1 - \alpha^2) + \varepsilon(3\alpha - \varepsilon - 2)}{2\gamma}\}, \qquad i = 1, 2, 3.$$

In summary, for generic α and ε the Peirce spectrum of A except for 0 and 1 contains five distinct Peirce numbers.[1] It is also straightforward to see that the syzygies, for example in the form (20.21), hold for the obtained algebra spectrum.

Remark 20.5.9 In the exceptional cases $\varepsilon = 1$ and $\varepsilon = 2\alpha - 1$, there exists an infinite family of (non isolated) idempotents and for $\gamma = 0$ there are 2-nilpotent elements.

[1] A three dimensional generic algebra may a priori have $14 = (2^3 - 1) \times (3 - 1)$ distinct eigenvalues except 0 and 1.

20.6 Metrized Algebras

Note that many examples discussed in Introduction are *metrized* algebras: they obey a non-trivial bilinear form $\langle x; y \rangle$ which associates with multiplication, i.e.

$$\langle xy; z \rangle = \langle x; yz \rangle \quad \text{for all } x, y, z \in A, \tag{20.48}$$

(cf. [5], [15, p. 453]). The standard examples are Lie algebras with the Killing form and Jordan algebras with the generic trace bilinear form. We say that A is a *Euclidean* metrized algebra if the bilinear form

$$\langle x; y \rangle$$

is positive definite. Sometimes an associative bilinear form is also called a trace form [29].

The condition (20.48) is very strong and implies that the multiplication operator $L_x : y \to xy$ is self-adjoint for any $x \in A$:

$$\langle L_x y; z \rangle = \langle y; L_x z \rangle, \qquad \forall x, y, z \in V. \tag{20.49}$$

In particular, all idempotents are automatically semisimple. If additionally A is an algebra over $\mathbb{R}$ then L_x has a real spectrum for any x.

20.6.1 Algebras of Cubic Forms

The category of metrized algebras is in a natural correspondence with the category of cubic forms on vector spaces with a distinguished inner product. Since the latter is an analytic object, it has many computational advances and is a useful tool for constructing diverse examples of non-associative algebras. Below, we briefly recall the correspondence, see also [22, 33, 34].

Let V be an inner product vector space, i.e. a vector space over K endowed with a non-singular symmetric K-bilinear form $\langle x; y \rangle$. Given a cubic form $u(x)$ on V we define the multiplication by duality:

$$xy := \text{ the unique element satisfying } \langle xy; z \rangle = u(x, y, z) \text{ for all } z \in V \tag{20.50}$$

where

$$u(x, y, z) = u(x + y + z) - u(x + y) - u(x + z) - u(y + z) + u(x) + u(y) + u(z)$$

is the linearization of u. Since $\langle \cdot; \cdot \rangle$ is nonsingular, such an element xy exists and unique. Thus defined algebra is denoted by $V(u)$ if the bilinear form $\langle ; \rangle$ is obvious.

Note also that the trilinear form $u(x, y, z)$ is symmetric, hence *the algebra* $V(u)$ *is always commutative.* It follows from the homogeneity that the cubic form u is recovered by

$$6u(x) = u(x, x, x) = \langle x; x^2 \rangle. \tag{20.51}$$

Moreover, in this setting, the directional derivative (or the first linearization of u) is expressed by

$$\partial_y u(x) = \tfrac{1}{2} u(x, x, y) =: u(x; y) \tag{20.52}$$

hence the multiplication is recovered explicitly by

$$xy = D^2 u(x) y = D^2 u(y) x, \tag{20.53}$$

where $D^2 u(x)$ is the Hessian matrix of u at x. In particular, the gradient of $u(x)$ is essentially the square of the element x (in $V(u)$):

$$Du(x) = \tfrac{1}{2} xx = \tfrac{1}{2} x^2. \tag{20.54}$$

The following correspondence is an immediate corollary of the definitions.

Proposition 20.6.1 *Given a vector space* V *with a non-singular symmetric bilinear form* $\langle \cdot; \cdot \rangle$*, there exists a canonical bijection between the vector space of all cubic forms on* V *and commutative metrized algebras with the multiplication* $(x, y) \to xy$ *uniquely determined by* (20.50)*.*

Proposition 20.6.2 *The metrized algebra* $V(u)$ *is a zero algebra if and only if* $u \equiv 0$.

Proof Let $A = V(u)$ be a non-zero metrized algebra. Suppose by contradiction that $u(x) \equiv 0$, then $\langle x; x^2 \rangle \equiv 0$ and the polarization yields $\langle xy; z \rangle \equiv 0$ for all $x, y, z \in V$, thus $xy = 0$ implying $AA = \{0\}$, a contradiction. In the converse direction, if $u(x) \not\equiv 0$ then $u(x_0) = \frac{1}{6} \langle x_0; x_0^2 \rangle \neq 0$ for some x_0, which obviously yields $x_0 x_0 \neq 0$, hence $V(u)$ is a non-zero algebra. □

Proposition 20.6.3 *If* A *is a nonzero Euclidean metrized algebra then* $\mathrm{Idm}(A) \neq \emptyset$*.*

Proof Let $S = \{x \in V : \langle x; x \rangle = 1\}$ for the unit hypersphere S in V. Then S is compact in the standard Euclidean topology on V. By Proposition 20.6.2, the cubic form $u(x) = \frac{1}{6} \langle x^2; x \rangle \not\equiv 0$. Since u is continuous as a function on S, it attains its global maximum value there, say at some point $y \in V$, $\langle y; y \rangle = 1$. Since u is an odd function, the maximum value $u(y)$ is strictly positive. We have the stationary equation $0 = \partial_x u|_y$ whenever $x \in V$ satisfies the tangential condition $\langle x; y \rangle = 0$. Thus, using (20.52) we obtain

$$0 = \partial_x u|_y = 3u(y; x) = \tfrac{1}{2} \langle y^2; x \rangle$$

which implies immediately that y and y^2 are parallel, i.e. $y^2 = ky$, for some $k \in \mathbb{R}^\times$ (observe also that $k > 0$ by virtue of $0 < u(y) = \langle y^2; y\rangle = k\langle y; y\rangle$). Scaling y appropriately, namely setting $c = y/k$ yields $c^2 = c$. □

Definition 20.6.4 An idempotent $c \in V(u)$ constructed in the course of proof of Lemma 20.6.3 will be called *extremal*.

In other words, the set of extremal idempotents coincide with the set of suitably normalized global maximum points of the (degree zero homogeneous) function $\langle x; x^2\rangle\langle x; x\rangle^{-3/2}$. It follows that if c is an extremal idempotent and c_1 is an arbitrary idempotent then

$$\frac{1}{\|c\|} = \frac{\langle c; c^2\rangle}{\langle c; c\rangle^{3/2}} \ge \frac{\langle c_1; c_1^2\rangle}{\langle c_1; c_1\rangle^{3/2}} = \frac{1}{\|c_1\|},$$

i.e. the extremal idempotents have the minimal possible norm among all idempotents in A.

Remark 20.6.5 See [19] for a topological proof of the existence of an idempotent element in nonassociative algebras, and also [20] for further generalizations.

20.6.2 The Exceptional Property of $\frac{1}{2}$ in Euclidean Metrized Algebras

Proposition 20.6.6 *Let A be a Euclidean metrized algebra. Then for any extremal idempotent c there holds*

$$\sigma(c) \subset (-\infty, \tfrac{1}{2}].$$

In particular, 1 *is a simple eigenvalue of L_c.*

Proof Then $f(x) = \frac{\langle x^2; x\rangle}{|x|^3}$ is a homogeneous of degree zero function which is smooth outside the origin. We have

$$\partial_y f|_x = \frac{3(\langle x^2; y\rangle|x|^2 - \langle x^2; x\rangle\langle x; y\rangle)}{|x|^5}, \tag{20.55}$$

implying $\frac{1}{3}Df(x) = \frac{x^2|x|^2 - \langle x^2; x\rangle x}{|x|^5}$. Arguing similarly, we have for the second derivative

$$\tfrac{1}{3}\partial_z Df(x) = \frac{2|x|^4 xz - 3|x|^2(x^2\langle x; z\rangle + \langle x^2; z\rangle x) - \langle x^2; x\rangle|x|^2 z + 5\langle x; z\rangle\langle x^2; x\rangle x)}{|x|^7},$$

hence

$$\tfrac{1}{3}D^2 f(x) = \frac{2|x|^4 L_x - \langle x^2; x\rangle |x|^2 - 3|x|^2(x \otimes x^2 + x^2 \otimes x) + 5\langle x^2; x\rangle x \otimes x}{|x|^5}. \tag{20.56}$$

Now let c be an extremal idempotent of V. Then $c^2 = c$ and $D^2 f(c) \le 0$. The second condition implies

$$2L_c - 1 \le c \otimes c.$$

Since L_c is self-adjoint and $\mathbb{R}c$ is an invariant subspace of L_c: $L_c = 1$ on $\mathbb{R}c$, the orthogonal complement $c^\perp = \{x \in V : \langle x; c\rangle = 0\}$ is an invariant subspace too. Indeed, if $\langle x; c\rangle = 0$ then

$$\langle L_c x; c\rangle = \langle x; L_c c\rangle = \langle x; c\rangle = 0.$$

Therefore, using $c \otimes c = 0$ on $c^\perp = \{x \in V : \langle x; c\rangle = 0\}$ we obtain $2L_c - 1 \le 0$ there, which yields the desired conclusion. □

The presence of the exceptional value $\frac{1}{2}$ in the spectrum of a metrized algebra implies that the corresponding Peirce subspace possesses a certain fusion rule.

Proposition 20.6.7 *If c is an extremal idempotent and $\frac{1}{2} \in \sigma(c)$ then $\langle z^2; z\rangle = 0$ for all $z \in A_c(\frac{1}{2})$. In other words, the following fusion rule holds:*

$$A_c(\tfrac{1}{2})A_c(\tfrac{1}{2}) \subset A_c(\tfrac{1}{2})^\perp. \tag{20.57}$$

Proof Let $z \in A_c(\frac{1}{2})$. Using (20.55) and (20.56) for $x = c$ yields the directional derivatives respectively

$$\frac{\partial f}{\partial z}\Big|_{x=c} = \frac{\partial^2 f}{\partial z^2}\Big|_{x=c} = 0.$$

Therefore, since c is a local maximum point of $f(x) = \frac{\langle x^2; x\rangle}{|x|^3}$, we have for the higher derivatives $\frac{\partial^3 f}{\partial z^3}\big|_{x=c} = 0$ and $\frac{\partial^4 f}{\partial z^4}\big|_{x=c} \ge 0$. Using (20.56) we have

$$\frac{1}{3}\frac{\partial^2 f}{\partial z^2} = \frac{2|x|^2\langle x; z^2\rangle - 6|x|^2\langle x; z\rangle\langle x^2; z\rangle + 5\langle x^2; x\rangle\langle x; z\rangle^2 - \langle x^2; x\rangle |z|^2}{|x|^7}.$$

Differentiating the latter identity and substituting $x = c$ yields by virtue of $c^2 = c$ and $\langle c; z\rangle = 0$ that

$$\frac{1}{6}\frac{\partial^3 f}{\partial z^3}\Big|_{x=c} = \frac{\langle z^2; z\rangle}{|x|^3} = 0.$$

Then polarization of $\langle z^2; z\rangle = 0$ in $A_c(\frac{1}{2})$ yields the desired fusion rule (20.57). □

The exceptionality of $\frac{1}{2}$ has also been recently discussed in [36].

20.6.3 Some Examples of Algebras of Cubic Forms

Below, we consider some concrete examples of cubic forms on the Euclidean space $\mathbb{R}^n$ and the corresponding metrized algebras. Recall that in this case, all idempotents are semi-simple: roots of the characteristic polynomial $\chi_c(t)$ are always real and the corresponding orthogonal decomposition in eigen-subspaces is the Peirce decomposition associated with c.

Example 20.6.8 Let us consider the algebra of cubic form $u_1 = \frac{1}{6}(x_1^3 + \ldots + x_n^3)$. The algebra multiplication is determined by virtue of (20.53):

$$xy = (x_1 y_1, \ldots, x_n y_n),$$

i.e. the algebra $V(u_1)$ is reducible and isomorphic to the product $\mathbb{R} \times \ldots \times \mathbb{R}$. It is easy to see that $V(u_1)$ has exactly 2^n idempotents which coincide with the vertices of the unit cube in $\mathbb{R}^n$:

$$c_x = (x_1, \ldots, x_n), \quad \text{where } x_i = 0 \text{ or } 1,$$

The spectrum consists of two different values: $\lambda \in \{0, 1\}$. For each idempotent,

$$\sigma(c_x) = \{1^m, 0^{n-m}\}, \quad \text{where } m = x_1 + \ldots + x_n.$$

Example 20.6.9 Let us consider a perturbation of the algebra $V(u_1)$ of the cubic form from Sect. 20.6.8 for $n = 3$. Then $V(u_1) \cong \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the total spectrum (counting total multiplicities over all $8 = 2^3$ idempotents in $V(u_1)$)

$$\Big(\underbrace{0}_{12}, \underbrace{1}_{12}\Big)$$

Define a perturbation by adding the term $= \varepsilon x_1 x_2 x_3$, $\varepsilon \in \mathbb{R}$:

$$u_{1,\varepsilon}(x) = \varepsilon x_1 x_2 x_3 + \tfrac{1}{6}(x_1^3 + x_2^3 + x_3^3).$$

Then one can show that for $\varepsilon \notin \{\pm\frac{1}{2}, \frac{1}{4}, 1\}$ the new algebra is also generic and the corresponding total spectrum is

$$\Big(\underbrace{0}_{3}, \underbrace{\varepsilon}_{3}, \underbrace{\varepsilon}_{3}, \underbrace{\frac{2\varepsilon(1-\varepsilon)}{1-2\varepsilon+4\varepsilon^2}}_{3}, \underbrace{1}_{7}, \underbrace{\frac{1-\varepsilon}{1+2\varepsilon}}_{2}, \underbrace{\frac{1-2\varepsilon-2\varepsilon^2}{1-2\varepsilon+4\varepsilon^2}}_{3}\Big)$$

Another interesting case when $V(u_{1,\varepsilon})$ is still generic, is when $\varepsilon = 1$. Then the total spectrum becomes much smaller:

$$(\underbrace{0}_{8}, \underbrace{-1}_{6}, \underbrace{1}_{10})$$

In this case the algebra contains the maximal number (totally 8) idempotents with resp. spectrum

$$(0^3) \oplus (0^2, 1) \oplus 3 \cdot (0, -1, 1) \oplus 3 \cdot (-1, 1, 1)$$

Example 20.6.10 Let $u_2 = \frac{1}{2}x_1(x_3^2 - x_4^2) + ix_2x_3x_4$. The algebra $V(u_2)$ over $\mathbb{C}$ has 9 (of $16 = 2^4$ maximally possible) isolated nonzero idempotents, (normalized) 2-nilpotents that lie in the two dimensional subspace $x_3 = x_4 = 0$. *All idempotents have the same Peirce spectrum*:

$$-\tfrac{1}{4} - \tfrac{\sqrt{7}}{4},\ -\tfrac{1}{2},\ -\tfrac{1}{4} + \tfrac{\sqrt{7}}{4},\ 1$$

Note also that the trace $\operatorname{tr} L_c = 0$.

Example 20.6.11 Let us consider the algebra $V(u)$ of the cubic form

$$u(x) = \tfrac{1}{6}(3x_1^2 + 3x_2^2 - (4k^2 - 2)x_3^2)x_3, \qquad x \in \mathbb{R}^3,$$

where $k \in \mathbb{R}^\times$. Then except for $c_0 = (0, 0, -\frac{1}{4k^2-2})$, all real nonzero idempotents lie on the circle

$$c = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 = k^2,\ x_3 = \tfrac{1}{2}\}.$$

There are no nontrivial 2-nilpotents in $V(u)$. The idempotents c lying on the circle have the *same* Peirce spectrum: $\sigma(c) = \{1, \frac{1}{2}, \frac{1}{2} - 2k^2\}$, while the spectrum $\sigma(c_0) = \{1, -\frac{1}{4k^2-2}, -\frac{1}{4k^2-2}\}$.

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Part V
Discrete Clifford Analysis

Chapter 21
Relativistic Wave Equations on the Lattice: An Operational Perspective

Nelson Faustino

Dedicated to Professor Wolfgang Sprößig on the occasion of his 70th birthday

Abstract This paper presents an operational framework for the computation of the discretized solutions for relativistic equations of Klein-Gordon and Dirac type. The proposed method relies on the construction of an evolution-type operador from the knowledge of the *Exponential Generating Function* (EGF), carrying a *degree lowering operator* $L_t = L(\partial_t)$. We also use certain operational properties of the discrete Fourier transform over the n-dimensional *Brioullin zone* $Q_h = \left(-\frac{\pi}{h}, \frac{\pi}{h}\right]^n$—a toroidal Fourier transform in disguise—to describe the discrete counterparts of the *continuum* wave propagators, $\cosh(t\sqrt{\Delta - m^2})$ and $\dfrac{\sinh(t\sqrt{\Delta - m^2})}{\sqrt{\Delta - m^2}}$ respectively, as discrete convolution operators. In this way, a huge class of discretized time-evolution problems of differential-difference and difference-difference type may be studied in the spirit of hypercomplex variables.

Keywords Discretized Dirac equations · Discrete Fourier transform · Discretized Klein-Gordon equations · Exponential generating function · Wave propagators

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21.1 Introduction

21.1.1 The State of Art

Some decades ago Gürlebeck and Sprössig have shown on their book [22] that the theory of finite difference potentials offers the possibility to study the solution of boundary value problems from the knowledge of the discrete fundamental solution, when a finite difference approximation of the Dirac operator is considered. Such method was sucessfully applied on the papers [13, 14] to compute numerically the solution of boundary value problems.

There are already some recent contributions that recognizes that the theory of finite difference potentials presented by Gürlebeck and Sprössig on their book may also be used to describe the solution of boundary value problems on half-lattices [5, 6] by a discrete version of the Hilbert transform.

It is almost well-known that the [discrete] Hilbert transform is nothing else than a Riesz type transform in disguise (cf. [3, 4]), that may be derived formally from the subordination formula

$$(-\Delta_h)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \exp\left(t\Delta_h\right) t^\alpha \frac{dt}{t}$$

in the limit $\alpha \to \frac{1}{2}$ (cf. [7, section 6]). Hereby Δ_h denotes the star-Laplacian operator on the lattice $h\mathbb{Z}^n$ (that will be introduced later on Sect. 21.1.2 via Eq. (21.2)). Thereby, the solutions of Riemann-Hilbert type problems may be recovered by the solution of a time-evolution problem of Cauchy-Riemann type (cf. [7, Proposition 6]).

A great deal of work has been done recently by Dattoli and his collaborators to extend the operational framework to relativistic wave equations of Dirac type (cf. [2, 10–12]). At the same time there has been interest in studying evolution problems in the context of hypercomplex variables, namely discretized variants for the heat equation (cf. [1]) and for the Cauchy-Kovaleskaya extension (cf. [8]).

21.1.2 Problem Setup

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, $\mathbf{e}_{n+1}, \mathbf{e}_{n+2} \ldots, \mathbf{e}_{2n}$ be an orthogonal basis of the Minkowski space-time $\mathbb{R}^{n,n}$, and $C\ell_{n,n}$ the Clifford algebra of signature (n,n) generated from the set of graded anti-commuting relations

$$\begin{aligned} &\mathbf{e}_j\mathbf{e}_k + \mathbf{e}_k\mathbf{e}_j = -2\delta_{jk}, && 1 \leq j,k \leq n \\ &\mathbf{e}_j\mathbf{e}_{n+k} + \mathbf{e}_{n+k}\mathbf{e}_j = 0, && 1 \leq j,k \leq n \\ &\mathbf{e}_{n+j}\mathbf{e}_{n+k} + \mathbf{e}_{n+k}\mathbf{e}_{n+j} = 2\delta_{jk}, && 1 \leq j,k \leq n. \end{aligned} \tag{21.1}$$

Here we recall that the linear space isomorphism provided by the linear extension of the mapping $\mathbf{e}_{j_1}\mathbf{e}_{j_2}\ldots\mathbf{e}_{j_r} \mapsto dx_{j_1}dx_{j_2}\ldots dx_{j_r}$, with $1 \leq j_1 < j_2 < \ldots < j_r \leq 2n$, allows us to show that the resulting algebra has dimension 2^{2n} and it is isomorphic to the exterior algebra $\bigwedge(\mathbb{R}^{n,n})$ (cf. [31, Chapter 3]) so that $\mathbf{e}_J = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\ldots\mathbf{e}_{j_r}$ corresponds to a basis of $C\ell_{n,n}$. For $J = \varnothing$ (empty set) we use the convention $\mathbf{e}_\varnothing = 1$. In particular, any vector $(x_1, x_2 \ldots, x_n)$ of $\mathbb{R}^n$ may be represented in terms of the linear combination $x = \sum_{j=1}^n x_j\mathbf{e}_j$ carrying the basis elements $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ with signature $(0, n)$, whereas the translations $(x_1, x_2, \ldots, x_j \pm \varepsilon, \ldots, x_n)$ on the lattice $\varepsilon\mathbb{Z}^n \subset \mathbb{R}^n$ with mesh width $\varepsilon > 0$ may be represented in terms of the displacements $x \pm \varepsilon\mathbf{e}_j$.

Along this paper we develop our results to lattices of the form

$$\mathbb{R}^n_{h,\alpha} := (1-\alpha)h\mathbb{Z}^n \oplus \alpha h\mathbb{Z}^n, \text{ with } \quad h > 0 \text{ and } \quad 0 < \alpha < \frac{1}{2}.$$

Here we would like to stress that $\mathbb{R}^n_{h,\alpha}$ contains $h\mathbb{Z}^n$. Indeed, for any x with membership in $h\mathbb{Z}^n$ may be uniquely rewritten as $x = (1-\alpha)x+\alpha x$, with $(1-\alpha)x \in (1-\alpha)h\mathbb{Z}^n$ and $\alpha x \in \alpha h\mathbb{Z}^n$.

The class of discrete multivector functions $\mathbb{R}^n_{\alpha,h} \to \mathbb{C} \otimes C\ell_{n,n}$ and $\mathbb{R}^n_{\alpha,h} \times T \to \mathbb{C} \otimes C\ell_{n,n}$ that are considered on the sequel admit one of the following representations:

$$\Phi(x) = \sum_{r=0}^n \sum_{|J|=r} \phi_J(x)\mathbf{e}_J, \quad \text{with } \mathbf{e}_J = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\ldots\mathbf{e}_{j_r}$$

$$\Psi(x,t) = \sum_{r=0}^n \sum_{|J|=r} \psi_J(x,t)\mathbf{e}_J, \text{ with } \mathbf{e}_J = \mathbf{e}_{j_1}\mathbf{e}_{j_2}\ldots\mathbf{e}_{j_r}.$$

Hereby $|J|$ denotes the cardinality of J. The scalar-valued functions $\Phi(x)$ resp. $\Psi(x,t)$ are thus represented as $\Phi(x) = \phi(x)\mathbf{e}_\emptyset$ resp. $\Psi(x,t) = \psi(x,t)\mathbf{e}_\emptyset$, whereas the vector-fields

$$(\phi_1(x), \phi_2(x), \ldots, \phi_n(x)) \text{ resp. } (\psi_1(x,t), \psi_2(x,t), \ldots, \psi_n(x,t))$$

of $\mathbb{R}^n$ are described through the ansatz

$$\Phi(x) = \sum_{j=1}^n \phi_j(x)\mathbf{e}_j \text{ and } \Psi(x,t) = \sum_{j=1}^n \psi_j(x,t)\mathbf{e}_j,$$

respectively.

The subscript notations $\phi_J(x)$ and $\psi_J(x,t)$ are adopted to denote the complex-valued functions $\mathbb{R}^n_{h,\alpha} \to \mathbb{C}$ resp. $\mathbb{R}^n_{h,\alpha} \times T \to \mathbb{C}$ carrying the multivector basis $\mathbf{e}_J$.

