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Simulation Methods

23.1 Monte Carlo Simulation

In the calculation of the value at risk by means of Monte Carlo simulations, all of the risk factors influencing a portfolio are simulated over the liquidation period δt as stochastic processes satisfying, for example, Eq. 2.17 or even more general processes of the form 2.19. The value at risk as a function of the risk factors themselves are taken into complete consideration using Eq. 21.16 sometimes neglecting the drift in the simulation if the liquidation period is short:

$$\begin{aligned}\mathrm{VaR}_{\mathrm{long}}(c) &\approx NS(t)\left[1-\exp\left(+Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right]\\ \mathrm{VaR}_{\mathrm{short}}(c) &\approx -NS(t)\left[1-\exp\left(-Q_{1-c}^{\mathrm{N}(0,1)}\sigma\sqrt{\delta t}\right)\right]\ .\end{aligned}$$

As explained in Sect. 21.2, the value at risk of a long position in an underlying is only then equal to that of a short position if the drift is neglected *and* the linear approximation has been used. Since the linear approximation is usually not assumed in the Monte Carlo method, the VaR value of a long position will not equal that of a short position on the same underlying.

In carrying out the simulation, it will be taken into consideration that the risk factors are not independent of one another, but are *correlated*. This has already been demonstrated in Sect. 11.3.2 for the case of *two* correlated underlying prices. Processes of the form 2.17 involve Wiener processes whose stochastic components are coupled as given by the covariance matrix 21.22.

H.-P. Deutsch, M. W. Beinker, *Derivatives and Internal Models*, Finance and Capital Markets Series, https://doi.org/10.1007/978-3-030-22899-6_23

In other words, the logarithmic changes in the risk factors are multivariate normally distributed with the covariance matrix 21.22. In this way, market scenarios (combinations of all risk factors) possibly occurring up to the end of the liquidation period (up to time T) are simulated. The *portfolio* values at the conclusion of the liquidation period are then computed on the basis of all these simulated market scenarios. With this information, the distribution of the potential *portfolio* values at the conclusion of the liquidation period can be approximated. The value at risk can then be obtained through the statistical evaluation of this portfolio value distribution.

The advantage of the Monte Carlo simulation compared to other methods (such as the variance-covariance method) is that for the portfolio valuation for each market scenario we can in principle use the same valuation methods as for determining the portfolio's *current* value. No additional approximations for the *valuation* of the financial instruments in the portfolio need to be made (called *full valuation*). In principal, the same valuation methods used for the daily valuation could also be used for the value at risk calculation Nevertheless, it is in practice often not possible to use the same (computationally intensive) routines for both the valuation of a portfolio with respect to, for example, 10,000 scenarios as for (one single) determination of the portfolio's current value, the *mark-to-market*. It is therefore often necessary to use simpler and less precise methods for the revaluation of financial instruments with respect to the Monte Carlo scenarios. The statistical error arising in connection with such simulation methods is also unavoidable since only a finite number of scenarios can be simulated and thus only mean values rather than expectations can be computed (see Sect. 31.2 for more on this subject).

23.1.1 The Risk Factors as Correlated Random Walks

A random number generator usually produces single, uncorrelated random numbers. However, with the methods described in Sect. 21.5.3, via the Cholesky decomposition $\mathbf{A}$ of the covariance matrix, uncorrelated random numbers can be transformed into correlated ones. This can be exploited when carrying out Monte Carlo simulations:

With the help of the Cholesky decomposition $\mathbf{A}$ of the covariance matrix standard normally distributed random variables X_j are transformed into components of a multivariate normally distributed random vector Y_j, for $j = 1, 2, \ldots, n$ whose covariances are given in Eq. 21.22:

$$Y_i = \sum_{j=1}^{n} A_{ij} X_j \ .$$

The random walks of the risk factors expressed in terms of the random variables Y_j are

$$d \ln S_j(t + \delta t) = \mu_j \, \delta t \; + Y_j \; .$$

Similar to Eq. 11.2, the values of the risk factors for a scenario simulated to occur at the end of the time interval δt are

$$\begin{aligned} \ln S_j(t + \delta t) &= \ln S_j(t) + \mu_j \, \delta t \; + Y_j \\ S_j(t + \delta t) &= e^{Y_j} e^{\mu_j \, \delta t} S_j(t) \quad j = 1, \; \ldots \; , n \; , \end{aligned} \tag{23.1}$$

where here, as has received mention on numerous occasions, the drifts μ_j are often neglected in the analysis.

