An algorithmic approach
to characterizations of admissibles

Bruno Durand$^{(\boxtimes)}$ and Grégory Lafitte$^{(\boxtimes)}$

LIRMM, CNRS, Université de Montpellier, 161 rue Ada, 34090 Montpellier, France
{bruno.durand,gregory.lafitte}@lirmm.fr

Abstract. Sacks proved that every admissible countable ordinal is the first admissible ordinal relatively to a real. We give an algorithmic proof of this result for constructibly countable admissibles. Our study is completed by an algorithmic approach to a generalization of Sacks' theorem due to Jensen, that finds a real relatively to which a countable sequence of admissibles, having a compatible structure, constitutes the sequence of the first admissibles. Our approach deeply involves infinite time Turing machines. We also present different considerations on the constructible ranks of the reals involved in coding ordinals.

Introduction

One motivation for ordinal computability (see, *e.g.*, [5]) is to find new proofs for theorems in constructible and descriptive set theory. Such proofs may yield extra information and give a further perspective. Examples so far include the proof by Koepke and Seyfferth of the existence of incomparable α-degrees using α-Turing machines in [26], Koepke's proof of the continuum hypothesis in L using ordinal Turing machines in [25] and the proof by Schlicht and Seyfferth of Shoenfield's absoluteness theorem in [34], also via ordinal Turing machines.

In this paper, we prove a constructive version of a theorem of Sacks, along with a strengthening thereof due to Jensen, the latter using *infinite time Turing machines* (ITTMs). These were invented by Hamkins and Kidder and first introduced in [17]. A topic that has received particular attention is the issue of *clockability*. This subject is of special concern for us as it shows some deep relations with admissible ordinals.

Admissible ordinals correspond to levels of the constructible hierarchy that are closed enough to carry computabilities, in other words closed under Σ_1 definability; the reference text is Barwise's book [1]. This notion has several equivalent definitions stated in rather different terms and involves many deep properties witnessed by ITTMs.

Sacks' theorem is very important to understand countable admissible ordinals. It states that for any countable admissible ordinal α, there exists a real r such that α is the first non-recursive ordinal relatively to r (written $\omega_1^{\mathrm{CK},r}$, this ordinal is admissible and the first admissible $> \omega$ relatively to r). For the first admissibles, this statement is rather clear: the first admissible is ω_1^{CK}, the next one is $\omega_1^{\mathrm{CK},r_1}$ where r_1 is a code of lowest constructible rank for ω_1^{CK}, and so on. The situation is more complex with the first *recursively inaccessible* ordinal (an admissible which is a limit of admissibles). Indeed a computability construction is required to transform a sequence of admissibles co-final in this admissible into an adequate real. The situation is much more complex for larger ordinals because of coding problems. For successor admissibles, those that are not recursively inaccessible, our idea is to use a code for the admissible just below, but questions arise to know whether such a code exists and if it exists, in which constructibility level. For recursively inaccessibles, the situation is even more subtle since the ω-sequence of admissibles cofinal in this ordinal must also be definable, and various cases arise depending on the different gaps in which the ordinals of the sequence might appear.

When one has proven Sacks' theorem, it is rather natural to try to obtain that any two admissibles become the first and the second admissibles relatively to a real. Once again the case where the ordinals are small is not difficult (although it requires to work both with Turing and hyperarithmetic reducibilities). For transfinite sequences of admissibles, the situation is more complex since our goal is that this sequence coincides with an initial segment of the admissibles using only an oracle r—which one can see as some kind of translation function. We provide an elementary proof of Jensen's generalization to Sacks' theorem for sequences of constructibly countable admissibles, the ordinal type of which is bounded by the *supremum* of the clockable times of ITTMs.

We begin our paper by describing various gaps and ranks concerning definability/coding issues, and then describe our algorithmic and computability approach to proving versions of these two theorems.

1 Gaps and ranks

In the different results of this paper, in particular in Theorem 3, we are interested in finding the *simplest* reals, simplest in the sense that they appear at the lowest possible rank in the constructible hierarchy. This approach is naturally linked to various gaps and ranks described in this section.

1.1 The constructible hierarchy

We recall Gödel's constructible hierarchy $(L_\alpha : \alpha \in \mathrm{On})$ and an accompanying *ad hoc* hierarchy $\mathcal{J}_\alpha$ in which we can find Jensen's master codes for levels where new reals appear. The reader is referred to [22,10,21] for more on this hierarchy.

