# 28

# Active Management and Benchmarking

## 28.1 The Capital Asset Pricing Model

One of the most important results of the previous chapter is the last line in Eq. 27.31. It can be written in the form

$$\widehat{\mathbf{R}} = \boldsymbol{\beta}_m \widehat{R}_m \ . \qquad (28.1)$$

Based on this equation, the expected excess returns of all assets are *explained* solely by their betas with respect to the optimal portfolio and the return of the optimal portfolio itself. This is very remarkable. Within the framework of classical Mean-Variance portfolio theory there is not much freedom for the individual assets. The optimal portfolio drives everything. The question now remains, what this optimal portfolio exactly is. The following argument leads to an identification[1]:

If all investors have the same information on all assets (in particular concerning expected returns, variances and covariances) and if all investors behave optimally (in the sense that they invest in such a way that their mean/variance-ratio is maximal), then every investment will lie on the capital market line. If Inequality 26.59 holds, than each investor will invest in a mixture of the optimal portfolio $V_m$ and the risk free money market account. Investments with a variance lower than $\sigma_m$ are long in the risk free investment while investments with a variance higher than $\sigma_m$ are leveraged, i.e., they

[1]Under the assumption that there are no transaction costs, taxes or other "friction".



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borrow from the money market. Adding up all investments of all market participants, the total amount borrowed must of course equal the total amount of lending, i.e., the grand total money market position of all market participants must equal zero. Therefore, the grand total of all investments of all market participants is the optimal portfolio. On the other hand, the grand total of all investments of all market participants is "the market" itself. Thus, **the optimal portfolio is the market itself**, which is why it is also called the *market portfolio*.

One can also look at this from a different angle: you can always do a linear regression for the (historical) asset excess returns $\widehat{r}_i$ on (historical) market excess returns, i.e.

$$\widehat{r}_i = \alpha_{i,m} + \beta_{i,m}\widehat{r}_m + \epsilon_i \text{ for } i = 1, \ldots, M \ , \tag{28.2}$$

where the $\epsilon_i$ are the error terms which by definition have zero expectation and are uncorrelated with $\widehat{r}_m$. Calculating expected values and comparing this with Eq. 28.1, we immediately see that the CAPM requires the *alpha* of each asset to be zero. In the CAPM, no additional return is expected for any asset above that produced by the asset's beta and the market!

The same holds for any *portfolio* of assets, since in the regression everything is linear.

$$\underbrace{\sum_i w_i\widehat{r}_i}_{\widehat{r}_V} = \underbrace{\sum_i w_i\alpha_{i,m}}_{\alpha_{V,m}} + \underbrace{\sum_i w_i\beta_{i,m}\widehat{r}_m}_{\beta_{V,m}} + \underbrace{\sum_i w_i\epsilon_i}_{\epsilon_V} \ ,$$

$$\widehat{r}_V = \alpha_{V,m} + \beta_{V,m}\widehat{r}_m + \epsilon_V \ . \tag{28.3}$$

Since the $\alpha_{i,m}$ should all be zero, so must $\alpha_{V,m}$. Thus, **in the CAPM, the alpha of any portfolio is zero**. No portfolio can beat the market in the long run. This is not a new insight really, but in fact trivial: We *constructed* the optimal portfolio to have the best possible mean/variance ratio. Then we argued that the optimal portfolio should have the same position weights as the market. Thus, the market has the best possible mean/variance-ratio. Therefore, no other portfolio has a better mean/variance ratio because if it did we would have taken this portfolio as the optimal portfolio.

The "market" contains everything one could possibly invest in: equity, debt, commodities, real estate, etc., even art. Most often, this is not a feasible concept for practical situations, so instead, "the market" is divided into several (sub-)markets like the German Bond Market, the US Mid Cap Market or the Japanese Real Estate Market, to name a few of many hundreds or even thousands. But even these smaller markets are usually too large to handle, so *Indices* are defined as representatives of those markets. Examples of such Indices are the S&P 500, the German DAX, the Japanese Nikkei, etc. They contain a limited number of assets (with weights defined and calculated by the respective index provider) and serve as a proxy of the market they are supposed to represent. Indices are often used as *benchmarks* to measure the performance of a portfolio. In the following we will use the subscript $B$ for the benchmark. The benchmark portfolio $V_B$ is supposed to be a proxy for the market portfolio $V_m$ (of whatever market we are interested in), but it is usually *not* identical to the market portfolio. In the following, we will carefully distinguish between benchmark and market portfolio properties using subscripts $B$ and $m$, respectively.

## 28.2 Theory of Efficient Markets

At the heart of *active portfolio management* lies the hope of a portfolio manager that he can outperform the market, despite the fact that the CAPM says this is impossible. That does not necessarily mean, however, that the portfolio manager does not believe in the CAPM. He rather doubts the assumptions of the CAPM, in particular the one that everybody has the same information. The information edge the manager may believe to have on some assets causes him to cast asset return expectations that differ from the "consensus returns" of Eq. 28.1. This then results in different position weights, even if the manager uses the same mean/variance optimization as in the CAPM. If his return estimates turn out right he will indeed outperform the market. However, efficient market theory (at least in its strong form) states that this is still not possible in the long run. *Efficient market theory* comes in three varieties:

- The *weak* efficient market theory states: Investors cannot outperform the market using historical price and volume data, only.
- The *semi strong* efficient market theory states: Investors cannot outperform the market using publicly available information, only (i.e., historical price and volume data, fundamental data, recommendations published by analysts, etc.)
- The *strong* efficient market theory states: Investors can never outperform the market. Market prices contain all relevant information.

A couple of observations support the view that at most the weak efficient market theory is in fact valid in the real world. For example, the long-term success of some value investors apparently excludes the semi-strong and strong theory. Also, not all market participants have equal access to all available informations, otherwise insider trading and manipulation of market indices (like in the Libor scandal) wouldn't be possible. In addition, we know from behavioural finance that even professional asset managers oft act quite irrational. Finally, institutional investors are bound to obey regulatory requirements, which sometimes stands in the way of rational decisions.

## 28.3 Benchmarking Against an Index

Nonetheless the main goal of so-called *active* asset management is "to beat the market", i.e., to gain a positive alpha against a benchmark (in most cases an index) which represents the market to be beaten. To assess how well an asset has been doing compared to the *benchmark* one can do a regression of the historical asset returns against the historical benchmark returns as in Eq. 28.2.

$$\widehat{r_i} = \alpha_{i,B} + \beta_{i,B}\widehat{r}_B + \epsilon_i \text{ for } i = 1, \ldots, M \ .$$

Taking the expectation (e.g., as a historical average) we get the relation between asset and benchmark returns that serves as the definition of $\boldsymbol{\alpha}_B$, the vector of the assets alphas with respect to the benchmark:

$$\widehat{\mathbf{R}} = \boldsymbol{\alpha}_B + \boldsymbol{\beta}_B \widehat{R}_B \ . \tag{28.4}$$

If the benchmark perfectly represented the market and if the CAPM were perfectly true, this would collapse to Eq. 28.1, i.e., the assets alphas would all vanish.

