8

Integral Forms and Analytic Solutions in the Black-Scholes World

In addition to Assumptions 1, 2, 3, 4, 5 and 6 from Chap. 4 required to set up the differential equation in Chap. 7, we will now further simplify our model by assuming that the parameters involved (interest rates, dividend yields, volatility) are constant (Assumptions 9, 11 and thus 7 from Chap. 4) despite the fact that these assumptions are quite unrealistic. These were the assumptions for which Fischer Black and Myron Scholes derived their famous analytic expression for the price of a plain vanilla option, the Black-Scholes option pricing formula.[1]

For this reason, we often speak of the *Black-Scholes world* when working with these assumptions. In the Black-Scholes world, solutions of the Black-Scholes differential equation (i.e., option prices) for some payoff profiles (for example for plain vanilla calls and puts) can be given in closed form. We will now present two elegant methods to derive such closed form solutions.

[1]The mathematician Louis Bachelier was actually the first to derive analytical expressions for the valuation of options in 1900 [7]. However, Bachelier's derivation is based on other assumptions and his work has been forgotten for a long time. Only through the work of Black and Scholes and nonetheless because of the availability of computers, the use of mathematical formulas and methods has become a market standard for the valuation of derivatives.

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8.1 Option Prices as Solutions of the Heat Equation

The first and perhaps most natural approach would be to take advantage of the constant parameter assumptions to transform the Black-Scholes equation into the heat equation 7.22 as presented above. Since the solution to the heat equation is known and given by Eq. 7.21 we simply need to write the initial condition corresponding to the desired financial instrument in terms of the variables x and τ, and, in accordance with Eq. 7.23, transform the solution u back in terms of the original financial variables. We now demonstrate this technique using a plain vanilla call as an example.

Expressing the payoff profile of the call in the variables of the heat equation gives

$$\begin{aligned} & P(S) = \max(S-K, 0) \\ \Longrightarrow \quad & u_0(x) = P(Ke^x) = \max(Ke^x - K, 0) = K\max(e^x - 1, 0)\ . \end{aligned}$$

Substituting this into Eq. 7.21 immediately yields the solution in integral form for this initial condition

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4\tau}\, u_0(y)\, dy\ . \qquad (8.1)$$

This *integral form* is not only valid for plain vanilla calls but for arbitrary European payoff profiles and the resulting initial conditions u_0. The integral can be computed numerically using, for example, the Monte Carlo method. Decades of research on numerical methods for computing integrals can be taken advantage of here.

In the case of the plain vanilla call, however, it is in fact possible to obtain a closed *analytical* form of the solution of the above integral. Substituting the initial condition for the call into the above equation yields

$$\begin{aligned} u(x,\tau) &= \frac{K}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} e^{-(x-y)^2/4\tau} \max(e^y - 1, 0)\, dy \\ &= \frac{K}{2\sqrt{\pi\tau}} \int_{0}^{\infty} e^{-(x-y)^2/4\tau} (e^y - 1)\, dy \\ &= \frac{K}{2\sqrt{\pi\tau}} \int_{0}^{\infty} e^{y-(x-y)^2/4\tau}\, dy - \frac{K}{2\sqrt{\pi\tau}} \int_{0}^{\infty} e^{-(x-y)^2/4\tau}\, dy\ . \end{aligned}$$

The second of the two integrals above can be computed after making the change in variable $\widetilde{y} \equiv (x-y)/\sqrt{2\tau}$ implying $dy = -\sqrt{2\tau}d\widetilde{y}$. The integral bounds must then be transformed as follows: $0 \to x/\sqrt{2\tau}$ and $\infty \to -\infty$. The integral has now become an integral over the density function of the standard normal distribution:

$$K\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x/\sqrt{2\tau}} e^{-\widetilde{y}^2/2}\, d\widetilde{y} = K\,\mathrm{N}\left(\frac{x}{\sqrt{2\tau}}\right)\,,$$

where $\mathrm{N}(x)$ denotes the cumulative normal distribution given by Eq. A.49. The first of the above two integrals can be calculated by completing the squares as follows

$$y - \frac{(x-y)^2}{4\tau} = \tau + x - \frac{(y-x-2\tau)^2}{4\tau}$$

This transforms the integral into

$$\frac{K}{\sqrt{2\pi}\sqrt{2\tau}}e^{\tau+x}\int_0^{\infty} e^{-\frac{(y-x-2\tau)^2}{2\sqrt{2\tau}^2}}\, dy$$