Definition 1. $L_0 = \mathcal{J}_0 = \varnothing$, $L_{\alpha+1} = \mathrm{Def}(L_\alpha)$, $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$, $L = \bigcup_{\alpha \in \mathrm{On}} L_\alpha$, $\mathcal{J}_{\omega \cdot \xi} = L_\xi$, $\mathcal{J}_{\omega \cdot \xi + n} = \Delta_n(L_\xi)$, *where λ is a limit ordinal and $\mathrm{Def}(X)$, resp.*

$\Delta_n(X)$, *is the set of all subsets of* X *definable with parameters in* $\langle X, \in \rangle$ *by a first order formula,* resp. *by both* Σ_n *and* Π_n *first order formulae.*

We are interested in finding the lowest level of Gödel's constructible hierarchy where certain sets appear, in particular certain reals (subsets of ω). The least level α of L where a certain set A appears ($A \in L_{\alpha+1}$) is called the L-rank of A.

1.2   Gaps in the constructible hierarchy

H. W. Putnam and G. S. Boolos identified levels of L where no new reals appear.

Theorem 1. *There are arbitrarily long gaps in L where no new reals appear.*[1]

This leads to the notion of gaps in the constructible hierarchy: an ordinal α is in a *Putnam gap* (or L-gap) if no new reals appear in L_α. The proof of Theorem 1 uses the fact that if M is a countable elementary subset of $L_{\omega_2^L}$, then the image of ω_1^L under the Mostowski collapse of M is a very long Putnam gap. The idea behind such a Putnam gap is that an ordinal $\alpha < \omega_1^L$ starts a long gap if it is very similar to ω_1^L. When new reals appear at level $\alpha + 1$, a real coding all of L_α is one of them.

Lemma 1 ([2,3]). *If new reals appear at $\alpha + 1$, then among them is an arithmetical copy*[2] E_α *of* L_α.

If α is a Putnam-gap ordinal and if $r \in L_\alpha$ is a real coding a well-order on ω, then the order type of that well-order is less than α. Thus if α starts a Putnam-gap, then α is a limit ordinal and Marek and Srebrny [30] actually showed that an ordinal α starts a Putnam-gap if and only if $L_\alpha \models \text{ZFC}^- + V = \text{HC}$.

Gaps naturally appear both in the L and the $\mathcal{J}$ hierarchies. When new reals appear at a level, we call this level *an index*. Let $\ell_{\mathcal{J}}(\alpha)$ be the maximum β such that $[\alpha, \alpha + \beta)$ is a $\mathcal{J}$-gap. Thus α is a $\mathcal{J}$-gap ordinal if and only if $\ell_{\mathcal{J}}(\alpha) \neq 0$, and if α starts an $\mathcal{J}$-gap, $\ell_{\mathcal{J}}(\alpha)$ is the length of that $\mathcal{J}$-gap.

Special reals in $\mathcal{J}$ were identified by Jensen: a real r is a *master code* for $\mathcal{J}_\xi$ (or, for ξ) if $\{x \subseteq \omega : x \leqslant_T r\} = \mathcal{J}_{\xi+1} \cap \mathcal{P}(\omega)$.

Theorem 2 (Jensen). ξ *is a $\mathcal{J}$-index if and only if there is a master code for* ξ. *Furthermore, if r is a master code for ξ, then r' is the master code for $\xi + 1$.*

The gaps in the $\mathcal{J}$ hierarchy can be described in the following way (for more on master codes and these gaps, see [20,18,19,22,23]): let $\varsigma(\alpha)$ be the least strict upper bound on $\{\text{Ind}_{\mathcal{J}}(\xi) : \xi < \alpha\}$, where $\text{Ind}_{\mathcal{J}} : \omega_1^L \to \omega_1^L$ enumerates the $\mathcal{J}$-indices in increasing order; obviously $\alpha \leqslant \text{Ind}_{\mathcal{J}}(\alpha)$. $\alpha \leqslant \varsigma(\alpha)$ and $\alpha < \varsigma(\alpha)$

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[1] Let $\beta > \alpha$ be countable ordinals such that there is an elementary embedding $j : L_\beta \to L_{\omega_2}$ with *critical point* $cr(j) \geqslant \alpha$. For every $\gamma < cr(j)$, $L_{\omega_2} \models$ "No new reals appear between ranks ω_1 and $\omega_1 + \gamma$." No new reals thus appear between $cr(\text{j})$ and $cr(j) + \gamma$, by elementarity and absoluteness. *Cf.* [31,2,3,28].