The asset betas with respect to the Benchmark are given by the matrix $\mathbf{C}$ and the position weights $\mathbf{w}_B$ of the benchmark portfolio as in Eq. 26.23.

$$\boldsymbol{\beta}_B = \frac{1}{\sigma_B^2}\mathbf{C}\mathbf{w}_B = \frac{\mathbf{C}\mathbf{w}_B}{\mathbf{w}_B^T\mathbf{C}\mathbf{w}_B} \ . \tag{28.5}$$

Equations 28.4 and 28.5 directly imply[2] that the following combination of $\boldsymbol{\alpha}_B$, $\boldsymbol{\beta}_B$ and $\mathbf{C}$ is be zero:

$$\boldsymbol{\alpha}_B^T \mathbf{C}^{-1} \boldsymbol{\beta}_B = 0 \,. \tag{28.6}$$

The same considerations that lead us to Eq. 28.4 also work for the *portfolio*'s expected excess return

$$\widehat{R}_V = \mathbf{w}^T \widehat{\mathbf{R}} = \underbrace{\mathbf{w}^T \boldsymbol{\alpha}_B}_{\alpha_{V,B}} + \underbrace{\mathbf{w}^T \boldsymbol{\beta}_B}_{\beta_{V,B}} \widehat{R}_B \,. \tag{28.7}$$

Here, the portfolio beta[3] $\beta_{V,B}$ with respect to the benchmark follows from the regression analysis in the usual way[4] given by Eq. A.20:

$$\beta_{V,B} \equiv \frac{\operatorname{cov}(r_V, r_B)}{\operatorname{Var}(r_B)} = \frac{\operatorname{cov}(\widehat{r}_V, \widehat{r}_B)}{\operatorname{Var}(\widehat{r}_B)} \,. \tag{28.8}$$

The portfolio alpha $\alpha_{V,B}$ can also be determined by such an regression of the historical portfolio excess returns $\widehat{r}_V$ against the historical benchmark excess returns $\widehat{r}_B$. In this way one gets an *ex post* estimation of how the portfolio did in the past. The historical portfolio returns are of course not only a result of market movements but also of trading, i.e., position changes done by the portfolio manager. Therefore, for the estimation of the impact of portfolio

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[2]Right-multiplying the transpose of Eq. 28.4 by $\mathbf{C}^{-1}$ yields

$$\widehat{\mathbf{R}}^T \mathbf{C}^{-1} = \boldsymbol{\alpha}_B^T \mathbf{C}^{-1} + \widehat{R}_B \boldsymbol{\beta}_B^T \mathbf{C}^{-1} \,.$$

Right-multiplying this by$\boldsymbol{\beta}_B$ and then inserting Eq. 28.5 results in

$$\begin{aligned}
\boldsymbol{\alpha}_B^T \mathbf{C}^{-1} \boldsymbol{\beta}_B &= \widehat{\mathbf{R}}^T \mathbf{C}^{-1} \boldsymbol{\beta}_B - \widehat{R}_B \boldsymbol{\beta}_B^T \mathbf{C}^{-1} \boldsymbol{\beta}_B \\
&= \frac{1}{\sigma_B^2} \widehat{\mathbf{R}}^T \mathbf{C}^{-1} \mathbf{C} \mathbf{w}_B - \widehat{R}_B \frac{1}{\sigma_B^2} \mathbf{w}_B^T \mathbf{C} \mathbf{C}^{-1} \frac{1}{\sigma_B^2} \mathbf{C} \mathbf{w}_B \\
&= \frac{1}{\sigma_B^2} \underbrace{\mathbf{w}_B^T \widehat{\mathbf{R}}}_{\widehat{R}_B} - \frac{1}{\sigma_B^4} \widehat{R}_B \underbrace{\mathbf{w}_B^T \mathbf{C} \mathbf{w}_B}_{\sigma_B^2} = 0 \,.
\end{aligned}$$

[3]The Beta appearing here is in terms of returns, not in terms of relative price changes as for instance in Eq. 26.23. For Beta this does not make any difference, since the factors $\delta t$ appearing due to Eq. 21.27 are the same in numerator and denominator and therefore cancel out.

[4]Covariances and variances do not change upon adding a constant $r_f$. Therefore, working with returns or with *excess* returns doesn't make any difference.

changes, it is important to get *ex ante* estimates for alpha and beta over the *next* holding period of the *current* portfolio with its *current* holdings. To get these, we take the *historical* information of the *assets* and the *current* position weights in the *portfolio* to produce ex ante estimates of portfolio properties, like for instance alpha and beta. This can be done by using the *historical* alphas and betas of the *assets* with respect to the benchmark together with the *current weights* of those assets in the portfolio:

$$\alpha_{V,B} = \sum_{i=1}^{M} w_i \alpha_{i,B} = \mathbf{w}^T \boldsymbol{\alpha}_B \ , \quad \beta_{V,B} = \sum_{i=1}^{M} w_i \beta_{i,B} = \mathbf{w}^T \boldsymbol{\beta}_B \ . \qquad (28.9)$$

The historical alphas and betas of the assets are determined via the regressions given in Eq. 28.2. One might also use other estimates for the asset alphas and betas in this ex ante estimation of the portfolio alpha and beta (e.g., an asset manager's personal opinion).

In the context of active portfolio management, not only alpha and beta but usually *all* quantities (return, risk, etc.) are defined *relative to the benchmark*: if the portfolio loses money but the benchmark performs even worse, the portfolio manager is still happy in this relative framework (this is not necessarily true for the investor). This is in stark contrast to the previous sections, where we considered the excess return and the risk of the portfolio *itself*, an approach sometimes called *Total Return Management*. Still, all of the above considerations in the Total Return framework can easily be recovered within the benchmark-relative framework by simply defining the benchmark to be the money market account.

There are two different ways commonly used by market participants to define everything relative to the benchmark: using *active* quantities and *residual* quantities. Let's start with the active quantities.

## 28.3.1 Active Portfolio Properties

The *active position* is defined as the deviation of the portfolio weights from the benchmark weights

$$\overline{\mathbf{w}} = \mathbf{w} - \mathbf{w}_B \ . \qquad (28.10)$$

Similarly, the difference between the cash positions of the portfolio and the benchmark is called the *active cash*. From Eq. 26.64, the active cash is simply

$$\text{Active Cash} = \left(1 - \mathbf{w}^T\mathbf{1}\right) - \left(1 - \mathbf{w}_B^T\mathbf{1}\right) = -\overline{\mathbf{w}}^T\mathbf{1}$$

The *active return* is defined as the difference between the portfolio return, Eq. 28.3, and the benchmark return.

$$\overline{r}_V := \overline{\mathbf{w}}^T\widehat{\mathbf{r}} = \widehat{r}_V - \widehat{r}_B = \alpha_{V,B} + (\beta_{V,B} - 1)\widehat{r}_B + \epsilon_V \ .$$

The *expected* active return is defined as the expected return of the portfolio with weights given by the active position

$$\begin{aligned}\overline{R}_V &= \mathrm{E}\left[\overline{r}_V\right] = \overline{\mathbf{w}}^T\widehat{\mathbf{R}} \\ &= \mathbf{w}^T\boldsymbol{\alpha}_B + (\mathbf{w}^T\boldsymbol{\beta}_B - 1)\widehat{R}_B \\ &= \alpha_{V,B} + (\beta_{V,B} - 1)\widehat{R}_B \end{aligned} \tag{28.11}$$

and the *active risk* is defined in terms of the variance of active returns:

$$\overline{\sigma}_V^2 := \delta t \ \mathrm{var}\left[\overline{r}_V\right] = \delta t \ \mathrm{var}\left[\epsilon_V\right] + (\beta_{V,B} - 1)^2\sigma_B^2 \ . \tag{28.12}$$

The last equation holds since the errors $\epsilon_V$ are by construction uncorrelated all other parameters. The term $(\beta_{V,B} - 1)$ appearing here is called *active beta*. The active risk is also called *Tracking Error* since it is a measure of how well the portfolio tracks the benchmark. We can of course also write the tracking error in the form of Eq. 26.17 as the risk of a portfolio with position weights $\overline{\mathbf{w}}$:

$$\overline{\sigma}_V^2 = \delta t \ \mathrm{var}\left[\overline{r}_V\right] = \delta t \ \mathrm{var}\left[\overline{\mathbf{w}}^T\widehat{\mathbf{r}}\right] = \overline{\mathbf{w}}^T\mathbf{C}\overline{\mathbf{w}} \ .$$