The remaining integral can now again be expressed in terms of the standard normal distribution. The necessary change in variable is $\widetilde{y} \equiv -(y-x-2\tau)/\sqrt{2\tau}$. Combining all these results the solution becomes

$$u(x,\tau) = K\,e^{x+\tau}\,\mathrm{N}\left(\frac{x}{\sqrt{2\tau}} + \sqrt{2\tau}\right) - K\,\mathrm{N}\left(\frac{x}{\sqrt{2\tau}}\right)\,.$$

Substituting now for the original variables using Eq. 7.23 gives

$$\begin{aligned} e^{r(T-t)}V(S,t) = K\, & e^{\ln(\frac{S}{K})+(r-q)(T-t)} \\ & \times \mathrm{N}\left(\frac{\ln(\frac{S}{K}) + (r-q-\frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}} + \sqrt{\sigma^2(T-t)}\right) \\ & - K\,\mathrm{N}\left(\frac{\ln(\frac{S}{K}) + (r-q-\frac{\sigma^2}{2})(T-t)}{\sqrt{\sigma^2(T-t)}}\right)\,. \end{aligned}$$

After multiplying by $e^{-r(T-t)}$, we finally obtain the famous Black-Scholes formula for the price of a European call:

$$V(S,t) = e^{-q(T-t)} S\,\mathrm{N}(x) - e^{-r(T-t)} K\ \mathrm{N}(x - \sigma\sqrt{(T-t)}) \tag{8.2}$$

where

$$x \equiv \frac{\ln(\frac{S}{K}) + (r-q)(T-t)}{\sqrt{\sigma^2(T-t)}} + \frac{1}{2}\sigma\sqrt{(T-t)}\ .$$

This formula is still generally valid, if interest rate and volatility are time-dependent, but deterministic. In this case, the constant interest rate and the constant volatility just need to be replaced by their average values $\tilde{r}$ and $\tilde{\sigma}$ with

$$\tilde{r} = \frac{1}{T-t}\int_t^T r(s)\,ds \qquad \text{and}$$

$$\tilde{\sigma} = \sqrt{\frac{1}{T-t}\int_t^T \sigma^2(s)\,ds}\ .$$

8.2 Option Prices and Transition Probabilities

We will now show how the foundations of stochastic analysis laid in Sect. 2.4 can be used to price options. In Sect. 7.2, we have seen that with the risk-neutral choice of drift, the prices of derivatives are given by the discounted expectation of the payoff profile, Eq. 7.16. This expectation is determined using the transition probabilities $p(S', t' \,|S, t)$. *If* these are known, calculating the price of the option reduces to simply calculating the integral. In the Black-Scholes world, i.e., for the simple process 2.23, the transition probabilities are given explicitly by Eq. 2.38 with $\widetilde{\mu} = \mu + \sigma^2/2$. Thus, Eq. 7.16 becomes the integral form for the price of an arbitrary derivative with an associated payoff profile $f(S, T)$:

$$\begin{aligned} V(S,t,T) &= B(t,T)\int_{-\infty}^{\infty} f(S',T)p(S',T\,|S,t)dS' \\ &= \frac{B(t,T)}{\sqrt{2\pi\sigma^2(T-t)}}\int_{-\infty}^{\infty} f(S',T)\exp\left\{\frac{-\left[\ln(S'/S)-\mu(T-t)\right]^2}{2\sigma^2(T-t)}\right\}\frac{dS'}{S'}\ . \end{aligned} \tag{8.3}$$

This integral can be computed numerically for arbitrary payoff profiles $f(S, T)$ and is equal to the price of the derivative for the risk-neutral choice of the drift as specified in Eq. 7.19.

