9

Binomial and Trinomial Trees

Binomial and *trinomial trees* are very intuitive and comparatively easy to implement tools to calculate prices and sensitivity parameters of derivatives while avoiding direct reference to the fundamental differential equations governing the price of the instrument. In practice, tree methods are applied occasionally only nowadays, since other methods, e.g. the finite difference methods (see Chap. 10), show significantly superior numerical features with respect to stability, accuracy, and flexibility. For pedagogical reasons however, it is useful to learn these tree methods, because of the illustrative and direct approach to the valuation of financial derivatives.

In addition to the usual assumptions when excluding arbitrage opportunities (Assumptions 1, 2, 3, and 5), non-stochastic interest rates and default risk (Assumptions 8 and 4) of Chap. 4 will be also assumed in the subsequent sections. These assumptions allow a general theory of binomial trees to be presented. In order to actually calculate option prices, the underlying must be assumed to behave according to a model. Thus, from Sect. 9.3 onwards, it will be assumed that the underlying can be modeled as a random walk with non-stochastic volatility, i.e., the additional Assumptions 7 and 10 from Chap. 4 will be made. Furthermore, we will assume that the underlying earns a dividend *yield* in accordance with Eq. 2.9 rather than discrete dividend payments.

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9.1 General Trees

9.1.1 Evolution of the Underlying and the Replicating Portfolio

Generally, in a tree procedure, the time span in question (the lifetime of the derivative) between t and T is divided into n time intervals of equal length dt:

$$T - t = n\, dt \ . \tag{9.1}$$

In each such time interval the underlying price $S(t)$ may increase in value to $u\,S$ (with $u > 1$) with a probability p', or it may decrease in value to dS (with $d < 1$) with a probability $(1 - p')$

$$S(t) \rightarrow \left\langle \begin{array}{l} S_u(t + dt) = u(t)S(t) \text{ with probability } p' \\ S_d(t + dt) = d(t)S(t) \text{ with probability } 1 - p' \ . \end{array} \right. \tag{9.2}$$

After three steps, for example, the price can take on $2^3 = 8$ possible values:

$$S(t) \rightarrow \left\langle \begin{array}{l} u(t)S(t) \left\langle \begin{array}{l} u(t+dt)u(t)S(t) \left\langle \begin{array}{l} u(t+2dt)u(t+dt)u(t)S(t) \\ d(t+2dt)u(t+dt)u(t)S(t) \end{array} \right. \\ d(t+dt)u(t)S(t) \left\langle \begin{array}{l} u(t+2dt)d(t+dt)u(t)S(t) \\ d(t+2dt)d(t+dt)u(t)S(t) \end{array} \right. \end{array} \right. \\ d(t)S(t) \left\langle \begin{array}{l} u(t+dt)d(t)S(t) \left\langle \begin{array}{l} u(t+2dt)u(t+dt)d(t)S(t) \\ d(t+2dt)u(t+dt)d(t)S(t) \end{array} \right. \\ d(t+dt)d(t)S(t) \left\langle \begin{array}{l} u(t+2dt)d(t+dt)d(t)S(t) \\ d(t+2dt)d(t+dt)d(t)S(t) \ . \end{array} \right. \end{array} \right. \end{array} \right.$$

Consider now a portfolio consisting of Δ underlyings and g monetary units in cash. If the dividend yield q earned in the time interval dt is paid, the cash is compounded at a risk-free rate r and the price of the underlying behaves as described above, the value of the portfolio after dt is given by

$$\Delta(t)S(t) + g(t) \rightarrow \left\langle \begin{array}{l} \Delta\,(t)u(t)S(t)B_q^{-1}(t) + g(t)B^{-1}(t) \\ \Delta\,(t)d(t)S(t)B_q^{-1}(t) + g(t)B^{-1}(t) \ . \end{array} \right. \tag{9.3}$$

Here, for the sake of simplifying the notation for the discount factors over a small time interval dt we have defined

$$B(t) := B(t, t + dt) \ , \quad B_q(t) := B_q(t, t + dt) \ . \tag{9.4}$$

In the following sections, Δ and g will be chosen so that the price of this portfolio behaves exactly as does the value of the derivative we wish to price. This value of the portfolio will then replicate the value of the derivative at each time point and is thus referred to as the *replicating portfolio.*

9.1.2 Evolution of the Derivative

If the underlying moves in accordance with Eq. 9.2 in the time interval dt, the price V of a derivative on this underlying evolves in accordance with

$$V(S, t) \rightarrow \left\langle \begin{array}{l} V(S_u, t + dt) \\ V(S_d, t + dt) \ , \end{array} \right.$$

where V^u and V^d represent the value of a derivative[1] whose underlying has a price of S_u and S_d, respectively. Setting the value of the portfolio (Eq. 9.3) equal to the value of the derivative at time t as well as after the next binomial step $t + dt$, we obtain the following three equations:

$$V(S, t) = \Delta(t)\, S(t) + g(t) \tag{9.5}$$

$$V(S_u, t + dt) = \Delta(t)\, u(t) S(t) B_q^{-1}(t) + g(t) B^{-1}(t) \tag{9.6}$$

$$V(S_d, t + dt) = \Delta(t)\, d(t) S(t) B_q^{-1}(t) + g(t) B^{-1}(t) \tag{9.7}$$

[1]In order to emphasize that this method is valid for all kinds of derivatives, we will continue to denote the value of the derivative with the letter V.

These can be easily rearranged[2] to express the number of underlyings, the cash amount and the value of the derivative at time t in terms of values at time $t+dt$:

$$\begin{aligned}\Delta(t) &= \frac{V(S_u, t+dt) - V(S_d, t+dt)}{[u(t) - d(t)]S(t)/B_q(t)} \\ g(t) &= \frac{u(t)\, V(S_d, t+dt) - d(t)\, V(S_u, t+dt)}{[u(t) - d(t)]/B(t)} \\ V(S,t) &= B(t)\,[p(t)V(S_u, t+dt) + (1-p(t))V(S_d, t+dt)] \qquad (9.8)\end{aligned}$$

where

$$p(t) = \frac{B_q(t)/B(t) - d(t)}{u(t) - d(t)} \,. \qquad (9.9)$$

In Eq. 9.8, we have succeeded in expressing the unknown value of the derivative at time t in terms of quantities known at time t, namely $B_q(t)$, $B(t)$, $u(t)$ and $d(t)$, (in Chap. 13, we will show how the values of u and d are determined) and the (likewise unknown) derivative values at time $t + dt$. The reader might ask what this has accomplished. This expression will in deed prove to be useful if the value of the derivative at a future time is known. Such a future time is, for example, the maturity date T of the derivative. At this time, the value of the derivative as a function of the underlying price is given explicitly by its payoff profile, and as such, is known. The strategy is thus to repeat the procedure described above until reaching a time at which the value of the option is known (in most cases, maturity T). This procedure will be demonstrated below.