The bold notations $\mathbf{f}$, $\mathbf{g}$, ..., $\boldsymbol{\Phi}$, $\boldsymbol{\Psi}$, ... and so on will be considered when we refer to discrete multivector functions with membership in the *complexified Clifford algebra* $\mathbb{C} \otimes C\ell_{n,n}$.

Our purpose here is centered around the study of relativistic wave equations of Klein-Gordon and Dirac-type on the space-time lattice $\mathbb{R}^n_{h,\alpha} \times T$ that exhibit a differential-difference or a difference-difference character. That includes time-evolution problems encoded by the discretized Klein-Gordon operator $L_t^2 - \Delta_h + m^2$, carrying the mass term $m > 0$.

Here and elsewhere

$$\Delta_h \Psi(x,t) = \sum_{j=1}^{n} \frac{\Psi(x + h\mathbf{e}_j, t) + \Psi(x - h\mathbf{e}_j, t) - 2\Psi(x,t)}{h^2} \tag{21.2}$$

denotes the discrete Laplacian on $h\mathbb{Z}^n \subset \mathbb{R}^n_{h,\alpha}$, and L_t a *degree-lowering operator*. The Dirac-Kähler discretizations D_ε on the lattice $\varepsilon\mathbb{Z}^n$, already studied in the author's recent papers [19, 20]:

$$\begin{aligned} D_\varepsilon \Psi(x,t) = & \sum_{j=1}^{n} \mathbf{e}_j \frac{\Psi(x + \varepsilon\mathbf{e}_j, t) - \Psi(x - \varepsilon\mathbf{e}_j, t)}{2\varepsilon} + \\ & + \sum_{j=1}^{n} \mathbf{e}_{n+j} \frac{2\Psi(x,t) - \Psi(x + \varepsilon\mathbf{e}_j, t) - \Psi(x - \varepsilon\mathbf{e}_j, t)}{2\varepsilon} \end{aligned} \tag{21.3}$$

as well as the pseudo-scalar γ of $C\ell_{n,n}$:

$$\gamma = \prod_{j=1}^{n} \mathbf{e}_{n+j} \mathbf{e}_j \tag{21.4}$$

are also considered with the aim of formulate a discrete counterpart for the time-evolution equation of Dirac type.

From now on let us take a close look for the *delta operators* L_t from an umbral calculus perspective (see e.g. [15, Chapter 1], [16, section 1], [17, subsection 2.1] and [18, subsection 1.2. & subsection 2.2] for an abridged version of Roman's book [26]). In case where $L_t = \partial_t$ and $T = [0, \infty)$, it is well-known (and easy to check) that the hypergeometric series representation of the following wave propagators (cf. [11, p. 704]):

$$\begin{aligned} \cosh(t\sqrt{\Delta_h - m^2}) &= {}_0F_1\left(\frac{1}{2}; \frac{t^2}{4}(\Delta_h - m^2)\right) \\ \frac{\sinh(t\sqrt{\Delta_h - m^2})}{\sqrt{\Delta_h - m^2}} &= t\, {}_0F_1\left(\frac{3}{2}; \frac{t^2}{4}(\Delta_h - m^2)\right) \end{aligned}$$

allows us to represent formally the null solutions of the differential-difference Klein-Gordon operator $\partial_t^2 - \Delta_h + m^2$ (cf. [9, Part I] & [30, Exercise 2.18]), while for the difference-difference evolution problem associated to the discretization $T = \{\frac{k\tau}{2} : k \in \mathbb{N}_0\}$ of the continuous time-domain $[0, \infty)$ (the lattice $\frac{\tau}{2}\mathbb{Z}_{\geq 0}$), and to the finite difference operator

$$L_t \Psi(x,t) = \frac{\Psi\left(x, t+\frac{\tau}{2}\right) - \Psi\left(x, t-\frac{\tau}{2}\right)}{\tau} \tag{21.5}$$

(difference-difference evolution problem) it can be easily verified that L_t admits the *formal Taylor series expansion* (cf. [18, Example 2.3])

$$L_t \Psi(x,t) = \frac{2}{\tau} \sinh\left(\frac{\tau}{2}\partial_t\right) \Psi(x,t).$$

Using the fact that first order differential and difference operators are particular cases of *shift-invariant operators* with respect to the exponentiation operator $\exp(s\partial_t)$:

$$L_t \exp(s\partial_t) = \exp(s\partial_t) L_t,$$

we can obtain an amalgamation of our approach to *delta operators* L_t, represented through the formal series expansion

$$L_t = \sum_{k=1}^{\infty} b_k \frac{(\partial_t)^k}{k!}, \quad \text{with} \quad b_k = [(L_t)^k t^k]_{t=0}$$

in the same order of ideas of [15, Chapter 1], [17, section 2] & [18, section 1 & section 2]. Indeed, from the combination of the *shift-invariant property* (cf. [26, Corollary 2.2.8]) with the isomorphism between the algebra of formal power series and the algebra of linear functionals associated to the ring of polynomials $\mathbb{R}[t]$ (cf. [26, Theorem 2.1.1]), the null solutions of the wave-type operator $L_t^2 - \Delta_h + m^2$ may be constructed from the *exponential generating function* (EGF)

$$\mathbf{G}(\mathbf{s},t) = \sum_{k=0}^{\infty} \frac{m_k(t)}{k!}\mathbf{s}^k, \ \mathbf{s} \in \mathbb{C} \otimes C\ell_{n,n} \ \& \quad t \in \mathbb{R} \tag{21.6}$$

associated to the Sheffer sequence $\{m_k(t) : k \in \mathbb{N}_0\}$ of $L_t := L(\partial_t)$.

More precisely, the following theorem (see Appendix for further details) goes beyond the Pauli matrices identity obtained in [11, p. 701]:

Theorem 21.1.1 *For the case where $\mathbf{s} = re^{i\phi}\omega$, with $-\pi < \phi \leq \pi$ and ω is an element of $C\ell_{n,n}$ satisfying $\omega^2 = +1$, the EGF $\mathbf{G}(\mathbf{s},t)$, defined through Eq. (21.6), satisfies*

$$\mathbf{G}(re^{i\phi}\omega, t) = \cosh\left(tL^{-1}(re^{i\phi})\right) + \omega \sinh\left(tL^{-1}(re^{i\phi})\right).$$

21.1.3 The Structure of the Paper

We turn next with the outline of the subsequent sections:

- In Sect. 21.2 we introduce, in a self-contained style, the basics of discrete Fourier analysis on the lattice $\mathbb{R}^n_{h,\alpha}$ (Sect. 21.2.1). Then we obtain an alternative factorization for the discretized Klein-Gordon operator (Sect. 21.2.2), based on the study of the Fourier multiplier underlying to the discrete Laplacian Δ_h defined by Eq. (21.2).
- In Sect. 21.3 we find some explicit representations underlying to the solution of the discretized versions of the Klein-Gordon (Theorem 21.3.1) and Dirac equation (Corollary 21.3.1) on the lattice $\mathbb{R}^n_{h,\alpha} \times T$. Here, the EGF representation obtained in Theorem 21.1.1 as well as the discrete Fourier analysis toolbox introduced in Sect. 21.2 play a central role.
- In Sect. 21.4 we study applications and generalizations for the results obtained in Sect. 21.3. We start to find explicit representations for difference-difference evolution problems of Klein-Gordon and Dirac type on the lattice $\mathbb{R}^n_{h,\alpha} \times \frac{\tau}{2}\mathbb{Z}_{\geq 0}$ by means of hypersingular integral representations involving Chebyshev polynomials of first and second kind (cf. [23]), and of fractional integral representations associated to a class of *generalized Mittag-Leffler functions* (cf. [28, Chapter 1]). We also establish a comparison with the approaches considered in references [1, 7, 8] (Sect. 21.4.2). In the end, we exploit the characterization obtained in Sect. 21.3 to fractional operators of Riesz type (Sect. 21.4.3).
- In Sect. 21.5 we outline the main results of the paper and discuss further directions of research.

21.2 Discrete Fourier Analysis Toolbox

21.2.1 Discrete Fourier Transform vs. Spaces of Tempered Distributions

Let us define by $\ell_p(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n}) := \ell_p(\mathbb{R}^n_{h,\alpha}) \otimes \left(\mathbb{C} \otimes C\ell_{n,n}\right)$ $(1 \leq p \leq \infty)$ the *right Banach-module* endowed by the Clifford-valued sesquilinear form (cf. [21, p. 533])

$$\langle \mathbf{f}(\cdot, t), \mathbf{g}(\cdot, t)\rangle_{h,\alpha} = \sum_{x\in\mathbb{R}^n_{h,\alpha}} h^n\, \mathbf{f}(x, t)^\dagger \mathbf{g}(x, t), \tag{21.7}$$

and by $\mathcal{S}(\mathbb{R}^n_{h,\alpha}; C\ell_{n,n}) := \mathcal{S}(\mathbb{R}^n_{h,\alpha}) \otimes \left(\mathbb{C} \otimes C\ell_{n,n}\right)$ the space of *rapidly decaying functions* $\mathbf{f}(\cdot, t)$ ($t \in T$ is fixed) with values on $\mathbb{C} \otimes C\ell_{n,n}$, defined through the semi-norm condition

$$\sup_{x\in\mathbb{R}^n_{h,\alpha}} (1+\|x\|^2)^M \;\|\mathbf{f}(x,t)\| < \infty$$

for any $\mathbb{R}$-valued constant $M < \infty$.

Here and elsewhere, the symbol † denotes the †-*conjugation* operation $\mathbf{a} \mapsto \mathbf{a}^\dagger$ on the *complexified Clifford algebra* $\mathbb{C}\otimes C\ell_{n,n}$, defined as

$$\begin{aligned}&(\mathbf{ab})^\dagger = \mathbf{b}^\dagger \mathbf{a}^\dagger\\ &(a\mathbf{e}_J)^\dagger = \overline{a_J}\,\mathbf{e}^\dagger_{j_r}\ldots \mathbf{e}^\dagger_{j_2}\mathbf{e}^\dagger_{j_1} \quad (1\leq j_1<j_2<\ldots<j_r\leq 2n)\,,\\ &\mathbf{e}^\dagger_j = -\mathbf{e}_j \quad \text{and} \quad \mathbf{e}^\dagger_{n+j} = \mathbf{e}_{n+j} \quad (1\leq j\leq n)\end{aligned} \tag{21.8}$$

whereas $\|\cdot\|$—the norm of the *complexified Clifford algebra* $\mathbb{C}\otimes C\ell_{n,n}$—is defined by the square condition $\|\mathbf{a}\|^2 = \mathbf{a}^\dagger\mathbf{a}$.

In the same order of ideas of [27, Exercise 3.1.7], under the seminorm constraint

$$\sup_{x\in\mathbb{R}^n_{h,\alpha}} (1+\|x\|^2)^{-M} \;\|\mathbf{g}(x,t)\| < \infty$$

the mapping $\mathbf{f}(\cdot,t) \mapsto \langle \mathbf{f}(\cdot,t), \mathbf{g}(\cdot,t)\rangle_{h,\alpha}$ defines the set of all *continuous linear functionals* with membership in $\mathcal{S}(\mathbb{R}^n_{h,\alpha};\mathbb{C}\otimes C\ell_{n,n})$. The underlying family of distributions $\mathbf{g}(\cdot,t):\mathbb{R}^n_{h,\alpha}\to\mathbb{C}\otimes C\ell_{n,n}$ (for a fixed $t\in T$) belong to

$$\mathcal{S}'(\mathbb{R}^n_{h,\alpha};\mathbb{C}\otimes C\ell_{n,n}) := \mathcal{S}'(\mathbb{R}^n_{h,\alpha})\otimes\left(\mathbb{C}\otimes C\ell_{n,n}\right),$$

the multivector counterpart of the *space of tempered distributions* on the lattice $\mathbb{R}^n_{h,\alpha}$.

Let us now take a close look to the *discrete Fourier transform*, defined as follows:

$$(\mathcal{F}_{h,\alpha}\mathbf{g})(\xi,t) = \begin{cases} \dfrac{h^n}{(2\pi)^{\frac{n}{2}}} \displaystyle\sum_{x\in\mathbb{R}^n_{h,\alpha}} \mathbf{g}(x,t)e^{ix\cdot\xi} & \text{for } \xi\in Q_h \\ 0 & \text{for } \xi\in\mathbb{R}^n\setminus Q_h \end{cases}. \tag{21.9}$$

Here $Q_h = \left(-\frac{\pi}{h},\frac{\pi}{h}\right]^n$ stands for the n-dimensional *Brioullin zone* representation of the n-torus $\mathbb{R}^n/\frac{2\pi}{h}\mathbb{Z}^n$, as already depicted on Rabin's seminal paper [25].

With the aid of the *Fourier coefficients* (cf. [22, subsection 5.2.1])

$$\widehat{\mathbf{g}}_{h,\alpha}(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{Q_h} (\mathcal{F}_{h,\alpha}\mathbf{g})(\xi,t)e^{-ix\cdot\xi}d\xi \tag{21.10}$$

we are able to derive, in a natural way, the isometric isomorphism

$$\mathcal{F}_{\alpha,h} : \ell_2(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n}) \to L_2(Q_h; \mathbb{C} \otimes C\ell_{n,n})$$

with inverse $(\mathcal{F}^{-1}_{h,\alpha}\mathbf{g})(x,t) = \widehat{\mathbf{g}}_{h,\alpha}(x,t)$.

Here and elsewhere $L_2(Q_h; \mathbb{C} \otimes C\ell_{n,n}) := L_2(Q_h) \otimes \left(\mathbb{C} \otimes C\ell_{n,n}\right)$ denotes the $\mathbb{C} \otimes C\ell_{n,n}$-*Hilbert module* endowed by the sesquilinear form

$$\langle \mathbf{f}(\cdot,t), \mathbf{g}(\cdot,t)\rangle_{Q_h} = \int_{Q_h} \mathbf{f}(\xi,t)^\dagger \mathbf{g}(\xi,t) d\xi. \tag{21.11}$$

Moreover, we can mimic the construction provided by [27, Exercise 3.1.15] & [27, Definition 3.1.25] to show that $\mathcal{S}(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$ is dense in $\ell_2(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$, and that $C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})$ is embedded on $C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})'$, the space of $\mathbb{C} \otimes C\ell_{n,n}$-valued distributions over Q_h.

As a consequence, we uniquely extend the *discrete Fourier transform* (21.9) as a mapping $\mathcal{F}_{h,\alpha} : \mathcal{S}'(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n}) \to C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})$ by the *Parseval type relation*, involving the sesquilinear forms (21.7) and (21.11) (cf. [27, Definition 3.1.27]):

$$\langle \mathcal{F}_{h,\alpha}\mathbf{f}(\xi,t), \mathbf{g}(\cdot,t)\rangle_{Q_h} = \left\langle \mathbf{f}(\cdot,t), \widehat{\mathbf{g}}_{h,\alpha}(\cdot,t)\right\rangle_{h,\alpha},$$

underlying to $\mathbf{f}(\cdot,t) \in \mathcal{S}'(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$ and $\mathbf{g}(\cdot,t) \in C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})$.

With the construction furnished above we can naturally define the convolution between a *discrete distribution* $\mathbf{f}(\cdot,t)$ with membership in $\mathcal{S}'(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$, and a *discrete function* $\Phi(x)$ with membership in $\mathcal{S}(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$:

$$\left(\mathbf{f}(\cdot,t) \star_{h,\alpha} \Phi\right)(x) = \sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n \Phi(y) \mathbf{f}(y-x,t) \tag{21.12}$$

via the duality condition

$$\left\langle \mathbf{f}(\cdot,t) \star_{h,\alpha} \Phi, \mathbf{g}(\cdot,t) \right\rangle_{h,\alpha} = \langle \mathbf{f}(\cdot,t), \widetilde{\Phi} \star_{h,\alpha} \mathbf{g}(\cdot,t) \rangle_{h,\alpha}, \ \widetilde{\Phi}(x) = [\Phi(-x)]^\dagger,$$

for all $\mathbf{g}(\cdot,t) \in \mathcal{S}(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$.

Also, the multiplication of a *continuous distribution* $\mathcal{U} \in C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})'$ by a function $\mathcal{F}_{h,\alpha}\Phi(\xi)$ with membership in $C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})$ can be defined via the sesquilinear identity

$$\left\langle \left(\mathcal{F}_{h,\alpha}\Phi\right)\mathcal{U}, \mathcal{F}_{h,\alpha}\mathbf{g}(\cdot,t) \right\rangle_{Q_h} = \left\langle \mathcal{U}, \left(\mathcal{F}_{h,\alpha}\Phi\right)^\dagger \left(\mathcal{F}_{h,\alpha}\mathbf{g}(\cdot,t)\right) \right\rangle_{Q_h}.$$

As in [7, p. 123], the following *discrete convolution formula* property

$$\mathcal{F}_{h,\alpha}\left[\mathbf{f}(\cdot,t) \star_{h,\alpha} \Phi\right] = \left(\mathcal{F}_{h,\alpha}\mathbf{f}(\cdot,t)\right)\left(\mathcal{F}_{h,\alpha}\Phi\right) \tag{21.13}$$

that holds at the level of distributions, yields as an immediate consequence of the sequence of identities

$$\begin{aligned}\langle\, \mathcal{F}_{h,\alpha}\left[\mathbf{f}(\cdot,t)\star_{h,\alpha}\Phi\right],\mathbf{g}(\cdot,t)\,\rangle_{Q_h} &= \langle\, \mathbf{f}(\cdot,t)\star_{h,\alpha}\Phi, \mathcal{F}_{h,\alpha}^{-1}[\mathbf{g}(\cdot,t)]\,\rangle_{h,\alpha}\\ &= \langle\, \mathbf{f}(\cdot,t), \widetilde{\Phi}\star_{h,\alpha}\mathcal{F}_{h,\alpha}^{-1}[\mathbf{g}(\cdot,t)]\,\rangle_{h,\alpha}\\ &= \left\langle\, \mathbf{f}(\cdot,t), \mathcal{F}_{h,\alpha}^{-1}\left(\mathcal{F}_{h,\alpha}\widetilde{\Phi}\,\mathbf{g}(\cdot,t)\right)\,\right\rangle_{h,\alpha}\\ &= \left\langle\, \mathcal{F}_{h,\alpha}\mathbf{f}(\cdot,t), \mathcal{F}_{h,\alpha}\widetilde{\Phi}\,\mathbf{g}(\cdot,t)\,\right\rangle_{h,\alpha}\\ &= \langle\, \left(\mathcal{F}_{h,\alpha}\mathbf{f}(\cdot,t)\right)\left(\mathcal{F}_{h,\alpha}\Phi\right),\mathbf{g}(\cdot,t)\,\rangle_{Q_h}.\end{aligned}$$

21.2.2 Discrete Dirac-Kähler vs. Discrete Laplacian

Let us take now a close look to the Fourier multiplier of $\mathcal{F}_{h,\alpha}\circ(-\Delta_h)\circ\mathcal{F}_{h,\alpha}^{-1}$ encoded by the discrete Laplacian (21.2). First, we observe that for $-h\leq\varepsilon\leq h$ the Clifford-valued sesquilinear form $\langle\cdot,\cdot\rangle_{h,\alpha}$ satisfies the summation property over $\mathbb{R}^n_{h,\alpha}$ (cf. [21, p. 536]):

$$\sum_{x\in\mathbb{R}^n_{h,\alpha}} h^n\,\mathbf{f}(x,t)^{\dagger}\mathbf{g}(x+\varepsilon\mathbf{e}_j,t)=\sum_{x\in h\mathbb{Z}^n} h^n\,\mathbf{f}(x-\varepsilon\mathbf{e}_j,t)^{\dagger}\mathbf{g}(x,t).$$

In particular, for the substitutions

$$\mathbf{f}(x,t)\to e^{-ix\cdot\xi}\text{ and }\mathbf{g}(x,t)\to\Psi(x,t)$$

we can conclude that the translation action $x\mapsto\Psi(x+\varepsilon\mathbf{e}_j,t)$ over $\mathcal{S}(\mathbb{R}^n_{h,\alpha};\mathbb{C}\otimes C\ell_{n,n})$ gives rise to the property

$$\mathcal{F}_{h,\alpha}\Psi(\cdot+\varepsilon\mathbf{e}_j,t)=e^{-i\varepsilon\xi_j}\mathcal{F}_{h,\alpha}\Psi(\cdot,t). \tag{21.14}$$

Therefore $\mathcal{F}_{h,\alpha}(\Delta_h\Psi)(\xi,t)=-d_h(\xi)^2\mathcal{F}_{h,\alpha}\Psi(\xi,t)$ (cf. [22, Subsection 5.2.2]), where

$$d_h(\xi)^2=\frac{4}{h^2}\sum_{j=1}\sin^2\left(\frac{h\xi_j}{2}\right) \tag{21.15}$$

stands for the Fourier multiplier of $\mathcal{F}_{h,\alpha}\circ(-\Delta_h)\circ\mathcal{F}_{h,\alpha}^{-1}$.

Next, we observe that the sequence of identities

$$\frac{4}{h^2}\sin^2\left(\frac{h\xi_j}{2}\right)=\frac{1}{h^2}\left(1-e^{-ih\xi_j}\right)\left(1-e^{ih\xi_j}\right)=\left|\frac{1-e^{-ih\xi_j}}{h}\,e^{ih\theta_j}\right|^2$$

hold for every $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, and $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ with membership in $Q_h = \left(-\frac{\pi}{h}, \frac{\pi}{h}\right]^n$ so that (21.15) may be rewritten as

$$d_h(\xi)^2 = \sum_{j=1}^{n} \left| \frac{1 - e^{-ih\xi_j}}{h} \, e^{ih\theta_j} \right|^2 .$$

In particular, under the choice $\theta_j = (1-\alpha)\xi_j$ the above identity may be expressed in terms of the complex numbers

$$z_{h,\alpha}(\xi_j) = \frac{e^{i(1-\alpha)h\xi_j} - e^{-i\alpha h\xi_j}}{h}, \text{ with } -\pi < h\xi_j \leq \pi \quad \& \quad 0 < \alpha < \frac{1}{2}.$$

Moreover, from the set of basic identities

$$\begin{aligned}
|z_{h,\alpha}(\xi_j)|^2 &= \frac{1}{2}\left(z_{h,\alpha}(\xi_j) z_{h,\alpha}(\xi_j)^\dagger + z_{h,\alpha}(\xi_j)^\dagger z_{h,\alpha}(\xi_j) \right) \\
&= \left(\frac{z_{h,\alpha}(\xi_j) + z_{h,\alpha}(\xi_j)^\dagger}{2} \right)^2 - \left(\frac{z_{h,\alpha}(\xi_j) - z_{h,\alpha}(\xi_j)^\dagger}{2} \right)^2 \\
&= \left(\frac{\cos(\alpha h\xi_j) - \cos((1-\alpha)h\xi_j)}{h} \right)^2 + \left(\frac{\sin(\alpha h\xi_j) + \sin((1-\alpha)h\xi_j)}{h} \right)^2 \\
&= \left(\mathbf{e}_{n+j} \frac{\cos(\alpha h\xi_j) - \cos((1-\alpha)h\xi_j)}{h} \right)^2 + \\
&+ \left(-i\mathbf{e}_j \frac{\sin(\alpha h\xi_j) + \sin((1-\alpha)h\xi_j)}{h} \right)^2
\end{aligned}$$

one readily has that the Clifford-vector-valued function

$$\begin{aligned}
\mathbf{z}_{h,\alpha}(\xi) = \sum_{j=1}^{n} -i\mathbf{e}_j \frac{\sin((1-\alpha)h\xi_j) + \sin(\alpha h\xi_j)}{h} + \\
+ \sum_{j=1}^{n} \mathbf{e}_{n+j} \frac{\cos(\alpha h\xi_j) - \cos((1-\alpha)h\xi_j)}{h}
\end{aligned} \tag{21.16}$$

satisfies the square condition $\mathbf{z}_{h,\alpha}(\xi)^2 = d_h(\xi)^2$.

