14

Interest Rates and Term Structure Models

So far, with only few exceptions (e.g. Sect. 8.3.3), we have considered interest rates as being deterministic or even constant. This directly contradicts to the simple existence of interest rate options. If interest rates were deterministic and hence predictable with certainty for all future times, we would know for certain if a given interest rate option is either worthless (because it is out-of-the-money) or otherwise would be a simple forward contract (if it is in-the-money) and could be priced by simple cash flow discounting.

Nevertheless, interest rates may assumed to be deterministic, if the derivative underlying are shares (or some other asset class like FX or commodities) and if the term to expiry is rather short (e.g. less than 3 years), since in such a case, the value of an equity derivative is much more sensitive to changes of the underlying share price than to changes of the interest rate level. Also, the volatility of share prices is often much higher than interest rate volatilities.[1] Empirical studies (e.g. see [109, 145]) also demonstrate that different effects of "false" assumptions (in particular, the assumed equality of forward and futures prices) tend to cancel out each other.

Methods for interest rate modeling and valuation of interest rate options have been for more than 20 years and area of active research in financial mathematics. *Term structure models* are used to model the stochastic changes observed in interest rate curves, similar to the way stock prices and exchange rates are modeled with an underlying stochastic process $S(t)$.[2]

[1]As we will see later, e.g. in Sect. 30.1 ff, there is more beyond Black-Scholes.

[2]The Black-76 model for simple interest rate options can be derived as a special case of the more complex Heath-Jarrow-Morton term structure model.

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In contrast to a stock price or exchange rate $S(t)$, an interest rate $R(t, T)$ depends on *two* time variables, namely on the time t at which the interest rate is being considered *and* on the maturity T of the term over which the interest is to be paid. Usually, interest is paid at a higher rate for longer terms $\tau := T - t$ (i.e., later maturities) than for shorter ones. For a fixed time t (today), the set of all interest rates for the various maturity dates T form a curve called the *term structure*. Theoretically, each point in this curve is a stochastic variable associated with a single maturity. The term structure is thus theoretically a continuum of infinitely many stochastic variables. From a practical point of view, these processes are naturally very strongly correlated; the interest rate over a term of 3 years and 1 day is (almost) the same as that over a term of 3 years and 0 days and so on. In practice, market participants therefore consider only finitely many terms whose lengths lie well distinct from one another (for example, terms $\tau = T - t$ of 1 day, 1 month, 3 months, 6 months, 9 months, 1 year, 2 years, 3 years, 5 years, 10 years and so on), depending on which liquid market quotes have been used to build up the curve. Motivated by the results of principle component analysis (see for instance Sect. 34.2) which show that well over 90% of the dynamics of the yield curve can be explained by just one or two stochastic factors, most term structure models go a step further and reduce the number of factors driving the stochastic evolution of the entire term structure to just a few stochastic variables (e.g. 1 or 2 for simple term structure models). These models are referred to as 1-factor or 2-factor, for example, depending on the number of stochastic variables.

14.1 Instantaneous Spot Rates and Instantaneous Forward Rates

Many simple 1-factor models are based on a stochastic process of the form specified in Eq. 2.19, where the stochastic factor is usually assumed to be a very short term rate, called *instantaneous* interest rate. These rates take the form of either an *instantaneous spot rate* (also called the *instantaneous short rate*) or an *instantaneous forward rate*.[3] The terms $\tau = T - t$ belonging to these rates are infinitesimally short, i.e., we consider the limit $\tau \to 0$, or equivalently $T \to t$.

[3]This approach is to be differentiated from *market rate models* (which are also used to be called *Brace-Gatarek-Musiela models* or (*BGM models* for short). For such models, forward interest rates over longer periods (e.g. the 3-months LIBOR rate), but distinct start dates, are modeled (see Sect. 14.13 and [19]).

In the following discussion, we adopt the convention of continuous compounding usually observed in the literature. This allows the instantaneous rates to be defined quite easily. The *instantaneous spot rate* $r(t)$ is defined by

$$e^{-r(t)dt} := \lim_{dt \to 0} B(t, t + dt)$$
$$r(t) = -\lim_{dt \to 0} \frac{\ln B(t, t + dt)}{dt} \,. \tag{14.1}$$

The *instantaneous forward rate* $f(t, T)$ is defined by

$$e^{-f(t,T)dT} := \lim_{dT \to 0} B(T, T + dT \mid t) = \lim_{dT \to 0} \frac{B(t, T + dT)}{B(t, T)}$$
$$f(t, T) = -\lim_{dT \to 0} \frac{1}{dT} \ln \frac{B(t, T + dT)}{B(t, T)} = -\lim_{dT \to 0} \frac{\ln B(t, T + dT) - \ln B(t, T)}{dT} \,,$$

where Eq. 2.7 has been used. Thus

$$f(t, T) = -\frac{\partial \ln B(t, T)}{\partial T} \,. \tag{14.2}$$

Integrating this equation over dT and making use of the fact that $B(t, t) = 1$ yields

$$\int_t^T f(t, s)ds = -\int_t^T \frac{\partial \ln B(t, s)}{\partial s} ds = -\ln B(t, T) + \ln \underbrace{B(t, t)}_{1} = -\ln B(t, T) \,,$$

and therefore

$$B(t, T) = \exp\left[-\int_t^T f(t, s)ds\right] \,. \tag{14.3}$$

This, together with Eq. 2.7, implies that the forward rate over a finite time interval of length $T' - T$ is the *average* of the instantaneous forward rates over this interval:

$$R(T, T' \big| \, t) = -\frac{1}{T' - T} \ln \frac{B(t, T')}{B(t, T)} = \frac{1}{T' - T} \int_T^{T'} f(t, s)ds \,. \tag{14.4}$$

The definitions of instantaneous spot and forward rate imply that both are interest rates with continuous compounding (see also Table 2.5).

The relation between the forward rates and zero bond yields (spot rates over finite time intervals of length $T-t$) can be established as well by inserting the explicit form of the discount factor for continuous compounding into Eq. 14.2 and taking the derivative with respect to T:

$$f(t,T) = -\frac{\partial\,(\ln\exp[-R(t,T)(T-t)])}{\partial T} = \frac{\partial\,[R(t,T)(T-t)]}{\partial T}\ ,$$

And after application of the product rule

$$f(t,T) = R(t,T) + (T-t)\frac{\partial R(t,T)}{\partial T}\ . \tag{14.5}$$

The forward rates are greater than the spot rates for $\partial R(t,T)/\partial T > 0$, i.e., for term structures (interest rate term structure = spot rates $R(t,T)$ as a function of T) whose values increase with T.

Finally, we make note of the relationship between the instantaneous forward rate and the *instantaneous* spot rate. This is simply

$$r(t) = \lim_{T\to t} f(t,T)\ . \tag{14.6}$$

In anticipation of the following sections we stress here that all of the above equations hold in any arbitrary probability measure, i.e., irrespective of any choice of numeraire, since they have been derived directly from the definitions of the instantaneous interest rates.

14.2 Important Numeraire Instruments

As was shown in Chap. 13 in great detail, the value V of an arbitrary, tradable interest rate instrument[4] normalized with an arbitrary, tradable financial instrument Y is a process $Z = V/Y$ which, according to Eq. 13.15, is a martingale

$$Z(t) = \mathrm{E}_t^Y\,[Z(u)] \qquad \forall u \geq t\ .$$

[4]In spot rate models, every interest rate instrument can be interpreted as a derivative V on the underlying $S(t) = r(t)$ or $S(t) = \ln r(t)$.

The martingale measure at time t with respect to which the expectation E_t^Y is calculated depends on the choice of the numeraire instrument Y. In principle, any arbitrary tradable instrument can be used as the *numeraire*. An appropriate choice of numeraire, however, is a deciding factor in enabling an elegant solution of specific problems to be found. Two numeraire instruments are particularly popular (not only for term structure models but for other models as well). These will be introduced in the following sections.

14.2.1 The Risk-Neutral Measure

We define $\beta(t_0, t)$ as the value of a *bank account* or *money market account*. This is the value one monetary unit (for example, 1 euro) has at time t, if it was invested at a time $t_0 < t$ and was subsequently always compounded at the current spot rate, with the interest earnings being immediately reinvested in the same account at the current spot rate. Intuitively, we could imagine reinvesting ever decreasing interest payments earned over ever shorter interest periods, the number of interest periods finally approaching infinity. The value of such an account can be written in terms of the instantaneous spot rate as

$$\beta(t_0, t) = \exp\left[\int_{t_0}^{t} r(s)ds\right] . \tag{14.7}$$

In the *risk-neutral measure* this bank account, Eq. 14.7, is used as the numeraire.

$$Y(t) = \beta(t_0, t) = \exp\left[\int_{t_0}^{t} r(s)ds\right] \qquad \text{for arbitrary } t_0 \leq t .$$

This numeraire has the advantage of satisfying the important property specified in Eq. 13.34. Since

$$\frac{dY(t)}{dt} = r(t)Y(t) .$$

we have

$$dY(t) = Y(t+dt) - Y(t) = \left(e^{r(t)dt} - 1\right) Y(t) \approx r(t)Y(t)dt$$

Both $r(t)$ and $Y(t)$ are known at time t (even if they are stochastic variables and as such not yet known for the future time $t + dt$). This implies that the process $m(t) = r(t)Y(t)$ is previsible as required in Eq. 13.34.

With this numeraire the martingale property Eq. 13.15 becomes

$$\frac{V(t)}{\beta(t_0, t)} = \mathrm{E}_t^{\beta}\left[\frac{V(u)}{\beta(t_0, u)}\right] \quad \forall u \geq t \geq t_0 \, .$$

Note that the initial time point t_0 in the money account can be chosen at will. For every t_0 we obtain another, different risk-neutral measure. Setting $t_0 = t$ and using $\beta(t, t) = 1$ reduces the above expression to

$$V(t) = \mathrm{E}_t^{\beta}\left[\frac{V(u)}{\beta(t, u)}\right] = \mathrm{E}_t^{\beta}\left[e^{-\int_t^u r(s)ds} V(u)\right] \quad \forall u \geq t \, , \tag{14.8}$$

which directly yields the price of the financial instrument. In words:

With respect to the risk-neutral measure, today's value of a financial instrument is equal to the expectation of the discounted future value.

This is *not* the same as the discounted future expectation. In this measure, the discounting is performed *first* and *then* the expectation is calculated. Discounting means *division* by the numeraire.

We take as an example, the value at time t of a zero-coupon bond with maturity u, i.e., we set $V(t) = B(t, u)$ and consequently $V(u) = B(u, u) = 1$, in Eq. 14.8 to obtain

$$B(t, u) = \mathrm{E}_t^{\beta}\left[\frac{1}{\beta(t, u)}\right] = \mathrm{E}_t^{\beta}\left[e^{-\int_t^u r(s)ds}\right] \, . \tag{14.9}$$

The bond price is the expectation with respect to the risk neutral measure of the reciprocal of the bank account. Comparing this with Eq. 14.3, which always holds, yields the relationship between the instantaneous forward rates and the future instantaneous spot rates with respect to the risk-neutral measure

$$e^{-\int_t^u f(t,s)ds} = \mathrm{E}_t^{\beta}\left[e^{-\int_t^u r(s)ds}\right] \, .$$

The *future price* of an instrument V is $V(u)$ for $u > t$ and is unknown at time t. It is well known that this must be distinguished from the *forward price* $V(t,u)$, which is known at time t since it follows from arbitrage arguments (see Eq. 6.1). If the instrument under consideration pays no dividends between t and u, the forward price is

$$V(t,u) = \frac{V(t)}{B(t,u)} \quad \text{with } u > t \; . \tag{14.10}$$

The forward price, Eq. 14.10, of an instrument with respect to the risk-neutral measure is given by

$$\frac{V(t)}{B(t,u)} = \frac{1}{B(t,u)} \mathrm{E}_t^{\beta}\left[\frac{V(u)}{\beta(t,u)}\right] = \mathrm{E}_t^{\beta}\left[e^{-\int_t^u (r(s)-f(t,s))ds} V(u)\right] \quad \forall u \geq t \; , \tag{14.11}$$

where Eq. 14.3 was used, exploiting the fact the instantaneous forward rates are by definition known at time t and thus can be included in or taken out of the expectation operator as desired.

14.2.2 The Forward-Neutral Measure

For the *forward-neutral measure* or T-emphterminal measure (to emphasize the fixed end date T), a zero bond is used as the numeraire (see Eq. 14.3)

$$Y(t) = B(t,T) = \exp\left[-\int_t^T f(t,s)ds\right] \quad \text{for arbitrary } T > t \; .$$

This numeraire has the property prescribed in Eq. 13.34 as well:

$$\frac{dY(t)}{dt} = f(t,t)Y(t) = r(t)Y(t) \; .$$

Both $r(t)$ and $Y(t)$ are known values at time t, implying the previsibility of $m(t) = r(t)Y(t)$ as required in Eq. 13.34.

With this choice, the martingale property equation 13.15 becomes

$$\frac{V(t)}{B(t,T)} = \mathrm{E}_t^{T}\left[\frac{V(u)}{B(u,T)}\right] \quad \forall T \geq u \geq t \; .$$

The index T of E represents here always the maturity of the zero bond numeraire. Observe that T can be selected arbitrarily. For each choice of T, we obtain a different normalizing factor and thus another forward-neutral measure. Setting $T = u$ and using the fact that $B(u, u) = 1$ we obtain the following price for a financial instrument

$$V(t) = B(t, u)\mathrm{E}_t^u\left[V(u)\right] = e^{-\int_t^u f(t,s)ds}\mathrm{E}_t^u\left[V(u)\right] \quad \forall u \geq t\ . \qquad (14.12)$$

> Today's value of a financial instrument is equal to the discounted expectation of its future value taken with respect to the forward neutral measure.

The expectation is *first* taken with respect to this measure and *then* discounted. Discounting means *multiplication* by the numeraire.

The forward price, Eq. 14.10, for an interest rate instrument is, with respect to this measure, exactly equal to the expected future price, hence the name "forward-neutral"

$$\frac{V(t)}{B(t, u)} = \mathrm{E}_t^u\left[V(u)\right] \quad \forall u \geq t\ . \qquad (14.13)$$

14.3 The Special Case of Deterministic Interest Rates

The reader may be somewhat confused, since in earlier chapters (see in particular Eq. 9.20), prices of financial instruments were always calculated by discounting the future expectation using $B(t, T)$, which, from the above discussion, would indicate that the *forward*-neutral measure was used in the calculations. The measure was, however, always referred to as the *risk*-neutral measure. In those chapters however, interest rates were always assumed to be completely deterministic (or in many cases even constant). We will now show that for deterministic interest rates, the forward-neutral and the risk-neutral measures are identical.

Consider a portfolio consisting of a zero bond $B(t, T)$ and a loan made at time t to finance the purchase of the zero bond. The interest rate on this loan is floating and is always equal to the current spot rate for each interest period δt. After one such period δt, the loan debt will have grown to $B(t, T)\exp(r(t)\delta t)$. The portfolio thus constructed has no value at time t

and the evolution of the entire portfolio over one time step δt is known exactly at time t, and therefore involves no risk. Because arbitrage is not possible and no risk is taken, the portfolio must have no value for all later times as well. Thus, at time $t + \delta t$ the value of the zero bond must be equal to the value of the loan

$$B(t + \delta t, T) = B(t, T)\exp(r(t)\delta t) \ .$$

So far, everything is as it was as for stochastic interest rates. On the right-hand side of the above equation there appear only terms which are known at time t. The difference comes in the next time step: for deterministic interest rates, the spot rates at all later times $u > t$ are known at time t as well. After *two* interest periods, the credit debt will have grown to $B(t, T)\exp(r(t)\delta t)\exp(r(t + \delta t)\delta t)$ with a *known* interest rate $r(t + \delta t)$. Again, no risk is taken since the rate $r(t + \delta t)$ is already known at time t. Therefore, because the market is arbitrage free, the portfolio must still be worthless at time $t + 2\delta t$ and the loan must therefore still equal the value of the zero bond. Proceeding analogously over n time steps, we obtain the value of the zero bond as

$$B(t + n\delta t, T) = B(t, T)\exp\left[\sum_{i=0}^{n-1}(r(t + i\delta t)\delta t)\right] \ .$$

Taking the limit as $\delta t \to 0$, the value of the bond at time $u := t + n\delta t$ is

$$B(u, T) = B(t, T)\exp\left[\int_t^u r(s)ds\right] \quad \text{with} \quad u \geq t \ .$$

This holds for every $u \geq t$, in particular for $u = T$. Thus, observing that $B(T, T) = 1$, we obtain the price of a zero bond for deterministic interest rates:

$$B(t, T) = \exp\left[-\int_t^T r(s)ds\right] = \frac{1}{\beta(t, T)} \ , \tag{14.14}$$

where in the last step the definition of a bank account, Eq. 14.7, was used. Hence, for deterministic interest rates, the numeraire $B(t, T)$ associated to the forward-neutral measure is equal to the reciprocal $1/\beta(t, T)$ of the numeraire of the risk-neutral measure. Substituting this into Eq. 14.8, we obtain

$$V(t) = \mathrm{E}_t^{\beta}\left[\frac{V(u)}{\beta(t, u)}\right] = \mathrm{E}_t^{\beta}\left[B(t, u)V(u)\right] = B(t, u)\mathrm{E}_t^{\beta}\left[V(u)\right] \ ,$$

where in the last step we have made use of the fact that $B(t, u)$ is known at time t, is therefore not stochastic and can be factored out of the expectation. The price of a financial instrument must, however, be independent of the measure used in its computation. In consequence, comparison of this equation with Eq. 14.12 immediately yields the equation

$$B(t,u)\mathrm{E}_t^u\left[V(u)\right] = V(t) = B(t,u)\mathrm{E}_t^\beta\left[V(u)\right] ,$$

and thus $\mathrm{E}_t^u\left[V(u)\right] = \mathrm{E}_t^\beta\left[V(u)\right]$. This implies that both measures are identical if interest rates are deterministic.

