A Note on the Ordinal Analysis of $\mathbf{RCA_0} + \mathrm{WO}(\sigma)$

Lorenzo Carlucci[1(✉)], Leonardo Mainardi[1], and Michael Rathjen[2]

[1] Department of Computer Science, University of Rome I, Rome, Italy
carlucci@di.uniroma1.it, l.mainardi.90@gmail.com
[2] School of Mathematics, University of Leeds, Leeds LS2 9JT, UK
M.Rathjen@leeds.ac.uk

Abstract. We fill an apparent gap in the literature by giving a short and self-contained proof that the ordinal of the theory $\mathbf{RCA_0} + \mathrm{WO}(\sigma)$ is σ^ω, for any ordinal σ satisfying $\omega \cdot \sigma = \sigma$ (e.g., ω^ω, ω^{ω^ω}, ε_0). Theories of the form $\mathbf{RCA_0} + \mathrm{WO}(\sigma)$ are of interest in Proof Theory and Reverse Mathematics because of their connections to a number of well-investigated combinatorial principles related to various subsystems of arithmetic.

1 Introduction

Well-ordering statements are commonly used in Proof Theory and Reverse Mathematics as measures of strength of a theorem or a theory. For example a number of interesting theorems is known to be equivalent to $\mathrm{WO}(\omega^\omega)$ or $\mathrm{WO}(\omega^{\omega^\omega})$ over the theory $\mathbf{RCA_0}$ (see, e.g., [3,7,9,15]). It is then natural to ask what is the proof-theoretic ordinal of the theories $\mathbf{RCA_0} + \mathrm{WO}(\omega^\omega)$, $\mathbf{RCA_0} + \mathrm{WO}(\omega^{\omega^\omega})$ and, in general, $\mathbf{RCA_0} + \mathrm{WO}(\alpha)$. While it is well-known that the ordinal of $\mathbf{RCA_0}$ is ω^ω, the answer for the other theories is not so immediate and occasionally some confusion arises.[1] Even the standard argument for showing that ω^ω is an upper bound on the proof-theoretic ordinal of $\mathbf{RCA_0}$ is somewhat indirect in that it hinges on the characterization of the provably recursive functions of $\mathbf{RCA_0}$ rather than only on the computation of its proof-theoretical ordinal. A proper direct treatment approach to determining the proof-theoretic ordinal of theories of the form $\mathbf{RCA_0}+\mathrm{WO}(\alpha)$ seems to be missing from the literature. The closest match is Sommer's [17] model-theoretical treatment of first-order theories with transfinite induction restricted to various formula-classes and ordinals strictly below ε_0.

In this paper we show that, if σ is an ordinal satisfying $\omega \cdot \sigma = \sigma$, the proof-theoretic ordinal of the theory $\mathbf{RCA_0} + \mathrm{WO}(\sigma)$ is σ^ω. Examples of relevant σs

---

[1] For example, in proving that a Π_1^1-version of Ramsey's Theorem called the Adjacent Ramsey Theorem is equivalent to $\mathrm{WO}(\varepsilon_0)$ over $\mathbf{RCA_0}$, [4] Lemma 2.2 makes use of the false equivalence, over $\mathbf{RCA_0}$, between $\mathrm{WO}(\varepsilon_0)$ and the Π_1^1-soundness of $\mathbf{ACA_0}$. The presentation in the later [5] avoids this pitfall but establishes a slightly different result.

are ω^ω, ω^{ω^ω} etc. and ε_0. This should be contrasted with the fact that the ordinal of $\mathbf{ACA}_0 + \mathrm{WO}(\varepsilon_0)$ is the much larger ε_1, as can be gleaned from the proof-theoretical analysis of transfinite induction over Peano Arithmetic (the original proof seems to be in [8]).

Essentially, we show that the first-order part of $\mathbf{RCA}_0 + \mathrm{WO}(\sigma)$ is the theory $\mathbf{I}\Sigma_1$ plus the scheme of transfinite induction up to σ restricted to Π_1 formulas, which we denote by $\mathrm{TI}(\sigma, \Pi_1)$. We give an ordinal analysis of the latter theory augmented by a generic unary predicate symbol U and then show that the theories prove the same Π_1^1-statements, where a Π_1^1-sentence $\forall X F(X)$ with $F(X)$ being arithmetic is identified with $F(\mathsf{U})$ in the first-order context of $\mathrm{TI}(\sigma, \Pi_1)$.

For the remainder of the paper, we fix an ordinal σ such that $\omega \cdot \sigma = \sigma$. The ordinal σ is assumed to be represented in a natural ordinal representation system. We denote by $\lhd$ the primitive recursive ordering on the ordinals smaller that σ to distinguish it from the usual ordering on the naturals.