The same holds for the expected active return in Eq. 28.11 when compared to the expected portfolio return in Eq. 26.3. Indeed, many concepts introduced in the previous sections carry over to the benchmark-relative framework by simply replacing the weights $\mathbf{w}$ by the active weights $\overline{\mathbf{w}}$. For instance, in analogy to Eq. 26.19 we can define the *marginal active risk* as

$$\frac{\partial\overline{\sigma}_V}{\partial\overline{\mathbf{w}}} = \frac{\mathbf{C}\overline{\mathbf{w}}}{\sqrt{\overline{\mathbf{w}}^T\mathbf{C}\overline{\mathbf{w}}}} = \frac{\mathbf{C}\overline{\mathbf{w}}}{\sqrt{\delta t \ \mathrm{var}\left[\epsilon_V\right] + (\beta_{V,B} - 1)^2\sigma_B^2}}$$

and attribute the active risk to the individual positions as in Eqs. 26.20 or 26.21.

$$\overline{\sigma}_V = \overline{\mathbf{w}}^T \frac{\partial \overline{\sigma}_V}{\partial \overline{\mathbf{w}}} = \sum_{i=1}^{M} \overline{w}_i \frac{\partial \overline{\sigma}_V}{\partial \overline{w}_i} \ .$$

The amount of active risk attributed to the $i$th asset is thus

$$\overline{w}_i \frac{\partial \overline{\sigma}_V}{\partial \overline{w}_i} = \frac{\overline{w}_i^2 \sum_{k=1}^{M} C_{ki}}{\sqrt{\delta t^2 \operatorname{var}[\epsilon_V] + (\beta_{V,B} - 1)^2 \sigma_B^2}} \quad \text{for } i = 1, \ldots, M \ .$$

The percentage of active risk attributed to the $i$th asset is this number divided by $\overline{\sigma}_V$.

### 28.3.2 Residual Portfolio Properties

While the above *active* quantities focus on the differences between the portfolio and the benchmark itself, the *residual* quantities focus on difference between portfolio properties and the properties implied by the CAPM. To be specific, the *residual (excess) return* is defined as the difference between the portfolio excess return and its excess return implied by the CAPM via Eq. 28.1.

$$\widetilde{r}_V := \widehat{r}_V - \beta_{V,B} \widehat{r}_B = \alpha_{V,B} + \epsilon_V \ . \qquad (28.13)$$

Correspondingly, the *residual risk,* is defined as the volatility of the residual return:

$$\widetilde{\sigma}_V^2 := \delta t \operatorname{var}[\widetilde{r}_V] = \delta t \operatorname{var}[\epsilon_V] \ . \qquad (28.14)$$

According to Eq. 28.4, the *expectation* of the residual return is equal to the portfolio's *alpha*.

$$\widetilde{R}_V = \mathrm{E}[\widetilde{r}_V] = \mathbf{w}^T \boldsymbol{\alpha}_B = \alpha_{V,B} \ . \qquad (28.15)$$

The *Information Ratio* is defined as the expected residual (excess) return per residual risk and is therefore the benchmark-relative analogue to the Sharpe

Ratio, Eq. 27.22.

$$\widetilde{\gamma}_V \equiv \frac{\widetilde{R}_V}{\widetilde{\sigma}_V} = \frac{\alpha_{V,B}}{\sqrt{\delta t \,\mathrm{var}\left[\epsilon_V\right]}} \,. \tag{28.16}$$

The residual position weights producing the above residual return and residual risk are

$$\widetilde{\mathbf{w}} = \mathbf{w} - \beta_{V,B}\mathbf{w}_B \,. \tag{28.17}$$

With these weights, we can write

$$\widetilde{R}_V = \widetilde{\mathbf{w}}^T \widehat{\mathbf{R}} \quad \text{and} \quad \widetilde{\sigma}_V^2 = \widetilde{\mathbf{w}}^T \mathbf{C} \widetilde{\mathbf{w}} \,. \tag{28.18}$$

and define the *marginal* residual risk in analogy to Eq. 26.19 as

$$\frac{\partial \widetilde{\sigma}_V}{\partial \widetilde{\mathbf{w}}} = \frac{\mathbf{C}\widetilde{\mathbf{w}}}{\sqrt{\widetilde{\mathbf{w}}^T \mathbf{C} \widetilde{\mathbf{w}}}} = \frac{\mathbf{C}\widetilde{\mathbf{w}}}{\sqrt{\delta t \,\mathrm{var}\left[\epsilon_V\right]}} \,. \tag{28.19}$$

In analogy to Eq. 26.20, the residual risk attributed to the $i$th position is

$$\widetilde{w}_i \frac{\partial \widetilde{\sigma}_V}{\partial \widetilde{w}_i} = \frac{\widetilde{w}_i^2 \sum_{k=1}^M C_{ki}}{\sqrt{\delta t \,\mathrm{var}\left[\epsilon_V\right]}} \quad \text{for} \quad i = 1, \ldots, M \,. \tag{28.20}$$

The percentage $\widetilde{A}_i$ of residual risk attributed to the $i$th asset is this number divided by $\widetilde{\sigma}_V$.

## 28.4 Benchmark and Characteristic Portfolios

We have shown that attributes and characteristic portfolios prove to be very useful tools in the "total return" framework. Motivated by this we will now define some attributes and their characteristic portfolios in the benchmark-relative framework. In Chap. 27 we analyzed the properties (i.e., leverage, return, variance, etc.) of the characteristic portfolio for any arbitrary vector **a** of asset attributes. We will now add to this list the characteristic portfolio's *residual* properties as well as its alpha and beta with respect to the benchmark.

The ex ante estimators for alpha and beta are given by Eq. 28.9 with the vectors of *asset* alphas and betas as in Eq. 28.4. With the weights of the characteristic portfolio given by Eq. 27.4, the characteristic portfolio's alpha and beta with respect to the benchmark are

$$\alpha_{a,B} \equiv \mathbf{w}_a^T \boldsymbol{\alpha}_B = \frac{\mathbf{a}^T \mathbf{C}^{-1} \boldsymbol{\alpha}_B}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}}$$
$$\beta_{a,B} \equiv \mathbf{w}_a^T \boldsymbol{\beta}_B = \frac{\mathbf{a}^T \mathbf{C}^{-1} \boldsymbol{\beta}_B}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}} . \tag{28.21}$$

Let's look now at the residual properties of the characteristic portfolio. With Eq. 27.4, the residual weights defined in Eq. 28.17 are

$$\begin{aligned}\widetilde{\mathbf{w}}_a &= \mathbf{w}_a - \beta_{a,B} \mathbf{w}_B \\ &= \frac{\mathbf{C}^{-1}\mathbf{a}}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}} - \frac{\mathbf{a}^T \mathbf{C}^{-1} \boldsymbol{\beta}_B}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}} \mathbf{w}_B \\ &= \frac{\mathbf{C}^{-1}\mathbf{a}}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}} - \frac{\mathbf{a}^T \mathbf{C}^{-1} \boldsymbol{\beta}_B}{\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}} \frac{\mathbf{C}^{-1} \boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T \mathbf{C}^{-1} \boldsymbol{\beta}_B} ,\end{aligned} \tag{28.22}$$

where in the last step we have used the implied benchmark weights which will be introduced below in Eq. 28.28. According to Eq. 28.18, residual return and residual risk are then

$$\begin{aligned}\widetilde{R}_a &\equiv \widetilde{\mathbf{w}}^T \widehat{\mathbf{R}} = \widehat{R}_a - \sigma_a^2 \frac{\mathbf{a}^T \mathbf{C}^{-1} \boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T \mathbf{C}^{-1} \boldsymbol{\beta}_B} \boldsymbol{\beta}_B^T \mathbf{C}^{-1} \widehat{\mathbf{R}}, \\ \widetilde{\sigma}_a^2 &\equiv \widetilde{\mathbf{w}}^T \mathbf{C} \widetilde{\mathbf{w}} = \sigma_a^2 \left[ 1 - \frac{\left(\mathbf{a}^T \mathbf{C}^{-1} \boldsymbol{\beta}_B\right)^2}{\left(\boldsymbol{\beta}_B^T \mathbf{C}^{-1} \boldsymbol{\beta}_B\right) \left(\mathbf{a}^T \mathbf{C}^{-1} \mathbf{a}\right)} \right] ,\end{aligned} \tag{28.23}$$

where we have used Eqs. 27.6 and 27.7. Note that the Cauchy-Schwarz inequality, Eq. 26.37, implies that $\widetilde{\sigma}_a^2$ is always larger than zero except for those attributes which are multiples[5] of $\boldsymbol{\beta}_B$.