Let us now continue with the class of discrete Dirac-Kähler operators D_ε introduced in (21.3). By means of the †-conjugation (21.8), one can also define formally the conjugation of D_ε as follows:

$$\begin{aligned}
D_\varepsilon^\dagger \Psi(x,t) = \sum_{j=1}^{n} -\mathbf{e}_j \frac{\Psi(x + \varepsilon\mathbf{e}_j, t) - \Psi(x - \varepsilon\mathbf{e}_j, t)}{2\varepsilon} + \\
+ \sum_{j=1}^{n} \mathbf{e}_{n+j} \frac{2\Psi(x,t) - \Psi(x + \varepsilon\mathbf{e}_j, t) - \Psi(x - \varepsilon\mathbf{e}_j, t)}{2\varepsilon}.
\end{aligned}$$

Here we notice that the combination of the summation property (21.14) with the †-conjugation properties $\mathbf{e}_j^\dagger = -\mathbf{e}_j$ and $\mathbf{e}_{n+j}^\dagger = \mathbf{e}_{n+j}$ $(j = 1, 2, \ldots, n)$ shows in turn that D_ε and $D_\varepsilon^\dagger$ are self-adjoint with respect to Clifford-valued sesquilinerar form $\langle \cdot, \cdot \rangle_{h,\alpha}$ defined by Eq. (21.7), since

$$\langle D_\varepsilon \mathbf{f}(\cdot, t), \mathbf{g}(\cdot, t)\rangle_{h,\alpha} = \langle \mathbf{f}(\cdot, t), D_\varepsilon \mathbf{g}(\cdot, t)\rangle_{h,\alpha}$$
$$\langle D_\varepsilon^\dagger \mathbf{f}(\cdot, t), \mathbf{g}(\cdot, t)\rangle_{h,\alpha} = \langle \mathbf{f}(\cdot, t), D_\varepsilon^\dagger \mathbf{g}(\cdot, t)\rangle_{h,\alpha}.$$

As a consequence of the above construction, the Clifford-vector-valued function $\mathbf{z}_{h,\alpha}(\xi)$ defined by (21.16) is the Fourier multiplier of the operator

$$\mathcal{F}_{h,\alpha} \circ ((1-\alpha)D_{(1-\alpha)h} - \alpha D_{\alpha h}^\dagger) \circ \mathcal{F}_{h,\alpha}^{-1}.$$

Therefore, for every $0 < \alpha < \frac{1}{2}$ the mapping property

$$D_{h,\alpha} : \mathcal{S}(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n}) \to C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})$$

stands for the Dirac-Kähler type operator

$$D_{h,\alpha} := (1-\alpha)D_{(1-\alpha)h} - \alpha D_{\alpha h}^\dagger, \tag{21.17}$$

as well as the discrete Laplacian splitting $\left(D_{h,\alpha}\right)^2 = -\Delta_h$.

Furthermore, the factorization property

$$-\Delta_h + m^2 = (D_{h,\alpha} - m\gamma)^2 \tag{21.18}$$

yields from the set of anti-commuting relations

$$\gamma \mathbf{e}_j + \mathbf{e}_j \gamma = 0 \;\; \gamma \mathbf{e}_{n+j} + \mathbf{e}_{n+j}\gamma = 0 \;\; \gamma^2 = +1, \tag{21.19}$$

carrying the Clifford basis elements $\mathbf{e}_j, \mathbf{e}_{n+j}$ $(j = 1, 2, \ldots, n)$, and the pseudo-scalar γ defined by Eq. (21.4) (cf. [19, Proposition 3.1]).

Remark 21.2.1 (Towards a Fractional Regularization of Discrete Dirac Operators) We would like to stress here that the discretizations D_ε and $D_{h,\alpha}$, given by Eqs. (21.3) and (21.17) respectively, are interrelated by the limit formula

$$\lim_{\alpha \to 0} D_{h,\alpha} = D_h.$$

On the other hand, the limit property

$$\lim_{\alpha \to \frac{1}{2}} D_{h,\alpha} = \frac{D_{h/2}^+ + D_{h/2}^-}{2}$$

involving the finite difference Dirac operators $D^{\pm}_{h/2}$ of forward/backward type:

$$D^{+}_{h/2}\Psi(x,t)=\sum_{j=1}^{n}\mathbf{e}_j\frac{\Psi\left(x+\frac{h}{2}\mathbf{e}_j,t\right)-\Psi\left(x,t\right)}{h/2}$$

$$D^{-}_{h/2}\Psi(x,t)=\sum_{j=1}^{n}\mathbf{e}_j\frac{\Psi\left(x,t\right)-\Psi\left(x-\frac{h}{2}\mathbf{e}_j,t\right)}{h/2}.$$

shows us that $D_{h,\alpha}$ may also be seen as a fractional regularization for the discrete Dirac operators on the lattice $\frac{h}{2}\mathbb{Z}^n$, already considered in the series of papers [16, 18–20].

Remark 21.2.2 (The Lattice Fermion Doubling Gap) Since the Fourier multipliers $\mathbf{z}_{h,\frac{1}{2}}(\xi)$ of $\mathcal{F}_{h,\alpha}\circ\left(\frac{D^{+}_{h/2}+D^{-}_{h/2}}{2}\right)\circ\mathcal{F}^{-1}_{h,\alpha}$ share the same set of zeros of the Fourier multiplier $d_h(\xi)^2$ of $\mathcal{F}_{h,\alpha}\circ(-\Delta_h)\circ\mathcal{F}^{-1}_{h,\alpha}$ defined in terms of Eq. (21.15), we can conclude that the spectrum doubling of $D_{h,\alpha}$ only occurs on the limit $\alpha\to\frac{1}{2}$ (cf. [25, p. 323]).

At this stage, we have obtained from a multivector perspective that Rabin's homological approach [25, Section 6], based on the geometry of the n-torus $\mathbb{R}^n/\frac{2\pi}{h}\mathbb{Z}^n$, also works on $\mathbb{R}^n_{h,\alpha}$ for the discretized Dirac operators $D_{h,\alpha}$. This is indeed a direct consequence of the so-called *lattice fermion doubling* gap, formulated by Nielsen & Ninomiya (cf. [24]). We also refer to [19, subsection 1.1 & section 4] for further details regarding the discussion of Nielsen–Ninomiya *no-go result*.

21.3 Solution of Discretized Time-Evolution Problems

21.3.1 Discretized Klein-Gordon Equations

In this section we study the solutions of the second-order evolution problems of the type

$$\begin{cases} L_t^2\Psi(x,t)=\Delta_h\Psi(x,t)-m^2\Psi(x,t)\ , & (x,t)\in\mathbb{R}^n_{h,\alpha}\times T \\ \Psi(x,0)=\Phi_0(x) & ,\ x\in\mathbb{R}^n_{h,\alpha} \\ \left[L_t\Psi(x,t)\right]_{t=0}=\Phi_1(x) & ,\ x\in\mathbb{R}^n_{h,\alpha}\end{cases}\tag{21.20}$$

on the space-time domain $\mathbb{R}^n_{h,\alpha}\times T$, from an umbral calculus perspective.

In terms of the *discrete Fourier transform* (21.9), the formulation of the time-evolution problem (21.20) on the momentum space $Q_h \times T$ reads as

$$\begin{cases} L_t^2 \left[\mathcal{F}_{h,\alpha}\Psi(\xi,t)\right] = -\left(d_h(\xi)^2+m^2\right)\mathcal{F}_{h,\alpha}\Psi(\xi,t) & ,(\xi,t)\in Q_h\times T \\ \mathcal{F}_{h,\alpha}\Psi(\xi,0)=\mathcal{F}_{h,\alpha}\Phi_0(\xi) & ,\xi\in Q_h \\ \left[L_t\mathcal{F}_{h,\alpha}\Psi(\xi,t)\right]_{t=0}=\mathcal{F}_{h,\alpha}\Phi_1(\xi) & ,\xi\in Q_h \end{cases} \tag{21.21}$$

With the aid of the hiperbolic functions $(s,t)\mapsto \cosh(tL^{-1}(s))$ and $(s,t)\mapsto \sinh(tL^{-1}(s))$ obtained in Theorem A.1 (see Appendix) we can describe the solution of the discretized Klein-Gordon equation (21.20) as a *discrete convolution* on $\mathbb{R}^n_{h,\alpha}$, endowed by the kernel functions

$$\begin{aligned} K_0(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{Q_h}\cosh\left(t\ L^{-1}\left(i\sqrt{d_h(\xi)^2+m^2}\right)\right)e^{-ix\cdot\xi}\,d\xi \\ K_1(x,t) &= \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{Q_h}\frac{\sinh\left(t\ L^{-1}\left(i\sqrt{d_h(\xi)^2+m^2}\right)\right)}{i\sqrt{d_h(\xi)^2+m^2}}e^{-ix\cdot\xi}\,d\xi. \end{aligned} \tag{21.22}$$

More precisely, for the discretized wave propagators defined in terms of the *discrete convolution formulae*

$$\begin{aligned} \cosh\left(t\ L^{-1}\left(\sqrt{\Delta_h-m^2}\right)\right)\Phi(x) &= \sum_{y\in\mathbb{R}^n_{h,\alpha}} h^n\Phi(y)K_0(x-y,t) \\ \frac{\sinh\left(t\ L^{-1}\left(\sqrt{\Delta_h-m^2}\right)\right)}{\sqrt{\Delta_h-m^2}}\Phi(x) &= \sum_{y\in\mathbb{R}^n_{h,\alpha}} h^n\Phi(x)K_1(x-y,t) \end{aligned} \tag{21.23}$$

we are able to mimic the so-called wave Duhamel formula *in continuum* (cf. [30, Exercise 2.22] & [30, p. 71]). That corresponds to the following theorem:

Theorem 21.3.1 *Let Φ_0 and Φ_1 be two Clifford-valued functions membership in $\mathcal{S}(\mathbb{R}^n_{h,\alpha};\mathbb{C}\otimes C\ell_{n,n})$, and K_0, K_1 be the kernel functions defined by the integral formulae (21.22). Then we have the following:*

(i) *The function*

$$\begin{aligned} \mathcal{F}_{h,\alpha}\Psi(\xi,t) &= \cosh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2+m^2}\right)\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi)+ \\ &+\ \frac{\sinh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2+m^2}\right)\right)}{i\sqrt{d_h(\xi)^2+m^2}}\mathcal{F}_{h,\alpha}\Phi_1(\xi) \end{aligned} \tag{21.24}$$

solves the time-evolution problem (21.21).

(ii) *The ansatz*

$$\Psi(x,t) = \cosh\left(tL^{-1}\left(\sqrt{\Delta_h - m^2}\right)\right)\Phi_0(x) + \\ + \frac{\sinh\left(tL^{-1}\left(\sqrt{\Delta_h - m^2}\right)\right)}{\sqrt{\Delta_h - m^2}}\Phi_1(x) \tag{21.25}$$

solves the discretized Klein-Gordon equation (21.20).

Proof **Proof of (i):**

In the shed of Theorem 21.1.1 let us now take a close look to the ansatz functions of the type

$$\mathcal{F}_{h,\alpha}\Psi(\xi,t) = \mathbf{G}(e^{i\phi}r\omega,t)\mathcal{F}_{h,\alpha}\Phi_+(\xi) + \mathbf{G}(-e^{i\phi}r\omega,t)\mathcal{F}_{h,\alpha}\Phi_-(\xi). \tag{21.26}$$

From the eigenvalue property (21.44) one readily obtains that (21.26) satisfies the equation

$$L_t^2\left[\mathcal{F}_{h,\alpha}\Psi(\xi,t)\right] = -\left(d_h(\xi)^2 + m^2\right)\mathcal{F}_{h,\alpha}\Psi(\xi,t)$$

whenever $\phi = \frac{\pi}{2}$, $r = \sqrt{d_h(\xi)^2 + m^2}$ and $\omega^2 = +1$.

It is also straightforward to see that the initial conditions of the evolution problem (21.21) lead to the system of equations

$$\mathcal{F}_{h,\alpha}\Phi_0(\xi) = \mathcal{F}_{h,\alpha}\Phi_+(\xi) + \mathcal{F}_{h,\alpha}\Phi_-(\xi)$$
$$\mathcal{F}_{h,\alpha}\Phi_1(\xi) = i\omega\sqrt{d_h(\xi)^2 + m^2}\ \mathcal{F}_{h,\alpha}\Phi_+(\xi) - i\omega\sqrt{d_h(\xi)^2 + m^2}\ \mathcal{F}_{h,\alpha}\Phi_-(\xi).$$

Solving the above system of equations in order to $\mathcal{F}_{h,\alpha}\Phi_\pm(\xi)$, it readily follows that

$$\mathcal{F}_{h,\alpha}\Phi_\pm(\xi) = \frac{1}{2}\mathcal{F}_{h,\alpha}\Phi_0(\xi) \pm \frac{\omega}{2i\sqrt{d_h(\xi)^2 + m^2}}\mathcal{F}_{h,\alpha}\Phi_1(\xi).$$

Therefore, we can recast (21.26) as

$$\mathcal{F}_{h,\alpha}\Psi(\xi,t) = \frac{\mathbf{G}\left(i\omega\sqrt{d_h(\xi)^2 + m^2},t\right) + \mathbf{G}\left(-i\omega\sqrt{d_h(\xi)^2 + m^2},t\right)}{2}\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \\ +\omega\frac{\mathbf{G}\left(i\omega\sqrt{d_h(\xi)^2 + m^2},t\right) - \mathbf{G}\left(-i\omega\sqrt{d_h(\xi)^2 + m^2},t\right)}{2i\sqrt{d_h(\xi)^2 + m^2}}\mathcal{F}_{h,\alpha}\Phi_1(\xi).$$

Moreover, from Theorem 21.1.1 we immediately get that Eq. (21.27) is equivalent to Eq. (21.24).

Proof of (ii):

By applying the *discrete Fourier transform* $\mathcal{F}_{h,\alpha}$ to both sides of (21.23), the **Proof of (ii)** follows straightforwardly from the *discrete convolution property* (21.13), and from the fact that the function $\Psi(x,t)$ defined by Eq. (21.25) is also a solution of the equation

$$L_t^2\left[\mathcal{F}_{h,\alpha}\Psi(\xi,t)\right] = -\left(d_h(\xi)^2 + m^2\right)\mathcal{F}_{h,\alpha}\Psi(\xi,t).$$

■

21.3.2 Discretized Dirac Equations

Let us now look to discretized version of the Dirac equation

$$\begin{cases} -iL_t\Psi(x,t) = \left(D_{h,\alpha} - m\gamma\right)\Psi(x,t)\ , (x,t)\in\mathbb{R}^n_{h,\alpha}\times T \\ \Psi(x,0) = \Phi_0(x) \qquad\qquad\qquad\qquad , x\in\mathbb{R}^n_{h,\alpha} \end{cases}, \tag{21.27}$$

carrying the discrete Dirac operator $D_{h,\alpha}$ introduced via Eq. (21.17).

From the framework developed on the previous sections and on Appendix, the solution of (21.27) can be easily found. In concrete, the formal solution of (21.27) provided by the operational formula

$$\Psi(x,t) = \mathbf{G}\left(iD_{h,\alpha} - im\gamma, t\right)\Phi_0(x)$$

is a direct consequence of the set of identities

$$\begin{aligned} -iL_t\left[\mathcal{F}_{h,\alpha}\Psi(\xi,t)\right] &= \left(\mathbf{z}_{h,\alpha}(\xi) - m\gamma\right)\mathcal{F}_{h,\alpha}\Psi(\xi,t) \\ \mathbf{G}\left(i\mathbf{z}_{h,\alpha}(\xi) - im\gamma, 0\right) &= 1 \\ L_t\mathbf{G}\left(i\mathbf{z}_{h,\alpha}(\xi) - im\gamma, t\right) &= i\left(\mathbf{z}_{h,\alpha}(\xi) - m\gamma\right)\mathbf{G}\left(i\mathbf{z}_{h,\alpha}(\xi) - im\gamma, t\right). \end{aligned}$$

underlying to the Fourier multiplier $\mathbf{z}_{h,\alpha}(\xi)$ of $\mathcal{F}_{h,\alpha}\circ D_{h,\alpha}\circ\mathcal{F}_{h,\alpha}^{-1}$ (see Eq. (21.16)), and the EGF $\mathbf{G}(\mathbf{s},t)$ (see Eq. (21.6)). Then, the following corollary is rather obvious:

Corollary 21.3.1 *Let Φ_0 be a function with membership in $\mathcal{S}(\mathbb{R}^n_{h,\alpha};\mathbb{C}\otimes C\ell_{n,n})$. Then, under the first order condition*

$$[L_t\Psi(x,t)]_{t=0} = i\left(D_{h,\alpha} - m\gamma\right)\Phi_0(x)$$

the ansatz (21.25) solves the discretized Dirac equation (21.27).

Proof First, we recall that the square relation

$$(\mathbf{z}_{h,\alpha}(\xi) - m\gamma)^2 = d_h(\xi)^2 + m^2$$

involving the Fourier multipliers $d_h(\xi)^2$ and $\mathbf{z}_{h,\alpha}(\xi)$, defined by Eqs. (21.15) resp. (21.16), yields as a direct consequence of the factorization property (21.19) involving the Clifford generators $\mathbf{e}_j, \mathbf{e}_{n+j}$ $(j = 1, 2, \ldots, n)$, and the pseudo-scalar γ defined by Eq. (21.4). Then

$$\omega = \frac{i\mathbf{z}_{h,\alpha}(\xi) - im\gamma}{i\sqrt{d_h(\xi)^2 + m^2}}$$

is a unitary vector of $\mathbb{C} \otimes C\ell_{n,n}$ satisfying the property $\omega^2 = +1$. Thus, from Theorem 21.1.1

$$\mathbf{G}\left(i\mathbf{z}_{h,\alpha}(\xi) - im\gamma, t\right) = \cosh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right) + \\ + \frac{\sinh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right)}{i\sqrt{d_h(\xi)^2 + m^2}}\left(i\mathbf{z}_{h,\alpha}(\xi) - im\gamma\right).$$

Moreover, from the property

$$\mathcal{F}_{h,\alpha}[i(D_{h,\alpha} - m\gamma)\Phi_0](\xi) = \left(i\mathbf{z}_{h,\alpha}(\xi) - im\gamma\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi)$$

it readily follows from direct application of statement **(ii)** of Theorem 21.3.1 that $\mathbf{G}\left(iD_{h,\alpha} - im\gamma, t\right)\Phi_0(x)$—a formal solution of the discretized Dirac equation (21.27)—equals to the ansatz (21.24), whenever $\Phi_1(x) = i(D_{h,\alpha} - m\gamma)$ $\Phi_0(x)$. ■

Remark 21.3.1 (The Zassenhaus Formula Gap) The framework that we have considered here to describe formally the solutions of the discretized Dirac equation (21.27) may be seen as a multivector extension of the framework obtained in terms of Pauli matrices by Dattoli and his collaborators on the papers [10, 11]. The major difference here lies in fact that we have considered the EGF provided by Theorem 21.1.1 to rid the limitations associated to the operational representation of Dirac type propagators by means of the Zassenhaus formula (cf. [10, p. 8]).

21.4 Further Applications

21.4.1 A Space-Time Fourier Inversion Formula Based on Chebyshev Polynomials

Let us now discuss a difference-difference version of the evolution problems (21.20) and (21.27) on the lattice $\mathbb{R}^n_{h,\alpha} \times \frac{\tau}{2}\mathbb{Z}_{\geq 0}$, associated to the finite diference operator (21.5) defined on Sect. 21.1.2. From direct application of Theorem 21.3.1 it can be easily seen that

$$\mathcal{F}_{h,\alpha}\Psi(\xi,t) = \cos\left(\frac{2t}{\tau}\sin^{-1}\left(\frac{\tau}{2}\sqrt{d_h(\xi)^2+m^2}\right)\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \\ + \frac{\sin\left(\frac{2t}{\tau}\sin^{-1}\left(\frac{\tau}{2}\sqrt{d_h(\xi)^2+m^2}\right)\right)}{\sqrt{d_h(\xi)^2+m^2}}\mathcal{F}_{h,\alpha}\Phi_1(\xi)$$

solves the second-order time-evolution problem (21.21) on the momentum space $Q_h \times \mathbb{Z}_{\geq 0}$ (see statement **(i)**).

Here we recall that from the inverse trigonometric relation $\sin^{-1}(z) = \cos^{-1}(\sqrt{1-z^2})$, that holds for every $0 \leq z \leq 1$, we can recognize that the above identity may be expressed in terms of Chebyshev polynomials of first and second kind (cf. [23, p. 170]) for values of τ satisfying the following condition:

$$0 \leq \sqrt{d_h(\xi)^2+m^2} \leq \frac{2}{\tau}.$$

That is,

$$\mathcal{F}_{h,\alpha}\Psi(\xi,t) = T_{\frac{2t}{\tau}}\left(\sqrt{1-\frac{\tau^2}{4}\left(d_h(\xi)^2+m^2\right)}\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \\ + \frac{\tau}{2}U_{\frac{2t}{\tau}-1}\left(\sqrt{1-\frac{\tau^2}{4}\left(d_h(\xi)^2+m^2\right)}\right)\mathcal{F}_{h,\alpha}\Phi_1(\xi), \tag{21.28}$$

with $T_k(\lambda) = \cos(k\cos^{-1}(\lambda))$ resp. $U_{k-1}(\lambda) = \dfrac{\sin(k\cos^{-1}(\lambda))}{\sqrt{1-\lambda^2}}$.

We recall here that the Chebyshev polynomials of first and second kind, T_k resp. U_{k-1}, admit the following Cauchy principal value representations (cf. [23, subsection 4.1]):

$$\begin{aligned}\int_{-1}^{1} \frac{U_{k-1}(s)}{s-\lambda}(1-s^2)^{\frac{1}{2}}ds &= -\pi T_k(\lambda)\\ \int_{-1}^{1} \frac{T_k(s)}{s-\lambda}(1-s^2)^{-\frac{1}{2}}ds &= \pi U_{k-1}(\lambda).\end{aligned} \tag{21.29}$$

In particular, for the change of variable $s = \cos\left(\frac{\omega\tau}{2}\right)$ $(0 \leq \omega \leq \frac{2\pi}{\tau})$ the sequence of identities

$$\begin{aligned}T_{\frac{2t}{\tau}}(\lambda) &= -\frac{\tau}{2\pi}\int_0^{\frac{2\pi}{\tau}} \frac{\sin\left(\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2}\right)-\lambda}\sin(\omega t)d\omega\\ &= \frac{\tau}{4\pi}\int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \frac{-i\sin\left(\frac{\omega\tau}{2}\right)}{\cos\left(\frac{\omega\tau}{2}\right)-\lambda}e^{-i\omega t}d\omega\\ U_{\frac{2t}{\tau}-1}(\lambda) &= \frac{\tau}{2\pi}\int_0^{\frac{2\pi}{\tau}} \frac{1}{\cos\left(\frac{\omega\tau}{2}\right)-\lambda}\cos(\omega t)d\omega\\ &= \frac{\tau}{4\pi}\int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \frac{1}{\cos\left(\frac{\omega\tau}{2}\right)-\lambda}e^{-i\omega t}d\omega.\end{aligned} \tag{21.30}$$

follow straightforwardly from parity arguments carrying the conjugation of the complex exponential function $e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$ (cf. [23, p. 173]).