Equation 9.8 holds for European derivatives since it is implicitly assumed that the option still exists after a time step has been taken. In order to account for the possibility of exercising early as in the case of derivatives with American features, the derivative's value as given in Eq. 9.8 is compared with its intrinsic

[2]Subtracting Eq. 9.7 from Eq. 9.6 yields

$$V_S^u - V_S^d = \Delta(t)\,(u-d)S(t)B_q^{-1}(t)\,.$$

This allows us to isolate the $\Delta(t)$ term easily. Multiplying Eq. 9.7 by u, and Eq. 9.6 by d and subtracting the results yields

$$u\,V_S^d - d\,V_S^u = (u-d)g(t)B^{-1}(t)\,.$$

This can be readily solved for $g(t)$. Substituting the expressions thus obtained for $\Delta(t)$ and $g(t)$ into Eq. 9.5 yields the value of the derivative $V_S(t, T, K)$.

value at each node in the tree. Then the larger of the two values is taken as the derivative price at that node. For instance for American calls and puts with payoff profiles $S(t) - K$ and $K - S(t)$, respectively, Eq. 9.8 would be replaced by

$$C_S(t) = \max\left\{B(t)\left[p(t)C_S^u(t+dt) + (1-p(t))C_S^d(t+dt)\right], S(t) - K\right\}$$
$$P_S(t) = \max\left\{B(t)\left[p(t)P_S^u(t+dt) + (1-p(t))P_S^d(t+dt)\right], K - S(t)\right\} .$$

9.1.3 Forward Contracts

The evolution of the replicating portfolio consisting of underlyings and cash in a bank account is described by Eq. 9.3. According to Eq. 6.6, the evolution of a futures position is given by

$$V(S,t) \to \left\langle \begin{array}{l} V(S_u, t+dt) = S_u(t+dt, T) - S(t,T) \\ V(S_d, t+dt) = S_d(t+dt, T) - S(t,T) . \end{array} \right.$$

Here, it is not the value of the future at time t which is unknown (this is equal to zero since $K = S(t,T)$), but the forward price of the underlying $S(t,T)$. Setting the portfolio equal to the future at both time t and at the next time in the binomial tree $t + dt$ yields three equations:

$$0 = V(S,t) = \Delta(t)\, S(t) + g(t) \tag{9.10}$$

$$S_u(t+dt, T) - S(t,T) = V(S_u, t+dt) = \Delta(t)\, u(t)S(t)B_q^{-1}(t) + g(t)B^{-1}(t) \tag{9.11}$$

$$S_d(t+dt, T) - S(t,T) = V(S_d, t+dt) = \Delta(t)\, d(t)S(t)B_q^{-1}(t) + g(t)B^{-1}(t) \tag{9.12}$$

With the help of these three equations, the number of underlyings, the money in the bank account and the forward price at time t can (making use of the expression p defined in Eq. 9.9) be expressed[3] in terms of the forward price

[3] Subtracting Eq. 9.11 from Eq. 9.12 yields

$$S^u(t+dt, T) - S^d(t+dt, T) = \Delta(t)\,(u-d)S(t)B_q^{-1}(t) .$$

at time $t + dt$:

$$\Delta(t) = \frac{S_u(t+dt,T) - S_d(t+dt,T)}{[u(t)-d(t)]S(t)B_q(t)} \tag{9.13}$$

$$g(t) = -\frac{S_u(t+dt,T) - S_d(t+dt,T)}{[u(t)-d(t)]B_q(t)}$$

$$S(t,T) = p(t)S_u(t+dt,T) + (1-p(t))S_d(t+dt,T)\ .$$

9.2 Recombining Trees

9.2.1 The Underlying

If the parameters u and d are independent of time[4]

$$u(t + j\,dt) = u(t) \equiv u\ \ \forall j \quad , \quad d(t + j\,dt) = d(t) \equiv d\ \ \forall j$$

then obviously $udS(t) = duS(t)$ holds, i.e., an upward move followed by a downward move results in the same underlying price as a downward move followed by an upward move. Thus the tree is forced to *recombine*. This significantly reduces the number of possible nodes, making the computation much more efficient. Such a *recombining binomial tree* has the form depicted in Fig. 9.1. The probability for a *single* path ending at $S(T) = u^j d^{n-j} S(t)$ is

$$p'^j (1-p')^{n-j}\ .$$

The number of all paths ending at $S(T) = u^j d^{n-j} S(t)$ can be deduced from permutation laws and is given by the *binomial coefficient*

$$\binom{n}{j} \equiv \frac{n!}{j!(n-j)!}\ .$$

This allows us to easily isolate $\Delta(t)$. Because of Eq. 9.10, $g(t) = -\Delta(t)S(t)$ holds, which immediately yields $g(t)$ if $\Delta(t)$ is known. Substituting the expressions for $\Delta(t)$ and $g(t)$ into Eq. 9.11 or Eq. 9.12 yields, after a simple calculation, the forward price $S(t,T)$.

[4]It is possible to construct recombining trees with time-dependent u and d, if at the same time some other constraint, e.g. constant time steps, is dropped.

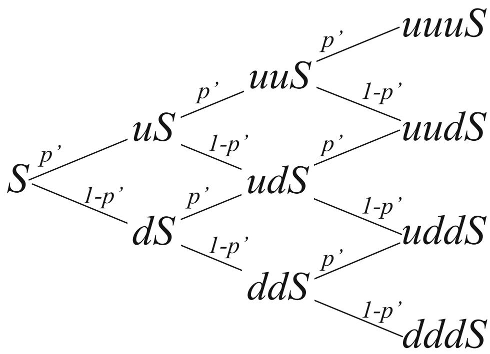

Fig. 9.1 The first steps in a recombining binomial tree

The probability in the above tree of arriving at a value $S(T) = u^j d^{n-j} S(t)$ regardless of the path taken to get there is equal to the number of such paths multiplied by the probability of realizing such a path.

$$P\left[S(T) = u^j d^{n-j} S(t)\right] = \binom{n}{j} p'^j (1-p')^{n-j} = B_{n,p'}(j) \; . \qquad (9.14)$$

This is the definition of the probability density function of the *binomial distribution*, see Sect. A.4.2. This is how the binomial distribution enters into the binomial trees.