For some special payoff profiles, the integral can even be solved analytically, or reduced to an expression in terms of known functions. We demonstrate this using the concrete example of a plain vanilla call option with payoff profile $f(S', T) = \max(S' - K, 0)$. For this payoff profile, the integral can be written as

$$V(S,t,T) = \frac{B(t,T)}{\sqrt{2\pi\sigma^2(T-t)}} \int_K^\infty (S' - K)\exp\left\{\frac{-\left[\ln(S'/S) - \mu(T-t)\right]^2}{2\sigma^2(T-t)}\right\} \frac{dS'}{S'} .$$

The substitution $u := \ln(S'/S)$ simplifies the integral to

$$\begin{aligned} V(S,t,T) &= \frac{B(t,T)}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\ln(K/S)}^\infty (Se^u - K)\exp\left\{\frac{-[u - \mu(T-t)]^2}{2\sigma^2(T-t)}\right\} du \qquad (8.4) \\ &=: B(t,T)SI_1 - B(t,T)KI_2 . \end{aligned}$$

Both integrals I_1 and I_2 can be easily calculated. In the first integral we complete the square in the argument of the exp-function:

$$\begin{aligned} u - \frac{[u-\mu(T-t)]^2}{2\sigma^2(T-t)} &= -\frac{u^2 - 2\mu(T-t)u + \mu^2(T-t)^2 - 2\sigma^2(T-t)u}{2\sigma^2(T-t)} \\ &= -\frac{\left[u - \left(\mu+\sigma^2\right)(T-t)\right]^2 - \left(\mu+\sigma^2\right)^2(T-t)^2 + \mu^2(T-t)^2}{2\sigma^2(T-t)} \\ &= -\frac{\left[u - \left(\mu+\sigma^2\right)(T-t)\right]^2 - \left(\sigma^4 + 2\mu\sigma^2\right)(T-t)^2}{2\sigma^2(T-t)} \\ &= \frac{-\left[u - \left(\mu+\sigma^2\right)(T-t)\right]^2}{2\sigma^2(T-t)} + \left(\mu + \frac{\sigma^2}{2}\right)(T-t) . \end{aligned}$$

Thus, the first integral becomes

$$\begin{aligned} I_1 &\equiv \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\ln(K/S)}^{\infty} \exp\left\{ u - \frac{[u-\mu(T-t)]^2}{2\sigma^2(T-t)} \right\} du \\ &= \frac{e^{\left(\mu+\sigma^2/2\right)(T-t)}}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\ln(K/S)}^{\infty} \exp\left\{ \frac{-\left[u-\left(\mu+\sigma^2\right)(T-t)\right]^2}{2\sigma^2(T-t)} \right\} du \ . \end{aligned}$$

With the substitution

$$\begin{aligned} y &:= -\frac{u-\left(\mu+\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \Rightarrow \\ \frac{dy}{du} &= -1/\sqrt{\sigma^2(T-t)} \Rightarrow du = -\sqrt{\sigma^2(T-t)}dy \end{aligned}$$

the upper and lower limits of integration become

$$\begin{aligned} y_{\text{upper}} &= -\frac{\infty-\left(\mu+\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} = -\infty \\ y_{\text{lower}} &= -\frac{\ln(K/S)-\left(\mu+\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} = \frac{\ln(S/K)+\left(\mu+\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \ . \end{aligned}$$

Exchanging the upper and lower limits results in a change in the sign of the integral. This is compensated for by the sign of du. Combining the above results, I_1 becomes after this substitution

$$\begin{aligned} I_1 &= e^{\left(\mu+\sigma^2/2\right)(T-t)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(S/K)+(\mu+\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}} \exp\left\{ \frac{-y^2}{2} \right\} dy \\ &= e^{\left(\mu+\sigma^2/2\right)(T-t)} \mathrm{N}\left(\frac{\ln(S/K)+\left(\mu+\sigma^2\right)(T-t)}{\sqrt{\sigma^2(T-t)}} \right) , \end{aligned}$$

where, as usual, N denotes the cumulative standard normal distribution.