The fundamental difference between the general case and deterministic interest rates is that the price of a zero bond in Eq. 14.14 is given by the *future spot rates*. Of course, the general equation 14.3 stating that the price of a zero bond is given by the *current forward rates* continues to hold. We might suspect that this is closely related to the fact that the future instantaneous spot rates must equal the current instantaneous forward rates if interest rates are deterministic. This is in fact the case since the derivative of Eq. 14.14 with respect to T gives

$$r(T) = -\frac{\partial}{\partial T}\ln B(t,T) .$$

Comparing this with Eq. 14.2, which holds in general, yields

$$f(t,T) = r(T) \quad \forall t, T \quad \text{with } t \leq T .$$

Thus, if interest rates are deterministic, the instantaneous forward rates are indeed equal to the (known) future instantaneous spot rates. Since the right-hand side of this equation is not dependent on t, this must be true for the left-hand side as well. Hence, if interest rates are deterministic, the instantaneous forward rates are independent of the present time t.

14.4 Tradable and Non-tradable Variables

As was emphasized at the end of the Sect. 13.2, both the numeraire Y and the financial instrument V *must* be **tradable** instruments in order for the martingale property Eq. 13.1 to hold. Otherwise, potential arbitrage opportunities cannot be exploited by trading and in this way fail to violate the assumption of an arbitrage-free market. This realization, which may seem trivial from our modern perspective, was in the past by no means trivial. In fact, there exist pricing methods which assume that non-tradable variables have the martingale property. The mistake made in doing so is then (approximately) corrected after the fact by making a so-called *convexity adjustment* (see Sect. 14.5).

One example of a non-tradable variable is the *yield* of a tradable instrument whose price is a non-linear function of its yield. Take, for example, a zero bond B with lifetime τ. The yield r of the zero bond depends non-linearly on its price (except if the linear compounding convention has been adopted; see Table 2.5). For example, for simple compounding the price of the zero bond is $B = (1 + r\tau)^{-1}$. The zero bond is obviously a tradable instrument. Therefore its future expectation taken with respect to the forward-neutral measure as in Eq. 14.13 must be equal to its forward price.

$$\begin{aligned} B(T, T+\tau|t) &= \mathrm{E}_t^T\left[B(T, T+\tau)\right] \\ &= \mathrm{E}_t^T\left[\frac{1}{1+r(T, T+\tau)\tau}\right] \geq \frac{1}{1+\mathrm{E}_t^T\left[r(T, T+\tau)\right]\tau} \,. \end{aligned}$$

The lower-equal sign in the above expression follows from Jensen's inequality.[5] In general, the expectation of the price is *not* equal to the price calculated with the expectation of the yield because of the non-linearity in the relation between the price and the yield. How big the difference is depends on the applied interest rate term structure model.

On the other hand, the forward rate for the time period between T and $T + \tau$ is by definition equal to the yield of a forward zero bond over this period; for linear compounding explicitly:

$$\frac{1}{1+r_f(T, T+\tau|t)\tau} = \frac{B(t, T+\tau)}{B(t, T)} \equiv B(T, T+\tau|t) \,. \qquad (14.15)$$

Comparing this with the above expression for the forward price of the bond gives

$$\begin{aligned} &\frac{1}{1+r_f(T, T+\tau|t)\tau} \geq \frac{1}{1+\mathrm{E}_t^T\left[r(T, T+\tau)\right]\tau} \\ \Longrightarrow \quad & r_f(T, T+\tau|t) \leq \mathrm{E}_t^T\left[r(T, T+\tau)\right] \,. \end{aligned}$$

[5] For convex function f and stochastic variable X, Jensen's inequality states that $f(\mathrm{E}[X]) \leq \mathrm{E}[f(X)]$. The above equation follows from the convexity (all points $f(x)$ with $a < x < b$ lie below a straight line through points $f(a)$ and $f(b)$) of function $f(x) = 1/(1 + ax)$.

Therefore, the forward rate can *not* simply (i.e. model independently) be equalized with the future expectation of the spot rate taken with respect to the forward-neutral measure. Because of Eq. 14.12, zero bond yields are no martingales with respect to the forward-neutral measure. Since *all tradable* instruments must be martingales with respect to the forward-neutral measure, zero bond yields can *not* be tradable.

What about products depending linearly on the forward rate? With Eq. 14.15 we have

$$r_f(T, T+\tau|t) = \frac{1}{\tau}\left(\frac{B(t,T)}{B(t,T+\tau)} - 1\right) . \tag{14.16}$$

On the other hand, we could express the zero bond $B(t, T)$ in terms of the expectation in the $(T + \tau)$-forward measure

$$\frac{B(t,T)}{B(t,T+\tau)} = \mathrm{E}_t^{T+\tau}\left[\frac{B(T,T)}{B(T,T+\tau)}\right] . \tag{14.17}$$

Comparison of Eq. 14.16 with Eq. 14.17 yields directly

$$r_f(T, T+\tau|t) = \mathrm{E}_t^{T+\tau}\left[r_f(T, T+\tau|t)\right] = \mathrm{E}_t^{T+\tau}\left[r(T, T+\tau)\right] . \tag{14.18}$$

That's the fundamental reason why, for example, in *forward rate agreements* or *caplets* and *floorlets* (which, of course, are tradable), the difference between the future LIBOR and the strike is not paid out at the LIBOR fixing at the beginning of the relevant interest period but at the *end* of that period (see Sects. 15.2 and 18.6.3,[6] respectively). This has the effect that we are not really dealing with a forward contract (or an option) on a future interest rate, but rather with a forward contract (or an option) on future *zero bonds,* i.e., on *tradable* instruments. The same holds for *swaps*, *caps* and *floors,* which are nothing other than a series of forward rate agreements, caplets and floorlets, respectively, strung together.

[6]In practice, a FRA pays out at maturity (i.e. at the beginning of the interest rate period) the present value of a virtual future payment at the end of the period. This reduces the credit default risk. If, as we did here, credit default risk is neglected, or if the trade is sufficiently collateralized, this does not have a significant impact.

14.5 Convexity Adjustments

According to Eq. 14.12, today's price $V(t)$ of a financial instrument normalized with respect to the forward-neutral numeraire (or if interest rates are considered deterministic) is equal to its discounted future expectation. If the future time for which the expectation is calculated is chosen to be the maturity date T of the instrument, then the price of the instrument is equal to the discounted future expectation of its *payoff profile* $V(T)$. For instruments whose payoff profiles are *linear* functions of the underlying S, i.e., $V(S,T) = a + bS$, the expectation of the *payoff profile* is equal to the payoff profile of the expectation of the *underlying*:

$$
\begin{aligned}
\mathrm{E}[V(S,T)] = \mathrm{E}[a+bS] &= \int_{-\infty}^{\infty}[a+bS]\,p(S)dS = a\int_{-\infty}^{\infty}p(S)dS + b\int_{-\infty}^{\infty}S\,p(S)dS \\
&= a + b\mathrm{E}_t^T[S] \ , \qquad (14.19)
\end{aligned}
$$

where p denotes the probability density of the pertinent martingale measure.

Although the prices of most instruments (for example, the zero bond) are non-linear functions of their underlyings, there do exist transactions with linear payoff profiles for which Eq. 14.19 holds, for example forward contracts. The payoff profile of a forward contract with maturity T and delivery price K is known to be $S(T) - K$. The expectation of this payoff profile is simply the expectation of the underlying less the delivery price:

$$
\mathrm{E}[V(S,T)] = \mathrm{E}[S(T) - K] = \mathrm{E}[S(T)] - K \ .
$$

If the underlying S of the forward contract is itself a *tradable* instrument (for example, a stock), we can now go a step further. With respect to the forward-neutral normalization (or in the case of deterministic interest rates), the future expectation of any tradable instrument is equal to its current forward price in accordance with Eq. 14.13, i.e., $\mathrm{E}_t^T[S(T)] = S(t,T)$. Furthermore, Eq. 14.12 states that with respect to this measure, today's price is equal to the discounted future expectation. So in summary (and in agreement with Eq. 6.5):

$$
\begin{aligned}
V(S,t) &= B(t,T)\mathrm{E}_t^T[V(S,T)] \\
&= B(t,T)\left(\mathrm{E}_t^T[S(T)] - K\right) \\
&= B(t,T)(S(t,T) - K) \ .
\end{aligned}
$$

In the first of the above equations, we used Eq. 14.12. In the second line we made use of the fact that the payoff profile is a linear function of the underlying and hence, that Eq. 14.19 holds, while in the third equation, we finally used of the namesake property of the forward-neutral measure, Eq. 14.13.

The mistake made in some traditional pricing methods corresponds to precisely this last step, i.e., simply replacing the future expectation of the underlying as in Eq. 14.13 with the forward price of the underlying even if the underlying is *not* a tradable instrument (and as such, is not a martingale with respect to the forward-neutral measure and does not satisfy Eq. 14.13). This mistake is then corrected (approximately) after the fact by a *convexity adjustment*.

The convexity adjustment is defined as the difference between the future expectation of the underlying (with respect to the forward-neutral measure) and the forward price of the underlying

$$\text{Convexity Adjustment} \equiv \mathrm{E}_t^T\left[S(T)\right] - S(t,T)\ . \tag{14.20}$$

For interest rate underlyings, it is often the case that convexity adjustment need to be taken into account. As shown in Sect. 14.4, zero bond yields are no tradable instruments, because the relation between those yields and related prices of the (tradable) instruments are non-linear, according to Table 2.5:

$$B(t,t+\tau) = \begin{cases} \exp(-r\tau) & \text{continuous} \\ (1+r)^{-\tau} & \text{discrete} \\ (1+r\tau)^{-1} & \text{simple}\ . \end{cases} \tag{14.21}$$

We now want to determine an approximation for the convexity adjustment of the yield of such a zero bond, i.e., we want to determine

$$\text{Convexity Adjustment} = \mathrm{E}_t^T\left[r(T,T+\tau)\right] - r_f(T,T+\tau|\,t)\ .$$

Here we have written out explicitly the time dependence of the zero bond yields: $r(T, T+\tau)$ is the (unknown future) spot rate at time T for a period of length τ starting at time T. The forward rate for that same period, as known at time t, is denoted by $r_f(T, T+\tau|\,t)$. To find the expectation of the unknown future spot rate we expand the (also unknown) future bond price formally written as function of r as $B(T, T+\tau) = B(r, T, T+\tau)$ as a Taylor series up to second order around the (known) forward rate $r_f = r(T, T+\tau|t)$:

$$B(r,T,T+\tau) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[r(T,T+\tau) - r_f(T,T+\tau|t)\right]^n \left.\frac{\partial^n B(r,T,T+\tau)}{\partial r^n}\right|_{r=r_f}$$

$$\approx B_{r_f} + \left[r - r_f\right] \left.B'\right|_{r=r_f} + \frac{\left[r-r_f\right]^2}{2} \left.B''\right|_{r=r_f} ,$$

where in the last line we have dropped all time arguments for ease of notation. We now calculate the forward-neutral expectation of this bond price:

$$\underbrace{\mathrm{E}_t^T\left[B\right]}_{B_r} \approx B_{r_f} + \left.B'\right|_{r=r_f} \mathrm{E}_t^T\left[r - r_f\right] + \frac{1}{2} \left.B''\right|_{r=r_f} \underbrace{\mathrm{E}_t^T\left[(r-r_f)^2\right]}_{\approx \mathrm{var}[r]} . \tag{14.22}$$

Here, we used the fact that B and its derivatives are evaluated at r_f which is known at time t. Therefore, B and its derivative can be factored out of the expectation.

On the left-hand side appears the expectation of the bond price with respect to the forward-neutral measure. This is, since bonds are tradable, exactly the forward bond price and, by the definition of the forward rate r_f, is identical to $B(r_f, T, T+\tau)$.

The expectation $\mathrm{E}_t^T\left[(r-r_f)^2\right]$ is approximately equal to the variance of $r(T, T+\tau)$ (it would be exactly this variance if $r_f = \mathrm{E}_t^T[r]$). To express this variance in terms of values known at time t, the variance is approximated by the variance of the *forward rate*, i.e.:

$$\mathrm{E}_t^T\left[(r-r_f)^2\right] \approx \mathrm{Var}\left[r\right] \approx \mathrm{Var}\left[r_f\right] = r_f^2\sigma_f^2\left(T-t\right) . \tag{14.23}$$

Here the volatility σ_f of the forward rate r_f, called the *forward volatility*, appears. This is (at least in principle) known at time t.

Substituting all of the above into Eq. 14.22 yields

$$0 \approx \left.B'\right|_{r=r_f} \left(\mathrm{E}_t^T\left[r\right] - r_f\right) + \frac{1}{2} \left.B''\right|_{r=r_f} r_f^2\sigma_f^2\left(T-t\right) .$$

This can now be solved for the desired expectation of the future spot rate:

$$\mathrm{E}_t^T\left[r\right] \approx r_f \underbrace{- \frac{1}{2} r_f^2\sigma_f^2\left(T-t\right)\frac{\left.B''\right|_{r=r_f}}{\left.B'\right|_{r=r_f}}}_{\text{convexity adjustment}} . \tag{14.24}$$

Within these approximations, i.e., by expanding the bond price up to second order (see Eq. 14.22) and with the approximations in Eq. 14.23, the future expectation of the interest rate can thus be approximated by the forward rate adjusted by the amount:

$$
\begin{aligned}
&\mathrm{E}_t^T\left[r(T, T+\tau)\right] - r_f(T, T+\tau \mid t) \\
&\approx -\frac{1}{2} r_f(T, T+\tau \mid t)^2 \sigma_f^2 (T-t) \frac{\left.B''(r, T, T+\tau)\right|_{r=r_f}}{\left.B'(r, T, T+\tau)\right|_{r=r_f}} \\
&= \begin{cases} \frac{1}{2} r_f(T, T+\tau \mid t)^2 \sigma_f^2 (T-t)\tau & \text{continuous} \\ \frac{1}{2} r_f(T, T+\tau \mid t)^2 \sigma_f^2 (T-t) \frac{\tau(\tau+1)}{1+r_f} & \text{discrete} \\ r_f(T, T+\tau \mid t)^2 \sigma_f^2 (T-t) \frac{\tau}{1+\tau r_f} & \text{simple.} \end{cases}
\end{aligned}
\tag{14.25}
$$

Here, the convexity adjustments for all compounding conventions listed in Eq. 14.21 have been explicitly calculated.

14.5.1 In-Arrears Swaps

As an example, we consider a forward contract on an interest rate index (zero bond yield). As was mentioned at the end of Sect. 14.4, no convexity correction is required for a standard forward rate agreement, if the payment of the interest rate (difference), which is fixed at the start of the period, is at the *end* of the period (resp. if the present value of the virtual cash flow at the end of the period is paid at the start of the period). Effectively, such contracts are forward contracts on (tradable) zero bonds. The same holds for plain vanilla swaps since such a swap can be interpreted as a portfolio of forward rate agreements.

But there are swaps for which the difference between the future interest rate index, e.g. LIBOR, and the fixed side is paid at the same time when the LIBOR rate is fixed. Or, to put it the other way round: the interest payable is determined only at the time when payment is to be made, i.e., at the *end* of the corresponding interest period. Such instruments are called *In-Arrears Swaps*. Here indeed the underlying is directly the LIBOR rate which is *not* a tradable instrument. If we nevertheless wish to price such an instrument as if we could replace the future expectation of the underlying (with respect to the forward-neutral measure) with the forward rate, the resulting error must be corrected, at least approximately, by the convexity adjustment.