9.2.2 The Binomial Distribution for European Derivatives

In addition to the assumptions made at the beginning of Chap. 9 and the one just made, namely that the parameters u and d are constant over time, we will henceforth assume that the yields (interest rates and dividends) are constant over time as well, i.e., that Assumptions 9 and 12 from Chap. 4 hold[5]:

$$B(t + j\,dt) = B(t) \quad \forall j \; , \qquad B_q(t + j\,dt) = B_q(t) \quad \forall j \; . \qquad (9.15)$$

In consequence, the parameter p defined in Eq. 9.9 is time independent as well:

$$p(t + j\,dt) = p(t) = p \quad \forall j \; .$$

[5]On both sides we use here again the short notation defined in Eq. 9.4.

Equation 9.8 holds not only at time t but for other times as well, for example at time $t + dt$. This is true for both V_S^u and V_S^d:

$$V(S_u, t + dt) = B(t)\,[p\,V(S_{uu}, t + 2dt) + (1 - p)\,V(S_{ud}, t + 2dt)]$$
$$V(S_d, t + dt) = B(t)\,[p\,V(S_{ud}, t + 2dt) + (1 - p)\,V(S_{dd}, t + 2dt)]\ .$$

Substitution into Eq. 9.8 leads to an expression for $V(S, t)$ as a function of the derivative price at time $t + 2dt$. Analogous expressions can be obtained for V_S^{uu}, V_S^{ud}, etc. This recursive procedure performed iteratively for $n = (T - t)/dt$ binomial steps gives

$$\begin{aligned} V(S,t) &= B(t,T)\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}V\left(u^j d^{n-j}S(t),T\right) \\ &= B(t,T)\sum_{j=0}^{n}B_{n,p}(j)V\left(u^j d^{n-j}S(t),T\right)\ , \end{aligned} \tag{9.16}$$

where the second line was obtained by observing that $\binom{n}{j}p^j(1-p)^{n-j}$ corresponds to a binomial probability density $B_{n,p}(j)$ but with parameters n and p (*not* with p' as in Eq. 9.14).

Thus the value of the derivative at time t has been expressed as a sum over its values at a later time T. If this time T is chosen to be the maturity then the value of the derivative at time t is written in terms of its payoff profile. This reads explicitly for European calls and puts

$$c_s(t) = B(t,T)\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\max\left\{0, u^j d^{n-j}S(t) - K\right\}$$
$$p_s(t) = B(t,T)\sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}\max\left\{0, K - u^j d^{n-j}S(t)\right\}\ .$$

Because of the maximum function appearing in the summand, the sum for the call is effectively taken over the values of j for which $u^j d^{n-j}S(t)$ is larger than K. This condition can be written as $(u/d)^j > d^{-n}K/S$. Taking the logarithm of both sides yields the equivalent condition

$$j > \ln\left(\frac{K}{S(t)d^n}\right) / \ln\left(\frac{u}{d}\right)\ .$$

The sum is taken over whole numbers j. The smallest whole number greater than the right-hand side in the above inequality is

$$y = 1 + \text{Trunc}\left(\ln(\frac{K}{S(t)d^n})/\ln(\frac{u}{d}) \right) , \tag{9.17}$$

where the function "Trunc" is defined as the greatest whole number smaller than the argument (decimal values are simply truncated and not rounded). The number y defined in Eq. 9.17 is thus the lower limit in the sum for the call (correspondingly, the sum for the put is taken over the whole numbers j ranging from 0 to the upper limit $y - 1$). The value of a call is thus

$$c_S(t) = S(t)B(t,T)\sum_{j=y}^{n}\binom{n}{j}p^j(1-p)^{n-j}u^j d^{n-j} - KB(t,T)\underbrace{\sum_{j=y}^{n}\binom{n}{j}p^j(1-p)^{n-j}}_{B_{n,p}(j\geq y)} .$$

According to Eq. A.42, the last sum is the probability that a binomially distributed random variable (where $B_{n,p}$ denotes the binomial distribution with parameters n and p) is greater than or equal to y. Under Assumption 9.15 that yields are constant, i.e., $B_r(t) = B_r$ independent of t, we can write

$$B(t,T) = \prod_{k=0}^{n-1} B(t+kdt) = B^n \quad , \quad B_q(t,T) = B_q^n \ .$$

Now the first sum can be represented as a binomial probability as well:

$$\begin{aligned} c_S(t) &= S(t)B_q(t,T)\sum_{j=y}^{n}\binom{n}{j}p^j u^j \frac{B^n}{B_q^n}(1-p)^{n-j}d^{n-j} \\ &\quad - KB(t,T)\sum_{j=y}^{n}\binom{n}{j}p^j(1-p)^{n-j} \\ &= S(t)B_q(t,T)\underbrace{\sum_{j=y}^{n}\binom{n}{j}\hat{p}^j(1-\hat{p})^{n-j}}_{B_{n,\hat{p}}(j\geq y)} - KB(t,T)\underbrace{\sum_{j=y}^{n}\binom{n}{j}p^j(1-p)^{n-j}}_{B_{n,p}(j\geq y)} \end{aligned} \tag{9.18}$$

where[6]

$$\hat{p} = u\,\frac{B}{B_q}p \Rightarrow 1-\hat{p} = d\frac{B}{B_q}(1-p)\ .$$

The value of a European put can be determined analogously with the help of the binomial distribution. So the prices of European options expressed in terms of binomial distributions are

$$\begin{aligned} c_s(t) &= B_q(t,T)\,S(t)B_{n,\hat{p}}(j\geq y) - B(t,T)K\,B_{n,p}(j\geq y) \\ p_s(t) &= -B_q(t,T)\,S(t)\left[1-B_{n,\hat{p}}(j\geq y)\right] + B(t,T)\,K\left[1-B_{n,p}(j\geq y)\right]\ . \end{aligned} \tag{9.19}$$

Note the similarity to the famous Black-Scholes equation (see, for example Eq. 8.6 or Eq. 8.7). The difference is that the *binomial* distribution appears in the above expression in place of the *normal* distribution. In Sect. 9.4, we will see that the binomial distribution for infinitesimally small intervals dt converges towards a normal distribution and thus the binomial model approaches the Black-Scholes model in the limit $dt \to 0$.