The second integral can be computed after making the substitution $y := -(u - \mu(T-t))/\sqrt{\sigma^2(T-t)}$:

$$
\begin{aligned}
I_2 &\equiv \frac{1}{\sqrt{2\pi\sigma^2(T-t)}} \int_{\ln(K/S)}^{\infty} \exp\left\{\frac{1}{2}\frac{-[u-\mu(T-t)]^2}{\sigma^2(T-t)}\right\} du \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{\ln(S/K)+\mu(T-t)}{\sqrt{\sigma^2(T-t)}}} \exp\left\{-\frac{y^2}{2}\right\} dy \\
&= \mathrm{N}\left(\frac{\ln(S/K)+\mu(T-t)}{\sqrt{\sigma^2(T-t)}}\right)
\end{aligned}
$$

The generalization of Eq. 7.19 for the risk-neutral choice of drift in arbitrary compounding methods is (see Eq. 9.25)

$$
\mu(T-t) := \ln\left(\frac{B_q(t,T)}{B(t,T)}\right) - \frac{\sigma^2}{2}(T-t)\}
$$

with $B_q(t,T) = \exp(-q(T-t))$. This simplifies the integrals further to

$$
I_1 = \frac{B_q(t,T)}{B(t,T)}\mathrm{N}(x) \quad I_2 = \mathrm{N}\left(x - \sigma\sqrt{T-t}\right) .
$$

where x is, as usual, given by

$$
x = \frac{\ln(\frac{B_q(t,T)S}{B(t,T)K}) + \frac{1}{2}\sigma^2(T-t)}{\sqrt{\sigma^2(T-t)}} . \tag{8.5}
$$

Collecting these results, we obtain the price of a plain vanilla call as

$$
\begin{aligned}
V(S,t,T) &= B(t,T)SI_1 - B(t,T)KI_2 \\
&= B_q(t,T)S(t)\,\mathrm{N}(x) - B(t,T)K\,\mathrm{N}\left(x - \sigma\sqrt{T-t}\right) .
\end{aligned} \tag{8.6}
$$

Again, this is the famous *Black-Scholes option pricing formula* and corresponds exactly to Eq. 8.2 for continuous compounding.

8.3 Compilation of Black-Scholes Option Prices for Different Underlyings

8.3.1 Options on the Spot Price

The above derivation holds for a call on an underlying with a continuous compounding dividend *yield*. However, in reality, dividends are paid as discrete amount at only a few days per year, often once a year only. Such a discrete dividend payment could either be modeled as a fixed absolute amount (absolute discrete dividend) or as an amount relative to the spot price at the ex-dividend date (relative discrete dividend). A simple approach for taking discrete dividends into account is to adjust the spot price of the underlying by subtracting the value of the dividend for the considered time period according to Eq. 2.9. The adjusted spot price can be modeled like an underlying that does not pay any dividends. The payoff profile and the Black-Scholes value are summarized here for puts and calls. The payoff profiles at time T are:

$$
\begin{aligned}
c_S(T,T,K) &= \max\{0, S(T)-K\}\\
p_S(T,T,K) &= \max\{0, K-S(T)\}\ .
\end{aligned}
$$

The Black-Scholes option prices at time t are:

$$
\begin{aligned}
c_S(t,T,K) &= \quad \tilde{S}(t,T)\mathrm{N}(\ x) - K\,B(t,T)\mathrm{N}(\ x-\sigma\sqrt{T-t})\\
p_S(t,T,K) &= -\tilde{S}(t,T)\mathrm{N}(-x) + K\,B(t,T)\mathrm{N}(-x+\sigma\sqrt{T-t})
\end{aligned}
\tag{8.7}
$$

where

$$
x = \frac{\ln\left(\frac{\tilde{S}(t,T)}{K\,B(t,T)}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} = \frac{\ln\left(\frac{S(t,T)}{K}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\ .
$$

The application of this famous *Black-Scholes option pricing formula* to option portfolios is demonstrated in detail in the Excel workbook BlackScholesModel.xls from the download section [50]. In this workbook, the derivatives of the option price with respect to its parameters, called the *Greeks*, are also computed in anticipation of Chap. 12. The workbook can be used as a complete option calculator (the fields colored yellow are the input fields).

