Bounded Reducibility for Computable Numberings

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Abstract. The theory of numberings gives a fruitful approach to studying uniform computations for various families of mathematical objects. The algorithmic complexity of numberings is usually classified via the reducibility $\leq$ between numberings. This reducibility gives rise to an upper semilattice of degrees, which is often called the Rogers semilattice. For a computable family S of c.e. sets, its Rogers semilattice $R(S)$ contains the $\leq$-degrees of computable numberings of S. Khutoretskii proved that $R(S)$ is always either one-element, or infinite. Selivanov proved that an infinite $R(S)$ cannot be a lattice.

We introduce a bounded version of reducibility between numberings, denoted by $\leq_{bm}$. We show that Rogers semilattices $R_{bm}(S)$, induced by $\leq_{bm}$, exhibit a striking difference from the classical case. We prove that the results of Khutoretskii and Selivanov cannot be extended to our setting: For any natural number $n \geq 2$, there is a finite family S of c.e. sets such that its semilattice $R_{bm}(S)$ has precisely $2^n - 1$ elements. Furthermore, there is a computable family T of c.e. sets such that $R_{bm}(T)$ is an infinite lattice.

1 Introduction

Uniform computations for families of mathematical objects constitute a classical line of research in computability theory. Formal methods for studying such computations are provided by the theory of numberings. The theory goes back to the seminal article of Gödel [17], where an effective numbering of first-order formulae was used in the proof of the incompleteness theorems. One of the first results, which gave rise to the systematic study of numberings, was obtained by

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Kleene [25]: he gave a construction of a universal partial computable function. After that, the foundations of the modern theory of numberings were developed by Kolmogorov and Uspenskii [26,36] and, independently, by Rogers [34].

Let $\mathcal{S}$ be a countable set. A *numbering* of $\mathcal{S}$ is a surjective map ν from ω onto $\mathcal{S}$. A standard tool for measuring the algorithmic complexity of numberings is provided by the notion of *reducibility* between numberings: A numbering ν is *reducible* to another numbering μ (denoted by $\nu \leq \mu$) if there is total computable function $f(x)$ such that $\nu(x) = \mu(f(x))$ for all $x \in \omega$. In other words, there is an effective procedure which, given a ν-index of an object from $\mathcal{S}$, computes a μ-index for the same object. In general, however, the goal is for f to be a readily understandable function, so that we can actually obtain some information from the reduction.

In this paper, we consider only families $\mathcal{S}$ containing subsets of ω, i.e., we always assume that $\mathcal{S} \subset P(\omega)$ and $\mathcal{S}$ is countable.

Let Γ be a complexity class (e.g., Σ^0_1, d-Σ^0_1, Σ^0_n, or Π^1_n). A numbering ν of a family $\mathcal{S}$ is *Γ-computable* if the set $\{\langle x, n\rangle : x \in \nu(n)\}$ belongs to the class Γ. We say that a family $\mathcal{S}$ is *Γ-computable* if it has a Γ-computable numbering.

Following the literature, the term *computable numbering* will be used as a synonym of a Σ^0_1-computable numbering. In particular, a *computable family* is a family with a Σ^0_1-computable numbering.

In a standard recursion-theoretical way, the notion of reducibility between numberings give rise to the *Rogers upper semilattice* (or *Rogers semilattice* for short) of a family $\mathcal{S}$: For a given complexity class Γ, this semilattice contains the degrees of all Γ-computable numberings of $\mathcal{S}$. Here two numberings have the same degree if they are reducible to each other, see Sect. 2 for the formal details.

There is a large body of literature on Rogers semilattices of computable families. To name only a few, computable numberings were studied by Badaev [4,5], Ershov [11,12], Friedberg [14], Goncharov [18,19], Lachlan [27,28], Mal'tsev [29], Pour-El [33], and many other researchers. Note that computable numberings are closely connected to algorithmic learning theory (see, e.g., the recent papers [1,9,23]). For a survey of results and bibliographical references on computable numberings, the reader is referred to the seminal monograph [12] and the articles [3,6,13].

Goncharov and Sorbi [21] started developing the theory of generalized computable numberings: In particular, this area includes investigations of Γ-computable numberings. The approach of [21] proved to be fruitful for classifying Rogers semilattices in hyperarithmetical hierarchy [3,8,32] and the Ershov hierarchy [7,20,22,31].