For the sake of simplicity, we consider only one period of an In-Arrears Swap, an *In-Arrears Forward Rate Agreement (FRA)*[7] so to speak. This has a principal N, a fixed interest rate K and extends over a period from T to $T+\tau$. A potential compensation payment is calculated by means of simple compounding and flows at the beginning of this period, i.e., directly at time T when the LIBOR rate $r(T, T+\tau)$ is fixed. Such an FRA has a payoff profile given by

$$V(r(T, T+\tau), T) = N\tau\,[r(T, T+\tau) - K]\;. \tag{14.26}$$

This is a *linear* function of the underlying r. The expectation of the payoff profile is thus simply

$$\mathrm{E}\,[V(r(T, T+\tau), T)] = N\tau\mathrm{E}\,[r(T, T+\tau)] - N\tau K\,, \tag{14.27}$$

and its value today is this expectation, taken with respect to the forward-neutral measure, discounted back to today

$$V(r(T, T+\tau), t) = B(r, t, T)N\tau\left(\mathrm{E}_t^T\,[r(T, T+\tau)] - K\right)\;.$$

Up to now, all the equations are exact. The calculation of the future expectation is now performed either using a term structure model or an approximation by means of the convexity adjustments given in Eq. 14.24:

$$\begin{aligned}
&V(r(T, T+\tau), t)\\
&= B(r, t, T)N\tau\left(\mathrm{E}_t^T\,[r(T, T+\tau)] - K\right)\\
&\approx B(r, t, T)N\tau\left(r_f(T, T+\tau|\,t)\right.\\
&\qquad\qquad\qquad \left. -\frac{1}{2}r_f^2(T, T+\tau|\,t)\sigma_f^2\,(T-t)\frac{B''(r, T, T+\tau)\big|_{r=r_f}}{B'(r, T, T+\tau)\big|_{r=r_f}} - K\right)\\
&= B(r, t, T)N\tau\left(r_f(T, T+\tau|\,t) + r_f^2(T, T+\tau|\,t)\sigma_f^2\,(T-t)\frac{\tau}{1+\tau r_f} - K\right)\;.
\end{aligned} \tag{14.28}$$

[7] An In-Arrears Swap is simply a portfolio consisting of such FRAs.

The convexity adjustment for simple compounding has been used in the last step, since the specified instrument prescribes this compounding convention. Thus for r_f, we must use the forward rate *with respect to simple compounding* as well.

Again, we emphasize that above considerations hold only for those instruments whose payoff profiles are *linear* functions of the underlying. Equation 14.19 holds only in this case and only than does the expectation of the *underlying* come into play. For instruments with *non*-linear payoff profiles, on the other hand, the expectation of the *payoff profile* must be calculated directly. For example, for a plain vanilla call the expectation of the payoff profile is, in contrast to Eq. 14.19

$$\mathrm{E}\left[\max\left\{S(T)-K,0\right\}\right]=\int_{-\infty}^{\infty}\max\left\{S-K,0\right\}p(S)dS$$
$$=\int_{K}^{\infty}(S-K)p(S)dS\neq\max\left\{\mathrm{E}\left[S(T)\right]-K,0\right\}\ .$$

14.5.2 Money Market Futures

The above example of a In-Arrears FRA may appear a bit academic to the reader. In reality though, *money market futures* are among the most actively traded interest derivatives and are nothing other than In-Arrears FRAs traded on an exchange, most commonly based on the 3-month LIBOR or the EURIBOR.

A money market future with a nominal N, a fixed rate K over a period from T to $T+\tau$ yields (theoretically) at maturity T a compensation payment calculated using simple compounding as in Eq. 14.26, thus

$$V(T)=N\tau\left[r(T,T+\tau)-K\right]\ .$$

Since this instrument has a *future-styled* payment mode, the changes in the position's value do not remain unrealized until maturity T, but are *immediately* realized on a margin account. As explained in Sect. 6.1.4, this has the effect that today's value of a futures position is equal to the future expectation of the payoff profile *without discounting*. This expectation of the future payoff profile is given by Eq. 14.27. The value of a money market future at time t is then, in contrast to Eq. 14.28, simply:

$$V(t)=N\tau\left(\mathrm{E}_t^T\left[r(T,T+\tau)\right]-K\right)\ . \tag{14.29}$$

This is obviously equal to zero when $K = \mathrm{E}[r(T, T+\tau)]$. The fixed rate K of a money market future is in fact always chosen so that the value of the contract is zero at the time when the contract is entered into. The fixed rate K of a money market futures contracted at time t thus gives directly the information on the opinion of the market on the value of $\mathrm{E}[r(T, T+\tau)]$, in other words, on the *future expectation* of the interest rate. Since the interest rate is not a tradable instrument, this is *not* equal to the forward rate.

Money market futures are often used in constructing spot rate curves for maturities ranging from approximately three months to two years (we refer the reader to Part VI). To construct spot rate curves from such contracts, the *forward* rates associated with these transactions are needed, since we can use these to calculate the spot rates quite easily by utilizing Eq. 2.6, for example. To determine the forward rate $r_f(T, T+\tau \mid t)$ from the market's opinion on the future LIBOR *expectation* obtained from quotes on money market futures, the convexity adjustment must be *subtracted* from the expectation as in Eq. 14.20 to obtain

$$\begin{aligned} r_f(T, T+\tau \mid t) &= \mathrm{E}_t^T\left[r(T, T+\tau)\right] - \text{Convexity Adjustment} \\ &\approx \mathrm{E}_t^T\left[r(T, T+\tau)\right] \\ &\quad + \frac{1}{2} r_f^2(T, T+\tau \mid t)\sigma_f^2 (T-t) \frac{B''(r, T, T+\tau)\big|_{r=r_f}}{B'(r, T, T+\tau)\big|_{r=r_f}} \\ &= \mathrm{E}_t^T\left[r(T, T+\tau)\right] - r_f^2(T, T+\tau \mid t)\sigma_f^2 (T-t) \frac{\tau}{1+\tau r_f}\,, \end{aligned} \tag{14.30}$$

where in the last step the approximation in Eq. 14.25 for the convexity adjustment for the linear compounding convention is used. This is a non-linear equation which can be solved numerically for the unknown r_f.

As has been often emphasized, the convexity adjustments presented above are only approximations since they derive from an approximation of the theoretical value of $\mathrm{E}_t^T\left[r(T, T+\tau)\right]$, see Eq. 14.24). The future expectation $\mathrm{E}_t^T\left[r(T, T+\tau)\right]$ can, however, be calculated by other means, for example, using a term structure model. We then obtain *another* expression for $\mathrm{E}_t^T\left[r(T, T+\tau)\right]$, and not Eq. 14.24. Since the forward rates are determined solely from arbitrage considerations (independent of any model) as in Eq. 2.6, this implies that also another expression for the convexity adjustment is obtained. The convexity adjustment is thus dependent on the method (the term structure model) being used. To ensure consistency, the convexity adjustment for the money market futures used in constructing the spot rate curves should be consistent with the term structure model applied for pricing.

Quotation for Money Market Futures

In Europe, money market futures are traded primarily on the *LIFFE*. Futures on the 3-month LIBOR in pound sterling (*short sterling future*), in US dollars (*euro dollar future*), in euros (*euro EUR future*) and in Swiss franc (*euro Swiss future*) are available for trade on this exchange. Futures in euros on the 3-month EURIBOR and on the 1-month EURIBOR are available for trade on the *EUREX*.

Money market futures are quoted in a way which takes some getting used to. Not the delivery price$\mathrm{E}_t^T\,[r(T, T+\tau)]$ is quoted, i.e., the fixed interest rate for which the future has no value, but rather

$$\text{Quote}_{\text{Money Market Future}} = 100\% - \mathrm{E}_t^T\,[r(T, T+\tau)]\ .$$

A quote of 96.52%, for example, for a money market future means that in the opinion of the market, the expectation (in the forward neutral measure) for the future 3-month rate is $\mathrm{E}[r(T, T+\tau)] = 3.48\%$. The value at time t of a futures position contracted at time $t = 0$ with $K = \mathrm{E}_0^B\,[r(T, T+\tau)]$ is given, according to Eq. 14.29, by

$$\begin{aligned} V(t) &= N\tau\left(\mathrm{E}_t^T\,[r(T, T+\tau)] - \mathrm{E}_0^T\,[r(T, T+\tau)]\right) \\ &= N\tau\underbrace{\left(1 - \mathrm{E}_0^T\,[r(T, T+\tau)]\right)}_{\text{Quote at Time } t=0} - N\tau\underbrace{\left(1 - \mathrm{E}_t^T\,[r(T, T+\tau)]\right)}_{\text{Quote at Time } t}\ . \end{aligned}$$

A money market future on a 3-month LIBOR (i.e., $\tau = 1/4$) with a nominal amount of $N = 1{,}000{,}000$ euros which was agreed to at a quoted price of 96.52% and which is currently quoted as 95.95% is thus valued at

$$\begin{aligned} V(t) &= 1.000.000\ \text{EUR} \times \tfrac{1}{4}\,(96,52\% - 95,95\%) \\ &= 1.425\ \text{EUR}\ . \end{aligned}$$

This amount is deposited in a margin amount. A change in the value of this position is directly reflected by a corresponding daily adjustment in the balance of the margin account.

14.6 Arbitrage-Free Interest Rate Trees Grid (Tree) Models

Construction of arbitrage-free *tree models* begins with the assumption that the martingale property Eq. 13.15 holds, justifying the name arbitrage free.[8] An appropriate normalizing factor (numeraire) is selected and the integrals necessary for the computation of the expectations (with respect to the chosen measure) are discretized in a tree-structure. This procedure will be demonstrated explicitly in this section for *1-factor short rate models*, i.e., for models which have the instantaneous short rate, defined in Eq. 14.1, as their one and only stochastic driver.

We will use the risk-neutral measure and the associated numeraire, which is the bank account. The price of every interest rate instrument is then given by Eq. 14.8. We discretize first with respect to time by partitioning the time axis in intervals of length δt taking this length to be so small that the (stochastic) short rate can be assumed to be constant over this interval. Then Eq. 14.8 for $u = t + \delta t$ becomes

$$\begin{aligned} V(t) &= \mathrm{E}_t^{\beta}\left[e^{-\int_t^{t+\delta t} r(s)ds} V(t+\delta t)\right] \\ &\approx \mathrm{E}_t^{\beta}\left[e^{-r(t)\delta t} V(t+\delta t)\right] \\ &= e^{-r(t)\delta t} \mathrm{E}_t^{\beta}\left[V(t+\delta t)\right] \\ &= B(t, t+\delta t)\, \mathrm{E}_t^{\beta}\left[V(t+\delta t)\right] \; . \end{aligned} \tag{14.31}$$

In calculating the integral, we have made use of the assumption that r is approximately constant on the interval of integration. Variables which are known at time t can be factored out of the expectation. According to the last equation, the risk-neutral price and the forward-neutral price (see Eq. 14.12) cannot be distinguished from one another over the very *short* time interval δt. This is in agreement with Sect. 14.3, since r is taken to be constant (in particular, deterministic) over this short time interval. The equality, however, does not hold for longer time spans, since r changes randomly from one time interval δt to the next. Thus, globally, we are still within the framework of the risk-neutral measure even though the local equations may look "forward neutral".

[8]More traditional models, known as *equilibrium models*, will not be investigated here. Our discussion will be restricted to arbitrage-free pricing methods.

14.6.1 Backward Induction

What we have accomplished up to this point is to factor the numeraire (the bank account) out of the expectation. It now remains to calculate the expectation of the instrument's price. To this end, we discretize the continuous range of V after a time step of length δt into finitely many values, i.e., starting from the value of V at time t, the value should be allowed to take on only finitely many different values after the next time step. Allowing *two* different values generates a *bi*nomial tree, *three* a *tri*nomial tree, etc. Since the financial instrument V under consideration is an interest rate instrument, the different values potentially taken on by V at time $t+\delta t$ result directly from the different possible interest rate term structures which might exist at $t + \delta t$. Since one of the model assumptions was that the evolution of the entire interest rate curve is driven by the instantaneous short rate, it follows that the different values attained by V are ultimately determined by the values this short rate can take on.

We will work below with binomial trees. We assume that the short rate increases to the value r_u or decreases to the value r_d with a probability p and $1-p$, respectively. We have more than one possibility at our disposal to ensure that the martingale property is satisfied (and thus eliminating arbitrage). Either we fix the values r_u and r_d and selects the probability p accordingly so that the market is governed by an arbitrage-free measure (this, for example, is done in finite difference methods where the grid is given at the onset of the analysis), or we specify the probability p first and subsequently select appropriate values for r_u and r_d. We will take the second path in our discussion here. We set

$$p = 1/2 \tag{14.32}$$

and determine the value of the short rate (i.e., the discount factors) on all nodes of the tree so that the short rate process as described by the tree guarantees arbitrage freedom at time t. The binomial tree with $p = 1/2$ allows the expectation in Eq. 14.31 to be written as

$$\begin{aligned} V(t) &\approx B(t, t + \delta t)\, \mathrm{E}_t^{\beta}\left[V(t + \delta t)\right] \\ &\approx B(t, t + \delta t)\ \left[p\, V(r_u, t + \delta t) + (1 - p)\, V(r_d, t + \delta t)\right] \\ &= B(t, t + \delta t)\ \left[\frac{1}{2} V(r_u, t + \delta t) + \tfrac{1}{2} V(r_d, t + \delta t)\right] \\ &= B(t, t + \delta t)\ \left[\tfrac{1}{2} V_u + \tfrac{1}{2} V_d\right]\ , \end{aligned} \tag{14.33}$$

where in the last step, the short form notation $V_u := V(r_u, t + \delta t)$, etc. has been introduced.

One time step later, the short rate branches again (and in consequence, the price of the financial instrument does as well): $r_u \to r_{u\,u}$ with probability p, and $r_u \to r_{u\,d}$ with probability $1 - p$, and similarly for r_d. In addition, note that at time $t + \delta t$, two *different* discount factors $B(t + \delta t, t + 2\delta t)$ appear, according to whether the short rate rose to r_u or fell to r_d in the previous step. To emphasize the difference, the discount factors are indexed with the associated short rate. V_u, for example, is then expressed as

$$\begin{aligned} V_u &\approx B_{r_u}(t + \delta t, t + 2\delta t)\, \mathrm{E}_t^{\beta}\, [V_u(t + 2\delta t)] \\ &\approx B_{r_u}(t + 1\delta t, t + 2\delta t) \left[\frac{1}{2} V(r_{uu}, t + 2\delta t) + \frac{1}{2} V(r_{ud}, t + 2\delta t)\right] \\ &= B_u \left[\frac{1}{2} V_{uu} + \frac{1}{2} V_{ud}\right], \end{aligned}$$

where in the last step the short form notation $V_{uu} := V(r_{u\,u}, t + 2\delta t)$, analogous to that introduced above, has been used, and also the short form notation $B_u := B_{r_u}(t + 1\delta t, t + 2\delta t)$. Analogously,

$$V_d \approx B_d \left[\frac{1}{2} V_{du} + \frac{1}{2} V_{dd}\right].$$

Substituting this into Eq. 14.33, the price $V(t)$ given by a binomial tree with two steps can be calculated as

$$V(t) \approx B(t, t + \delta t) \left[\frac{1}{2} B_u \left[\frac{1}{2} V_{uu} + \frac{1}{2} V_{ud}\right] + \frac{1}{2} B_d \left[\frac{1}{2} V_{du} + \frac{1}{2} V_{dd}\right]\right]. \tag{14.34}$$

Proceeding analogously, $V_{dd} \approx B_{dd} \left[\frac{1}{2} V_{ddu} + \frac{1}{2} V_{ddd}\right]$, etc., the price at time t calculated from the prices three time steps later is given by

$$\begin{aligned} V(t) \approx B(t, t + \delta t)\, &[\frac{1}{2} B_u\, [\frac{1}{2} B_{uu}\, [\frac{1}{2} V_{uuu} + \frac{1}{2} V_{uud}] + \frac{1}{2} B_{ud}\, [\frac{1}{2} V_{udu} + \frac{1}{2} V_{udd}]] \\ &+ \frac{1}{2} B_d\, [\frac{1}{2} B_{du}\, [\frac{1}{2} V_{du\,u} + \frac{1}{2} V_{dud}] + \frac{1}{2} B_{dd}\, [\frac{1}{2} V_{ddu} + \frac{1}{2} V_{ddd}]]], \end{aligned} \tag{14.35}$$

and so on. Calculating backwards through the tree (*backward induction*) is completely analogous to the treatment of options on stocks and exchange rates with binomial trees as described in Chap. 9.

In order to ensure that any arbitrage opportunity has been eliminated, the short rates r_u, r_d, r_{uud} (and in consequence, the discount factors B_u, B_d, B_{uud}), etc. must be chosen so that the *prices calculated* using the tree actually agree with the *market prices* of traded instruments. In particular, *zero bonds* (and thus today's term structure) must be exactly reproduced. However, at each time point $t + i\delta t$, there are just as many unknowns as a (non-recombinant) binomial tree has nodes, namely 2^i starting with $r_{uu\ldots u}$ continuing through all permutations of up and down moves until $r_{dd\ldots d}$. The number of unknowns increases exponentially with the number of time steps! So many conditions cannot conceivably be generated by the market prices of tradable instruments. In particular, no such functional relation between the number of interest rate instruments on the market and the number of time steps (depending only on the numerical implementation) in a tree could possibly exist. Therefore, based on this principle, another condition must first be established preventing the exponential growth of the number of unknowns with respect to increasing i.