8.3.2 Options on the Forward Price

The underlying is now not the spot price $S(t)$ but the forward price $S(t, T')$ for a time $T' \geq T$, where T is the maturity date of the option. Options on the forward price refer to either futures or forwards. If physical settlement was agreed, exercising the option yields in both cases to the payment of the value of the underlying forward contract. In addition, the exerciser of the option goes long (in the case of a call) or short (in the case of a put) in the forward contract concerned.[2] Since the value of a futures and forward position according to Eqs. 6.6 and 6.5, respectively, are different, the payoff profile and thus the value of options on these contracts are different as well.

Options on Futures

Upon maturity at time T of the option, the value of the future with a maturity date $T' \geq T$ is paid if this value is positive. The payoff profiles are

$$\begin{aligned} c_F(T, T, K) &= \max\left\{0, F_S(T, T', K)\right\} = \max\left\{0, S(T, T') - K\right\} \\ p_F(T, T, K) &= \max\left\{0, -F_S(T, T', K)\right\} = \max\left\{0, K - S(T, T')\right\} \ . \end{aligned}$$

A method often used to find the Black-Scholes price is to transform the payoff profile into a payoff profile of a known option. For this reason, we write the payoff profile of the call as

$$c_F(T, T, K) = \frac{B_q(T, T')}{B(T, T')} \max\left\{0, S(T) - \frac{B(T, T')}{B_q(T, T')} K\right\} \ .$$

where we have used Eq. 6.1 for the forward price at time T (for the case of a dividend yield q). Thus, the price of a call on a future with strike price K can be written as the price of B_q/B calls on the spot price with strike price KB/B_q. The argument for the put is completely analogous. A call on a *future* with strike price K thus has the same payoff profile as B_q/B calls on the *spot* with strike price KB/B_q. Thanks to of Eq. 8.7, the price of an option on

[2] But of course with the then valid forward price as the delivery price so that the forward contract—as always—has zero value when entered into.

the spot price is known. Substituting KB/B_q for the strike price and using Eq. 2.7 for B and B_q yields:

$$\begin{aligned} c_S(t,T,K) &= \frac{B_q(T,T')}{B(T,T')}\left[B_q(t,T)S(t)\mathrm{N}(x')\right. \\ &\quad \left. - B(t,T)K\frac{B(T,T')}{B_q(T,T')}\mathrm{N}(x'-\sigma\sqrt{T-t})\right] \\ &= \frac{B_q(t,T')}{B(T,T')}S(t)\mathrm{N}(x') - B(t,T)K\mathrm{N}(x'-\sigma\sqrt{T-t})\ . \end{aligned}$$

Here, x' corresponds to the x in Eq. 8.7 with the modified strike price:

$$\begin{aligned} x' &= \frac{\ln\left(\frac{B_q(t,T)S(t)}{B(t,T)\left[K\,B(T,T')/B_q(T,T')\right]}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t} \\ &= \frac{\ln\left(\frac{B_q(t,T')S(t)}{B(t,T')K}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\ . \end{aligned}$$

Now, again with the help of Eq. 6.1, the spot price $S(t)$ is written in terms of the actual underlying, namely the forward price. Using the expression for B in Eq. 2.7 finally gives the Black-Scholes price for options on futures:

$$\begin{aligned} c_F(t,T,K) &= B(t,T)\left[S(t,T')\mathrm{N}(x') - K\,\mathrm{N}(x'-\sigma\sqrt{T-t})\right] \\ p_F(t,T,K) &= B(t,T)\left[-S(t,T')\mathrm{N}(-x') + K\,\mathrm{N}(-x'+\sigma\sqrt{T-t})\right] \end{aligned} \tag{8.8}$$

where

$$x' = \frac{\ln\left(\frac{S(t,T')}{K}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\ .$$

Comparing this with the corresponding prices for options on the spot prices, Eq. 8.7, for the special case of an underlying whose dividend yield is exactly equal to the risk-free interest rate, in other words where (from Eq. 2.9) $\tilde{S}(t, T) = B(t, T)S(t)$, we obtain the following "cookbook" recipe:

> Plain vanilla option on futures can be priced like options on the spot price of an (artificial) underlying whose spot price is equal to $S(t, T')$ and whose dividend yield is equal to the risk free rate r.