With the above results, Information Ratio and marginal residual risk contributions can easily be calculated from their respective Definitions 28.16

[5] For such attributes we have $\widetilde{\sigma}_a^2 = 0$. Thus, the characteristic portfolios of such attributes have no residual risk.

and 28.19 for any characteristic portfolio.

$$\widetilde{\gamma}_a = \frac{\widetilde{R}_a}{\widetilde{\sigma}_a} = \frac{\sigma_a^2}{\widetilde{\sigma}_a}\mathbf{a}^T\mathbf{C}^{-1}\widehat{\mathbf{R}} - \frac{\sigma_a^2}{\widetilde{\sigma}_a}\frac{\mathbf{a}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}\boldsymbol{\beta}_B\mathbf{C}^{-1}\widehat{\mathbf{R}}$$

$$\frac{\partial\widetilde{\sigma}_a}{\partial\widetilde{\mathbf{w}}} = \frac{\mathbf{C}\widetilde{\mathbf{w}}}{\widetilde{\sigma}_a} = \frac{\sigma_a^2}{\widetilde{\sigma}_a}\mathbf{a} - \frac{\sigma_a^2}{\widetilde{\sigma}_a}\frac{\mathbf{a}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}\boldsymbol{\beta}_B\ . \tag{28.24}$$

The residual risk attributions follow directly from definition 28.20.

### 28.4.1 The Fully Invested Minimal Risk Portfolio

As a first example, we will apply the above results to the characteristic portfolio for the attribute $\mathbf{L} = \mathbf{1}$ introduced in Sect. 27.2, i.e., for the fully invested minimal risk portfolio. Its alpha and beta with respect to the benchmark are

$$\alpha_{L,B} = \frac{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\alpha}_B}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}\ , \quad \beta_{L,B}\frac{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}\ .$$

Its residual weights are

$$\widetilde{\mathbf{w}}_L = \frac{\mathbf{C}^{-1}\mathbf{1}}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}} - \frac{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}}\frac{\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}\ .$$

and the portfolio's residual properties are

$$\widetilde{R}_a = \widehat{R}_L - \sigma_L^2\frac{\mathbf{a}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\widehat{\mathbf{R}}$$

$$\widetilde{\sigma}_L^2 = \sigma_L^2\left[1 - \frac{\left(\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B\right)^2}{\left(\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B\right)\left(\mathbf{1}^T\mathbf{C}^{-1}\mathbf{1}\right)}\right]\ .$$

Information Ratio and marginal residual risk contributions are

$$\widetilde{\gamma}_L = \frac{\sigma_L^2}{\widetilde{\sigma}_L}\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}} - \frac{\sigma_L^2}{\widetilde{\sigma}_L}\frac{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}\boldsymbol{\beta}_B\mathbf{C}^{-1}\widehat{\mathbf{R}}$$

$$\frac{\partial\widetilde{\sigma}_a}{\partial\widetilde{\mathbf{w}}} = \frac{\sigma_L^2}{\widetilde{\sigma}_L}\mathbf{1} - \frac{\sigma_L^2}{\widetilde{\sigma}_L}\frac{\mathbf{1}^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\boldsymbol{\beta}_B}\boldsymbol{\beta}_B\ .$$

As usual, the residual risk attributions now follow directly from definition 28.20.

We will postpone the discussion of the other two important characteristic portfolios already introduced above, namely the portfolio for the attribute $\mathbf{A} = \widehat{\mathbf{R}}$ of Sect. 27.3 and the market portfolio of Sect. 27.4 until we have introduced two new portfolios, namely the characteristic portfolios for alpha and beta.

### 28.4.2 The Characteristic Portfolio for Beta

Let the attribute vector $\mathbf{b}$ be the beta of the assets with respect to the benchmark as in Eq. 28.4

$$b_i := \beta_{i,B} \equiv \frac{\text{cov}(\widehat{r}_i, \widehat{r}_B)}{\text{var}(\widehat{r}_B)} = \frac{\delta t}{\sigma_B^2} \sum_{k=1}^{M} w_{B,k} \underbrace{\text{cov}(\widehat{r}_i, \widehat{r}_k)}_{C_{ik}/\delta t}$$

or in vector notation

$$\mathbf{b} = \boldsymbol{\beta}_B = \frac{\mathbf{C}\mathbf{w}_B}{\mathbf{w}_B^T \mathbf{C}\mathbf{w}_B} \tag{28.25}$$

According to Eq. 27.1, the exposure of *any* portfolio $V$ to this particular attribute is

$$b_V = \mathbf{w}^T \mathbf{b} = \frac{\mathbf{w}^T \mathbf{C}\mathbf{w}_B}{\mathbf{w}_B^T \mathbf{C}\mathbf{w}_B} \ .$$

Comparing this with the *portfolio* beta $\beta_{V,B}$ with respect to the benchmark, i.e., with

$$\begin{aligned}\beta_{V,B} &\equiv \frac{\text{cov}(\widehat{r}_V, \widehat{r}_B)}{\text{var}(\widehat{r}_B)} = \frac{\delta t}{\sigma_B^2} \sum_{k=1}^{M} w_{B,k} \, \text{cov}(\widehat{r}_V, \widehat{r}_k) \\ &= \frac{\delta t}{\sigma_B^2} \sum_{k=1}^{M} w_{B,k} \sum_{i=1}^{M} w_i \underbrace{\text{cov}(\widehat{r}_i, \widehat{r}_k)}_{C_{ik}/\delta t} = \frac{\mathbf{w}^T \mathbf{C}\mathbf{w}_B}{\mathbf{w}_B^T \mathbf{C}\mathbf{w}_B}\end{aligned} \tag{28.26}$$

we find that the exposure of any portfolio $V$ to the attribute $\mathbf{b}$ is equal to the portfolio beta with respect to the benchmark:

$$b_V = \mathbf{w}^T \mathbf{b} = \beta_{V,B} \ .$$

Let's investigate now the *characteristic* portfolio $V_b$ for this particular attribute, i.e., the minimum risk portfolio with unit exposure $b_V = 1$. According to Eqs. 27.4 and 28.25, this portfolio has the following weights

$$\mathbf{w}_b = \frac{\mathbf{C}^{-1}\mathbf{b}}{\mathbf{b}^T \mathbf{C}^{-1}\mathbf{b}} = \left(\mathbf{w}_B^T \mathbf{C} \mathbf{w}_B\right) \frac{\mathbf{C}^{-1} \left(\mathbf{C}\mathbf{w}_B\right)}{\left(\mathbf{w}_B^T \mathbf{C}\right) \mathbf{C}^{-1} \left(\mathbf{C}\mathbf{w}_B\right)} = \mathbf{w}_B \ .$$

Thus, the characteristic portfolio for the attribute "beta to a benchmark" is the benchmark itself:

$$\mathbf{w}_b = \mathbf{w}_B \ . \tag{28.27}$$

Therefore, among all portfolios with $\beta_{V,B} = 1$, the benchmark itself has minimum risk.