Requiring that the tree be *recombinant* for all financial instruments with path-independent payoff profiles presents itself as a good candidate for the above mentioned condition, i.e., that $V_{ud} = V_{du}$, $V_{uud} = V_{udu} = V_{duu}$, etc. This can only happen if the tree for the underlying also recombines. A recombinant binomial tree is known to have only $i + 1$ nodes after i steps. This fact will be accounted for in our notation. The nodes of the tree at which we arrive having traveled upwards i times and downwards j times is uniquely determined by the ordered pair (i, j), irrespective of the order in which these upward and downward steps were taken. This is because the tree is recombinant. Therefore we denote the value of the financial instrument at this node with $V(i, j)$ and the nodes themselves with the ordered pairs (i, j) as presented in Fig. 14.1. For example, $V(1, 2) = V_{udd} = V_{dud} = V_{ddu}$, etc.

We use the same notation for the short rate and for the zero bonds evaluated at the nodes:

$$\begin{aligned}
(i, j) &= \text{Node after } i \text{ up moves and } j \text{ down moves} \\
r(i, j) &= \text{Instantaneous short rate at the node } (i, j) \\
V(i, j) &= \text{Value of an interest rate instrument at the node } (i, j) \\
B(i, j) &= \exp\{-r(i, j)\delta t\} \ . \qquad (14.36)
\end{aligned}$$

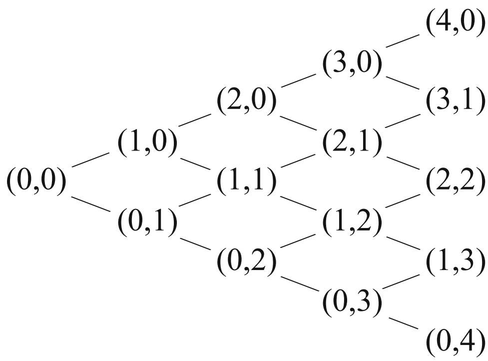

Fig. 14.1 The binomial tree with the indexing showing the number of up and down moves required to get to the nodes starting from node $(0, 0)$. For instance it takes two up moves and one down move to get to the node $(2, 1)$

In particular, $B(t, t + \delta t) = B(0, 0)$. In this notation, the prices of (path independent) financial instruments can be written after one binomial step in a recombinant tree as

$$\begin{aligned} V(t) &= V(0,0) \\ &\approx V(1,0)\frac{1}{2}B(0,0) \\ &\quad + V(0,1)\frac{1}{2}B(0,0) \end{aligned} \tag{14.37}$$

Of course, this equation holds not only for the node (0, 0) but for any arbitrary node (i, j) in the tree

$$V(i, j) \approx B(i, j)\left[\frac{1}{2}V(i+1, j) + \frac{1}{2}V(i, j+1)\right] \tag{14.38}$$

This equation applied to $V(1, 0)$ and $V(0, 1)$ yields the price after two binomial steps

$$\begin{aligned} V(t) \approx\ & V(2,0)\frac{1}{4}B(0,0)\ B(1,0) \\ &+ V(1,1)\frac{1}{4}B(0,0)\ [B(1,0) + B(0,1)] \\ &+ V(0,2)\frac{1}{4}B(0,0)\ B(0,1) \end{aligned} \tag{14.39}$$

and after three

$$
\begin{aligned}
V(t) \approx\ & V(3,0)\frac{1}{8}B(0,0)B(1,0)B(2,0) \\
& + V(2,1)\frac{1}{8}B(0,0)\left[B(1,0)B(2,0) + B(1,1)\left[B(1,0) + B(0,1)\right]\right] \\
& + V(1,2)\frac{1}{8}B(0,0)\left[B(0,1)B(0,2) + B(1,1)\left[B(1,0) + B(0,1)\right]\right] \\
& + V(0,3)\frac{1}{8}B(0,0)B(0,1)B(0,2)
\end{aligned}
\tag{14.40}
$$

and so on. The expressions were purposely written in terms of the prices at the last node in each branch.

14.6.2 Forward Induction and Green's Functions

Although backward induction was simplified greatly by the requirement that the tree be recombinant, we are still not yet prepared to calculate an instrument's price since the instantaneous discounting factors $B(i, j)$ at the nodes $(i, j) \neq (0, 0)$ remain unknown. Before we construct a tree for the instantaneous discount factors (i.e., for the short rate, see Eq. 14.36) via a procedure, known as *forward induction*, on the basis of arbitrage considerations, we introduce a class of extremely useful "artificial" instruments. One such artificial instrument whose value at time t is denoted by $G(i, j)$ pays by definition one monetary unit if and only if the underlying (the short rate) attains the tree node (i, j) at time $t + (i + j)\delta t$. This is the node at which we arrive having traveled upwards i times and downward j times, regardless of the order in which the upward and downward moves occurred as the tree is recombinant. $G(i, j)$ then is the value at time t of a single monetary unit paid out at one single node of the tree, namely at node (i, j). In this sense, G is the system's reaction to a perturbation of magnitude one at a single point in the system. This is analogous to the Green's functions in physics. We will therefore refer to G as a *Green's function*. By definition,

$$
G(0,0) \equiv 1\ . \tag{14.41}
$$

In order to establish further values of the Green's function, we just have to set the value $V(i, j)$ in the above Eqs. 14.37, 14.39 and 14.40 equal to one at exactly one node and zero on all the other nodes. Equation 14.37 then yields

$$G(1,0) = \frac{1}{2}B(0,0) = G(0,1) \tag{14.42}$$

From Eq. 14.39 we obtain

$$\begin{aligned}
G(2,0) &= \frac{1}{4}B(0,0)\ B(1,0) = \frac{1}{2}G(1,0)B(1,0)\\
G(0,2) &= \frac{1}{4}B(0,0)\ B(0,1) = \frac{1}{2}G(0,1)B(0,1)\\
G(1,1) &= \frac{1}{4}B(0,0)\ [B(1,0) + B(0,1)] = \frac{1}{2}G(1,0)B(1,0) + \frac{1}{2}G(0,1)B(0,1)
\end{aligned}$$

and finally Eq. 14.40 gives

$$\begin{aligned}
G(3,0) &= \frac{1}{2}G(2,0)B(2,0)\\
G(0,3) &= \frac{1}{2}G(0,2)B(0,2)\\
G(2,1) &= \frac{1}{2}G(2,0)B(2.0) + \frac{1}{2}G(1,1)B1,1)\\
G(1,2) &= \frac{1}{2}G(0,2)B(0,2) + \frac{1}{2}G(1,1)B(1,1)\ .
\end{aligned}$$

The following general recursion relation can be easily verified (this will be proven for an even more general case later)

$$\begin{aligned}
G(i,j) &= \frac{1}{2}G(i,j-1)B(i,j-1) + \frac{1}{2}G(i-1,j)B(i-1,j) \quad \text{for } i>0, j>0\\
G(i,0) &= \frac{1}{2}G(i-1,0)B(i-1,0)\\
G(0,j) &= \frac{1}{2}G(0,j-1)B(0,j-1)
\end{aligned} \tag{14.43}$$

The prices of all path-independent interest rate instruments can be represented as linear combinations of the Green's function evaluated at diverse nodes on the tree since each payment profile of the form $f(r, T)$ can be distributed on the nodes (i, j) as appropriate:

$$f(r, T) \rightarrow f_T(i, j) := f(r(i, j), T) \quad \text{with} \quad t + (i + j)\delta t = T \quad \forall i, j \ .$$

The value at time t of each individual payment $f_T(i, j)$ at node (i, j) is naturally equal to the Green's function belonging to this node (which has a value of exactly one monetary unit) multiplied by the number of monetary units that are to be paid, i.e., multiplied by $f_T(i, j)$. The total value of the financial instrument $V(t)$ with this payoff profile is then simply

$$V(t, T) = \sum_{(i,j)} f_T(i, j)\, G(i, j) \quad \text{with} \quad t + (i + j)\delta t = T \quad \forall i, j \ ,$$

where $\sum_{(i,j)}$ denotes "the sum over the nodes (i, j)". This can be immediately generalized to instruments with payoff profiles defined on *arbitrary* nodes (which need not all lie in the set of nodes corresponding to a time T)

$$V(t) = \sum_{(i,j)} f(i, j)\, G(i, j) \quad \text{for arbitrary payoff profiles } f(i, j) \ . \tag{14.44}$$

Path independence here is therefore *not* the restriction that the payoff profile depend only on a time point T. The payoff profile can depend on (the interest rate at) all possible nodes at all times, not however, on the path taken to arrive at these nodes (since this information is not available in the recombinant tree).

As a simple example, we consider the value of a zero bond $B(t, T)$, an instrument that pays one monetary unit at time T, regardless of the state of the underlying

$$\begin{aligned} B(t, T) &= \sum_{(i,j)} G(i, j) \ \text{ for } \ t + (i + j)\delta t = T \qquad (14.45) \\ &= \sum_{i=0}^{n} G(i, n - i) \quad \text{for} \quad n = \frac{T - t}{\delta t} \ . \end{aligned}$$

This *theoretical* price has to exactly match the *market* price of the zero bond to prevent arbitrage. If the current term structure is available (for example, because it has been constructed on the basis of traded benchmark bonds for some maturities and interpolations in between) this term structure yields the *market* price for all zero bonds $B(t, T)$. In particular it yields the market prices for all those zero bonds maturing at times $t + i\delta t$ corresponding to the time steps of the binomial tree. Those have to be matched by Eq. 14.45. Hence, we obtain one single condition for each time step $t + i\delta t$. For example, for $i = 1$

$$
\begin{aligned}
B(t, t + 1\delta t) &= \sum_{(i,j)} G(i, j) \ \text{ with } \ i + j = 1 \\
&= G(1, 0) + G(0, 1) \\
&= B(0, 0) \ ,
\end{aligned}
$$

where in the last step, the Green's function as in Eq. 14.42 is introduced into the equation. This simply checks for consistency: the discount factor at the first node of our tree (the right-hand side) must be equal to the market price for the zero bond with maturity $T = t + \delta t$ (left-hand side). It becomes more interesting at the next maturity date:

$$
\begin{aligned}
B(t, t + 2\delta t) &= \sum_{(i,j)} G(i, j) \ \text{ with } \ i + j = 2 \\
&= G(2, 0) + G(1, 1) + G(0, 2) \\
&= \frac{1}{2} G(1, 0) B(1, 0) + \frac{1}{2} G(1, 0) B(1, 0) \\
&\quad + \frac{1}{2} G(0, 1) B(0, 1) + \frac{1}{2} G(0, 1) B(0, 1) \\
&= G(1, 0) B(1, 0) + G(0, 1) B(0, 1) \ .
\end{aligned}
$$

We have used Eq. 14.43 to calculate backwards from the Green's function at the time corresponding to $i + j = 2$ to the previous time step, i.e., to the time for which $i + j = 1$. The Green's function $G(1, 0)$ at the earlier time step $t + 1\delta t$ is already known (see Eq. 14.42). The resulting equation is the arbitrage condition for the instantaneous discount factors $B(1, 0)$ and $B(0, 1)$ at the tree nodes $(1, 0)$ and $(0, 1)$. This procedure can be generalized to n time steps. In accordance with Eq. 14.45 we write

$$
B(t, t + n\delta t) = \sum_{i=0}^{n} G(i, n - i) \ .
$$

At this point we separate the boundary terms from the rest of the sum, since these obey another recursion in Eq. 14.43. Now applying Eq. 14.43, we see that

$$\begin{aligned} B(t,t+n\delta t) &= G(0,n) + \sum_{i=1}^{n-1} G(i,n-i) + G(n,0) \\ &= \frac{1}{2}G(0,n-1)B(0,n-1) + \frac{1}{2}\sum_{i=1}^{n-1} G(i,n-i-1)B(i,n-i-1) \\ &\quad + \frac{1}{2}\sum_{i=1}^{n-1} G(i-1,n-i)B(i-1,n-i) + \frac{1}{2}G(n-1,0)B(n-1,0)\;. \end{aligned}$$

The boundary terms combine this expression into a simple sum. To see this, we use the index $k = i - 1$ in the second sum

$$\begin{aligned} B(t,t+n\delta t) &= \frac{1}{2}G(0,n-1)B(0,n-1) + \frac{1}{2}\sum_{i=1}^{n-1} G(i,n-i-1)B(i,n-i-1) \\ &\quad + \frac{1}{2}\sum_{k=0}^{n-2} G(k,n-k-1)B(k,n-k-1) + \frac{1}{2}G(n-1,0)B(n-1,0)\;. \end{aligned}$$

Both boundary terms now have exactly the form needed to extend the index range in both sums to 0 through $n - 1$. This means that the no arbitrage requirement can be represented as a simple recursion formula by means of the Green's function:

$$B(t,t+n\delta t) = \sum_{i=0}^{n-1} G(i,n-i-1)B(i,n-i-1)\;. \tag{14.46}$$

Remember: on the left we have the market price of a zero bond, while on the right we have the values to be determined for the interest rate tree, i.e., the instantaneous discount factors at the nodes. The values on the right-hand side are all in terms of nodes corresponding to the time point

$$t + [i + (n-i-1)]\,\delta t = t + (n-1)\delta t\;,$$

i.e., the time step prior to the maturity $t + n\delta t$ of the zero bond on the left. The *Green's function* evaluated at this earlier time point has already been calculated in the previous iteration step so that Eq. 14.46 represents an arbitrage condition for the instantaneous *discount factors* at the nodes at time point $t + (n - 1)\delta t$. Having determined the discount factors, we can use the recursion for the Green's function, Eq. 14.43, to determine the values of the Green's function at the time point $t + n\delta t$. These are subsequently used again in the arbitrage condition, Eq. 14.46, to calculate the next discount factors for the time step $t + n\delta t$. These are then used to calculate the next Green's function values for $t + (n + 1)\delta t$ utilizing Eq. 14.43 and so on. In this way, an arbitrage-free interest rate tree is constructed using *forward* induction from today into the future. Nevertheless, Eq. 14.46 contains several (to be precise n) unknown discount factors which cannot be uniquely determined by only *one* arbitrage condition (except, of course, for the case $n = 1$). In other words, the exact reproduction of the term structure at time t is already attained singly from the fact the instantaneous discount factors and the Green's function at each node satisfy Eqs. 14.46 and 14.43. But this does not fix the *numerical* values of all instantaneous discount factors—or short rates $r(i, j)$. Additional information (as we will see below, the volatility) extracted from the market as well as an explicit specification of a stochastic process of the form 2.19 for the short rate is needed to fix the numerical values of $r(i, j)$. Before we start on this point however, we will first introduce a few more concepts which hold in general, i.e., for every interest rate tree, irrespective of the specification of a specific stochastic process.

14.7 Market Rates vs. Instantaneous Rates

The valuation of financial instruments using the Green's function as in Eq. 14.44 above is only possible for those instruments whose payoff profiles are functions of the *instantaneous* short rate.[9] This is not usually the case for *traded* instruments, however. Typical interest rate underlyings for traded instruments are 3-month or 6-month LIBOR rates, which belong to longer time periods. Choosing such a long period as time distant in the tree, the calculation would no longer be accurate enough for many applications. In contrast to the chapters on stock or FX options, the stochastic process being simulated (the short rate) does *not* describe the evolution of the underlying (the 3-month LIBOR, for example) of the instrument to be priced! This problem can be overcome in

[9]Or for financial instruments whose payoff profile is independent of the short rate, such as a zero bond, for which $f(i, j) = 1$ for $t + (i + j)\Delta t = T$ and $f(i, j) = 0$ otherwise.

some cases, as for floaters and forward rate agreements, since, as will be seen in Eqs. 15.21 or 15.4, their prices at time t can be interpreted as a combination of zero bond prices and as such can be priced *exactly* using Eq. 14.46. However, the decomposition into zero bonds will not be possible for options such as caps and floors on the 3-month LIBOR whose general payoff functions are given by Eq. 18.2 and, adopting the usual market practice of using simple compounding over the cap period, through Eq. 18.2 (or when interpreted as bond options by Eq. 18.6). In such cases, before the payoff profile at nodes (m, n) with $t + (m + n)\delta t = T$ can be calculated, the value of the *underlying* on each of these nodes must first be determined. The question now is: how do we calculate the underlying of interest (for example, a 3-month rate, a 6-month rate, a swap rate, etc.) at all nodes corresponding to the exercise date T from the stochastic process (the tree) of the instantaneous short rate?

As soon as the payoff profile at all nodes is known, the value of all (path independent) financials instruments at time t, i.e., at the node $(0, 0)$, are directly given by Eq. 14.44. As with the node $(0, 0)$, the value of all instruments (in particular, of all zero bonds and thus all interest rates over arbitrary interest periods) would be known at an *arbitrary* node (m, n) if the "Green's functions" were known for *that* node.