[2] E_α is an *arithmetical copy* of L_α if there is one-one function f from L_α to ω (and onto the field of E_α) such that $\forall x, y \in L_\alpha$, $x \in y \iff \langle f(x), f(y) \rangle \in E_\alpha$.

iff $\mathrm{Ind}_{\mathcal{J}}(\alpha) > \alpha$ and α does not start a $\mathcal{J}$-gap. $\varsigma(\alpha) \neq \mathrm{Ind}_{\mathcal{J}}(\alpha)$ iff α starts a $\mathcal{J}$-gap. $\mathrm{Ind}_{\mathcal{J}}(\alpha) = \varsigma(\alpha) + \ell_{\mathcal{J}}(\varsigma(\alpha))$. $\varsigma(\lambda)$ is a $\mathcal{J}$-gap ordinal iff $\varsigma(\lambda)$ is admissible iff λ is admissible and locally countable ($\models \forall a \, (a \text{ is countable})$). If α starts an L-gap, then α also starts a $\mathcal{J}$-gap. If α starts a $\mathcal{J}$-gap, then α is the *supremum* of L-indices and α starts an L-gap iff $\ell_{\mathcal{J}}(\alpha) \geqslant \omega$. Moreover if α starts a $\mathcal{J}$-gap and $\ell_{\mathcal{J}}(\alpha) \geqslant n$, then α is Σ_n-admissible.

1.3   Definability and coding

There are some other gaps that we call *definability gaps*, which are obviously linked to Putnam gaps: there are countable ordinals α, υ such that α is not definable in L_υ.[3] It is possible to characterize the least such definability gap: the ordinal υ_0, which is the least υ such that there is an ordinal α not definable in L_υ, can be characterized as the least η such that there exists an ordinal $\delta < \eta$ such that $L_\delta \prec L_\eta$ ($\prec$ means *being an elementary submodel*).[4]

   Now we would like to know for a particular real where it first appears, especially for a real coding a countable ordinal.

Definition 2. *Let α be a countable ordinal. α is* definable at (level)γ *if α is definable without parameters in L_γ. α is* codable at (level)γ *if a real appears in $L_{\gamma+1}$ coding α. The* code-rank *of α ($< \omega_1^L$), code-rk(α), is the least γ such that α is codable at γ. α is* countable at (level)γ *if $L_\gamma \models$ "α is countable".*

   If $\alpha < \omega_1^L$, then it is codable at some level. And if α is countable at β, then α is codable and definable at β.

Lemma 2. *For every countable ordinal α, there exists a countable β such that α is definable at β.*[5]

   By Löwenheim-Skolem and a combination of footnotes 5 and 3, there exists an ordinal $\alpha < \omega_1^L$ which is definable at a β, then not definable at a $\beta' > \beta$, *etc.* There is actually an upper bound for the ordinals that remain definable from

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[3] There exists α such that $L_\alpha \prec L_{\omega_1}$, α is thus not definable in L_{ω_1}. There is a countable $\upsilon > \alpha$ such that $L_\upsilon \prec L_{\omega_1}$, and α is already not definable in L_υ.

[4] υ_0 is clearly $\leqslant$ the least such η, η_0, since whenever one has $L_\alpha \prec L_\beta$, α is not definable in L_β. Now, suppose that $\upsilon_0 < \eta_0$, in other words, for all $\delta < \upsilon_0$, $L_\delta \not\prec L_{\upsilon_0}$. Now, by Löwenheim-Skolem there is a countable elementary submodel of L_{υ_0}. Take the $\subseteq$-least such model M. By the Condensation Lemma, there is an $\alpha < \upsilon_0$ and an isomorphism j such that the Mostowski collapse of M is isomorphic to L_α via j. j cannot be trivial as this would mean that $L_\alpha \prec L_\delta$, although $\delta < \upsilon_0$ and υ_0 is the least such ordinal. We can thus consider κ, the critical point[1] of j. Since $L_\alpha \cong M \prec L_{\upsilon_0}$, $L_\kappa \prec L_{j(\kappa)}$. But then κ cannot be definable in $L_{j(\kappa)}$, and thus $\upsilon_0 \leqslant j(\kappa)$. But $j(\kappa) < \upsilon_0$, contradiction.