Equation 28.27 shows a way how to imply the weights of the individual assets in the benchmark, if those weights are not known to the investors: From historical time series determine the asset betas with respect to the benchmark and the asset's covariances. Then the *implied benchmark weights* are given by

$$\mathbf{w}_B = \frac{\mathbf{C}^{-1}\boldsymbol{\beta}_B}{\boldsymbol{\beta}_B^T \mathbf{C}^{-1}\boldsymbol{\beta}_B} \tag{28.28}$$

The portfolio risk and return and the asset betas follow from Eqs. 27.6, 27.7 and 27.9 as usual:

$$\sigma_b^2 = \frac{1}{\boldsymbol{\beta}_B^T \mathbf{C}^{-1}\boldsymbol{\beta}_B} \ , \quad \widehat{R}_b = \frac{\boldsymbol{\beta}_B^T \mathbf{C}^{-1}\widehat{\mathbf{R}}}{\boldsymbol{\beta}_B^T \mathbf{C}^{-1}\boldsymbol{\beta}_B} \ , \quad \boldsymbol{\beta}_b = \boldsymbol{\beta}_B \ . \tag{28.29}$$

Finally, the marginal risk contribution, Eq. 27.10, of each asset and the percentage of risk attributed to each asset, Eq. 27.11, are:

$$\frac{\partial \sigma_b}{\partial w_{b,i}} = \sigma_b \beta_{i,B} \ , \quad A_{b,i} = w_{b,i}\beta_{i,B} \quad \text{for } i = 1, \ldots, M \ .$$

According to Eqs. 28.21 and 28.6, the portfolio's alpha and beta with respect to the benchmark are simply zero and one

$$\alpha_{b,B} = 0\ , \quad \beta_{b,B} = 1\ . \tag{28.30a}$$

Since the portfolio is the benchmark itself, all above properties also describe the benchmark portfolio $V_B$. Moreover, the residual weights all vanish. This can also be verified explicitly by setting $\mathbf{a} = \beta_B$ in Eq. 28.22. Therefore residual risk and return are both zero, and the Information Ratio is undefined.

### 28.4.3 The Characteristic Portfolio for Alpha

Let's now choose an attribute vector $\mathbf{a}$ to be the alpha of the assets with respect to the benchmark, i.e., according to Eq. 28.4

$$\mathbf{a} := \boldsymbol{\alpha}_B = \widehat{\mathbf{R}} - \boldsymbol{\beta}_B \widehat{R}_B\ . \tag{28.31}$$

According to Eqs. 27.1 and 28.9, the exposure of *any* portfolio $V$ to this particular attribute is the (ex ante estimate for the) portfolio alpha:

$$a_V = \mathbf{w}^T\mathbf{a} = \sum_{i=1}^{M} w_i \alpha_{i,B} = \alpha_{V,B}\ .$$

The *characteristic* portfolio $V_a$ for this particular attribute is the minimum risk portfolio with unit exposure $a_V = 1$, i.e., with a portfolio alpha of 1. According to Eqs. 27.4, 27.6 and 27.9, the *characteristic* portfolio for the attribute $\mathbf{a}$ has the following weights, variance, return and betas:

$$\begin{aligned}
\mathbf{w}_a &= \frac{\mathbf{C}^{-1}\boldsymbol{\alpha}_B}{\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\boldsymbol{\alpha}_B} = \sigma_a^2\mathbf{C}^{-1}\boldsymbol{\alpha}_B \\
\sigma_a^2 &= \frac{1}{\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\boldsymbol{\alpha}_B} \\
\widehat{R}_a &= \sigma_a^2\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\widehat{\mathbf{R}} = \frac{\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\boldsymbol{\alpha}_B} \\
\frac{\mathbf{C}\mathbf{w}_a}{\sigma_a^2} &= \boldsymbol{\beta}_a = \boldsymbol{\alpha}_B\ .
\end{aligned} \tag{28.32}$$

Thus, the asset betas with respect to this *characteristic* portfolio $V_a$ are equal to the asset alphas with respect to the benchmark portfolio $V_B$. By contrast, the *portfolio* alpha and beta of $V_a$ with respect to the benchmark are simply one and zero[6] as follows directly[7] from Eqs. 28.21 and 28.6

$$\alpha_{a,B}=1\ ,\quad \beta_{a,B}=0\ . \tag{28.33}$$

Since $\beta_{a,B}=\rho_{a,B}\sigma_a/\sigma_B$ this implies that portfolio $V_a$ is totally uncorrelated with the benchmark:

$$\rho_{a,B}=0\ .$$

With the asset betas as in Eq. 28.32, the marginal risk contribution, Eq. 26.25, of each asset and the percentage of risk attributed to each asset, Eq. 26.26, are:

$$\frac{\partial\sigma_a}{\partial w_{a,i}}=\sigma_a\alpha_{i,B}\ ,\quad A_{a,i}=w_{a,i}\alpha_{i,B}\quad \text{for}\quad i=1,\ldots,M\ . \tag{28.34}$$

Let's look now at the residual quantities of this portfolio. Since $\beta_{a,B}=0$ we have

$$\widetilde{\mathbf{w}}_a=\mathbf{w}_a-\beta_{a,B}\mathbf{w}_B=\mathbf{w}_a$$

and therefore

$$\widetilde{R}_a=\widehat{R}_a \text{ und } \widetilde{\sigma}_a^2=\sigma_a^2\ . \tag{28.35}$$

[6] $V_a$ usually contains long and short positions.

[7] The fact that the portfolio beta is zero can also be derived very elegantly by applying Eq. 27.12 to the characteristic portfolios $V_a$ and $V_b$:

$$b_a\sigma_b^2=a_b\sigma_a^2\ .$$

Here $a_b:=\mathbf{w}_b^T\mathbf{a}=\mathbf{w}_b^T\boldsymbol{\alpha}_B=\alpha_{b,B}$ is the portfolio alpha of portfolio $V_b$ with respect to the benchmark. However, as shown in Eq. 28.27, portfolio $V_b$ is the benchmark itself and has of course zero alpha w.r.t. to itself, since a regression like Eq. 28.3 of any variable onto itself always yields $\alpha=0$ and $\beta=1$. Thus $b_a$ must vanish as well, since both variances $\sigma_a^2$ and $\sigma_b^2$ are positive. But $b_a$, being the exposure of $V_a$ to attribute $b$, is the portfolio Beta of $V_a$ with respect to the benchmark, $b_a:=\mathbf{w}_a^T\mathbf{b}=\mathbf{w}_a^T\boldsymbol{\beta}_B=\beta_{a,B}$.

Moreover, the marginal residual risk contribution of each asset and the percentage of residual risk attributed to each asset are the same as in Eq. 28.34.

$$\frac{\partial\widetilde{\sigma}_a}{\partial\widetilde{w}_{a,i}} = \sigma_a\alpha_{i,B}\,, \quad \widetilde{A}_{a,i} = w_{a,i}\alpha_{i,B} \quad \text{for} \quad i = 1, \ldots, M\ .$$

Note that because of Eq. 28.15 we have

$$\widetilde{R}_a = \alpha_{a,B} = \mathbf{w}_a^T \boldsymbol{\alpha}_B = 1 \tag{28.36}$$

where the last step follows because $V_a$ is the characteristic portfolio for $\boldsymbol{\alpha}_B$. Thus, portfolio $V_a$ not only has $\alpha_{a,B} = 1$ but it also has expected excess return $\widehat{R}_a = 1$. It therefore has the same expected excess return as the characteristic portfolio $V_A$ for attribute $\widehat{\mathbf{R}}$ discussed in Sect. 27.3. However, both portfolios are not necessarily the same: among all portfolios with $\mathbf{w}^T\widehat{\mathbf{R}} = 1$ there are some which also fulfill $\mathbf{w}^T\boldsymbol{\alpha}_B = 1$. However, among the other portfolios with $\mathbf{w}^T\widehat{\mathbf{R}} = 1$ there could well be some with less risk than portfolio $V_a$. Thus, portfolio $V_a$ is not necessarily the minimum risk portfolio with excess return equal one, i.e., not necessarily equal to $V_A$.