14.7.1 Arrow-Debreu Prices

Arrow-Debreu prices (*ADP*s for short) are generalized Green's functions whose reference point is a fixed but arbitrary node (m, n) rather than the origin node $(0, 0)$. The Arrow-Debreu price $G_{m,n}(i, j)$ is the value at node (m, n) of an instrument paying one monetary unit at node (i, j). The Arrow-Debreu prices at the node $(m = 0, n = 0)$ are, of course, simply the values of the Green's function introduced above:

$$G(i, j) = G_{0,0}(i, j) .$$

It follows immediately from the geometry of the tree that a monetary unit at node (i, j) can generate non-zero prices at a node (m, n) only if (i, j) is attainable when starting from the node (m, n). We know that m up moves have already occurred at node (m, n). These up moves cannot be undone, even if the following steps consist only of down moves since the index m merely counts the number of up moves having been made up to this point. This implies that for all nodes (i, j) attainable from the starting node (m, n), the condition $i \geq m$ must hold. Likewise, n down moves have already occurred at node

(m, n) which cannot be reversed. As before, this means that the condition $j \geq n$ must hold for all nodes (i, j) which are attainable from the starting point (m, n). All other Arrow-Debreu prices must be zero:

$$\begin{aligned} G_{m,n}(i, j) &= 0 \quad \forall m > i \\ G_{m,n}(i, j) &= 0 \quad \forall n > j \\ G_{i,j}(i, j) &= 1 \quad \forall i, j \ , \end{aligned} \tag{14.47}$$

where the last property is included solely for the sake of completeness, being in itself trivial: one monetary unit at node (i, j) is, of course, worth exactly one monetary unit at this node. This corresponds to property Eq. 14.41 of the Green's function.

A further fundamental property of Arrow-Debreu prices follows from Eq. 14.38, which holds for the value of any instrument at any node in the tree. Setting either $V(i+1, j) = 1$ and $V(i, j+1) = 0$ or $V(i+1, j) = 0$ and $V(i, j+1) = 1$, yields the *instantaneous* Arrow-Debreu prices, i.e., the Arrow-Debreu prices over a time step of length δt

$$G_{i,j}(i+1, j) = \frac{1}{2}B(i, j) = G_{i,j}(i, j+1) \ . \tag{14.48}$$

The instantaneous Arrow-Debreu prices are thus half the instantaneous discount factors. This accomplishes the first step in the calculation of the Arrow-Debreu prices. We now merely require a generalization of the recursion relation given in Eq. 14.43 in order to determine all subsequent Arrow-Debreu prices. To this end, consider one monetary unit at node (i, j) at time $t + (i + j)\delta t$. Because of the binomial structure of the tree, non-zero Arrow-Debreu prices are generated by this monetary unit at two nodes in the previous time slice, that is at nodes $(i, j-1)$ and $(i-1, j)$. The *sum* of the ADPs of *both* of these prices at a still earlier node (m, n) must then be equal to the Arrow-Debreu price at node (m, n) of the whole, original monetary unit at node (i, j). This is illustrated in Fig. 14.2. This means that the Arrow-Debreu prices obey the following recursion:

$$G_{m,n}(i, j) = G_{m,n}(i, j-1)G_{i,j-1}(i, j) + G_{m,n}(i-1, j)G_{i-1,j}(i, j) \ .$$

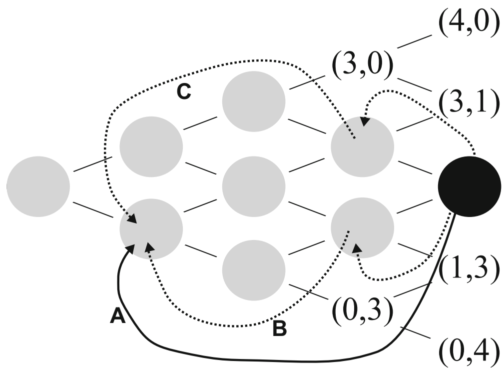

Fig. 14.2 A monetary unit (black dot) at node $(2, 2)$ generates ADPs at nodes $(2, 1)$ and $(1, 2)$ and also at all the earlier 'striped' nodes. The monetary unit must have the same influence (shown as line **A**) on a striped node, e.g. on node $(0, 1)$, as both ADPs it has induced at nodes $(2, 1)$ and $(1, 2)$ together (shown as lines **B** and **C**)

Substituting the corresponding discount factors for both of the instantaneous Arrow-Debreu prices as given in Eq. 14.48, the recursion relation becomes

$$G_{m,n}(i,j) = G_{m,n}(i,j-1)\frac{1}{2}B(i,j-1) + G_{m,n}(i-1,j)\frac{1}{2}B(i-1,j) \text{for } i \geq m,\ \ j \geq n\ . \tag{14.49}$$

Together with the fundamental properties in Eq. 14.47, this recursion determines uniquely all Arrow-Debreu prices. For example, for $m = i$ or $n = j$ Eq. 14.47 implies that one of the Arrow-Debreu prices on the right-hand side is equal to zero.[10] This allows the recursion to be carried out explicitly for these cases:

$$\begin{aligned} G_{i,n}(i,j) &= G_{i,n}(i,j-1)\frac{1}{2}B(i,j-1) = \frac{1}{2^{j-n}}\prod_{k=1}^{j-n} B(i,j-k) \\ G_{m,j}(i,j) &= G_{m,j}(i-1,j)\frac{1}{2}B(i-1,j) = \frac{1}{2^{i-m}}\prod_{k=1}^{i-m} B(i-k,j)\ . \end{aligned} \tag{14.50}$$

[10] For $m = i$ we have $G_{i,n}(i-1, j) = 0$ and for $n = j$ we have $G_{m,j}(i, j-1) = 0$.

In particular, both of these equations hold at the boundary of the tree. The first equation with $i = 0$ (i.e., no up move) represents the *lower* boundary, the second with $j = 0$ (i.e., no down move) the *upper* boundary of the tree.

With these Arrow-Debreu prices, the value of any arbitrary financial instrument with a payoff profile given by $f(i, j)$ at any *arbitrary* node (m, n) is, analogous to Eq. 14.44, simply

$$V(m, n) = \sum_{(i, j)} G_{m,n}(i, j)\, f(i, j)\ . \tag{14.51}$$

In particular, the value $B_\tau(m, n)$ of a zero bond at the node (m, n), whose time to maturity at this node is given by τ, is likewise given by Eq. 14.51 where i and j satisfy the following three conditions

$$\begin{aligned} i + j &= m + n + \tau/\delta t \\ i &\geq m \\ j &\geq n\ . \end{aligned}$$

The first of the three conditions characterizes all nodes corresponding to the time $m + n + \tau/\delta t$, the maturity of the zero bond. The payoff profile $f(i, j)$ of the zero bond equals one at precisely these nodes and zero elsewhere. This condition allows j to be expressed in terms of i. The limits in the sum appearing in Eq. 14.51 can be specified explicitly by the other two conditions which, as a result of Eq. 14.47 must always be satisfied: the lower limit is $i \geq m$, the upper limit follows from $m+n+\tau/\delta t-i = j \geq n$ which can be rewritten as $i \leq m + \tau/\delta t$. The value of the zero bond at node (m, n) with a time to maturity of τ is thus given explicitly by

$$B_\tau(m, n) = \sum_{i=m}^{m+\tau/\delta t} G_{m,n}(i, m + n + \tau/\delta t - i)\ . \tag{14.52}$$

Thus, at each node of the tree, a complete (future) term structure (i.e., future interest rates for arbitrary times to maturity) can be constructed from Arrow-Debreu prices, since the interest rate at node (m, n) for any arbitrary time to maturity τ is by definition (for continuous compounding) given by

$$r_\tau(m, n) = -\frac{1}{\tau} \ln B_\tau(m, n)\ . \tag{14.53}$$

With simple compounding, which is the common convention for typical interest rate indexes like 3-months LIBOR, it follows analogously:

$$r_\tau(m,n) = \frac{1}{\tau}\left(B_\tau^{-1}(m,n) - 1\right) \ , \tag{14.54}$$

so that we are now in a position to price any arbitrary derivative on an underlying whose value can be derived from the term structure.

14.7.2 Pricing Caplets Using Arrow-Debreu Prices

In anticipation of Part III, where caps and floors will be defined, we will demonstrate explicitly how the price of a caplet on a 3-month rate with principal N, strike rate K, exercise time (maturity, to be assumed to be equal to forward rate fixing time) T and payment date $T' = T + \tau$ (with $\tau = 3$ months) can be expressed solely in terms of Arrow-Debreu prices. Observe that the stochastic process for the short rate need *not* be specified in order to do so. What will be shown here holds for *any arbitrary* arbitrage-free short rate model.

Consistent with Eq. 18.2 for the payoff profile of a caplet, we will adopt the market convention of using *simple compounding* over a single caplet period (in this case 3 months), which can be calculated by using Eq. 14.54. The 3-month rate at which the payoff profile is to be evaluated at each node (m,n) with $t + (m+n)\delta t = T$, or equivalently, $n = (T-t)/\delta t - m$ is thus:

$$\begin{aligned} r_\tau(m, a-m) &= \frac{1}{\tau}\left[B_\tau(m,a-m)^{-1} - 1\right] \\ &= \left(b\delta t \sum_{i=m}^{m+b} G_{m,a-m}(i, a+b-i)\right)^{-1} - \frac{1}{b\delta t} \\ &\text{for all } 0 \le m \le a \ , \end{aligned} \tag{14.55}$$

where we have defined

$$a := (T-t)/\delta t \ , \quad b := \tau/\delta t \ .$$

This is the underlying of our caplet at all nodes relevant to the caplet's payoff profile. According to the payoff profile in Eq. 18.2, the values of the caplet at exercise date T, i.e., at nodes $(m, a-m)$ with $0 \leq m \leq a$, are given by

$$f(m, a-m) = \tau N\, B_\tau(m, a-m) \max\left\{r_\tau(m, a-m) - K, 0\right\} .$$

Both the discount factor B_τ appearing in the payoff profile as well as the underlying (the interest rate r_τ) can be expressed in terms of the Arrow-Debreu prices:

$$\begin{aligned} f(m, a-m) = N \sum_{i=m}^{m+b} G_{m,a-m}(i, a+b-i) \\ \times \max\left\{\left(\sum_{i=m}^{m+b} G_{m,a-m}(i, a+b-i)\right)^{-1} - 1 - \tau K, 0\right\} . \end{aligned}$$

Equation 14.44 now directly yields the value of this payoff profile, i.e., the caplet value, at time t

$$\begin{aligned} c^{\text{cap}}(T, T+\tau, K\,|t) &= \sum_{m=0}^{a} G(m, a-m)\, f(m, a-m) \\ &= N \sum_{m=0}^{a} G_{0,0}(m, a-m) \sum_{i=m}^{m+b} G_{m,a-m}(i, a+b-i) \\ &\quad \times \max\left\{\left(\sum_{i=m}^{m+b} G_{m,a-m}(i, a+b-i)\right)^{-1} - 1 - \tau K, 0\right\} . \end{aligned}$$

The price of the caplet has now been expressed completely in terms of the Arrow-Debreu prices. Since these prices can be determined using the recursion relations 14.49 and 14.50 with initial values given by Eqs. 14.47 and 14.48, this procedure can be used to price any interest rate instrument.

Practical Implementation of Arrow-Debreu Prices

Arrow-Debreu prices have four indices, two for the position of the cash flow in the tree and two for the position at which the effect of the cash flow is felt. The numerical implementation thus requires in principle the construction of a four-dimensional field, which leads to computer memory problems for somewhat finer trees as well as performance problems in the computation of all the ADPs. It will prove to be unnecessary in most cases to compute all of the ADPs. In the above case of caplets, for example, it is only necessary to compute the ADPs of the form $G_{m,a-m}(i, a+b-i)$ with the fixed parameters a and b. Thus, only two of the four above mentioned indices are free, namely m and i. Consequently, it is entirely sufficient to determine the two-dimensional field

$$\widetilde{G}_{m,i} := G_{m,a-m}(i, a+b-i)$$

thereby reducing the numerical effort involved considerably. The effort is now only the same as that needed in the determination of the Green's function.

14.8 Explicit Specification of Short Rate Models

Up to this point, the pricing procedure has remained quite general in that no specific term structure model has been used. None of the relations introduced above can as yet be used to calculate an explicit numerical value since, as mentioned above, the arbitrage condition in Eq. 14.46 is by no means sufficient to determine all the (unknown) discount factors appearing in this equation. From now on, we suppose that the term structure model is of the general form 2.19 where we restrict ourselves to functions $b(r, t)$ of the form $b(r, t) = b(t)r^{\beta}$, i.e. models of the form

$$dr(t) = a\,(r, t)\,dt + b(t)r^{\beta}\,dW\ . \tag{14.56}$$

If the exponent $\beta = 0$,

$$dr(t) = a\,(r, t)\,dt + b(t)\,dW\ , \tag{14.57}$$

we end up with a *normal model* or *Gaussian model*. In the special case that parameter a is also independent of r, the model is called *Ho-Lee Model*. Such models with normally distributed short rate have the advantage that they are very easy to implement (for example, as a tree). The Ho-Lee model

$$dr(t) = a(t)dt + b(t)\,dW$$

for constant b is even analytically tractable (for this special case the model was originally invented [97]). It used to be seen as a disadvantage that normal models allow for negative interest rates and that the volatility is absolute rather than relative with respect to the forward rates. However, with the advent of real negative interest rates for EUR and CHF, these features turned out to be clear advantages, at least for the EUR and CHF currency area. Normal models also tend to be easier calibrated in markets with low, but positive interest rates, avoiding the tendency to push the interest rate level away from zero.

The second class of short rate models does not specify a stochastic process for the short rate itself, but for its logarithm:

$$d\ln r(t) = a\,(r, t)\,dt + b(r, t)\,dW \tag{14.58}$$

These models are thus referred to as *lognormal models*. In the special case of the volatility b being independent of r, this process is called the *Black–Derman-Toy model*, which has been especially designed for the binomial tree, as well as the *Black Karasinski model*. Sections 14.11.2 discusses short rate models in more detail. Such lognormal models have the disadvantage that they are much more difficult to implement and cannot be solved analytically. Negative interest rates cannot occur in these models and the parameter b represents the relative volatility, simplifying fitting this model to relative volatilities observed in markets with strictly positive interest rates. Such models can be transformed into models for r via the Ito formula,[11] Eq. 2.21:

$$dr(t) = \left[r(t)a\,(r, t) + \frac{1}{2}b^2(r, t)r(t)\right]dt + r(t)\,b(r, t)\,dW\ . \tag{14.59}$$

The numerical evaluation of lognormal models is easier to implement when written in this form.

14.8.1 The Effect of Volatility

According to the Girsanov Theorem (see Sect. 13.4), the drift $a(r, t)$ is uniquely determined by the probability measure used. We fixed this probability measure "by hand" when we required the condition specified in Eq. 14.32 to be satisfied. In doing so, we implicitly fixed the drift as well. The drift

[11] For this, we need to consider the following: $\ln r(t)$ in Eq. 14.58 corresponds to S in Eq. 2.21. For the function f we take $f(S) = e^S$ (since this is exactly r). The partial derivatives appearing in Eq. 2.21 are then simply $\partial f/\partial t = 0$ and $\partial f/\partial S = \partial^2 f/\partial S^2 = f = r$.

can thus no longer be given as "input" into the specification of the stochastic process. Since different normal and lognormal models differ only in their drift, this implies that a special choice for Eq. 14.32 effectively belongs to a special choice of the normal resp. lognormal model, namely the Ho Lee and Black Derman Toy model. Only the volatility remains as a parameter through which market information (in addition to the bond prices) may enter into our model for the determination of the discount factors in Eq. 14.46. We will now show that if the volatility is given for the time step n, exactly $n-1$ conditions are generated on the n instantaneous discount factors at time n. Taken together with the arbitrage condition given by Eq. 14.46, requiring that the observed market price of the zero bond with maturity $T = t + n\delta t$ be reproduced by the model, we have exactly as many conditions as unknowns and the interest rate tree can be uniquely constructed.

In general, from the viewpoint of the node (i, j), the variance of a variable x (we can think of x as representing, for example, the short rate r or its logarithm $\ln r$) is caused by its possible two different values in the next time step, either x_u or x_d. The expectation and variance for random variables of this type are given by[12]

$$\begin{aligned} \mathrm{E}[x] &= px_u + (1-p)x_d \\ \mathrm{Var}[x] &= p(1-p)(x_u - x_d)^2 \ , \end{aligned} \tag{14.60}$$

In particular, for the case $p = 1/2$ the variance is given by $\mathrm{Var}[x] = (x_u - x_d)^2/4$.

On the other hand, for models of the form given by Eqs. 14.57 and 14.58, the variance of the stochastic variable x (with $x = r$ resp. $x = \ln r$) over an interval of time of length δt is given by $b(r,t)^2\delta t$.