To generate a complete market scenario for the time $t + \delta t$ a random number Y_j for each risk factor is required. The portfolio is then revaluated at the value date $t + \delta t$ on the basis of this scenario. We thus obtain a simulated portfolio value at the end of the liquidation period. The approach in a Monte Carlo simulation in risk management is summarized below. This type of simulation is sometimes called *structured Monte Carlo*.

23.1.2 Structured Monte Carlo

Simulation

- Generate n standard normally distributed, uncorrelated random numbers X_j, one for each risk factor.
- Generate correlated random numbers Y_j, $j = 1, 2, \ldots, n$ using the equation

  $$Y_i = \sum_{j=1}^{n} A_{ij} X_j \; .$$

  The elements of the matrix $\mathbf{A}$ are given through the Cholesky decomposition of the covariance matrix in accordance with Eq. 21.41.
- Using these Y_j, the risk factors for the simulated scenario at the end of a time interval δt are calculated via Eq. 23.1. If the portfolio contains path-dependent derivatives, it is not possibly to simply jump to the end of the liquidation period in a single step if the liquidation period is longer than

one day. Smaller steps are necessary to simulate the paths of the risk factors up to the conclusion of the liquidation period similar to Eq. 11.1, instead of simulating directly with Eq. 11.2.

- Perform a new valuation of the portfolio with respect to these simulated risk factors. If computationally possible a *full valuation* is preferable.[1]

Thus, one single market scenario is simulated and the portfolio is re-valued with respect to this single scenario. This simulation is now repeated (for example, 10,000 times) in order to generate numerous scenarios and a portfolio value for each of these scenarios. Finally, the statistical evaluation is performed.

Evaluation

The change in the value of the portfolio observed in the i-th simulated scenario will be denoted by δV_i, the vector containing all risk factor values in the i-th simulated scenario by $\mathbf{S}_i$. We let m denote the number of simulations and n the number of relevant risk factors. For every simulated scenario, the induced simulated value change of the portfolio is the difference between the portfolio's value with respect to the simulated scenario and its current value:

$$\delta V_i = V\left(\mathbf{S}_i(t + \delta t)\right) - V\left(\mathbf{S}(t)\right) \quad \text{with} \quad i = 1, \ \ldots \ , m \ . \tag{23.2}$$

We thus obtain m simulated value changes. The value at risk of the Monte Carlo simulation is the minimum of these δV_i, where a certain number of the least favorable value changes are ignored dependent on the desired confidence level. For 95% confidence, for example, these are 5% of the least favorable value changes. For 10,000 simulated scenarios, for example, the 500 worst scenarios are ignored. We denote by δV_{1-c} the most favorable of the value changes which are ignored at a level of confidence c. The value at risk of the portfolio is now the least favorable result among the set of results remaining after those $(1 - c)\%$ least favorable simulations have been removed from consideration or equivalently, the least favorable value greater than V_{1-c}:

$$\mathrm{VaR}_V(c) = -\min_i \left\{\delta V_i \ \mid \delta V_i > \delta V_{1-c}\right\} \quad \text{with} \quad i = 1, \ldots, m \ . \tag{23.3}$$

[1]If the portfolio valuation requires for example Monte Carlo *pricing* methods for some (exotic) financial instruments, these additional Monte Carlo simulations (for pricing) have to run *inside* the simulation loop for the VaR calculation. Clearly this may lead to unacceptably large computation times.

With a confidence c, the portfolio will depreciate in value by no more than this value at risk by the end of the liquidation period.

23.2 Historical Simulation

Historical simulations are performed by investigating historical time series with the objective of identifying market changes which have actually occurred in the past and using these changes to compute the value at risk. The covariance matrix in Eq. 21.22 is not necessary for a historical simulation nor is it necessary to assume that the risk factors behave as random walks with constant yields and volatilities or even that they behave as random walks at all! This freedom from model assumptions is the primary advantage of this method.