As another example of the above procedure we demonstrate how the forward price can be determined by iterating Eq. 9.13 for $n = (T-t)/dt$ binomial steps

$$S(t,T) = \sum_{j=0}^{n}\binom{n}{j}p^j(1-p)^{n-j}u^j d^{n-j}S(t)\ ,$$

[6]Writing $S^u = uS$ and $S^d = dS$ in the third equation in 9.13 and using Eq. 6.1 for $S(t,T)$ yields

$$\frac{B_q}{B}S(t) = p\,u\,S(t) + (1-p)\,d\,S(t)\ .$$

Dividing by the left-hand side gives

$$1 = p\frac{B}{B_q}u + (1-p)\frac{B}{B_q}d\ .$$

It then follows immediately that

$$1-\hat{p} \equiv 1-u\frac{B}{B_q}p = d\frac{B}{B_q}(1-p)\ .$$

where the spot price $S(T)$ for the forward contract after n steps is substituted into the equation, since the maturity date T will have been reached after this time. Under the assumption of constant interest rates and dividend yields, we obtain

$$\begin{aligned}S(t,T) &= \frac{B_q(t,T)}{B(t,T)}S(t)\sum_{j=0}^{n}\binom{n}{j}p^j u^j \frac{B^n}{B_q^n}(1-p)^{n-j}d^{n-j}\\ &= \frac{B_q(t,T)}{B(t,T)}S(t)\sum_{j=0}^{n}\binom{n}{j}\hat{p}^j(1-\hat{p})^{n-j} = \frac{B_q(t,T)}{B(t,T)}S(t)\underbrace{B_{n,\hat{p}}(j\geq 0)}_{1}\,,\end{aligned}$$

corresponding to the result in Eq. 6.1 obtained solely on the basis of arbitrage considerations.

9.2.3 A Third Contact with the Risk-Neutral World

Neither for derivatives (see Eq. 9.8) nor for forward prices (see Eq. 9.13) does the probability p' for the underlying S to increase to S_u (see, for example Eq. 9.2) enter into any equation. The *valuation* of derivatives (or forward prices) does not depend on the probability that the underlying rises or falls! Instead, it depends on the value p as defined in Eq. 9.9. If we could interpret p as a kind of "artificial probability", and if for pricing purposes we could put ourselves into an "artificial world" in which, after one step, the price of the underlying is given by S_u with this "probability" p (and not with the probability p'as in the real world), then the probability for the underlying to decrease to S_d over one time step would be $(1-p)$ in this "artificial world". The expression in brackets in Eq. 9.8 would then just be the *expectation* of the derivative price *one* time step later with respect to the probability p in the artificial world. This holds for *many* time steps as well, since the binomial density $B_{n,p}(j)$ appearing in Eq. 9.16 is the probability in the *artificial* world for the underlying to arrive at the value $S(T)=u^j d^{n-j}S(t)$; just as Eq. 9.14 was this probability in the *real* world. The sum in Eq. 9.16 over all the derivative values is then the *expectation* of the derivative value at the future time T in the artificial world.

We can summarize our observations in the following way: In an artificial world, where the probability of an up-move is p (and not p' as in the real world), **today's price of a derivative is the expectation of its future price discounted back to today**.

$$V(S,t) = B(t,T)\,\mathrm{E}_p\,[V(S,T)] \text{ Derivat (z. B. Option) auf } S\,. \tag{9.20}$$

The notation "$\mathrm{E}_p[]$" here means: "expectation with respect to the probability p".

Likewise, the forward price of the underlying in Eq. 9.13 is exactly the expectation of the underlying's price at the future time T with respect to the probability p:

$$S(t,T) = \mathrm{E}_p\,[S(T)] \text{ Forward Price of } S\,. \tag{9.21}$$

In the case of forward prices (which are not tradable financial *instruments* themselves) the expectation is not discounted.

By substituting Eq. 6.1 for $S(t,T)$ into the above equation it follows that the dividend-adjusted spot price of the underlying (which is a tradable financial instrument) likewise can be expressed as the *discounted* expectation with respect to this probability

$$\tilde{S}(t,T) = B(t,T)\,\mathrm{E}_p\,[S(T)] \text{ Spot price } S\,. \tag{9.22}$$

As in Sects. 7.1.3 and 7.2.1, it does not matter if the underlying is expected to rise or fall in the real world. This plays no role in the valuation of derivatives on the underlying. The valuation is independent of the expected changes in the underlying. In contrast to the real world where investors are compensated for taking the risk of investing in an underlying by the underlying's mean return, this mean return doesn't play any role at all in the artificial world used for pricing derivatives. This artificial world is thus neutral to the risk inherent in the underlying and is therefore called the *risk-neutral world* and the probability p is called the *risk-neutral probability.* We are again confronted with the *risk neutrality* described in Sects. 7.1.3 and 7.2.1.

This risk neutrality is caused by the fact that the option buyer would hedge himself against the risk of an unfavorable development of the option underlying by entering into a portfolio of shares and cash, which replicates the option pay off at expiry. Therefore, he has eliminated the risk and his total position is risk neutral. The cost for the derivative is identical to the cost of this replication strategy. Would the seller value the derivative differently, an arbitrage opportunity would arise, since the potential buyer could apply the replication strategy in order to earn a risk-less profit. Likewise, because of this effect, different market participants would be able to agree on the same option price independently of their estimate of the future development of the underlying. Based on these arbitrage considerations making use of the fact that options and futures can be replicated by a portfolio consisting of the underlying and risk-free assets, it follows that a purely objective, risk-neutral probability p for an up-move in a risk-neutral world exists, eliminating any subjectivity (see Eq. 9.9). On the other hand, a derivative buyer, who does plan to hedge his derivative position by means of a strategy replicating the pay off profile (and also does not need this derivative to hedge some other existing positions), would rely on a valuation of the derivative based on his subjective judgment about the future market development. He would indeed seek his advantage in making a profit by means of the difference between risk neutral valuation and the real world development. Of course, this strategy would not be risk-free anymore.

To see how powerful Eqs. 9.20, 9.21 and 9.22 are, we have to be more specific. We now choose a stochastic process for the underlying. In what follows, we will assume that the relative changes of $S(t)$ behave as a random walk as in Eq. 2.17, i.e., we will rely on Assumption 7 from Chap. 4. But Eq. 2.17 was established to model the behavior of S in the *real* world.[7] We will show later,[8] however, that if S performs a random walk in the real world, it *also* performs a random walk in the risk-neutral world. The distribution and first moments of such a random walk are those given in Table 2.7 at the end of Sect. 2.3. Thus, the underlying is lognormally distributed with expectation

$$\langle S(T)\rangle = S(t)e^{(\mu+\sigma^2/2)(T-t)} \ . \qquad (9.23)$$

In the risk-neutral world (i.e., in the world we need for pricing) this expectation has to be equal to the expectation $\mathrm{E}_p\left[S(T)\right]$ with respect to the risk-neutral probability p:

$$\langle S(T)\rangle \stackrel{!}{=} \mathrm{E}_p\left[S(T)\right] \ .$$

[7] A "risk-neutral" world was never mentioned in the vicinity of Eq. 2.17, nor in the whole of Chap. 2.