Because of Eqs. 28.36, 28.35, and 28.32, the information ratio of the characteristic portfolio $V_A$ is

$$\widetilde{\gamma}_a = \frac{\widetilde{R}_a}{\widetilde{\sigma}_a} = \frac{1}{\widetilde{\sigma}_a} = \frac{1}{\sigma_a} = \sqrt{\boldsymbol{\alpha}_B^T \mathbf{C}^{-1} \boldsymbol{\alpha}_B}\ . \tag{28.37}$$

This can also be read in the following way: the total risk as well as the residual risk of this portfolio are both equal to the reciprocal of the Information Ratio.

Portfolio $V_a$ has minimum risk, and because of Eq. 28.35, it also has minimum *residual* risk among all portfolios $V$ with $\alpha_{V,B} = 1$. Thus, portfolio $V_a$ maximizes the Information ratio Eq. 28.37 among those portfolios. To show that portfolio $V_a$ has maximal Information Ratio among *all* portfolios (and not only among those with $\alpha_{V,B} = 1$), we will now show that for any arbitrary portfolio $V$ with weights $\mathbf{w}$, there is a portfolio with the *same* information ratio and a portfolio alpha equal to one. We find this portfolio by constructing its weights: the residual position weights of portfolio $V$ are given by Eq. 28.17. Because of Eq. 28.18, the Information Ratio (defined in Eq. 28.16) for any portfolio $V_\lambda$ with residual weights $\lambda\widetilde{\mathbf{w}}$ is the same for any $\lambda > 0$. The weights

$\mathbf{w}_\lambda$ of this portfolio follow directly from the requirement $\widetilde{\mathbf{w}}_\lambda = \lambda\widetilde{\mathbf{w}}$:

$$\begin{aligned} \widetilde{\mathbf{w}}_\lambda &= \lambda\widetilde{\mathbf{w}} \\ \mathbf{w}_\lambda - \beta_{\lambda,B}\mathbf{w}_B &= \lambda\left(\mathbf{w} - \beta_{V,B}\mathbf{w}_B\right) \\ \mathbf{w}_\lambda &= \lambda\mathbf{w} + (\beta_{\lambda,B} - \lambda\beta_{V,B})\mathbf{w}_B \ . \end{aligned} \tag{28.38}$$

We can adjust $\lambda$ such that the alpha of portfolio $V_\lambda$ becomes one:

$$1 \stackrel{!}{=} \alpha_{\lambda,B} = \widetilde{R}_\lambda = \lambda\widetilde{R} = \lambda\alpha_{V,B} \implies \lambda = \frac{1}{\alpha_{V,B}} \ .$$

Thus, for any given portfolio $V$ with weights $\mathbf{w}$ there is indeed a portfolio $V_\lambda$ with the *same* information ratio and a portfolio alpha equal to one. The weights of this portfolio can be obtained by solving the equation[8]

$$\mathbf{w}_\lambda = \frac{1}{\alpha_{V,B}}\mathbf{w} + (\beta_{\lambda,B} - \frac{\beta_{V,B}}{\alpha_{V,B}})\mathbf{w}_B \ .$$

This proves that portfolio $V_a$, having maximal Information Ratio among all portfolios $V$ with $\alpha_{V,B} = 1$, has indeed maximal Information Ratio among *all* portfolio.

$$\widetilde{\gamma}_{\max} = \widetilde{\gamma}_a = \frac{1}{\widetilde{\sigma}_a} = \frac{1}{\sigma_a} \ . \tag{28.39}$$

This Equation is the benchmark-relative analogue to Eq. 27.23.

Choosing the arbitrary portfolio $V$ in Eq. 28.38 to be the characteristic portfolio $V_a$ itself (and observing Eq. 28.33) we get the most general form a portfolio with the same (i.e., maximal) Information Ratio as $V_a$ may have:

$$\begin{aligned} \mathbf{w}_\lambda &= \lambda\mathbf{w}_a + \beta_{\lambda,B}\mathbf{w}_B \ \text{ mit } \lambda > 0 \\ &\implies \widetilde{\gamma}_\lambda = \widetilde{\gamma}_a \ . \end{aligned} \tag{28.40}$$

[8]According to Eq. 28.26, $\beta_{\lambda,B}$ also contains the weights $\mathbf{w}_\lambda$:

$$\beta_{\lambda,B} = \frac{\mathbf{w}_\lambda^T\mathbf{C}\mathbf{w}_B}{\mathbf{w}_B^T\mathbf{C}\mathbf{w}_B} \ .$$

Therefore solving for $\mathbf{w}_\lambda$ can only be done numerically.

Equation 28.40 provides a handy method for *constructing* portfolios with maximum Information ratio using the characteristic portfolios $V_a$ and $V_b = V_B$. Furthermore, it provides a powerful tool to *check* whether a given portfolio has maximum IR. We will come back to this point later.

To complete this section, we will now show that a relation similar to Eq. 27.25 for Sharpe Ratios also holds in the benchmark-relative framework for Information Ratios. With Eq. 28.32 and the general Eq. 28.9, the alpha of any portfolio with weights $\mathbf{w}$ can be expressed in terms of covariances with the characteristic portfolio $V_a$ as

$$\alpha_{V,B} = \mathbf{w}^T \boldsymbol{\alpha}_B = \frac{\mathbf{w}^T \mathbf{C} \mathbf{w}_a}{\sigma_a^2} = \frac{\delta t \operatorname{cov}(r_V, r_a)}{\sigma_a^2} = \frac{\delta t \operatorname{cov}(\widehat{r}_V, \widehat{r}_a)}{\sigma_a^2} = \frac{\delta t \operatorname{cov}(\widetilde{r}_V, \widetilde{r}_a)}{\sigma_a^2} ,$$

where the second-to-last step is trivial, since subtracting a constant $r_f$ doesn't change the covariances. The last step, however, is not trivial and only holds because portfolio $V_a$ has zero beta with respect to the benchmark.[9] With this alpha, the IR of portfolio $V$ can be written as[10]

$$\widetilde{\gamma}_V = \frac{\widetilde{R}_V}{\widetilde{\sigma}_V} = \frac{\alpha_{V,B}}{\widetilde{\sigma}_V} = \frac{\delta t \operatorname{cov}(\widetilde{r}_V, \widetilde{r}_a)}{\widetilde{\sigma}_V \sigma_a^2} = \underbrace{\frac{1}{\widetilde{\sigma}_a}}_{\widetilde{\gamma}_a} \underbrace{\frac{\delta t \operatorname{cov}(\widetilde{r}_V, \widetilde{r}_a)}{\widetilde{\sigma}_V \widetilde{\sigma}_a}}_{\operatorname{corr}(\widetilde{r}_V, \widetilde{r}_a)} ,$$

where in the second step we have used Eq. 28.15 and in the last step we have used Eqs. 28.35 and 28.37. Therefore, the IR of any portfolio $V$ is given by the (maximum) IR of the characteristic portfolio $V_a$ and the correlation between the *residual* returns of those two portfolios:

$$\widetilde{\gamma}_V = \widetilde{\gamma}_a \operatorname{corr}(\widetilde{r}_V, \widetilde{r}_a) \quad \text{for all portfolios } V . \tag{28.41}$$

This equation is the benchmark-relative analogue to Eq. 27.25.