Thus, based on (21.30) one finds that (21.28) admits the integral representation formula

$$\mathcal{F}_{h,\alpha}\Psi(\xi,t) = \frac{\tau}{4\pi}\int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \frac{-i\sin\left(\frac{\omega\tau}{2}\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \frac{\tau}{2}\mathcal{F}_{h,\alpha}\Phi_1(\xi)}{\cos\left(\frac{\omega\tau}{2}\right) - \sqrt{1-\frac{\tau^2}{4}\left(d_h(\xi)^2+m^2\right)}}e^{-i\omega t}d\omega. \tag{21.31}$$

Thereby, in the view of the Fourier inversion formula for $\mathcal{F}_{h,\alpha}$ provived by (21.10), the *discrete convolution formula* provided by statement **(iii)** of Theorem 21.3.1 may be reformulated as a *space-time Fourier inversion formula* over $Q_h \times \left(-\frac{2\pi}{\tau}, \frac{2\pi}{\tau}\right]$ so that $\Psi(x,t)$ equals to

$$\frac{\tau}{2(2\pi)^{\frac{n}{2}+1}}\int_{Q_h}\int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \frac{-i\sin\left(\frac{\omega\tau}{2}\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \frac{\tau}{2}\mathcal{F}_{h,\alpha}\Phi_1(\xi)}{\cos\left(\frac{\omega\tau}{2}\right) - \sqrt{1-\frac{\tau^2}{4}\left(d_h(\xi)^2+m^2\right)}}e^{-i(\omega t + x\cdot\xi)}d\omega d\xi.$$

Moreover, through the substitution $\mathcal{F}_{h,\alpha}\Phi_1(\xi) = i(\mathbf{z}_{h,\alpha}(\xi) - m\gamma)\mathcal{F}_{h,\alpha}\Phi_0(\xi)$ on the right-hand side of the above equality, we recognize that the above integral representation over $Q_h \times \left(-\frac{2\pi}{\tau}, \frac{2\pi}{\tau}\right]$ also fulfils for the discretized Dirac equation (21.27) (see statement **(ii)** of Corollary 21.3.1).

Remark 21.4.1 (Connection with the Discrete Cauchy-Kovaleskaya Extension) The solution of the discretized Dirac equation that we have considered here for the finite difference operator L_t defined by Eq. (21.5) resembles the construction considered by Constales and De Ridder on the paper [8], from a discrete Fourier analysis perspective. Here we recall that from the isomorphism (cf. [25])

$$Q_h \times \left(-\frac{2\pi}{\tau}, \frac{2\pi}{\tau}\right] \cong \left(\mathbb{R}^n/\frac{2\pi}{h}\mathbb{Z}^n\right) \times \left(\mathbb{R}/\frac{4\pi}{\tau}\mathbb{Z}\right),$$

the resulting integral representation formula may be interpreted as a *space-time toroidal Fourier transform* (cf. [27, section 3 of Part II]).

21.4.2 Connection with the Discrete Heat Semigroup

In this subsection we will explore the connection between the solutions of the discretized Klein-Gordon and Dirac equations obtained in Sect. 21.4.1, and the solution of the differential-difference heat equation

$$\begin{cases} \partial_s \Psi(x,s) = \Delta_h \Psi(x,s) \ , (x,s) \in \mathbb{R}^n_{h,\alpha} \times [0,\infty) \\ \Psi(x,0) = \Phi(x) \qquad\qquad , x \in \mathbb{R}^n_{h,\alpha} \end{cases} \tag{21.32}$$

by means of the *discrete heat semigroup* $\{\exp(s\Delta_h)\}_{s\geq 0}$. Before we proceed, we will revisit the construction of the *discrete heat kernel* obtained by Baaske et al. in [1]. Along the same lines as in [7, section 2] one can show that $\exp(s\Delta_h)$ may be expressed in terms of the *discrete convolution formula*

$$\exp(s\Delta_h)\Phi(x) = \sum_{y\in\mathbb{R}^n_{h,\alpha}} h^n \Phi(y) K(x-y,s), \tag{21.33}$$

involving the kernel function

$$K(x,s) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{Q_h} e^{-sd_h(\xi)^2} e^{-ix\cdot\xi} d\xi. \tag{21.34}$$

On the other hand, in view of integral representation formula for the *modified Bessel functions of the first kind* $I_k(u)$:

$$I_k(u)=\frac{1}{\pi}\int_0^{\pi} e^{u\cos(\theta)}\cos(k\theta)d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{u\cos(\theta)}e^{-ik\theta}d\theta,$$

and the identity associated to the Fourier multipliers (21.15):

$$d_h(\xi)^2=\sum_{j=1}^{n}\frac{2}{h^2}\left(1-\cos(h\xi_j)\right)$$

we thereby obtain the closed formula

$$K(x,s)=\frac{(2\pi)^{\frac{n}{2}}}{h^n}e^{-\frac{2ns}{h^2}}I_{\frac{x_1}{h}}\left(\frac{2s}{h^2}\right)I_{\frac{x_2}{h}}\left(\frac{2s}{h^2}\right)\ldots I_{\frac{x_n}{h}}\left(\frac{2s}{h^2}\right), \tag{21.35}$$

after the change of variables $\xi_j=\frac{\theta_j}{h}$ $(-\pi<\theta_j\leq\pi)$ on (21.34).

Next, let us turn again our attention to the *space-time Fourier inversion formula* (21.31) derived on Sect. 21.4.1. Starting from the Laplace transform identity (cf. [28, p. 21] & [29, p. 282])

$$\int_0^{\infty} e^{p\lambda^2}p^{\beta-1}E_{\alpha,\beta}\left(sp^{\alpha}\right)dp=\frac{\lambda^{-2\beta}}{1-s\lambda^{-2\alpha}},\ \ \Re(\lambda^2)>|s|^{\frac{1}{\alpha}}\ \&\ \Re(\beta)>0 \tag{21.36}$$

involving the *generalized Mittag-Leffler functions* (cf. [21, subsection 4.2])

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\beta+\alpha k)},\ \text{for}\ \Re(\alpha)>0\ \&\ \Re(\beta)>0,$$

we realize that

$$\frac{1}{\cos\left(\frac{\omega\tau}{2}\right)-\sqrt{1-\frac{\tau^2}{4}\left(d_h(\xi)^2+m^2\right)}}=$$

$$=-\int_0^{\infty} e^{-\frac{p\tau^2}{4}d_h(\xi)^2}\,\frac{E_{\frac{1}{2},\frac{1}{2}}\left(\cos\left(\frac{\omega\tau}{2}\right)\sqrt{p}\right)}{\sqrt{p}}\,e^{p\left(1-\frac{\tau^2}{4}m^2\right)}dp$$

so that (21.31) is equivalent to

$$\mathcal{F}_{h,\alpha}\Psi(\xi,t) = -\frac{\tau}{2(2\pi)^{\frac{n}{2}+1}} \int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \int_0^\infty e^{-\frac{p\tau^2}{4}d_h(\xi)^2} \frac{E_{\frac{1}{2},\frac{1}{2}}\left(\cos\left(\frac{\omega\tau}{2}\right)\sqrt{p}\right)}{\sqrt{p}} \times$$

$$\times \left(-i\sin\left(\frac{\omega\tau}{2}\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \frac{\tau}{2}\mathcal{F}_{h,\alpha}\Phi_1(\xi)\right) e^{p\left(1-\frac{\tau^2}{4}m^2\right)} e^{-i\omega t} dp d\omega.$$

By making again use of the inversion formula (21.10) associated to $\mathcal{F}_{h,\alpha}$ and after some straightforward simplifications involving the interchanging on the order of integration, there holds

$$\Psi(x,t) = \int_0^\infty \exp\left(\frac{p\tau^2}{4}\Delta_h\right)[\Phi(x,t;p)]dp, \tag{21.37}$$

with

$$\Phi(x,t;p) = -\frac{\tau}{4\pi}\int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \left(-i\sin\left(\frac{\omega\tau}{2}\right)\Phi_0(x) + \frac{\tau}{2}\Phi_1(x)\right) \times$$

$$\times \frac{E_{\frac{1}{2},\frac{1}{2}}\left(\cos\left(\frac{\omega\tau}{2}\right)\sqrt{p}\right)}{\sqrt{p}} e^{p\left(1-\frac{\tau^2}{4}m^2\right)} e^{-i\omega t} d\omega.$$

Here one notice that the substitution $s = \frac{u\tau^2}{4}$ on (21.32) reveals that the action $\exp\left(\frac{p\tau^2}{4}\Delta_h\right)[\Phi(x,t;p)]$ corresponds to the *discrete convolution formula* (21.33), written in terms of the *discrete heat kernel* $K\left(x, \frac{p\tau^2}{4}\right)$. The later may be computed from (21.35) as a product of the *modified Bessel functions of the first kind* $I_k(u)$. Thus, as in [7] the solution of the discretized Klein-Gordon equation on the lattice $\mathbb{R}^n_{h,\alpha} \times \frac{\tau}{2}\mathbb{Z}_{\geq 0}$ may be recovered from the *discrete heat semigroup*. On the other hand, from the set of identities (cf. [29, p. 281])

$$E_{\alpha,\beta}(u) = \frac{1}{u}E_{\alpha,\beta-\alpha}(u) - \frac{1}{u}\frac{1}{\Gamma(\beta-\alpha)}$$

$$E_{\frac{1}{2},1}(u) = e^{u^2}\operatorname{erfc}(-u)$$

involving the *generalized Mittag-Leffler functions*, and the *complementary error function*

$$\operatorname{erfc}(-u) = \frac{2}{\sqrt{\pi}}\int_{-u}^\infty e^{-q^2}dq$$

we realize, after a short computation by means of the trigonometric identities ($t \in \tau\mathbb{Z}_{\geq 0}$ & $-\frac{2\pi}{\tau} < \omega \leq \frac{2\pi}{\tau}$):

$$1 + \cos^2\left(\frac{\omega\tau}{2}\right) = 2\cos(\omega\tau)$$
$$\sin\left(\frac{\omega\tau}{2}\right)\cos\left(\frac{\omega\tau}{2}\right) = \frac{1}{2}\sin(\omega\tau)$$
$$\sin\left(\frac{2\pi t}{\tau}\right) = 0,$$

and on parity arguments that the function $\Phi(x, t; p)$, defined as above, equals to

$$\Phi(x, t; p) = -\frac{\tau}{8\pi}\int_{-\frac{2\pi}{\tau}}^{\frac{2\pi}{\tau}} \left(-i\sin(\omega\tau)\Phi_0(x) + \tau\cos\left(\frac{\omega\tau}{2}\right)\Phi_1(x)\right) \times$$
$$\times \operatorname{erfc}\left(-\cos\left(\frac{\omega\tau}{2}\right)\sqrt{p}\right) e^{p\left(2\cos(\omega\tau) - \frac{\tau^2}{4}m^2\right)} e^{-i\omega t} d\omega.$$

Remark 21.4.2 One notice here that the Laplace operational identity (21.37) slightly differs from the one considered in [7, section 2] for the operational representation of an analogue for the Poisson type semigroup. Here we have considered a *time Fourier inversion formula* over $\left(-\frac{2\pi}{\tau}, \frac{2\pi}{\tau}\right]$, that results from the identity

$$\frac{1}{\sqrt{p}}\left(E_{\frac{1}{2},\frac{1}{2}}\left(\cos\left(\frac{\omega\tau}{2}\right)\sqrt{p}\right) + \frac{1}{\sqrt{\pi}}\right) =$$
$$= \cos\left(\frac{\omega\tau}{2}\right) e^{p\cos^2\left(\frac{\omega\tau}{2}\right)} \operatorname{erfc}\left(-\cos\left(\frac{\omega\tau}{2}\right)\sqrt{p}\right)$$

involving the *complementary error function* $\operatorname{erfc}(-u)$, instead of the subordination formula

$$e^{-\beta t} = \frac{t}{2\sqrt{\pi}}\int_0^\infty \frac{e^{-\frac{t^2}{4v}}}{v^{\frac{3}{2}}} e^{-v\beta^2} dv \quad (\beta > 0)$$

endowed by the kernel of the Weierstraß transform.

21.4.3 A Discrete Fractional Calculus Insight

In Sect. 21.3 we have shown that a simple operational substitution $s \to \sqrt{\Delta_h - m^2}$ on the functions

$$\cosh(tL^{-1}(s)) \text{ and } \frac{\sinh(tL^{-1}(s))}{s}$$

allows us to express, in a simple way, the solutions of the discretized Klein-Gordon and Dirac equations, (21.20) resp. (21.27). Such characterization may be reformulated in terms of the fractional operators

$$\left(-\Delta_h + m^2\right)^{-\alpha} \Phi(y) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tm^2} \exp\left(t\Delta_h\right)\left[\Phi(y)\right]\, t^\alpha \frac{dt}{t} \quad (21.38)$$

for values $0 < \alpha < \frac{1}{2}$, in spite of the right-hand side of (21.17) does not absolutely converges for $\alpha = \frac{1}{2}$ in the *massless limit* $m \to 0$. This is due to the fact that the Fourier multiplier $\left(d_h(\xi)^2\right)^{-\frac{1}{2}}$ of $\mathcal{F}_{h,\alpha} \circ (-\Delta_h)^{-\frac{1}{2}} \circ \mathcal{F}_{h,\alpha}^{-1}$ does not belong to the space $C^\infty(Q_h; \mathbb{C} \otimes C\ell_{n,n})$.

The following result, which complements the construction of Theorem 21.3.1, provides us an alternative way to obtain a solution for the discretized Klein-Gordon equation (21.20) as a *discrete convolution formula* endowed by the kernel functions

$$\begin{aligned} K_0^{(\alpha)}(x,t) &= \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{Q_h} (d_h(\xi)^2 + m^2)^\alpha \cosh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right) e^{-ix\cdot\xi} d\xi \\ K_1^{(\alpha)}(x,t) &= \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{Q_h} (d_h(\xi)^2 + m^2)^\alpha \frac{\sinh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right)}{i\sqrt{d_h(\xi)^2 + m^2}} e^{-ix\cdot\xi} d\xi. \end{aligned} \quad (21.39)$$

That corresponds to statement **(i)** of Theorem 21.4.1. Moreover, with the aid of the lattice discretization

$$\mathcal{R}_{h,\alpha} = \left(D_{h,\alpha} - m\gamma\right)(-\Delta_h + m^2)^{-\alpha} \quad (21.40)$$

of the *Riesz type operator* $(D - m\gamma)(-\Delta + m^2)^{-\alpha}$ on $\mathbb{R}^n_{h,\alpha}$ (cf. [3]), we are able to recover the solution of the Dirac equation (21.27). Such construction corresponds essentially to statements **(iii)** and **(iv)** of Theorem 21.4.1.

Theorem 21.4.1 *Let $\mathcal{R}_{h,\alpha}$ be the discretized Riesz transform defined by Eq. (21.40), and $K_0^{(\alpha)}$, $K_1^{(\alpha)}$ the kernel functions defined through the integral equations (21.39).*

Under the condition that Φ_0 *and* Φ_1 *belong to* $\mathcal{S}(\mathbb{R}^n_{h,\alpha}; \mathbb{C} \otimes C\ell_{n,n})$*, we have the following:*

(i) *The ansatz function* $\Psi(x,t)$ *defined by means of the discrete convolution formula*

$$\begin{aligned}\Psi(x,t) = &\sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n (-\Delta_h + m^2)^{-\alpha} \Phi_0(y)\; K_0^{(\alpha)}(x-y,t) \\ &+ \sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n (-\Delta_h + m^2)^{-\alpha} \Phi_1(y)\; K_1^{(\alpha)}(x-y,t)\end{aligned} \tag{21.41}$$

solves the discretized Klein-Gordon equation (21.20).

(ii) *The ansatz function (21.41) solves the discretized Dirac equation (21.27) whenever*

$$(-\Delta_h + m^2)^{-\alpha} \Phi_1(x) = i \mathcal{R}_{h,\alpha} \Phi_0(x).$$

(iii) *The inverse of the Riesz type operator* $\mathcal{R}_{h,\alpha}$ *is given by*

$$\left(\mathcal{R}_{h,\alpha}\right)^{-1} = (D_{h,\alpha} - m\gamma)(-\Delta_h + m^2)^{\alpha-1}.$$

(iv) *If* $\Psi_0(x,t) = \boldsymbol{P}_t[\Phi_0(x)]$ *and*

$$\Psi_1(x,t) = \mathbf{P}_t[(D_{h,\alpha} - m\gamma)(-\Delta_h + m^2)^{-1} \Phi_1(x)]$$

are two independent solutions of the discretized Dirac equation (21.27), generated by the discrete convolution operator

$$\begin{aligned}\mathbf{P}_t[\Phi(x)] = &\sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n (-\Delta_h + m^2)^{-\alpha} \Phi(y)\; K_0^{(\alpha)}(x-y,t) \\ &+ \sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n\; i \mathcal{R}_{h,\alpha} \Phi(y)\; K_1^{(\alpha)}(x-y,t),\end{aligned} \tag{21.42}$$

then the function

$$\Psi(x,t) = \frac{\Psi_0(x,t) + \Psi_0(x,-t)}{2} + \frac{\Psi_1(x,t) - \Psi_1(x,-t)}{2i}$$

solves the discretized Klein-Gordon equation (21.20).

Proof The proof of Theorem 21.4.1 follows the same train of thought of the proof of Theorem 21.3.1 and Corollary 21.3.1. To avoid an overlap between the proof of these results we present only an abridged version of it, by sketching only the main ideas:

Proof of (i):

From the Laplace transform identity

$$(d_h(\xi)^2 + m^2)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-tm^2} e^{-td_h(\xi)^2} \, t^\alpha \frac{dt}{t}$$

there holds the following identity for the operator $(-\Delta_h + m^2)^{-\alpha}$ defined through Eq. (21.38):

$$\mathcal{F}_{h,\alpha}[(-\Delta_h + m^2)^{-\alpha}\Phi](\xi) = (d_h(\xi)^2 + m^2)^{-\alpha}\mathcal{F}_{h,\alpha}\Phi(\xi).$$

Therefore, the sequence of identities

$$\begin{aligned}
\mathcal{F}_{h,\alpha}\Psi(\xi,t) &= \mathcal{F}_{h,\alpha}[(-\Delta_h + m^2)^{-\alpha}\Phi_0(\xi)]\, \mathcal{F}_{h,\alpha}K_0^{(\alpha)}(\xi,t) + \\
&+ \mathcal{F}_{h,\alpha}[(-\Delta_h + m^2)^{-\alpha}\Phi_1(\xi)]\, \mathcal{F}_{h,\alpha}K_1^{(\alpha)}(\xi,t) \\
&= (d_h(\xi)^2 + m^2)^{-\alpha}\mathcal{F}_{h,\alpha}\Phi_0(\xi)\, \mathcal{F}_{h,\alpha}K_0^{(\alpha)}(\xi,t) + \\
&+ (d_h(\xi)^2 + m^2)^{-\alpha}\mathcal{F}_{h,\alpha}\Phi_1(\xi)\, \mathcal{F}_{h,\alpha}K_1^{(\alpha)}(\xi,t) \\
&= \cosh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right)\mathcal{F}_{h,\alpha}\Phi_0(\xi) + \\
&+ \frac{\sinh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right)}{i\sqrt{d_h(\xi)^2 + m^2}}\mathcal{F}_{h,\alpha}\Phi_1(\xi)
\end{aligned}$$

yield straightforwardly from application of the *discrete convolution property* (21.13) underlying to the discrete Fourier transform $\mathcal{F}_{h,\alpha}$, and from the standard identities involving the wave kernels (21.39):

$$\mathcal{F}_{h,\alpha}K_0^{(\alpha)}(\xi,t) = (d_h(\xi)^2 + m^2)^{\alpha}\cosh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right)$$

$$\mathcal{F}_{h,\alpha}K_1^{(\alpha)}(\xi,t) = (d_h(\xi)^2 + m^2)^{\alpha}\frac{\sinh\left(tL^{-1}\left(i\sqrt{d_h(\xi)^2 + m^2}\right)\right)}{i\sqrt{d_h(\xi)^2 + m^2}}.$$

Thus, $\mathcal{F}_{h,\alpha}\Psi(\xi,t)$ is a solution of the evolution problem (21.21), and whence, the ansatz (21.41) solves the discretized Klein-Gordon equation (21.20).

Proof of (ii):

The proof that the condition $(-\Delta_h + m^2)^{-\alpha}\Phi_1(x) = i\mathcal{R}_{h,\alpha}\Phi_0(x)$ gives rise to the solution of the discretized Dirac equation (21.27) is an immediate consequence of Corollary 21.3.1 and of statement **(i)** of Theorem 21.4.1.

Proof of (iii):
By noting that the Clifford vector

$$\mathbf{r}_{h,\alpha}(\xi) = (\mathbf{z}_{h,\alpha}(\xi) - m\gamma)(d_h(\xi)^2 + m^2)^{-\alpha}$$

corresponds to the Fourier multiplier of $\mathcal{F}_{h,\alpha} \circ \mathcal{R}_{h,\alpha} \circ \mathcal{F}_{h,\alpha}^{-1}$, and the Clifford vector

$$\mathbf{s}_{h,\alpha}(\xi) = (\mathbf{z}_{h,\alpha}(\xi) - m\gamma)(d_h(\xi)^2 + m^2)^{\alpha-1}$$

to the inverse of $\mathbf{r}_{h,\alpha}(\xi)$, the proof that $\mathcal{S}_{h,\alpha} = (D_{h,\alpha} - m\gamma)(-\Delta_h + m^2)^{\alpha-1}$ equals to $\left(\mathcal{R}_{h,\alpha}\right)^{-1}$ follows straightforwardly from the set of identities

$$\mathcal{F}_{h,\alpha}\left[\mathcal{S}_{h,\alpha}\mathcal{R}_{h,\alpha}\Phi\right](\xi) = \mathcal{F}_{h,\alpha}\left[\mathcal{R}_{h,\alpha}\mathcal{S}_{h,\alpha}\Phi\right](\xi) = \mathcal{F}_{h,\alpha}\Phi(\xi).$$

Proof of (iv):
First, we recall that the splitting formulae

$$\frac{\mathbf{P}_t[\Phi(x)] + \mathbf{P}_{-t}[\Phi(x)]}{2} = \sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n(-\Delta_h + m^2)^{-\alpha}\Phi(y)\ K_0^{(\alpha)}(x-y,t)$$
$$\frac{\mathbf{P}_t[\Phi(x)] - \mathbf{P}_{-t}[\Phi(x)]}{2i} = \sum_{y \in \mathbb{R}^n_{h,\alpha}} h^n\ \mathcal{R}_{h,\alpha}\Phi(y)\ K_1^{(\alpha)}(x-y,t)$$

yield from the parity properties involving the hyperbolic functions cosh (even function) and sinh (odd function). By noting also that

$$(D_{h,\alpha} - m\gamma)(-\Delta_h + m^2)^{-1}\Phi_1(y) = (-\Delta_h + m^2)^{-\alpha}\mathcal{R}_{h,1-\alpha}\Phi_1(x),$$

the proof that the function

$$\Psi(x,t) = \frac{\Psi_0(x,t) + \Psi_0(x,-t)}{2} + \frac{\Psi_1(x,t) - \Psi_1(x,-t)}{2i}$$

provides a solution for the discretized Klein-Gordon equation (21.20) is rather immediate, since $\Psi(x,t)$ coincides with the ansatz (21.41). ■

Remark 21.4.2 (A Poisson Semigroup Counterpart) Statement **(iii)** of Theorem 21.4.1 may be seen as an hypercomplex analogue for the differential-difference Cauchy-Riemann equations, complementary to the one obtained in [7, Theorem 3.] in terms of Poisson semigroup based representations. In accordance with the discussion depicted in the end of Sect. 21.4.2, we can also see that the nonexistence of self-adjointness property for the discrete Dirac operators $D_{h,\alpha}$ (see Remark 21.2.2) is not an obstacle to this approach.

21.5 Outlook of the Main Results

This paper provides us a guideline to extend substancial part of the framework already done in [1, 2, 7, 8, 10, 12] to the differential-difference and to the difference-difference setting as well. To our best knowledge, there has been no literature paying attention to the description of the solutions of time-evolution problems as a blending between the continuous and the discrete side.

To mimic the construction depicted on [30, Chapter 2] (see Theorem 21.3.1, Corollary 21.3.1 and Theorem 21.4.1) we adopted in Sect. 21.2 the *toroidal Fourier framework* proposed by Ruzhansky and Turunen on their book [27, Part II, Chapter 3] to exploit the framework proposed by Gürlebeck and Sprössig in [22, subsection 5.2] to lattices of the type $\mathbb{R}^n_{h,\alpha} = (1-\alpha)h\mathbb{Z}^n + \alpha h\mathbb{Z}^n$ ($h > 0$ and $0 < \alpha < \frac{1}{2}$), based on the one-to-one correspondence between the so-called *n-Brioullin zone* $Q_h = \left(-\frac{\pi}{h}, \frac{\pi}{h}\right]^n$ and the toroidal manifold $\mathbb{R}^n/\frac{2\pi}{h}\mathbb{Z}^n$ (cf. [25]). We also propose in Sect. 21.2 a *fractional regularization* for discrete Dirac operators acting on the lattices $h\mathbb{Z}^n$ ($\alpha \to 0$) and $\frac{h}{2}\mathbb{Z}^n$ ($\alpha \to \frac{1}{2}$), based on construction of a wide class of Fourier multipliers with values on the Clifford algebra with signature (n, n).