[8] We will explicitly show this in great detail and on a much more fundamental basis in Chap. 13 when we discuss the famous Girsanov theorem.

Substituting Eqs. 9.22 and 9.23 into this requirement completely determines the drift μ of the underlying in the risk-neutral world, i.e., the drift to be used for pricing:

$$S(t)e^{(\mu+\sigma^2/2)(T-t)} \stackrel{!}{=} \frac{\tilde{S}(t,T)}{B(t,T)}$$
$$\Leftrightarrow \qquad \mu \stackrel{!}{=} \frac{1}{T-t}\ln\left(\frac{\tilde{S}(t,T)}{S(t)B(t,T)}\right) - \frac{\sigma^2}{2} .$$

Using the forward price equation 9.21 instead, we obtain the drift from the ratio of forward price to spot price:

$$S(t)e^{(\mu+\sigma^2/2)(T-t)} \stackrel{!}{=} S(t,T)$$
$$\Leftrightarrow \qquad \mu \stackrel{!}{=} \frac{1}{T-t}\ln\left(\frac{S(t,T)}{S(t)}\right) - \frac{\sigma^2}{2} . \tag{9.24}$$

With a dividend yield q and continuous compounding, we obtain, for example,

$$\mu = \frac{1}{T-t}\ln\left(\frac{B_q(t,T)}{B(t,T)}\right) - \frac{\sigma^2}{2} = r - q - \frac{\sigma^2}{2} , \tag{9.25}$$

where the first equality holds as a result of the assumed dividend yield and the second is valid for continuous compounding. But this is exactly Eq. 7.19.

As was pointed out in Eq. 2.30, the drift μ is exactly equal to the expected return of the underlying. This means that the expected return in the risk-neutral world (i.e., with respect to the probability p) is objectively given through the risk-free interest rate and the dividends (or through the ratio of forward price to spot price) and the volatility, independent of an investor's opinion as to whether the price will rise or fall. The parameter μ thus determined is called the *risk-neutral yield* or the *risk-neutral drift*.

We note for later reference that all this holds for any arbitrary time span $T-t$, for instances also for one time step dt in a binomial tree:

$$\mu\, dt = \ln\left(\frac{B_q(t)}{B(t)}\right) - \frac{\sigma^2}{2}dt . \tag{9.26}$$

9.3 Random Walk and Binomial Parameters

Risk neutrality is the essential link connecting the stochastic model describing the underlying with the pricing method used for a derivative. This will be demonstrated for the binomial model. The parameters u and d must first be determined before the binomial model can be applied in pricing derivatives. The choice of these parameters has a significant influence on derivative and forward prices thus calculated. To make a reasonable choice of u and d further assumptions concerning the behavior of the underlying must be made, i.e., a stochastic process for the underlying must be specified. We will again assume that the relative changes of $S(t)$ behave as a random walk as in Eq. 2.17 and are therefore normally distributed with moments of the form given in Table 2.7 at the end of Sect. 2.3. On the other hand we know from the previous sections that in a binomial tree the underlying S is distributed according to the binomial distribution, see Eq. 9.14, where for pricing purposes we have to replace the real world probability p' by the risk-neutral "probability" p. We will now relate the random walk parameters μ and σ to the parameters u and d of the binomial tree by *matching the moments* of the random walk distribution to the moments of the binomial distribution. We will ensure that we work in the risk-neutral world (i.e., that we determine the parameters needed for pricing) by using p as defined in Eq. 9.9 for the binomial tree and by using the risk-neutral drift defined in Eq. 9.25 for the random walk.

In the following derivation, we will assume that the parameters u and d are constant over time until maturity. After j up-moves and $n-j$ down-moves, the final value $S(T)$ and thus the logarithm of the relative price change is

$$S(T) = u^j d^{n-j} S(t) \quad \Rightarrow \quad \ln\left(\frac{S(T)}{S(t)}\right) = j \ln\left(\frac{u}{d}\right) + n \ln(d) \; .$$

$S(T)$ (and thus j) is binomially distributed in our binomial model and from Eq. A.44, it follow that $\langle j \rangle = np$ and $\operatorname{var}(j) = np(1-p)$.

For the random walk model, on the other hand, the expectation and variance of the logarithmic changes of S are equal to the drift and the square of the volatility, each multiplied by the time difference $T-t$ (see the first column of Table 2.7). Thus matching the first two moments of the random walk distribution to the distribution induced by the binomial tree yields

$$\mu(T-t) = \left\langle \ln\left(\frac{S(T)}{S(t)}\right)\right\rangle = \underbrace{\langle j\rangle}_{np} \ln\left(\frac{u}{d}\right) + n\ln(d)$$
$$\sigma^2(T-t) = \mathrm{var}\left(\ln\left(\frac{S(T)}{S(t)}\right)\right) = \underbrace{\mathrm{var}\,(j)}_{np(1-p)} \left(\ln\left(\frac{u}{d}\right)\right)^2 . \tag{9.27}$$

Because $dt = (T-t)/n$ this can be written as

$$\mu dt = p\ln\left(\frac{u}{d}\right) + \ln(d)$$
$$\sigma^2 dt = p(1-p)\left(\ln\left(\frac{u}{d}\right)\right)^2 .$$

Now we use Eq. 9.9 for the risk-neutral probability p and Eq. 9.26 for the risk-neutral drift μ to establish a system of two (non-linear!) equations for the two unknown binomial parameters u and d:

$$\ln\left(\frac{B_q}{B}\right) - \frac{\sigma^2}{2}dt = \frac{B_q/B-d}{u-d}\ln\left(\frac{u}{d}\right) + \ln(d)$$
$$\sigma^2 dt = \frac{\left(B_q/B-d\right)\left(u-B_q/B\right)}{(u-d)^2}\left(\ln\left(\frac{u}{d}\right)\right)^2 . \tag{9.28}$$

There exist several closed form solutions to this system which are exact up to linear order in dt. One such solution is given by

$$u = \frac{B_q}{B}e^{-(\sigma^2/2)dt+\sigma\sqrt{dt}} \quad , \quad d = \frac{B_q}{B}e^{-(\sigma^2/2)dt-\sigma\sqrt{dt}} . \tag{9.29}$$