[5] Consider $\kappa = \aleph_\alpha$. κ is definable as the greatest cardinal in L_{κ^+}. (Here κ^+ denotes the least ordinal of cardinality greater than κ.) And thus α is also definable in L_{κ^+}. Löwenheim-Skolem's theorem, in conjunction with Mostowski's lemma and the Condensation Lemma, provides the countable β such that α is definable in L_β.

some point on (they have been called *memorable*[6] ordinals): ordinals α for which there exists β such that for any countable $\gamma \geqslant \beta$, α is still definable at γ.[7]

1.4 Clockability

Computations by ITTMs are intimately related to admissibles since they are in some sense universal tools for Σ_1 functions. In particular, an ordinal α is said to be *clockable* if there exists an ITTM that halts exactly in time α on input 0. ITTMs can also write a real coding an ordinal. In this case the ordinal is said to be *writable*. See [17] for more on these notions. A very powerful theorem by Welch [36] asserts that the *supremum* of clockable ordinals is the same as the *supremum* of the writable ordinals. In our paper, we denote this ordinal by λ_∞.[8]

Writable ordinals have no gaps: α is writable if and only if $\alpha < \lambda_\infty$. The situation is different with clockability: there are gaps inside clockable ordinals. The study of these gaps is very interesting and has been carried out through many papers (see in particular the seminal [17,37]), among them we refer to [6] since it contains all the considerations on gaps needed in the present paper. We can summarize the situation in terms of clockable gaps as follows. The situation resembles Putnam gaps with a major difference: for a clockable gap, the starting point is also a limit ordinal, but its size is always a limit ordinal. Furthermore, they are deeply related to admissibles. The properties we most often use in the present paper are that all starting points of gaps are admissibles, that no admissible is clockable, but some admissibles can be strictly inside a gap (see [6]). Moreover, if α starts a gap, the ordinal type of clockable ordinals below α is exactly α.

The structure of admissible ordinals can be relativized to a real without any major change. With ITTMs we can use the standard oracle definition of Turing machines (with a special oracle tape), or even see the oracle real as an input.

Some other considerations are important for us. If an ordinal can be written by an ITTM in time γ then it is codable at level γ. The writing time of a recursive ordinal is exactly ω and if the ordinal α is not recursive, its writing time is exactly the *supremum* of the ends of those gaps that start no later than α. For admissibles, the situation is simple: their writing time is exactly the end of the gap they belong to. A detailed proof of these results can be found in [27,11]. A consequence of these results is that for every clockable $\alpha < \lambda_\infty$ which does not end a clockable gap, we have that the code-rank of α is $< \alpha$ and bounded by its writing time.

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[6] *Cf.* https://mathoverflow.net/questions/259100/memorable-ordinals.

[7] Any countable τ such that $L_\tau \prec L_{\omega_1}$ is such an upper bound: if α is definable at β, take δ above τ and β such that $L_\delta \prec L_{\omega_1}$. We then have $L_\tau \prec L_\delta \prec L_{\omega_1}$. α is thus definable at δ, since δ is above β, and also at τ. τ is therefore above α and any other definable ordinal. In fact, the least non-memorable ordinal τ_0 is the least ordinal τ with uncountably many elementary extensions $L_\tau \prec L_\gamma$. (*Cf.* footnote 6).

[8] We use Barwise's convention for admissibles: $\tau_0 = \omega$, $\tau_1 = \omega_1^{\text{CK}}$, ..., τ_α is the α-th admissible. Note that there exist admissibles α such that $\alpha = \tau_\alpha$. Such is the case for λ_∞, but it is not the first one.

If we set a (sufficiently closed) bound for computation times of the ITTMs, we obtain nice computation models. If the bound is chosen as ω, we obtain classical Turing machines; if the bound is chosen to be any admissible ordinal, then the model is well defined, keeping the same universal machine for all bounds. Therefore, if α is admissible, we can define $\leqslant_\alpha$ as the ITTM reducibilities with computations bounded by admissible time α. The reducibilities $\leqslant_\alpha$ and $\leqslant_\beta$ are the same over reals if and only if α and β belong to the same clockable gap. If α and β do not belong to the same clockable gap and $\alpha < \beta$ then $\leqslant_\alpha$ is a refinement of $\leqslant_\beta$.

2   Sacks' theorem revisited

Theorem 3 (Sacks). *For every admissible countable ordinal α, there exists a real r such that $\alpha = \omega_1^{\mathrm{CK},r}$.*

Gerald E. Sacks first proved Theorem 3 in [32] by a forcing argument. Friedman and Jensen [13] gave an alternative proof, which does not make use of forcing but involves infinitary logic. See also Chong and Yu [8, Theorem 5.4.12] for a proof using Steel forcing.

In the proof of our version of Sacks' theorem, Theorem 4, we need to be able to construct reals with *ad hoc* properties. We choose to isolate this construction in the following lemma.