2   Ordinal Analysis of $\mathbf{I}\Sigma_1 + \mathbf{TI}(\sigma, \Pi_1)$

The language of $\mathbf{T}_\sigma := \mathbf{I}\Sigma_1 + \mathrm{TI}(\sigma, \Pi_1)$ is the language of Primitive Recursive Arithmetic, $\mathbf{PRA}$, augmented by a unary predicate symbol U. Especially we assume that there is a binary surjective coding function $\langle \cdot, \cdot \rangle$ with inverses $(\cdot)_0$, $(\cdot)_1$. The order relation for the ordering on σ will be denoted by the same symbol $\lhd$ used to denote the corresponding primitive recursive relation. Bounded quantifiers $\forall x \leq t$ and $\exists x \leq t$ will be treated as quantifiers in their own right. Formulas containing only bounded quantifiers are called Δ_0-*formulas*. For our proof-theoretic purposes, $\mathbf{T}_\sigma$ will be formalized in a one-sided sequent calculus, using negation normal forms following [14] (this is also known as the Tait-calculus [18]). $\mathbf{T}_\sigma$ has the usual axioms pertaining to primitive recursive functions and predicates. A noteworthy feature is that transfinite induction on σ for Π_1-formulas is expressed via the rule

$$\frac{\Theta, \; \exists z \, ((z)_0 \lhd a \, \wedge \, \neg F((z)_1, (z)_0)), \; \forall x F(x, a)}{\Theta, \; F(t, s)} \tag{1}$$

where a is an eigenvariable, $F(x, a)$ is Δ_0, t, s are arbitrary terms, and Θ is an arbitrary finite set of formulas.

Observe that we do not need Σ_1-induction as an extra induction principle as it follows from $\mathrm{TI}(\sigma, \Pi_1)$, since $\mathbf{I}\Pi_1$ entails $\mathbf{I}\Sigma_1$.

In order to perform partial cut eliminations, we define the degree, $|A|$, of a formula A as follows:

– $|A| = |\neg A| = 0$ if A is Δ_0.

If A is not Δ_0 and of one of the forms below, then:

– $|A_0 \wedge A_1| = |A_0 \vee A_1| = \max(|A_0|, |A_1|) + 1$;
– $|\forall x F(x)| = |\exists x F(x)| = |F(0)| + 1$;
– $|\forall x \leq t \, F(x)| = |\exists x \leq t \, F(x)| = |F(0)| + 2$.

As the rule (1) introduces a Δ_0-formula and the main formulas of axioms are Δ_0 as well, we can easily eliminate cuts of degree greater than 0. We use the notation $\mathbf{T}_\sigma \vdash^{\underline{m}}_{k} \Gamma$ to convey that Γ is deducible in $\mathbf{T}_\sigma$ by a deduction of length at most m such that all cuts occurring in this deduction are with cut formulas of a degree $< k$. Thus $\mathbf{T}_\sigma \vdash^{\underline{m}}_{1} \Gamma$ means that there is deduction in which all cut formulas (if any) are Δ_0-formulas.

Theorem 1. $\mathbf{T}_\sigma \vdash^{\underline{n}}_{r+1} \Gamma \ \Rightarrow \ \exists m \, \mathbf{T}_\sigma \vdash^{\underline{m}}_{1} \Gamma$.

Proof. By the usual cut elimination method of Gentzen's Hauptsatz.

2.1   Embedding $\mathbf{T}_\sigma$ in an Infinitary System

Next we embed $\mathbf{T}_\sigma$ into an infinitary system, called $\mathbf{PA}_\omega$, with ω-rule (basically the same as the system Z_∞ in [14]; a definition of $\mathbf{PA}_\omega$ in a two-sided Gentzen calculus can be found in [10]). The formulas of $\mathbf{PA}_\omega$ are the closed formulas of $\mathbf{T}_\sigma$, i.e. formulas without free variables. We shall assign a rank, $|A|_{\Delta_0}$ to a formula A of $\mathbf{PA}_\omega$ as follows:

(i) $|A|_{\Delta_0} = 0$ if A is atomic or a negated atom.
(ii) $|A_0 \wedge A_1|_{\Delta_0} = |A_0 \vee A_1|_{\Delta_0} = \max(|A_0|_{\Delta_0}, |A_1|_{\Delta_0}) + 1$.
(iii) $|\exists x \leq t \, F(x)|_{\Delta_0} = |\forall x \leq t \, F(x)|_{\Delta_0} = |F(0)|_{\Delta_0} + 1$.
(iv) $|\exists x \, F(x)|_{\Delta_0} = |\forall x \, F(x)|_{\Delta_0} = \max(\omega, |F(0)|_{\Delta_0} + 1)$.

Note that $|A|_{\Delta_0} < \omega$ exactly when A is Δ_0, and $|\exists x \, F(x)|_{\Delta_0} = |\forall x \, F(x)|_{\Delta_0} = \omega$ when $F(0)$ is Δ_0.

Definition 1. For a natural number n we use $\bar{n}$ to denote the nth numeral, that is the term obtained from the term $\bar{0}$ for zero by adding the successor function symbol n-times in front of it. The terms of $\mathbf{PA}_\omega$ are closed and thus can be evaluated to a number. For a term t let $t^\mathbb{N}$ be the number n such that t evaluates to n (in the following we occasionally refer to $t^\mathbb{N}$ by t).

The axioms of $\mathbf{PA}_\omega$ are sequents of two kinds. Let Γ be a finite set of formulas of $\mathbf{PA}_\omega$.