From the results obtained in Sect. 21.4 (namely Theorem 21.4.1) we believe that the proposed approach does not offer only a wise strategy to determine discrete counterparts for the results provided by the papers [1, 2, 7, 8, 10, 12]. In the shed of the fractional calculus formulation proposed recently by Bernstein (cf. [3, 4]), it may also provides us a meaningful way to generalize the results of [5, 6], where only the properties of the *discrete Fourier transform* depicted in [22, subsection 5.2] were taken into account.

As a whole, the fractional integration approach combined with a *space-time Fourier inversion type formula* has been revealed as an exceptionally well-suited tool to represent, in an operational way, the discrete convolution representations underlying to the solutions of an equation of Klein-Gordon type. As we have noticed on Sect. 21.4.2, this is ultimately due to the possibility of describe the superposition of the wave-type propagators

$$\cosh(tL^{-1}(\Delta_h - m^2)) \text{ resp. } \frac{\sinh(tL^{-1}(\sqrt{\Delta_h - m^2}))}{\sqrt{\Delta_h - m^2}}$$

in terms of its Fourier-Laplace multipliers endowed by the *discrete heat kernel* $\exp\left(\frac{p\tau^2}{4\tau}\Delta_h\right)$ (see, for instance, Remark 21.4.2). For a general overview of this framework, we refer to [28, Chapter 5].

Summing Up The discrete Fourier transform framework brings into the representation of the solution for an evolution type equation over the momentum space $Q_h \times T$. In the future it can also be useful to look in depth for the associated hypersingular operator representations on the *space-time toroidal manifold* $(\mathbb{R}^n/\frac{2\pi}{h}\mathbb{Z}^n) \times (\mathbb{R}/\frac{4\pi}{\tau}\mathbb{Z})$ (see, for instance, Remark 21.4.1). Although the Fourier-Laplace type multipliers are a little trickier to compute, due to the fractional calculus technicalities, the

Fourier modes $e^{-i(\omega t+x\cdot\xi)}$ associated to the space-time Fourier inversion formula are more easier to treat on the space of tempered distributions, in comparison to the description of fractional integro-differential operators on the manifold $(\mathbb{R}^n/h\mathbb{Z}^n)\times[0,\infty)$ (see, for instance, subsection 24.10 of [28, Chapter 5]).

Appendix: The Exponential Generating Function Connection

In the paper [18] we have considered the exponential generating function (EGF) $\exp(tL^{-1}(s))$ to derive hypercomplex formulations for Appell sets and *Exponential Generating Function* (EGF). With the aim of construct umbral counterparts for the wave type propagators

$$\cosh\left(t\sqrt{\Delta_h-m^2}\right) \text{ resp.} \frac{\sinh\left(t\sqrt{\Delta_h-m^2}\right)}{\sqrt{\Delta_h-m^2}},$$

we consider here a wise adaptation of [15, Corollary 1.1.15] for $\cosh(tL^{-1}(s))$ and $\sinh(tL^{-1}(s))$. That corresponds to the following result:

Theorem A.1 *The formal series representation of* $\cosh(tL^{-1}(s))$ *and* $\sinh(tL^{-1}(s))$ *determined by the delta operator* $L_t=L(\partial_t)$ *are given by*

$$\cosh(tL^{-1}(s))=\sum_{k=0}^{\infty}\frac{m_{2k}(t)}{(2k)!}s^{2k} \text{ and } \sinh(tL^{-1}(s))=\sum_{k=0}^{\infty}\frac{m_{2k+1}(t)}{(2k+1)!}s^{2k+1},$$

where $\{m_k(t) \ : \ k\in\mathbb{N}_0\}$ *is a basic polynomial sequence associated to* L_t.

Proof of Theorem A.1 First, we recall that from [15, Corollary 1.1.15], the exponentiation operator $\exp(s\partial_t)$ may be formally represented as

$$\exp(t\partial_s)=\sum_{k=0}^{\infty}\frac{m_k(t)}{k!}L(\partial_s)^k.$$

Then, we have

$$\cosh(t\partial_s)=\frac{\exp(t\partial_s)+\exp(-t\partial_s)}{2}=\sum_{k=0}^{\infty}\frac{m_{2k}(t)}{(2k)!}L(\partial_s)^{2k}$$

$$\sinh(t\partial_s)=\frac{\exp(t\partial_s)-\exp(-t\partial_s)}{2}=\sum_{k=0}^{\infty}\frac{m_{2k+1}(t)}{(2k+1)!}L(\partial_s)^{2k+1}.$$

By applying the isomorphism theorem (cf. [26, Theorem 2.2.1]) one get a one-to-one correspondence between $\cosh(t\partial_s)$ resp. $\sinh(t\partial_s)$ with the formal power series expansions

$$\begin{aligned}\cosh(ts) &= \sum_{k=0}^{\infty} \frac{m_{2k}(t)}{(2k)!} L(s)^{2k} \\ \sinh(ts) &= \sum_{k=0}^{\infty} \frac{m_{2k+1}(t)}{(2k+1)!} L(s)^{2k+1}.\end{aligned} \tag{21.43}$$

The conclusion of Theorem A.1 for $\cosh(tL^{-1}(s))$ and $\sinh(tL^{-1}(s))$ then follows from the substitution $s \to L^{-1}(s)$ on both sides of (21.43). ■

Let us now take a close look for the *Exponential Generating Functions* (EGF) $\mathbf{G}(\mathbf{s}, t)$ of hypercomplex type defined by means of Eq. (21.6).

It is quite easy to see that the condition $m_0(t) = 1$ and the lowering properties $L_t m_k(t) = k m_{k-1}(t)$ ($k \in \mathbb{N}$) lead us naturally to the condition $\mathbf{G}(\mathbf{s}, 0) = 1$, and to the eigenvalue property

$$L_t \mathbf{G}(\mathbf{s}, t) = \mathbf{s}\mathbf{G}(\mathbf{s}, t). \tag{21.44}$$

Also, it is worth stressing that Theorem A.1 may me extended/generalized for hypercomplex variables. In particular, Theorem 21.1.1 corresponds to a wise generalization of Theorem A.1. The proof proceeds as follows:

Proof of Theorem 21.1.1: First, we recall that the even resp. odd part of $\mathbf{G}(\mathbf{s}, t)$ may be expressed as

$$\begin{aligned}\frac{\mathbf{G}(\mathbf{s}, t) + \mathbf{G}(-\mathbf{s}, t)}{2} &= \sum_{k=0}^{\infty} \frac{m_{2k}(t)}{(2k)!} \mathbf{s}^{2k} \\ \frac{\mathbf{G}(\mathbf{s}, t) - \mathbf{G}(-\mathbf{s}, t)}{2} &= \sum_{k=0}^{\infty} \frac{m_{2k+1}(t)}{(2k+1)!} \mathbf{s}^{2k+1}.\end{aligned}$$

Finally, from direct application of Theorem A.1, we recognize that for the Clifford numbers of the form

$$\mathbf{s} = re^{i\phi}\omega, \text{ with } r \geq 0, \quad -\pi < \phi \leq \pi \;\; \& \;\; \omega^2 = +1$$

the above set of identities equals to

$$\frac{\mathbf{G}(re^{i\phi}\omega,t)+\mathbf{G}(-re^{i\phi}\omega,t)}{2}=\cosh(tL^{-1}\left(re^{i\phi}\right))$$
$$\frac{\mathbf{G}(re^{i\phi}\omega,t)-\mathbf{G}(-re^{i\phi}\omega,t)}{2}=\omega\sinh(tL^{-1}\left(re^{i\phi}\right)),$$

since $\mathbf{s}^{2k} = \left(re^{i\phi}\right)^{2k}\left(\omega^2\right)^k = \left(re^{i\phi}\right)^{2k}$ and $\mathbf{s}^{2k+1} = \mathbf{s}\mathbf{s}^{2k} = \left(re^{i\phi}\right)^{2k+1}\omega$, concluding in this way that

$$\mathbf{G}(re^{i\phi}\omega,t)=\cosh\left(tL^{-1}\left(re^{i\phi}\right)\right)+\omega\sinh\left(tL^{-1}\left(re^{i\phi}\right)\right),$$

as desired. ■

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Chapter 22
Cauchy-Pompeiu Formula for Discrete Monogenic Functions

Guangbin Ren and Zeping Zhu

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract For the integral theory of discrete monogenic functions, we establish a new version of Cauchy-Pompeiu formula via the notions 'discrete boundary measure' and 'discrete normal vector'. It shares the same form with the continuous version of Cauchy-Pompeiu formula in contrast to the original Cauchy-Pompeiu formula in discrete Clifford analysis. It has applications in the boundary theory of discrete monogenic functions. We can thus set up the discrete Sokhotski-Plemelj formula and provide an equivalent characterization of the Dirichlet problem with the discrete Dirac operator in terms of the eigenvectors of certain operator.

Keywords Discrete monogenic function · Cauchy-Pompeiu formula · Sokhotski-Plemelj formula

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22.1 Introduction

The Cauchy-Pompeiu formula is the cornerstone of the integral theory in the complex analysis [12]. It has been generalized to high dimensions in Clifford analysis [1, 9]. Just as in the complex analysis, the high dimensional Cauchy-Pompeiu formula plays a vital role in the boundary behaviour of monogenic functions which are high dimensional analogs of holomorphic functions.

Over recent decades, there has been increasing interest in the discretization of harmonic analysis and complex analysis, because it has important applications in pure mathematics, physics, and computer science. Taking the discrete complex analysis as an example, it has an elegant interaction with statistical physics as demonstrated by Smirnov in [13]. He proved the convergence of the scaling limit of specific discrete analytic quantities, the so-called pre-holomorphic fermions. This reveals the important result of conformal invariance of 2-D Ising models.

Motivated by the numerical treatment of the potential theory and boundary value problems [7, 8], the discrete Clifford analysis is developed which is a high dimensional generalization of discrete complex analysis (see, e.g., [2–5, 10]). Brackx et al. [2] established a discrete integral theory based on their version of discrete Cauchy-Pompeiu formula. Unfortunately, there is a term called residue in their formula which may cause difficulties in applications. It is worth trying to find a better formula without a residue.

The purpose of this article is to establish a new version of discrete Cauchy-Pompeiu formula without a residue. To achieve this goal, we need to introduce notions 'discrete boundary measure' and 'discrete normal vector'. In our new version of the Cauchy-Pompeiu formula, the two kernels related to the surface measure and the volume measure turn out to be different, in contrast to the classical version in [2]. Moreover, our formula shares the same form with the continuous version of Cauchy-Pompeiu formula in contrast to the original Cauchy-Pompeiu formula in discrete Clifford analysis,

This new formula is very helpful to study the boundary behaviour of discrete monogenic functions. It allows us to establish the Sokhotski-Plemelj formula for the discrete Dirac operator, which leads to a solvability criterion for the Dirichlet problem of the discrete Dirac operator.

Now we come to state our main theorems. We leave the detail notations to the next section.

Let D_m^h be the discrete Dirac operator on the grid $\mathbb{Z}_h^m$ with $m = 2, 3, \cdots$, and E_m^h the fundamental solution of D_m^h. Let S and $\vec{n} = (n_1^+, n_1^-, \cdots, n_m^+, n_m^-)$ stand for the discrete boundary measure and the discrete outward normal vector on the discrete boundary ∂B of a given subset B of $\mathbb{Z}_h^m$, respectively. The surface Cauchy-Pompeiu kernel is defined by

$$\mathcal{K}_m^h(x, y) := -\sum_{l=1}^{m} \left(E_m^h(he_l - x + y) n_l^-(x) e_l^+ + E_m^h(-he_l - x + y) n_l^+(x) e_l^- \right),$$

where $\{e_k^{\pm}\}_{k=1}^m$ is the Witt basis of the complex Clifford algebra $\mathbb{C}_{2m}$.

Theorem 22.1.1 (Cauchy-Pompeiu) *Let B be a bounded subset of $\mathbb{Z}_h^m$. Then for any function $f : \overline{B} \longrightarrow \mathbb{C}_{2m}$ we have*

$$\chi_B(y)f(y) = \int_{\partial B} \mathcal{K}_m^h(x, y)f(x)dS(x) + \int_B E_m^h(y-x)D_m^h f(x)dV_m^h(x). \qquad (22.1)$$

Denote by $\mathcal{F}_{\partial B}^h$ the discrete Cauchy integral operator. Associated with $\mathcal{F}_{\partial B}^h$, we construct the operator $\mathcal{S}_{\partial B}^h : F(\partial B, \mathbb{C}_{2m}) \longrightarrow F(\partial B, \mathbb{C}_{2m})$ via

$$\mathcal{S}_{\partial B}^h f(y) := 2\int_{\partial B} \mathcal{K}_m^h(x, y)\big(f(x) - f(y)\big)dS(x) + f(y).$$

Here $F(U, \mathbb{C}_{2m})$ is the space of $\mathbb{C}_{2m}$-valued functions defined on $U \subset \mathbb{Z}_h^m$. In the following theorem, the discrete boundary ∂B is separated into the inner and outer layers $\partial^{\pm} B$, defined by

$$\partial^+ B := B \bigcap \partial B, \qquad \partial^- B := B \setminus \partial B.$$

Theorem 22.1.2 (Sokhotski-Plemelj) *Let B be a bounded set in $\mathbb{Z}_h^m$. Then for any $f \in F(\partial B, \mathbb{C}_{2m})$ we have*

$$\mathcal{F}_{\partial B}^h f(y) = \frac{1}{2}\big(\pm f(y) + \mathcal{S}_{\partial B}^h f(y)\big), \qquad \forall\, y \in \partial^{\pm} B.$$

Based on the discrete Sokhotski-Plemelj formula, we can provide an equivalent characterization for the Dirichlet problem of the discrete Dirac operator in terms of the eigenvectors of the operator $\mathcal{S}_{\partial B}^h$.

Theorem 22.1.3 (Dirichlet Problem) *Let B be a bounded subset of $\mathbb{Z}_h^m$ and $f \in F(\partial B, \mathbb{C}_{2m})$ be any given function. Then*

(i) *f is the boundary values of a discrete monogenic function defined on $\overline{B}$ if and only if*

$$\mathcal{S}_{\partial B}^h f(y) = f(y), \qquad \forall\, y \in \partial B.$$

(ii) *f is the boundary values of a discrete monogenic function defined on $\overline{\mathbb{Z}_h^m \setminus B}$ which vanishes at infinity if and only if*

$$-\mathcal{S}_{\partial B}^h f(y) = f(y), \qquad \forall\, y \in \partial B.$$

22.2 Basic Concepts

Discrete Clifford analysis is a function theory built on the m-dimensional grid $\mathbb{Z}_h^m$. Here

$$\mathbb{Z}_h^m := (h\mathbb{Z})^m$$

with $m \geqslant 2$ and $h > 0$. This grid is embedded into the complex Clifford algebra $\mathbb{C}_{2m}$ by identifying the point $(x_1, \cdots, x_m)$ with the vector $x_1e_1 + \cdots + x_me_m$, where e_k is the generators of the algebra subject to the following identities:

$$e_k^2 = -1 \quad (1 \leqslant k \leqslant 2m), \quad e_le_n = -e_ne_l \quad (1 \leqslant l < n \leqslant 2m).$$

22.2.1 Discrete Monogenicity

Definition 22.2.1 ([2]) The discrete Dirac operator on $\mathbb{Z}_h^m$ is defined by

$$D_m^h := \sum_{k=1}^{m} e_k^+ \partial_k^{+,h} + e_k^- \partial_k^{-,h}.$$

Here $\partial_k^{+,h}$ and $\partial_k^{-,h}$ stand for the forward and backward difference operators along the direction e_k , i.e.,

$$\partial_k^{+,h} f(x) = \frac{f(x + he_k) - f(x)}{h},$$
$$\partial_k^{-,h} f(x) = \frac{f(x) - f(x - he_k)}{h},$$

and $\{e_k^{\pm}\}_{k=1}^m$ is the Witt basis of $\mathbb{C}_{2m}$, given by

$$e_k^{\pm} = \frac{1}{2}(e_k \pm ie_{k+m}).$$

Definition 22.2.2 Let B be a subset of $\mathbb{Z}_h^m$. Its discrete closure and interior are defined respectively as

$$\overline{B} := B \cup \partial B, \qquad B^\circ := B \setminus \partial B,$$

where ∂B is the discrete boundary of B consisting of every point $x \in \mathbb{Z}_h^m$ whose neighborhood

$$N(x) := \{x, x \pm he_1, x \pm he_2, \cdots, x \pm he_m\}$$

has some points inside B and some other points outside B, i.e.,

$$\partial B := \left\{ x \in \mathbb{Z}_h^m : N(x) \cap B \neq \emptyset \quad and \quad N(x) \setminus B \neq \emptyset \right\}.$$

A function is said to be discrete monogenic if it is annihilated by the discrete Dirac operator D_m^h.

Definition 22.2.3 ([2]) A function $f : \overline{B} \longrightarrow \mathbb{C}_{2m}$ with $B \subset \mathbb{Z}_h^m$ is said to be discrete monogenic on B if for any $x \in B$ we have $D_m^h f(x) = 0$.

22.2.2 Discrete Boundary Measure and Discrete Outward Normal Vector

In our previous work on the discrete quaternionic functions, we has established an integral theory based on the concepts of discrete boundary measure and discrete outward normal vector on the grid $\mathbb{Z}_h^4$ [11]. In this paper, we shall use the same method to give a new version of Cauchy-Pompeiu formula for the discrete monogenic functions.

Definition 22.2.4 Let B be a subset of $\mathbb{Z}_h^m$. The discrete boundary measure S on ∂B is defined as

$$S(U) = \sum_{x \in U} s(x), \quad \forall\, U \subset \partial B,$$

where $s : \partial B \longrightarrow \mathbb{R}$ is the density function

$$s = \frac{h^m}{2} \sqrt{\sum_{k=1}^{m} \left((\partial_k^{+,h} \chi_B)^2 + (\partial_k^{-,h} \chi_B)^2 \right)}, \tag{22.2}$$

and χ_B stands for the characteristic function of B.

Definition 22.2.5 The discrete outward normal vector at a boundary point of $B \subset \mathbb{Z}_h^m$ is a vector

$$\vec{n} = (n_1^+, n_1^-, n_2^+, n_2^-, \cdots, n_m^+, n_m^-),$$

defined by

$$n_l^{\pm} = \frac{-2\, \partial_l^{\pm,h} \chi_B}{\sqrt{\sum_{k=1}^{m} \left((\partial_k^{+,h} \chi_B)^2 + (\partial_k^{-,h} \chi_B)^2 \right)}}, \qquad l = 1, \cdots, m.$$

It is obvious that the Euclidean norm of $\vec{n}$ is always equal to 2 on ∂B.

Denote by V_m^h the Haar measure on the discrete group $\mathbb{Z}_h^m$. More precisely,

$$V_m^h(U) = \sum_{x \in U} h^m, \qquad \forall\, U \subset \mathbb{Z}_h^m.$$

Theorem 22.2.6 (Divergence Theorem) *Let B be a bounded subset of $\mathbb{Z}_h^4$. For any function $f : \overline{B} \longrightarrow \mathbb{R}$, we have*

$$\int_{\partial B} f n_k^{\pm} dS = \int_B \partial_k^{\mp,h} f dV_m^h, \quad (i = 1, \cdots, m).$$

Proof This theorem has been proved in the case $m = 4$, the generalization to the other dimensions is quit trivial. One may refer to Section 3 in [11] about the details in the 4-dimensional case. □

22.3 Discrete Integral Formulae

In this section we rebuild the integral formulae for the discrete monogenic functions.

We define the discrete vector in $\mathbb{C}_{2m}$ associated with the discrete outward normal vector $\vec{n}$ as

$$\hat{n} := \sum_{k=1}^{m} (e_k^+ n_k^- + e_k^- n_k^+).$$

It plays the same role in the integral theory as its continuous counterpart $\sum_{k=1}^{m} e_k n_k$.

Theorem 22.3.1 (Cauchy's Theorem) *Let B be a bounded subset of $\mathbb{Z}_h^m$. If $f : \overline{B} \longrightarrow \mathbb{C}_{2m}$ is discrete monogenic in B, then the following identity holds true*

$$\int_{\partial B} \hat{n} f dS = 0.$$

Proof According to the discrete divergence theorem (Theorem 22.2.6), we have

$$\int_{\partial B} \hat{n} f dS = \int_B D_m^h f dV_m^h.$$

Since f is discrete monogenic, the proof is completed. □

Theorem 22.3.2 (Morera's Theorem) *Let B be a subset of $\mathbb{Z}_h^m$. A function $f : \overline{B} \longrightarrow \mathbb{C}_{2m}$ is discrete monogenic if*

$$\int_{\partial U} \hat{n} f dS = 0,$$

for any bounded subset U of B.

Proof Again Theorem 22.2.6 yields

$$\int_U D_m^h f dV_m^h = \int_{\partial U} \hat{n} f dS = 0.$$

This means

$$\sum_{x \in U} D_m^h f(x) = 0.$$

Hence by choosing U as each single point subset of B, we obtain $D_m^h f$ vanishes on B. □

Denote V_m by the Lebesgue measure on $[-\pi, \pi]^m$. In [2, 3, 6] the fundamental solution of the discrete Dirac operator has been extensively studied.

Definition 22.3.3 ([2]) The fundamental solution of the difference operator D_m^h is defined as

$$E_m^h(x) = \frac{1}{h^{m-1}} E_m\left(\frac{x}{h}\right).$$

Here

$$E_m(x) = \frac{1}{(2\pi)^m} \int_{[-\pi,\pi]^m} \frac{\sum_{k=1}^m (e_k^+ \xi_{+k} + e_k^- \xi_{-k})}{4 \sum_{k=1}^m \sin^2 \frac{\xi_k}{2}} e^{-i \sum\limits_{l=1}^{m} \xi_l x_l} dV_m(\xi)$$

with

$$\xi_{\pm k} = \mp(1 - e^{\mp i \xi_k}).$$

Denote the discrete Dirac delta function on $\mathbb{Z}_h^m$ by $\delta_0^{h,m}$, i.e.,

$$\delta_0^{h,m}(x) = \begin{cases} h^{-m}, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

E_m^h is called the fundamental solution of D_m^h because it satisfies

$$D_m^h E_m^h = E_m^h D_m^h = \delta_0^{h,m} \qquad \text{in } \mathbb{Z}_h^m.$$

We restate Theorem 22.1.1 as follows:

Let B be a bounded subset of $\mathbb{Z}_h^m$. Then for any function $f : \overline{B} \longrightarrow \mathbb{C}_{2m}$ we have

$$\chi_B(y)f(y) = \int_{\partial B} \mathcal{K}_m^h(x,y)f(x)dS(x) + \int_B E_m^h(y-x)D_m^h f(x)dV_m^h(x). \tag{22.3}$$

Here the Cauchy-Pompeiu kernel on the boundary ∂B is given by

$$\mathcal{K}_m^h(x,y) := -\sum_{l=1}^m \big(E_m^h(he_l - x + y)n_l^-(x)e_l^+ + E_m^h(-he_l - x + y)n_l^+(x)e_l^-\big).$$

Proof of Theorem 22.1.1 Applying Theorem 22.2.6 to the boundary integral

$$\int_{\partial B} \mathcal{K}_m^h(x,y)f(x)dS(x),$$

we obtain

$$\begin{aligned}\int_{\partial B} \mathcal{K}_m^h(x,y)f(x)dS(x) = -\int_B \sum_{l=1}^m \partial_l^{+,h}\big(E_m^h(he_l - \cdot + y)e_l^+ f\big)(x) \\ + \partial_l^{-,h}\big(E_m^h(-he_l - \cdot + y)e_l^- f\big)(x)dV_m^h(x).\end{aligned} \tag{22.4}$$

By simple calculation, we have

$$\partial_l^{\pm,h}\big(E_m^h(\pm he_l - \cdot + y)e_l^{\pm} f\big) = E_m^h(-\cdot + y)e_l^{\pm}(\partial_l^{\pm,h} f) - (\partial_l^{\pm,h} E_m^h)(-\cdot + y)e_l^{\pm} f.$$

Hence

$$\begin{aligned}&\sum_{l=1}^m \partial_l^{+,h}\big(E_m^h(he_l - \cdot + y)e_l^+ f\big)(x) + \partial_l^{-,h}\big(E_m^h(-he_l - \cdot + y)e_l^- f\big)(x) \\ =&E_m^h(-x+y)D_m^h f(x) - (E_m^h D_m^h)(-x+y)f(x) \\ =&E_m^h(-x+y)D_m^h f(x) - \delta_0^{h,m}(-x+y)f(x).\end{aligned}$$

Therefore, substituting the identity above into (22.4), we obtain

$$\int_{\partial B} \mathcal{K}_m^h(x, y) f(x) dS(x) = -\int_B E_m^h(y-x) D_m^h f(x) - \delta_0^{h,m}(y-x) f(x) dV_m^h(x).$$

It completes the proof. □

Corollary 22.3.4 *If f is discrete monogenic on a bounded subset B of $\mathbb{Z}_h^m$, then*

$$\chi_B(y) f(y) = \int_{\partial B} \mathcal{K}_m^h(x, y) f(x) dS(x).$$

We observe that $\mathcal{K}_m^h(x, \cdot)$ is discrete monogenic away from the boundary ∂B.