Lemma 3. *For any countable set of reals $\{r_i : i \in \omega\}$ such that for every i, $r_i' \leqslant_T r_{i+1}$, there exists a real r such that for all i, $r_i \leqslant_T r$, but[9] $\bigoplus_i r_i \not\leqslant_T r$. Moreover $r \leqslant_T (\bigoplus_i r_i)'$.*

Proof. We first prove the lemma when for every i, $r_i'' \leqslant_T r_{i+1}$, then we adapt the proof to the hypothesis $r_i' \leqslant_T r_{i+1}$. Please note that it is not straightforward (we cannot just take every other real in the sequence) since it might be possible that $\bigoplus_i r_{2i} <_T \bigoplus_i r_i$. This cannot be the case for the sequences constructed in our use of this lemma but we prefer to formulate the lemma and prove it in its most general form.

The real r needs to code the r_i's in such a way that for every i, r_i can be computed from r but not uniformly, as $\bigoplus_i r_i$ would then be $\leqslant_T$-below r. We consider that r is a mapping from ω^2 to $\{0, 1\}$ that contains r_i in the column $\mathfrak{b}(i)$. At the same time, we make sure that $\bigoplus_i r_i \not\leqslant_T r$ by adding just the needed supplementary information to hide the r_i's.

Construction: r is constructed as $\bigcup_i o_i$ where the o_i's are compatible (infinite) oracles that represent the left part of r up to the column $\mathfrak{b}(i)$, the rest is empty (at 0). We now assume that o_i has been built and that we know $\mathfrak{b}(i)$, and we give the construction for o_{i+1} and $\mathfrak{b}(i+1)$.

We now consider all the computations $\varphi_i^\tau(\langle i + 1, j \rangle)$ for every j and every finite extension τ of o_i. We look for the first (τ, j) such that it either *(i)* converges

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[9] The infinite join, $\bigoplus_i r_i$, of the r_i's is defined as $\{\langle i, j \rangle : j \in r_i\}$.

and is $\neq r_{i+1}(j)$, or *(ii)* for every extension of τ, it diverges. $\mathfrak{b}(i+1)$ is then taken to be greater $(+1)$ than [case *(i)*] the maximum between $\mathfrak{b}(i)$ and the greatest column reached during the computation, [case *(ii)*] the greatest column reached in the enumeration of the extensions of o_i; which is the column from which every extension will make the computation diverge on j.

o_{i+1} is the found τ, in which we add r_{i+1} at column $\mathfrak{b}(i+1)$. Thus o_{i+1} contains o_i, r_{i+1} in column $\mathfrak{b}(i+1)$, and some extra *finite* information. $\square_{Construction}$

We now show, by induction, that the construction works: we suppose that o_i has been constructed and that $o_i \leqslant_T r'_i$. We show that o_{i+1} is built such that $o_{i+1} \leqslant_T r'_{i+1}$. In the construction of o_{i+1} from o_i, we end up either in case *(i)* or *(ii)*: by *reductio ad absurdum*, suppose that both cases are false. We then

–a– either have convergence for every j such that they all converge to $r_{i+1}(j)$. But then we would have $r_{i+1} \leqslant_T o_i$, which is impossible since $o_i \leqslant_T r'_i$,
–b– or there are some j's where it diverges and all extensions end up making it converge to $r_{i+1}(j)$, but then r_{i+1} would be recursively enumerable in o_i, which is also impossible since $r_{i+1} \geqslant_T r''_i$.

Looking closely at the construction, one observes that $o_{i+1} \leqslant_T r'_{i+1}$.

Now, for every i, $r_i \leqslant_T r$ (from a finite information, $\mathfrak{b}(\mathrm{i})$). And if we had $\bigoplus_i r_i \leqslant_T r$, there would be an e such that for all i, j, $\varphi_e^r(\langle i, j\rangle) = r_i(j)$. Take $i = e + 1$, we observe a contradiction:

Case *(i)*: $\varphi_e^r(\langle e+1, j\rangle) \neq r_{e+1}(j)$ for the found j,
Case *(ii)*: $\varphi_e^r(\langle e+1, j\rangle)$ diverges for every extension. But r is an extension of o_{i+1}, which implies that φ_e^r diverges on the found j.

As we have that for every i, $o_i \leqslant_T r'_i$, we get that $r = \bigcup_i o_i \leqslant_T (\bigoplus_i r_i)'$.