In the paper, we introduce the following *bounded version* of the reducibility between numberings:

Definition 1.1. *Let ν and μ be numberings. We say that ν is bm-*reducible *to μ if there is a total computable function $f(x)$ with the following properties:*

(a) for every $x \in \omega$, we have $\nu(x) = \mu(f(x))$;
(b) for every $y \in \omega$, the preimage $f^{-1}(y)$ is a finite set.

We write $f : \nu \leq_{bm} \mu$ if a function f bm-reduces ν to μ.

The notation $\leq_{bm}$ is a tribute to the little-known paper of Maslova [30]. She introduced a bounded version of m-reducibility on sets: Suppose that $f(x)$ is a computable function, A and B are subsets of ω. Then $f : A \leq_{bm} B$ iff $f : A \leq_m B$ and f satisfies the condition (b) above [30, Definition 1].

Nowadays various types of reductions are commonly used to study properties of mathematical structures (e.g., in Borel reducibility theory [15,16] or in the theory of ceers [2]). Following this line of research, we are introducing the reducibility $\leq_{bm}$, and we aim to investigate the complexity of the corresponding Rogers semilattices and their structural properties.

One would expect that investigating Rogers semilattices under *bm*-reducibility makes little or no difference for most of the known results on numberings. Quite strikingly, this is *not the case*. In the paper, we illustrate this by considering two algebraic properties of Rogers semilattices.

Historically, the first two major problems on Rogers semilattices were raised by Ershov [10] (see also [6] for a detailed discussion): Let $\mathcal{R}$ be the Rogers semilattice of a computable family $\mathcal{S}$.

Problem A. What is a possible cardinality of $\mathcal{R}$?

Problem B. Can $\mathcal{R}$ be a lattice?

In the classical case (i.e. for the reducibility $\leq$), the problems were solved in 1970s:

A. Khutoretskii [24] proved that $\mathcal{R}$ either has only one element, or is countably infinite.
B. Selivanov [35] proved that an infinite $\mathcal{R}$ cannot be a lattice.

Unexpectedly, the theorems of Khutoretskii and Selivanov *cannot* be extended to the case of *bm*-reducibility. We obtain the following results:

A′. For every natural number $n \geq 2$, there is a finite family of c.e. sets such that its Rogers semilattice under *bm*-reducibility has cardinality $2^n - 1$. A similar result is proved for infinite computable families.
B′. There is a computable family of c.e. sets such that its Rogers semilattice under *bm*-reducibility is an infinite lattice.

These results witness that the *bm*-reducibility of numberings is an interesting object of study in itself.

The outline of the paper is as follows. Section 2 contains the necessary preliminaries and some general observations about *bm*-reducibility. In Sect. 3, we obtain an infinite lattice under *bm*-reducibility (Result B′). Section 4 deals with the possible cardinalities of semilattices under *bm*-reducibility (Result A′). Section 5 contains further discussion.

2 Preliminaries and General Facts

In all the sections of the paper, except the last one, we consider only computable numberings.

Suppose that ν is a numbering of a family $\mathcal{S}_0$, and μ is a numbering of a family $\mathcal{S}_1$. Note that the condition $\nu \leq \mu$ always implies that $\mathcal{S}_0 \subseteq \mathcal{S}_1$. Clearly, if $\nu \leq_{bm} \mu$, then $\nu \leq \mu$.

Numberings ν and μ are *equivalent* (denoted by $\nu \equiv \mu$) if $\nu \leq \mu$ and $\mu \leq \nu$. The *bm-equivalence* $\equiv_{bm}$ is defined in a similar way. The numbering $\nu \oplus \mu$ of the family $\mathcal{S}_0 \cup \mathcal{S}_1$ is defined as follows:

$$(\nu \oplus \mu)(2x) = \nu(x), \quad (\nu \oplus \mu)(2x+1) = \mu(x).$$

The following fact is well-known (see, e.g., Proposition 3 in [12, p. 36]): If $\trianglelefteq \in \{\leq, \leq_{bm}\}$ and ξ is a numbering of a family $\mathcal{T}$, then

$$(\nu \trianglelefteq \xi \,\&\, \mu \trianglelefteq \xi) \;\Leftrightarrow\; (\nu \oplus \mu \trianglelefteq \xi).$$