Inserting this into Eq. 9.9 for the risk-neutral probability yields

$$p = \frac{e^{(\sigma^2/2)dt} - e^{-\sigma\sqrt{dt}}}{e^{+\sigma\sqrt{dt}} - e^{-\sigma\sqrt{dt}}} = \frac{e^{(\sigma^2/2)dt} - e^{-\sigma\sqrt{dt}}}{2\ \sinh(\sigma\sqrt{dt})} ,$$

where we have used the definition of the hyperbolic sine function in the last step. Using Eq. 9.26 (which is equivalent to $\exp(\mu\, dt) = \exp\left(-\sigma^2 dt/2\right) B_q/B$) we can bring the parameters u and d into a more intuitive form using the risk-neutral drift:

$$u = e^{\mu\,dt + \sigma\sqrt{dt}} \quad , \quad d = e^{\mu\,dt - \sigma\sqrt{dt}} \; .$$

Thus, in this solution the drift only appears in the parameters u and d, while the probability p is determined solely from knowledge of the volatility. For small values of dt and assuming continuous compounding, the Taylor series representation of the exponential function expanded up to linear terms in dt gives

$$u \approx 1 + \sigma\sqrt{dt} + (q - r)\,dt \ , \quad d \approx 1 - \sigma\sqrt{dt} + (q - r)\,dt \ , \quad p \approx 1/2 \; . \tag{9.30}$$

Since in this solution both the volatility and the risk neutral drift appear in u and d, we must assume constant volatilities and because of Eq. 9.25 also constant yields and dividends, i.e., Assumptions 9, 11 and 12 from Chap. 4, to ensure that the parameters u and d are constant over time and, in consequence, that the tree recombines.

Another frequently used solution of Eq. 9.28 for which it suffices to assume constant volatilities (Assumption 11) is

$$u = e^{+\sigma\sqrt{dt}} \quad , \quad d = e^{-\sigma\sqrt{dt}} \Rightarrow p = \frac{B_q/B - e^{-\sigma\sqrt{dt}}}{2\ \sinh(\sigma\sqrt{dt})} = \frac{e^{(q-r)dt} - e^{-\sigma\sqrt{dt}}}{2\ \sinh(\sigma\sqrt{dt})} \ , \tag{9.31}$$

where the last step is of course only valid for continuous compounding. In this solution, the volatility alone completely determines the parameters u and d. Observe that $u(t) = 1/d(t)$ holds. As long as the volatility is constant (allowing the parameters u and d to remain constant over time), the tree is recombining since the starting price is recovered after an up-move followed by a down-move:

$$u(t)\,d(t + dt) = u(t + dt)d(t) = u(t)/u(t) = 1 \; .$$

The ease in the construction and analysis of binomial trees resulting from this relation prompts us to utilize the solution given by Eq. 9.31 exclusively in the remainder of this book whenever we use binomial trees. For small time intervals dt, the Taylor series representation of the exponential function expanded up to and including terms of linear order in dt yields the following approximations for u, d and p:

$$u \approx 1+\sigma\sqrt{dt}+\frac{\sigma^2}{2}dt,\quad d \approx 1-\sigma\sqrt{dt}+\frac{\sigma^2}{2}dt\;,\quad p \approx \frac{1}{2}\left(1+\frac{\mu}{\sigma}\sqrt{dt}\right)\;,$$

where again the risk-neutral drift μ is used to simplify the last expression.

A detailed demonstration of the application of binomial trees for the valuation of an option portfolio is provided in the Excel workbook BINOMIAL-TREE.XLS. In anticipation of Chap. 12, the evaluation of the *Greeks* (derivatives of the option price with respect to its parameters) using binomial trees also receives attention. This workbook can be used as a small but fully functioning option calculator (as always, the yellow fields are the input fields).

9.4 The Binomial Model with Infinitesimal Steps

In this section, the Black-Scholes option pricing formula is derived directly from the binomial model for European options as given by Eq. 9.19. A deeper insight into the relationship between these two important methods in option pricing (finding solutions to a differential equation on the one hand and the calculation of (discounted) expectations on the other) can be gained from an understanding of this derivation. The reader less interested in mathematics may choose to continue on to the next section.

A classical result from statistics, the *Moivre-Laplace theorem,* states that the binomial distribution converges towards a normal distribution as the number n of the observed trials approaches infinity. The statement of the theorem in integral form can be expressed as

$$B_{n,p}\left(a \le \frac{j-np}{\sqrt{np(1-p)}} \le b\right) \stackrel{n\to\infty}{\longrightarrow} \mathrm{N}(b)-\mathrm{N}(a)=\frac{1}{\sqrt{2\pi}}\int_a^b e^{-z^2/2}dz\;.$$

The left-hand side of the equation denotes the probability that the standardized form of a *binomially distributed* random variable j (i.e., j less its expectation divided by its standard deviation) will lie in the interval between a and b, while the right-hand side is simply the probability that a *standard normally distributed* random variable will take on values lying within the same interval.

We can exploit this theorem to see what happens to Eq. 9.19 as the time interval dt converges towards zero, i.e., as the number n of steps in the binomial tree approaches infinity. This will be demonstrated for a call, the procedure for a put being completely analogous.

We need, for example, to determine the limit of $B_{n,p}(y \leq j)$ in Eq. 9.19 For arbitrary constants f and g with $g > 0$, the probability that $y \leq j$ is of course equal to the probability that $(y - f)/g \leq (j - f)/g$. We take advantage of this fact[9] to manipulate $B_{n,p}(y \leq j)$ into a suitable form to apply the Moivre-Laplace theorem:

$$\begin{aligned} B_{n,p}(y \leq j) &= B_{n,p}\left(\frac{y - np}{\sqrt{np(1-p)}} \leq \frac{j - np}{\sqrt{np(1-p)}} \leq \infty\right) \\ &\xrightarrow{n\to\infty} \underbrace{\mathrm{N}(\infty)}_{1} - \mathrm{N}\left(\frac{y - np}{\sqrt{np(1-p)}}\right) \\ &= \mathrm{N}\left(\frac{np - y}{\sqrt{np(1-p)}}\right) , \end{aligned}$$

where in the last step the symmetry property of the normal distribution, Eq. A.54, is used.

Equations 9.27 and 9.17 deliver the necessary elements for computing $(np - y)/\sqrt{np(1-p)}$:

$$np = \frac{1}{\ln(u/d)}\left(\mu(T-t) - n\ln(d)\right) , \quad \sqrt{np(1-p)} = \frac{1}{\ln(u/d)}\sigma\sqrt{T-t}$$

and

$$y = \frac{\ln\left(\frac{K}{S(t)d^n}\right)}{\ln(u/d)} + \varepsilon = \frac{\ln\left(\frac{K}{S(t)}\right) - n\ln(d) + \varepsilon\ln(u/d)}{\ln(u/d)} \quad \text{mit } 0 < \varepsilon \leq 1 .$$

Here, ε represents the difference between $\ln(K/Sd^n)/\ln(u/d)$ and the smallest whole number greater than this value. We will show immediately that the term $\varepsilon\ln(u/d)$ becomes arbitrarily small. Substituting accordingly yields the argument for the standard normal distribution above:

$$\frac{np - y}{\sqrt{np(1-p)}} = \frac{\ln(S(t)/K) + \mu(T-t) - \varepsilon\ln(u/d)}{\sigma\sqrt{T-t}} .$$

[9] With the choice $f = np$ und $g = \sqrt{np(1-p)}$.