[9] This can be derived from first principles with the help of Eq. 28.33:

$$\begin{aligned} \operatorname{cov}(\widetilde{r}_V, \widetilde{r}_a) &= \operatorname{cov}(\widehat{r}_V - \beta_{V,B}\widehat{r}_B, \widehat{r}_a - \underbrace{\beta_{a,B}}_{0}\widehat{r}_B) \\ &= \operatorname{cov}(\widehat{r}_V - \beta_{V,B}\widehat{r}_B, \widehat{r}_a) \\ &= \operatorname{cov}(\widehat{r}_V, \widehat{r}_a) - \beta_{V,B} \operatorname{cov}(\widehat{r}_B, \widehat{r}_a) \\ &= \operatorname{cov}(\widehat{r}_V, \widehat{r}_a) - \beta_{V,B}\, \sigma_b^2 \underbrace{\beta_{a,B}}_{0} . \end{aligned}$$

[10] To get the factors $\delta t$ right, observe Eqs. 26.16, 26.18 and 21.31.

## 28.5 Relations Between Sharpe Ratio and Information Ratio

To find out how the Sharpe Ratio used in Total Return Management is related to the Information Ratio used in the benchmark-relative framework, we will analyze the two characteristic portfolios with maximum Sharpe Ratio, i.e., portfolios $V_A$ of Sect. 27.3 and portfolio $V_m$ of Sect. 27.4, in the benchmark framework.

### 28.5.1 The Market Portfolio

We will begin by applying Eq. 28.40 to check if the market portfolio of Sect. 27.4 has maximum IR. To bring its weights $\mathbf{w}_m$ into a form similar to Eq. 28.40, we write the asset returns in Eq. 28.4 with Eqs. 28.32 for $\boldsymbol{\alpha}_B$ and 28.29 for $\boldsymbol{\beta}_B$:

$$\widehat{\mathbf{R}} = \boldsymbol{\alpha}_B + \boldsymbol{\beta}_B \widehat{R}_B = \frac{\mathbf{C}\mathbf{w}_a}{\sigma_a^2} + \frac{\mathbf{C}\mathbf{w}_B}{\sigma_B^2}\widehat{R}_B \ . \tag{28.42}$$

Expressing now the asset returns on the left hand side in terms of the market portfolio as in Eq. 27.31, we get

$$\frac{\mathbf{C}\mathbf{w}_m}{\sigma_m^2}\widehat{R}_m = \frac{\mathbf{C}\mathbf{w}_a}{\sigma_a^2} + \frac{\mathbf{C}\mathbf{w}_B}{\sigma_B^2}\widehat{R}_B \ .$$

Solving for $\mathbf{w}_m$ shows that the market portfolio can be written as a combination of the benchmark portfolio $V_B$ and the characteristic portfolio $V_a$.

$$\begin{aligned}
\mathbf{w}_m &= \frac{\sigma_m^2}{\sigma_a^2}\frac{1}{\widehat{R}_m}\mathbf{w}_a + \frac{\sigma_m^2}{\sigma_B^2}\frac{\widehat{R}_B}{\widehat{R}_m}\mathbf{w}_B \\
&= \frac{\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\boldsymbol{\alpha}_B}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\,\mathbf{w}_a + \frac{\boldsymbol{\beta}_B^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}\,\mathbf{w}_B \\
&= \underbrace{\frac{\boldsymbol{\alpha}_B^T\mathbf{C}^{-1}\boldsymbol{\alpha}_B}{\mathbf{1}^T\mathbf{C}^{-1}\widehat{\mathbf{R}}}}_{\lambda}\mathbf{w}_a + \underbrace{\left(\boldsymbol{\beta}_B^T\mathbf{w}_m\right)}_{\beta_{m,B}}\mathbf{w}_B \ ,
\end{aligned} \tag{28.43}$$

Here we have used Eqs. 27.31, 28.29 and 28.32 in the second step and Eq. 27.30 in the last step. Although it looks very similar, $\lambda$ is *not* the portfolio alpha of the market portfolio.[11] Nonetheless, Eq. 28.43 has indeed the form of Eq. 28.40 with $\lambda > 0$ as long as $\widehat{R}_m > 0$. Therefore, as is always the case when the market portfolio is involved, everything only works if Inequality 26.59 holds.[12]

**If Inequality 26.59 holds, then the market portfolio has maximal Information Ratio (in addition to having maximal Sharpe Ratio)**.

$$\widetilde{\gamma}_m = \widetilde{\gamma}_a = \widetilde{\gamma}_{\max} \,. \tag{28.44}$$

Because of Eq. 28.41, this means that the residual returns of the market portfolio are fully correlated with the residual returns of portfolio $V_a$.

$$\operatorname{corr}(\widetilde{r}_m, \widetilde{r}_a) = 1 \,.$$

From the above expression for $\mathbf{w}_m$, we find that the *residual* weights of the market portfolio can be expressed in terms of the weights of portfolio $V_a$ as

$$\widetilde{\mathbf{w}}_m \equiv \mathbf{w}_m - \beta_{m,B}\mathbf{w}_B = \frac{\boldsymbol{\alpha}_B^T \mathbf{C}^{-1} \boldsymbol{\alpha}_B}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \mathbf{w}_a = \frac{\sigma_m^2}{\sigma_a^2} \frac{1}{\widehat{R}_m} \mathbf{w}_a \,. \tag{28.45}$$

With these weights, the residual risk of the market portfolio becomes

$$\widetilde{\sigma}_m = \sqrt{\widetilde{\mathbf{w}}_m^T \mathbf{C} \widetilde{\mathbf{w}}_m} = \frac{\sigma_m^2}{\sigma_a^2} \frac{1}{\widehat{R}_m} \sqrt{\mathbf{w}_a^T \mathbf{C} \mathbf{w}_a} = \frac{\sigma_m^2}{\sigma_a \widehat{R}_m} \,. \tag{28.46}$$

Similarly, the residual return of the market portfolio is

$$\widetilde{R}_m = \widetilde{\mathbf{w}}_m^T \widehat{\mathbf{R}} \equiv \frac{\sigma_m^2}{\sigma_a^2} \frac{1}{\widehat{R}_m} \mathbf{w}_a^T \widehat{\mathbf{R}} = \frac{\sigma_m^2}{\sigma_a^2} \frac{\widehat{R}_a}{\widehat{R}_m} \,. \tag{28.47}$$

[11] The Alpha of the market portfolio is given by

$$\alpha_{m,B} = \boldsymbol{\alpha}_B^T \mathbf{w}_m = \frac{\boldsymbol{\alpha}_B^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\mathbf{1}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}} \,.$$

[12] This is usually (but unfortunately not always) the case, since the expected return of the market portfolio should be above the risk free rate as a compensation for the risk of the market portfolio.

Inserting these results into Eq. 28.16 we find the *Information* Ratio of the market portfolio to be the same as the *Sharpe* Ratio of portfolio $V_a$.

$$\widetilde{\gamma}_m = \frac{\widetilde{R}_m}{\widetilde{\sigma}_m} = \frac{\widehat{R}_a}{\sigma_a} = \gamma_a$$

or, written with $\widetilde{\gamma}_{\max}$ from Eq. 28.44:

$$\widetilde{\gamma}_{\max} = \widetilde{\gamma}_m = \gamma_a \ . \tag{28.48}$$

where in the last step we have used Eq. 28.44. Therefore, the (maximum) Information Ratio of portfolio $V_a$ is equal to its Sharpe Ratio. However, this is not the *maximum* Sharpe Ratio. The relation between the maximum Sharpe Ratio $\gamma_m$ (see Eq. 27.33) and the *maximum* Information Ratio $\widetilde{\gamma}_m = \widetilde{\gamma}_a$ (see Eq. 28.37) is

$$\frac{\gamma_{\max}}{\widetilde{\gamma}_{\max}} = \frac{\gamma_m}{\widetilde{\gamma}_a} = \frac{\sigma_a \widehat{R}_m}{\sigma_m} = \frac{\sigma_m}{\widetilde{\sigma}_m} \ ,$$

where we have used Eq. 28.46 in the last step. **The maximum Sharpe Ratio is as much larger than the maximum Information Ratio as the total risk of the market portfolio is larger than its residual risk**.

$$\gamma_{\max} = \frac{\sigma_m}{\widetilde{\sigma}_m} \widetilde{\gamma}_{\max} \ . \tag{28.49}$$

With $\gamma_{\max} = \gamma_m = \widehat{R}_m / \sigma_m$ and $\widetilde{\gamma}_{\max} = \widetilde{\gamma}_m = \widetilde{R}_m / \widetilde{\sigma}_m$, this means that the ratio of excess and residual return of the market portfolio equals the ratio of the *squares* of its risk and residual risk:

$$\frac{\widehat{R}_m}{\widetilde{R}_m} = \frac{\sigma_m^2}{\widetilde{\sigma}_m^2} = \frac{\text{var}\,[r_m]}{\text{var}\,[\epsilon_m]} \ , \tag{28.50}$$

where in the last step we have used the original definition 28.14 of the residual risk.