Let $\mathcal{S}$ be a computable family of c.e. sets. By $Com^0_1(\mathcal{S})$ we denote the set of all computable numberings of $\mathcal{S}$. Suppose that $\sim$ is the equivalence relation induced by a preorder $\trianglelefteq \in \{\leq, \leq_{bm}\}$. Since the relation $\sim$ is a congruence on the structure $(Com^0_1(\mathcal{S}); \trianglelefteq, \oplus)$, we use the same symbols $\trianglelefteq$ and $\oplus$ on numberings of $\mathcal{S}$ and on $\sim$-equivalence classes of these numberings.

The quotient structure $Q_{\sim}(\mathcal{S}) := (Com^0_1(\mathcal{S})/{\sim}; \trianglelefteq, \oplus)$ is an upper semilattice. We say that $Q_{\sim}(\mathcal{S})$ is the *Rogers semilattice* of the family $\mathcal{S}$ under the reducibility $\trianglelefteq$. For the sake of convenience, we use the following notations:

$$\mathcal{R}_m(\mathcal{S}) := Q_{\equiv}(\mathcal{S}); \quad \mathcal{R}_{bm}(\mathcal{S}) := Q_{\equiv_{bm}}(\mathcal{S}).$$

Note that $card(\mathcal{R}_m(\mathcal{S})) \leq card(\mathcal{R}_{bm}(\mathcal{S}))$.

Numberings ν and μ are *computably isomorphic* if $\nu = \mu \circ f$, where f is a computable permutation of ω. If ν is a numbering, then by η_ν we denote the corresponding equivalence relation on ω:

$$m \;\eta_\nu\; n \;\Leftrightarrow\; \nu(m) = \nu(n).$$

A numbering ν is *decidable* if the relation η_ν is computable. Numbering ν is *Friedberg* if η_ν is the identity relation.

In our proofs, we will often refer to the following simple fact about *bm*-reducibility:

Lemma 2.1. *Suppose that ν and μ are numberings, and $\nu(x) = \mu(y)$. If the class $[x]_{\eta_\nu}$ is infinite and $[y]_{\eta_\mu}$ is finite, then $\nu \not\leq_{bm} \mu$.*

Proof. Assume that $f\colon \nu \leq_{bm} \mu$. Since $\nu(x) = \mu(y)$, we have $f^{-1}([y]_{\eta_\mu}) = [x]_{\eta_\nu}$. By the pigeonhole principle, there is an element $z \in [y]_{\eta_\mu}$ such that $f^{-1}(z)$ is infinite, which contradicts the definition of *bm*-reducibility. $\square$

It is well-known that any decidable numbering ν of a family $\mathcal{S}$ induces a *minimal* element in the semilattice $\mathcal{R}_m(\mathcal{S})$. It is easy to show that a similar result fails for the structure $\mathcal{R}_{bm}(\mathcal{S})$:

Corollary 2.1. *Suppose that $\mathcal{S}$ is a computable infinite family, and ν is a decidable, computable numbering of $\mathcal{S}$. Then ν is minimal in $\mathcal{R}_{bm}(\mathcal{S})$ if and only if for every $x \in \omega$, the class $[x]_{\eta_\nu}$ is finite.*

For reasons of space, the proof of Corollary 2.1 is omitted.

A countable family $\mathcal{S}$ of sets is *discrete* if there is a family of finite sets $\mathcal{F}$ with the following properties:

- for any $X \in \mathcal{F}$, there is at most one $W \in \mathcal{S}$ with $X \subseteq W$;
- for every $W \in \mathcal{S}$, there is at least one $X \in \mathcal{F}$ such that $X \subseteq W$.

A family $\mathcal{S}$ is *effectively discrete* if it is discrete, and for the witnessing family $\mathcal{F}$, there is a strongly computable sequence of finite sets $(F_i)_{i\in\omega}$ such that $\mathcal{F} = \{F_i : i \in \omega\}$.