Now let us adapt the proof to the hypothesis $r'_i \leqslant_T r_{i+1}$: instead of coding r_i in the column $\mathfrak{b}(i)$ we code r_{2i} in $\mathfrak{b}(i)$ and r_{2i+1} in $\mathfrak{b}(i) + 1$. The oracle o_i is thus defined until the $\mathfrak{b}(i) + 1$ column and the finite extension which is defined uses r_{2i+3} instead of r_{i+1} since we have that $r_{2i+3} \geqslant_T o'_i$. Thus we have non-recursivity relative to oracle r only on the r_{2i+1} terms of $\bigoplus_i r_i$ but it is sufficient for our lemma. $\square$

The construction can be slightly enhanced to make r and $\bigoplus_i r_i$ incomparable, by introducing witnesses for $r \not\leqslant_T \bigoplus_i r_i$. We just need to be careful to introduce only a finite number of witnesses on every column. The following theorem provides an explicit construction of a weaker version of Sacks' result.

Theorem 4. *For every admissible constructibly countable ordinal α, there exists a real r such that $\alpha = \omega_1^{\mathrm{CK},r}$.*

Proof. First, we solve the easy case, where α is a *successor admissible*, *i.e.*, admissible but not a limit of admissibles: $\alpha = \beta^+$ for some admissible β.

We first assume that α and β are not codable at the same level. There exist (many) reals that code β. For any such real r, we have that $\alpha \leqslant \omega_1^{\mathrm{CK},r}$ and $\omega_1^{\mathrm{CK},r}$ is admissible. Among those reals, we choose r_β of least constructible rank γ.

If $\alpha < \omega_1^{\mathrm{CK},r_\beta}$, then α is recursive in r_β and codable at level γ. Therefore, β and α are codable at the same level, which contradicts the hypothesis.

We now assume that α and β have the same code-rank (in L). Let γ then be the least level of $\mathcal{J}$ where a code for α appears, and thus also where one appears for β. New reals appear at this level and among them, there is a master code for γ. From this master code, we can extract a code r for β of minimal Turing degree. α is not recursive in r, since α is admissible. And thus, $\alpha = \omega_1^{\mathrm{CK},r}$.

Now assume that α is *recursively inaccessible*. As $\alpha < \omega_1^L$, we have $\alpha = \lim_{n<\omega} \alpha_n$, where the α_n's are admissible, the α_n's are codable at the code-rank of α, and α is admissible relative[10] to $\{\alpha_n : n < \omega\}$. For each n, r_n is chosen as a "simplest" code for α_n. To precise the meaning of "simplest", there are two cases: either they are all of the same code-rank than α, or they can be chosen to have strictly increasing code-ranks.

In the latter case, by the admissibility hypothesis on the structure, we choose the r_n's of least code-rank so that the $\bigoplus_n r_n \equiv_T r_\alpha$. In other terms, we do not add extra information in the precise chosen sequence.

In the former case, α and all α_n's are of code-rank γ. As a code for α, we choose a real r_α extracted from a master code of $\mathcal{J}$ for the least level where a code for α appears. From r_α we can define integer indices $i_0, i_1, \ldots$ such that in the order that codes α in r_α, α_n is at index i_n. Thus, the order for α truncated at level i_n is a code for α_n and is represented by a real r_n. We can remark that 1's in r_n are also 1's in r_{n+1} while some 0's in r_n become 1's in r_{n+1}. By the admissibility hypothesis on the structure, the infinite join of the r_n's is Turing-equivalent to r_α.

In both cases, we also have that $r_n' \leqslant_T r_{n+1}$, since the α_n's are admissible. The real r we look for is obtained directly by Lemma 3. As r_α is not recursive in r, and because of the hypothesis on the sequence of the α_n's, r_n is recursive in r and α is the least ordinal which is not recursive in r. α is thus $\omega_1^{\mathrm{CK},r}$. $\square$

This proof provides properties that go beyond the statement of Theorem 4. For instance, if $\alpha = \omega_1^{\mathrm{CK},r}$ for $r \in L_\alpha$ then $\alpha = \beta^+$, *i.e.*, is a successor admissible, and there is γ, such that $\beta < \gamma < \alpha$ and new reals appear in L_γ, *i.e.*, the interval (β, α) is not inside a coding gap. An analogous result can be found in [7].

Note that since Lemma 3 gives $r \leqslant_T (\bigoplus_i r_i)'$, our proof provides a certain minimality property of the constructed real. In general, it can be expressed in terms of hyperarithmetic[11] minimality, exactly as the refinement that Sacks obtained in [33, Theorem 4.26] of his original theorem. But in the case where the admissible is a successor admissible, we get an improved optimality result:

Theorem 5. *For every admissible $\alpha < \omega_1^L$, there exists a real r such that $\omega_1^{\mathrm{CK},r} = \alpha$, and for every real $s <_h r$, $\omega_1^{\mathrm{CK},s} < \alpha$. Moreover if α is a successor admissible, for every real $s <_T r$, $\omega_1^{\mathrm{CK},s} < \alpha$.*

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[10] An ordinal α is *admissible relative to a set of ordinals A* if $\langle L_\alpha [A]; A \cap \alpha \rangle$ is an admissible structure.