Theorem 22.3.5 *For any given $x \in \partial B$, the kernel $\mathcal{K}_m^h(x, \cdot)$ is discrete monogenic on $\mathbb{Z}_h^m \setminus N(x)$.*

Proof It follows from the fact that $E_m^h(\pm h e_l - x + \cdot)$ is discrete monogenic on $\mathbb{Z}_h^m \setminus N(x)$. □

22.4 Boundary Behaviour of Discrete Monogenic Functions

22.4.1 Discrete Sokhotski-Plemelj Formula

Firstly, we introduce some basic notations. Let $F(X, \mathbb{C}_{2m})$ stand for the space of $\mathbb{C}_{2m}$-valued functions defined on $X \subset \mathbb{Z}_h^m$.

Definition 22.4.1 Let B be a bounded set in $\mathbb{Z}_h^m$. The operator

$$\mathcal{F}_{\partial B}^h : F(\partial B, \mathbb{C}_{2m}) \longrightarrow F(\mathbb{Z}_h^m, \mathbb{C}_{2m}),$$

defined by

$$\mathcal{F}_{\partial B}^h f(y) := \int_{\partial B} \mathcal{K}_m^h(x, y) f(x) dS(x),$$

is called the discrete Cauchy integral operator. The operator

$$\mathcal{T}_B^h : F(B, \mathbb{C}_{2m}) \longrightarrow F(\mathbb{Z}_h^m, \mathbb{C}_{2m}),$$

defined by

$$\mathcal{T}_B^h f(y) := \int_B E_m^h(y-x) f(x) dV_m^h(x),$$

is called the discrete Teodorescu operator.

As their continuous counterparts, the discrete Teodorescu operator $\mathcal{T}_B^h$ is a right inverse of the discrete Dirac operator D_m^h. Namely, for any $f \in F(B, \mathbb{C}_{2m})$,

$$D_m^h \mathcal{T}_B^h f = f.$$

Remark 22.4.2 With these notations the discrete Cauchy-Pompeiu formula in Theorem 22.1.1 can be rewritten in the form

$$\mathcal{F}_{\partial B}^h f(y) + \mathcal{T}_B^h (D_m^h f)(y) = \chi_B(y) f(y).$$

Associated with the discrete Cauchy integral $\mathcal{F}_{\partial B}^h$, we introduce an important integral operator

$$\mathcal{S}_{\partial B}^h : F(\partial B, \mathbb{C}_{2m}) \longrightarrow F(\partial B, \mathbb{C}_{2m})$$

defined by

$$\mathcal{S}_{\partial B}^h f(y) := 2 \int_{\partial B} \mathcal{K}_m^h(x, y)\big(f(x) - f(y)\big) dS(x) + f(y).$$

Now we come to prove the discrete Sokhotski-Plemelj formula in Theorem 22.1.2.

Proof of Theorem 22.1.2 From the definition of $\mathcal{S}_{\partial B}^h$, it follows immediately that for any $y \in \partial B$

$$\begin{aligned} \mathcal{S}_{\partial B}^h f(y) &= 2\mathcal{F}_{\partial B}^h (f - f(y))(y) + f(y) \\ &= 2\mathcal{F}_{\partial B}^h f(y) - 2f(y)\mathcal{F}_{\partial B}^h(1)(y) + f(y). \end{aligned}$$

On the other hand, taking $f = 1$ in Remark 22.4.2 yields that

$$\mathcal{F}_{\partial B}^h(1)(y) = \begin{cases} 1, & y \in \partial^+ B, \\ 0, & y \in \partial^- B. \end{cases}$$

Therefore, we conclude that

$$\begin{aligned} \mathcal{F}_{\partial B}^h f(y) - \frac{1}{2}\mathcal{S}_{\partial B}^h f(y) &= f(y)\mathcal{F}_{\partial B}^h(1)(y) - \frac{1}{2} f(y) \\ &= \begin{cases} +\frac{1}{2} f(y), & y \in \partial^+ B, \\ -\frac{1}{2} f(y), & y \in \partial^- B. \end{cases} \end{aligned}$$

Namely,

$$\mathcal{F}^h_{\partial B} f(y) = \frac{1}{2}\big(\pm f(y) + \mathcal{S}^h_{\partial B} f(y)\big), \qquad \forall\, y \in \partial^{\pm} B.$$

It completes the proof. □

22.4.2 Characterizations of Boundary Values of Discrete Monogenic Functions

In this section we shall give a characterization for the boundary data of discrete monogenic functions.

In the work on the discrete Hardy space [3], a pointwise estimate for the fundamental solution was given as

$$|E^h_m(x)| \leqslant \frac{C}{|x|^{m-1} + h^{m-1}} + \frac{Ch}{|x|^m + h^m}, \qquad \forall\, x \in \mathbb{Z}^m_h, \tag{22.5}$$

where C is a positive constant independent of h and x. For more details, see Lemma 2.8 in [3]. We remark that the inequality is only proved in the case $m = 3$ but the approach is valid in other dimensions.

The following Cauchy formula for discrete monogenic functions vanishing at infinity is very useful.

Lemma 22.4.3 *Let B be a bounded subset of $\mathbb{Z}^m_h$. Then for any discrete monogenic function $u : \overline{\mathbb{Z}^m_h \setminus B} \longrightarrow \mathbb{C}_{2m}$ vanishing at infinity, we have*

$$-\mathcal{F}^h_{\partial B} u(y) = \chi_{\mathbb{Z}^m_h \setminus B}(y) u(y).$$

Proof For convenience, we identity the function u with its zero extension into the whole lattice $\mathbb{Z}^m_h$. Set

$$\Omega_n = \{0, \pm h, \pm 2h, \cdots, \pm nh\}^m$$

and apply Remark 22.4.2 to the bounded discrete domains Ω_n and B. Then we get

$$\mathcal{F}^h_{\partial \Omega_n} u = \chi_{\Omega_n} u - \mathcal{T}^h_{\Omega_n}(D^h_m u)$$

and

$$\mathcal{F}^h_{\partial B} u = \chi_B u - \mathcal{T}^h_B(D^h_m u).$$

Hence

$$\mathcal{F}^h_{\partial\Omega_n} u - \mathcal{F}^h_{\partial B} u = \chi_{\Omega_n \setminus B} u - \mathcal{T}^h_{\Omega_n \setminus B}(D^h_m u).$$

Since $D^h_m u$ vanishes on $\Omega_n \setminus B$, we have

$$\mathcal{T}^h_{\Omega_n \setminus B}(D^h_m u) \equiv 0,$$

which indicates

$$\mathcal{F}^h_{\partial\Omega_n} u - \mathcal{F}^h_{\partial B} u = \chi_{\Omega_n \setminus B} u.$$

It is apparent that as n tends to $+\infty$, the right side of this identity converges to $\chi_{\mathbb{Z}^m_h \setminus B} u$. We also claim that the first item in the left side converges to zero as n tends to $+\infty$, so that

$$-\mathcal{F}^h_{\partial B} u = \chi_{\mathbb{Z}^m_h \setminus B} u.$$

Now we come to prove that $\mathcal{F}^h_{\partial\Omega_n} u$ converges to zero point-wisely as n tends to $+\infty$. Because Ω_n has a quite simple structure, its discrete boundary can be easily found. Indeed,

$$\partial\Omega_n = \bigcup_{k=1}^{m} \left\{ x \in \mathbb{Z}^m_h : |x_k| = nh, (n+1)h \text{ and } |x_j| \leqslant nh \ (j \neq k) \right\}.$$

On the other hand, by direct calculation the discrete boundary measure S on $\partial\Omega_n$ satisfies the estimate

$$S(\partial\Omega_n) \leqslant C h^{m-1} (n+1)^{m-1},$$

where C is a positive constant independent of n. Consequently,

$$\begin{aligned}
\left| \mathcal{F}^h_{\partial\Omega_n} u(y) \right| &= \left| \int_{\partial\Omega_n} \mathcal{K}^h_m(x,y) u(x) dS(x) \right| \\
&\leqslant \max_{\partial\Omega_n} |u| \max_{x \in \partial\Omega_n} |\mathcal{K}^h_m(x,y)| S(\partial\Omega_n) \\
&\leqslant C h^{m-1} \max_{\partial\Omega_n} |u| \left(\max_{x \in \partial\Omega_n} |\mathcal{K}^h_m(x,y)| (n+1)^{m-1} \right).
\end{aligned}$$

It is easy to see that $\partial\Omega_n$ moves towards infinity as n tends to $+\infty$. Hence $\max\limits_{\partial\Omega_n} |u|$ converges to zero by assumption. Then it remains to show

$$\max_{x \in \partial\Omega_n} |\mathcal{K}^h_m(x,y)| n^{m-1} = O(1).$$

Recall that the discrete kernel $\mathcal{K}_m^h$ can be expressed via the fundamental solution E_m^h and the discrete normal vector $\vec{n}$ as

$$\mathcal{K}_m^h(x,y) := -\sum_{l=1}^{m} E_m^h(he_l - x + y)n_l^-(x)e_l^+ + E_m^h(-he_l - x + y)n_l^+(x)e_l^-.$$

Since

$$\sum_{l=1}^{m}(n_l^+)^2 + (n_l^-)^2 = 4,$$

we have

$$|\mathcal{K}_m^h(x,y)| \leqslant 2\sum_{l=1}^{m}|E_m^h(he_l - x + y)| + |E_m^h(-he_l - x + y)|.$$

Let y be fixed. In virtue of (22.5) there exists a constant C such that for any $x \in \mathbb{Z}_h^m$ with $|x| > |y| + h$ we have

$$|\mathcal{K}_m^h(x,y)| \leqslant \frac{C}{(|x|-|y|-h)^{m-1} + h^{m-1}} + \frac{Ch}{(|x|-|y|-h)^m + h^m}$$

so that, when n is sufficiently large,

$$\max_{x\in\partial\Omega_n} |\mathcal{K}_m^h(x,y)| \leqslant \frac{C}{(nh-|y|-h)^{m-1} + h^{m-1}} + \frac{Ch}{(nh-|y|-h)^m + h^m}.$$

Therefore,

$$\max_{x\in\partial\Omega_n} |\mathcal{K}_m^h(x,y)|n^{m-1} = O(1).$$

It completes the proof. □

Let B be a bounded subset of $\mathbb{Z}_h^m$ and $f \in F(\partial B, \mathbb{C}_{2m})$ be any given function, not identically zero. Theorem 22.1.3 can be restated as follows:

(i) The nonzero function $f \in F(\partial B, \mathbb{C}_{2m})$ is the boundary values of a discrete monogenic function defined on $\overline{B}$ if and only if f is an eigenvector of the operator $\mathcal{S}_{\partial B}^h : F(\partial B, \mathbb{C}_{2m}) \longrightarrow F(\partial B, \mathbb{C}_{2m})$ with eigenvalue 1.
(ii) The nonzero function $f \in F(\partial B, \mathbb{C}_{2m})$ is the boundary values of a discrete monogenic function defined on $\overline{\mathbb{Z}_h^m \setminus B}$ which vanishes at infinity if and only if f is an eigenvector of the operator $\mathcal{S}_{\partial B}^h : F(\partial B, \mathbb{C}_{2m}) \longrightarrow F(\partial B, \mathbb{C}_{2m})$ with eigenvalue -1.

Proof of Theorem 22.1.3 By Remark 22.4.2, for any function $u : \mathbb{Z}_h^m \longrightarrow \mathbb{C}_{2m}$ we have

$$\mathcal{F}_{\partial B}^h u(y) = \begin{cases} u(y) - \mathcal{T}_B^h(D_m^h u)(y), & y \in \partial^+ B, \\ -\mathcal{T}_B^h(D_m^h u)(y), & y \in \partial^- B. \end{cases}$$

Now we let u be an arbitrary extension of f from ∂B into $\mathbb{Z}_h^m$. By Theorem 22.1.2 we then obtain

$$\mathcal{S}_{\partial B}^h f(y) = u(y) - 2\mathcal{T}_B^h(D_m^h u)(y), \qquad y \in \partial B. \tag{22.6}$$

(i) Assume that

$$\mathcal{S}_{\partial B}^h f(y) = f(y), \qquad \forall\, y \in \partial B.$$

Thus one can easily see that

$$f(y) = u(y) - \mathcal{T}_B^h(D_m^h u)(y), \qquad y \in \partial B.$$

Hence f represents the boundary data of the function

$$F(y) := u(y) - \mathcal{T}_B^h(D_m^h u)(y), \qquad y \in \overline{B},$$

which is discrete monogenic since

$$D_m^h F(y) = D_m^h u(y) - (D_m^h \mathcal{T}_B^h)(D_m^h u)(y) = D_m^h u(y) - D_m^h u(y) = 0, \qquad y \in B.$$

(ii) Assume that f represents the boundary data of a discrete monogenic function $u : \overline{B} \to \mathbb{C}_{2m}$. Then it follows immediately from (22.6) that

$$\mathcal{S}_{\partial B}^h f(y) = u(y) - 2\mathcal{T}_B^h(D_m^h u)(y) = u(y) = f(y), \qquad y \in \partial B.$$

Therefore we conclude that f is the boundary values of a discrete monogenic function defined on $\overline{B}$ if and only if

$$\mathcal{S}_{\partial B}^h f(y) = f(y), \qquad \forall\, y \in \partial B.$$

(iii) Assume that

$$\mathcal{S}_{\partial B}^h f(y) = -f(y), \qquad \forall\, y \in \partial B.$$

(22.6) yields

$$f(y) = \mathcal{T}_B^h(D_m^h u)(y), \qquad y \in \partial B.$$

Thus f represents the boundary data of the function

$$G(y) := \mathcal{T}_B^h(D_m^h u)(y), \qquad y \in \overline{\mathbb{Z}_h^m \setminus B}.$$

Moreover, since

$$D_m^h \mathcal{T}_B^h u(y) = \chi_B(y)u(y), \qquad y \in \mathbb{Z}_h^m$$

holds true for any function $u : B \to \mathbb{C}_{2m}$, we have

$$D_m^h G(y) = (D_m^h \mathcal{T}_B^h)(D_m^h u)(y) = 0, \qquad y \in \mathbb{Z}_h^m \setminus B.$$

Namely, G is discrete monogenic. On the other hand, (22.5) says

$$|E_m^h(x)| \leqslant \frac{C}{|x|^{m-1} + h^{m-1}} + \frac{Ch}{|x|^m + h^m}. \tag{22.7}$$

Therefore,

$$\begin{aligned}
&|\mathcal{T}_B^h(D_m^h u)(y)| \\
&\leqslant V_m^h(B) \max_B |D_m^h u| \max_{x\in B} |E_m^h(y-x)| \\
&\leqslant V_m^h(B) \max_B |D_m^h u| \left(\frac{1}{(|y|-\rho)^{m-1} + h^{m-1}} + \frac{Ch}{(|y|-\rho)^m + h^m} \right),
\end{aligned}$$

where $\rho = \max_{x\in B} |x|$. This means $\mathcal{T}_B^h(D_m^h u)$ vanishes at infinity.

(iiii) Assume that f represents the boundary data of a discrete monogenic function $u : \overline{\mathbb{Z}_h^m \setminus B} \to \mathbb{C}_{2m}$ vanishing at infinity. By Lemma 22.4.3, we thus have

$$-\mathcal{F}_{\partial B}^h u(y) = \chi_{\mathbb{Z}_h^m \setminus B}(y)u(y), \qquad y \in \mathbb{Z}_h^m.$$

Then it follows from Theorem 22.1.2 that

$$\chi_{\mathbb{Z}_h^m \setminus B}(y)u(y) = -\mathcal{F}_{\partial B}^h u(y) = \frac{1}{2}\big(\mp f(y) - \mathcal{S}_{\partial B}^h f(y)\big), \qquad \forall\, y \in \partial^{\pm} B.$$

Hence, according to the definitions of $\partial^{\pm} B$ we have

$$\mathcal{S}_{\partial B}^h f(y) = -f(y), \qquad y \in \partial B.$$

Therefore, we conclude that f is the boundary values of a discrete monogenic function defined on $\overline{\mathbb{Z}_h^m \setminus B}$, which vanishes at infinity, if and only if

$$\mathcal{S}_{\partial B}^h f(y) = -f(y), \qquad y \in \partial B.$$

It completes the proof.

□

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Chapter 23
A Useful Transformation for Solving the Discrete Beltrami Equation and Reducing a Difference Equation of Second Order to a System of Equations of First Order

Angela Hommel

Dedicated to Wolfgang Sprößig on the occasion of his 70th birthday

Abstract In the plane elliptic equations of second order with constant coefficients can be transformed into an 2×2-equation system of first order by using a coordinate transform. The system can be described by methods of complex analysis. Equations similar to Vekua equations are obtained. From the classical theory it is well known that by using the coordinate transformation the Beltrami equation is fulfilled. The solution of this equation can be represented by the help of the Π-operator. This paper shows how similar results can be obtained in the discrete case. In this case the functions are defined on an uniform lattice with step size h. The aim is not only an approximation of the classical operators. The most important fact is that the discrete operators should preserve the basic properties. Especially a discrete Π-operator is defined which can be used to describe the solution of the discrete Beltrami equation.

Keywords Discrete Π operator · Discrete Beltrami equation

Mathematics Subject Classification (2010) Primary 39A70; Secondary 39A12, 32A10

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23.1 The Transformation in the Classical Theory

In the continuous case equations of the form $a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} = f$ can be rewritten in the form $u_{\omega_0,\omega_0} + u_{\omega_1,\omega_1} = \tilde{f}$ by applying the transformation

$$\frac{\partial \omega_0}{\partial x} = \frac{b}{\sqrt{\Delta}}\frac{\partial \omega_1}{\partial x} + \frac{c}{\sqrt{\Delta}}\frac{\partial \omega_1}{\partial y}$$
$$\frac{\partial \omega_0}{\partial y} = -\frac{a}{\sqrt{\Delta}}\frac{\partial \omega_1}{\partial x} - \frac{b}{\sqrt{\Delta}}\frac{\partial \omega_1}{\partial y}$$

with $\Delta = ac - b^2 > 0$. Many statements from analysis and geometry, such as the problem of conforming a surface to a plane, lead to this problem. A further development and application of the method on issues of quasi conformal mappings can be found in the works of B. W. Bojarski.

Using the factorization of the Laplace operator each solution of the problem

$$\begin{pmatrix} u_{\omega_0} - v_{\omega_1} \\ u_{\omega_1} + v_{\omega_0} \end{pmatrix} = \begin{pmatrix} T_1^2[\tilde{f}, g] \\ T_2^2[\tilde{f}, g] \end{pmatrix} \tag{23.1.1}$$

fulfills the Poisson equation mentioned above, if $T^2 = \begin{pmatrix} T_1^2 \\ T_2^2 \end{pmatrix}$ is the right inverse operator with the property $\begin{pmatrix} \frac{\partial}{\partial \omega_0} & \frac{\partial}{\partial \omega_1} \\ -\frac{\partial}{\partial \omega_1} & \frac{\partial}{\partial \omega_0} \end{pmatrix} \begin{pmatrix} T_1^2[\tilde{f}, g] \\ T_2^2[\tilde{f}, g] \end{pmatrix} = \begin{pmatrix} \tilde{f} \\ g \end{pmatrix}$. If Eq. (23.1.1) is splitted into its two components, a system of equations of first order is obtained. On the other hand it should be noted that by the help of the ansatz

$$\begin{pmatrix} u_{\omega_0} - v_{\omega_1} \\ u_{\omega_1} + v_{\omega_0} \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} u \\ -v \end{pmatrix} \tag{23.1.2}$$

a solution of the problem $u_{\omega_0\omega_0} + u_{\omega_1\omega_1} = (a^2 + b^2)u$ is described. Concretely follows from

$$u_{\omega_0\omega_0} - v_{\omega_1\omega_0} = a\,u_{\omega_0} + b\,v_{\omega_0} \quad \text{and} \quad u_{\omega_1\omega_1} + v_{\omega_0\omega_1} = b\,u_{\omega_1} - a\,v_{\omega_1}$$

by summation

$$u_{\omega_0\omega_0} + u_{\omega_1\omega_1} = a\,(u_{\omega_0} - v_{\omega_1}) + b\,(u_{\omega_1} + v_{\omega_0}) = a\,(au + bv) + b(bu - av)\,.$$

Using this ansatz the relation to Vekua equations becomes obvious.

23.2 The Discrete Case and the Discrete Beltrami Equation

Now the idea of transformation is generalized to the discrete situation. From now (m_1, m_2) is used instead of (m_1h, m_2h) in all arguments to simplify the formulas. Applying forward differences $D_h^j u(k) = h^{-1}(u(k + b_j) - u(k))$ and backward differences $D_h^{-j} u(k) = h^{-1}(u(k) - u(k - b_j))$ for $j \in \{1, 2\}$, $k = (k_1, k_2)$, $b_1 = (1, 0)$ and $b_2 = (0, 1)$ it follows

$$u_{m_1} = D_h^1 u(\omega_0(m_1 - 1, m_2), \omega_1(m_1, m_2)) = u_{\omega_0} \cdot D_h^{-1}\omega_0 + u_{\omega_1} \cdot D_h^1\omega_1$$

with

$$u_{\omega_0} = \frac{u(\omega_0(m_1, m_2), \omega_1(m_1+1, m_2)) - u(\omega_0(m_1-1, m_2), \omega_1(m_1+1, m_2))}{\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2)}$$
$$u_{\omega_1} = \frac{u(\omega_0(m_1-1, m_2), \omega_1(m_1+1, m_2)) - u(\omega_0(m_1-1, m_2), \omega_1(m_1, m_2))}{\omega_1(m_1+1, m_2) - \omega_1(m_1, m_2)}.$$

Another approximation is obtained if $u_{m_2} = D_h^2 u(\omega_0(m_1, m_2 - 1), \omega_1(m_1, m_2))$ is chosen as starting point.