3 Lattices

Let $\mathcal{S}$ be a computable family of c.e. sets. Here we show that the Rogers semilattice $\mathcal{R}_{bm}(\mathcal{S})$ can be an infinite lattice.

Theorem 3.1. *Consider a family $\mathcal{S} := \{\{k\} : k \in \omega\}$. Then the structure $\mathcal{R}_{bm}(\mathcal{S})$ is isomorphic to the lattice of all Π^0_2 sets (under inclusion).*

Proof. Let ν be a computable numbering of the family $\mathcal{S}$. We define a set

$$\mathrm{Inf}(\nu) := \{k \in \omega : \text{the set } \{k\} \text{ has infinitely many } \nu\text{-numbers}\}.$$

It is not hard to show that $\mathrm{Inf}(\nu)$ is a Π^0_2 set.

Lemma 3.1. *Let X be an arbitrary Π^0_2 set. Then there is a computable numbering μ of the family $\mathcal{S}$ such that* $\mathrm{Inf}(\mu) = X$.

Proof. Choose a computable predicate $R(e, y)$ with the following property: for any $e \in \omega$,

$$e \in X \;\Leftrightarrow\; \exists^\infty y R(e, y).$$

W.l.o.g., one may assume that $R(e, 0)$ is true for every e. Fix a computable injective function $f : \omega \to \omega^2$ such that $range(f) = R$.

For $n \in \omega$, we define $\mu(n) := \{e_n\}$, where $f(n) = (e_n, y_n)$. It is not hard to show that μ is a computable numbering of the family $\mathcal{S}$. Furthermore, the following holds:

(a) If $e \in X$, then there are infinitely many n with $R(e, y_n)$ true. For each such n, we have $\mu(n) = \{e\}$.
(b) If $e \notin X$, then there are only finitely many numbers n with $\mu(n) = \{e\}$.

Therefore, we deduce that $\mathrm{Inf}(\mu) = X$. □

Lemma 3.1 implies that in order to prove the theorem, it is sufficient to establish the following fact:

Lemma 3.2. *Let ν and μ be computable numberings of the family $\mathcal{S}$. Then $\nu \leq_{bm} \mu$ if and only if* $\mathrm{Inf}(\nu) \subseteq \mathrm{Inf}(\mu)$.

Proof. Assume that $\nu \leq_{bm} \mu$. Then Lemma 2.1 shows the following: If the set $\{k\}$ has infinitely many ν-numbers, then $\{k\}$ also has infinitely many μ-numbers. Hence, $\mathrm{Inf}(\nu) \subseteq \mathrm{Inf}(\mu)$.

Now suppose that $\mathrm{Inf}(\nu) \subseteq \mathrm{Inf}(\mu)$. We build a *bm*-reduction $f\colon \nu \leq_{bm} \mu$.

For a number $e \in \omega$, we choose effective enumerations $\{a_i\}_{i\in I}$ and $\{b_j\}_{j\in J}$ (without repetitions), which enumerate the set of all ν-numbers of $\{e\}$ and the set of all μ-numbers of $\{e\}$, respectively. Here we assume that $I = \bigcup_{s\in\omega} I[s]$ and $J = \bigcup_{s\in\omega} J[s]$, where all $I[s]$ and $J[s]$ are finite initial segments of ω, $I[0] = J[0] = \{0\}$, $I[s] \subseteq I[s+1]$, $J[s] \subseteq J[s+1]$, and $card(I[s+1] \setminus I[s]) \leq 1$. Moreover, $\{I[s]\}_{s\in\omega}$ and $\{J[s]\}_{s\in\omega}$ are strongly computable sequences of finite sets.

The desired function f is built in stages.

Stage 0. Set $f(a_0) = b_0$.

Stage $s+1$. If $I[s+1] = I[s]$, then proceed to the next stage. Otherwise, find n such that $I[s+1] \setminus I[s] = \{n\}$. Let k be the greatest number with $b_k \in f(I[s])$. Consider the following two cases:

1. If $k+1 \in J[s]$, then define $f(a_n) := b_{k+1}$.
2. If $k+1 \notin J[s]$, then set $f(a_n) := b_k$.

Note that the described procedure is effective, uniformly in $e \in \omega$. Thus, it is easy to see that f is a total computable function such that $f\colon \nu \leq \mu$.