[11] Recall that $Y \leqslant_h X$ if Y is hyperarithmetic in X, that is if Y is ITTM-computable in some bounded recursive ordinal length of time ($< \omega_1^{\mathrm{CK},X}$).

This naturally leads to the study of the least constructible rank of the real that defines a countable admissible ordinal *via* Sacks' result. The *Sacks-rank* of a countable ordinal α, Sacks-rk(α), is the least γ such that there is a real r in $L_{\gamma+1}$ such that $\omega_1^{\mathrm{CK},r} = \alpha$. Thanks to our construction, some structural properties can be proved concerning Sacks ranks: for every countable ordinal $\alpha < \omega_1^L$, if α is a successor admissible ($=\beta^+$), we have that Sacks-rk(α) = code-rk(β), and otherwise (α is recursively inaccessible) that Sacks-rk(α) = code-rk(α).

3   Jensen's theorem revisited

Ronald B. Jensen and Harvey Friedman gave in [13] a model-theoretic proof of Sacks' theorem (*cf.* Theorem 3). Jensen had formulated the *model existence* theorem and applied it to provide an alternative proof. This method could not be applied to prove Jensen's theorem, which is a generalization of Sacks' theorem to a sequence of admissibles. He had to use proper class forcing over admissible sets. A proof of Jensen's theorem can be found in his unpublished manuscript [23, Chapter 6, Theorem 4] and also in Simpson and Weitkamp's [35].

It has to be noted that one needs to be careful when stating the hypothesis of this theorem: choose for example a recursively inaccessible for α_ω and $\{\alpha_i : i \in \omega\}$ a sequence of admissibles cofinal in α_ω. Relative to any oracle, the *supremum* of the first ω admissibles cannot be admissible: to see that, design an ITTM looking for clockable gaps and make it halt at the *sup* of the ω first gaps (*cf.* [6]). The hypothesis proposed by Jensen, which solves this difficulty, asserts that every admissible in the considered sequence is still admissible relative[10] to admissibles of the sequence below it.[12] This is the hypothesis that we use in our version of the theorem (Theorem 6).

We first present a computability lemma that provides a real that verifies some computability specifications in terms of ITTM reducibilities ($\leqslant_{\tau_\alpha}$ means ITTM-computable in time $< \tau_\alpha$). The proof of this lemma is a direct generalization of the proof of Lemma 3.

Lemma 4. *For any ordinal $\gamma < \lambda_\infty$, for any sequence of reals $(r_\alpha : \alpha < \omega \cdot \gamma)$, such that for every α, $r'_\alpha <_T r_{\alpha+1}$, there exists $r : \gamma \to \mathcal{P}(\omega)$ such that for all $\beta < \gamma$, for all $i < \omega$, $r_{\omega\cdot\beta+i} \leqslant_{\tau_\beta} r(\beta)$, but $\bigoplus_i r_{\omega\cdot\beta+i}$ and $r(\beta)$ are $\leqslant_{\tau_\beta}$-incomparable, and $\bigoplus_i r_{\omega\cdot\beta+i}$ and $r(\beta+1)$ are $\leqslant_{\tau_\beta}$-incomparable; and for any limit ordinal $\lambda < \omega \cdot \gamma$ and any $\alpha < \lambda$, $r_\alpha \leqslant_{\tau_\lambda} r(\lambda)$, $\bigoplus_{\alpha<\lambda} r_\alpha$ and $r(\lambda)$ are $\leqslant_{\tau_\lambda}$-incomparable. Moreover for all $\beta < \gamma$, $r(\beta) \leqslant_T \left(\bigoplus_i r_{\omega\cdot\beta+i}\right)'$ and for any limit ordinal $\lambda < \omega \cdot \gamma$, $r(\lambda) \leqslant_T \left(\bigoplus_{\alpha<\lambda} r_\alpha\right)'$.*

$\bigoplus_{\alpha<\lambda} r_\alpha$ is defined as follows: first note that if λ is a recursive ordinal, we can use a standard bi-recursive encoding of ω^2 in ω. Otherwise, we form the

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[12] This hypothesis carries the ideas of progressivity of the sequence and indiscernibility by first order properties: in the list of admissibles $\langle \tau_\beta : \beta < \lambda_\infty \rangle$, the sequence of indices that correspond to the considered sequence does not contain too much information in itself.