The *marginal* contributions of the assets to the residual risk of the market portfolio are by definition

$$\frac{\partial \widetilde{\sigma}_m}{\widetilde{\mathbf{w}}_m} = \frac{\mathbf{C}\widetilde{\mathbf{w}}}{\sqrt{\widetilde{\mathbf{w}}^T \mathbf{C}\widetilde{\mathbf{w}}}} = \frac{\mathbf{C}\widetilde{\mathbf{w}}}{\widetilde{\sigma}_m} \ .$$

With the above results 28.45 and 28.46, this can be written as

$$\frac{\partial \widetilde{\sigma}_m}{\widetilde{\mathbf{w}}_m} = \frac{\sigma_m^2}{\sigma_a^2}\frac{1}{\widehat{R}_m}\frac{\mathbf{C}\mathbf{w}_a}{\widetilde{\sigma}_m} = \frac{\mathbf{C}\mathbf{w}_a}{\sigma_a} \ .$$

We can now apply Eqs. 28.32 to write the marginal residual risk contributions to the market portfolio in several useful forms.

$$\frac{\partial \widetilde{\sigma}_m}{\widetilde{\mathbf{w}}_m} = \sigma_a \boldsymbol{\beta}_a = \sigma_a \boldsymbol{\alpha}_B \ . \tag{28.51}$$

With Eq. 28.37 for $\sigma_a$ we get a particularly insightful result:

$$\boldsymbol{\alpha}_B = \widetilde{\gamma}_{\max}\frac{\partial \widetilde{\sigma}_m}{\widetilde{\mathbf{w}}_m} \ . \tag{28.52}$$

**The asset alphas are proportional to the assets' marginal residual risk contributions to the market portfolio, with the maximum Information Ratio being the constant of proportionality.**

## 28.5.2 The Characteristic Portfolio of the Excess Return

Let's now analyze the excess return's characteristic portfolio $V_A$ with its properties given by Eq. 27.24. Since $V_A$ is closely related to the market portfolio via Eq. 27.29, we can directly use Eq. 28.43 to express $V_A$ in terms of the characteristic portfolios for alpha and beta as

$$\begin{aligned}
\mathbf{w}_A &= \frac{\sigma_A^2}{\sigma_L^2}\widehat{R}_L \mathbf{w}_m = \frac{\sigma_m^2}{\sigma_L^2}\frac{\widehat{R}_L}{\widehat{R}_m}\left[\frac{\sigma_A^2}{\sigma_a^2}\mathbf{w}_a + \frac{\sigma_A^2}{\sigma_B^2}\widehat{R}_B \mathbf{w}_B\right] \\
&= \frac{\sigma_A^2}{\sigma_a^2}\mathbf{w}_a + \frac{\sigma_A^2}{\sigma_B^2}\widehat{R}_B \mathbf{w}_B = \frac{\boldsymbol{\alpha}_B^T \mathbf{C}^{-1}\boldsymbol{\alpha}_B}{\widehat{\mathbf{R}}^T \mathbf{C}^{-1}\widehat{\mathbf{R}}}\,\mathbf{w}_a + \underbrace{\left(\boldsymbol{\beta}_B^T \mathbf{w}_A\right)}_{\beta_{A,B}}\mathbf{w}_B \ ,
\end{aligned}$$

where Eqs. 27.18, 27.24, 28.29 and 28.32 have been used. This again has the form of Eq. 28.40 with $\lambda > 0$ for any positive definite Matrix **C**. Thus, **the excess return's characteristic portfolio has maximal Information Ratio (in addition to having maximal Sharpe Ratio)**.

$$\widetilde{\gamma}_A = \widetilde{\gamma}_a = \widetilde{\gamma}_{\max} \ .$$

This holds in *every* situation, even if Inequality 26.59 does not hold. The *residual* weights of portfolio $V_A$ can now be expressed in terms of the weights of portfolio $V_a$ as

$$\widetilde{\mathbf{w}}_A \equiv \mathbf{w}_a - \beta_{A,B}\mathbf{w}_B = \frac{\sigma_A^2}{\sigma_a^2}\mathbf{w}_a \ .$$

With these weights, the residual variance of the portfolio becomes

$$\widetilde{\sigma}_m = \sqrt{\widetilde{\mathbf{w}}_A^T \mathbf{C} \widetilde{\mathbf{w}}_A} = \frac{\sigma_A^2}{\sigma_a^2}\sqrt{\mathbf{w}_a^T \mathbf{C} \mathbf{w}_a} = \frac{\sigma_A^2}{\sigma_a^2}\sigma_a \ .$$

Similarly, the residual return of the portfolio is

$$\widetilde{R}_A = \widetilde{\mathbf{w}}_A^T \widehat{\mathbf{R}} = \frac{\sigma_A^2}{\sigma_a^2}\mathbf{w}_a^T \mathbf{R} = \frac{\sigma_A^2}{\sigma_a^2}\widehat{R}_a \ .$$

Inserting these results into Eq. 28.16, we find the *Information* Ratio of the portfolio to be the same as the *Sharpe Ratio* of portfolio $V_a$.

$$\widetilde{\gamma}_A = \frac{\widetilde{R}_A}{\widetilde{\sigma}_A} = \frac{\widehat{R}_a}{\sigma_a} = \gamma_a \ .$$

This is no surprise since we already established $\widetilde{\gamma}_A = \widetilde{\gamma}_a$ and we know from Eq. 28.48 that $\gamma_a = \widetilde{\gamma}_A$.

The relation between the maximum Sharpe Ratio $\gamma_A$ (see Eq. 27.23) and the maximum Information Ratio $\widetilde{\gamma}_A = \widetilde{\gamma}_a$ (see Eq. 28.37) is

$$\frac{\gamma_A}{\widetilde{\gamma}_a} = \sqrt{\frac{\widehat{\mathbf{R}}^T \mathbf{C}^{-1} \widehat{\mathbf{R}}}{\boldsymbol{\alpha}_B^T \mathbf{C}^{-1} \boldsymbol{\alpha}_B}} = \frac{\sigma_a}{\sigma_A} \ .$$

Thus the relation between the maximum Sharpe Ratio and the maximum Information Ratio can also be expressed in terms of the two volatilities $\sigma_a$ and $\sigma_A$. Together Eq. 28.49, the relationships between maximum Sharp Ratio and maximum Information Ratio are in summary:

$$\frac{\sigma_m}{\widetilde{\sigma}_m} = \frac{\gamma_{\max}}{\widetilde{\gamma}_{\max}} = \frac{\sigma_a}{\sigma_A} \ . \tag{28.53}$$

The first equality, involving the market portfolio, is only true if Inequality 26.59 holds, while the second equality holds in *every* situation.