In both solutions 9.29 and 9.31 the ratio u/d converges towards 1 as $n \to \infty$ (i.e., $dt \to 0$), thus by continuity, $\ln(u/d)$ converges towards zero. Thus, the limit for infinitely many binomial steps becomes

$$\frac{np - y}{\sqrt{np(1-p)}} \overset{n\to\infty}{\longrightarrow} \frac{\ln(S(t)/K) + \mu(T-t)}{\sigma\sqrt{T-t}} = x - \sigma\sqrt{T-t}\ .$$

Using now Eq. 9.25 for the risk-neutral drift we can write

$$\frac{np - y}{\sqrt{np(1-p)}} \overset{n\to\infty}{\longrightarrow} x - \sigma\sqrt{T-t}$$

where we have defined the abbreviation x as in Eq. 8.5:

$$x = \frac{1}{\sigma\sqrt{T-t}} \ln\left(\frac{S(t)\,B_q(t,T)}{K\,B(t,T)}\right) + \frac{1}{2}\sigma\sqrt{T-t}\ .$$

The limit of $B_{n,p}(y \le j)$ is thus established. Proceeding analogously, we can calculate the limit of the other binomial probability in Eq. 9.19. In summary, for an infinite number of binomial steps in a finite time interval, the behavior of the binomial distribution is given by

$$B_{n,p}(y \le j) \overset{n\to\infty}{\longrightarrow} \mathrm{N}\left(x - \sigma\sqrt{T-t}\right)\ ,\quad B_{n,\hat{p}}(y \le j) \overset{n\to\infty}{\longrightarrow} \mathrm{N}(x)\ .$$

Using these convergence relations, we obtain the value of a call as the number of binomial steps approaches infinity to be

$$c_s(t) \overset{n\to\infty}{=} S(t)B_q(t,T)\mathrm{N}(x) - K\,B(t,T)\mathrm{N}(x - \sigma\sqrt{T-t})\ . \tag{9.32}$$

This is in complete agreement with Eq. 8.6 and is thus (again!) the famous *Black-Scholes option pricing formula.*

9.4.1 Components of the Black-Scholes Option Pricing Formula

In the above section the Black-Scholes formula was derived from Eqs. 9.18 and 9.19. We can see from this derivation that the cumulative normal distribution found next to the discounted strike price $B(t,T)K$ is the risk-neutral probability for the price of the underlying to be larger than the strike price.

Table 9.1 Interpretation of the various components in the Black-Scholes option pricing formulae for plain vanilla calls and puts. All probabilities mentioned are risk-neutral. The lower four terms are required for replicating (hedge) calls resp. puts

$\mathrm{N}(x-\sigma\sqrt{T-t})$	Risk-neutral exercise probability for call
$\mathrm{N}(-x+\sigma\sqrt{T-t})$	Risk-neutral exercise probability for put
$KB(t,T)$	Present value (PV) of cash flow at exercise
$B_q(t,T)\mathrm{N}(x)$	Number of underlyings to buy for replicating the call
$B_q(t,T)\mathrm{N}(-x)$	Number of underlyings to buy for replicating the put
$KB(t,T)\mathrm{N}(x-\sigma\sqrt{T-t})$	Amount to be borrowed for call replication
$KB(t,T)\mathrm{N}(-x+\sigma\sqrt{T-t})$	Amount to be borrowed for put replication

This is referred to as the *risk-neutral exercise probability*. A comparison with the replicating portfolio in Eq. 9.3 shows that the number $\Delta(t)$ of underlyings needed to replicate the option is given by the factor next to $S(t)$ in Eq. 9.32, namely $B_q(t,T)\mathrm{N}(x)$, while the amount $g(t)$ of money in the bank account is given by the second summand in Eq. 9.32. The intuitive interpretations of these values in the Black-Scholes formulae for puts and calls are collected in Table 9.1.

9.5 Trinomial Trees

Trinomial trees present us with an alternative method to binomial trees. The form of a trinomial tree is represented graphically in Fig. 9.2. The jth step of the tree at time t_j is connected, not with two other nodes in the next step (as was the case for the binomial tree), but with three. The price paid for this additional degree of freedom is additional computational effort. The advantage is that a trinomial tree can always be constructed in such a way that it recombines and in addition, achieves the same degree of accuracy as the binomial tree with fewer time steps. The trinomial tree has $2j+1$ nodes after j steps where the time t is indexed with $j=0$. The length of the time steps may vary. The value of the underlying at the ith node after j steps is denoted by S_{ji} where

$$i=-j,-j+1,\ldots,j-1,j\ .$$

Each node at time step j branches into three nodes at time step $j+1$, with a probability being associated with each of these branches.[10] Starting from S_{ji},

[10] All probabilities appearing in this context are risk-neutral probabilities.

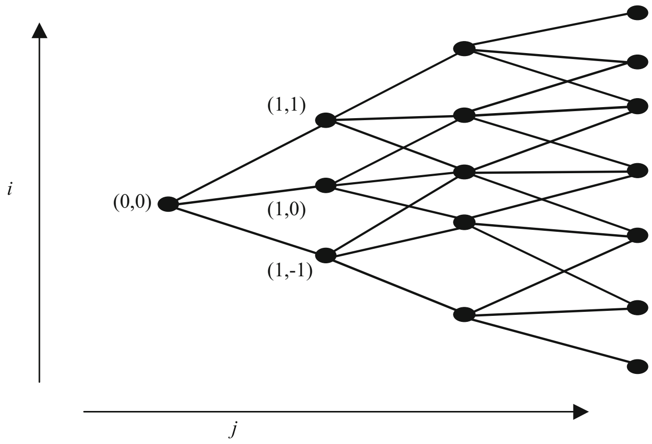

Fig. 9.2 A simple trinomial tree

we denote by p^+ the probability that the underlying will increase from S_{ji} to the value $S_{j+1,i+1}$ at time step $(j + 1)$. Correspondingly, denote by p^0 and p^- the probabilities that the underlying at time step $(j + 1)$ will take on the values $S_{j+1,i}$ and $S_{j+1,i-1}$, respectively. Each of these probabilities must be ≥ 0 and ≤ 1. In addition:

$$p^+ + p^0 + p^- = 1 \,. \tag{9.33}$$

For the *bi*nomial tree,[11] a portfolio can be constructed consisting of a position Δ in the underlying and money g in a bank account whose value replicates the option price exactly at each time step. This is also possible for trinomial trees. Though, because of the addition of the *third* attainable value for the underlying after one time step, just *two* parameters Δ and g are not sufficient to determine a replicating portfolio that replicates exactly all *three* possible states after one time step. Therefore, we need to make a further choice before all parameters could be determined unequivocally. Because of this additional degree of freedom, trinomial trees can be more flexibly adopted for different purposes.