The second derivatives are calculated by using the product rules

$$D_h^1(uv) = u(m_1 + 1, m_2)\, D_h^1 v + v\, D_h^1 u \quad \text{and}$$
$$D_h^{-1}(uv) = u(m_1 - 1, m_2) D_h^{-1} v + v\, D_h^{-1} u.$$

Let

$$u_{m_1 m_1} = u_{\omega_0}(m_1 + 1, m_2) D_h^1 D_h^{-1}\omega_0 + D_h^{-1}\omega_0\, D_h^1 u_{\omega_0}$$
$$+ u_{\omega_1}(m_1 - 1, m_2) D_h^{-1} D_h^1 \omega_1 + D_h^1\omega_1\, D_h^{-1} u_{\omega_1}$$

with $D_h^1 u_{\omega_0} = u_{\omega_0\,\omega_0}\, D_h^{-1}\omega_0 + u_{\omega_0\,\omega_1}\, D_h^1\omega_1$ and

$$u_{\omega_0\,\omega_0} = \frac{u(\omega_0(m_1+1, m_2), \omega_1(m_1+2, m_2) - u(\omega_0(m_1, m_2), \omega_1(m_1+2, m_2))}{(\omega_0(m_1+1, m_2) - \omega_0(m_1, m_2))(\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2))}$$
$$+ \frac{-u(\omega_0(m_1, m_2), \omega_1(m_1+2, m_2)) + u(\omega_0(m_1-1, m_2), \omega_1(m_1+2, m_2))}{(\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2))(\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2))}$$

$$u_{\omega_0\,\omega_1} = \frac{u(\omega_0(m_1, m_2), \omega_1(m_1+2, m_2) - u(\omega_0(m_1-1, m_2), \omega_1(m_1+2, m_2))}{(\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2))(\omega_1(m_1+1, m_2) - \omega_1(m_1, m_2))}$$
$$+ \frac{-u(\omega_0(m_1, m_2), \omega_1(m_1+1, m_2)) + u(\omega_0(m_1-1, m_2), \omega_1(m_1+1, m_2))}{(\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2))(\omega_1(m_1+1, m_2) - \omega_1(m_1, m_2))}$$

as well as $D_h^{-1}u_{\omega_1} = u_{\omega_1\omega_0}D_h^{-1}\omega_0 + u_{\omega_1\omega_1}D_h^1\omega_1$ with

$$u_{\omega_1\,\omega_0} = \frac{u(\omega_0(m_1-1,m_2),\omega_1(m_1+1,m_2)-u(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2))}{(\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2))(\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2))}$$
$$+\frac{-u(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2))+u(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2))}{(\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2))(\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2))}$$

$$u_{\omega_0 1,\omega_1} = \frac{u(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2)-u(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2))}{(\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2))(\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2))}$$
$$+\frac{-u(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2))+u(\omega_0(m_1-2,m_2),\omega_1(m_1-1,m_2))}{(\omega_1(m_1,m_2)-\omega_1(m_1-1,m_2))(\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2))}.$$

Analogously can be defined

$$\begin{aligned} u_{m_1m_2} &= u_{\omega_0}(m_1+1,m_2)D_h^2D_h^{-1}\omega_0 + D_h^{-1}\omega_0\, D_h^2u_{\omega_0}\\ &\quad + u_{\omega_1}(m_1-1,m_2)D_h^{-2}D_h^1\omega_1 + D_h^1\omega_1\, D_h^{-2}u_{\omega_1}\end{aligned}$$

$$\begin{aligned} u_{m_2m_1} &= u_{\omega_0}(m_1+1,m_2)D_h^1D_h^{-2}\omega_0 + D_h^{-2}\omega_0\, D_h^1u_{\omega_0}\\ &\quad + u_{\omega_1}(m_1-1,m_2)D_h^{-1}D_h^2\omega_1 + D_h^2\omega_1\, D_h^{-1}u_{\omega_1}\end{aligned}$$

$$\begin{aligned} u_{m_2m_2} &= u_{\omega_0}(m_1+1,m_2)D_h^2D_h^{-2}\omega_0 + D_h^{-2}\omega_0\, D_h^2u_{\omega_0}\\ &\quad + u_{\omega_1}(m_1-1,m_2)D_h^{-2}D_h^2\omega_1 + D_h^2\omega_1\, D_h^{-2}u_{\omega_1}\end{aligned}$$

with $D_h^2u_{\omega_0} = u_{\omega_0\omega_0}D_h^{-2}\omega_0 + u_{\omega_0\omega_1}D_h^2\omega_1$ and $D_h^{-2}u_{\omega_1} = u_{\omega_1\omega_0}D_h^{-2}\omega_0 + u_{\omega_1\omega_1}D_h^2\omega_1$.

To lead back the problem

$$\begin{aligned} f &= a\,u_{m_1m_1} + b\,u_{m_1m_2} + b\,u_{m_2m_1} + c\,u_{m_2m_2}\\ &= a\,u_{\omega_0}(m_1+1,m_2)D_h^1D_h^{-1}\omega_0 + a\,D_h^{-1}\omega_0(u_{\omega_0\omega_0}D_h^{-1}\omega_0 + u_{\omega_0\omega_1}D_h^1\omega_1)\\ &\quad + a\,u_{\omega_1}(m_1-1,m_2)D_h^{-1}D_h^1\omega_1 + a\,D_h^1\omega_1(u_{\omega_1\omega_0}D_h^{-1}\omega_0 + u_{\omega_1\omega_1}D_h^1\omega_1)\\ &\quad + b\,u_{\omega_0}(m_1+1,m_2)D_h^2D_h^{-1}\omega_0 + b\,D_h^{-1}\omega_0(u_{\omega_0\omega_0}D_h^{-2}\omega_0 + u_{\omega_0\omega_1}D_h^2\omega_1)\\ &\quad + b\,u_{\omega_1}(m_1-1,m_2)D_h^{-2}D_h^1\omega_1 + b\,D_h^1\omega_1(u_{\omega_1\omega_0}D_h^{-2}\omega_0 + u_{\omega_1\omega_1}D_h^2\omega_1)\\ &\quad + b\,u_{\omega_0}(m_1+1,m_2)D_h^1D_h^{-2}\omega_0 + b\,D_h^{-2}\omega_0(u_{\omega_0\,\omega_0}D_h^{-1}\omega_0 + u_{\omega_0\,\omega_1}\,D_h^1\omega_1)\end{aligned}$$

$$+b\, u_{\omega_1}(m_1-1,m_2) D_h^{-1} D_h^2 \omega_1 + b\, D_h^2\omega_1 (u_{\omega_1\omega_0} D_h^{-1}\omega_0 + u_{\omega_1\omega_1} D_h^1\omega_1)$$
$$+c\, u_{\omega_0}(m_1+1,m_2) D_h^2 D_h^{-2} \omega_0 + c\, D_h^{-2}\omega_0 (u_{\omega_0\omega_0} D_h^{-2}\omega_0 + u_{\omega_0\omega_1} D_h^2\omega_1)$$
$$+c\, u_{\omega_1}(m_1-1,m_2) D_h^{-2} D_h^2 \omega_1 + c\, D_h^2\omega_1 (u_{\omega_1\omega_0} D_h^{-2}\omega_0 + u_{\omega_1\omega_1} D_h^2\omega_1)$$

to the equation $u_{\omega_0\omega_0} + u_{\omega_1\omega_1} = \tilde{f}$ the following identities must be true:

$$\begin{aligned} & a\,(D_h^{-1}\omega_0)^2 + b\,(D_h^{-1}\omega_0)(D_h^{-2}\omega_0) + b\,(D_h^{-2}\omega_0)(D_h^{-1}\omega_0) + c\,(D_h^{-2}\omega_0)^2 \\ = {} & a\,(D_h^1\omega_1)^2 + b\,(D_h^1\omega_1)(D_h^2\omega_1) + b\,(D_h^2\omega_1)(D_h^1\omega_1) + c\,(D_h^2\omega_1)^2 \end{aligned}$$

$$\begin{aligned} 0 = {} & a\,(D_h^{-1}\omega_0)(D_h^1\omega_1) + b\,(D_h^{-1}\omega_0)(D_h^2\omega_1) + b\,(D_h^{-2}\omega_0)(D_h^1\omega_1) \\ & + c\,(D_h^{-2}\omega_0)(D_h^2\omega_1) \end{aligned}$$

$$0 = a\, D_h^1 D_h^{-1}\omega_0 + b\, D_h^2 D_h^{-1}\omega_0 + b\, D_h^1 D_h^{-2}\omega_0 + c\, D_h^2 D_h^{-2}\omega_0$$

$$0 = a\, D_h^{-1} D_h^1\omega_1 + b\, D_h^{-2} D_h^1\omega_1 + b\, D_h^{-1} D_h^2\omega_1 + c\, D_h^{-2} D_h^2\omega_1\ .$$

In the next step it is proved that all these equations are fulfilled if the transformation

$$\begin{aligned} D_h^{-1}\omega_0 &= \frac{b}{\sqrt{\Delta}}\, D_h^1\omega_1 + \frac{c}{\sqrt{\Delta}}\, D_h^2\omega_1 \\ -D_h^{-2}\omega_0 &= \frac{a}{\sqrt{\Delta}}\, D_h^1\omega_1 + \frac{b}{\sqrt{\Delta}}\, D_h^2\omega_1 \end{aligned} \tag{23.2.1}$$

with $\Delta = ac - b^2 > 0$ is applied. At the same time it follows from this transformation

$$\begin{aligned} D_h^2\omega_1 &= \frac{a}{\sqrt{\Delta}}\, D_h^{-1}\omega_0 + \frac{b}{\sqrt{\Delta}}\, D_h^{-2}\omega_0 \\ -D_h^1\omega_1 &= \frac{b}{\sqrt{\Delta}}\, D_h^{-1}\omega_0 + \frac{c}{\sqrt{\Delta}}\, D_h^{-2}\omega_0\,. \end{aligned} \tag{23.2.2}$$

Let's start with the first equation. It holds

$$\begin{aligned} & a\,(D_h^{-1}\omega_0)^2 + b\,(D_h^{-1}\omega_0)(D_h^{-2}\omega_0) + b\,(D_h^{-2}\omega_0)(D_h^{-1}\omega_0) + c\,(D_h^{-2}\omega_0)^2 \\ = {} & a\left(\frac{b}{\sqrt{\Delta}}\, D_h^1\omega_1 + \frac{c}{\sqrt{\Delta}}\, D_h^2\omega_1\right)^2 + c\left(-\frac{a}{\sqrt{\Delta}}\, D_h^1\omega_1 - \frac{b}{\sqrt{\Delta}}\, D_h^2\omega_1\right)^2 \\ & + 2b\left(\frac{b}{\sqrt{\Delta}} D_h^1\omega_1 + \frac{c}{\sqrt{\Delta}} D_h^2\omega_1\right)\left(-\frac{a}{\sqrt{\Delta}} D_h^1\omega_1 - \frac{b}{\sqrt{\Delta}} D_h^2\omega_1\right) \end{aligned}$$

$$= (D_h^1\omega_1)^2\left(\frac{ab^2}{\Delta} - \frac{2ab^2}{\Delta} + \frac{ca^2}{\Delta}\right) + (D_h^2\omega_1)^2\left(\frac{ac^2}{\Delta} - \frac{2b^2c}{\Delta} + \frac{b^2c}{\Delta}\right)$$
$$+(D_h^1\omega_1)(D_h^2\omega_1)\left(\frac{2abc}{\Delta} - \frac{2b^3}{\Delta} - \frac{2abc}{\Delta} + \frac{2abc}{\Delta}\right)$$
$$= a(D_h^1\omega_1)^2\left(\frac{ac-b^2}{\Delta}\right) + 2b(D_h^1\omega_1)(D_h^2\omega_1)\left(\frac{ac-b^2}{\Delta}\right) + c(D_h^2\omega_1)^2\left(\frac{ac-b^2}{\Delta}\right)$$
$$= a(D_h^1\omega_1)^2 + 2b(D_h^1\omega_1)(D_h^2\omega_1) + c(D_h^2\omega_1)^2\,.$$

The right-hand side of the second equation can be written in the form

$$a\,(D_h^{-1}\omega_0)(D_h^1\omega_1) + b\,(D_h^{-1}\omega_0)(D_h^2\omega_1) + b\,(D_h^{-2}\omega_0)(D_h^1\omega_1) + c\,(D_h^{-2}\omega_0)(D_h^2\omega_1)$$
$$= a(D_h^{-1}\omega_0)\left(-\frac{b}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 - \frac{c}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right) + b(D_h^{-1}\omega_0)\left(\frac{a}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 + \frac{b}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)$$
$$+b(D_h^{-2}\omega_0)\left(-\frac{b}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 - \frac{c}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right) + c(D_h^{-2}\omega_0)\left(\frac{a}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 + \frac{b}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)$$
$$= (D_h^{-1}\omega_0)^2\left(-\frac{ab}{\sqrt{\Delta}} + \frac{ab}{\sqrt{\Delta}}\right) + (D_h^{-1}\omega_0)(D_h^{-2}\omega_0)\left(-\frac{ac}{\sqrt{\Delta}} + \frac{b^2}{\sqrt{\Delta}} - \frac{b^2}{\sqrt{\Delta}} + \frac{ac}{\sqrt{\Delta}}\right)$$
$$+(D-h^{-2}\omega_0)^2\left(-\frac{bc}{\sqrt{\Delta}} + \frac{bc}{\sqrt{\Delta}}\right) = 0\,.$$

Similar it can be proved in the last equations

$$a\,D_h^1 D_h^{-1}\omega_0 + b\,D_h^2 D_h^{-1}\omega_0 + b\,D_h^1 D_h^{-2}\omega_0 + c\,D_h^2 D_h^{-2}\omega_0$$
$$= a D_h^1\left(\frac{b}{\sqrt{\Delta}}\,D_h^1\omega_1 + \frac{c}{\sqrt{\Delta}}\,D_h^2\omega_1\right) + b D_h^2\left(\frac{b}{\sqrt{\Delta}}\,D_h^1\omega_1 + \frac{c}{\sqrt{\Delta}}\,D_h^2\omega_1\right)$$
$$+b D_h^1\left(-\frac{a}{\sqrt{\Delta}}\,D_h^1\omega_1 - \frac{b}{\sqrt{\Delta}}\,D_h^2\omega_1\right) + c D_h^2\left(-\frac{a}{\sqrt{\Delta}}\,D_h^1\omega_1 - \frac{b}{\sqrt{\Delta}}\,D_h^2\omega_1\right)$$
$$= D_h^1 D_h^1\omega_1\left(\frac{ab}{\sqrt{\Delta}} - \frac{ab}{\sqrt{\Delta}}\right) + D_h^1 D_h^2\omega_1\left(\frac{ac}{\sqrt{\Delta}} + \frac{b^2}{\sqrt{\Delta}} - \frac{b^2}{\sqrt{\Delta}} - \frac{ac}{\sqrt{\Delta}}\right)$$
$$+D_h^2 D_h^2\omega_1\left(\frac{bc}{\sqrt{\Delta}} - \frac{bc}{\sqrt{\Delta}}\right) = 0$$

by using the relation $D_h^1 D_h^2 \omega_1 = D_h^2 D_h^1 \omega_1$ and

$$
\begin{aligned}
& a\, D_h^{-1} D_h^1 \omega_1 + b\, D_h^{-2} D_h^1 \omega_1 + b\, D_h^{-1} D_h^2 \omega_1 + c\, D_h^{-2} D_h^2 \omega_1 \\
&= a D_h^{-1} \left(-\frac{b}{\sqrt{\Delta}}\, D_h^{-1} \omega_0 - \frac{c}{\sqrt{\Delta}}\, D_h^{-2} \omega_0\right) + b D_h^{-2} \left(-\frac{b}{\sqrt{\Delta}}\, D_h^{-1} \omega_0 - \frac{c}{\sqrt{\Delta}}\, D_h^{-2} \omega_0\right) \\
&\quad + b D_h^{-1} \left(\frac{a}{\sqrt{\Delta}}\, D_h^{-1} \omega_0 + \frac{b}{\sqrt{\Delta}}\, D_h^{-2} \omega_0\right) + c D_h^{-2} \left(\frac{a}{\sqrt{\Delta}}\, D_h^{-1} \omega_0 + \frac{b}{\sqrt{\Delta}}\, D_h^{-2} \omega_0\right) \\
&= D_h^{-1} D_h^{-1} \omega_0 \left(-\frac{ab}{\sqrt{\Delta}} + \frac{ab}{\sqrt{\Delta}}\right) + D_h^{-1} D_h^{-2} \omega_0 \left(-\frac{ac}{\sqrt{\Delta}} - \frac{b^2}{\sqrt{\Delta}} + \frac{b^2}{\sqrt{\Delta}} + \frac{ac}{\sqrt{\Delta}}\right) \\
&\quad D_h^{-2} D_h^{-2} \omega_0 \left(-\frac{bc}{\sqrt{\Omega}} + \frac{bc}{\sqrt{\Omega}}\right) = 0
\end{aligned}
$$

by using the relation $D_h^{-1} D_h^{-2} \omega_0 = D_h^{-2} D_h^{-1} \omega_0$.

Equations of the form $u_{\omega_0\omega_0} + u_{\omega_1\omega_1} = \tilde{f}$ can be solved analogously to the continuous case, because in the discrete theory a factorization of the Laplace operator is possible, too.

We now turn to the following question: Is a corresponding procedure similar to the ansatz (23.1.2) available in the discrete theory? In order to investigate this fact, the second derivatives with respect to ω_0 and ω_1 are rewritten. It holds

$$
\begin{aligned}
u_{\omega_0\omega_0} &= \frac{u_{\omega_0}(\omega_0(m_1+1, m_2), \omega_1(m_1+1, m_2)) - u_{\omega_0}(\omega_0(m_1, m_2), \omega_1(m_1+1, m_2))}{\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2)} \\
u_{\omega_0\omega_1} &= \frac{u_{\omega_0}(\omega_0(m_1, m_2), \omega_1(m_1+1, m_2)) - u_{\omega_0}(\omega_0(m_1, m_2), \omega_1(m_1, m_2))}{\omega_1(m_1+1, m_2) - \omega_1(m_1, m_2)} \\
u_{\omega_1\omega_0} &= \frac{u_{\omega_1}(\omega_0(m_1, m_2), \omega_1(m_1, m_2)) - u_{\omega_1}(\omega_0(m_1-1, m_2), \omega_1(m_1, m_2))}{\omega_0(m_1, m_2) - \omega_0(m_1-1, m_2)} \\
u_{\omega_1\omega_1} &= \frac{u_{\omega_1}(\omega_0(m_1-1, m_2), \omega_1(m_1, m_2)) - u_{\omega_1}(\omega_0(m_1-1, m_2), \omega_1(m_1-1, m_2))}{\omega_1(m_1+1, m_2) - \omega_1(m_1, m_2)} .
\end{aligned}
$$

The following system of equations of first order is the discrete analogon to the ansatz (23.1.2).

$$
\begin{aligned}
& u_{\omega_0}(\omega_0(m_1+1, m_2), \omega_1(m_1+1, m_2)) - v_{\omega_1}(\omega_0(m_1, m_2), \omega_1(m_1, m_2)) \\
&= a\, u(\omega_0(m_1, m_2), \omega_1(m_1+2, m_2)) + b\, v(\omega_0(m_1, m_2), \omega_1(m_1+2, m_2))
\end{aligned}
$$

$$
\begin{aligned}
& u_{\omega_1}(\omega_0(m_1-1, m_2), \omega_1(m_1, m_2)) + v_{\omega_0}(\omega_0(m_1, m_2), \omega_1(m_1+1, m_2)) \\
&= b\, u(\omega_0(m_1-2, m_2), \omega_1(m_1+1, m_2)) - a\, v(\omega_0(m_1-2, m_2), \omega_1(m_1+1, m_2)) .
\end{aligned}
$$

Based on these equations it follows

$$
\begin{aligned}
& u_{\omega_0\omega_0} - v_{\omega_1\omega_0} \\
&= \frac{u_{\omega_0}(\omega_0(m_1+1,m_2),\omega_1(m_1+1,m_2)) - u_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1+1,m_2))}{\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)} \\
&\quad - \frac{v_{\omega_1}(\omega_0(m_1,m_2),\omega_1(m_1,m_2)) - v_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2))}{\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)} \\
&= \frac{a\,u(\omega_0(m_1,m_2),\omega_1(m_1+2,m_2)) + b\,v(\omega_0(m_1,m_2),\omega_1(m_1+2,m_2))}{\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)} \\
&\quad \frac{-a\,u(\omega_0(m_1-1,m_2),\omega_1(m_1+2,m_2)) - b\,v(\omega_0(m_1-1,m_2),\omega_1(m_1+2,m_2))}{\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)} \\
&= a\,\frac{u(\omega_0(m_1,m_2),\omega_1(m_1+2,m_2)) - u(\omega_0(m_1-1,m_2),\omega_1(m_1+2,m_2))}{\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)} \\
&\quad +b\,\frac{v(\omega_0(m_1,m_2),\omega_1(m_1+2,m_2)) - v(\omega_0(m_1-1,m_2),\omega_1(m_1+2,m_2))}{\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)} \\
&= a\,u_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1+1,m_2)) + b\,v_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1+1,m_2))
\end{aligned}
$$

as well as

$$
\begin{aligned}
& u_{\omega_1\omega_1} + v_{\omega_0\omega_1} \\
&= \frac{u_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2)) - u_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1-1,m_2))}{\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)} \\
&\quad + \frac{v_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1+1,m_2)) - v_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1,m_2))}{\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)} \\
&= \frac{b\,u(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2)) - a\,v(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2))}{\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)} \\
&\quad + \frac{-b\,u(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2)) + a\,v(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2))}{\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)} \\
&= b\,\frac{u(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2)) - u(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2))}{\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)} \\
&\quad -a\,\frac{v(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2)) - v(\omega_0(m_1-2,m_2),\omega_1(m_1,m_2))}{\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)} \\
&= b\,u_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2)) - a\,v_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2))\,.
\end{aligned}
$$

All these terms can be summed up to

$$
\begin{aligned}
& u_{\omega_0\omega_0} - v_{\omega_1\omega_0} + u_{\omega_1\omega_1} + v_{\omega_0\omega_1} \\
&= a\,u_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1+1,m_2)) + b\,v_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1+1,m_2)) \\
&\quad + b\,u_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2)) - a\,v_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2))
\end{aligned}
$$

$$\begin{aligned}
&= a\,(u_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1{+}1,m_2)) - v_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2)))\\
&\quad + b\,(v_{\omega_0}(\omega_0(m_1,m_2),\omega_1(m_1{+}1,m_2)) + u_{\omega_1}(\omega_0(m_1-1,m_2),\omega_1(m_1,m_2)))\\
&= a(a\,u(\omega_0(m_1-1,m_2),\omega_1(m_1{+}2,m_2)) + b\,v(\omega_0(m_1-1,m_2),\omega_1(m_1{+}2,m_2)))\\
&\quad + b(b\,u(\omega_0(m_1-2,m_2),\omega_1(m_1{+}1,m_2)) - a\,v(\omega_0(m_1-2,m_2),\omega_1(m_1{+}1,m_2)))\\
&= a^2\,u(\omega_0(m_1-1,m_2),\omega_1(m_1{+}2,m_2)) + b^2\,u(\omega_0(m_1-2,m_2),\omega_1(m_1{+}1,m_2))\\
&\quad + ab\,v(\omega_0(m_1-1,m_2),\omega_1(m_1{+}2,m_2)) - ab\,v(\omega_0(m_1-2,m_2),\omega_1(m_1{+}1,m_2))\,.
\end{aligned}$$

Of course the equation in the discrete version is more complicated and especially the summands associated with the function v can not be removed because of the use of different mesh points. But if h tends to zero the relation to the classical differential equation is recognizable. More detailed information with respect to the convergence behaviour are possible, if the expression

$$ab\,v(\omega_0(m_1,m_2),\omega_1(m_1,m_2))$$

is subtracted and then added on the right-hand side in the sense of an additive zero. By this way the differences

$$ab\,v(\omega_0(m_1-1,m_2),\omega_1(m_1+2,m_2)) - ab\,v(\omega_0(m_1,m_2),\omega_1(m_1,m_2))$$

and

$$ab\,v(\omega_0(m_1,m_2),\omega_1(m_1,m_2)) - ab\,v(\omega_0(m_1-2,m_2),\omega_1(m_1+1,m_2))$$

arise which can be considered as difference derivatives multiplied by h, even if the mesh points are not directly adjacent. A remarkable fact is that a system of equations of first order exists, which is very important for the problem of second order even if many neighboring lattice points are included.