[12] Substituting the expectation into the definition of the variance defined as the expectation of the squared deviation from the expectation gives

$$\begin{aligned} \mathrm{Var}[x] &= \mathrm{E}\left[(x - \mathrm{E}[x])^2\right] \\ &= p\,(x_u - \mathrm{E}[x])^2 + (1-p)\,(x_d - \mathrm{E}[x])^2 \\ &= p\,(x_u - px_u - (1-p)x_d)^2 + (1-p)\,(x_d - px_u - (1-p)x_d)^2 \ . \end{aligned}$$

Multiplying out and collecting terms yields the desired expression.

14.8.2 Normal Models

For models of the form in Eq. 14.57 the variance of the short rate at node (i, j) must satisfy:

$$
\begin{aligned}
& b(i, j)\sqrt{\delta t} = \sqrt{\mathrm{Var}\left[r(i, j)\right]} = \frac{1}{2}\left[r(i+1, j) - r(i, j+1)\right] \\
\Longrightarrow \quad & r(i+1, j) = r(i, j+1) + 2b(i, j)\sqrt{\delta t}\ . \qquad (14.61)
\end{aligned}
$$

This enables us to establish a recursion formula (after performing the substitution $i + 1 \rightarrow i$) for the instantaneous discount factors at all nodes corresponding to the time slice $n = i + j$

$$
\begin{aligned}
B(i, j) &= \exp\left\{-r(i, j)\delta t\right\} \\
&= \exp\left\{-\left[r(i-1, j+1) + 2b(i-1, j)\sqrt{\delta t}\right]\delta t\right\} \\
&= \exp\left\{-2b(i-1, j)\sqrt{\delta t}\delta t\right\}\exp\left\{-r(i-1, j+1)\delta t\right\} \\
&= \exp\left\{-2b(i-1, j)\sqrt{\delta t}\delta t\right\}B(i-1, j+1)\ ,
\end{aligned}
$$

or

$$
B(i, j) = \alpha(i-1, j)B(i-1, j+1) \quad \text{mit} \quad \alpha(i, j) = \exp\left\{-2b(i, j)\delta t^{3/2}\right\}\ . \qquad (14.62)
$$

Recursive substitution into this equation allows each instantaneous discount factor in the nodes corresponding to this time slice to be expressed in terms of a single discount factor at the "lowest" node $(0, i + j)$:

$$
\begin{aligned}
B(i, j) &= \alpha(i-1, j)B(i-1, j+1) \\
&= \alpha(i-1, j)\alpha(i-2, j+1)B(i-2, j+2) \\
&= \alpha(i-1, j)\alpha(i-2, j+1)\alpha(i-3, j+2)B(i-3, j+3) \\
&= \cdots \\
&= B(0, j+i)\prod_{k=1}^{i}\alpha(i-k, j+k-1)\ .
\end{aligned}
$$

Observe that only α-parameters (volatility information) in the time slice $i - k + j + k - 1 = i + j - 1$ are required for the determination of this value, i.e., information from the *previous* time step. Volatility information for the value of the interest rate at the node actually being calculated (i.e., for the spacing of the nodes at the time slice $(i + j)$) is *not* required; these values are determined with volatility information from the immediately proceeding time! This (perhaps counter-intuitive) property is of course nothing other than the *previsibility* which we always require for the coefficients of dW and dt in all models of the form 2.19 or 13.17. At this point now, we get an intuitive picture (see also Fig. 14.3) what it means that a and b in Eq. 2.19 are previsible processes. This previsibility is the deeper mathematical reason for why we are able calculate anything at all.

Using this expression for $B(i, j)$ with $j = n - i - 1$ and substituting it into the arbitrage condition, Eq. 14.46, we obtain the arbitrage condition for the discount factor $B(0, n - 1)$ at the lowest node of the time slice $(n - 1)$:

$$B(t, t + n\delta t) = B(0, n - 1) \sum_{i=0}^{n-1} G(i, n - i - 1) \prod_{k=1}^{i} \alpha(i - k, n - i + k - 2) ,$$

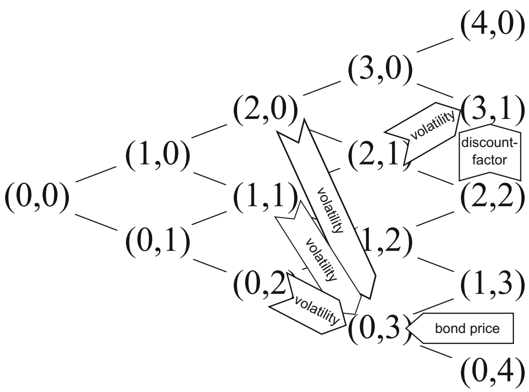

Fig. 14.3 Flow of information when constructing the short rate tree. The discount factor at the lowest node needs the market price of the zero bond maturing one time step *later* and *all* volatility information from one time step *earlier*. For the other discount factors in the time slice it suffices to know the discount factors already calculated at lower nodes in the same time slice and the volatility at the neighboring node one time step earlier

which, after a simple change of index $n \rightarrow (n+1)$, yields the arbitrage condition for the discount factor $B(0, n)$ at the lowest node of time slice n:

$$B(t, t+(n+1)\delta t) = B(0, n) \sum_{i=0}^{n} G(i, n-i) \prod_{k=1}^{i} \alpha(i-k, n-i+k-1) \; . \tag{14.63}$$

The left-hand side is the given *market* price of a zero bond with maturity at time slice $(n+1)$, which must be reproduced using zero bonds and the Green's function on time slice n. The α's on the right-hand side are all defined on time slice $(n-1)$ and are given by the volatility governing the process at this time point. This condition now actually contains only one unknown and can be solved easily for $B(0, n)$:

$$B(0, n) = \frac{B(t, t+(n+1)\delta t)}{\sum_{i=0}^{n} G(i, n-i) \prod_{k=1}^{i} \alpha(i-k, n-i+k-1)} \; .$$

From this single value $B(0, n)$ we obtain all further instantaneous discount factors in this time slice by repeated application of Eq. 14.62. Note that we find ourselves at time n at this point of the iteration. The instantaneous discount factors in the time slice n are determined from the price of the bond maturing at time $(n+1)$ and volatility information from the immediately preceding time step $(n-1)$. This is illustrated in Fig. 14.3.

Having established the instantaneous discount factors, an expression for the instantaneous short rates results immediately from Eqs. 14.36 and 14.62

$$\begin{aligned} r(i, j) &= -\frac{1}{\delta t} \ln B(i, j) = -\frac{\ln B(i-1, j+1)}{\delta t} - \frac{\ln \alpha(i-1, j)}{\delta t} \\ &= r(i-1, j+1) + 2b(i-1, j)\sqrt{\delta t} \\ &= \ldots \\ &= r(0, j+i) + 2\sqrt{\delta t} \sum_{k=1}^{i} b(i-k, j+k-1) \; . \end{aligned} \tag{14.64}$$

For volatility structures depending only on time (but not on the interest rate), the volatility values on time slice n are all identical. We can therefore set them equal to the volatility at the lowest node $(0, n)$. Thus

$$b(i, j) = b(0, i + j) = b(0, n) \equiv \sigma(t + n\delta t)$$
$$\alpha(i, j) = \alpha(0, i + j) = \alpha(0, n) \ \ \forall i, j \text{mit } \ i + j = n \ , \tag{14.65}$$

and Eq. 14.62 reduces to

$$\begin{aligned} B(i, j) &= \alpha(0, n-1) B(i-1, j+1) \\ &= \alpha^i(0, n-1)\, B(0, n) \ \text{ mit } \ i + j = n \ . \end{aligned}$$

Equation 14.61 then implies that the short rate in the tree at time slice n changes from node to node by a constant term $2\sigma\sqrt{\delta t}$

$$\begin{aligned} r(i, j) &= r(i-1, j+1) + 2\sigma(t + (n-1)\delta t)\sqrt{\delta t} \\ &= r(0, n) + 2\, i\, \sigma(t + (n-1)\delta t)\sqrt{\delta t}\ . \end{aligned} \tag{14.66}$$

The arbitrage condition for this discount factor at the lowest node of time slice n reduces to

$$B(t, t + (n+1)\delta t) = B(0, n) \sum_{i=0}^{n} G(i, n-i) \alpha^i(0, n-1)\ .$$

14.8.3 Lognormal Models

An analogous procedure for lognormal models can be obtained as follows: in models of the form specified in Eq. 14.58, $b(i, j)^2 \delta t$ is the variance of the *logarithm* of the short rate. Therefore, as seen from the node (i, j), we have

$$b(i, j)\sqrt{\delta t} = \sqrt{\mathrm{Var}\left[\ln r(i, j)\right]} = \frac{1}{2}\left[\ln r(i+1, j) - \ln r(i, j+1)\right] = \frac{1}{2}\ln\left(\frac{r(i+1, j)}{r(i, j+1)}\right.$$
$$\Longrightarrow \quad r(i+1, j) = r(i, j+1) \exp\left\{2b(i, j)\sqrt{\delta t}\right\} \tag{14.67}$$

This enables us to establish a recursion formula[13] (after performing the substitution $i + 1 \rightarrow i$) for the instantaneous discount factors at all nodes

[13] Here we use the property $\exp\{ax\} = (\exp\{x\})^a$ of the exponential function with $a = e^{2b(i-1, j)\sqrt{\delta t}}$.

in time slice $n = i + j$

$$B(i, j) = \exp\{-r(i, j)\delta t\} = \exp\left\{-r(i-1, j+1)e^{2b(i-1,j)\sqrt{\delta t}}\delta t\right\}$$
$$= \left[\exp\{-r(i-1, j+1)\delta t\}\right]^{e^{2b(i-1,j)\sqrt{\delta t}}} = B(i-1, j+1)^{e^{2b(i-1,j)\sqrt{\delta t}}} ,$$

or

$$B(i, j) = B(i-1, j+1)^{\alpha(i-1,j)} \quad \text{mit} \;\; \alpha(i, j) = \exp\left\{+2b(i, j)\delta t^{1/2}\right\} . \tag{14.68}$$

This is quite similar to the corresponding equation for normal models (Eq. 14.62), but since the volatility information enters the equation as an exponent rather than a factor in the recursion, the structure is somewhat more complicated. Furthermore, a comparison with Eq. 14.62 reveals a sign change in the expression for α. Nevertheless, a simple trick enables us to find a structure which is quite similar to that given in Eq. 14.62. Taking logarithms in Eq. 14.68 yields a recursion relation for the *logarithm* of the discount factors having the same structure as Eq. 14.62:

$$\ln B(i, j) = \alpha(i-1, j) \ln B(i-1, j+1) .$$

This recursion can be carried out explicitly giving

$$\begin{aligned}
\ln B(i, j) &= \alpha(i-1, j) \ln B(i-1, j+1) \\
&= \alpha(i-1, j)\alpha(i-2, j+1) \ln B(i-2, j+2) \\
&= \alpha(i-1, j)\alpha(i-2, j+1)\alpha(i-3, j+2) \ln B(i-3, j+3) \\
&= \cdots \\
&= [\ln B(0, j+i)] \prod_{k=1}^{i} \alpha(i-k, j+k-1) ,
\end{aligned}$$

and allowing the instantaneous discount factors in time slice n to be written as a function of the "lowest" node $(0, i + j)$:

$$\begin{aligned}
B(i, j) &= \exp\left\{[\ln B(0, j+i)] \prod_{k=1}^{i} \alpha(i-k, j+k-1)\right\} \\
&= B(0, j+i)^{\prod_{k=1}^{i} \alpha(i-k, j+k-1)} .
\end{aligned}$$

Using this $B(i, j)$ for $j = n - i - 1$ in the arbitrage condition Eq. 14.46 we obtain, after performing an index transformation $n \to (n+1)$, the arbitrage condition for the discount factor $B(0, n)$ at the lowest node in time slice n.

$$B(t, t + (n+1)\delta t) = \sum_{i=0}^{n} G(i, n-i) \exp\left\{[\ln B(0, n)] \prod_{k=1}^{i} \alpha(i-k, n-i+k-1)\right\} \tag{14.69}$$

$$= \sum_{i=0}^{n} G(i, n-i) B(0, n)^{\prod_{k=1}^{i} \alpha(i-k, n-i+k-1)} .$$

This can only be solved *numerically* for $B(0, n)$ using, for example, the well-known *Newton-Raphson method*.[14] From this value $B(0, n)$ we obtain all further instantaneous discount factors in this time slice by repeated application of Eq. 14.68. We now have a similar situation as for normal models, see Fig. 14.3: the discount factors in the time slice n are calculated from the price of the zero bond maturing in the *following* time slice $(n+1)$ and from the volatility information from the immediately *preceding* time slice $(n-1)$.

Having established the instantaneous discount factors, the short rates follow immediately from Eqs. 14.36 and 14.68.

$$\begin{aligned} r(i, j) &= -\frac{1}{\delta t} \ln B(i, j) = -\alpha(i-1, j) \frac{\ln B(i-1, j+1)}{\delta t} \\ &= r(i-1, j+1)\alpha(i-1, j) = \ldots \\ &= r(0, j+i) \prod_{k=1}^{i} \alpha(i-k, j+k-1) . \end{aligned} \tag{14.70}$$

[14] To solve a non-linear equation of the form $f(x) = 0$, the *Newton-Raphson method* uses the following iteration to find the points were the function f equals zero: having an estimate x_i for a zero of f, a better estimate is obtained from the formula

$$x_{i+1} = x_i - f(x_i) \left(\left. \frac{\partial f}{\partial x} \right|_{x=x_i} \right)^{-1} .$$

We usually start the procedure with a rough estimate x_0 and iterate until the difference between x_{i+1} and x_i is sufficiently small for the required purpose. The iteration sequence converges if

$$\left| f \frac{\partial^2 f / \partial^2 x}{(\partial f / \partial x)^2} \right| < 1$$

holds in a neighborhood of the zero of f. This can always assumed to be the case in our applications.

Exact Reproduction of the Term Structure with the Lognormal Model

Alternatively, we can proceed from the process for the short rate in the lognormal model obtained from Ito's lemma, Eq. 14.59, and derive all equations exactly in the form obtained for the normal model, in particular Eq. 14.63, the only difference being that α is then given by

$$\alpha(i, j) = \exp\left\{-2b(i, j)r(i, j)\delta t^{3/2}\right\} . \tag{14.71}$$

The $r(i, j)$ needed for this equation were calculated in the previous iteration step, so that all terms in Eq. 14.63 are known. Thus, this equation can be solved *analytically* for the lowest bond, even though we are working within the context of the lognormal model.

Both of these two possible methods, i.e., the standard method of solving Eq. 14.69 by means of the Newton-Raphson method, as well as the more elegant method through Eqs. 14.59 and 14.63 with α as in Eq. 14.71, are demonstrated in detail in the Excel workbook TermStructureModels.xlsx from the download section [50].

With the interest rate independency of the volatility structure Eq. 14.65 holds again. Thus, Eq. 14.62 reduces to

$$\begin{aligned} B(i, j) &= B(i-1, j+1)^{\alpha(0,n-1)} \\ &= B(0, n)^{\alpha^i(0,n-1)} \quad \text{with } i + j = n \end{aligned}$$

where $\alpha(i, j) = \exp\left\{2\sigma(t + n\delta t)\sqrt{\delta t}\right\}$. Equation 14.67 implies that the short rate is simply to be multiplied by a constant factor $\alpha(0, n-1)$ when moving from one node to the next in time slice n of the tree, thus

$$\begin{aligned} r(i, j) &= r(i-1, j+1)\alpha(0, n-1) \\ &= r(0, n)\alpha^i(0, n-1) , \end{aligned} \tag{14.72}$$

and the arbitrage condition for the discount factor at the lowest node of the time slice n reduces to

$$B(t, t + (n+1)\delta t) = \sum_{i=0}^{n} G(i, n-i)B(0, n)^{\alpha^i(0,n-1)} .$$

14.9 The Example Program TermStructureModels.xlsm

14.9.1 Construction of Interest Rate Trees and Option Pricing

We now demonstrate explicitly how the interest rate tree is constructed for a given term structure, or equivalently, for given market prices of zero bonds,

$$B(t, t + i\delta t) \text{ for } \quad i = 1 \ldots n$$

and given volatilities

$$b(r, t + i\delta t) \text{ for } \quad i = 0 \ldots n - 1 \; .$$

The calculation of option prices from this interest rate tree—once it has been constructed—has already been discussed in Sect. 14.7.2 in detail for any arbitrary arbitrage-free short rate term structure model using caplets as an example.