If the future matures at the same date as the option, i.e., if $T' = T$, then there is (because of Eq. 6.1) no difference in either the payoff profile or the price of the option on the future and the option on the spot price. In this case Eq. 8.8 (i.e., pricing options using the forward price, even if it is an option on the spot price) is referred to as the *Black-76 model*.

In summary, if either $T = T'$ or $q = r$, there is no difference in the option on a futures contract and the option on a spot price.

Options on Forwards

On the maturity date T of the option, the value of the forward maturing on $T' \geq T$ will be paid to the holder of a call if this value is positive. This is different from the value of the future since in the case of a forward, the value is discounted from maturity T' back to T (see Eq. 6.5). The payoff profiles are thus:

$$
\begin{aligned}
c_f(T, T, K) &= \max\left\{0, f_S(T, T', K)\right\} = B(T, T') \max\left\{0, S(T, T') - K\right\} \\
p_f(T, T, K) &= \max\left\{0, -f_S(T, T', K)\right\} = B(T, T') \max\left\{0, K - S(T, T')\right\} .
\end{aligned}
$$

Comparing this with the payoff profiles for options on futures shows that an option on a forward corresponds to B options on the future. Therefore the Black-Scholes prices can be immediately obtained from Eq. 8.8

$$
\begin{aligned}
c_f(t, T, K) &= B(t, T') \left[S(t, T')\mathrm{N}(x') - K\,\mathrm{N}(x' - \sigma\sqrt{T-t})\right] \\
p_f(t, T, K) &= B(t, T') \left[-S(t, T')\mathrm{N}(-x') + K\,\mathrm{N}(-x' + \sigma\sqrt{T-t})\right]
\end{aligned}
\tag{8.9}
$$

with $x\prime$ defined as in Eq. 8.8. The difference between this and Eq. 8.8 is that for options on *forwards* the discounting is done from the maturity of the *forward* contract T', whereas for options on *futures* the discounting is done from the maturity of the *option* T.

If the forward matures on the same date as the option, $T' = T$, there is no difference in the payoff profile or in the Black-Scholes price between an option on the forward and an option on the spot price. In this case, the prices of an option on the spot, an option on a future and an option on a forward are all equal.

8.3.3 Options on Interest Rates

Forward Volatilities

In the derivation of the Black-Scholes equation for options on the forward price, it has been assumed that volatility remained constant throughout. Therefore, in Eq. 8.8 and Eq. 8.9 the volatility of the spot price is used, though the underlying is the forward price. That is because the volatility of the forward price and the spot price are the same if the volatility is constant. This model is commonly referred to as the *Black-76 model*.

The Black-76 model is commonly used especially when the underlying S is an interest rate or an interest rate instrument (like a bond, for instance). It can be shown that the Black-76 model holds even when the Black-Scholes assumptions are weakened somewhat. The underlying process must not necessarily be a random walk with constant volatility. It is sufficient that the logarithm of the underlying $S(T)$ at option maturity is normally distributed. The variance of the distribution of $\ln(S(T))$ will be written as

$$\text{var}\,[\ln S(T)] = \sigma(T)^2 T$$

The parameter $\sigma(T)$ is called the *forward volatility*. It is the volatility of the underlying price $S(T)$ at maturity T. Because the Black-76 model "lives" in the Black-Scholes world, interest rates are assumed to be non-stochastic. Therefore the forward price $S(t, T)$ and the future price $S(T)$ are equal if the underlying S is an interest rate or an interest rate instrument.[3] At an earlier time $t < T$ we can thus use the forward price $S(t, T)$ for $S(T)$ and the current volatility *of this forward price* for the forward volatility.

[3]This will be shown explicitly in Sect. 14.3.