(i) Let $R(t_1, \ldots, t_r)$ be an atomic formula, where R is a relation symbol for a primitive recursive relation $R^\mathbb{N}$. If $R^\mathbb{N}(t_1^\mathbb{N}, \ldots, t_r^\mathbb{N})$ is true, then

$$\Gamma, R(t_1, \ldots, t_r)$$

is an axiom. If $R^\mathbb{N}(t_1^\mathbb{N}, \ldots, t_r^\mathbb{N})$ is false, then

$$\Gamma, \neg R(t_1, \ldots, t_r)$$

is an axiom.
(ii) If $s^\mathbb{N} = t^\mathbb{N}$ holds for terms s and t, then

$$\Gamma, \mathsf{U}(s), \neg \mathsf{U}(t)$$

is an axiom.

The ω-rule is the following rule: If $\Gamma, F(\bar{n})$ is deducible for all n, then $\Gamma, \forall x\, F(x)$ is the conclusion.

Similarly to derivations in $\mathbf{T}_\sigma$, we will use the notation $\mathbf{PA}_\omega \vdash^{\alpha}_{\beta} \Gamma$ to convey that Γ is deducible in $\mathbf{PA}_\omega$ by a deduction of height at most α such that all cuts occurring in this deduction are with cut formulas of $|\cdot|_{\Delta_0}$-rank $< \beta$.

Lemma 1 (Reduction Lemma). *If* $|B|_{\Delta_0} = \omega$, $\mathbf{PA}_\omega \vdash^{\alpha}_{\omega} \Gamma, B$ *and* $\mathbf{PA}_\omega \vdash^{\beta}_{\omega} \Gamma, \neg B$, *then*

$$\mathbf{PA}_\omega \vdash^{\alpha\#\beta}_{\omega} \Gamma$$

where $\alpha\#\beta$ *denotes the* natural *or* Hessenberg sum *of* α *and* β.

Proof. Standard.

Theorem 2 (Embedding Theorem). *If* $\mathbf{T}_\sigma \vdash^{m}_{1} \Gamma$, *then* $\mathbf{PA}_\omega \vdash^{\sigma^m}_{\omega} \Gamma^*$, *where* Γ^* *is the result of assigning closed terms to all free variables in* Γ *(the same term to the same variable).*

Proof. We proceed by induction on m. We only need to pay attention to the case where the last inference is an instance of the rule (1). So let $\Gamma = \Theta, F(t,s)$ and assume $\mathbf{T}_\sigma \vdash^{m_0}_{1} \Lambda$ with $m_0 < m$ and $\Lambda = \Theta, \exists z\ ((z)_0 \lhd a \wedge \neg F((z)_1,(z)_0)), \forall x\, F(x,a)$.

Let $*$ be an assignment. Inductively we have for all closed terms q that

$$\mathbf{PA}_\omega \vdash^{\sigma^{m_0}}_{\omega} \Theta^*, \exists z\,((z)_0 \lhd q \wedge \neg F^*((z)_1,(z)_0)), \forall x\, F^*(x,q). \tag{2}$$

We use transfinite induction on α for α in the field of $\lhd$ to show that:

$$\mathbf{PA}_\omega \vdash^{\sigma^{m_0}\cdot\omega\cdot(\alpha+1)}_{\omega} \Theta^*, \forall x\, F^*(x,\bar{\alpha}) \tag{3}$$

By the induction hypothesis, we have:

$$\mathbf{PA}_\omega \vdash^{\sigma^{m_0}\cdot(\omega\cdot(\eta+1))}_{\omega} \Theta^*, F^*(s',\bar{\eta})$$

for every $\eta \lhd \alpha$ and arbitrary closed term s', yielding

$$\mathbf{PA}_\omega \vdash^{\sigma^{m_0}\cdot(\omega\cdot\alpha)+1}_{\omega} \Theta^*, \bar{\eta} \lhd \bar{\alpha} \to F^*(s',\bar{\eta})$$

via an inference ($\vee$). If r is a closed term such that $r^{\mathbb{N}}$ is different from all η preceding α, then $\neg r \lhd \bar{\alpha}$ is an axiom, and thus, via an inference ($\vee$), we arrive at $\mathbf{PA}_\omega \vdash^{1}_{0} \Theta^*, r \lhd \bar{\alpha} \to F^*(s',r)$. Thus from the above we conclude that

$$\mathbf{PA}_\omega \vdash^{\sigma^{m_0}\cdot(\omega\cdot\alpha)+1}_{\omega} \Theta^*, (\bar{k})_0 \lhd \bar{\alpha} \to F^*((\bar{k})_1,(\bar{k})_0)$$

holds for all k, so that, via an application of the ω-rule, we get:

$$\mathbf{PA}_\omega \vdash^{\sigma^{m_0}\cdot(\omega\cdot\alpha)+2}_{\omega} \Theta^*, \forall z\,((z)_0 \lhd \bar{\alpha} \to F^*((z)_1,(z)_0)). \tag{4}$$

Applying the Reduction Lemma 1 to (2) and (4) yields:

$$\mathbf{PA}_\omega \left|\frac{\sigma^{m_0}\#(\sigma^{m_0}\cdot(\omega\cdot\alpha))+2}{\omega}\right. \Theta^*, \forall x\, F^*(x, \bar\alpha). \tag{5}$$

From (5) we finally get:

$$\mathbf{PA}_\omega \left|\frac{\sigma^{m_0}\cdot(\omega\cdot(\alpha+1))}{\omega}\right. \Theta^*, \forall x\, F^*(x, \bar\alpha)\,,$$

confirming (3).