Assume that there is a number y such that the set $f^{-1}(y)$ is infinite. Suppose that $\mu(y) = \{e\}$ and $y = b_k$. The description of the construction implies that there is a number n such that for all $m \geq n$, we have $m \in I$ and $f(a_m) = b_k$. Thus, $k+1 \notin J[s]$ for every s. We deduce that $e \in \mathrm{Inf}(\nu) \setminus \mathrm{Inf}(\mu)$, which contradicts our original assumption. Therefore, the function f provides a *bm*-reduction from ν onto μ. Lemma 3.2 is proved.

This concludes the proof of Theorem 3.1. □

The proof of Theorem 3.1 can be easily modified to obtain the following:

Corollary 3.1. *Let $\mathcal{S} = \{A_i : i \in \omega\}$ be a computable family of c.e. sets. Suppose that there is a strongly computable sequence of finite sets $(F_i)_{i\in\omega}$ such that $F_i \subseteq A_i$ and $F_i \not\subseteq A_j$, for all $i \neq j$. Then the structure $\mathcal{R}_{bm}(\mathcal{S})$ is isomorphic to the lattice of all Π^0_2 sets.*

4 Cardinalities of Rogers Semilattices

Here we attack Problem A from the introduction. For finite families $\mathcal{S}$, we obtain a complete description of possible cardinalities of $\mathcal{R}_{bm}(\mathcal{S})$ (Subsect. 4.1). In Subsect. 4.2, we show that the cardinalities from the previous subsection can also be realized via infinite families $\mathcal{S}$. In order to prove this, we give a computable infinite family $\mathcal{T}$ such that all its computable numberings are computably isomorphic (Theorem 4.2). We also provide two sufficient conditions for $\mathcal{R}_{bm}(\mathcal{S})$ being infinite.

First, we recall the following classical result:

Lemma 4.1 (folklore). *Let $\mathcal{S}$ be a computable family of c.e. sets. If $\mathcal{S}$ contains sets A and B such that $A \subsetneq B$, then the semilattice $\mathcal{R}_m(\mathcal{S})$ is infinite. In particular, this implies that $\mathcal{R}_{bm}(\mathcal{S})$ is also infinite.*

4.1 Finite Families

Theorem 4.1. *Suppose that $\mathcal{S}$ is a finite family of c.e. sets. If $\mathcal{S}$ contains precisely n sets, then $card(\mathcal{R}_{bm}(\mathcal{S}))$ is either equal to $2^n - 1$ or countably infinite. Furthermore, for $n \geq 2$, both these cases can be realized.*

Proof. Let $\mathcal{S} = \{A_1, A_2, \dots, A_n\}$ be a family of c.e. sets. If there are numbers $i \neq j$ with $A_i \subsetneq A_j$, then by Lemma 4.1, the semilattice $\mathcal{R}_{bm}(\mathcal{S})$ is infinite.

Assume that $A_i \not\subseteq A_j$ for all $i \neq j$. Now it is sufficient to show that the structure $\mathcal{R}_{bm}(\mathcal{S})$ contains precisely $2^n - 1$ elements.

Note that the family $\mathcal{S}$ is effectively discrete. Indeed, for every $i \neq j$, choose an element $a_{i,j} \in A_i \setminus A_j$, and define the set $F_i := \{a_{i,j} : j \neq i\}$. It is easy to see that for any i and k, the condition $F_k \subseteq A_i$ holds iff $k = i$. Therefore, if ν is an arbitrary computable numbering of $\mathcal{S}$, then for all $i \leq n$ and $x \in \omega$, the following conditions are equivalent:

$$\nu(x) = A_i \;\Leftrightarrow\; F_i \subseteq \nu(x) \;\Leftrightarrow\; \text{for every } j \neq i,\; F_j \not\subseteq \nu(x).$$

This implies that the numbering ν is decidable.