The independence from model assumptions is at the expense of involved data management. While only three values provide sufficient statistical information about the past behavior of two risk factors (both volatilities and the correlation between the two) for variance-covariance and Monte Carlo methods, entire time series of prices for all risk factors relevant to the portfolio must be kept for a historical simulation to be performed. For example, the closing prices of every underlying for the previous 250 days. For two underlyings, this amounts to 500 values in comparison to just the 3 required for the methods mentioned above. Often, these 3 parameter are estimated based on historical data (if the data is not delivered by an external vendor). In this case, the Variance-Covariance and the Monte Carlo methods require the storage and maintenance of historical data as well. From the historical time series, the value changes $\delta S_j(\delta t_i)$ of all risk factors S_j over time intervals δt_i with the same length as the liquidation period are determined over the entire available history of the risk factors[2]:

$$\delta S_j(\delta t_i) = S_j(t - i\,\delta t + \delta t) - S_j(t - i\,\delta t) \quad \text{with} \quad i = 1, \ldots, m \;\; ; \;\; j = 1, \ldots, n \,. \tag{23.4}$$

For example, the time series over 250 days yields 249 daily changes or 240 changes for a liquidation period of 10 days.[3]

The historical risk factor changes applied to the price of the risk factors at time t (today) provide m different scenarios. For reasons of consistency,

[2]The number of available liquidation periods obtained from the historical time span for which data is available is denoted by m, the number of relevant risk factors again by n.

[3]In the second case, the liquidation periods overlap resulting in auto-correlations.

the *relative* changes are often be applied to today's price rather than the absolute changes.[4] For each scenario i the thus induced value change δV_i of the portfolio V is computed. This can be accomplished with a full valuation of the portfolio. However, in practice it is often the case that a simple linear (delta valuation) or quadratic (delta-gamma valuation) approximation as in Eq. 22.4 is performed.

In this way, m "historical" value changes $\delta V\,(t_i)$ are generated from the past time series data. The value at risk of a historical simulation is now the minimum of all δV_i, where—similar to the Monte Carlo simulation—unfavorable changes in the portfolio's value falling outside a previously specified confidence interval are ignored. For a confidence level of 95%, for example, and a history consisting of 250 days, the 12 worst out of the 249 portfolio value changes are not considered when finding the minimum over daily changes.

If δV_{1-c} denotes the most favorable change among the ignored value changes at a confidence level of c, then the value at risk of the portfolio is the least favorable portfolio change greater than V_{1-c}:

$$\mathrm{VaR}_V(c) = -\min_i \{\delta V_i \mid \delta V_i > \delta V_{1-c}\} \quad \text{with} \quad i = 1, \ldots, m\ .$$

With a confidence c, the portfolio at the end of the liquidation period depreciates by an amount no larger than this value at risk.

At this point we can clearly see the greatest disadvantage of this method: the weak statistical information on the basis of which the probabilistic conclusions such as confidence levels are drawn. Despite the effort in data management of all relevant historical time series, usually only a dozen (in the above example) or so values remain for the final analysis, namely those falling below the lower boundary of the confidence interval. The probabilistic conclusion is drawn on the basis of these few values. In contrast, the statistical basis deriving from 10,000 Monte Carlo simulation runs is approximately 50 times larger (of course, this statistical advantage of the Monte Carlo method is at the expense of assuming that the risk factors are random walks). In addition, the results could be biased because of autocorrelation effects due to the overlapping time intervals.

A further disadvantage of historical simulations is the following effect: For each change of position in a portfolio (after each transaction), the new

[4] A historical change for example, a 12 point change in the DAX index which stood at 1200 at the outset of a liquidation period is quite different from a 12 point change when the DAX is at 7000. A relative change of 1% (i.e., 70 points) is therefore more suitable. This is not necessarily the best choice for all possible risk factors, though. For interest rates, absolute shifts are also applied frequently.

portfolio (and its value changes) must be recalculated for all 250 days. In doing so, it may happen that *another* historical risk factor change affects the make-up of the set falling outside the confidence interval for the new portfolio so that suddenly the value at risk based on *another scenario* is relevant. Thus, a trader, after entering into a transaction intended to optimize the VaR according to the *original* scenario, is then informed of a value at risk computed on the basis of *another* scenario. This greatly increases the difficulty of evaluating the success of the transaction.