If a term q has the property that $q^{\mathbb{N}}$ is not in the field of $\lhd$ then one can directly infer from (2) that

$$\mathbf{PA}_\omega \left|\frac{\sigma^{m_0}}{\omega}\right. \Theta^*, \forall x\, F^*(x, q). \tag{6}$$

The reason for this is that if the formula $\exists z\,((z)_0 \lhd q \wedge \neg F^*((z)_1, (z)_0))$ figures as the main formula of an inference in this derivation its minor formula is of the form $(p)_0 \lhd q \wedge \neg F^*((p)_1, (p)_0)$. The latter formula conjunctively contains a false atomic formula. Such a formula can always be erased from the derivation. Formally, of course, this has to be proved by a separate induction on the ordinal of the derivation.

(3) and (6) now yield

$$\mathbf{PA}_\omega \left|\frac{\sigma^m}{\omega}\right. \Theta^*, F^*(t, s)$$

for all closed terms t and s, since $\omega\cdot(\alpha+1) \lhd \sigma$ on account of $\omega\cdot\sigma = \sigma$.

2.2   Eliminating Cuts with Δ_0-Formulas

The next step is to eliminate cuts with Δ_0-formulas that are not atomic.

Lemma 2. *Let $0 < n < \omega$ and suppose $\mathbf{PA}_\omega \left|\frac{\alpha}{n+1}\right. \Gamma$. Then $\mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha}{n}\right. \Gamma$.*

Proof. We proceed by induction on α. The crucial case is when the last inference was a cut of rank n with cut formulas $A, \neg A$. Note that A is not an atomic formula. We then have $\mathbf{PA}_\omega \left|\frac{\alpha_0}{n+1}\right. \Gamma, A$ and $\mathbf{PA}_\omega \left|\frac{\alpha_0}{n+1}\right. \Gamma, \neg A$ for some $\alpha_0 < \alpha$. The induction hypotheses furnishes us with

$$\mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha_0}{n}\right. \Gamma, A \quad\text{and}\quad \mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha_0}{n}\right. \Gamma, \neg A\,. \tag{7}$$

Let A be of the form $\exists x \le t\, F(x)$. Then $\neg A$ is the formula $\forall x \le t\, \neg F(x)$. From (7) we obtain

$$\mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha_0}{n}\right. \Gamma, F(\bar 0), \dots, F(\bar p) \quad\text{and}\quad \mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha_0}{n}\right. \Gamma, \neg F(\bar k) \tag{8}$$

for all $k \le p$, where p is the numerical value of t. As the formulas $F(\bar k), \neg F(\bar k)$ have rank $< n$, we can employ $(p+1)$-many cuts to (8) to arrive at $\mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha_0+p+1}{n}\right. \Gamma$. Thus we have $\mathbf{PA}_\omega \left|\frac{\omega\cdot\alpha}{n}\right. \Gamma$ as $\omega\cdot\alpha_0 + p + 1 < \omega\cdot\alpha$. A similar argument works when A is of either form $A_0 \wedge A_1$ or $A_0 \vee A_1$.

Corollary 1. *If* $\mathbf{PA}_\omega \vdash^{\alpha}_{\omega} \Gamma$ *then* $\mathbf{PA}_\omega \vdash^{\omega^\omega \cdot \alpha}_{1} \Gamma$.

Proof. We use induction on α. The only interesting case arises when the last inference is a cut with a formula A of rank $k > 0$. Then we have $\mathbf{PA}_\omega \vdash^{\alpha_0}_{\omega} \Gamma, A$ and $\mathbf{PA}_\omega \vdash^{\alpha_0}_{\omega} \Gamma, \neg A$ for some $\alpha_0 < \alpha$. The induction hypothesis yields $\mathbf{PA}_\omega \vdash^{\omega^\omega \cdot \alpha_0}_{1} \Gamma, A$ and $\mathbf{PA}_\omega \vdash^{\omega^\omega \cdot \alpha_0}_{1} \Gamma, \neg A$. Hence $\mathbf{PA}_\omega \vdash^{\omega^\omega \cdot \alpha_0 + 1}_{k+1} \Gamma$. Applying Lemma 2 k times we arrive at $\mathbf{PA}_\omega \vdash^{\omega^k \cdot (\omega^\omega \cdot \alpha_0 + 1)}_{1} \Gamma$. As $\omega^k \cdot (\omega^\omega \cdot \alpha_0 + 1) = \omega^\omega \cdot \alpha_0 + \omega^k \leq \omega^\omega \cdot \alpha$ we also have $\mathbf{PA}_\omega \vdash^{\omega^\omega \cdot \alpha}_{1} \Gamma$ as desired.

Note that $\sigma \geq \omega^\omega$ since $\omega \cdot \sigma = \sigma$.

Corollary 2. *Let* $m > 0$. *If* $\mathbf{PA}_\omega \vdash^{\sigma^m}_{\omega} \Gamma$ *then* $\mathbf{PA}_\omega \vdash^{\sigma^{m+1}}_{1} \Gamma$.