Let D be a non-empty subset of $\{1, 2, \dots, n\}$, and d be the least number from D. We define a decidable numbering μ_D of the family $\mathcal{S}$ as follows:

$$\mu_D(x) = A_{x+1}, \quad \text{for } x < n;$$

$$\mu_D(\langle i, j\rangle) = \begin{cases} A_i, & \text{if } i \in D, \\ A_d, & \text{otherwise,} \end{cases} \quad \text{where we assume that } \langle i, j\rangle \geq n.$$

It is not hard to establish the following properties:

(a) Lemma 2.1 implies that for finite sets $D \neq E$, we have $\mu_D \not\equiv_{bm} \mu_E$.
(b) Consider an arbitrary computable numbering ν of $\mathcal{S}$. We define a non-empty set $D_\nu := \{i : A_i$ has infinitely many ν-numbers$\}$. Using the decidability of ν, one can obtain that $\nu \equiv_{bm} \mu_{D_\nu}$.

These properties show that the cardinality of $\mathcal{R}_{bm}(\mathcal{S})$ is equal to the number of non-empty subsets of the set $\{1, 2, \dots, n\}$. Thus, $card(\mathcal{R}_{bm}(\mathcal{S})) = 2^n - 1$. □

4.2 Infinite Families

First, we build infinite computable families $\mathcal{S}$ with finite semilattices $\mathcal{R}_{bm}(\mathcal{S})$. Recall that numberings ν and μ are *computably isomorphic* if there is a computable permutation g of ω such that $\nu = \mu \circ g$. We establish the following fact:

Theorem 4.2. *There is an infinite computable family $\mathcal{T}$ such that any two computable numberings of $\mathcal{T}$ are computably isomorphic. In particular, the semilattice $\mathcal{R}_{bm}(\mathcal{T})$ contains only one element.*

Proof. In the proof of Theorem 3.3 from [2], Andrews and Sorbi built a uniform sequence $(E_i)_{i\in\omega}$ of computably enumerable equivalence relations on ω (or *ceers* for short) with the following properties: Each E_i has infinitely many equivalence classes, and if a c.e. set W intersects infinitely many E_i-classes, then it intersects *every* E_i-class.

Fix such a ceer $E := E_0$. Note that every E-class is non-computable: Indeed, if a class $[x]_E$ is computable, then the c.e. set $\omega \setminus [x]_E$ intersects all but one E-classes.

We define the desired family $\mathcal{T}$ by arranging its computable numbering: For $x \in \omega$, set $\theta(x) := [x]_E$.

Lemma 4.2. *1. The family $\mathcal{T}$ is effectively discrete.*
2. Let $\mathcal{S} \subsetneq \mathcal{T}$. If $\mathcal{S}$ is infinite, then it does not have computable numberings.
3. If $\nu \in Com_1^0(\mathcal{S})$, then every set $A \in \mathcal{S}$ has infinitely many ν-numbers.

Proof. (1) If $A \neq B$ are sets from $\mathcal{T}$, then $A \cap B = \emptyset$. Thus, the sequence of finite sets $(\{k\})_{k\in\omega}$ witnesses the effective discreteness of the family $\mathcal{T}$.

(2) Assume that ν is a computable numbering of an infinite family $\mathcal{S} \subsetneq \mathcal{T}$. Then the c.e. set $W := \bigcup_{n\in\omega} \nu(n)$ intersects infinitely many E-classes, but it does not intersect all E-classes. This contradicts the choice of the ceer E.

(3) Suppose that A has only finitely many ν-numbers. W.l.o.g., one may assume that there is a natural number n_0 such that $\nu(x) = A$ iff $x \leq n_0$. Then a numbering $\mu(x) := \nu(x + n_0 + 1)$ is a computable numbering of the family $\mathcal{T} \setminus \{A\}$, which contradicts the previous item of the lemma.

□

We say that a numbering ν is 1-*reducible* to a numbering μ (denoted by $\nu \leq_1 \mu$) if there is an injective, total computable function $f(x)$ such that $\nu = \mu \circ f$. The following analogue of Myhill Isomorphism Theorem is known (see, e.g., Corollary 2 in [12, p. 208]): If $\nu \leq_1 \mu$ and $\mu \leq_1 \nu$, then ν is computably isomorphic to μ.