- Time step $i = 0$
  According to Eq. 14.47, or in particular according to Eq. 14.41, the Green's function at the node (0, 0) is simply

$$G(0, 0) = 1 \; .$$

  From Eq. 14.46, it then follows that the discount factor at node (0, 0) can be obtained directly from the market price $B(t, t + 1\delta t)$ of the zero bond:

$$B(t, t + 1\delta t) = G(0, 0)B(0, 0) = B(0, 0) \; .$$

- Time step $i = 1$
  The discount factors just computed together with the Green's function evaluated at the node (0, 0) are substituted into the recursion relation Eq. 14.43 to determine the values of the Green's function at the next nodes:

$$G(1, 0) = \frac{1}{2} G(0, 0)B(0, 0)$$
$$G(0, 1) = \frac{1}{2} G(0, 0)B(0, 0) \; .$$

These values, the volatility information from the previous time slice and the price of the zero bond maturing at the next time slice are all used to obtain the discount factor at the lower boundary which here is the node (0, 1). With Eq. 14.63 for the normal model, the discount factor at this node is obtained as

$$B(0,1) = \frac{B(t, t+2\delta t)}{G(0,1) + G(1,0)\alpha(0,0)} .$$

Application of the recursion relation in Eq. 14.62 yields the other discount factor required for this step

$$B(1,0) = \alpha(0,0)B(0,1) .$$

Analogously, the arbitrage condition for the discount factor on the lower boundary in the *log*normal model is, according to Eq. 14.69,

$$B(t, t+2\delta t) = G(0,1)B(0,1) + G(1,0)B(0,1)^{\alpha(0,0)} ,$$

which can only be solved numerically (using the Newton Raphson method, for example) for $B(0,1)$. Having solved the equation, the next discount factor for this time slice can be obtained immediately using Eq. 14.68:

$$B(1,0) = B(0,1)^{\alpha(0,0)} .$$

- Time step $i = 2$
  The discount factors just calculated together with the Green's function are substituted into the recursion 14.43 to obtain the values of the Green's function evaluated at the nodes corresponding to this time step

$$G(1,1) = \frac{1}{2}G(1,0)B(1,0) + \frac{1}{2}G(0,1)G(0,1)$$
$$G(2,0) = \frac{1}{2}G(1,0)B(1,0)$$
$$G(0,2) = \frac{1}{2}G(0,1)B(0,1) .$$

Using these values as well as the volatility information from the previous time slice and the market price of the zero bond maturing at the next time slice, Eq. 14.63 for the normal model can be applied to obtain the discount factor at the lower boundary, i.e., at node (0, 2)

$$\begin{aligned} B(t, t + 3\delta t) &= B(0, 2) \sum_{i=0}^{2} G(i, 2 - i) \prod_{k=1}^{i} \alpha(i - k, 1 - i + k) \\ &= B(0, 2)G(0, 2) \\ &+ B(0, 2)G(1, 1)\alpha(0, 1) \\ &+ B(0, 2)G(2, 0)\alpha(1, 0)\alpha(0, 1)\ , \end{aligned}$$

which yields for $B(0, 2)$

$$B(0, 2) = \frac{B(t, t + 3\delta t)}{G(0, 2) + G(1, 1)\alpha(0, 1) + G(2, 0)\alpha(1, 0)\alpha(0, 1)}\ .$$

Applying the recursion formula 14.62 yields the next discount factors belonging to this time step

$$\begin{aligned} B(1, 1) &= \alpha(0, 1)B(0, 2) \\ B(2, 0) &= \alpha(1, 0)B(1, 1)\ . \end{aligned}$$

Analogously for the *log*normal model, the arbitrage condition Eq. 14.69 for the discount factors on the lower boundary is used to obtain

$$B(t, t + 3\delta t) = G(0, 2)B(0, 2) + G(1, 1)B(0, 2)^{\alpha(0,1)} + G(2, 0)B(0, 2)^{\alpha(1,0)\alpha(0,1)}\ ,$$

where again we can be solve for $B(0, 2)$ numerically. Once $B(0, 2)$ is known, the additional discount factors for this time step can be computed immediately using Eq. 14.68:

$$\begin{aligned} B(1, 1) &= B(0, 2)^{\alpha(0,1)} \\ B(2, 0) &= B(1, 1)^{\alpha(1,0)}\ . \end{aligned}$$

This procedure is repeated until the entire tree has been constructed up to maturity $T + \tau$ (maturity of the derivative to be priced plus the lifetime of the underlying). After this has been done, all required Arrow-Debreu prices can be determined using Eqs. 14.47, 14.48 and 14.49 and finally, these ADPs are used in pricing the derivative as demonstrated in Sect. 14.7.2 for caplets.

This is all demonstrated explicitly for normal models (referred to there as Ho-Lee models) and lognormal models (referred to as Black-Derman-Toy models) in the Excel workbook TermStructureModels.xlsm from the download section [50]. To emphasize that the interest rate tree is independent of the derivative being priced, the structure of the Visual Basic code is extremely modular: first, a short rate tree is generated. This remains in the main memory of the computer until it is used to compute the value of any desired underlying dependent on the term structure (for example a 3-month zero bond yield or a swap rate, etc.) using Arrow-Debreu prices and subsequently pricing any chosen (path-independent) derivative on that underlying with a European payoff mode. Plain vanilla caplets and floorlets on the 3-month rate serve as examples, with an explicit demonstration of their valuation being included in the workbook.

14.9.2 Absolute and Relative Volatilities

As already mentioned, the volatility input for lognormal models has to have the form of a *relative* volatility (for example, 14% of the current underlying value). For normal models, an *absolute* volatility (for example, 0.75 percentage *points*[15]) is required.

If, for comparison, we would like to transform absolute volatilities into relative ones and vice versa, an appropriate reference rate is required in addition. Here, the *forward rate of the underlying at the maturity of the derivative being priced* is a natural choice. This is especially true if we intend to compare the volatilities with the Black-76 volatility, since it is exactly this forward rate which is used as the underlying in the Black-76 model.[16] Therefore the Black-76 volatility quoted in the market belongs to this forward rate. For these reasons, this forward rate has been selected as the multiplicative factor in the Excel workbook TermStructureModels.xlsm.

As can be seen from Eq. 14.59, the factor of the stochastic term differs by the additional multiplication with the short rate. The product of relative

[15]At an interest rate of 6% this would correspond to a relative volatility of 14%.

[16]As emphasized in Eq. 14.55, the forward rate for the caplet period τ within *linear* compounding serves as the underlying of a cap in the payoff profile. This has already been accounted for in the term structure model through Eq. 14.55. To obtain the correct forward rate as input for the *Black-76* model in the Excel workbook TermStructureModels.xlsm from the download section [50] from the current term structure (which holds for continuous compounding), we first determine the forward *discount factors* using Eq. 2.7. From those discount factors the desired forward rates for linear compounding are given by $r = \left(B^{-1} - 1\right)/\tau$.

volatility and short rate is equivalent to the absolute volatility in a Gaussian or normal model. Since the short rate is different at each note for a given time slice, the absolute volatility is different at each node in a log-normal model, while it was the same at each node in a normal model. This is the substantial difference between both models. Indeed, it is enough to change a single line in the implementation of the Gaussian model to transform it into the implementation of a log-normal model according to Eq. 14.59.

14.9.3 Calibration of Volatilities

The current term structure and volatility structure are required as input for the construction of an interest rate tree. In practice, the volatilities $b(r, t + i\delta t)$ themselves are generally not known. Only the prices of options traded on the market (caps, floors, swaptions, etc.) can be observed directly. The tree must then be constructed, leaving the volatilities unspecified as free parameters which are then adjusted until the observed market prices of the options are reproduced by the model. Fitting the parameters $b(r, t + i\delta t)$ to the market prices in this manner is referred to as the *calibration* of the model.

There are many ways of performing such a calibration. For instance we can—exactly as was done when reproducing the market prices of zero bonds—reproduce the market prices of the options stepwise through the tree, beginning with the shortest option lifetimes and proceeding through to the longest. This procedure is demonstrated explicitly in the Excel workbook TermStructureModels.xlsm from the download section [50] where the prices of a strip of caplets at the 3-month rate are available, whose underlyings (the respective 3-month rates) cover the time span under consideration without overlap, i.e.

$$T_{k+1} = T_k + \tau \ .$$

where T_k is the maturity of the kth caplet and τ is the lifetime of the underlying. The calibration starts by assuming that the volatility in the tree between times t and $T_1 + \tau$ is constant,

$$b(r, t + i\delta t) = b(T_1) \quad \text{for all } r \text{ and all } i \text{ with } t \leq t + i\delta t \leq T_1 + \tau \ .$$

This constant volatility is adjusted (using Newton-Raphson) until the price computed using the tree equals the known market price of the first caplet.

To reproduce the second known caplet price, we assume that the volatility remains equal to the (just calibrated) $b(T_1)$ for the time between t and $T_1 + \tau$, and is constant (although still unknown) at all nodes for times between $T_1 + \tau$ and $T_2 + \tau$:

$$b(r, t + i\delta t) = b(T_2) \quad \text{for all } r \text{ and all } i \text{ with } T_1 + \tau < t + i\delta t \leq T_2 + \tau \ .$$

This constant volatility is then adjusted (again using Newton-Raphson) so that the price for the second caplet computed using the tree is equal to the observed market price of this caplet. To reproduce the third caplet price we assume, as above, that the (just calibrated) volatilities computed for the time span from t to $T_2 + \tau$ continue to be valid, and adjust the volatility at all nodes associated to the times between $T_2 + \tau$ and $T_3 + \tau$ until the price of the third caplet computed using the tree matches the observed market price of this caplet, and so on.[17] By this method, we obtain a piecewise constant function for the volatility[18] as a function of time, independent of r.

Such a calibration process yields different volatility values $b(r, t + i\delta t)$ for each different term structure model. We therefore refer to the respective volatilities by the name of the model with we are working, for example *Ho-Lee volatilities*, *Black-Derman-Toy volatilities*, *Hull-White volatilities*, etc. For this reason, it is generally not possible to simply take the Black-76 volatilities as the input values for $b(r, t + i\delta t)$. The tiresome process of calibrating to the observed option *prices* is in most cases unavoidable.

The market quotes option prices often in terms of log-normal volatilities (or, recently, normal model vols), from which prices could be calculated by applying the Black-Scholes resp. Black'76 or Bachelier (for normal vols) formula. Since all other input parameters, e.g. underlying interest rates, are commonly known, the Black-76 model is simply a translation algorithm for moving between the two different ways of quoting the option price, i.e. price vs. volatility.[19] An advantage of quoting volatilities rather than prices is that volatilities change less frequently than other parameters like interest rates. Also, volatilities offer an illustrative, easy to interpret way of comparing options with different strikes and terms to expiry. Therefore, the Black-76 model (or the

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[17] Alternatively, the volatilities can be (simultaneously) calibrated using a least squares fit. We then minimize the sum of the quadratic differences between the calculated and the traded option prices by varying the volatilities.

[18] Naturally, this procedure can be extended if a "caplet-price-surface", i.e. caplet prices with different times to maturity *and* different strikes, is available to obtain a calibrated volatility surface as a function of time *and moneyness* (the relative or absolute difference between underlying and strike).

[19] Though, in some cases, market makers quote both, prices and volatilities, at the same time.

equivalent normal model) is better understood as a quoting vehicle rather than a pricing model. The procedure behind the quoting is as follows: the market maker calculates the price of an interest rate option which she wishes to quote to the market using the term structure model of her choice (maybe calibrated to other liquid option prices). Before quoting, this price is first translated into a volatility using the Black-76 model. It is this volatility that is quoted on the market. The volatility quoted is simply that which, when used as an input parameter in the Black-76 model, reproduces the price calculated with the bank's (perhaps very complicated and proprietary) term structure model.

The calibration to given (Black-76) caplet (or floorlet) prices are demonstrated in the Excel Workbook TermStructureModels.xlsm from the download section [50]. The calibration to other instruments such as swaption prices is considerably more complicated in its implementation, but is based on the same principles:

1. Calculation of all required Arrow-Debreu prices from the existing tree.
2. Generation of the necessary underlyings, namely the swap rate under consideration, from the Arrow-Debreu prices.
3. Calculation of the payoff profiles of the swaption at the nodes corresponding to the swaption maturity date based on the value of the underlying (swap rate) and with the help of the Arrow-Debreu prices
4. Calculation of the swaption prices at node (0, 0) by discounting the payoff profile with the Green's function
5. Adjusting the volatility in the interest rate tree until the swaption price computed using the tree agrees with the price quoted on the market.

14.10 Monte Carlo on the Tree

In the discussion above, it was emphasized that only path-independent derivatives in the sense of Eq. 14.44 could be priced using the methods introduced in this chapter since the trees recombine and as a result information on the history of the short rate path is lost. However, by combining the presented trees with Monte-Carlo simulations, it becomes possible to price path-dependent derivatives with a pay off profile depending on the past realizations of the interest rate This kind of Monte-Carlo simulation works as follows:

- First observe that the tree needs to be generated (and calibrated!) only once (as described above) and then stays in the memory of the computer.

- The short rate paths are then simulated by jumping randomly from node to node in the tree, always proceeding one time step further with each jump.
- Simulating the jumps from node to node with the appropriate transition probabilities (in our case $p = 1/2$ for up as well as for down moves, see Eq. 14.32) of the chosen martingale measure (in our case the risk-neutral measure, see Eq. 14.31) ensures that the simulated paths already have the correct probability weight needed for pricing financial instruments. This procedure is called *importance sampling*.[20]
- For each simulated path of the short rate the corresponding path of the underlying (for instance 3-month LIBOR) must be calculated using Arrow-Debreu prices.
- At the end of each simulated path the payoff of the derivative resulting from the underlying having taken this path is calculated.
- This payoff is then discounted back to the current time t (i.e., to node $(0, 0)$) at the short rates *along the simulated path*, since after all, we still are within the risk-neutral measure, see Eq. 14.31.
- After many (usually several thousand) paths have been simulated, the (several thousand) generated discounted payoff values can then simply be averaged to yield an estimation for the risk-neutral expectation of the discounted payoff. Here the very simple arithmetic average (with equal weights) of the discounted payoff values can be used without worrying about the correct probability weight of each payoff value since the payoff values (more precisely the paths which generated the payoff values) have already been *simulated* with the correct (risk-neutral) probability. This is (besides the effectiveness in sampling the phase space) another great advantage of importance sampling.
- According to Eq. 14.8, the risk-neutral expectation calculated in this way is directly the desired derivative price.

[20] Doing a move with its associated probability only ensures a very effective sampling of the *phase space* (the set of all possible values of the simulated variables): phase space regions (values of the simulated variables) which have low probabilities (and therefore contribute only little to the desired averages of whatever needs to be measured by the simulation) are only visited with low probability (i.e. rarely) while phase space regions with high probabilities (which contribute a lot to the desired averages) are visited with high probability (i.e. often). Because of this feature *importance sampling* is heavily used in thousands of Monte-Carlo applications, especially in physics, meteorology and other sciences which rely on large-scale simulations.

14.11 The Drift in Term Structure Models

At this stage, it makes sense to reconcile the practical computations in Chap. 14 with the profound concepts from the related theory presented in Chap. 13. We first consider the relationship between the underlying instantaneous interest rates and tradable instruments such as zero bonds. We defer proceeding with the short rate models introduced above to take a brief excursion to the models in which the instantaneous *forward* rate plays the role of the underlying.

14.11.1 Heath-Jarrow-Morton Models

We see from the fundamental Eqs. 14.2 or 14.3 and 14.5 that the instantaneous *forward* rates can be used to generate the prices $B(t, T)$ of all zero bonds as well as the entire interest rate term structure $R(t, T)$. The three descriptions of the term structure, the zero bond prices $B(t, T)$, the zero bond yields $R(t, T)$ and the instantaneous forward rates $f(t, T)$ are equivalent. Of these three variables, only a single one needs to be chosen to be modeled by a general stochastic process of the form specified in Eq. 2.19. We take, for example, the forward rates $f(t, T)$ to be modeled by a process of the form:

$$df(t, T) = a(t, T)\, dt + b(t, T)\, dW \quad \text{with} \quad dW = X\sqrt{dt},\ X \sim \mathrm{N}(0, 1)\ . \tag{14.73}$$

As mentioned previously, all bond prices and thus the entire term structure can be generated from the solution of this equation. An entire class of term structure models, the *Heath-Jarrow-Morton models* (*HJM models* for short) take this approach of employing the forward rates as the driving factor of the term structure.

Like all interest rates, the instantaneous forward rates are not tradable (see Sect. 14.4). In Chap. 13, and in particular in Sects. 13.4 and 13.5, a detailed discussion can be found on how to proceed when the underlying is not tradable; we choose a tradable instrument U whose price $U(S, t)$ is a *function* of the underlying. Then all of the results shown in Chap. 13 hold:

- The Harrison-Pliska Theorem establishes the *uniqueness* (in complete markets) of the probability measure with respect to which the prices of tradable financial instruments normalized with an arbitrarily chosen, tradable numeraire instrument Y are martingales.
- According to the Girsanov Theorem, this implies that there is only *one single underlying drift* which may be used for pricing.

- In addition, the drift of the underlying in the real world plays absolutely no role in the world governed by the martingale measure if the numeraire instrument Y satisfies the property 13.34 (which is always the case).