In general, the historical simulation is carried out by simulating the same risk factors, which are required for the risk-neutral valuation of the portfolio, i.e. volatilities implied from quoted option prices (if the volatilities are not quoted directly anyway) instead of historical volatilities based on historical time series of, e.g. quoted share prices. Therefore, the value at risk is based on the portfolio's simulated *risk-neutral* present values, based on *real-world* historical changes of the underlying risk factors.

23.3 Crash and Stress Testing: Worst Case Scenarios

Each value at risk concept introduced up to this point yields the *potential loss* in the course of a liquidation period and the *probability* with which no bigger loss occurs. The confidence levels most commonly used are 95% or 99%. This means that losses amounting to the value at risk or higher actually occur between 2 and 12 times per year, as they actually should. Otherwise the model on the basis of which these probabilities are derived is incorrect. The value at risk can thus not be considered a worst case scenario, but rather as part of daily business: losses of this magnitude must occur on average once a month at a confidence level of 95%! Accordingly, the value of these losses must be kept below an acceptably small limit.

In order to obtain a measure of a portfolio's risk should a catastrophe occur, a *worst case* or *crash scenario* is constructed by hand through the explicit specification of all risk factors influencing the portfolio. The portfolio is then revaluated on the basis of this market scenario. Such a scenario could, for example, be the financial crisis 2008/2009. The difference between the calculated value and the current portfolio value is the "value at risk" of the portfolio with respect to the crash-scenario. Obviously, this value expresses the potential loss as a result of the crash but no information is available about the *probability* of such an event.

A further method used to get a feeling for the risk of a portfolio in the case of rare and very unfavorable market developments is to use 6 or 8 standard deviations from the expectation as the boundary of the confidence interval rather than the usual 1.65 or 2.36, i.e. to consider selected, extreme scenarios. This approach is sometimes referred to as a *stress test.* In this way, a potential loss is obtained as well as a *theoretical* probability that a loss of this magnitude is incurred. For example, for a standard normal distribution, the probability of a loss of more than six standard deviations in a one-sided confidence interval is approximately one to one billion:

$$1 - \frac{1}{\sqrt{2\pi}} \int_{-6}^{\infty} e^{-x^2/2} dx \approx 9,86610^{-10} \approx 10^{-9} .$$

No great importance should be attached to such probability statements since the random walk assumptions, constant volatilities and correlations, for instance, are in all probability no longer satisfied when such events occur. As a rule, market scenarios of this type change the correlations drastically and the volatilities explode. Thus, de facto, stress tests, like crash tests, provide information on the potential loss involved without specifying the probability of such an event.

23.4 Advantages and Disadvantages of the Commonly Used Value at Risk Methods

In Table 23.1 the advantages and disadvantages of the VaR methods introduced above are summarized. A "+" in the method column indicates that this particular method has the advantage of the property associated with the corresponding row. A "−" means that it has the disadvantage of the corresponding row. No entry indicates that the method does not have the property of the row under consideration. A symbol in parentheses means that the indicated property is usually assumed in the application of the method but that the advantage or disadvantage is not, in *principle*, characteristic of the method.

Table 23.1 The pros and cons of the most common value at risk methods

	Variance-covariance	Monte Carlo	Historical simulation
Models risk factors as random walks	$-$	$-$	
Assumes constant vol & correlation	$-$	$(-)$	
Requires historical time series			$-$
Requires vol & correlations	$(-)$	$(-)$	
Neglects the mean yield	$-$		
Linear proxy for risk factors	$-$		$(-)$
Linear proxy for prices (delta valuation)	$-$	$-$	$-$
Full valuation		$+$	$+$
Specified scenarios		$+$	$+$
Specified probabilities	$+$	$+$	
Based on large data sets		$+$	
Valid for long liquidation periods		$+$	
Takes vega risk into account		$(+)$	$+$
Takes theta risk into account		$+$	$+$