Therefore, it is sufficient to show that for any $\nu, \mu \in Com_1^0(\mathcal{T})$, we have $\nu \leq_1 \mu$. A desired 1-reducibility $f\colon \nu \leq_1 \mu$ can be built in stages. At a stage s, we find an element k enumerated into the c.e. set $\nu(s)$. After that, we search for a number m such that $m \notin range(f[s])$ and $k \in \mu(m)$. Such a number m exists by the third item of Lemma 4.2. Moreover, it is easy to see that $\mu(m) = \nu(s)$. Thus, we set $f(s) := m$ and proceed to the next stage. Theorem 4.2 is proved. □

Corollary 4.1. *For any natural number $n \geq 1$, there is a computable infinite family $\mathcal{S}$ such that card$(\mathcal{R}_{bm}(\mathcal{S})) = 2^n - 1$.*

Proof. Consider the family $\mathcal{T}$ from the theorem above. If $n = 1$, then one can just choose $\mathcal{S} := \mathcal{T}$.

Suppose that $n \geq 2$. Choose a finite family $\mathcal{V}$ from Theorem 4.1 such that $card(\mathcal{R}_{bm}(\mathcal{V})) = 2^n - 1$. Then the desired family $\mathcal{S}$ contains the following sets: For any $A \in \mathcal{T}$, we add the set $\{2x : x \in A\}$ into $\mathcal{S}$. For every $B \in \mathcal{V}$, we put the set $\{2y + 1 : y \in B\}$. It is not difficult to show that for this $\mathcal{S}$, the Rogers bm-semilattice contains precisely $2^n - 1$ elements. □

The next two propositions give sufficient conditions for a semilattice $\mathcal{R}_{bm}(\mathcal{S})$ being infinite.

Proposition 4.1. *Let $\mathcal{S}$ be a computable infinite family. Suppose that there is a computable numbering ν of $\mathcal{S}$ with the following property: there are infinitely many sets $A \in \mathcal{S}$ such that the set $\nu^{-1}[A] = \{x \in \omega : \nu(x) = A\}$ is computable. Then the semilattice $\mathcal{R}_{bm}(\mathcal{S})$ is infinite.*

Proof (sketch). Suppose that $A_0, A_1, \dots, A_n$ are distinct sets from $\mathcal{S}$ such that $\nu^{-1}[A_i]$, $i \leq n$, are computable. For $i \leq n$, fix the least number m_i such that $\nu(m_i) = A_i$. W.l.o.g., we may assume that $m_i > 0$. We define computable numberings

$$\mu(x) := \begin{cases} \nu(x), & \text{if } \nu(x) \notin \{A_0, A_1, \dots, A_n\}, \\ A_i, & \text{if } x = m_i, \\ \nu(0), & \text{otherwise;} \end{cases}$$

$$\theta_i(2x) := \mu(x), \quad \theta_i(2x+1) := A_i, \ i \leq n.$$

Lemma 2.1 implies that the numberings θ_i, $i \leq n$, are pairwise incomparable under bm-reducibility. Therefore, the semilattice $\mathcal{R}_{bm}(\mathcal{S})$ is infinite. □

Corollary 4.2. *If an infinite family $\mathcal{S}$ has a decidable, computable numbering, then the semilattice $\mathcal{R}_{bm}(\mathcal{S})$ is infinite.*

Recall that an infinite set $X \subset \omega$ is *immune* if there is no infinite c.e. set W with $W \subseteq X$. A set $Y \subseteq \omega$ is *co-immune* if its complement is immune.

Proposition 4.2. *Let $\mathcal{S}$ be a computable infinite family. Suppose that there is a computable numbering ν of $\mathcal{S}$ with the following property: there are infinitely many sets A from $\mathcal{S}$ such that $\nu^{-1}[A]$ is co-immune. Then the semilattice $\mathcal{R}_{bm}(\mathcal{S})$ contains an infinite antichain.*

Proof (sketch). Given a set A from $\mathcal{S}$, we define a computable numbering

$$\mu_A(2x) := \nu(x), \quad \mu_A(2x+1) := A.$$

Suppose that A and B are distinct sets from $\mathcal{S}$ such that both $\nu^{-1}[A]$ and $\nu^{-1}[B]$ are co-immune. Assume that $f\colon \mu_A \leq_{bm} \mu_B$. Then the set $V := \{f(2x+1)/2 : x \in \omega\}$ is an infinite c.e. subset of $\omega \setminus \nu^{-1}[B]$, which contradicts the co-immunity of $\nu^{-1}[B]$. Thus, μ_A and μ_B are incomparable under bm-reducibility. Therefore, the semilattice $\mathcal{R}_{bm}(\mathcal{S})$ contains an infinite antichain. □

5 Further Discussion

First, we briefly discuss related results on hyperarithmetical numberings.