Proof. Corollary 1 yields $\mathbf{PA}_\omega \vdash^{\omega^\omega \cdot \sigma^m}_{1} \Gamma$. Thus the desired conclusion follows as $\omega^\omega \cdot \sigma^m \leq \sigma \cdot \sigma^m = \sigma^{m+1}$.

3   Upper Bounds for the Provable Well-Orderings of $\mathbf{T}_\sigma$

The results of the previous section can be utilized to determine the ordinal rank of provable well-orderings of $\mathbf{T}_\sigma$. Let $\prec$ be a primitive recursive ordering. $\prec$ is said to be a *provable well-ordering* of $\mathbf{T}_\sigma$ if $\mathbf{T}_\sigma$ proves that $\prec$ is a total linear ordering and

$$\mathbf{T}_\sigma \vdash \mathrm{WO}(\prec)$$

where $\mathrm{WO}(\prec)$ stands for the formula

$$\forall v [\forall u \prec v \mathsf{U}(u) \rightarrow \mathsf{U}(v)] \rightarrow \forall v \mathsf{U}(v).$$

Assuming $\mathbf{T}_\sigma \vdash \mathrm{WO}(\prec)$, as a consequence of Theorems 1, 2 and Corollary 2 we then have

$$\mathbf{PA}_\omega \vdash^{\sigma^m}_{1} \forall v [\forall u \prec v \mathsf{U}(u) \rightarrow \mathsf{U}(v)] \rightarrow \forall v \mathsf{U}(v) \tag{9}$$

for some $m > 0$. There are several ways of obtaining an upper bound for the order-type of $\prec$ in terms of the length of a cut-free deduction of $\mathrm{WO}(\prec)$ (see e.g. [13, Theorem 23.1], [19, Theorem 3.6], [6, Theorem 2.27]) which ultimately go back to Gentzen. Schütte [13, Theorem 23.1] obtains particularly sharp bounds. He shows that the length α of a cut-free derivation of transfinite induction along an ordering $\prec$ provides an upper bound for the ordinal rank of $\prec$ if $\omega \cdot \alpha = \alpha$. For our purpose, however, we need to extract bounds from deductions that still have cuts with formulas $\mathsf{U}(s), \neg \mathsf{U}(s)$.[2] We could first eliminate these remaining cuts, however, we would get bounds of the form 2^{σ^m}, and these are too high for our purpose of showing that σ^ω is the proof-theoretic ordinal of $\mathbf{T}_\sigma$. To overcome

---

[2] They may also contain cuts with formulas $R(t_1, \ldots, t_k), \neg R(t_1, \ldots, t_k)$, where R is a symbol for a primitive recursive predicate. But these are entirely harmless.

this obstacle we shall draw on a technique that the third author has used for many years. To this end we extend $\mathbf{PA}_\omega$ by yet another infinitary rule $\text{Prog}_\prec$ due to Schütte [11, p. 384] called *Progressionsregel* ($\text{Prog}_\prec$ was also used in [12, p. 214] and in [14]):

$$\frac{\Gamma, \mathsf{U}(\bar{m}) \;\; \text{for all } m \prec n}{\Gamma, \mathsf{U}(s)} \tag{10}$$

whenever s is a closed term with value n.

Let $\text{PROG}_\prec$ be an abbreviation for $\forall v[\forall u \prec v\, \mathsf{U}(u) \rightarrow \mathsf{U}(v)]$. The rule $\text{Prog}_\prec$ has the effect of making $\text{PROG}_\prec$ provable. We shall refer by $\mathbf{PA}_\infty^*$ to the extension of $\mathbf{PA}_\omega$ by the rule $\text{Prog}_\prec$.

Lemma 3

$$\mathbf{PA}_\infty^* \;\vdash^{\alpha}_1\; \neg\text{PROG}_\prec, \Gamma \;\Rightarrow\; \mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha}_1\; \Gamma. \tag{11}$$

Proof. We proceed by induction on α. If $\neg\text{PROG}_\prec$ was not the main formula of the last inference then the desired result follows immediately by applying the inductive assumption to its premises and subsequently reapplying the same inference. Thus suppose that $\neg\text{PROG}_\prec$ was the main formula of the last inference. Then

$$\mathbf{PA}_\infty^* \;\vdash^{\alpha_0}_1\; \neg\text{PROG}_\prec, \forall u \prec s\, \mathsf{U}(u) \wedge \neg\mathsf{U}(s), \Gamma \tag{12}$$

for some $\alpha_0 \lhd \alpha$. The induction hypothesis yields

$$\mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha_0}_1\; \forall u \prec s\, \mathsf{U}(u) \wedge \neg\mathsf{U}(s), \Gamma. \tag{13}$$

for some s. Using inversion for $(\wedge)$, $(\forall)$ and $(\vee)$ we arrive at

$$\mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha_0}_1\; \Gamma, \neg\bar{n} \prec s, \mathsf{U}(\bar{n}) \tag{14}$$

for all n, and

$$\mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha_0}_1\; \neg\mathsf{U}(s), \Gamma. \tag{15}$$

Since $\mathbf{PA}_\infty^* \;\vdash^0_0\; \Gamma, \bar{n} \prec s$ holds for all n with $n \prec s^{\mathbb{N}}$, we can apply cuts and the rule $\text{Prog}_\prec$ to (14) to arrive at

$$\mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha_0+2}_1\; \Gamma, \mathsf{U}(s). \tag{16}$$

Applying Cut to (16) and (15) yields

$$\mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha_0+3}_1\; \Gamma \tag{17}$$

and hence

$$\mathbf{PA}_\infty^* \;\vdash^{3\cdot\alpha}_1\; \Gamma. \tag{18}$$

Corollary 3

$$\mathbf{PA}^*_\infty \left|\frac{\sigma^m}{1}\right. \mathsf{U}(\bar{n})$$

for all n.