The existence of a function $U(S,t)$ relating the underlying S to a tradable instrument U is the deciding factor for the validity of the above results. If the underlying is the instantaneous forward rate $f(t,T)$, such a functional relation to a tradable instrument exists, namely Eq. 14.3 for the zero bond price $B(t,T)$, and thus the results found in Chap. 13 hold for the Heath-Jarrow-Morton models. In particular, for every complete market, there exists for each numeraire instrument exactly one single underlying drift which may be used for pricing. Therefore, as far as pricing is concerned, the HJM model is uniquely determined by the specification of the volatility term $b(t,T)$ in Eq. 14.73 (which models the process in the real world). The Girsanov Theorem implies that the transition from the real world into the world governed by the martingale measure only effects a (in this case unique) change in the drift; the volatility term $b(t,T)$ is invariant under this transformation. Indeed, the forward rate process in the risk-neutral measure corresponding to the real world process in Eq. 14.73 has the following appearance [92]:

$$df(t,T) = \left[b(t,T) \int_t^T b(t,s)ds \right] dt + b(t,T)\, d\widetilde{W} \; ,$$

Here $d\widetilde{W}$ denotes the standard Brownian motion with respect to the martingale measure. The coefficient of dt appearing in square brackets is the drift with respect to the martingale measure. This formula shows *explicitly* that for HJM models, the entire model (including the drift) is uniquely specified through the volatility $b(t,T)$. The drift to be used in the valuation is *unique,* in complete agreement with the general statements made in Chap. 13.

14.11.2 Short Rate Models

As opposed to the HJM models, the term structure models in Sect. 14.6 make use of the instantaneous *spot* rate defined in Eq. 14.1 as the driving factor. However, a one-to-one mapping between the spot rates and the zero bond prices does *not* exist and in consequence, *no* one-to-one mapping between the spot rates and the term structure $R(t,T)$ can exist either. The instantaneous spot rates are not sufficient to generate the term structure. This is indicated by the fact that the instantaneous spot rate $r(t)$ is a function of a single time variable t in contrast to the processes $B(t, T)$, $R(t, T)$ (and $f(t, T)$ as well!), which are functions of *two* time variables. Taking the limit $dt \to 0$ in the definition of the instantaneous short rate in Eq. 14.1 results in the loss of the second argument (and thus in the loss of the corresponding information). This can be seen explicitly in Eq. 14.6. For this reason, there is *no* analogy to Eq. 14.3 relating the instantaneous spot rates directly to the zero bond prices. The best possible alternative available is to determine the bond prices from the *expectations* of the short rates (see for example Eq. 14.9), but not directly as a *function* of the short rates. This has significant consequences:

The results presented in Chap. 13, in particular those in Sects. 13.4 and 13.5 (regarding the martingale measure, unique drift, etc.), can be shown for non-tradable underlyings only if there exists a tradable instrument whose price process is a *function* of the underlying. This *direct functional relationship* between the underlying (the instantaneous spot rate) and a tradable instrument is *missing* in short rate models. Or from the view point of the Harrison-Pliska Theorem: since $r(t)$ contains less information than $f(t, T)$ or $B(t, T)$, the market is not complete for short rate models. For this reason, the martingale measure in short rate models is *not uniquely determined* by fixing the numeraire instrument. Thus, the Girsanov theorem asserts that we retain the freedom of choosing from various drift terms in the model. The information lost in the transition shown in Eq. 14.6, for example, must be reinserted into the model "by hand". This is accomplished by directly specifying a drift in the world governed by the *martingale measure.* This is the essential difference in the models here compared to those encountered in the previous chapters, where the drift was always specified in the *real* world. The situation for short rate models is different; the drift is specifically chosen for the world governed by the *martingale measure* rather than for the real world. Only through this drift specification is the martingale measure uniquely determined in short rate models.

Frequently, it is seen as a requirement that the has drift shows an effect called *mean reversion* which is observed in the evolution of interest rates but not seen in e.g., stock prices. Interest rates do not rise or fall to arbitrarily high or low levels but tend to oscillate back and forth about a long-term mean. This can be modeled with a drift in the functional form $\mu - vr$ for some $v > 0$: for values of r small enough so that $vr < \mu$ holds, the drift is positive and, consequently, r tends on average toward larger values. Conversely, for values of r large enough so that $vr > \mu$ holds, the drift is negative and r tends to drift toward smaller values on average. The interest rate thus tends to drift toward a mean value μ at a rate ν. Naturally, a stochastic component driven by $\sim dW$ is superimposed onto this deterministic movement. An example of a *mean reversion model* is the *Hull-White model* [104]. This model specifies the following stochastic process for the short rate with respect to the *risk-neutral martingale measure:*

$$dr(t) = [\mu(t) - v(t)r]\, dt + \sigma(t)\, dW \qquad (14.74)$$

This is only one of many examples. Several well-known models, each named after their respective "inventors" can be distinguished from one another, after having separated them into categories of normal and lognormal models, essentially through the form of their drift. The best known representatives of these models are summarized in the following list:

- Normal models $dr(t) = a\,(r, t)dt + b(r, t)\, dW$
  - Stationary models $b(r, t) = \sigma$
    - Arbitrage-free models
      - Hull-White $a\,(r, t) = \mu(t) - vr$ (mean reverting)
      - Ho-Lee $a\,(r, t) = \mu(t)$
    - Equilibrium models (not arbitrage free because of too few degrees of freedom)
      - Vasiceck $a\,(r, t) = \mu - vr$ (mean reverting)
      - Rendleman-Barter $a\,(r, t) = \mu$
  - Non-stationary models $b(r, t) = \sigma(t)$
- Lognormal models $d \ln r(t) = a\,(r, t)dt + b(r, t)\, dW$
  - Stationary models $b(r, t) = \sigma$
  - Non-stationary models $b(r, t) = \sigma(t)$
    - Arbitrage-free models
      - Black-Karasinski $a\,(r, t) = \mu(t) - v(t) \ln r$ (mean reverting)
      - Black-Derman-Toy $a\,(r, t) = \mu(t) - \frac{\partial\sigma(t)/\partial t}{\sigma(t)} \ln r$

The models are called either *stationary*[21] or *non-stationary* depending on whether or not the volatility is assumed to be a function of time.[22] The models

[21]This is not to be confused with the definition of stationary time series in Chap. 32.

[22]Ho-Lee and Hull-White are often applied for time-dependent volatilities. Their inventors, however, originally assumed constant volatilities.

allowing for neither a time dependence nor an r dependence in the drift or volatility terms (Vasiceck, Rendleman-Barter) obviously cannot reproduce the current term structure arbitrage-free. For these models, only a "best fit" can be obtained (for example by minimizing the *root mean square error*). These models are referred to as *equilibrium models*. Since equilibrium models are not arbitrage free, they will receive no further attention in this book. Nowadays they are no longer very important in practice.

The volatility of dr is independent of r in normal models and proportional to r in lognormal models (see, Eq. 14.59). An intermediate scheme between these two possibilities is the *Cox-Ingersoll-Ross* model [42] for which the volatility is assumed to be proportional to $\sqrt{r}$.

$$dr = (\mu - vr)dt + \sigma\sqrt{r}dW \ .$$

Note that all these processes are modeled in the *risk-neutral martingale measure* directly, i.e., should be used directly in the valuation of the financial instrument under consideration without first performing a drift transformation through an application of the Girsanov Theorem. The volatility term is invariant under the Girsanov transformation. This implies that the volatility taken for the valuation is the same as that observed in the real world. The form taken on by the drift, however, is a result of the particular choice of the measure (coordinate system) established for pricing through an application of the Girsanov Theorem and as such, a rather artificial construct. It is thus not readily apparent why a specific form of drift (for example mean reversion) should be modeled in a specific (for example risk-neutral) artificial world (dependent on the selection of a particular numeraire) when our intuitive conception of the drift actually pertains to the *real* world.

Or more precisely: according to the Girsanov Theorem, the process modeled with respect to the martingale measure differs from the real world process by a previsible process $\gamma(r,t)$. This previsible process is *arbitrary* (with the restriction that it must satisfy the boundedness condition $\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma(r,t)dt\right)\right] < \infty$). Therefore the choice of model with respect to a martingale measure provides as good as no information about the drift of the short rate in the *real* world. For example, the model given by Eq. 14.74 has a mean reversion in the world governed by the martingale measure, but has the form

$$dr(t) = [\mu(t) - v(t)r(t) + \gamma(r,t)]\ dt + \sigma(t)\, dW$$

in the real world with a (practically) *arbitrary* previsible process $\gamma(r, t)$. It is therefore by no means clear that this process shows any mean reversion in the real world.

After having placed such emphasis on the necessity of specifying a drift, i.e., a martingale measure, explicitly in short rate models (despite the results in Chap. 13 the martingale measure is not unique here, even when the numeraire has been fixed), the attentive reader will surely have asked why only the volatility but no drift information has entered as input into our explicit computations in Sect. 14.8. For both the normal and the lognormal models, the interest rate trees were constructed in their entirety and no drift information from the specific stochastic processes was needed at any point.

This stems from the condition in Eq. 14.32, which we introduced "by hand" for the sake of simplicity; the probability of an up move was simply *set* to $p = 1/2$. Through this choice, we have explicitly selected one particular measure from the family of arbitrage-free martingale measures belonging to the risk-neutral numeraire instrument (the bank account). This fixed the drift in accordance with the Girsanov Theorem. We can also see this fact explicitly since the drift can be quite simply determined from the generated tree. As was shown in Sect. 14.8 for the variance, it follows from the general equation 14.60 for a binomial tree with $p = 1/2$ that the expectations of the short rate as seen from the node (i, j) are

$$
\begin{aligned}
a(i, j)\delta t &= \mathrm{E}\left[r(i, j)\right] = \tfrac{1}{2}\left[r(i+1, j) + r(i, j+1)\right] && \text{normal model} \\
a(i, j)\delta t &= \mathrm{E}\left[\ln r(i, j)\right] = \tfrac{1}{2}\left[\ln r(i+1, j) + \ln r(i, j+1)\right] && \text{log-normal model,}
\end{aligned}
$$

from which the drift at each node in the tree can be immediately determined since all the short rates have already been established (the tree has already been built).

For example, Eq. 14.66 can be used to compute the drift in a normal model with r-independent volatility as

$$
a(i, j)_{\text{normal}} = \frac{r(0, i+j+1)}{\delta t} + (2i+1)\frac{\sigma(t + (i+j)\delta t)}{\sqrt{\delta t}} \, .
$$

Analogously, the drift for *log*-normal models with r-independent volatility $b(0, n) = \sigma(t + n\delta t)$ can be computed explicitly, employing Eq. 14.72 and α as in Eq. 14.68:

$$\begin{aligned} a(i, j)_{\text{log-normal}} &= \frac{1}{2\delta t}\left[\ln r(i+1, j) + \ln r(i, j+1)\right] \\ &= \frac{1}{\delta t}\left[\ln r(0, i+j+1) + (i + \frac{1}{2})\ln \alpha(0, i+j)\right] \\ &= \frac{\ln r(0, i+j+1)}{\delta t} + (2i+1)\frac{\sigma(t + (i+j)\delta t)}{\sqrt{\delta t}} \; . \end{aligned}$$

Had the probability of an up move not been fixed at 1/2, a free parameter p would remain unspecified in Eq. 14.60 which (if dependent on time and perhaps on r as well) would allow for many different drift functions.

14.12 Short Rate Models with Discrete Compounding

The discount factor over a single time period used in this chapter was always of the form $B(t, t + \delta t) = e^{-r(t)\delta t}$ (see for example Eq. 14.36). Intuitively, in view of Eq. 2.3, this means that interest has been paid infinitely often in the reference period δt, and that these payments were then immediately reinvested at the same rate. Strictly speaking, this contradicts the concept of a tree model, for which time has been *discretized* into intervals of positive length δt, implying by definition that nothing can happen in between these times. To be consistent, we should have therefore used *discrete* compounding, allowing the payment and immediate reinvestment of interest solely after each δt. Then, only in the limiting case $\delta t \to 0$ will the discount factor for continuous compounding be obtained. If we wish to be consistent, we would therefore have to write

$$B(t, t + \delta t) = \frac{1}{1 + r(t)\delta t} \underset{\delta t \to 0}{\longrightarrow} e^{-r(t)\delta t} \; .$$

Despite the inconsistency, the discount factor for continuous compounding is commonly used in the literature. In *this* section, we will collect and present the differences caused by using discrete rather than continuous compounding and, in doing so, show how short rate models with discrete compounding can be treated. The discount factor in Eq. 14.36 has the following form when adopting the convention of discrete compounding

$$B(i, j) = \frac{1}{1 + r(i, j)\delta t} . \tag{14.75}$$

Formulating recursion equations as in Eqs. 14.62 or 14.68 for these discount factors $B(i, j)$ in a time slice or for the lowest zero bond in a time slice (see Eqs. 14.63 and 14.69) is quite awkward. Such conditions are more easily formulated for the *interest rate* $r(i, j)$. All recursion relations for the short rate follow from the arbitrage condition for the Green's function, Eq. 14.46. This expression for the instantaneous discount factor in the form of Eq. 14.75 is

$$B(t, t + n\delta t) = \sum_{i=0}^{n-1} \frac{G(i, n - i - 1)}{1 + r(i, n - i - 1)\delta t} . \tag{14.76}$$

14.12.1 Normal Models

If the short rate $r(i, j)$ in Eq. 14.75 is governed by a stochastic process of the form given in Eq. 14.57, then Eq. 14.61 holds. The recursion relation for the short rate in the normal model is given by Eq. 14.64:

$$\begin{aligned} r(i, j) &= r(i - 1, j + 1) + 2b(i - 1, j)\sqrt{\delta t} \\ &= r(0, j + i) + 2\sqrt{\delta t} \sum_{k=1}^{i} b(i - k, j + k - 1) . \end{aligned} \tag{14.77}$$

Substituting this into Eq. 14.76 with $j = n - i - 1$ and performing the transformation $n \to n + 1$, we obtain a condition analogous to Eq. 14.63 for the interest rate $r(0, n)$ at the lowest node in the time slice n

$$B(t, t + (n + 1)\delta t) = \sum_{i=0}^{n} \frac{G(i, n - i)}{1 + r(0, n)\delta t + 2\delta t^{3/2} \sum_{k=1}^{i} b(i - k, n - i + k - 1)} .$$

This equation can only be solved numerically for $r(0, n)$. Once this value is known, Eq. 14.77 provides all other $r(i, j)$ on the time slice n. From the $r(i, j)$, the discount factors can then be calculated immediately using Eq. 14.75. Note that in the case of discrete compounding, the arbitrage condition, Eq. 14.76, can no longer be solved analytically, not even in the context of the normal model.

14.12.2 Lognormal Models

If the short rate $r(i, j)$ in Eq. 14.75 is governed by a stochastic process of the form 14.58, then Eq. 14.67 holds. The recursion relation for the short rate in the lognormal model is given by Eq. 14.70:

$$\begin{aligned} r(i+1, j) &= r(i, j+1) \exp\left\{2b(i, j)\sqrt{\delta t}\right\} \\ &= r(0, j+i) \prod_{k=1}^{i} \alpha(i-k, j+k-1) \end{aligned} \tag{14.78}$$

with $\alpha(i, j)$ as defined in Eq. 14.68. Substituting this into Eq. 14.76 with $j = n - i - 1$, and performing the transformation $n \to n + 1$, we obtain the condition on the interest rate $r(0, n)$ at the lowest node in the time slice n analogous to Eq. 14.69

$$B(t, t + (n+1)\delta t) = \sum_{i=0}^{n} \frac{G(i, n-i)}{1 + r(0, n)\delta t \prod_{k=1}^{i} \alpha(i-k, n-i+k-1)} .$$

Again, this can only be solved for $r(0, n)$ numerically. Once this has been done, the other values $r(i, j)$ in the time slice n can be calculated immediately using Eq. 14.78. The discount factors are finally determined from the $r(i, j)$ using Eq. 14.75.

14.13 Other Interest Rate Models

The world of interest rate models is not limited to the models presented so far. Instead, there are a couple of other models used in practice. For example, a further, quite important class of models are market models, which in contrast to Heath Jarrow Morton models use forward rates for finite periods as driving factors. The *LIBOR market model* plays here an especially dominant role. Such a model may use, e.g., 6M LIBOR rates as driving factors. For a time horizon of 30 years, a model with 60 non-overlapping forward rates as stochastic factors could be constructed. Each forward rate follows a log-normal (or, alternatively, normal) process, which are not independent, but correlated with each other. In general, the number of independent stochastic drivers is reduced by means of a principal component analysis (see Chap. 34) to a few factors only (typically, 3 or 4). Each forward rate is a martingale in its own T measure, where T is the end date of the period the interest rate belongs to. By application of a unique numeraire to all 60 forward rates, a drift term is implied by the change of numeraire. For an extensive discussion of the LIBOR market model see for example [4, 21].