Normally Versus Log Normally Distributed Interest Rates

The Black-76 model is applied to options on interest rate instruments such as bonds as well as to options which depend directly on interest rates, such as caps and floors. For $S(t, T)$ we take either the forward price of the underlying instrument (for example, of the bond) or the forward rate of the reference interest rate (3-month LIBOR rate in 6 months, for example). In both cases the application of the Black-76 model implicitly assumes that each underlying (bond price or interest rate) is lognormally distributed. However, both assumptions contradict each other. Yields are the logarithms of relative price changes. If, in the case of a *bond* option, it is assumed that these logarithms are normally distributed, the interest rates (=yields) cannot simultaneously be *log*normally distributed. This is, however, common market practice: bond options are priced under the assumption that bond prices are lognormally distributed, i.e., that interest rates are normally distributed. On the other hand, caps, floors and collars are priced under the assumption that interest rates are lognormally distributed (similarly, swaptions are priced under the assumption of lognormal swap rates).

Though, this inconsistency does not imply that the option prices are also inconsistent, since the used volatilities are different (bond price volatility vs. interest rate volatility). As will be shown in Sect. 30.3.3, price volatility and yield volatility are related to one another through the modified duration, see Eq. 30.18 (in linear approximation).

Now the question arises, which of both assumptions is indeed correct? Actually, both assumptions are neither completely wrong nor fully true. Empirical analysis of interest rate time series shows that the distribution of interest rates depends on the interest rate level (e.g., see [161]). For low or very high interest rate levels (i.e. lower than 1.2% or greater than 5.6%),[4] the distribution of interest rate is lognormal, while within these boundaries the distribution is normal. Common interest rate models, modeling the time evolution of the full interest rate term structure, with consistent modeling of e.g. Caps as well as bond options, usually assume either normally or lognormally distributed interest rates. For a more detailed discussion of the pros and cons of various models see Chap. 14.

[4]However, own studies conclude that it is not possible with statistical significance to discriminate between the lognormally and normally distributed interest rate assumption; at least with a advent of negative interest rates the (unmodified) lognormally distribution assumptions has to be ruled out for low interest rate levels.

Here, it should be enough to note that simple interest rate options are prices by means of Black-75 and that the according underlying is assumed to be lognormally distributed. In Case of Caps, Floors and Collars, the underlying is a forward interest rate, in the case of bond options it is the bond price and in the case of swaptions the (forward) swap rate (the assumption of lognormally distributed swap rates, which is a weighted sum of forward rates, is inconsistent with the assumption of lognormally distributed forward rates as well as with the assumption of normally distributed forward rates).

The Black-76 formula could best be understood as a vehicle to price option in an intuitively, simple form (by expressing the price in terms of the Black-76 volatility). Then, the inconsistent assumptions do not cause any trouble, if the markets are kept thoroughly separated and not mixed up.

With negative interest rates in some markets, the Black-76 is increasingly replaced by normal models or shifted log normal models (i.e. assuming that rates plus an offset are lognormally distributed).

Black-76 Model for Interest Rate Options

With the above interpretations of the input parameters and under the assumptions described above, Eq. 8.8 yields the Black-76 model for interest rate options. Explicitly:

$$
\begin{aligned}
c(t,T,K) &= B(t,T)\left[\;\; S(t,T)\mathrm{N}(\;\;x) - K\,\mathrm{N}(\;\;x - \sigma\sqrt{T-t})\right] \\
p(t,T,K) &= B(t,T)\left[-S(t,T)\mathrm{N}(-x) + K\,\mathrm{N}(-x + \sigma\sqrt{T-t})\right]\,,
\end{aligned}
\tag{8.10}
$$

where

$$
x = \frac{\ln\left(\frac{S(t,T)}{K}\right)}{\sigma\sqrt{T-t}} + \frac{1}{2}\sigma\sqrt{T-t}\,.
$$

Here, $S(t,T)$ is either the forward rate of the underlying interest rate or forward price of an underlying interest rate *instrument* (like a bond, for example). This equation forms the basis for pricing interest rate option *in the Black-Scholes world*. Formally, the difference between this and Eq. 8.8 is that the forward price with respect to the option's maturity is used, i.e., $T = T'$. As mentioned after Eq. 8.8, there is no difference in this case between an option on a forward price and an option on a spot price. We could just as well work with Eq. 8.7. The only subtlety involved is that the *forward volatility* or the volatility of the forward rate should be used.