Proof. Follows from (9) and Lemma 3. Note that $m > 0$.

For a closed numerical term s we denote by $|s|_\prec$ the ordinal $\{|\bar{n}|_\prec \mid \bar{n} \prec s \text{ is true}\}$.

Proposition 1. *Assume that the sequent $\neg\mathsf{U}(t_1), \ldots, \neg\mathsf{U}(t_r), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q)$ is not an axiom and $s_1 \preceq \ldots \preceq s_q$ holds. Then*

$$\mathbf{PA}^*_\infty \left|\frac{\alpha}{1}\right. \neg\mathsf{U}(t_1), \ldots, \neg\mathsf{U}(t_r), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q)$$

implies

$$|s_1|_\prec < \omega \cdot \alpha. \tag{19}$$

Proof. Let $\neg\mathsf{U}(\boldsymbol{t})$ be an abbreviation for $\neg\mathsf{U}(t_1), \ldots, \neg\mathsf{U}(t_r)$. In the above we allow $r = 0$ in which case $\neg\mathsf{U}(\boldsymbol{t})$ is the empty sequent.

We proceed by induction on α. As the sequent is not an axiom it must have been inferred. The only two possibilities are applications of $\mathrm{Prog}_\prec$ or cuts with atomic formulas.

Case 1: The last inference was $\mathrm{Prog}_\prec$. Then there is a term s_j and $\alpha_0 \lhd \alpha$ such that $\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg\mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q), \mathsf{U}(\bar{n})$ for all $\bar{n} \prec s_j$. As $s_1 \preceq s_j$ this also holds for all $\bar{n} \prec s_1$. The induction hypothesis yields that

$$|\bar{n}|_\prec < \omega \cdot \alpha_0$$

holds for those $\bar{n} \prec s_1$ for which the sequent is not an axiom. Since by Definition 1 (ii)

$$\neg\mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q), \mathsf{U}(\bar{n}) \tag{20}$$

is an axiom only if $\bar{n}$ has the same value as some $t_1, \ldots, t_r$, then there are only finitely many n for which (20) is an axiom. Thus $|s_1|_\prec < \omega \cdot \alpha_0 + \omega$, whence $|s_1|_\prec < \omega \cdot \alpha$.

Case 2: The last inference was a cut with cut formulas $\mathsf{U}(p), \neg\mathsf{U}(p)$, i.e., we have

$$\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg\mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q), \mathsf{U}(p) \tag{21}$$

$$\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg\mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q), \neg\mathsf{U}(p) \tag{22}$$

for some $\alpha_0 < \alpha$ and closed term p. If the sequent from (22) is not an axiom, the induction hypothesis applied to that derivation yields $|s_1|_\prec < \omega \cdot \alpha_0$. If it is an axiom, there is an s_j such that p and s_j evaluate to the same numeral, and hence $s_1 \preceq p$. So in this case the induction hypothesis applied to (21) yields $|s_1|_\prec < \omega \cdot \alpha_0$.

Case 3: The last inference was a cut with cut formulas $R(u_1, \ldots, u_p), \neg R(u_1, \ldots, u_p)$ for a symbol R for a primitive recursive relation. Then we have

$$\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg \mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q), R(u_1, \ldots, u_p) \qquad (23)$$

$$\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg \mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q), \neg R(u_1, \ldots, u_p) \qquad (24)$$

for some $\alpha_0 < \alpha$. If $R(u_1, \ldots, u_p)$ is true it follows from (24) that we also have

$$\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg \mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q)$$

and hence the induction hypothesis yields $|s_1|_\prec < \omega \cdot \alpha_0$.

Likewise, if $R(u_1, \ldots, u_p)$ is false it follows from (23) that we also have

$$\mathbf{PA}^*_\infty \left|\frac{\alpha_0}{1}\right. \neg \mathsf{U}(\boldsymbol{t}), \mathsf{U}(s_1), \ldots, \mathsf{U}(s_q)$$

and hence the induction hypothesis yields $|s_1|_\prec < \omega \cdot \alpha_0$.

Corollary 4

(i)

$$\mathbf{PA}^*_\infty \left|\frac{\alpha}{1}\right. \mathsf{U}(s) \implies |s|_\prec < \omega \cdot \alpha.$$

(ii)

$$\mathbf{PA}_\omega \left|\frac{\beta}{1}\right. \mathrm{WO}(\prec) \implies |\prec| \le \omega \cdot 3 \cdot \beta$$

where $|\prec|$ stands for the ordinal rank of $\prec$.

Proof. (i) is an immediate consequence of Proposition 1.
(ii) follows from (i) and Lemma 3.

In sum, it follows that the ordinal rank of $\prec$ is not bigger than σ^m, and hence σ^ω is an upper bound for the proof-theoretic ordinal of $\mathbf{T}_\sigma$.