Now the Beltrami equation is considered. Its solution is based on the coordinate transform introduced above in the continuous as well as in the discrete case.

Starting with the classical case, the equation

$$\partial_{\overline{z}} w + q(z)\,\partial_z w = 0 \quad \text{with} \quad \partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right), \quad \partial_z = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right)$$

and $\omega = \omega_0 + i\omega_1$ is called Beltrami equation. In this notation the complex function

$$q(z) = q_0 + iq_1 = \frac{a-c+2ib}{a+c+2\sqrt{\Delta}} = \frac{a-\sqrt{\Delta}+ib}{a+\sqrt{\Delta}-ib} \qquad \text{with } \Delta = ac - b^2 > 0$$

is related to the constant coefficients in the differential equation

$$a\,u_{xx} + 2b\,u_{xy} + c\,u_{yy} = f.$$

It is easy to prove that the Beltrami equation is fulfilled using the classical transform. Accordingly, in the discrete theory the difference equation

$$\begin{aligned}\begin{pmatrix}0\\0\end{pmatrix} &= \frac{1}{2}\begin{pmatrix}D_h^{-1} & -D_h^2\\ D_h^{-2} & D_h^1\end{pmatrix}\begin{pmatrix}\omega_0\\ \omega_1\end{pmatrix} \\ &+\frac{1}{2}\begin{pmatrix}q_0 & -q_1\\ q_1 & q_0\end{pmatrix}\begin{pmatrix}D_h^1 & D_h^2 - hD_h^{-1}D_h^2\\ -D_h^{-2}-hD_h^1 D_h^{-2} & D_h^{-1}\end{pmatrix}\begin{pmatrix}\omega_0((m_1-1),m_2)\\ \omega_1((m_1+1),m_2)\end{pmatrix}\end{aligned} \tag{23.2.3}$$

is called *discrete Beltrami equation*, where the operators

$$D^{1h} = \begin{pmatrix}D_h^{-1} & -D_h^2\\ D_h^{-2} & D_h^1\end{pmatrix} \quad \text{and} \quad D^{2h} = \begin{pmatrix}D_h^1 & D_h^2\\ -D_h^{-2} & D_h^{-1}\end{pmatrix}$$

have the property $D^{1h}D^{2h} = D^{2h}D^{1h} = I_2\,\Delta_h$ with the 2×2 identity matrix I_2 and the discrete Laplacian $\Delta_h = D_h^1 D_h^{-1} + D_h^2 D_h^{-2}$. Functions u with $D^{1h}u = 0$ are called *discrete holomorphic*. Beside the operator D^{2h}, a shift operator is used in order to manage the problems of different mesh points.

It will be proved that the discrete Beltrami equation is fulfilled using the transformation (23.2.2) as well as the relations

$$\begin{aligned}&D_h^1\omega_0(m_1-1,m_2)\\ &= h^{-1}(\omega_0(m_1,m_2)-\omega_0(m_1-1,m_2)) = D_h^{-1}\omega_0(m_1,m_2)\end{aligned}$$

$$\begin{aligned}&(D_h^2 - hD_h^{-1}D_h^2)\omega_1(m_1+1,m_2)\\ &= D_h^2\omega_1(m_1+1),m_2) - D_h^2((m_1+1,m_2) + D_h^2\omega_1(m_1 h, m_2 h)\\ &= D_h^2\omega_1(m_1 h, m_2 h)\end{aligned}$$

$$\begin{aligned}&(-D_h^{-2} - hD_h^1 D_h^{-2})\omega_0(m_1-1,m_2)\\ &= -D_h^{-2}\omega_0(m_1-1,m_2) - D_h^{-2}\omega_0(m_1,m_2) + D_h^{-2}\omega_0(m_1-1,m_2)\\ &= -D_h^{-2}\omega_0(m_1,m_2)\end{aligned}$$

$$\begin{aligned}&D_h^{-1}\omega_1(m_1+1,m_2)\\ &= h^{-1}(\omega_1(m_1+1,m_2)-\omega_1(m_1,m_2)) = D_h^1\omega_1(m_1,m_2)\,.\end{aligned}$$

In the first component it holds

$$
\begin{aligned}
&\frac{1}{2}(D_h^{-1}\omega_0 - D_h^2\omega_1) + \frac{1}{2}\,\frac{a-c}{a+c+2\sqrt{\Delta}}(D_h^{-1}\omega_0 + D_h^2\omega_1)\\
&-\frac{1}{2}\,\frac{2b}{a+c+2\sqrt{\Delta}}(-D_h^{-2}\omega_0 + D_h^1\omega_1)\\
={}& \frac{1}{2}\left(D_h^{-1}\omega_0 - \frac{a}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 - \frac{b}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)\\
&+\frac{1}{2}\,\frac{a-c}{a+c+2\sqrt{\Delta}}\left(D_h^{-1}\omega_0 + \frac{a}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 + \frac{b}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)\\
&-\frac{1}{2}\,\frac{2b}{a+c+2\sqrt{\Delta}}\left(-D_h^{-2}\omega_0 - \frac{b}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 - \frac{c}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)\\
={}& \frac{1}{2(a+c+2\sqrt{\Delta})}\left(aD_h^{-1}\omega_0 - \frac{a^2}{\sqrt{\Delta}}D_h^{-1}\omega_0 - \frac{ab}{\sqrt{\Delta}}D_h^{-2}\omega_0 + cD_h^{-1}\omega_0\right.\\
&-\frac{ac}{\sqrt{\Delta}}D_h^{-1}\omega_0 - \frac{bc}{\sqrt{\Delta}}D_h^{-2}\omega_0 + 2\sqrt{\Delta}D_h^{-1}\omega_0 - 2aD_h^{-1}\omega_0 - 2bD_h^{-2}\omega_0\\
&+aD_h^{-1}\omega_0 + \frac{a^2}{\sqrt{\Delta}}D_h^{-1}\omega_0 + \frac{ab}{\sqrt{\Delta}}D_h^{-2}\omega_0 - cD_h^{-1}\omega_0 - \frac{ac}{\sqrt{\Delta}}D_h^{-1}\omega_0\\
&\left.-\frac{bc}{\sqrt{\Delta}}D_h^{-2}\omega_0 + 2bD_h^{-2}\omega_0 + \frac{2b^2}{\sqrt{\Delta}}D_h^{-1}\omega_0 + \frac{2bc}{\sqrt{\Delta}}D_h^{-2}\omega_0\right)\\
={}& \frac{1}{2(a+c+2\sqrt{\Delta})}\left(\left(-\frac{2ac}{\sqrt{\Delta}} + 2\sqrt{\Delta} + \frac{2b^2}{\sqrt{\Delta}}\right)D_h^{-1}\omega_0\right) = 0\,.
\end{aligned}
$$

In the second component it can be proved

$$
\begin{aligned}
&\frac{1}{2}(D_h^{-2}\omega_0 + D_h^1\omega_1) + \frac{1}{2}\,\frac{2b}{a+c+2\sqrt{\Delta}}(D_h^{-1}\omega_0 + D_h^2\omega_1)\\
&+\frac{1}{2}\,\frac{a-c}{a+c+2\sqrt{\Delta}}(-D_h^{-2}\omega_0 + D_h^1\omega_1)\\
={}& \frac{1}{2}\left(D_h^{-2}\omega_0 - \frac{b}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 - \frac{c}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)\\
&+\frac{1}{2}\,\frac{2b}{a+c+2\sqrt{\Delta}}\left(D_h^{-1}\omega_0 + \frac{a}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 + \frac{b}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)\\
&+\frac{1}{2}\,\frac{a-c}{a+c+2\sqrt{\Delta}}\left(-D_h^{-2}\omega_0 - \frac{b}{\sqrt{\Delta}}\,D_h^{-1}\omega_0 - \frac{c}{\sqrt{\Delta}}\,D_h^{-2}\omega_0\right)
\end{aligned}
$$

$$= \frac{1}{2(a+c+2\sqrt{\Delta})}\left(aD_h^{-2}\omega_0 - \frac{ab}{\sqrt{\Delta}}D_h^{-1}\omega_0 - \frac{ac}{\sqrt{\Delta}}D_h^{-2}\omega_0 + cD_h^{-2}\omega_0\right.$$
$$-\frac{bc}{\sqrt{\Delta}}D_h^{-1}\omega_0 - \frac{c^2}{\sqrt{\Delta}}D_h^{-2}\omega_0 + 2\sqrt{\Delta}D_h^{-2}\omega_0 - 2bD_h^{-1}\omega_0 - 2cD_h^{-2}\omega_0$$
$$+2bD_h^{-1}\omega_0 + \frac{2ab}{\sqrt{\Delta}}D_h^{-1}\omega_0 + \frac{2b^2}{\sqrt{\Delta}}D_h^{-2}\omega_0 - aD_h^{-2}\omega_0 - \frac{ab}{\sqrt{\Delta}}D_h^{-1}\omega_0$$
$$\left.-\frac{ac}{\sqrt{\Delta}}D_h^{-2}\omega_0 + cD_h^{-2}\omega_0 + \frac{bc}{\sqrt{\Delta}}D_h^{-1}\omega_0 + \frac{c^2}{\sqrt{\Delta}}D_h^{-2}\omega_0\right)$$
$$= \frac{1}{2(a+c+2\sqrt{\Delta})}\left(\left(-\frac{2ac}{\sqrt{\Delta}} + 2\sqrt{\Delta} + \frac{2b^2}{\sqrt{\Delta}}\right)D_h^{-2}\omega_0\right) = 0\,.$$

23.3 Definition of the Discrete Π-Operator

First results concerning the discretization of the Π operator and a discrete Beltrami equation are published in [1] and [2]. Now the theory is completed and more properties and relations are available.

Let $G \subset \mathbb{R}^2$ be a bounded domain and $G_h = G \cap \mathbb{R}_h^2$ be the corresponding discrete domain with $\mathbb{R}_h^2 = \{mh = (m_1h, m_2h)$ and $m_1, m_2 \in \mathbb{Z}\}$. In order to define a discrete Π-operator the scalar product

$$< u, v >= \sum_{mh\in G_h} h^2 \begin{pmatrix} u_0(m) \\ u_1(m) \end{pmatrix}^T \begin{pmatrix} v_0(m) \\ v_1(m) \end{pmatrix}$$

is considered. In all arguments the step size h is omitted in order to simplify the notation. Additionally $u(r) = v(r) = (0,0)^T$ is required in all mesh points rh outside the domain G_h. The right inverse operator $T^{1h} = (T_{h1}^1, T_{h2}^1)^T$ of the operator D^{1h} has the componentwise structure

$$(T_{hk}^1 u)(m) = \sum_{lh\in G_h} h^2 \begin{pmatrix} E_{hk1}^1(m-l) \\ E_{hk2}^1(m-l) \end{pmatrix}^T \begin{pmatrix} u_0(l) \\ u_1(l) \end{pmatrix}$$

with the discrete fundamental solution

$$E_h^1(m) = \begin{pmatrix} E_{h11}^1(m) & E_{h12}^1(m) \\ E_{h21}^1(m) & E_{h22}^1(m) \end{pmatrix}$$

and its components

$$E^1_{h11}(m) = \frac{1}{(2\pi)^2} \int\limits_{\xi \in Q_h} \frac{\xi^h_{-1}}{d^2}\, e^{-ih<m,\xi>} d\xi$$

$$E^1_{h12}(m) = \frac{1}{(2\pi)^2} \int\limits_{\xi \in Q_h} \frac{\xi^h_{-2}}{d^2}\, e^{-ih<m,\xi>} d\xi$$

$$E^1_{h21}(m) = \frac{1}{(2\pi)^2} \int\limits_{\xi \in Q_h} \frac{\xi^h_{2}}{d^2}\, e^{-ih<m,\xi>} d\xi$$

$$E^1_{h22}(m) = \frac{1}{(2\pi)^2} \int\limits_{\xi \in Q_h} \frac{-\xi^h_{1}}{d^2}\, e^{-ih<m,\xi>} d\xi \, .$$

Especially for $j \in \{1, 2\}$ the notation $Q_h = \{\xi \in \mathbb{R}^2 : -\pi/h < \xi_j < \pi/h\}$, $\xi^h_{-j} = h^{-1}(1 - e^{-ih\xi_j})$ and $\xi^h_j = h^{-1}(1 - e^{ih\xi_j})$ is used.

Now the aim is to find the adjoint operator $\overline{T^{1h}}$ with the property

$$< T^{1h}u, v >=< u, \overline{T^{1h}}v > .$$

In order to do this, the relation between the discrete fundamental solution $E^1_h(m)$ and the fundamental solution $E^2_h(m)$ corresponding to the difference operator D^{2h}, and the properties of $E^2_h(m)$ themself are studied. Especially it holds $E^1_{h11}(m) = E^2_{h22}(m)$, $E^1_{h12}(m) = -E^2_{h12}(m)$, $E^1_{h21}(m) = -E^2_{h21}(m)$ and $E^1_{h22}(m) = E^2_{h11}(m)$.

Using the substitution $\xi^{new}_1 = -\xi_1$ and $\xi^{new}_2 = -\xi_2$ it is possible to prove for each component

$$\frac{1}{(2\pi)^2} \int\limits_{\xi \in Q_h} \frac{\xi^h_{\pm l}}{d^2}\, e^{-ih<m-l,\xi>} d\xi = \frac{1}{(2\pi)^2} \int\limits_{\xi \in Q_h} \frac{\xi^h_{\mp l}}{d^2}\, e^{-ih<l-m,\xi>} d\xi$$

with $l \in \{1, 2\}$. By this way the properties

$$E^2_{h11}(m - l) = -E^2_{h22}(l - m) \quad \text{and} \quad E^2_{h12}(m - l) = E^2_{h21}(l - m)$$

are fulfilled. For the scalar product it can be proved

$$
\begin{aligned}
&< T^{1h}u, v > \\
&= \sum_{mh\in G_h} h^2 \Bigg[\sum_{lh\in G_h} h^2 \big(E^1_{h11}(m-l)\, u_0(l) + E^1_{h12}(m-l)\, u_1(l)\big)\, v_0(m) \\
&\qquad + \sum_{lh\in G_h} h^2 \big(E^1_{h21}(m-l)\, u_0(l) + E^1_{h22}(m-l)\, u_1(l)\big)\, v_1(m) \Bigg] \\
&= \sum_{mh\in G_h} h^2 \Bigg[\sum_{lh\in G_h} h^2 \big(E^2_{h22}(m-l)\, u_0(l) - E^2_{h12}(m-l)\, u_1(l)\big)\, v_0(m) \\
&\qquad + \sum_{lh\in G_h} h^2 \big(- E^2_{h21}(m-l)\, u_0(l) + E^2_{h11}(m-l)\, u_1(l)\big)\, v_1(m) \Bigg] \\
&= \sum_{lh\in G_h} h^2 \Bigg[\sum_{mh\in G_h} h^2 \big(- E^2_{h11}(l-m)\, v_0(m) - E^2_{h12}(l-m)\, v_1(m)\big)\, u_0(l) \\
&\qquad + \sum_{mh\in G_h} h^2 \big(- E^2_{h21}(l-m)\, v_0(m) - E^2_{h22}(l-m)\, v_1(m)\big)\, u_1(l) \Bigg] \\
&= < u, -T^{2h}v >,
\end{aligned}
$$

where $T^{2h} = (T^2_{h1}, T^2_{h2})^T$ with the components

$$
(T^2_{hk}u)(m) = \sum_{lh\in G_h} h^2 \begin{pmatrix} E^2_{hk1}(m-l) \\ E^2_{hk2}(m-l) \end{pmatrix}^T \begin{pmatrix} u_0(l) \\ u_1(l) \end{pmatrix} \qquad k \in \{1, 2\}
$$

is the right inverse operator of D^{2h} if all boundary values of the function u are equal to zero. Especially it holds $(D^{2h}(T^{2h}u))(mh) = u(mh)\ \forall mh \in G_h$. Here we proved $\overline{T^{1h}} = -T^{2h}$.

Analogous to the continuous case the operator $\Pi^{1h} = -D^{1h}\, T^{2h}$ is called *discrete* Π-*operator*. By repeating all steps a second operator $\Pi^{2h} = -D^{2h}\, T^{1h}$ can be defined. These definitions provide important properties in discrete theory. In particular, the operator is defined by the derivation of a right-inverse operator. Note that these definitions are not unique, as there are different factorizations of the discrete Laplace operator. In terms of a closed theory, it is important at this point that all proven properties build on each other. Further, for holomorphic functions, the following well-known properties can be proved:

Lemma 23.3.1 *For discrete holomorphic functions u with $D^{1h}u = 0$ and $u(r) = (0,0)^T$ on the boundary, by the factorization $D^{1h}\, D^{2h} = D^{2h}\, D^{1h} = I_2\, \Delta_h$ it follows*

$$D^{2h}\, \Pi^{1h} u = D^{2h}(-D^{1h}\, T^{2h} u) = -D^{1h}(D^{2h}\, T^{2h} u) = -D^{1h} u = 0\,.$$

If the property $D^{2h}u = 0$ is fulfilled, the relation $D^{1h}\, \Pi^{2h} u = 0$ is obtained.

23.4 The Solution of the Discrete Beltrami Equation Constructed by the Π-Operator

Now the operator Π^{2h} is used to describe the solution of the discrete Beltrami equation (23.2.3) from Sect. 23.2. In the first step the ansatz

$$\begin{pmatrix} W_0(m_1, m_2) \\ W_1(m_1, m_2) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} m_1 h \\ m_2 h \end{pmatrix} + \begin{pmatrix} (T^1_{h1} f)(m) \\ (T^1_{h2} f)(m) \end{pmatrix}$$

with the unknown function $f = (f_0, f_1)^T$ is substituted into the Beltrami equation. It should be noted that for $1 < p < 2$ and $q < \frac{2p}{2-p}$ the operator $T^{1h} : l_p(G_h) \to l_q(G_h)$ is bounded.

Using the operator $(\Pi^{2h} f)(m) = ((\Pi^2_{h1} f)(m), (\Pi^2_{h2} f)(m))^T$ it holds

$$\begin{aligned}
\begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \frac{1}{2} D^{1h} \left(\frac{1}{2} \begin{pmatrix} m_1 h \\ m_2 h \end{pmatrix} + \begin{pmatrix} (T^1_{h1} f)(m) \\ (T^1_{h2} f)(m) \end{pmatrix} \right) \\
&\quad + \frac{1}{2} \begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix} D^{2h} \left(\frac{1}{2} \begin{pmatrix} (m_1 - 1)h \\ m_2 h \end{pmatrix} + \begin{pmatrix} (T^1_{h1} f)(m_1 - 1, m_2) \\ (T^1_{h2} f)(m_1 + 1, m_2) \end{pmatrix} \right) \\
&\quad - \frac{h}{2} \begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix} \begin{pmatrix} D_h^{-1} D_h^2 \left(\frac{1}{2} m_2 h + (T^1_{h2} f)(m_1 + 1, m_2) \right) \\ D_h^1 D_h^{-2} \left(\frac{1}{2} (m_1 - 1)h + (T^1_{h1} f)(m_1 - 1, m_2) \right) \end{pmatrix} \\
&= \frac{1}{4} D^{1h} \begin{pmatrix} m_1 h \\ m_2 h \end{pmatrix} + \frac{1}{2} \begin{pmatrix} f_0(m_1, m_2) \\ f_1(m_1, m_2) \end{pmatrix} \\
&\quad + \frac{1}{2} \begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix} \left(\frac{1}{2} D^{2h} \begin{pmatrix} (m_1 - 1)h \\ m_2 h \end{pmatrix} - \begin{pmatrix} (\Pi^2_{h1} f)(m_1 - 1, m_2) \\ (\Pi^2_{h2} f)(m_1 + 1, m_2) \end{pmatrix} \right) \\
&\quad - \frac{h}{2} \begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix} \begin{pmatrix} D_h^{-1} D_h^2 \left(\frac{1}{2} m_2 h + (T^1_{h2} f)(m_1 + 1, m_2) \right) \\ D_h^1 D_h^{-2} \left(\frac{1}{2} (m_1 - 1)h + (T^1_{h1} f)(m_1 - 1, m_2) \right) \end{pmatrix} .
\end{aligned}$$

Based on the relations

$$D^{1h}\begin{pmatrix} m_1h \\ m_2h \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \frac{1}{2}D^{2h}\begin{pmatrix} (m_1-1)h \\ m_2h \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

the above equation becomes the structure

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} f_0(m_1,m_2) \\ f_1(m_1,m_2) \end{pmatrix} + \frac{1}{2}\begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix}\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} (\Pi^2_{h1}f)(m_1-1,m_2) \\ (\Pi^2_{h2}f)(m_1+1,m_2) \end{pmatrix}\right)$$
$$-\frac{h}{2}\begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix}\begin{pmatrix} D_h^{-1}D_h^2\left(\frac{1}{2}m_2h + (T^1_{h2}f)(m_1+1,m_2)\right) \\ D_h^1D_h^{-2}\left(\frac{1}{2}(m_1-1)h + (T^1_{h1}f)(m_1-1,m_2)\right) \end{pmatrix}.$$

Consequently the unknown function $f = (f_0, f_1)^T$ has to fulfill the equation

$$\begin{pmatrix} q_0 \\ q_1 \end{pmatrix} = -\begin{pmatrix} f_0(m_1,m_2) \\ f_1(m_1,m_2) \end{pmatrix} + \begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix}\begin{pmatrix} (\Pi^2_{h1}f)(m_1-1,m_2) \\ (\Pi^2_{h2}f)(m_1+1,m_2) \end{pmatrix}$$
$$+h\begin{pmatrix} q_0 & -q_1 \\ q_1 & q_0 \end{pmatrix}\begin{pmatrix} D_h^{-1}D_h^2\left(\frac{1}{2}m_2h + (T^1_{h2}f)(m_1+1,m_2)\right) \\ D_h^1D_h^{-2}\left(\frac{1}{2}(m_1-1)h + (T^1_{h1}f)(m_1-1,m_2)\right) \end{pmatrix}.$$

This equation approximates the well known equation $q = -f + q\Pi f$ from the classical theory, which was studied especially by Tricomi (see [3]). If the last summand is omitted for a moment (because it tends to zero for $h \to 0$) and the process of shifting the mesh points in the second summand is neglected too (which can be described with a shift operator of norm 1), then the question arises whether the operator $I_2 - q\,\Pi^{2h}$ is invertible. Looking at Banach's fixed point theorem, it is to prove that the norm of the operator Π^{2h} is bounded. Based on the property $D^{1h}(T^{1h}f)(mh) = f(mh)$ as well as $< w, D^{2h}s >= - < D^{1h}w, s >$ and $< w, D^{1h}s >= - < D^{2h}w, s >$ it follows

$$< \Pi^{2h}f, \Pi^{2h}g >=< -D^{2h}(T^{1h}f), -D^{2h}(T^{1h}g) >$$
$$= < D^{1h}D^{2h}(T^{1h}f), -T^{1h}g >=< D^{2h}D^{1h}(T^{1h}f), -T^{1h}g >$$
$$= < D^{2h}f, -T^{1h}g >=< f, D^{1h}(T^{1h}g) > = < f, g > .$$

This isometry is also valid in the special case $f = g$. Thus, the norm of the operator Π^{2h} is equal to 1 and the boundedness is guaranteed. With regard to q, it should be noted that assuming $\Delta = ac - b^2 \geq \Delta_0 > 0$, $a > 0$ the inequality $|q(z)| = \frac{(a-\sqrt{\Delta})^2+b^2}{(a+\sqrt{\Delta})^2+b^2} \leq q^* < 1$ with $q^* = \text{const}$ holds.

Finally it should be mentioned that the difference equation studied here has for fixed step size h no singularity, because of the special structure of the discrete fundamental solution. The fact, that the singular behavior appears only if h tends to zero is very beneficial for further examination of the equation.

First of all the explanations show that there are many similarities between the continuous and the discrete case of Beltrami equations. The results obtained can be used as a starting point for further extension of the discrete theory.

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