To do so, it is sufficient to choose the probabilities p^+, p^0 and p^- and the nodes S_{ji} so that the tree reflects the probability distribution of the underlying. Again, we assume that the underlying price is lognormally distributed (this

[11]The binomial tree usually assumes a constant time step dt. In the most general case, this assumption is not necessary.

corresponds to Assumption 7 from Chap. 4). The lognormal distribution is completely determined by two parameters, the expectation and the variance of the logarithm. Thus, two conditions are sufficient to adapt the trinomial tree to a lognormal distribution. Like in Eq. 9.23, the expectation for the lognormally distributed random variable after a time step of length dt starting from the node S_{ji} (with the risk-neutral drift Eq. 9.25) is given by

$$E\left[S(t_{j+1})\right] = S_{ji}e^{(r-q)dt} .$$

On the other hand, from the tree we have

$$E\left[S(t_{j+1})\right] = p^+S_{j+1,i+1} + p^0S_{j+1,i} + p^-S_{j+1,i-1} .$$

Setting these two expressions equal to one another yields one equation for the determination of the probabilities:

$$S_{ji}e^{(r-q)dt} = p^+S_{j+1,i+1} + p^0S_{j+1,i} + p^-S_{j+1,i-1} . \tag{9.34}$$

Analogously, taking the expression for the variance of the lognormal distribution shown in Table 2.7 at the end of Sect. 2.3 we have

$$\text{Var}\left[S(t_{j+1})\right] = S_{ij}^2 e^{2(r-q)dt}\left(e^{\sigma^2 dt} - 1\right) .$$

It is sometimes easier to work with the expectation of $S^2(t_{j+1})$ rather than the variance and such is the case here. With the help of Eq. A.7 we obtain this expectation as

$$\begin{aligned} E\left[S^2(t_{j+1})\right] &= S_{ji}^2 e^{2(r-q)dt}\left(e^{\sigma^2 dt} - 1\right) + S_{ji}^2 e^{2(r-q)dt} \\ &= S_{ji}^2 e^{2(r-q)dt} e^{\sigma^2 dt} . \end{aligned}$$

Expressed in terms of the probabilities for the trinomial tree, the same expectation is given by

$$E\left[S^2(t_{j+1})\right] = p^+S_{j+1,i+1}^2 + p^0S_{j+1,i}^2 + p^-S_{j+1,i-1}^2 .$$

Combining the two above expressions gives

$$S_{ji}^2 e^{2(r-q)dt} e^{\sigma^2 dt} = p^+S_{j+1,i+1}^2 + p^0S_{j+1,i}^2 + p^-S_{j+1,i-1}^2 . \tag{9.35}$$

Equations 9.33, 9.34 and 9.35 are sufficient to fit the trinomial tree to the lognormal distribution. Of course, only three of the six parameters p^+, p^0, p^-, $S_{j+1,i+1}$, $S_{j+1,i}$ and $S_{j+1,i-1}$ will be determined. In general, these will be the probabilities. The nodes can then be arbitrarily selected.

The actual option pricing now proceeds as for a binomial tree. V_{ji} denotes the price of the option at the ith node of the jth time step. We assume that the tree consists of N time steps with $j = 0, 1, \ldots, N$. To price a European option, the nodes are initialized with the payoff profile of the option at maturity $t_N = T$. In the case of a call option we have:

$$V_{Ni} = \max(S_{Ni} - X, 0) \ .$$

The calculation then *rolls backwards* through the tree. The option value is calculated iteratively for a time step using the values just calculated at the next time step starting with $j = N - 1$ and working back to $j = 0$:

$$V_{ji} = B(t_j, t_{j+1}) \left[p^+ V_{j+1,i+1} + p^0 V_{j+1,i} + p^- V_{j+1,i-1} \right] \ .$$

V_{00} is the present value of the option at time $t_0 = t$ (assuming that S_{00} is the price of the underlying at $t = t_0$). It should be emphasized that the model admits both time-dependent interest rates and volatilities. To take this into consideration either the nodes need to be selected accordingly or the probabilities must be made time-dependent. American options are treated in the same manner as they are treated in binomial trees. Barrier options should be calculated by choosing the nodes such that they lie *directly* on the barrier.

9.5.1 The Trinomial Tree as an Improved Binomial Tree

After *two* time steps a recombining binomial tree has exactly three distinct nodes. This is equal to the number of nodes in the trinomial tree after *one* step. Since the nodes of the trinomial tree can be freely chosen, it is possible to generate a *tri*nomial tree (with an even number of time steps) corresponding to any given recombining *bi*nomial tree. Such a trinomial tree yields the exact same results as the binomial tree, but in only half the time steps. This will be demonstrated for the binomial tree with parameters u, d, and p as given in Eq. 9.31 serving as an example. Starting from the node S_{ij}, we can choose the nodes at time $t_{j+1} = t_j + 2\,dt$ as follows:

$$S_{j+1,i+1} = u^2 S_{ij}$$
$$S_{j+1,i} = S_{ij}$$
$$S_{j+1,i-1} = d^2 S_{ij} \ .$$

where dt is the length of one time step in the binomial tree and hence $2\,dt$ is the length of one time step in the trinomial tree. The probabilities for the trinomial tree are easily obtained from the probability p in the binomial tree:

$$p^+ = p^2 \,, \quad p^0 = 2p(1-p) \,, \quad p^- = (1-p)^2 \,.$$

The values for the probabilities are consistent with those in Eqs. 9.33, 9.34 and 9.35.

The trinomial tree converges faster than the corresponding binomial tree because it requires only half as many steps. Approximately half of the nodes in the binomial tree need not be computed. This advantage is, however not quite as great as it may seem at first glance. It is known that the results of the binomial tree oscillate strongly when the number of time steps increases by one. The best results are obtained by averaging two calculations with N and $N + 1$ time steps (which doubles the required computation time). This trick cannot be exploited when using trinomial trees. Moreover, the parameters specified above are not an optimal choice for the trinomial tree.