equivalence classes of those ITTMs (represented by their indices in a standard enumeration) that halt in a given ordinal time δ. In each class we have ω integers. We encode $r_{\omega+\delta}$ at the abscisse given by the first ITTM that halts in time δ. Note that this encoding covers all clockable ordinals. When we enter gaps, we proceed in the same way but on the second ITTM that halts in time η, for the η-th non-clockable ordinal. Then we continue the construction for ω imbrications of gaps of gaps of gaps, *etc.* But we could have more than ω such imbrications. Thus, we use once again ITTM indices inside the sequence of ITTMs that halt at a given time to find the proper abscisse. Please note that the lengths and the ranks of the clockable gaps are co-final in λ_∞. Thus, for any $\lambda < \lambda_\infty$, we get a coding that is recursive for ITTMs bounded by time *supremum* of length of the possible gap and starting point. This construction of a transfinite join is compatible with Jensen's hypothesis on relative admissibility of the admissible ordinals of the considered sequence.

Our version of Jensen's theorem provides an explicit construction: we keep Jensen's hypothesis, but add λ_∞ as an upper bound on the *length* of the sequence and also require that the admissibles of the sequence are below ω_1^L.

Theorem 6. *Let $\gamma < \lambda_\infty$. If $\langle \alpha_\beta : \beta < \gamma \rangle$ is a sequence of constructibly countable admissibles such that for every $\delta < \gamma$, α_δ is admissible relative*[10] *to $\{\alpha_\beta : \beta < \delta\}$, then there is a real r such that α_β is the β-th r-admissible ordinal.*

Proof sketch. To start with, we would like to construct r such that $\alpha_0 = \omega_1^{\mathrm{CK},r}$. And of course, we would like to add much more things in r in order to get that α_1 becomes the next admissible, and so on. The situation is analogous to that of Sacks' theorem: if α_0 is a successor admissible, then we consider r_0 a code of its predecessor that we write in r (if we are in a definability gap, we proceed as we have done in the proof of Theorem 4). If α_0 is recursively inaccessible, then we encode in a special way an ω-sequence of codes $\langle r_{\beta_i} : i < \omega \rangle$ for admissibles $\langle \beta_i : i < \omega \rangle$ cofinal in α_0. We use Lemma 3 to get the *ad hoc* real r_0.

Now we would like to add some new information to encode also the real r_1 that makes α_1 the second admissible. But while doing this, we should not make α_0 computable. Thus we can make a special version of Lemma 3 where the reduction used is not the Turing reducibility, but the ITTM reducibility with time bounded by the first admissible (which is exactly the hyperarithmetic reducibility, $\leqslant_h$) and while doing this, we make sure that we still have $\alpha_0 \not\leqslant_T r_0 \oplus r_1$.

This construction gives the induction step when β is a successor ordinal. When β is limit, let us first assume that β is not strictly inside an ITTM-gap; thanks to the relative admissibility hypothesis of the sequence, α_β is recursively inaccessible if and only if τ_β is. We propose a single construction for the two cases concerning α_β, being recursively inaccessible or a successor admissible. We proceed as above, with the help of a more complex lemma. We consider ITTM-reducibilities bounded by the β-th admissible, and apply this to the ω-sequence extracted from admissibles below α_β if inaccessible, or to the real coding its predecessor if α_β is a successor admissible. The hypotheses of our improved lemma (Lemma 4) that provides r_β are verified thanks to the relative admissibility hypothesis and the constructibly countability of the considered ordinals.

The problem now is when the β-th admissible is inside an ITTM clockable gap. Indeed, if τ_η starts a clockable gap which contains τ_β, then $\leqslant_{\tau_\eta}$ is exactly the same reducibility than $\leqslant_{\tau_\beta}$. Note that in this case, α_η starts a clockable gap which contains α_β. We use a slightly modified version of Lemma 4 that uses α_η as an oracle from rank η on. If α_β is in a gap of a gap of a gap ... (of rank δ), then we modify analogously Lemma 4 adding as an oracle the starting points of the gaps of rank $< \delta$. □

We propose an application of our theorem to a bounded version of Solovay's problem, namely finding a real relatively to which the admissibles are the recursively inaccessibles. The solution to Solovay's problem by Sy D. Friedman (*cf.* [15]) proves that the sequence of recursively inaccessible ordinals below ω_1^L verify the relative admissibility hypothesis of Jensen, thus our construction works as is.

Theorem 7 (Solovay's problem below λ_∞). *If $\langle \iota_\beta : \beta < \lambda_\infty \rangle$ is the sequence of recursively inaccessible ordinals below λ_∞, then there is a real r such that ι_β is the β-th r-admissible ordinal.*

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