Let α be a computable ordinal such that $\alpha \geq 2$. Consider a family of Σ^0_α-sets $\mathcal{S}$, which has a Σ^0_α-computable numbering. Following the lines of Sect. 2, one can introduce the Rogers semilattices $\mathcal{R}^0_{\alpha;m}(\mathcal{S})$ and $\mathcal{R}^0_{\alpha;bm}(\mathcal{S})$, which are induced by the degrees of Σ^0_α-computable numberings of $\mathcal{S}$, under the reducibilities $\leq$ and $\leq_{bm}$, respectively.

Proposition 5.1. *Let $\alpha \geq 2$ be a computable successor ordinal. Suppose that $\mathcal{S}$ is a Σ^0_α-computable family such that $\mathcal{S}$ contains at least two elements. Then the Rogers semilattice $\mathcal{R}^0_{\alpha;bm}(\mathcal{S})$ is infinite, and it is not a lattice.*

Proof. Recall that $card(\mathcal{R}^0_{\alpha;m}(\mathcal{S})) \leq card(\mathcal{R}^0_{\alpha;bm}(\mathcal{S}))$. Goncharov and Sorbi [21, Theorem 2.1] proved that the semilattice $\mathcal{R}^0_{\alpha;m}(\mathcal{S})$ is infinite.

Furthermore, in [21, Proposition 2.8] the following result was obtained. If $\mathcal{S}$ is infinite, then one can build a uniform sequence $(\nu_i)_{i\in\omega}$ of Σ^0_α-computable numberings of $\mathcal{S}$ with the following property: If $i \neq j$, then there is no Σ^0_α-computable numbering μ of $\mathcal{S}$ such that $\mu \leq \nu_i$ and $\mu \leq \nu_j$.

This implies that for an infinite $\mathcal{S}$, both structures $\mathcal{R}^0_{\alpha;m}(\mathcal{S})$ and $\mathcal{R}^0_{\alpha;bm}(\mathcal{S})$ are not lower semilattices. Note that the results of Goncharov and Sorbi are formulated and proved only for finite ordinals α. Nevertheless, essentially the same proofs also work for infinite successor α.

Now assume that a Σ^0_α-computable family $\mathcal{S}$ is equal to $\{A_0, A_1, \dots, A_n\}$, and consider the following Σ^0_α-computable numberings of $\mathcal{S}$:

$$\nu_i(x) := \begin{cases} A_x, & \text{if } x \leq n, \\ A_i, & \text{otherwise.} \end{cases}$$

Lemma 2.1 shows that the numberings ν_i, $i \leq n$, are pairwise bm-incomparable. Moreover, it is not hard to show that for $i \neq j$, there is no numbering μ of the family $\mathcal{S}$ such that $\mu \leq \nu_i$ and $\mu \leq \nu_j$. Hence, $\mathcal{R}^0_{\alpha;bm}(\mathcal{S})$ is not a lattice. ◻

We note that the methods of [22] can be used to transfer the obtained existence results (such as Theorem 3.1 and Corollary 4.1) into non-limit levels of the Ershov hierarchy, see Theorems 2 and 17 in [22] for the details.

In conclusion, we formulate two problems that are left open.

Question 5.1. Let $\mathcal{S}$ be a computable infinite family of c.e. sets. Describe all possible cardinalities of the Rogers semilattice $\mathcal{R}_{bm}(\mathcal{S})$.

Note that all our examples of computable families $\mathcal{S}$ possess the following property: If $\mathcal{R}_{bm}(\mathcal{S})$ is an infinite lattice, then the structure $\mathcal{R}_m(\mathcal{S})$ has only one element.

Question 5.2. Is there a computable family $\mathcal{S}$ such that $\mathcal{R}_m(\mathcal{S})$ is infinite and $\mathcal{R}_{bm}(\mathcal{S})$ is a lattice?

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