Proposition 1 can also be shown via techniques in A. Beckmann's dissertation, notably his [1, 5.2.5 Boundedness Theorem] that also features in [2].

Turning to lower bounds, one can easily show, using external induction on n, that $\mathbf{T}_\sigma \vdash \mathrm{WO}(\sigma^n)$. This is a folklore result; details can be found in [17, Lemma 4.3]. As a consequence of the results gathered so far we have

Theorem 3. *The proof-theoretic ordinal of $\mathbf{I}\Sigma_1 + \mathrm{TI}(\sigma, \Pi_1)$ is σ^ω.*

It remains to transfer this result to our target theory $\mathbf{RCA}_0 + \mathrm{WO}(\sigma)$.

4 Π_1^1-Conservativity

We here prove that $\mathbf{RCA}_0 + \text{WO}(\sigma)$ is Π_1^1-conservative over $\mathbf{T}_\sigma$. More precisely, a Π_1^1-sentence $\forall X F(X)$ (with $F(X)$ being arithmetic) is identified with $F(\mathsf{U})$ in the first-order context of $\text{TI}(\sigma, \Pi_1)$. This is enough to apply our results from the previous section to conclude that the ordinal of $\mathbf{RCA}_0 + \text{WO}(\sigma)$ is σ^ω.

To prove the conservativity result, we proceed as follows. We start by showing that any model of $\mathbf{I}\Sigma_1 + \text{TI}(\sigma, \Pi_1)$ can be extended to a model of $\mathbf{RCA}_0 + \text{WO}(\sigma)$. The argument is essentially contained in Simpson [16], IX.1. By writing that $\mathbf{M}_1$ is an ω-submodel of $\mathbf{M}_2$ we mean that $\mathbf{M}_1 = (M_1, \mathcal{S}_1)$ and $\mathbf{M}_2 = (M_1, \mathcal{S}_2)$ where $\mathcal{S}_1 \subseteq \mathcal{S}_2$. In other words, the two models share the same first-order part M_1.

Lemma 4. *Let $\mathbf{M}$ be an L_2-structure which satisfies the axioms of $\mathbf{I}\Sigma_1 + \text{TI}(\sigma, \Pi_1)$. Then $\mathbf{M}$ is an ω-submodel of some model of $\mathbf{RCA}_0 + \text{WO}(\sigma)$.*

Proof. We first show that $\mathbf{M}$ can be extended to a model $\mathbf{M}'$ satisfying $\mathbf{RCA}_0$ and $\text{TI}(\sigma, \Delta_0^0)$ with the same first-order domain as $\mathbf{M}$. Then we show that such an extension also satisfies $\text{WO}(\sigma)$.

The ω-extension $\mathbf{M}'$ is defined exactly as in Simpson [16] Lemma IX.1.8, i.e., the second-order part is given by the Δ_1^0-definable sets of the base model $\mathbf{M}$. By Lemma IX.1.8 of [16] we have that $\mathbf{M}'$ satisfies $\mathbf{RCA}_0$.

Then, in order to check that $\text{TI}(\sigma, \Delta_0^0)$ is also satisfied, we use the first claim in Simpson's Lemma IX.1.8. Let φ be a Σ_0^0 formula with no free set variables and parameters in M'. Then, there exists a Π_1^0-formula φ_Π with the same free variables and parameters only in M such that φ and φ_Π are equivalent over M'. Thus, $\text{TI}(\sigma, \Pi_1^0)$ in $\mathbf{M}$ implies $\text{TI}(\sigma, \Delta_0^0)$ in $\mathbf{M}'$.

Finally, we show that M' also satisfies $\text{WO}(\sigma)$. If this were not the case, letting S be a set witnessing $\neg\text{WO}(\sigma)$, we would have that $\bar{S}$, i.e. the complement of S, witnesses the failure of an instance of $\text{TI}(\sigma, \Delta_0^0)$. More precisely: suppose that S is non-empty and has no $\lhd$-minimal element. Then $\exists x(x \in S)$. On the other hand, $\bar{S}$ is in M' (since any model of $\mathbf{RCA}_0$ is closed under Turing reducibility hence under complement) and $\forall x(\forall y(y \lhd x \to y \in \bar{S}) \to x \in \bar{S})$. Suppose in fact that for some x, $\forall y(y \lhd x \to y \in \bar{S})$ but $x \in S$. Then all $y \lhd x$ are not in S but x is in S and thus x is the minimum of S, contra our hypothesis. □

Remark 1. The proof of Lemma 4 above shows that if $\mathbf{RCA}_0 + \text{WO}(\sigma)$ proves $\forall X F(X)$ with $F(X)$ arithmetic, then $\mathbf{T}_\sigma$ proves the formula $F'(\mathsf{U})$ obtained from $F(X)$ by replacing expressions of the form '$t \in X$' by '$\mathsf{U}(t)$'.

Then, we proceed by showing that Lemma 4 gives a sufficient condition for Π_1^1-conservativity.

Lemma 5. *If T_1 and T_2 are theories in the language of second order arithmetic and every model of T_1 is an ω-submodel of a model of T_2 then T_2 is Π_1^1-conservative over T_1.*

Proof. If ψ is Π_1^1 and T_1 does not prove ψ, let $\mathbf{M}_1$ be a model of $T_1 + \neg\psi$. Then $\mathbf{M}_1$ is an ω-submodel of a model $\mathbf{M}_2$ of T_2. Then $\mathbf{M}_2$ is a model of $T_2 + \neg\psi$ and thus T_2 does not prove ψ. □

Theorem 4. $\mathbf{RCA}_0 + \mathrm{WO}(\sigma)$ *is* Π_1^1-*conservative over* $\mathbf{I}\Sigma_1 + \mathrm{TI}(\sigma, \Pi_1)$.

Proof. Follows immediately from Lemmas 4 and 5. □

Theorem 5. *The proof-theoretic ordinal of* $\mathbf{RCA}_0 + \mathrm{WO}(\sigma)$ *is* σ^ω.

Proof. The upper bound follows from Theorems 3 and 4. The lower bound follows from the observation that for each n $\mathbf{RCA}_0 + \mathrm{WO}(\sigma) \vdash \mathrm{WO}(\sigma^n)$. The proof, which we omit, is analogous to the proof that $\mathbf{RCA}_0 \vdash \mathrm{WO}(\omega^n)$, for each n. □

Remark 2. The Π_1^1-conservativity of $\mathbf{I}\Sigma_1 + \mathrm{TI}(\sigma, \Pi_1)$ over $\mathbf{RCA}_0 + \mathrm{WO}(\sigma)$ also holds and can be established by standard arguments. In particular one can prove that if $A(\mathsf{U})$ is provable in $\mathbf{T}_\sigma$ then $\forall X A^*(X)$ is provable in $\mathbf{RCA}_0 + \mathrm{WO}(\sigma)$, where '$A^*(X)$' is the result of first replacing '$\mathsf{U}(t)$' by '$t \in X$' and then translating the primitive recursive function and predicate symbols not belonging to the language of $\mathbf{RCA}_0$ as in [16], Definition IX.3.4.

References

1. Beckmann, A.: Separating fragments of bounded arithmetic. Ph.D. thesis, Universität Münster (1996)
2. Beckmann, A., Pohlers, W.: Applications of cut-free infinitary derivations to generalized recursion theory. Ann. Pure Appl. Logic **94**, 7–19 (1995)
3. Carlucci, L., Dehornoy, P., Weiermann, A.: Unprovability results involving braids. Proc. London Math. Soci. **102**(1), 159–192 (2011)
4. Friedman, H.: Adjacent Ramsey theory. Draft, August 2010. https://u.osu.edu/friedman.8/
5. Friedman, H., Pelupessy, F.: Independence of Ramsey theorem variants using ε_0. Proc. Am. Math. Soc. **144**, 853–860 (2016)
6. Friedman, H., Sheard, S.: Elementary descent recursion and proof theory. Ann. Pure Appl. Logic **71**, 1–45 (1995)
7. Hatzikiriakou, K., Simpson, S.G.: Reverse mathematics, Young diagrams, and the ascending chain condition. J. Symbolic Logic **82**, 576–589 (2017)
8. Kreisel, G., Lévy, A.: Reflection principles and their use for establishing the complexity of axiomatic systems. Zeitschrift für mathematische Logik und Grundlagen der Mathematik **14**, 97–142 (1968)
9. Kreuzer, A., Yokoyama, K.: On principles between Σ_1- and Σ_2-induction, and monotone enumerations. J. Math. Logic **16**, 1650004 (2016)

10. Rathjen, M.: The art of ordinal analysis. In: Sanz-Solé, M., Soria, J., Varona, J.L., Verdera, J. (eds.) Proceedings of the International Congress of Mathematicians, Madrid, 22–30 August 2006, pp. 45–69. European Mathematical Society (2006)
11. Schütte, K.: Beweistheoretische Erfassung der unendlichen Induktion in der Zahlentheorie. Mathematische Annalen **122**, 369–389 (1951)
12. Schütte, K.: Beweistheorie, 1st ed. Springer, Berlin (1960)
13. Schütte, K.: Proof Theory. Grundlehren der mathematischen Wissenschaften, vol. 225, 1st edn. Springer, Heidelberg (1977). https://doi.org/10.1007/978-3-642-66473-1
14. Schwichtenberg, H.: Proof theory: some applications of cut-elimination. In: Barwise, J. (ed.) Handbook of Mathematical Logic, pp. 867–895. North Holland, Amsterdam (1977)
15. Simpson, S.: Ordinal numbers and the Hilbert basis theorem. J. Symbolic Logic **53**, 961–974 (1988)
16. Simpson, S.: Subsystems of Second Order Arithmetic, 2nd edn. Cambridge University Press, New York (2009). Association for Symbolic Logic
17. Sommer, R.: Transfinite induction within Peano arithmetic. Ann. Pure Appl. Logic **76**, 231–289 (1995)
18. Tait, W.W.: Normal derivability in classical logic. In: Barwise, J. (ed.) The Syntax and Semantics of Infinitary Languages. LNM, vol. 72, pp. 204–236. Springer, Heidelberg (1968). https://doi.org/10.1007/BFb0079691
19. Takeuti, G.: Proof Theory, 2nd edn. North Holland, Amsterdam (1987)